Domains and constructions of Beilinson–Bloch height pairings

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Abstract

We organize the geometric domains of zero-curvature Green forms and normalized Beilinson–Bloch height pairings. Under explicit actual-form, numerical and reciprocity hypotheses, Chow covariance turns decompositions and model corrections into constructions on the exact support complement of a prescribed cycle. The resulting domains include products of arbitrary curves and surfaces, their specified motivic summands, Hilbert schemes of points on surfaces, and the surface-moduli motives furnished by Bülles. Other applications use low-dimensional diagonal support, Chow niveau, flat fibrations, and semistable models. We give the precise complementary-degree conditions for global heights, construct Chow identities for separated hypersurfaces and cyclic covers, and record the field and cover degrees in the comparison with classical abelian heights. The geometric proofs are given in full; the separately supplied analytic and model statements are identified at each use.

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1 Introduction

Let $K/\mathbb{Q}_p$ be finite, with a fixed absolute value, and let X/K be smooth, projective and geometrically integral of dimension n. For a rational codimension-r cycle Z, a zero-curvature Green form is required to satisfy

$$\int_{X^{\mathrm{an}}\setminus|Z|^{\mathrm{an}}} \gamma_Z \wedge d'd''\eta = -\int_{Z^{\mathrm{an}}} \eta, \qquad \eta \in B_c^{n-r,n-r}(X^{\mathrm{an}}), \tag{1.1}$$

where $B^{\bullet,\bullet}$ is Mihatsch's sheaf of ordinary actual δ-forms and $\gamma_Z \in B^{r-1,r-1}(X^{\mathrm{an}} \setminus |Z|^{\mathrm{an}})$ has the integrability required in this identity. The support $|Z|$ is the support of the specified cycle after equal components have been collected. All forms, tests and integrals in (1.1) use the original valued field K.

For complete intersections cut out by global functions, Mihatsch constructs such forms explicitly [24, Theorem 5.5]. The general projective problem requires a global compatibility condition. The companion obstruction theory [31, 32, 33] attaches to a Chow class a closed-form Bott–Chern class. Vanishing of this obstruction allows one to correct a curved Green form for the original cycle by a global ordinary form. The algebraic question is therefore to identify Chow sources on which this obstruction vanishes.

The present paper develops this question through Chow decompositions, semistable models, and explicit geometric constructions. Its analytic foundation is stated in Section 2; the corresponding internal proofs are identified in the accompanying dossier. We first describe the abelian transfer, which also provides a classical formula for the height. The required algebraic datum is literal. For smooth projective varieties V, W, a correspondence of degree a from V to W means an element of $\mathrm{CH}^{\dim V+a}(V \times W)_{\mathbb{Q}}$; it sends codimension r to codimension $r+a$. Write $\mathrm{CH}^r(V)_{\mathbb{Q},\mathrm{hom}}$ for the kernel of the geometric cycle class map, with homological triviality imposed on every geometric component when V is disconnected.

Definition 1.1. A smooth projective geometrically integral variety X/k has a *potential abelian Chow retraction* if there are a finite extension L/k, abelian varieties A_j/L, integers a_j, and rational correspondences

$$u_j \in \mathrm{CH}^{\dim X+a_j}(X_L \times_L A_j)_{\mathbb{Q}}, \qquad v_j \in \mathrm{CH}^{\dim A_j-a_j}(A_j \times_L X_L)_{\mathbb{Q}}$$

such that

$$\Delta_{X_L} = \sum_j v_j \circ u_j \quad \text{in } \mathrm{CH}^{\dim X}(X_L \times_L X_L)_{\mathbb{Q}}. \tag{1.2}$$

The zero-dimensional abelian variety $\operatorname{Spec} L$ is allowed.

Standing analytic hypotheses. Throughout this paper, assertions about actual Green forms use the obstruction, numerical-model and Chow-covariance hypotheses in Section 2. Assertions about the analytic height additionally use the reciprocity and comparison hypotheses in §2.3. Each further model theorem is used with the hypotheses stated at its point of application. This convention separates the geometric deductions proved here from their companion analytic foundation.

Theorem 1.2 (Chow-motivic transfer)**.** *Assume the local analytic hypotheses of Section 2. Let $K/\mathbb{Q}_p$ be finite with its fixed absolute value. Let X/K be smooth, projective and geometrically integral, and suppose that X has a potential abelian Chow retraction. Then*

$$\mathfrak{o}^r_{X,K}(z) = 0 \quad \textit{for every } z \in \mathrm{CH}^r(X)_{\mathbb{Q},\mathrm{hom}} \quad \textit{and every } r.$$

For $1 \le r \le \dim X$, every specified rational cycle Z representing such a class has an ordinary actual Green form on $X^{\mathrm{an}} \setminus |Z|^{\mathrm{an}}$ satisfying (1.1). For a fixed r, it is enough that the correspondences in (1.2) act as the identity on $\mathrm{CH}^r(X_L)_{\mathbb{Q},\mathrm{hom}}$.

Proof. See §3.2.

We apply this theorem to three classes of varieties. The block conditions in the following statement are geometric conditions over an algebraic closure.

Theorem 1.3 (Explicit families)**.** *Over a field k of characteristic zero, each of the following varieties has a potential abelian Chow retraction, as does every finite product of them and every smooth projective geometrically integral variety becoming isomorphic to such a product over a finite extension.*

(i) *A positive-dimensional hypersurface*

$$\sum_{i=1}^{b} f_i(x_{i0}, x_{i1}) + \sum_{j=1}^{a} c_j t_j^d = 0 \quad \text{in } \mathbb{P}_k^{2b+a-1}, \tag{1.3}$$

where $d \ge 2$, $2b + a \ge 3$, the variable blocks are disjoint, each $c_j \ne 0$, and each binary form f_i has a reduced zero divisor on $\mathbb{P}^1$.

(ii) *A cubic hypersurface in $N = 5b + a \ge 3$ variables, where $0 \le a \le 4$, given by*

$$\sum_{i=1}^{b} f_i(x_{i1}, \dots, x_{i5}) + g(y_1, \dots, y_a) = 0. \tag{1.4}$$

Each $f_i = 0$ is a smooth cubic threefold. For $a = 0$ the last term is omitted; for $a = 1$ it is cy_1^3 with $c \ne 0$; for $a \ge 2$ it defines a smooth cubic hypersurface in $\mathbb{P}^{a-1}$.

(iii) *The cyclic cover of $P = \mathbb{P}_k^n$, $n \ge 1$, defined by*

$$X(F; q, m) = \mathrm{Spec}_P \left(\bigoplus_{j=0}^{q-1} \mathcal{O}_P(-jm) \right), \qquad u^q = F, \tag{1.5}$$

where $q \ge 2$, $m \ge 1$, and F is a degree-qm polynomial in $n + 1 = 2b + a \ge 2$ variables, with disjoint reduced binary blocks and nonzero unary terms as in (1.3).

Under the local analytic hypotheses, over every finite $K/\mathbb{Q}_p$ these varieties satisfy the conclusion of Theorem 1.2 for every residue prime p.

The proofs are in §8.2, §9.1 and §10.1, respectively. Products are handled by a tensor identity of Chow correspondences. The cubic motive in (ii) is the abelian-type motive of Laterveer [27, Theorem 5]; the work needed here is its finite algebraic descent and its use in the original valued-field theory. The construction in (i) follows the controlled join of Shioda–Katsura [26, Remark 1.10].

Theorem 1.4 (Normalized height formula)**.** *Assume the local and global analytic hypotheses of Section 2. Let F be a number field and let X/F be smooth, projective and geometrically integral of dimension $n \ge 1$, with a potential abelian Chow retraction. The canonical normalized analytic height of the companion height theory is defined on*

$$\mathrm{CH}^r(X)_{\mathbb{Q},\mathrm{hom}} \times \mathrm{CH}^{n+1-r}(X)_{\mathbb{Q},\mathrm{hom}}, \qquad 1 \le r \le n. \tag{1.6}$$

For any factorization (1.2) over a finite E/F,

$$h_X^{\mathrm{an}}(z,w) = \frac{1}{[E:\mathbb{Q}]} \sum_j \left\langle u_{j*} z_E, {}^t v_{j*} w_E \right\rangle_{A_j,\mathrm{Kun}}. \tag{1.7}$$

Here the classical abelian pairing has the normalization of Künnemann [25]. The two image codimensions are $r+a_j$ and $\dim A_j+1-r-a_j$. Terms in zero or impossible homological degrees are omitted. The value is independent of the factorization and the finite field of definition.

Proof. See §13.1.

1.1 The geometric domains

For a smooth projective variety V/K, write $\mathsf{V}(V)$ for

$$\mathrm{CH}^r(V)_{\mathbb{Q},\text{geom hom}} \subseteq \ker \mathfrak{o}^r_{V,K} \quad \text{for every } r.$$

Here and below, the local assertions use the companion results stated in Section 2. A number-field variety has a *full height domain* if both complete groups in (1.6) are in the domain of the companion normalized pairing for every r.

Theorem 1.5 (Geometric domains)**.** *Assume the analytic hypotheses of Section 2 and the model inputs specified in the cited sections. Let $K/\mathbb{Q}_p$ be finite. Each of the following smooth projective geometrically integral varieties has property V.*

(i) *A variety of dimension at most three; a geometrically uniruled fourfold; or a variety of dimension at most five whose universal geometric rational CH_0 is supported in dimension at most three.*

(ii) *A product*

$$T \times \prod_a C_a \times \prod_b S_b,$$

where the C_a are arbitrary smooth projective curves, the S_b are arbitrary smooth projective surfaces, and T has potentially good projective reduction. The coefficient T may be a point.

(iii) *A variety whose rational Chow motive, after a finite extension, is a retract of a finite direct sum of Tate twists of the varieties in* (ii)*. In particular this includes products of point Hilbert schemes of arbitrary surfaces with curves and the allowed coefficient T, as well as the surface-moduli motives specified in Section 5.*

(iv) *A variety in one of the full-degree Chow-niveau, complete-intersection, or flat-fibration cases of Section 6, or satisfying an all-degree numerical model criterion of Section 11.*

For every specified geometrically homologically trivial cycle, property V supplies a form satisfying (1.1) on its exact support complement. For a number-field variety to which one of these criteria applies at every finite completion, the normalized pairing has a full height domain.

The proofs of (i), (ii)–(iii), and (iv) occupy Section 4, Section 5, and Section 6–Section 11, respectively. The global assertion follows from Lemma 4.2. Geometric constructions can be base changed to all completions; a reduction condition on T is imposed separately at each finite place.

The distinction between a local degree range and a global height range is useful even when all degrees are not covered. If $S \subseteq \{1,\dots,n\}$ is the set of local degrees established at every finite place, then the full height pairs occur in

$$S \cap (n+1-S). \tag{1.8}$$

For example, small diagonal support gives local degrees $r \leq 2$ or $r \geq n-2$; for $n \geq 6$ the corresponding full height degrees are $r \leq 2$ or $r \geq n-1$.

On an arbitrary smooth projective variety, the same formalism applies to subgroups. In particular algebraically trivial cycles lie in the local obstruction kernel in every degree. This gives the pairing on the two algebraically trivial groups described in Section 12.

