Arithmetic Lefschetz and Integral Heights
Proofs and Reductions on Products of Curves

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Abstract

For a product of pointed curves, the tensor Chow projectors turn arithmetic hard Lefschetz into an exact vanishing condition on the all-h^1 groups of its subproducts. A primitive sum-of-squares identity then reduces positivity and integral finite-height questions to the central degree. In that degree, a self-transpose Chow projector separates a Néron–Tate summand from a canonical Griffiths pairing. We prove that central integral Northcott is equivalent to injectivity of the algebraic Abel–Jacobi map and integral Northcott on this residual quotient. We also construct residual arithmetic regulators and prove criteria for the common integral lattice and for the fields of definition needed in a Northcott argument. The height statements concern a system satisfying the four explicit axioms below.

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1 Introduction

A height pairing on homologically trivial cycles is expected to satisfy arithmetic analogues of hard Lefschetz and Hodge–Riemann. Finiteness of integral height balls asks for additional arithmetic control. On a product of curves, these questions can be separated using its exact tensor Chow decomposition: the position of a Lefschetz string is geometric, its signed height is central, and its integral coefficients have one fixed denominator bound.

Let F be a number field and let

$$X=\prod_{i=1}^{n} C_i, \qquad o_i \in C_i(F), \qquad H_i = p_i^*[o_i], \qquad \ell=\sum_{i=1}^{n} a_i H_i \quad (a_i \in \mathbb{Z}_{>0}).$$

Each C_i is smooth, projective and geometrically connected. For a finite extension E/F, write $V^r(X_E)=\mathrm{CH}^r(X_E)_{\mathrm{hom},\mathbb{Q}}$. For $S\subseteq\{1,\dots,n\}$, put

$$Y_S=\prod_{i\in S} C_i, \qquad \Pi_S=\bigotimes_{i\in S}(\Delta_{C_i}-o_i\times C_i - C_i\times o_i),$$

$$U_S^c(E)=\Pi_{S*}V^c(Y_{S,E}).$$

The empty subproduct has zero homologically trivial Chow groups. A nonempty core U_S^c is central when $|S|=2c-1$.

1.1 Heights and integral classes

Fix an algebraic closure $\overline{F}$ of F and an embedding into $\mathbb{C}$. Homological triviality is defined by geometric rational Betti cohomology. All Chow groups below have rational coefficients unless otherwise indicated. We regard $V^r(X_E)$ as a subspace of $V^r(X_{\overline{F}})$: restriction is injective because a rational equivalence descends to a finite extension and trace composed with restriction multiplies by its degree.

Definition 1.1. A curve-normalized height system consists of pairings on all smooth projective varieties used below, including the subproducts and the source curves of correspondences. For such a variety X of dimension n, these are

$$h_X : V^r(X_E)\times V^{n+1-r}(X_E)\longrightarrow\mathbb{R}$$

over every finite extension E/F, satisfying:

(H1) bilinearity over $\mathbb{Q}$ and symmetry under exchange of the complementary degrees;

(H2) invariance under extension in the absolute normalization, and covariance under isomorphisms, including Galois conjugation;

(H3) correspondence adjunction

$$h_X(\Gamma_* z, w) = h_Y(z, {}^t\Gamma_* w);$$

(H4) on a pointed curve C, the form $-h_C$ on $\mathrm{Pic}^0(C)_{\mathbb{Q}}$ is the positive Néron–Tate form for the canonical principal polarization, in the corresponding Poincaré normalization.

These hypotheses will be understood in every statement involving h. A real positive form means positive definiteness after tensoring the rational group over a fixed number field with $\mathbb{R}$. For codimension r on X, the symbol $\mathrm{AJ}_{\mathbb{Q}}$ denotes the complex Abel–Jacobi map with target $J^r(X_{\mathbb{C}})(\mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let $\Lambda^r_X(E)$ be the image in $V^r(X_{\overline{F}})$ of the integral-coefficient cycles over E with zero geometric rational cycle class. Thus torsion and rational equivalence are removed before counting. For a fixed rational projector P, we use its projected integral image $P_*\Lambda^r_X(E)$; for an integral projector this equals $\Lambda^r_X(E) \cap P_* V^r(X_E)$.

Definition 1.2. For compatible integral images $\Lambda(E)$ in one geometric rational group and a quadratic form q, bounded-degree Northcott means that, for every $B \in \mathbb{R}$ and integer $D \geq 1$,

$$\{\xi \in \bigcup_{[E:F] \leq D} \Lambda(E) : q(\xi) \leq B\}$$

is finite. Fixed-field Northcott means the same assertion with E fixed. Bounded denominators mean the union of $d^{-1}\Lambda(E)$ for $1 \leq d \leq d_0$, with d_0 fixed.

Northcott for integral images implies the bounded-denominator version by scaling. A fixed initial extension F_0/F is harmless for bounded-degree assertions: after replacing E by EF_0, $[EF_0 : F_0] \leq [E : F]$, the integral image remains integral, and the absolute height is unchanged.

1.2 The principal reduction

For an odd subproduct $|S| = 2c - 1$, set

$$A_{S,E} = U^c_S(E) \cap \mathrm{CH}^c(Y_{S,E})_{\mathrm{alg},\mathbb{Q}}, \qquad G_{S,E} = U^c_S(E)/A_{S,E}.$$

Here algebraic triviality is geometric. The construction in Proposition 3.2 supplies actual self-transpose projectors $e_S \leq \Pi_S$ and $R_S = \Pi_S - e_S$. After one finite extension F_0/F, these projectors are defined for all odd subproducts. The e_S-summand is represented by a fixed abelian variety, and

$$R_S A_{S,E} = \ker(\mathrm{AJ}_{\mathbb{Q}}|_{A_{S,E}}), \qquad R_S U^c_S(E)/R_S A_{S,E} \simeq G_{S,E}.$$

Orthogonalization defines a canonical residual pairing $h_{G,S}$ on $G_{S,E}$; write $q_{G,S}(\xi) = (-1)^c h_{G,S}(\xi, \xi)$. Let $\Lambda_{G,S}(E)$ be the image of $\Pi_{S*}\Lambda^c_{Y_S}(E)$ in this quotient.

For $1 \leq r \leq \lfloor (n+1)/2 \rfloor$, define

$$P^r_\ell(X_E) = \ker\big(\ell^{n+2-2r} : V^r(X_E) \longrightarrow V^{n+2-r}(X_E)\big),$$
$$q_{\ell,r}(z) = (-1)^r h_X(z, \ell^{n+1-2r} z).$$

Its integral image is $\Lambda^r_X(E) \cap P^r_\ell(X_E)$.

Theorem 1.3 (The product and central reductions)**.** *Let X, ℓ, F_0 be as above, and let h be a curve-normalized height system. The following conditions are equivalent.*

(i) For every finite E/F_0 and every $1 \le r \le \lfloor (n+1)/2 \rfloor$, the map

$$\ell^{n+1-2r} : V^r(X_E) \xrightarrow{\sim} V^{n+1-r}(X_E)$$

is an isomorphism; and $q_{\ell,r}$ has bounded-degree Northcott on the primitive integral images, in every such degree.

(ii) The following three assertions hold for the subproducts of X:

(A) $U_S^c(E) = 0$ for every finite E/F_0 whenever $|S| \neq 2c-1$.