1.2 Proof strategy and relation to earlier work

The local transfer has three steps. First, Künnemann's exact complex on a strictly semistable abelian model produces a cycle extension whose restriction to every special-fiber component is homologically trivial. The companion numerical criterion then annihilates the actual obstruction. Second, the diagonal identity expresses an original Chow class as a sum of images of homologically trivial abelian classes. Chow-kernel covariance transfers their vanishing and the full finite degree descends it to K. Third, the original curved Green pair is corrected on X itself. This last step retains the original support.

For the separated-variable families, a single join step has source $\mathrm{Bl}_C(A \times B)$ with a smooth codimension-two center C. Both the ambient product and the center have lower-complexity Chow factorizations. The exact blowup identity and the degree retraction therefore give a recursive factorization for the target. For cyclic covers, the additional map from an ordinary hypersurface is finite flat everywhere, of rank m. Its full rank is the factor in the Chow identity and in the height formula.

The geometry of abelian-type motives and the classical abelian height are inputs to this argument. Künnemann's height already has correspondence functoriality and finite-extension compatibility [25, §1]. Our height formula records its realization by the companion ordinary-form theory and the precise scalar in the explicit Chow constructions. The companion references in Section 2 separate this analytic input from the algebraic arguments proved here.

After the analytic input and abelian transfer, we treat the endpoint degrees, small diagonal support, products, Chow niveau, and flat fibrations. The explicit Chow identities then give separated hypersurfaces and cyclic covers. The final sections discuss model criteria, global families, and the normalized height formulas.

2 The analytic hypotheses

2.1 Ordinary forms and the cycle obstruction

Fix K as above and set $D = d'd''$. For a smooth projective K-variety, the actual Bott–Chern group used in the companion theory is

$$H^{r,r}_{\mathrm{BC}}(X^{\mathrm{an}}) = \frac{\{\omega \in B^{r,r}(X^{\mathrm{an}}) : d'\omega = d''\omega = 0\}}{DB^{r-1,r-1}(X^{\mathrm{an}})}. \tag{2.1}$$

The real vector-space structure of this group is retained, and the cycle map below is $\mathbb{Q}$-linear.

Hypothesis 2.1 (Actual cycle obstruction)**.** *There is a Chow-invariant map*

$$\mathfrak{o}^r_{X,K} : \mathrm{CH}^r(X)_{\mathbb{Q}} \longrightarrow H^{r,r}_{\mathrm{BC}}(X^{\mathrm{an}}).$$

For a specified cycle Z, the construction gives an ordinary actual form G_Z on its exact support complement and a closed global ordinary form Ω_Z such that

$$D[G_Z] + \delta_Z = [\Omega_Z], \qquad \mathfrak{o}^r_{X,K}([Z]) = [\Omega_Z]. \tag{2.2}$$

The equality is tested against every complementary compact actual form, with the integrability supplied by the construction. If $\mathfrak{o}^r_{X,K}([Z]) = 0$, there is a global $\Psi \in B^{r-1,r-1}(X^{\mathrm{an}})$ with $D\Psi = \Omega_Z$, and $G_Z - \Psi$ satisfies (1.1) on the same complement.

This is the curved-pair and obstruction construction of [31], specifically the formulas PC7, OC1 and OC2 and their exact-support sufficient direction. The current notation in (2.2) expresses the integral identity of the supplied ordinary forms.

2.2 Numerical witnesses and Chow covariance

We state the numerical input in the finite constant-field case used in this paper. A pure operational representative of a codimension-r cycle on a regular model means a rational virtual vector bundle whose Chern character has that cycle in degree r and vanishes in all other degrees. Such a representative is part of the companion construction.

Hypothesis 2.2 (Finite numerical-model criterion)**.** *Let V/K be smooth projective and integral, let $1 \le r \le \dim V$, and let $z \in \mathrm{CH}^r(V)_{\mathbb{Q}}$. Suppose that a smooth integral finite constant-field cover $V' \to V$, of complete native degree $e > 0$, has a regular projective model carrying a pure operational representative of the pulled-back class z. Suppose that its degree-r character pairs to zero with every integral vertical r-cycle, with all scheme-theoretic fiber multiplicities and residue degrees retained. Then $\mathfrak{o}^r_{V,K}(z) = 0$.*

This is the finite-cover case of [32], Theorem `r12a:finite-numerical-cover`. It applies as well when the constant field of V is a finite extension of K: the model and its numerical degrees are then regarded over K°.

Write $\mathcal{K}^r_K(V) = \ker \mathfrak{o}^r_{V,K}$. The following covariance is an assertion about Chow classes.

Hypothesis 2.3 (Native Chow-kernel covariance)**.** *In the fixed K-theory, rational Chow correspondences between smooth projective varieties preserve $\mathcal{K}^*_K$, with their usual codimension shifts. In particular proper Chow pushforward preserves the kernel. For a finite extension L/K, the projection $\pi : V_L \to V$ satisfies*

$$\pi_* \pi^* z = [L : K] z.$$

Thus native-K obstruction zero of $\pi^* z$ implies obstruction zero of z. Finite actual trace and integration use the full degree $[L : K]$ and the absolute value extending that of K.

The source is [33], Theorem `r16cov:correspondences`, together with the full finite trace theorem FT16. If L/K has ramification index $e(L/K)$ and residue degree $f(L/K)$, then

$$[L : K] = e(L/K) f(L/K), \qquad c_L = -\log |\varpi_L| = c_K / e(L/K). \tag{2.3}$$

There is a useful distinction in this convention. If V is defined over L, its native obstruction $\mathfrak{o}_{V,K}$ is calculated on V regarded as a K-scheme. A correspondence over L is inserted into the open-and-closed diagonal factor

$$V \times_L W \hookrightarrow V \times_K W.$$

Composition on this factor is composition over L. This realizes an L-Chow identity inside the original K-theory without changing its coefficients.

2.3 Heights on the admitted Chow domain

For number fields, use the standard absolute values: the finite ones extend $|p| = p^{-1}$ and the archimedean ones are the usual absolute values, with the companion product-formula weights. For a number-field variety X/F, let $\mathcal{H}^r(X)$ be the subgroup of $\mathrm{CH}^r(X)_{\mathbb{Q},\mathrm{hom}}$ whose localizations at all finite places have zero actual obstruction.

Hypothesis 2.4 (Admitted-domain reciprocity and comparison)**.** *The companion constructions [34] define a canonical normalized real pairing*

$$h_X^{\mathrm{an}} : \mathcal{H}^r(X) \times \mathcal{H}^{n+1-r}(X) \longrightarrow \mathbb{R}.$$

It includes the archimedean terms, local residue identities, global reciprocity, and descent to rational equivalence. On admitted classes it is invariant under finite number-field extension. For an actual Chow correspondence $v : A \rightsquigarrow X$, its adjunction formula is

$$h_X^{\mathrm{an}}(v_*a, w) = h_A^{\mathrm{an}}(a, {}^t v_* w). \tag{2.4}$$

For an abelian variety A/E, the comparison theorem gives

$$h_A^{\mathrm{an}}(a, b) = [E : \mathbb{Q}]^{-1} \langle a, b \rangle_{A,\mathrm{Kun}}. \tag{2.5}$$

The three assertions are H16, HA16 and MM16 in the fixed companion version. Their original proof texts are supplied separately. In particular the height hypothesis includes independence of the Green choices, local reciprocity, good-place vanishing and rational-equivalence descent; it is a separate input from the existence of local Green forms.

3 Abelian varieties and Chow-motivic transfer

3.1 The local abelian model

Let R be a complete discrete valuation ring of characteristic-zero fraction field F and finite residue field k. Let $\mathcal{P}/R$ be regular, projective and strictly semistable of relative dimension d, with special fiber $Y = \bigcup_i D_i$. For a smooth projective k-variety D, put

$$A^q(D) = \mathrm{im}\big(\mathrm{CH}^q(D)_{\mathbb{Q}} \to H^{2q}_{\text{ét}}(D_{\bar{k}}, \mathbb{Q}_\ell(q))\big), \qquad \ell \neq \mathrm{char}\, k.$$

Using the signed restriction ρ and Gysin γ maps on the component intersections, define

$$A^q(Y) = \ker\big(A^q(Y^{(1)}) \xrightarrow{\rho} A^q(Y^{(2)})\big),$$
$$A_s(Y) = \mathrm{coker}\big(A^{d-s-1}(Y^{(2)}) \xrightarrow{\gamma} A^{d-s}(Y^{(1)})\big).$$

Here $Y^{(a)}$ is the disjoint union of the a-fold intersections.

For an abelian variety over F, Künnemann constructs such a model after a finite extension and proves that the natural homology map $A_s(Y) \to H^{\text{ét}}_{2s}(Y_{\bar{k}}, \mathbb{Q}_\ell(-s))$ is injective and that

$$A_{d+1-r}(Y) \xrightarrow{i^* i_*} A^r(Y) \xrightarrow{\cap [Y]} A_{d-r}(Y) \tag{3.1}$$

is exact at the middle term [25, Theorems 5.6 and 6.6]. The complete-DVR clause of the correction in [25, §5.8] is the one used here.

Lemma 3.1. *Assume the injectivity and exactness just stated. Every $z \in \mathrm{CH}^r(\mathcal{P}_F)_{\mathbb{Q},\mathrm{hom}}$ has an extension $\Theta_0 \in \mathrm{CH}^r(\mathcal{P})_{\mathbb{Q}}$ whose refined restriction to each D_i is geometrically homologically trivial.*

Proof. Choose a cycle extension Θ and let $a \in A^r(Y)$ be its component restrictions. The class $a \cap [Y]$ is the cycle-image class of the specialization of z. Its geometric homology class is zero because specialization commutes with the cycle class map. Injectivity gives $a \cap [Y] = 0$ in $A_{d-r}(Y)$, and exactness of (3.1) supplies $b \in A_{d+1-r}(Y)$ with $i^* i_* b = a$.

The map $\mathrm{CH}_{d+1-r}(Y)_{\mathbb{Q}} \to A_{d+1-r}(Y)$ is surjective. Lift b to β and set $\Theta_0 = \Theta - i_*\beta$. Its generic restriction is z, and its tuple of component cycle classes is zero. Since $A^r(Y)$ is a subspace of the direct sum of the component cycle-image groups, each component restriction has zero class. □

Proposition 3.2. *For every abelian variety A/K and every r, $\mathrm{CH}^r(A)_{\mathbb{Q},\mathrm{hom}} \subset \mathcal{K}^r_K(A)$. The same holds for an abelian variety over a finite L/K regarded as a K-scheme, with homological triviality on every geometric component.*

Proof. Choose the finite extension and the abelian model supplied above. By Lemma 3.1, the pulled-back cycle has an extension with homologically trivial component restrictions. These restrictions are numerically trivial. Its pure operational representative therefore has zero pairing with every integral vertical r-cycle: such a cycle lies in a special-fiber component, and the pairing is the numerical pairing of the refined restriction there. Hypothesis 2.2 gives the desired native obstruction zero.