(B) For every $|S| = 2c-1$ and finite E/F_0, the rational Abel–Jacobi map is injective on $A_{S,E}$.

(C) For every $|S| = 2c-1$, $q_{G,S}$ has bounded-degree Northcott on $\Lambda_{G,S}$.

Either condition implies real positive definiteness of every primitive form and nondegeneracy of every complementary height pairing over each fixed number field.

The proof is completed in Section 7. The three assertions in (ii) are formulated separately as conjectures in the companion programme [1]. The theorem identifies their combined consequence on the whole product.

1.3 The proof and the remaining arithmetic inputs

The proof has three stages. First, the exact Chow projectors identify each summand with

$$U_S^c(E) \otimes_{\mathbb{Q}} \mathbb{Q}[x_i : i \notin S]/(x_i^2).$$

Multiplication by ℓ acts on the square-zero factor. A noncentral core places the centre of its string in the wrong degree, forcing a failure of injectivity or surjectivity. This proves the Lefschetz equivalence in Section 2.3.

Second, adjunction pairs only complementary point and fundamental-class indices. In a central block, with $m = n - |S|$, $t = r - c$, $I = \{1, \dots, n\} \setminus S$, and $z_S = \sum_{|T|=t} u_T x_T$ primitive, it gives

$$q_{\ell,r}(z_S) = (m-2t)! \prod_{i \in I} a_i \sum_{|T|=t} (-1)^c h_{Y_S}(u_T, u_T). \tag{1.1}$$

The exact sign and factorial are proved in Section 2.4. This identity on a central block uses no off-central vanishing hypothesis.

Third, the central projector separates the classical Néron–Tate summand and the residual Griffiths group. The algebraic Abel–Jacobi kernel is precisely the algebraic height radical. One fixed correspondence detects the integral classes on the visible summand, and one denominator clears the residual projector. These facts prove the central Northcott criterion in Section 3.3. Integral Northcott itself forces real positive definiteness by the rounding argument of Lemma 5.1.

The later sections address the arithmetic inputs to residual Northcott. The actual residual projector isolates one Hochschild–Serre row and defines a regulator $G_E \to H^1(E, V_p)$. Weight -1 makes the entire target finite-dimensional; scalar-extended injectivity would then give finite rational rank. The lattice criteria identify the further common-denominator condition.

Finally, trace and Hermite–Minkowski show that fixed-field Northcott upgrades to bounded-degree Northcott exactly when the fields of the bounded-height classes ramify in one finite set.

The arithmetic Lefschetz and primitive positivity assertions fit the standard conjectural framework recorded in [10, Conjecture 2.3.1]. The regulator construction uses the continuous-cohomology framework of [8, 4]. The results below give the reductions, constructions and arithmetic criteria in this setting.

2 Lefschetz strings and primitive heights

The tensor projectors of the curves give exact Chow summands. In each summand, the product polarization acts on a square-zero polynomial algebra. Its Lefschetz strings detect the central degree.

Theorem 2.1 (Arithmetic Lefschetz reduction)**.** *Fix positive integers a_i and $\ell = \sum_i a_i H_i$. For every finite E/F, the following conditions are equivalent:*

(i) $U_S^c(E) = 0$ *for every S, c with $|S| \neq 2c-1$.*

(ii) For $1 \leq r \leq \lfloor (n+1)/2 \rfloor$,

$$\ell^{n+1-2r} : V^r(X_E) \xrightarrow{\sim} V^{n+1-r}(X_E).$$

In particular, condition (ii) for one choice of positive integers a_i is equivalent to condition (ii) for every such choice, with the points o_i fixed.

2.1 Exact Chow blocks

On a pointed curve put

$$\pi_0 = o \times C, \qquad \pi_2 = C \times o, \qquad \pi_1 = \Delta_C - \pi_0 - \pi_2.$$

They are orthogonal projectors, with ${}^t\pi_0 = \pi_2$ and ${}^t\pi_1 = \pi_1$. For a partition $[n] = S \sqcup T \sqcup U$, define

$$P_{S,T,U} = \bigotimes_{i \in S} \pi_{1,i} \otimes \bigotimes_{i \in T} \pi_{2,i} \otimes \bigotimes_{i \in U} \pi_{0,i}.$$

Let j_T insert the point $[o_i]$ on factors in T and the fundamental class $[C_i]$ on factors in U. Its inverse on this projector image is the fixed correspondence

$$b_T(z) = \Pi_{S*} p_{S*} \left(z \cdot \prod_{i \in U} H_i \right).$$

Indeed $b_T j_T = \Pi_S$ and $j_T b_T = P_{S,T,U}$. Therefore

$$V^r(X_E) = \bigoplus_{S,T,U} j_T U_S^{r-|T|}(E). \tag{2.1}$$

These identities are also valid on integral images. All intersections and pushforwards in b_T have integral coefficients.

Put $I = [n] \setminus S$, $m = |I|$, and

$$B_I = \mathbb{Q}[x_i : i \in I]/(x_i^2), \qquad \deg x_i = 1.$$

For $T \subset I$, write $a_T = \prod_{i\in T} a_i$. The identification

$$u \otimes x_T \longmapsto a_T j_T u \tag{2.2}$$

groups (2.1) as

$$V^*(X_E) = \bigoplus_{S,c} U_S^c(E) \otimes B_I[c],$$

where a vector in $U_S^c \otimes B_I^t$ has codimension $c + t$. Multiplication by $\ell = \sum_i a_i H_i$ becomes multiplication by $E_I = \sum_{i\in I} x_i$ on the second factor. On the other factors it vanishes because $H_i \pi_{1,i} = 0$ as a Chow correspondence. This last identity follows by composing $\mu_{[o_i]} = [o_i] \times [o_i]$ with the three curve projectors.

If z is integral, its coefficient u_T in (2.2) satisfies $a_T u_T \in \Lambda_{U_S^c}(E)$. Thus the single integer

$$N_a = \prod_{i=1}^{n} a_i \tag{2.3}$$

clears every coefficient denominator, in every degree and over every finite extension.

2.2 A square-zero Lefschetz calculation

On B_I, use the monomial basis x_T and define

$$Ex_T = \sum_{i\notin T} x_{T\cup\{i\}}, \qquad Fx_T = \sum_{i\in T} x_{T\setminus\{i\}}, \qquad Hx_T = (m - 2|T|)x_T.$$

Then $[F, E] = H$, $[H, F] = 2F$, and $[H, E] = -2E$. The operators E, F are adjoint for the positive inner product in which the monomials are orthonormal. Let $*x_T = x_{I\setminus T}$.

Lemma 2.2. *For $0 \le t \le m/2$,*

$$E^{m-2t} : B_I^t \xrightarrow{\sim} B_I^{m-t}, \qquad \ker(E^{m-2t+1}|_{B_I^t}) = \ker(F|_{B_I^t}).$$

For every rational vector space M and every $z \in M \otimes \ker(F|_{B_I^t})$,

$$E^{m-2t} z = (-1)^t (m - 2t)! * z. \tag{2.4}$$

Proof. The displayed commutators give the standard finite-dimensional $\mathfrak{sl}_2$ representation on B_I. More explicitly, if $Fv = 0$, $v \in B_I^t$, then

$$FE^j v = j(m - 2t - j + 1)E^{j-1} v.$$

Adjointness shows that the string generated by a nonzero such v has nonzero terms through $E^{m-2t}v$ and then terminates. Decomposing successively into $\ker F$ and $\operatorname{im} E$, which are orthogonal, gives the Lefschetz strings and the first two assertions.