If A is over L, choose its semistable extension over a finite M/L and regard the model over K°. Residue degrees multiply the numerical pairings, and ramification is included in the scheme-theoretic multiplicities. Their zero values are unchanged. The same criterion applies to the finite K-morphism $A_M \to A$. The degree-zero and out-of-range homological Chow groups vanish. □

3.2 Proof of the transfer theorem

Let L/K and (u_j, v_j) be a retraction as in (1.2), and let $z \in \mathrm{CH}^r(X)_{\mathbb{Q},\mathrm{hom}}$. Every $u_{j*}z_L$ is geometrically homologically trivial. By Proposition 3.2, it lies in the native group $\mathcal{K}_K^{r+a_j}(A_j)$. Applying Hypothesis 2.3 to v_j and summing the diagonal identity gives

$$z_L = \sum_j v_{j*} u_{j*} z_L \in \mathcal{K}^r_K(X_L).$$

As explained in §2.2, this is an identity in the original K-theory. The finite projection $\pi : X_L \to X$ gives $\pi_* z_L = [L : K] z$. Kernel covariance and rational division by the complete degree imply $\mathfrak{o}^r_{X,K}(z) = 0$.

For the specified cycle Z, choose the pair (G_Z, Ω_Z) of Hypothesis 2.1. Its vanishing class supplies a global Ψ with $D\Psi = \Omega_Z$. Thus $\gamma_Z = G_Z - \Psi$ has the required support, integrability and test identity. The same argument proves the fixed-degree version, since only the action of the diagonal on z_L was used. This proves Theorem 1.2.

4 Endpoint degrees and small diagonal support

4.1 Chow transfer and complementary degrees

Lemma 4.1 (Transfer through a Chow diagonal)**.** *Suppose that over a finite extension L/K there are smooth projective varieties V_j and correspondences of opposite degrees satisfying*

$$\Delta_{X_L} = \sum_j v_j \circ u_j \quad \text{in } \mathrm{CH}^{\dim X}(X_L \times_L X_L)_{\mathbb{Q}}.$$

Fix r. If every homologically trivial image $u_{j*}z_L$ is in the native obstruction kernel of V_j, then $z \in \mathcal{K}^r_K(X)$. In particular $\mathsf{V}(V_j)$ for every j implies $\mathsf{V}(X)$.

Proof. Cycle classes commute with correspondences, so the image classes are homologically trivial. Kernel covariance sends $v_{j*}u_{j*}z_L$ into $\mathcal{K}^r_K(X_L)$. Sum the diagonal and push down by $X_L \to X$. The result is $[L : K]z$, and rational division gives the conclusion. All correspondences over L are interpreted on the diagonal open-and-closed factor of the product over K, as in §2.2. Finally apply the curved-pair theorem to the specified representative of z. □

Lemma 4.2 (The two-degree criterion)**.** *Let X/F have dimension n, and let $S \subseteq \{1, \dots, n\}$. Suppose that at every finite place v, all geometric homological classes in degrees $r \in S$ have zero native obstruction. Then $\mathcal{H}^r(X) = \mathrm{CH}^r(X)_{\mathbb{Q},\mathrm{geom\,hom}}$ for $r \in S$. The full height pairing in degrees $(r, n+1-r)$ is therefore defined whenever $r \in S \cap (n+1-S)$.*

Proof. Geometric cycle classes commute with passage to F_v. Thus every global homologically trivial class in an asserted degree belongs to each local kernel defining $\mathcal{H}^r(X)$. The opposite inclusion is in the definition of $\mathcal{H}^r(X)$. Apply the companion pairing to both resulting equalities. □

Proposition 4.3 (Endpoint input)**.** *For a smooth projective n-fold over the original K, geometric homological divisors, zero-cycles and one-cycles have zero obstruction. Thus $\mathsf{V}(X)$ holds if $n \leq 3$.*

The divisor and zero-cycle assertions are the endpoint results in [11]. For one-cycles, the precise input is the regular-model curve correction followed by dominant obstruction descent, also recorded there. Its proof proceeds as follows. Choose a projective strictly semistable alteration $V \to X$, over a finite constant extension, of the same dimension as X. The pullback of a homologically trivial one-cycle remains a one-cycle. The curve correction produces a rational virtual bundle on the regular model whose generic degree-$(n-1)$ character is that class and whose degree on every vertical $(n-1)$-cycle is zero. The numerical-model criterion annihilates its obstruction; dominant descent gives the assertion on X. This uses the full degree of the alteration and leaves the final curved-pair correction on X.

4.2 Two orientations of the diagonal

Definition 4.4. A rational small-support diagonal for X, of dimension n, is an identity

$$\Delta_X = A + B, \qquad \operatorname{Supp} A \subseteq D \times X, \quad \operatorname{Supp} B \subseteq X \times T, \tag{4.1}$$

in rational Chow, where D is a divisor and $\dim T \leq 3$. The identity may be supplied after a finite extension of the ground field. The supports belong to actual cycle representatives of A, B.

Theorem 4.5 (Small diagonal support)**.** *If X has a small-support diagonal, its complete geometric homological Chow groups are in the local obstruction kernel for*

$$r \leq 2 \quad \text{or} \quad r \geq n-2. \tag{4.2}$$

It has property V if $n \leq 5$. Over a number field, the same geometric hypothesis gives a full height domain if $n \leq 5$; for $n \geq 6$ it gives the full pairs with $r \leq 2$ or $r \geq n-1$.

Proof. Resolve the components of D and T, with maps $a_i : D_i \to X$ and $b_j : T_j \to X$. A cycle supported on the image of a proper surjection can be lifted rationally: choose a closed point

of the fiber above its generic point, take its closure, and divide by the full generic degree. This also applies to cycles contained in exceptional images. It lifts (4.1) to correspondences

$$U_i \in \mathrm{CH}^{n-1}(D_i \times X)_{\mathbb{Q}}, \qquad V_j \in \mathrm{CH}^{t_j}(X \times T_j)_{\mathbb{Q}}, \quad t_j = \dim T_j \le 3.$$

The identity and its transpose act as

$$z = \sum_i U_{i*} a_i^* z + \sum_j b_{j*} V_{j*} z, \tag{4.3}$$

$$z = \sum_i a_{i*}\, {}^t U_{i*} z + \sum_j {}^t V_{j*} b_j^* z. \tag{4.4}$$

All intermediate classes are geometrically homologically trivial. In (4.3), the class on D_i has dimension $n-1-r \le 1$ when $r \ge n-2$. In (4.4), it has codimension $r-1 \le 1$ when $r \le 2$. These are the endpoint degrees. Every feasible class on T_j is an endpoint class because $t_j \le 3$; impossible degrees and homologically trivial rank classes are zero. Apply kernel covariance to the indicated orientation. If the identity is defined after a finite extension, use Lemma 4.1 to descend. The two local ranges cover every degree for $n \le 5$. Intersecting them with their complementary ranges proves the height assertions by Lemma 4.2. □

Proposition 4.6 (Universal zero-cycle support)**.** *Suppose that a closed subset $T \subset X_{\bar{k}}$, of dimension at most s, surjects onto $\mathrm{CH}_0(X_\Omega)_{\mathbb{Q}}$ for every algebraically closed extension $\Omega/\bar{k}$. Then after a finite extension of k there is an identity (4.1) with $\dim T \le s$.*

Proof. Apply the support hypothesis to the generic point over $\overline{\bar{k}(X)}$. A rational-equivalence certificate uses only finitely many points, curves and functions, so it is defined over a finite extension of $\bar{k}(X)$. Norming down and dividing by its degree gives the supported generic-point relation over $\bar{k}(X)$. Close the cycle in $X_{\bar{k}} \times T$ and spread the relation over an open subset of the first factor. Chow localization puts the remaining difference from Δ_X over its closed complement; contain this complement in a divisor. Finally descend the finite cycle-and-function certificate and the supports to a finite extension of k. □

Corollary 4.7 (Rationally connected fibrations)**.** *Let $f : X \to B$ be a surjective morphism of smooth projective varieties in characteristic zero. Suppose that over a nonempty open of B it is smooth with rationally connected geometric fibers. If $\dim B \le 3$, then Theorem 4.5 applies to X. In particular it gives all degrees if $\dim X \le 5$. Geometrically rationally connected varieties, including smooth Fano varieties, are examples with zero-dimensional support.*

Proof. The closure of a closed point of the generic fiber is a multisection $T \subset X$ of dimension $\dim B$. It is proper and surjective over B. After any algebraically closed extension, move a zero-cycle into the smooth rationally connected locus. Each of its points is rationally equivalent, inside its fiber, to a point of T in the same fiber. Rational connectedness gives this equality by chains of rational curves. The multisection therefore supplies universal geometric rational zero-cycle support. Apply Proposition 4.6 and Theorem 4.5. □

4.3 Projective bundles and dominant images

Proposition 4.8. *Property V is preserved by projective bundles, by blowups along smooth centers when both the ambient variety and the center have V, and by smooth projective dominant morphism images. The assertions allow a preliminary finite extension.*

Proof. The Chow-motive projective-bundle and blowup decompositions are finite direct sums of Tate twists of the indicated smooth varieties. Apply Lemma 4.1. For a projective dominant morphism $f : V \to X$ with relative dimension e, choose a relatively ample class a whose top degree on the generic fiber is $d > 0$. The correspondence $d^{-1} f_*(a^e \cdot -)$ is a left inverse to f^* by the projection formula. Its action gives a rational Chow retraction and hence the same transfer. The formula can be checked after product with any smooth projective test variety, so it is a Chow correspondence identity. Finite descent is as before. □

Corollary 4.9. *A geometrically uniruled smooth projective fourfold has* V*. So does any smooth projective dominant morphism image of a projective-bundle tower over a variety of dimension at most three, or over one of these uniruled fourfolds.*

Proof. After a finite extension, a uniruled fourfold is dominated rationally by $B \times \mathbb{P}^1$ for a smooth projective threefold B. Resolve this map in characteristic zero by smooth blowups of the four-dimensional source. Every nontrivial center has dimension at most two, hence has V. The source and its resolution have V by Proposition 4.8. The resolved dominant morphism transfers it to X. The remaining assertions follow successively from projective bundle and dominant-morphism transfer. □

5 Products of curves and surfaces and their Chow summands

5.1 The product-model input

The product theorem uses a numerical calculation on a particular semistable model. We state its algebraic content before applying the companion comparison with actual-form obstructions.

Let R be a complete discrete valuation ring with finite residue field. Take regular projective reduced strictly semistable models of finitely many curves and surfaces. Fix orders on their components and form the ordered staircase resolution of their product. Write its special fiber as $D = \bigcup_i D_i$, with $\dim D_i = d$, retaining each connected intersection separately. For an oriented edge $i \to j$, let $j_i : E_{ij} \to D_i$ and $j_j : E_{ij} \to D_j$ be the inclusions. Set

$$(\rho x)_{ij} = j_j^* x_j - j_i^* x_i, \qquad \sigma y = (j_{i*} y, -j_{j*} y), \qquad A = \sigma\rho.$$

These are cohomological actions of actual rational Chow correspondences. The off-diagonal entries of A have codimension $d+1$; its diagonal entry on D_i is

$$(\Delta_i)_* c_1(N_{D_i/\mathcal{D}}) = -(\Delta_i)_* \sum_{j \neq i} [E_{ij}].$$

Proposition 5.1 (Companion product-model theorem)**.** *For this ordered model there is an invertible prime ℓ and a matrix of rational Chow correspondences G, with entries of codimension $d-1$, for which*

$$A^H G^H A^H = A^H \tag{5.1}$$

on full geometric ℓ-adic cohomology, in every degree and with its Tate twist. This remains true after product with a smooth projective R-scheme, by replacing A, G with the exterior products of their correspondences and the coefficient's diagonal.