For the exact constant, write $z = \sum_{|T|=t} u_T x_T$. From $Fz = 0$ one obtains, for every $J \subset I$ with $|J| < t$,

$$\sum_{\substack{T\supset J\\ |T|=t}} u_T = 0.$$

This follows by looking at the coefficient of x_J in $F^{t-|J|}z$, which is $(t - |J|)!$ times the sum. If $K \subset I$ has size $m - t$ and $R_0 = I \setminus K$, inclusion–exclusion gives

$$\sum_{\substack{T\subset K\\ |T|=t}} u_T = \sum_{J\subset R_0} (-1)^{|J|} \sum_{\substack{T\supset J\\ |T|=t}} u_T = (-1)^t u_{R_0}.$$

The coefficient of x_K in $E^{m-2t}z$ is $(m - 2t)!$ times the left side. This proves (2.4), including $t = 0$. □

2.3 Proof of the arithmetic Lefschetz reduction

We prove Theorem 2.1. Suppose first that all the noncentral cores vanish. On a remaining block put $k = |S| = 2c-1$, $m = n-k$, and $t = r-c$. Then

$$n+1-2r = m-2t,$$

and the desired map is $1 \otimes E^{m-2t}$. It is an isomorphism by Lemma 2.2. If $t < 0$, both its source and target in this block vanish. Taking the direct sum proves (ii).

Conversely, suppose $U_S^c(E) \neq 0$ and put $d = k+1-2c$. The homologically trivial groups force $1 \leq c \leq k$. If $d > 0$, take $r = c$. The required map has exponent $m+d > m$, so it kills $U_S^c(E) \otimes 1$; it cannot be injective. If $d < 0$, take $r = k+1-c$. This satisfies $1 \leq r \leq \lfloor (n+1)/2 \rfloor$. The target has the nonzero component $U_S^c(E) \otimes x_I$ in degree $c+m$, whereas the source in this same block has polynomial degree $r-c = d < 0$ and is zero. Multiplication by ℓ preserves the block, so surjectivity fails. Hence $d = 0$ for every nonzero core. This proves (i).

2.4 Primitive heights and their exact constants

Adjunction and ${}^tP_{S,T,U} = P_{S,U,T}$ show that Chow blocks are height-orthogonal unless their point and fundamental-class indices are complementary and their S indices agree. In the complementary case,

$$h_X(j_T u, j_{I\setminus T} v) = h_{Y_S}(u, v). \tag{2.5}$$

Indeed the transpose of the first embedding composed with the second integrates one degree-one point in each omitted factor.

Define

$$P_\ell^r(X_E) = \ker\bigl(\ell^{n+2-2r} : V^r(X_E) \to V^{n+2-r}(X_E)\bigr)$$

for $1 \leq r \leq \lfloor (n+1)/2 \rfloor$.

Theorem 2.3 (Primitive sum of squares)**.** *Let $|S| = 2c-1$, $m = n-|S|$, and $0 \leq t = r-c \leq m/2$. Write a vector in the corresponding block as*

$$z_S = \sum_{|T|=t} u_T x_T, \qquad u_T \in U_S^c(E).$$

It is primitive exactly when $Fz_S = 0$. For such a vector,

$$q_{\ell,r}(z_S) := (-1)^r h_X(z_S, \ell^{n+1-2r} z_S) = (m-2t)!\, a_I \sum_{|T|=t} q_{S,c}(u_T), \tag{2.6}$$

where $a_I = \prod_{i \in I} a_i$ and $q_{S,c}(u) = (-1)^c h_{Y_S}(u, u)$.

Proof. The primitive exponent is $m-2t+1$, so the equivalence follows from Lemma 2.2. The same lemma gives $\ell^{n+1-2r} z_S = (-1)^t (m-2t)! * z_S$. In pairing z_S with $*z_S$, only the identical coefficient indices survive by (2.5). Their weights multiply to $a_T a_{I\setminus T} = a_I$. The remaining sign is $(-1)^{r+t} = (-1)^c$, since $r = c+t$. This proves the formula. □

Corollary 2.4. *Suppose all the noncentral subproduct cores vanish and all central forms $q_{S,c}$ are real positive definite over a fixed number field. Then the complementary height pairings on X are nondegenerate over that field, also after real extension, and $q_{\ell,r}$ is real positive definite on P_ℓ^r. Its associated bilinear form $b_{\ell,r}$ satisfies*

$$|b_{\ell,r}(z,w)|^2 \leq q_{\ell,r}(z) q_{\ell,r}(w).$$

If noncentral vanishing holds over every finite extension and each central integral image has bounded-degree Northcott, then so does $\Lambda_X^r(E) \cap P_\ell^r$ for $q_{\ell,r}$.

Proof. The decomposition into complementary blocks and the definiteness of their central forms prove nondegeneracy. Distinct S are orthogonal. Formula (2.6) therefore proves primitive definiteness. Cauchy–Schwarz follows by applying positivity to $q_{\ell,r}(z+tw)$, $t \in \mathbb{R}$.

For the bounded-degree assertion, central Northcott gives nonnegativity over each finite extension: a nonzero rational class of nonpositive height would have infinitely many integral multiples in a bounded height ball. For an integral primitive z of height at most B, every coefficient satisfies

$$q_{S,c}(u_T) \le \frac{B}{(m-2t)!\,a_I}, \qquad N_a u_T \in \Lambda_{U_S^c}(E).$$

Thus central Northcott and (2.3) give finitely many possibilities for every coefficient. There are only finitely many blocks, and the decomposition is injective. □

3 Central projectors and finite height balls

The all-h^1 central group has a curve-visible summand with a Néron–Tate height. We first construct its Chow projector and identify the algebraic radical. The integral finiteness criterion then follows by orthogonal decomposition and one denominator bound.

3.1 Curve-invisible cycles

Lemma 3.1. *Let C be a smooth projective curve and $\Gamma \in \mathrm{CH}^r(C \times X)_{\mathbb{Q}}$, with $\dim X = n$. If $\Gamma_* H^1(C) = 0$ in $H^{2r-1}(X)$, then*

$$^t\Gamma_* w = 0 \quad \text{in } \mathrm{Pic}^0(C)_{\mathbb{Q}} \qquad (w \in V^{n+1-r}(X)).$$

Hence $\Gamma_* \mathrm{Pic}^0(C)_{\mathbb{Q}}$ is in the left height radical.

Proof. By Poincaré duality, the transpose map from $H^{2n+1-2r}(X)$ to $H^1(C)$ is zero. Functoriality of the Abel–Jacobi map under Chow correspondences therefore makes the Abel–Jacobi class of ${}^t\Gamma_* w$ zero. On a curve, the rationalized Abel–Jacobi map is the identification with $\mathrm{Pic}^0(C)_{\mathbb{Q}}$. Vanishing after complex base change descends for divisor classes with rational coefficients. The final assertion is adjunction. The correspondence functoriality used here is also proved in [6, Lemma 4.11]. □

Algebraically trivial cycles are generated by images of degree-zero divisors on smooth projective curves. For geometric algebraic triviality over E, the finitely many data can be defined over a finite extension and then traced to E. This convention is used throughout.