The proof is [12, model splitting theorem], whose actual deletion and residual-core constructions are included in the proof dossier. The essential structure is as follows. Each component is a tower of blowups of closed strata in the full reduced boundary of a product

of smooth curve and surface components. For a subset I of the surface indices, coordinate deletion gives an actual proper birational map

$$f_I : D_{\boldsymbol{i}} \longrightarrow \left(\prod_{j \in I} X_{j,i_j} \right) \times W_{-I,\boldsymbol{i}_{-I}}, \tag{5.2}$$

where W_{-I} is the staircase component of the remaining factors. The refined surface projectors p_j onto H^2_{tr} annihilate cohomology supported on the boundary curves. Consequently the transported projectors

$$P_I = f_I^*((\boxtimes_{j \in I} p_j) \boxtimes 1) f_{I*}$$

satisfy $P_I = \prod_{j \in I} P_{\{j\}}$, commute with the incidence, and do so on exceptional summands as well. The proof of commutation uses support localization for the kernel of a birational pull-push operator in (5.2).

The simultaneous sectors retain the tensor factors t_2 selected by these projectors. The residual projector annihilates each generator which still contains a full-surface t_2 factor; the deletion proof establishes this equality in relative Chow before passing to localization. The surviving blowup and localization generators in the complementary factor are of abelian type. Thus the incidence operator on a sector is the identity on its retained transcendental factor tensored with an operator on one abelian carrier. Numerical semisimplicity and the one-prime comparison for that carrier give its inner inverse, which is tensored back with the retained factor. Finite-field choices descend by normalized averaging: the equation (5.1) is linear in G. This is the algebraic product-model statement used here.

Two consequences of (5.1) explain its role in the height problem. Let $\mathscr{A}^u$ be the space of actual algebraic cycle classes on the disjoint union of the D_i, and let $p : \mathscr{A}^u \twoheadrightarrow N^u(D^{(0)})_{\mathbb{Q}} \otimes \mathbb{Q}_\ell$ be the quotient to numerical classes. If $p(x)$ is in the numerical kernel of A, then

$$x^\flat = x - G^H A^H x \tag{5.3}$$

has the same numerical image and belongs to $\ker A^H$. Also, if an algebraic class α is in $\operatorname{im} A^H$, then $G^H \alpha$ is algebraic and $A^H G^H \alpha = \alpha$. Hence

$$\mathscr{A}^{u+1} \cap \operatorname{im} A^H = A^H \mathscr{A}^u. \tag{5.4}$$

The numerical-kernel lifting and actual-span saturation are the two inputs to the companion model criterion [13].

Theorem 5.2 (Arbitrary curve and surface products)**.** *Let $X_1, \dots, X_m/K$ be arbitrary smooth projective geometrically integral curves or surfaces. Let T/K be smooth projective geometrically integral with potentially good projective reduction. Then*

$$X = T \times \prod_{a=1}^{m} X_a$$

has $\bigvee$*. The assertion holds over every finite extension of K for every newly specified geometrically homologically trivial cycle there. Empty products and a point as coefficient are permitted.*

Proof. After a common finite extension choose projective strictly semistable alterations of the low-dimensional factors and a smooth projective model of T. Semistable repair after any intermediate ramified extension is part of the ordered-model construction in [12]. Apply Proposition 5.1 to the resulting witness. Weight–monodromy for curves and surfaces, the

purity of the good coefficient, and Künneth give the odd-weight condition in the companion model criterion. Equations (5.3) and (5.4) supply its two exactness conditions. The pulled-back obstruction therefore vanishes. Dominant and finite-degree descent give zero obstruction over K; the original curved pair gives the requested form.

For an arbitrary finite K'/K, the factors still have the same dimensions and the coefficient remains potentially good. Reapply the construction over K' to its newly specified cycle. This proves the assertion for new classes, as well as for pullbacks of old ones. □

Over a number field, the curves and surfaces are unrestricted. If a coefficient T is present, it is required to have potentially good projective reduction at every finite place. Applying Theorem 5.2 at all completions and using Lemma 4.2 gives the full height in every complementary degree, also after every finite number-field extension.

5.2 Abelian factors and the surface-generated Chow domain

Lemma 5.3. *Over a characteristic-zero field, an abelian variety becomes a rational Chow-motive retract of a product of curves after a finite extension.*

Proof. For an abelian variety of dimension $g \geq 2$, choose a smooth complete-intersection curve and a point c_0 on it after a finite extension. Translate by $-c_0$ and call the resulting pointed curve C; choose the complete intersection so that $C - C$ generates A. The addition map $C^g \to A$ is dominant. Indeed, the successive sums of C increase in dimension until they fill A: stabilization at a proper subvariety would make that subvariety stable under the group generated by C. Since source and target have dimension g, the map is generically finite, of some degree $d > 0$. Its graph satisfies $f_* f^* = d\Delta_A$, and $d^{-1} f_*$ is a Chow retraction. Dimension one is immediate; dimension zero gives the unit motive. □

Corollary 5.4. *Suppose that after a finite extension the rational Chow motive of X is a direct summand of a finite direct sum of Tate twists of products of curves and surfaces. Then $\mathsf{V}(X)$ holds. It also holds for $T \times X$ with a potentially good projective coefficient. These conclusions include products with arbitrary abelian factors.*

Proof. Choose the inclusion and retraction correspondences. Their composites sum to the diagonal. Tensor with the coefficient's diagonal when needed, apply Theorem 5.2 to each intermediate variety, and use Lemma 4.1. An abelian factor may first be replaced by its curve-product retraction from Lemma 5.3. □

5.3 Hilbert schemes of points on arbitrary surfaces

Let S be a smooth projective geometrically integral surface and let $H = S^{[m]}$. For a partition $\lambda = 1^{a_1} \cdots m^{a_m}$, put $l(\lambda) = \sum_i a_i$, $s_\lambda = m - l(\lambda)$, and $G_\lambda = \prod_i \mathfrak{S}_{a_i}$. On the smooth power $S^{l(\lambda)}$, use the rational projector

$$e_\lambda = |G_\lambda|^{-1} \sum_{g \in G_\lambda} \Gamma_g.$$

The theorem of de Cataldo–Migliorini [1, Theorem 6.2.1 and Remark 6.2.2] gives

$$h(S^{[m]}) \simeq \bigoplus_{\lambda \vdash m} (h(S^{l(\lambda)}), e_\lambda) \otimes \mathbb{L}^{s_\lambda}. \tag{5.5}$$

We use the covariant twist convention $\mathrm{CH}_a(h(V)(s)) = \mathrm{CH}_{a-s}(V)_{\mathbb{Q}}$.

Theorem 5.5. *For arbitrary smooth projective geometrically integral surfaces S_i, curves C_j, and positive integers m_i, the variety*

$$T \times \prod_j C_j \times \prod_i S_i^{[m_i]}$$

has V *if T is smooth projective geometrically integral and has potentially good projective reduction. Over a number field, it has a full height domain when this reduction condition holds at every finite place.*

Proof. Tensor the actual incidence correspondences in (5.5). For a tuple of partitions they factor through $T \times \prod_j C_j \times \prod_i S_i^{l(\lambda_i)}$. If $s = \sum_i s_{\lambda_i}$, this intermediate variety has dimension $\dim X - 2s$ and an original codimension-r class maps to codimension $r - s$. Its homological class is zero. Apply Theorem 5.2 in every feasible degree and then Lemma 4.1. The group in degree zero has no homological kernel, and out-of-range groups vanish. The global assertion follows at all completions. The correspondences commute with base change, so the same proof applies to all new classes after finite extension. □

5.4 Finite specialization of Chow certificates

Lemma 5.6. *Let X/k be smooth projective in characteristic zero. Suppose that over a field extension Ω/k its diagonal factors in rational Chow through finitely many products of smooth projective curves and surfaces. Then such a factorization exists over a finite extension E/k, with auxiliary curves and surfaces allowed to change.*

Proof. Choose cycle representatives for the correspondences and finitely many principal divisors witnessing the diagonal equality. All these data, together with the auxiliary varieties and their projective embeddings, descend to a field finitely generated over k. Spread them over a smooth integral finite-type k-scheme B. Shrink B so that the auxiliary families are smooth projective with geometrically integral fibers, the cycles have their expected codimensions, and the relative Chow equality holds. The last step follows either by spreading its principal-divisor certificate or by the direct-limit description of Chow groups over the function field.

Choose a closed point $b \in B$. Its residue field E is finite over k. Since B is smooth, refined pullback to b commutes with the smooth pullbacks, proper projections and diagonal intersections which define correspondence composition. The constant family $X \times B$ specializes to X_E, while the auxiliary families specialize to smooth curves and surfaces. Thus the diagonal certificate specializes with all its degree shifts. □

Theorem 5.7 (Complex certificates and surface moduli)**.** *Let X/F be smooth projective over a number field. If for one complex embedding $h(X_{\mathbb{C}})$ retracts through products of curves and surfaces in rational Chow motives, then X has a full height domain, also after every finite number-field extension. In particular this holds if $X_{\mathbb{C}}$ is a smooth projective moduli space appearing in Bülles's theorem: Gieseker-stable twisted sheaves, or stable objects for a generic stability condition, on a complex projective K3 or abelian surface. Finite products of these spaces and the varieties in Theorem 5.5 are allowed, with an everywhere potentially good projective coefficient.*

Proof. By Lemma 5.6, a finite extension E/F carries an actual diagonal certificate of the required kind. Complete it at a place above each finite place of F and apply Corollary 5.4; full-degree descent gives the original local kernel vanishing. The same construction over a compositum treats every new cycle after finite extension. Apply Lemma 4.2.

For the moduli spaces, [2, Theorem 0.1] supplies precisely an actual rational Chow retraction through powers of the underlying surface. The first assertion applies. For products, tensor the certificates; the intermediate varieties remain products of curves and surfaces. The surface's finite-dimensionality is not an assumption of the cited retraction theorem. □

6 Chow niveau, complete intersections, and flat fibrations

6.1 A residual motive of small dimension

We use covariant rational Chow motives, with $\mathrm{CH}_a(h(T)(u)) = \mathrm{CH}_{a-u}(T)_{\mathbb{Q}}$.

Theorem 6.1 (Chow-niveau transfer)**.** *Let X/k have dimension n. Suppose that after a finite extension there is an actual rational Chow decomposition*

$$h(X) \simeq Q \oplus (T, e, u), \qquad Q \text{ of abelian type}, \quad \dim T \le n - 2u, \tag{6.1}$$

where $u \ge 0$. Over a finite extension of $\mathbb{Q}_p$ this gives zero obstruction on the full geometric homological Chow groups for

$$r \le u + 1 \quad \text{or} \quad r \ge n - u - 1. \tag{6.2}$$

Every degree is covered if $n \le 2u + 3$. Over a number field the full height pairs are obtained in every degree in that case; otherwise they are obtained for

$$r \le u + 1 \quad \text{or} \quad r \ge n - u. \tag{6.3}$$

A sufficient hypothesis for (6.1) is a morphism $N \to h(X)$ from an abelian-type Chow motive which, on an algebraically closed universal domain Ω/k of infinite transcendence degree over the field of definition, surjects on algebraically trivial CH_i for every $i < u$.