3.2 The maximal curve-visible projector

Let $k = 2c-1$, $Y = \prod_{i=1}^k C_i$, and $\Pi = \bigotimes_i \pi_{1,i}$. Set

$$U_E = \Pi_* V^c(Y_E), \qquad A_E = U_E \cap \mathrm{CH}^c(Y_E)_{\mathrm{alg},\mathbb{Q}}, \qquad G_E = U_E/A_E.$$

For the product polarization $\ell_Y = \sum a_i [o_i]$, the actual Chow identity $\ell_Y \Pi = 0$ follows from $[o_i]\pi_{1,i} = 0$. Thus the geometric cohomology of Π is middle and primitive.

Let $W \subset \Pi H^k(Y, \mathbb{Q})$ be the span of all images

$$H^1(C) \longrightarrow \Pi H^k(Y, \mathbb{Q})$$

of codimension-c curve correspondences over finite extensions of F. Choose finitely many correspondences spanning W, and one field F_0 defining them and their source points. Write their matrix as $\Gamma = (\Gamma_j)_j$, with $\Gamma_j = \Pi\Gamma_j\pi_{1,j}$, and put $N = \bigoplus_j h^1(C_j)$, $J = \prod_j \mathrm{Pic}^0(C_j)$.

Proposition 3.2. *There are actual Chow projectors e, R over F_0, with*

$$e = {}^t e, \quad e^2 = e, \quad e\Pi = \Pi e = e, \quad R = \Pi - e,$$

such that the following hold for every finite E/F_0.

(i) The cohomological action of e is the orthogonal projection onto W. There is a fixed abelian variety B/F_0 with $eU_E \simeq B(E)_{\mathbb{Q}}$, through fixed Chow correspondences. The form $q_e = (-1)^c h_Y$ on this summand is a positive Néron–Tate form.

(ii) With $K_E = RA_E$,

$$K_E = \ker(\mathrm{AJ}_{\mathbb{Q}}|_{A_E}) = A_E \cap \mathrm{rad}(h_Y|_{U_E}) = \mathrm{rad}(h_Y|_{A_E}).$$

(iii) There is an orthogonal decomposition $U_E = eU_E \oplus RU_E$. The residual form on $RU_E/RA_E \simeq G_E$ is independent of the choice of maximal curve-visible projector. Write it as h_G and set $q_G = (-1)^c h_G$.

Proof. Set $a = {}^t\Gamma\Gamma$. Since $2c-k = 1$, its entries are divisor correspondences, so $a \in \mathrm{End}(N) = \mathrm{End}(J)_{\mathbb{Q}}$. This realization is faithful, and transposition corresponds to Rosati adjunction for the product principal polarization. Geometric Hodge–Riemann on the primitive subspace W implies that $(-1)^{c-1}a$ is positive semidefinite and $\ker a = \ker \Gamma$ on H^1. In particular a is semisimple. Choose a polynomial $f \in \mathbb{Q}[t]$ equal to t^{-1} on the nonzero spectrum and zero at zero, and set

$$a^+ = f(a), \qquad p = a^+ a, \qquad e = \Gamma a^{+t}\Gamma.$$

Faithfulness on N makes $a^+aa^+ = a^+$, $p^2 = p = {}^t p$ actual Chow identities. These identities prove the assertions about e and its cohomology. Let $B = pJ$, up to isogeny. The maps Γ and $a^{+t}\Gamma$ identify $B(E)_{\mathbb{Q}}$ with eU_E. Adjunction gives

$$h_Y(\Gamma x, \Gamma y) = h_{\mathrm{curves}}(x, ay).$$

The curve normalization and the positivity of $(-1)^{c-1}a$ on pJ give (i).

For every later curve correspondence Δ, the geometric H^1 action of $R\Delta$ is zero by maximality of W. Curve generation and Lemma 3.1 give $RA_E \subseteq \mathrm{rad}(h_Y|_{U_E})$. The Abel–Jacobi map is zero on RA_E by the same geometric action. Its restriction to eU_E is injective after rationalization: the corresponding map from B to the intermediate Jacobian has injective rational H_1, hence finite kernel. Since $eU_E \subseteq A_E$ and its height is nondegenerate, all equalities in (ii) follow.

The self-transpose projectors give the orthogonal decomposition. The radical assertion just proved makes the residual form well defined on G_E. Two orthogonalized lifts of the same Griffiths class differ by an element of A_E orthogonal to A_E. By (ii), their difference is in the full radical. Thus the residual form is independent of the lift and of e. □

The cohomological subspace W and its orthogonal complement are defined independently of the chosen spanning correspondences. Betti and étale comparison transports them to each $\mathbb{Q}_{\ell_0}$ realization. The algebraic construction of e in the proof uses geometric Hodge–Riemann and the faithful realization of h^1 endomorphisms; it is available independently of the height data.

Lemma 3.3. *The projected integral image in eU_E has bounded-degree Northcott for q_e, allowing any fixed denominator bound.*

Proof. Put $D = a^{+t}\Gamma$. These fixed curve correspondences satisfy $De = D$. One integer N clears their denominators, so an integral cycle z gives a genuine point $NDz \in J(E)$. The same point detects ez, even though ez may have rational coefficients.

On the fixed factor $pJ(\overline{F})_{\mathbb{Q}}$, the ambient Néron–Tate height of J is bounded above by a fixed positive multiple of q_e under the inverse D. This is comparison of two polarizations on B; passing to rational points removes torsion without changing either height. Thus a bound for $q_e(ez)$ bounds the height of the actual point NDz. Classical bounded-degree Northcott on J, followed by the injectivity of D on eU_E, proves the assertion. Fixed extra denominators are absorbed into N. See [2, Sections 9–10] for the height and Mordell–Weil facts used here. □

3.3 The exact central Northcott reduction

Continue with the central notation of Section 3.2. Let $\Lambda_U(E) = \Pi_*\Lambda_Y^c(E)$ and let $\Lambda_G(E)$ be its image in G_E. The quotient may be viewed in the geometric group: if an element defined over E becomes geometrically algebraically trivial, it is in A_E by definition. In particular, these rational quotients inject under field extension.

Theorem 3.4. *The following conditions are equivalent:*

(i) The form $q_U(z) = (-1)^c h_Y(z,z)$ has bounded-degree Northcott on Λ_U.

(ii) For every finite E/F_0, $K_E = \ker(\mathrm{AJ}_{\mathbb{Q}}|_{A_E}) = 0$, and q_G has bounded-degree Northcott on Λ_G.

There is also a fixed-field version, with all groups and the vanishing condition taken over one fixed E.

Proof. Suppose (i). If $0 \neq z \in K_E$, multiply by one integer to make z integral. All positive integer multiples of this class are distinct and have height zero, contradicting Northcott. Thus $K_E = 0$.

Choose an integer $M \geq 1$ such that MR is represented by an integral correspondence. For $\xi \in \Lambda_G(E)$, choose $z \in \Lambda_U(E)$ mapping to ξ. Then

$$MRz \in \Lambda_U(E), \qquad q_U(MRz) = M^2 q_G(\xi), \qquad [MRz] = M\xi.$$

Northcott for Λ_U bounds the possible classes MRz, and hence the possible ξ. This proves residual Northcott.

Conversely, assume (ii). Residual Northcott implies $q_G(\xi) > 0$ for every nonzero rational class $\xi \in G_E$. Indeed one multiple of ξ is in $\Lambda_G(E)$, and a nonpositive height would give infinitely many integral multiples of height at most zero. For every $z \in U_E$, orthogonality gives

$$q_U(z) = q_e(ez) + q_G([z]). \tag{3.1}$$

Both terms are nonnegative. If z is integral and $q_U(z) \leq B$, residual Northcott bounds the possibilities for $[z]$. The denominators of the fixed projector e, followed by Lemma 3.3, bound the possibilities for ez. Finally

$$z \longmapsto (ez, [z])$$

is injective: its kernel is $RA_E = K_E = 0$. This proves (i). The argument with E fixed is identical. □

The equivalence separates the algebraic Abel–Jacobi kernel from integral finiteness on the residual quotient. In fact, Northcott also forces real positive definiteness, by Lemma 5.1. This observation will combine the central equivalence with the primitive calculation in Theorem 1.3.

4 Residual arithmetic regulators

The residual Chow projector isolates a single geometric Hochschild–Serre row. This gives an arithmetic cycle map on the Griffiths quotient and a finite-dimensional target.

Theorem 4.1 (Residual arithmetic regulator)**.** *For every prime ℓ_0 and finite E/F_0, set*

$$V_{\ell_0} = R_* H^{2c-1}_{\text{ét}}(Y_{\overline{F}}, \mathbb{Q}_{\ell_0}(c)).$$

There is a natural $\mathbb{Q}$-linear map

$$a_{\ell_0,E} : G_E \longrightarrow H^1(E, V_{\ell_0})$$

induced by the continuous étale cycle class. It is independent of the curve-visible Chow lift under the common cohomological identification, and is compatible with restriction and trace. There is one finite set S_0 of places of F_0 such that inflation identifies

$$H^1(G_{E,S_E}, V_{\ell_0}) \simeq H^1(E, V_{\ell_0}),$$

where S_E consists of places above S_0 and ℓ_0. This whole target is finite-dimensional over $\mathbb{Q}_{\ell_0}$.

4.1 The single-row construction

We use continuous étale cohomology and its cycle class map as in [8, 4]. Fix a rational prime ℓ_0, and put

$$k = 2c - 1, \qquad V_{\ell_0} = R_* H^k_{\text{ét}}(Y_{\overline{F}}, \mathbb{Q}_{\ell_0}(c)).$$

The action of the actual Chow idempotent R on geometric cohomology is concentrated in degree k. Apply it to the correspondence-compatible Hochschild–Serre spectral sequence

$$H^p\big(E, H^q_{\text{ét}}(Y_{\overline{F}}, \mathbb{Q}_{\ell_0}(c))\big) \Longrightarrow H^{p+q}_{\text{cont}}(Y_E, \mathbb{Q}_{\ell_0}(c)).$$

Only the row $q = k$ survives. Hence there is a canonical identification

$$R_* H^{2c}_{\text{cont}}(Y_E, \mathbb{Q}_{\ell_0}(c)) \simeq H^1(E, V_{\ell_0}). \tag{4.1}$$

On this summand the Hochschild–Serre filtration satisfies $F^1 = H$ and $F^2 = 0$.

Proposition 4.2. *The assignment $z \mapsto \mathrm{cl}_{\ell_0}(R_* z)$, followed by (4.1), annihilates A_E. It therefore defines a map*

$$a_{\ell_0,E} : G_E \longrightarrow H^1(E, V_{\ell_0}).$$

The map is compatible with restriction, corestriction and correspondences preserving the indicated projectors. Under the common cohomological identification, it is independent of the maximal curve-visible projector e.

Proof. For a degree-zero divisor D on a pointed curve C, its continuous cycle class lies in the $\pi_{1,C}$-part of $H^2_{\text{cont}}(C, \mathbb{Q}_{\ell_0}(1))$. This part has only the Hochschild–Serre term $H^1(E, H^1(C_{\overline{F}}, \mathbb{Q}_{\ell_0}(1)))$. If $\Delta \in \mathrm{CH}^c(C \times Y)_{\mathbb{Q}}$ is a projected curve correspondence, $R\Delta\pi_{1,C}$ has zero geometric H^1 action by Proposition 3.2. By functoriality its map on the only associated graded term is zero. Its image in the target of (4.1) is consequently in $F^2 = 0$.

Curve generation proves the vanishing for algebraically trivial cycles after a finite extension L/E. It descends to E because $\mathrm{cor}_{L/E} \mathrm{res}_{L/E} = [L : E]$ on rational cohomology. Restriction is therefore injective. Compatibility with restriction, trace and the stated correspondences follows from the cycle map and the spectral sequence.

For independence, let e' be another maximal curve-visible lift. Its geometric action is the same orthogonal projector onto W. The Π-part of total degree $2c = k + 1$ also has only its first associated graded term. Thus equality of the geometric actions of e, e' gives equality of their actions on this whole continuous-cohomology group. The cycle classes of $(\Pi - e)z$ and $(\Pi - e')z$ coincide. □

4.2 Arithmetic models and weight-minus-one cohomology

In the notation G_{E,S_E}, all archimedean places are included in S_E; displayed conditions on S_E specify its finite places.

Proposition 4.3. *There is a finite set S_0 of finite places of F_0 such that, for every prime ℓ_0 and finite E/F_0,*

$$a_{\ell_0,E}(G_E) \subseteq H^1(G_{E,S_E}, V_{\ell_0}) \subseteq H^1(E, V_{\ell_0}),$$

where S_E consists of places above S_0 or ℓ_0. The first space is finite-dimensional over $\mathbb{Q}_{\ell_0}$.

Proof. Spread Y to a smooth proper scheme over $\operatorname{Spec} \mathcal{O}_{F_0}[1/S_0]$. Spread the finitely many cycle representatives and rational-equivalence witnesses defining the relative correspondence R and the identity $R^2 = R$. After enlarging S_0 once, these data all exist on this model. For a chosen ℓ_0, work over

$$U_{\ell_0} = \operatorname{Spec} \mathcal{O}_{F_0}[1/(S_0 \cup \{v : v \mid \ell_0\})].$$

The higher direct images of the smooth proper morphism are lisse. Since their R-parts vanish at the geometric generic point in degrees other than k, they vanish everywhere in those degrees. The surviving part is a lisse sheaf $\mathcal{V}_{\ell_0}$. Applying R to Leray yields

$$R_* H^{2c}_{\mathrm{cont}}(\mathcal{Y}, \mathbb{Q}_{\ell_0}(c)) \simeq H^1(U_{\ell_0}, \mathcal{V}_{\ell_0}).$$

Every generic-fibre cycle extends by closure to this regular model. Compatibility of cycle classes with restriction proves the factorization. Arithmetic étale cohomology finiteness gives finite $\mathbb{Q}_{\ell_0}$-dimension. Equivalently one may apply the model argument of [8, Proposition 2.5 and Corollary 2.6].