Proof. Let $m = \dim T$. The residual image of a codimension-r class has dimension $n - r - u$ and codimension

$$m - n + r + u \le r - u. \tag{6.4}$$

If $r \le u + 1$, this is a divisor degree or lower. If $r \ge n - u - 1$, the residual cycle dimension is at most one. The endpoint input applies in either case. If $n \le 2u + 3$, then $m \le 3$, so every residual degree is covered. The abelian-type summands are covered by Theorem 1.2. Their actual inclusion and retraction maps, followed by Lemma 4.1, prove the local assertion. Intersect its range with the range of $n + 1 - r$ to obtain (6.3).

For the sufficient hypothesis apply Vial's [28, Theorem 3.11] with algebraic equivalence and $M = h(X)$. The finite-dimensionality required there holds for N. Self-duality of $h(X)$ gives (6.1) over the field of definition of the morphism. Its additional curve and Tate summands are of abelian type. A negative residual dimension means that the residual summand is zero. □

In particular it suffices that $\mathrm{CH}_i(X_\Omega)_{\mathbb{Q},\mathrm{alg}} = 0$ for $0 \le i < u$: take $N = 0$. This criterion concerns the subgroup algebraically equivalent to zero, rather than a quotient by algebraic equivalence.

6.2 Small-degree Chow bounds

Let $X \subset \mathbb{P}^N$ be a smooth complete intersection of degrees $d_1 \ge \cdots \ge d_c \ge 2$, and put $n = N - c$.

Proposition 6.2. *Let $0 \le t \le n-1$. Suppose either*

$$(d_1 \ge 3 \text{ or } c \ge t+1), \qquad \sum_{a=1}^{c} \binom{d_a+t}{t+1} \le N, \tag{6.5}$$

or all $d_a = 2$, $1 \le c \le t$, and

$$c(t+2) \le N + c - t - 1. \tag{6.6}$$

Then Theorem 6.1 applies with $u = t+1$. For a hypersurface of degree $d \ge 3$, an alternative sufficient condition is

$$(j+1)(N-j)+1 \ge \binom{d+j+1}{j+1} \quad (0 \le j \le t). \tag{6.7}$$

In each case all degrees are covered if $n \le 2t+5$; otherwise the guaranteed full height degrees are $r \le t+2$ or $r \ge n-t-1$.

Proof. Theorem 4.6 of Esnault–Levine–Viehweg [4] gives $\mathrm{CH}_i(X_\Omega)_\mathbb{Q} = \mathbb{Q}$ for $0 \le i \le t$ under (6.5) or (6.6). In the hypersurface case, Hirschowitz–Iyer [5, Theorem 1.5] gives the same conclusion degree by degree under (6.7); the upper variety is projective space and its relevant Chow group is one-dimensional. A linear-section generator has nonzero projective degree. The algebraically trivial subgroup is therefore zero in all these degrees. Apply Theorem 6.1 with $u = t+1$. □

There is a further useful bound for mixed quadrics and cubics. Otwinowska's rewritten author version [6, Corollaire 1] gives the following statement for smooth complete intersections: if there are a quadrics and $b \ge 1$ cubics in $\mathbb{P}^N$, $N \le 15$, then $\mathrm{CH}_i(-)_\mathbb{Q}$ is generated by the ambient section class for

$$0 \le i \le t = \left\lfloor \frac{N-2a-3b}{3} \right\rfloor. \tag{6.8}$$

Its complex-field formulation gives the required universal-domain statement in characteristic zero. Indeed, descend a given cycle and its equations to a finitely generated field, embed that field in $\mathbb{C}$, and obtain the rational-equivalence certificate there. The certificate descends to a further finitely generated field, which embeds over the original field of definition into the universal domain. Its base change proves the original relation. The coefficient of the ambient class is fixed by projective degree.

Corollary 6.3 (Explicit complete-intersection domains)**.** *Every smooth geometrically integral variety in the following table has V locally and a full height domain over a number field. A presentation after a finite extension suffices.*

Type	*Dimension*	*Low-Chow parameter t*
Cubic hypersurface	$6, 7$	1
Cubic hypersurface	$8, 9$	2
Cubic hypersurface	11	3
Quartic hypersurface	7	1
Complete intersection $(2,3)$	$6, 7$	1
Complete intersection $(2,3)$	9	2
Complete intersection $(2,2,3)$	7	1
Complete intersection $(3,3)$	7	1

The cubic cases in dimension at most five already follow from Corollary 4.7.

Proof. For cubics the values in (6.7) are $4, 10, 20, 35$ for $j = 0, 1, 2, 3$. The requisite minimum ambient dimensions for $t = 1, 2, 3$ are $6, 9, 12$, respectively. These imply the first three rows. For the quartic row, $N = 8$ and the two inequalities are $9 \geq 5$ and $15 \geq 15$.

For $(2, 3)$ of dimension 6 or 7, the zero-cycle statement follows from (6.5). The upper quadric, which may be singular, has $\mathrm{CH}_2 = \mathbb{Q}$ by the all-intersection form of [4, Theorem 4.6] and (6.6). The distinct-largest-degree case of [5, Theorem 1.5], with $j = 1, c = 2$, has $\rho = 3N-19$ and $\rho+1 \geq N-2$ exactly when $N \geq 8$. Thus $\mathrm{CH}_1 = \mathbb{Q}$. The last three rows follow from (6.8): their ambient dimensions are $11, 10, 9$, respectively. In every row $n \leq 2t + 5$. □

6.3 The all-fiber criterion

Proposition 6.4 (Flat-fibration transfer)**.** *Let $f : Y \to B$ be flat, projective and surjective, with Y, B smooth projective and $\dim Y = n$, $\dim B = b$. Fix an integer $u \geq 1$ with $n - 2u \geq 0$. Suppose that for every geometric scheme fiber over a universal domain*

$$\mathrm{CH}_i(Y_\omega)_{\mathbb{Q}} = \mathbb{Q} \quad (0 \leq i < u).$$

Assume $\mathsf{V}(B)$ in the native local theory. Then the full local source vanishes for $r \leq u + 1$ or $r \geq n - u - 1$, and in all degrees if $n - 2u \leq 3$. If these base hypotheses hold at every finite place of a number-field model, the corresponding full height degrees are obtained by (1.8).

Proof. Vial's flat-fibration decomposition [3, Theorem 4.2] is an actual Chow decomposition

$$h(Y) \simeq \bigoplus_{j=0}^{n-b} h(B)(j) \oplus (T, e, u), \qquad \dim T = n - 2u. \tag{6.9}$$

The base summands are admitted by hypothesis. The residual-degree calculation (6.4) and the endpoint input give the stated local degrees. Transfer through the actual diagonal and apply Lemma 4.2 globally. □

Corollary 6.5 (Quadric and cubic fibers)**.** *The following flat projective surjective maps, with smooth projective geometrically integral total space and base, have V:*

(i) *Every geometric scheme fiber is a nonzero quadric of fixed dimension $q \geq 1$ in $\mathbb{P}^{q+1}$, and $1 \leq b \leq 3$. The same holds for $b = 4$, q odd, if the base has V; in particular the base may be $\mathbb{P}^4$.*

(ii) *Every geometric scheme fiber is a nonzero cubic hypersurface of fixed dimension q, and*

$$(q, b) = (5, 1), (5, 2), (6, 1), \text{ or } (8, 1).$$

The conclusions concern the entire geometric homological Chow groups. Over number fields the same geometric hypotheses give full height domains, with the base condition imposed at all finite places when $b = 4$.

Proof. A nonzero quadric scheme of dimension q over an algebraically closed characteristic-zero field has $\mathrm{CH}_i \simeq \mathbb{Z}$ for $i < q/2$. This includes rank-one quadrics: Chow groups are unchanged by reduction. For other singular quadrics, the cone description and the split-quadric cellular filtration give the same range; the cone localization proof is recorded in [14]. Take $u = \lceil q/2 \rceil$ in Proposition 6.4. Its residual dimension is b for even q and $b - 1$ for odd q, giving (i).

For cubics of dimension at least five, the all-intersection ELV bound gives $\mathrm{CH}_0 = \mathrm{CH}_1 = \mathbb{Q}$. For $q = 8$, the HI bound also gives $\mathrm{CH}_2 = \mathbb{Q}$. These cited bounds apply to the reduced singular

hypersurfaces in question. A nonreduced cubic has reduction a hyperplane or a union of two hyperplanes; the closed union Chow sequence gives the same low-degree groups. Thus the four cases have residual dimensions $2, 3, 3, 3$, respectively, using $u = 2, 2, 2, 3$. The base has dimension at most two and hence has V. Apply Proposition 6.4. □

6.4 A signed retraction for intersections of quadrics

Theorem 6.6 (Intersections of quadrics)**.** *A smooth complete intersection of c quadrics in $\mathbb{P}^N$, $1 \le c \le N-1$, has V in all degrees when*

$$c \le 4, \qquad \text{or} \qquad c = 5 \text{ and } N \text{ is even.}$$

For arbitrary c, the local degrees supplied are

$$r \le \lfloor N/2 \rfloor - c + 2 \quad \text{or} \quad r \ge \lceil N/2 \rceil - 2. \tag{6.10}$$

The full global height degrees outside the full case are $r \le \lfloor N/2 \rfloor - c + 2$ or $r \ge \lceil N/2 \rceil - 1$.

Proof. For $c = 1$, pass to a finite splitting field of the smooth quadric. Its cellular Chow decomposition is Tate, so Theorem 1.2 gives every degree. Assume $c \ge 2$. Write the equations as $Q_1, \dots, Q_c$ and let

$$Y = \{\textstyle\sum_j t_j Q_j = 0\} \subset \mathbb{P}^N \times \mathbb{P}^{c-1}.$$

It is a flat family of nonzero $(N-1)$-dimensional quadric schemes. The total space is smooth: at a possible singular point all Q_j vanish, while their independent conormal differentials on the smooth complete intersection make $\sum_j t_j dQ_j$ nonzero. Set $m = c - 1$, $E = X \times \mathbb{P}^m$, and write $j : E \hookrightarrow Y$, $p : E \to X$. The normal sequence is

$$0 \longrightarrow N_{E/Y} \longrightarrow p^*\mathcal{O}_X(2)^{\oplus c} \longrightarrow p^*\mathcal{O}_X(2) \otimes \mathcal{O}_{\mathbb{P}^m}(1) \longrightarrow 0.$$

Its degree-m Chern class has fiber-leading term $(-1)^m c_1(\mathcal{O}_{\mathbb{P}^m}(1))^m$, and fiber integration gives

$$p_* c_m(N_{E/Y}) = (-1)^m.$$

Consequently the correspondences

$$U = j_* p^*, \qquad V = (-1)^m p_* j^!, \qquad VU = 1 \tag{6.11}$$

give a retraction, by self-intersection. The calculation after every smooth projective product shows the same identity for Chow correspondences. Here U sends codimension r to $r + m$.

Apply Proposition 6.4 to $Y \to \mathbb{P}^{c-1}$, with $u = \lceil (N-1)/2 \rceil$. Its residual dimension is $c - 2$ if N is even, and $c - 1$ if N is odd. These are at most three in the asserted full cases. Substituting $r + m$ in the outer range and using (6.11) gives (6.10). Intersecting the two complementary ranges gives the height statement. □

Theorem 6.7 (Relative quadrics)**.** *Let B be smooth projective geometrically integral of dimension $b \ge 1$, and let $\mathcal{V}, \mathcal{F}$ be vector bundles of ranks $N+1, c$, with $c \ge 1$ and $N \ge c+1$. Let $\pi : \mathbb{P}_B(\mathcal{V}) \to B$ parametrize lines. Suppose that the zero scheme of a regular section of $\mathcal{O}(2) \otimes \pi^*\mathcal{F}$ is smooth projective geometrically integral, is flat over B, and has expected codimension c in every geometric fiber. If $\mathsf{V}(B)$, then it has V whenever*

$$b + c \le 5 \quad (N \text{ even}), \qquad b + c \le 4 \quad (N \text{ odd}).$$

In general the local degree range is

$$r \le \lfloor N/2 \rfloor - c + 2 \quad \text{or} \quad r \ge b + \lceil N/2 \rceil - 2.$$

Over a number field with the base admitted at every finite place, the corresponding full height range ends instead at $r \ge b + \lceil N/2 \rceil - 1$. The base condition is automatic for $1 \le b \le 3$.

Proof. Form $C = \mathbb{P}_B(\mathcal{F}^\vee)$ and the universal quadric $Y \to C$. Expected fiber codimension ensures that no nonzero coefficient vector gives the zero quadratic equation. Smoothness of the total space follows from the absolute differential of the regular section defining X; the base directions are included. The projective-bundle formula gives $\mathsf{V}(C)$. For $E = X \times_B C$, the same normal sequence and fiber integration as in (6.11) give a signed retraction with shift $c-1$. The residual dimension of the universal quadric is $b+c-2$ for even N and $b+c-1$ for odd N. Apply the flat-fibration theorem, transfer through the retraction, and use Lemma 4.2. □

7 Actual Chow identities

For the algebraic arguments, let k be any characteristic-zero field. Write $\mathcal{A}_k(V)$ for a diagonal factorization of V through abelian varieties over k itself, in the sense of (1.2) with $L = k$. Smooth disconnected varieties are interpreted by the direct sum of their components.

Lemma 7.1. *The property $\mathcal{A}_k$ is stable under finite products, finite disjoint unions of pure-dimensional components, isomorphisms and base extension. A pointed smooth projective geometrically integral curve has this property.*

Proof. For products, take exterior products of the two sets of correspondences, with the factors ordered as source and target. Composition commutes with exterior products, so

$$\Delta_{V\times W} = \Delta_V \boxtimes \Delta_W = \sum_{j,l} (v_j \boxtimes v_l) \circ (u_j \boxtimes u_l).$$

The intermediate variety $A_j \times A_l$ is abelian. This equality is in the Chow group of the product with itself. Disjoint unions and isomorphisms follow by inserting component maps and by conjugating with graphs; flat field extension preserves the identity.

For a pointed curve C, use $h(C) = \mathbf{1} \oplus h^1(C) \oplus \mathbb{L}$ and the identification of $h^1(C)$ with the corresponding summand of $h(J(C))$. These are Chow-motive decompositions [29, §3.2, Proposition 4.5 and Theorems 5.2–5.3]. Their inclusion and retraction maps, with the Tate terms through $\operatorname{Spec} k$, give the required identity. □

7.1 A codimension-two blowup

Let $j : C \hookrightarrow V$ be a smooth center of codimension two, and let

$$q : W = \operatorname{Bl}_C V \longrightarrow V, \qquad i : E \hookrightarrow W, \qquad p : E = \mathbb{P}_C(N_{C/V}) \longrightarrow C.$$

Set $\xi = c_1(\mathcal{O}_E(1))$. Then $N_{E/W} = \mathcal{O}_E(-1)$ and $p_*\xi = 1$.

Lemma 7.2. *There is an equality of rational Chow correspondences*

$$\Delta_W = q^*q_* - i_*p^*p_*i^*. \tag{7.1}$$

If $\mathcal{A}_k(V)$ and $\mathcal{A}_k(C)$ hold, so does $\mathcal{A}_k(W)$.

Proof. The Chow blowup formula [30, §6.7] writes every class $z \in \mathrm{CH}^r(W)_{\mathbb{Q}}$ uniquely as $z = q^*a + i_*p^*b$, where $b \in \mathrm{CH}^{r-1}(C)_{\mathbb{Q}}$. The projection and self-intersection formulas give

$$\begin{aligned} q_*q^*a &= a, & q_*i_*p^*b &= 0, \\ p_*i^*q^*a &= 0, & p_*i^*i_*p^*b &= -p_*(\xi p^*b) = -b. \end{aligned}$$

Consequently $a = q_*z$ and $b = -p_*i^*z$, which proves (7.1) as an operator identity.

To obtain a correspondence identity, repeat the argument after product with any smooth projective T. Flat product commutes with the blowup, and the same calculation applies to every Chow group of $T \times W$. Taking $T = W$ and applying the operator on the second factor to Δ_W proves the equality of correspondences. Finally insert the given factorizations of V and C. The center maps $-p_* i^* : W \rightsquigarrow C$ and $i_* p^* : C \rightsquigarrow W$ have degrees -1 and 1. □

7.2 Generically finite morphisms

Lemma 7.3. *Let $g : W \to X$ be a dominant generically finite morphism of degree d between smooth projective geometrically integral varieties of the same dimension. Then*

$$g_* g^* = d\Delta_X \tag{7.2}$$

as Chow correspondences. Thus $\mathcal{A}_k(W)$ implies $\mathcal{A}_k(X)$.

Proof. For every smooth projective T, the Chow projection formula gives $(g_T)_* g_T^* z = z \cdot (g_T)_*[T \times W] = dz$. Taking $T = X$ and $z = \Delta_X$ proves (7.2). If $\Delta_W = \sum_j v_j u_j$, then

$$\Delta_X = \sum_j (d^{-1} g_* v_j) \circ (u_j g^*). \tag{7.3}$$

The maps in each pair have opposite degrees. □

8 Separated binary hypersurfaces

Allow first $2b + a \geq 2$ in (1.3), so that zero-dimensional factors can occur in an induction. These factors are finite étale. For every binary block, Euler's identity shows that simultaneous vanishing of its derivatives at a nonzero vector would give a multiple projective root. A nonzero unary term has nonvanishing derivative away from zero in characteristic zero. The full gradient therefore vanishes only at the affine origin. The hypersurface is smooth; when its dimension is positive, connectedness of a projective hypersurface makes it geometrically integral.

8.1 The controlled join

Let $F(x_0, \dots, x_r)$ and $G(y_0, \dots, y_s)$ be homogeneous of degree d, with $r, s \geq 1$, such that X_F and X_G are smooth. Set

$$A = \{F + u^d = 0\}, \quad B = \{G + v^d = 0\}, \quad H = \{F + G = 0\}.$$

Lemma 8.1. *Over a field containing ζ with $\zeta^d = -1$, the rational map*

$$A \times B \dashrightarrow H, \qquad ((x, u), (y, v)) \longmapsto (vx, \zeta uy) \tag{8.1}$$

extends to a morphism $g : \mathrm{Bl}_{X_F \times X_G}(A \times B) \to H$ of generic degree d. The center is smooth of codimension two.

Proof. Substitution in the equation of H gives $v^d F(x) + \zeta^d u^d G(y) = -u^d v^d + u^d v^d = 0$. Near a base point, choose nonzero coordinates x_i, y_j and trivialize the projective line bundles. The base ideal of the displayed sections is then (u, v). These are local equations of transverse smooth divisors from the two factors. Their blowup makes the base ideal invertible and resolves the map, with center $X_F \times X_G$.

On the dense open of H where $F(X) = -G(Y) \neq 0$, normalize $u = v = 1$. A preimage has $x = \lambda X$, $y = \lambda\zeta^{-1}Y$, and $\lambda^d = -1/F(X)$. There are exactly d distinct geometric solutions. The exceptional divisor has smaller dimension and does not dominate this open. Hence g is generically finite of degree d. This is the controlled construction of [26, Remark 1.10], with its center and degree retained. □

8.2 The recursive diagonal

We prove that every separated binary/unary hypersurface has $\mathcal{A}$ after a finite extension. In dimension zero, split the finite étale scheme. In dimension one, choose a point after a finite extension and apply Lemma 7.1.

Suppose H has dimension $n \geq 2$. Choose one binary block as G; if all blocks are unary, combine two unary terms into G. The remaining polynomial F has $n \geq 2$ variables. In the join construction, the dimensions are

$$\dim A = n - 1, \quad \dim B = 1, \quad \dim X_F = n - 2, \quad \dim X_G = 0.$$

All four varieties belong to the same class. Induction, followed by one common finite extension, supplies actual factorizations for A, B, X_F and every component of X_G. We include the root needed for Lemma 8.1 in that extension.

The tensor identity gives factorizations for $V = A \times B$ and $C = X_F \times X_G$. Lemma 7.2 gives one for $W = \mathrm{Bl}_C V$, and Lemma 7.3 gives one for H. All branches of the induction have strictly smaller single-factor dimension; finitely many finite extensions suffice. Products and finite forms follow from Lemma 7.1. This proves part (i) of Theorem 1.3.

More explicitly, suppose $\Delta_V = \sum_j v_j^V u_j^V$ and $\Delta_C = \sum_l v_l^C u_l^C$. The maps for H are

$$u_j^H = u_j^V q_* g^*, \qquad v_j^H = d^{-1} g_* q^* v_j^V, \tag{8.2}$$

$$u_l^H = -u_l^C p_* i^* g^*, \qquad v_l^H = d^{-1} g_* i_* p^* v_l^C. \tag{8.3}$$

Substituting (7.1) and (7.2) verifies their sum is Δ_H.

Example 8.2. For $\lambda_i \in k \setminus \{2, -2\}$, the quartic sixfold

$$\sum_{i=0}^{3} \left(x_{2i}^4 + \lambda_i x_{2i}^2 x_{2i+1}^2 + x_{2i+1}^4\right) = 0 \quad \text{in } \mathbb{P}_k^7$$

belongs to the family. A binary block has a repeated projective root precisely when $\lambda_i = 2$ or -2. Over a finite extension of $\mathbb{Q}_2$, the same characteristic-zero calculation applies.

9 Split cubics and finite Chow descent

Laterveer proves that the cubic varieties (1.4) over $\mathbb{C}$ have Chow motives of abelian type [27, Theorem 5]. This is a conclusion about rational equivalence, and gives a retract of a finite sum of Tate twists of abelian motives. We need its finite field of definition, including the rational equivalences in the retraction.