For a finite extension E/F_0, base change this same model to $\mathcal{O}_E[1/S_E]$. This is a finite flat base change, preserves smoothness, and preserves the relative correspondence identities. Taking closures of cycles over E proves the assertion uniformly. In degree one, the étale cohomology of the base with this lisse sheaf identifies with $H^1(G_{E,S_E}, V_{\ell_0})$. □

The residual representation has a further useful property: its geometric weight is $2c - 1 - 2c = -1$. This strengthens the preceding image statement to one about the whole target.

Proposition 4.4 (Weight-minus-one cohomology)**.** *Let S_0 contain the bad-reduction places of Y/F_0. For every finite E/F_0, every prime ℓ_0, and every finite $v \notin S_E$,*

$$H^1(E_v, V_{\ell_0}) = 0.$$

Inflation is an isomorphism

$$H^1(G_{E,S_E}, V_{\ell_0}) \xrightarrow{\sim} H^1(E, V_{\ell_0}).$$

In particular, the full group on the right is finite-dimensional. For every stable free $\mathbb{Z}_{\ell_0}$-lattice $T \subset V_{\ell_0}$, inflation also identifies $H^1(G_{E,S_E}, T)$ with $H^1(E, T)$, and these groups are finitely generated over $\mathbb{Z}_{\ell_0}$.

Proof. At $v \notin S_E$, smooth proper base change makes inertia act trivially on V_{ℓ_0}. The eigenvalues of geometric Frobenius have complex absolute value $q_v^{-1/2}$, by [3, Theorem 1.6]. Consequently $V_{\ell_0}/(\mathrm{Fr}_v - 1)V_{\ell_0} = 0$. The ℓ_0-part of tame inertia is $\mathbb{Z}_{\ell_0}(1)$, and continuous homomorphisms to V_{ℓ_0} kill wild inertia and the other tame factors. Therefore

$$H^1(I_v, V_{\ell_0})^{\mathrm{Fr}_v} = V_{\ell_0}(-1)^{\mathrm{Fr}_v} = 0,$$

since the eigenvalues on the twist have absolute value $q_v^{1/2}$. Local inflation–restriction proves the first assertion.

For a global continuous cocycle, the restriction to every such inertia group is zero. Indeed its local class is zero, and inertia acts trivially, so even the local coboundaries restrict to zero on inertia. The kernel of $\mathrm{Gal}(\overline{F}/E) \to G_{E,S_E}$ is the closed normal subgroup generated by these inertia groups. It acts trivially on V_{ℓ_0}. If z is the cocycle and i is in one of these inertia groups, the cocycle identity gives $z(gig^{-1}) = (1 - gig^{-1})z(g) = 0$. The restriction of z to the kernel is thus a continuous homomorphism vanishing on all generators, hence on the whole kernel. It factors through G_{E,S_E}. Inflation is injective, proving the isomorphism.

For T, the inertia invariants are $T(-1)^{\mathrm{Fr}_v} = 0$, as they inject into the rational ones. The same global cocycle argument proves integral inflation. The local integral group may be the nonzero finite group $T/(\mathrm{Fr}_v - 1)T$; its restriction to inertia is still zero. Finally, in degree one the continuous-cohomology convention agrees with these continuous cocycles: compactness gives a bounded image in a scalar multiple of a stable lattice.

To prove finite generation, write $G = G_{E,S_E}$. The groups $H^i(G, T/\ell_0^a T)$ are finite by [7, Corollary I.4.15]. The inverse system of degree-zero groups is Mittag–Leffler, so

$$H^1(G, T) = \varprojlim_a H^1(G, T/\ell_0^a T)$$

is a compact pro-ℓ_0 module. The exact sequence for $0 \to T \xrightarrow{\ell_0} T \to T/\ell_0 T \to 0$ embeds its reduction modulo ℓ_0 in the finite group $H^1(G, T/\ell_0 T)$. The compact form of Nakayama's lemma therefore makes $H^1(G, T)$ finitely generated. Every continuous cocycle into V_{ℓ_0} takes values in $\ell_0^{-a}T$ for some a, and a coboundary can likewise be cleared of denominators. Hence

$$H^1(G, V_{\ell_0}) = H^1(G, T) \otimes_{\mathbb{Z}_{\ell_0}} \mathbb{Q}_{\ell_0},$$

which proves the rational finite-dimensionality. □

Propositions 4.2 and 4.4 prove Theorem 4.1. These ramification statements concern cohomology over E, rather than the ramification of E/F_0. The latter enters separately in Section 6.

Corollary 4.5. *If, for one prime ℓ_0, the scalar-extended map*

$$G_E \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell_0} \longrightarrow H^1(E, V_{\ell_0}) \tag{4.2}$$

is injective, then $\dim_{\mathbb{Q}} G_E < \infty$.

Proof. Its image lies in the finite-dimensional space of Proposition 4.3. Scalar extension preserves the dimension of a rational vector space. □

The scalar extension in (4.2) is essential: a finite-dimensional $\mathbb{Q}_{\ell_0}$-space has infinite dimension over $\mathbb{Q}$.

4.3 Local integral bounds

Proposition 4.6. *For every fixed E, ℓ_0, the subgroup $a_{\ell_0,E}(\Lambda_G(E))$ is contained in a finitely generated $\mathbb{Z}_{\ell_0}$-module in the rational cohomological target. If (4.2) is injective, then $\Lambda_G(E) \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell_0}$ is finitely generated over $\mathbb{Z}_{\ell_0}$, and $\Lambda_G(E)/\ell_0\Lambda_G(E)$ is finite.*

Proof. Choose one integer M clearing the Chow denominators of R. For an integral representative Z, $M \operatorname{cl}_{\ell_0}(RZ) = \operatorname{cl}_{\ell_0}((MR)Z)$ is in the image of the integral continuous cycle class map. This image is a finitely generated $\mathbb{Z}_{\ell_0}$-module by [4, Theorem 2.4]. Let M_{ℓ_0} be its image in rational continuous cohomology. The intersection

$$M_{\ell_0} \cap R_* H^{2c}_{\mathrm{cont}}(Y_E, \mathbb{Q}_{\ell_0}(c))$$

is again finitely generated and contains $M \operatorname{cl}_{\ell_0}(RZ)$. Applying the single-row identification to this intersection and dividing by M gives the claimed bound.

Under scalar-extended injectivity, the induced map on $\Lambda_G(E) \otimes \mathbb{Z}_{\ell_0}$ is injective. Its source is torsion-free over $\mathbb{Z}_{\ell_0}$, and its localization is $G_E \otimes \mathbb{Q}_{\ell_0}$, because $\Lambda_G(E)$ spans G_E. Its image is a submodule of a finitely generated $\mathbb{Z}_{\ell_0}$-module, hence is finitely generated. Reduction modulo ℓ_0 proves the last assertion. □

The integral image here lies in a rational quotient. Unrestricted integral continuous cycle class maps have torsion counterexamples over number fields in codimension three [9, Theorems 1.3–1.4]. Those torsion kernels and the torsion-free subgroup $\Lambda_G(E)$ are different objects.