Lemma 9.1. *Let $k_0/\mathbb{Q}$ be finitely generated, let V_0/k_0 be smooth projective, and fix an embedding $k_0 \hookrightarrow \mathbb{C}$. If $h(V_{0,\mathbb{C}})$ is of abelian type, then an actual abelian Chow retraction of V_0 is defined over a finite extension k_1/k_0. For every embedding $k_0 \hookrightarrow k$ into a characteristic-zero field, it therefore exists over a finite extension of k.*

Proof. Vial's descent theorem [28, Theorem 1 and §1.1] applies to the extension $\bar{k}_0 \subset \mathbb{C}$. It shows that $h(V_{0,\bar{k}_0})$ is of abelian type. By the definition of this category, it is a retract of a finite sum of Tate twists of motives of abelian varieties: tensor products remain abelian motives, and duality is absorbed by a Tate twist. Thus there are finitely many correspondences satisfying the diagonal identity over $\bar{k}_0$.

The varieties, group laws, cycles, and rational equivalences in that identity are all defined by finitely many coefficients. After clearing rational denominators, each rational equivalence is a finite sum of divisors of rational functions on integral subvarieties. These data descend to one finite subextension k_1/k_0. Equivalently, the finitely many equalities in the Chow groups hold at a finite stage of their directed limit over $\bar{k}_0$.

Extend $k_0 \hookrightarrow k$ to $k_1 \hookrightarrow \bar{k}$. Then $L = kk_1$ is finite over k, and flat base change carries the same Chow identity to L. □

9.1 Application to the block family

For a cubic over k as in (1.4), put its coefficients in a finitely generated subfield $k_0 \subset k$. The smoothness assumptions descend and remain true after embedding k_0 into $\mathbb{C}$. Lateveer's theorem applies there, and Lemma 9.1 gives the required finite Chow factorization over k. Tensoring the factorizations deals with products; transporting them through an actual isomorphism deals with finite forms. This proves part (ii) of Theorem 1.3.

For example, two arbitrary smooth cubic threefold blocks followed by a smooth binary cubic give a cubic tenfold in $\mathbb{P}^{11}$. Replacing the binary block by a smooth four-variable cubic gives a cubic twelvefold in $\mathbb{P}^{13}$. Their central cycle codimensions are 5 and 6. The two five-variable blocks need not themselves separate into binary terms.

10 Separated-branch cyclic covers

Use the data of (1.5), set $d = qm$, and put

$$Y = \{T^d = F(x_0, \dots, x_n)\} \subset \mathbb{P}^{n+1}_k.$$

Lemma 10.1. *The varieties $X = X(F; q, m)$ and Y are smooth, projective and geometrically integral of dimension n. The map*

$$f : Y \longrightarrow X, \qquad u = T^m, \tag{10.1}$$

is finite flat of rank m.

Proof. The block derivative calculation in Section 8 shows that $B = \{F = 0\} \subset \mathbb{P}^n$ is smooth and nonempty. On the chart $x_i \neq 0$, write $R_i = k[x_j/x_i : j \neq i]$ and $F_i = F/x_i^d$. The cover algebra is

$$B_i = R_i[u_i]/(u_i^q - F_i).$$

It is free of rank q over R_i. Off B, the derivative in u_i is nonzero; above B, the differential of F_i proves smoothness. Over $\bar{k}$, the valuation at any component of B gives F_i value one. Eisenstein's criterion shows that $U^q - F_i$ is irreducible over the function field. The chart algebras inject into this field, so the cover is geometrically integral. Being finite over $\mathbb{P}^n$, it is projective.

The polynomial $T^d - F$ has the separated-block form with one additional unary term. Thus Y is smooth and geometrically integral. Its projection to $\mathbb{P}^n$ has no base point. With

$t_i = T/x_i$, the source algebra is

$$R_i[t_i]/(t_i^{qm} - F_i) \simeq B_i[t_i]/(t_i^m - u_i). \tag{10.2}$$

The right side is free of rank m with basis $1, t_i, \dots, t_i^{m-1}$. The transition laws $t_j = (x_i/x_j)t_i$ and $u_j = (x_i/x_j)^m u_i$ glue these maps. This proves the rank everywhere, including the length-m fibers with equation $t_i^m = 0$ along the branch. □

10.1 The product diagonal

Each Y_i associated with a cover X_i belongs to the separated hypersurface family. Over one finite extension, choose its Chow factorization $\Delta_{Y_i} = \sum_{j_i} v_{i,j_i} u_{i,j_i}$. For a finite product put $Y = \prod_i Y_i$, $X = \prod_i X_i$, and $M = \prod_i m_i$. The product map $f : Y \to X$ is finite flat of rank M.

For $J = (j_1, \dots, j_s)$ set

$$A_J = \prod_i A_{i,j_i}, \qquad u_J = \boxtimes_i u_{i,j_i}, \qquad v_J = \boxtimes_i v_{i,j_i}.$$

The tensor identity gives $\Delta_Y = \sum_J v_J u_J$. Since $f_* f^* = M\Delta_X$, the maps

$$U_J = u_J f^*, \qquad V_J = M^{-1} f_* v_J \tag{10.3}$$

satisfy $\sum_J V_J U_J = \Delta_X$. The rank M is included once, and all exceptional signs from the factorization of Y are retained. Actual finite forms are handled by transport of this identity. This proves part (iii) of Theorem 1.3.

11 Model criteria and arithmetic families

11.1 Numerical corrections on a semistable witness

The numerical-model criterion of Hypothesis 2.2 is often applied through a projective dominant witness. In the general companion formulation [11], the witness carries a rational virtual bundle whose generic degree-r Chern character is the pulled-back class and whose degree on every vertical r-dimensional subvariety is zero. The native model-Chern detector then gives zero obstruction upstairs, and dominant detection gives zero obstruction downstairs. This formulation permits a witness of positive relative dimension as well as a generically finite one.

We record three ways of producing this numerical correction. The model calculations and their supporting residue sequences are given in [15]; all model conditions below refer to supplied projective models.

Proposition 11.1 (Scalar total boundary image)**.** *Let $R = K^\circ$ and let $\mathcal{X}/R$ be integral, regular, projective reduced strictly semistable, with smooth geometrically integral generic fiber, of relative dimension $n \geq 3$, with smooth geometrically integral components D_i, smooth closed strata, connected dual graph, and one relatively ample class H. Fix $1 \leq r \leq n$ and an invertible prime ℓ. Assume that at every vertex i the sum of Gysin images*

$$\sum_{j \neq i} \operatorname{im}\big(H^{2r-2}(E_{ij,\bar{k}}, \mathbb{Q}_\ell(r-1)) \longrightarrow H^{2r}(D_{i,\bar{k}}, \mathbb{Q}_\ell(r))\big)$$

has dimension one. Then every geometrically homologically trivial codimension-r cycle on the generic fiber has zero obstruction.

Proof. Choose a Chow extension Θ. The companion specialization calculation puts its component class in the displayed total image. Set

$$\lambda_i = \deg_k(i_i^!\Theta\, H_i^{n-r}), \qquad w_{ij} = \deg_k((H|_{E_{ij}})^{n-1}),$$

summing disconnected intersections with their native residue degrees. The specialization trace gives $\sum_i \lambda_i = 0$. The connected weighted graph has positive weights and hence a rational solution of

$$\sum_j w_{ij}(v_j - v_i) = \lambda_i.$$

The corrected extension $\Theta_{\mathrm{ad}} = \Theta - \sum_i v_i i_{i*} H_i^{r-1}$ has zero component class: in a one-dimensional total image its degree against H_i^{n-r} detects that class. Its pure operational representative is numerically zero on every vertical r-cycle. Apply the numerical model criterion. A constant change of all v_i adds the principal total-fiber class multiplied by H^{r-1} and hence changes nothing in Chow. □

One geometric instance is a supplied two-factor hypersurface degeneration

$$AB + \varpi G = 0 \quad \text{in } \mathbb{P}_R^{n+1}, \qquad \deg A = a, \quad \deg B = b, \quad \deg G = a + b. \tag{11.1}$$

Assume the two special hypersurfaces, their intersection, and $\Sigma = (A = B = G = 0)$ have the smooth transverse geometrically integral strata specified by these equations, and that the generic fiber is smooth geometrically integral. Blowing up (A, ϖ) gives components $\mathrm{Bl}_\Sigma(A = 0)$ and $(B = 0)$ meeting in $(A = B = 0)$. The blowup and restriction computation in [15] verifies the scalar-image condition in every degree. Thus these local hypersurfaces have V.

Proposition 11.2 (Positive primitive components)**.** *Let a smooth projective fourfold have, after a finite extension, a projective dominant four-dimensional witness with a regular flat projective model. Write its special fiber as $\sum_i m_i D_i$, with $m_i > 0$, reduced strict normal-crossings support and smooth closed strata. Suppose a relatively ample H makes*

$$\deg(xy) \quad \textit{positive definite on} \quad P_H^2(D_i) = \ker\big(H : N^2(D_i)_{\mathbb{Q}} \to N^3(D_i)_{\mathbb{Q}}\big)$$

for every component. Then the original fourfold has V.

Proof. The companion numerical correction leaves a canonical primitive residual R_i with $\sum_i m_i \deg(R_i^2) = 0$. This residual identity, including the native constant-field factors, is the primitive-residual theorem of [15]. Positivity and $m_i > 0$ give $R_i = 0$ for every i. The resulting operational lift is numerically admissible, so codimension two follows by witness descent. One-cycles and the endpoint degrees complete the fourfold. □

The verified component classes include abelian fourfolds, rank-two projective bundles over smooth threefolds, rank-three bundles over smooth surfaces, rank-four bundles over smooth curves, and $\mathbb{P}^4$. For the abelian case the positivity input is [7]; the projective-bundle cases follow from the numerical projective-bundle formula and divisor Hodge index. If $f : D' \to D_{k'}$ is finite surjective and $(D', f^*H_{k'})$ has the required positivity, then (D, H) has it by pullback and the degree formula.

Proposition 11.3 (Finite model symmetries)**.** *For a regular projective model of relative dimension $n \geq 4$, with the preceding multiplicity and stratum conventions, put*

$$P_H^2 = \bigoplus_i \ker\big(H^{n-3} : N^2(D_i)_{\mathbb{Q}} \to N^{n-1}(D_i)_{\mathbb{Q}}\big), \qquad Q(x, y) = \sum_i m_i \deg(x_i y_i H_i^{n-4}).$$

Let a finite group act on the model preserving the numerical class of H. If Q is nondegenerate and P_H^2 is multiplicity-free as a rational representation, every codimension-two homological class has zero obstruction. In dimension four, nondegeneracy is automatic and the conclusion gives V.

Proof. The image of the canonical residual map is an invariant totally isotropic submodule of P_H^2, by the equivariant residual theorem in [15]. Every rational simple representation of a finite group is self-dual, by an averaged positive rational form. Distinct simple summands are therefore orthogonal for Q. Multiplicity-freeness and nondegeneracy imply that each summand is nondegenerate, so no nonzero invariant totally isotropic submodule exists. The residual vanishes. Numerical correction and witness descent give the asserted degree. In dimension four, numerical duality and divisor Hodge index give the required nondegeneracy; the endpoints and one-cycles give all remaining degrees. □

The same proof applies to a sum of rational central isotypic projectors e when eP_H^2 has these properties, and gives the corresponding cycle subspace $e\,\mathrm{CH}^2_{\mathbb{Q},\mathrm{geom\,hom}}$. Such a subspace pairs globally with the admitted complementary group.