5 Integral discreteness and height gaps

5.1 Real positivity forced by finiteness

Lemma 5.1. *Let Λ be a subgroup spanning a rational vector space V, and let q be a real-valued quadratic form on V. If q has Northcott on Λ, its extension to $V \otimes_{\mathbb{Q}} \mathbb{R}$ is positive definite.*

Proof. It suffices to consider a finite-dimensional rational subspace and choose a rational basis $v_1, \dots, v_s$ in Λ. Let Q be the Gram matrix. A negative real direction would, by rational approximation and clearing denominators, give a nonzero $v \in \sum_i \mathbb{Z}v_i$ with $q(v) < 0$. Its integer multiples contradict Northcott. Thus Q is positive semidefinite.

If $0 \neq u \in \ker Q$, round the coordinates of tu to integers, obtaining $z_t \in \mathbb{Z}^s$ with uniformly bounded $z_t - tu$. Then

$$q\Big(\sum_i (z_t)_i v_i\Big) = (z_t - tu)^t Q (z_t - tu)$$

is uniformly bounded. The rounded vectors are unbounded as $t \to \infty$, hence take infinitely many distinct values. This again contradicts Northcott. Therefore every such finite Gram matrix is positive definite, which proves the assertion. □

In particular, residual integral Northcott already contains the real-positivity assertion for q_G. The converse requires an arithmetic condition on the integral image.

5.2 Finite-rank lattices

Theorem 5.2. *Let Λ be torsion-free of finite rank, $V = \Lambda \otimes \mathbb{Q}$, and suppose that q is real positive definite on $V \otimes \mathbb{R}$. The following are equivalent:*

(i) q *has Northcott on* Λ.

(ii) There is $\varepsilon > 0$ *such that* $q(\lambda) \geq \varepsilon$ *for every nonzero* $\lambda \in \Lambda$.

(iii) Λ *is finitely generated.*

(iv) For a rational basis $v_1, \ldots, v_s$ *of* V*, there is an integer* $N \geq 1$ *such that* $\Lambda \subset N^{-1} \sum_i \mathbb{Z} v_i$.

(v) The diagonal image of Λ *in* $V(\mathbb{A}_f)$ *is relatively compact.*

The zero group satisfies these statements with the gap condition interpreted vacuously.

Proof. Finiteness in a unit height ball gives a positive minimum, or the bound 1 if that ball contains no nonzero point. This proves (i) implies (ii). The gap makes Λ uniformly discrete for the Euclidean norm $\sqrt{q}$. Choose a rational basis in Λ and reduce elements of Λ modulo its $\mathbb{Z}$-span into a bounded fundamental parallelepiped. Uniform discreteness makes the set of representatives finite. Hence this span has finite index in Λ, proving (iii) and (iv). Conversely, a bounded subset of a Euclidean lattice is finite, so (iv) implies (i). Finite generation gives (iv) for any rational basis by clearing the finitely many coefficients.

Every compact subset of $V(\mathbb{A}_f)$ is contained in $N^{-1}\Lambda_0 \otimes \widehat{\mathbb{Z}}$ for some integer N and a fixed reference lattice Λ_0. This follows by a finite subcover from the open compact sets of that form. Conversely each such set is compact. Intersecting with $V(\mathbb{Q})$ proves the equivalence with (v). □

If $\Lambda \subset N^{-1}\Lambda_0$, and the least eigenvalue of the Gram matrix in a basis of Λ_0 is $\mu > 0$, one obtains the explicit estimate

$$\#\{\lambda \in \Lambda : q(\lambda) \leq B\} \leq \left(1 + 2\left\lfloor N\sqrt{B/\mu}\right\rfloor\right)^s \qquad (B \geq 0). \tag{5.1}$$

5.3 An all-prime saturation criterion

Theorem 5.3. *Let* V *be a finite-dimensional rational vector space,* $\Lambda_0 \subset V$ *a full lattice, and* $\Lambda \subset V$ *a subgroup. For every prime* p*, let*

$$D_p : V \otimes \mathbb{Q}_p \hookrightarrow W_p$$

be an injective linear map to a finite-dimensional space, and let $M_p \subset W_p$ *be a full* $\mathbb{Z}_p$*-lattice. Assume*

$$D_p(\Lambda) \subseteq M_p \quad \text{for every } p, \qquad D_p^{-1}(M_p) = \Lambda_0 \otimes \mathbb{Z}_p \quad \text{for almost every } p.$$

Then $\Lambda \subset N^{-1}\Lambda_0$ *for one integer* N*. Every real positive definite quadratic form has Northcott on* Λ.

Proof. Injectivity makes $L_p = D_p^{-1}(M_p)$ a $\mathbb{Z}_p$-lattice: it is open and compact in $V \otimes \mathbb{Q}_p$. For each exceptional prime choose $b_p \geq 0$ with $L_p \subset p^{-b_p}\Lambda_0 \otimes \mathbb{Z}_p$, and put $N = \prod p^{b_p}$. The coordinates of $N\lambda$ lie in $\mathbb{Z}_p$ for every p. Since $\mathbb{Q} \cap \prod_p \mathbb{Z}_p = \mathbb{Z}$, they are integers. Apply Theorem 5.2. □

For the residual regulator, a common multiple of the projector denominators can first be absorbed into D_p. Once scalar injectivity is available, Proposition 4.6 supplies local integral modules. The almost-everywhere saturation condition in Theorem 5.3 is the additional comparison with one common rational lattice.

Example 5.4. Let

$$\Lambda = \sum_{p\ mprime} \mathbb{Z}\,\frac{1}{p} \subset \mathbb{Q}, \qquad q(x) = x^2.$$

Its elements have squarefree denominators. For every prime p, $\Lambda \otimes \mathbb{Z}_p = p^{-1}\mathbb{Z}_p$, which is finite free of rank one. Nevertheless $q(1/p) = p^{-2} \to 0$, so Northcott fails. The integral maps $D_p(x) = px$ have inverse lattices larger than $\mathbb{Z}_p$ at infinitely many primes. Thus local finite generation at every prime does not provide the common denominator required by Theorem 5.2.

Already $\mathbb{Z}[1/p]$, with $q(x) = x^2$, defeats any finite collection of integral detectors at primes different from p. Another distinction is real definiteness: the form $(a - \sqrt{2}b)^2$ on $\mathbb{Z}^2$ is positive on nonzero rational vectors, but rounding along its real kernel produces infinitely many bounded-height lattice vectors.

5.4 A quantitative descent criterion

Theorem 5.5. *Let Λ be torsion-free and let $\sqrt{q}$ be the norm of a real positive quadratic form on $\Lambda \otimes \mathbb{R}$. Suppose, for some integer $m \geq 2$,*

$$s = |\Lambda/m\Lambda| < \infty, \qquad q(\lambda) \geq \varepsilon > 0 \quad (0 \neq \lambda \in \Lambda).$$

Then Λ is finitely generated and q has Northcott. More precisely, if $B \geq 0$, $j \in \mathbb{Z}_{\geq 0}$, and $m^{2j}\varepsilon > 4B$, then

$$\#\{\lambda \in \Lambda : q(\lambda) \leq B\} \leq s^j. \tag{5.2}$$

Proof. If x, y are in the height-B ball and $x - y = m^j z$, then $\|z\| \leq 2\sqrt{B}/m^j < \sqrt{\varepsilon}$, so $z = 0$. Reduction modulo $m^j\Lambda$ is injective on that ball. Torsion-freeness identifies the successive quotients $m^i\Lambda/m^{i+1}\Lambda$ with $\Lambda/m\Lambda$, proving (5.2).

Choose a finite set T of representatives modulo $m\Lambda$, and let $C = \max_{t\in T} \|t\|$. Writing $x = my + t$ gives $\|y\| \leq (\|x\| + C)/m$. Repetition brings y into any fixed ball of radius greater than $C/(m-1)$. This ball is finite by the first part; together with T, it generates Λ. This is the height-descent argument underlying weak Mordell–Weil; compare [2, Section 10.3]. □

Northcott by itself need not give finite rational rank. For example, $\Lambda = \bigoplus_{i\geq 1} \mathbb{Z}e_i$ with $q(\sum a_i e_i) = \sum 2^i a_i^2$ has finite height balls and infinite rank. Rational detection, integral discreteness and real positivity therefore retain distinct roles in the general program.

6 Fields of classes and bounded-degree finiteness

6.1 Descent by trace

Let $\mathcal{G}$ be a rational geometric Chow quotient with its discrete $\mathrm{Gal}(\overline{F}/F_0)$-action, and let $\Lambda(E) \subset \mathcal{G}$ be compatible integral images. We assume that restriction and trace preserve these images in their usual directions. Fixed rational Chow projectors and quotient by geometric algebraic equivalence have this property. Every element considered is defined over a finite extension, so its stabilizer is open. Define its minimal class field by

$$F_0(\xi) = \overline{F}^{\mathrm{Stab}_{\mathrm{Gal}(\overline{F}/F_0)}(\xi)}.$$

Lemma 6.1. *If $\xi \in \Lambda(E)$ and $K = F_0(\xi)$, then*

$$K \subseteq E, \qquad [E : K]\xi \in \Lambda(K).$$

In particular, if $[E : F_0] \leq D$, the required multiplier is at most D.

Proof. The inclusion $\mathrm{Gal}(\overline{F}/E) \subseteq \mathrm{Stab}(\xi)$ gives $K \subseteq E$. Take an integral cycle over E representing ξ in the indicated quotient and push it forward to K. Over $\overline{F}$, the resulting class is the sum over the K-embeddings of E. Every summand equals ξ, so the sum is $[E : K]\xi$. Fixed projectors commute with trace, and algebraic equivalence is preserved by it. □

The lemma concerns the field of a class in a rational quotient. It supplies the precise denominator even when no chosen integral cycle representative descends to that field.

6.2 The ramification criterion

Let q be a Galois-invariant quadratic form on $\mathcal{G}$. For $B \geq 0$ and an integer $D \geq 1$, consider

$$\mathcal{B}(B, D) = \{\xi \in \bigcup_{[E:F_0] \leq D} \Lambda(E) : q(\xi) \leq B\}.$$

Theorem 6.2. *Assume q has fixed-field Northcott on $\Lambda(K)$ for every finite K/F_0. Then bounded-degree Northcott is equivalent to the following condition: for every B, D, there is a finite set $S(B, D)$ of finite places of F_0 such that each field $F_0(\xi)/F_0$, with $\xi \in \mathcal{B}(B, D)$, is unramified outside $S(B, D)$.*

Proof. Suppose the ramification condition holds. By the finite-ramification form of Hermite–Minkowski, there are only finitely many extensions K/F_0 in $\overline{F}$ of degree at most D unramified outside $S(B, D)$; see [5, Theorem 10.2.2]. This form also follows by bounding the local discriminant exponents in fixed degree and applying discriminant finiteness.

For $\xi \in \mathcal{B}(B, D)$, Lemma 6.1 gives $m\xi \in \Lambda(K)$, where $K = F_0(\xi)$ is among these finitely many fields and $1 \leq m \leq D$. Moreover

$$q(m\xi) = m^2 q(\xi) \leq D^2 B.$$

Fixed-field Northcott leaves finitely many possibilities for $m\xi$. Multiplication by m is injective in the rational group, and only finitely many m occur. Thus the ball is finite. Conversely, if the ball is finite, the union of the ramification sets of its finitely many class fields is finite. □

An explicit discriminant estimate would give a quantitative route to the hypothesis. For example, with $\mathfrak{d}_{K/F_0}$ the relative discriminant ideal, an inequality

$$\log N_{F_0/\mathbb{Q}} \mathfrak{d}_{F_0(\xi)/F_0} \leq A_D q(\xi) + B_D \tag{6.1}$$

on integral classes of degree at most D would suffice. The qualitative ramification condition only asks for some finite set depending on B, D; it is weaker than (6.1).

6.3 Why uniform fixed-field estimates do not suffice

Example 6.3. Choose infinitely many distinct quadratic extensions K_i/F_0 with characters χ_i. Put

$$\mathcal{G} = \bigoplus_i \mathbb{Q} e_i, \qquad \sigma e_i = \chi_i(\sigma) e_i, \qquad \Lambda(E) = \bigoplus_{K_i \subseteq E} \mathbb{Z} e_i, \qquad q\left(\sum a_i e_i\right) = \sum a_i^2.$$

The lattices are compatible with restriction and trace. Every finite extension E has at most $[E : F_0] - 1$ distinct quadratic subfields: their compositum is multiquadratic, and the number of its quadratic subfields is one less than its degree. Consequently

$$\operatorname{rank} \Lambda(E) \leq [E : F_0] - 1, \qquad q(\lambda) \geq 1 \ (\lambda \neq 0), \qquad |\Lambda(E)/m\Lambda(E)| \leq m^{[E:F_0]-1}.$$

All fixed-field height balls are finite. Nevertheless the classes $e_i \in \Lambda(K_i)$ all have degree two and height one. Thus bounded-degree Northcott fails despite uniform degree-dependent bounds on ranks, gaps and Kummer quotients.

The fixed-S cohomological statement of Proposition 4.3 holds over a chosen field of definition. Theorem 6.2 instead bounds the fields themselves. This distinction is essential when using residual arithmetic regulators to study the conjectural Northcott property.

7 Proof of the principal reduction

Proof of Theorem 1.3. Suppose (ii). Assertion (A) and Theorem 2.1 give arithmetic hard Lefschetz. Assertions (B) and (C), together with Theorem 3.4, give central integral Northcott on every odd subproduct. By Lemma 5.1 the central forms are real positive definite. Corollary 2.4 now gives primitive integral Northcott and positivity on X, as well as nondegeneracy of the complementary height pairings.

Conversely, assume (i). Arithmetic hard Lefschetz gives off-central vanishing by Theorem 2.1, proving (A). For an odd subproduct $|S| = 2c - 1$, inflate $u \in U_S^c(E)$ by fundamental classes on all omitted factors. In the notation of Section 2.1, this is the integral injection

$$u \longmapsto j_{\varnothing} u$$

in degree $r = c$. With $m = n - |S|$, it is primitive and Theorem 2.3 gives

$$q_{\ell,c}(j_{\varnothing} u) = m! \prod_{i \notin S} a_i \, q_{S,c}(u).$$

Thus primitive Northcott on X gives central integral Northcott on every odd subproduct. By Theorem 3.4, this is equivalent to (B) and (C). The positivity and nondegeneracy assertions then follow from the first direction of the proof. □

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