11.2 Ordinary quadratic reduction

Theorem 11.4 (Companion ordinary-reduction theorem)**.** *Let $K/\mathbb{Q}_p$ be finite for any prime p, and let X/K be smooth projective geometrically integral of dimension at least three. Suppose X has an integral flat projective model whose geometric special fiber is integral and smooth apart from finitely many ordinary quadratic points, meaning that their projectivized geometric tangent cones are smooth quadrics. Then $\mathsf{V}(X)$ holds. The same conclusion holds if such a smooth projective dominant witness is supplied after a finite extension. The theorem reapplies to all new cycles over every finite valued extension.*

Proof. In odd residue characteristic the normal-form and successive reduced-point blowup calculation of [16] produces a regular reduced quadric forest after finite extension. Its numerical corrections are compatible with the incidence maps. The all-characteristic quadric-forest theorem in the same source handles odd-dimensional ordinary quadratic models at residue two.

For even $n \geq 4$ at residue two, use the henselian normal form of [9, Lemma 4(ii)]. A common finite extension splits all the finitely many node polynomials. Translating each by an integral root gives an actual pointed étale local equation

$$P(x_1, \dots, x_n) + z^2 + \epsilon \varpi^q z = 0, \qquad q \geq 1, \quad \epsilon \in R^\times,$$

with unimodular polar form for P, where R is the auxiliary field's valuation ring and ϖ its uniformizer. The common étale charts and exact saturated blowup calculations are proved in [16, common étale neighborhoods and native degenerate-node chains]. Each reduced-point blowup decreases q by one. After exactly q blowups the exceptional chain consists of $q-1$ copies of $\mathbb{P}_E(\mathcal{O}_E \oplus \mathcal{O}_E(-1))$ and a terminal smooth quadric, where

$$E = (\overline{P}(X) + Z^2 = 0) \subset \mathbb{P}^n, \qquad T = (\overline{P}(X) + Z^2 + \bar{\epsilon} ZS = 0) \subset \mathbb{P}^{n+1}.$$

All double strata are the smooth odd-dimensional quadric E; all multiplicities are one and the intermediate sections are disjoint. The same quadric-forest numerical criterion now applies.

In every case numerical witness descent uses the complete common field degree, and the original curved pair gives the form on the specified support complement. The dominant-witness version follows from the same detector. After a finite valued extension the ordinary

geometric special points persist, with possibly larger finite orders; apply the construction anew to every new cycle. □

Over a number field, good reduction or the ordinary alternative at every finite completion, also on the stated dominant witnesses, gives the full height pairing. Local auxiliary extensions are descended before this global construction and need not come from one fixed global extension.

11.3 Tate and abelian-type strata

Theorem 11.5 (Companion all-strata criterion)**.** *Let $Y, T/K$ be smooth projective geometrically integral. Suppose Y has a supplied regular projective reduced strictly semistable model whose every connected closed geometric stratum has a rational Chow motive of abelian type after finite constants. Assume weight–monodromy for Y in every degree and every invertible coefficient prime. If T has potentially good projective reduction, then $T \times Y$ has V, also for all new cycles after every finite valued extension.*

Proof. The ramified all-strata theorem in [17] preserves the stated Chow-motive condition on a projective semistable repair. Its toroidal calculation expresses new strata through blowups of old boundary strata and projective bundles over them. For a closed-boundary blowup the property holds in both directions; this permits the toroidal factorization steps. The character lattice of a new stratum is primitive, and its calculation retains the actual line-bundle transition functions.

For a new cycle over L/K, pass to a finite extension where T is good and the strata split. On one abelian carrier, numerical semisimplicity and its comparison prime supply an algebraic cohomological inner inverse to the boundary incidence. Tensor it with the diagonal of the coefficient, so the inverse acts on all mixed Künneth summands. Weight–monodromy gives the odd-weight condition, while the inverse gives numerical-kernel lifting and actual-span saturation. Apply the companion model criterion and descend with the full degree. The final curved-pair correction is on the original support of the newly specified cycle. □

Corollary 11.6 (Drinfeld uniformization)**.** *Let X_a be smooth projective algebraizations of $\Gamma_a \backslash \Omega_K^{d_a}$, where $\Gamma_a \subset \mathrm{PGL}_{d_a+1}(K)$ is torsion-free, discrete and cocompact, and $d_a \geq 1$. Then $T \times \prod_a X_a$ has V for every potentially good projective coefficient T, also over every finite extension.*

Proof. The projective semistable cover of a Drinfeld quotient has components and closed strata obtained by the successive rational-linear-subspace blowups of projective space. Their actual Chow motives are split Tate, by the blowup formula. Staircase products retain this property. Ito's theorem [8] gives weight–monodromy, and Künneth gives it for the product. Apply Theorem 11.5 and descend the finite cover. □

One may also include curve or surface factors supplied with projective strict semistable models all of whose closed strata are of Chow abelian type. The resulting placewise presentations give full global heights when available at every finite completion; the factors in those local presentations may depend on the place.

Theorem 11.7 (Totally degenerate strata)**.** *Suppose a smooth projective X/K has, over a finite extension L/K, a projective dominant generically finite morphism $Y \to X_L$, with Y smooth projective geometrically integral of dimension $\dim X$ and carrying a regular projective reduced strictly semistable model. On every connected geometric closed stratum D, require:*

(i) *Each integral Chow group is finitely generated. The compatible hyperplane class H from a projective embedding of the special fiber satisfies, for $2c \le \dim D$ and nonzero rational primitive x with $H^{\dim D-2c+1}x = 0$,*

$$(-1)^c \deg(xH^{\dim D-2c}x) > 0.$$

(ii) *For every ℓ different from the residue characteristic, odd geometric integral cohomology is torsion and*

$$(\mathrm{CH}^c(D)/\mathrm{tors}) \otimes \mathbb{Z}_\ell \simeq H^{2c}(D, \mathbb{Z}_\ell(c))/\mathrm{tors}.$$

Then $\mathsf{V}(X)$ holds. Smooth projective toric strata satisfy these hypotheses. A global full height domain follows if such witnesses, or another full local criterion, are supplied at every finite place.

Proof. These are conditions (a),(b) of Raskind–Xarles [10, Definition 1]. Define finitely many Chow generators and components over one finite residue extension and lift it unramifiedly. The cycle isomorphisms and Poincaré duality identify native numerical groups with even cohomology, compatibly with the actual incidence correspondences. Thus every numerical-kernel test lifts to a closed cohomological support test, by the companion component-detection theorem [21].

The cited Raskind–Xarles weight result kills the relevant odd local Galois groups. More precisely, for $W = H^{2t-1}(Y_{\bar{L}}, \mathbb{Q}_\ell(t))$ the inertia invariants have weights at most -1, and the inertia-H^1 term has weights at least 1. Hence $W^{G_L} = H^1(L, W) = 0$. The companion period comparison identifies the rational detector period with the local Abel–Jacobi cup-product against the closed support test. It vanishes in $\mathbb{Q}_\ell$, hence is zero as its original rational intersection number. The finite detector and exact-support reconstruction give the result. The global assertion applies Lemma 4.2 to these supplied local witnesses. □

11.4 A common arithmetic construction

The dense families below use actual local models, together with an arithmetic pencil argument. We give the latter separately to specify the field and density quantifiers.

Lemma 11.8 (Congruence pencils)**.** *Let a fixed integral flat projective ambient model with geometrically integral fibers and a section system be supplied over a number ring. Suppose a finite free coefficient module surjects onto the section system and its required residue spaces. Suppose the system has a nonempty fixed-member Lefschetz-pencil open, and that one can prescribe smooth sections at a finite set of places and an established local smoothing model at distinguished places. Require the chosen equations to be nonzero on every special ambient fiber, and the generic smooth-member open to consist of geometrically integral members. After a single finite extension realizing these finite residue data, the congruence-pencil construction gives a Zariski-dense set of smooth generic members with the prescribed local models, smooth reduction at the exceptional places, and smooth or ordinary-quadratic reduction at all remaining places.*

Proof. Choose a smooth reference section H_0 and include in a finite set all rational primes below its bad reductions, its zero special sections, and residue characteristic two. Prepare the desired residue sections after finite residue extensions. A monic polynomial with separable irreducible reductions of one common degree realizes all these extensions in one number field, unramified at the chosen primes. This field is fixed from now on.

In the free coefficient module impose the finite congruences for a second section H_1, using the square of each native maximal ideal to retain a prescribed first-order smoothing. For a

distinguished rational prime p unramified in the starting number field, this is the modulus p^2 used in the applications. Intersect this affine congruence lattice with the fixed-H_0 Lefschetz open. The intersection is Zariski dense. For a chosen H_1, spread its pencil outside a finite further set of primes. At those additional primes, choose the parameter to specialize to the already smooth reference member; at the originally prepared primes choose it to preserve the chosen residue member. Chinese remainders give infinitely many parameters with these requirements. Omit the finitely many singular generic members.

Outside the spreading set, the pencil has smooth or ordinary quadratic fibers. At every exceptional prime the selected residue fiber is smooth or is the prescribed established smoothing. The nonzero residue equations are non-zero-divisors on the integral ambient fibers. The relative Cartier criterion therefore makes the resulting divisors flat over the number ring. Generic geometric integrality follows from the prescribed open. In each application, nonzero residue equations are checked at the exceptional primes by the congruences and elsewhere by independence of the two pencil sections. Finally a polynomial vanishing on all constructed members vanishes on each permitted pencil, since that pencil contains infinitely many of them. Varying H_1 in the dense congruence open proves density in the stated section space. The finite spreading set may depend on H_1; the fixed field does not. □

11.5 Hypersurfaces and complete intersections

The following are the geometric conclusions of the local model calculations in [18]. Their arithmetic proofs use Lemma 11.8. Each constructed member has full heights on all new homological classes after every finite extension of its stated field.

Theorem 11.9 (Dense hypersurface families)**.** *For every $n \geq 3$ and $D \geq 3$, there is a Zariski-dense subset of the full degree-D hypersurface parameter space over $\mathbb{Q}$ consisting of smooth geometrically integral varieties with full height domains over every number field.*

For $n \geq 4$, fix positive $d_1, \dots, d_m$, $m \geq 3$, $D = \sum_i d_i$, and suppose

$$b_n = \sum_{|I|=n+1} \prod_{i \in I} d_i - \binom{m-1}{n} > 0.$$

There is such a dense subset with a distinguished product-profile degeneration of extreme monodromy rank $\operatorname{rk} N^n = b_n$*. For linear factors this gives $D \geq n+2$ and* $\operatorname{rk} N^n = \binom{D-1}{n+1}$.

Proof. For the ordinary family take

$$F_0 = \sum_{j=0}^{n+1} X_j^D, \qquad J_D = \sum_{j=0}^{n} X_j^{D-1} X_{j+1} + X_{n+1}^D.$$

At a prime dividing D, descending through the partial derivatives shows J_D is smooth. At every other prime F_0 is smooth. Choose an integral F_1 congruent to J_D at primes dividing D and in the fixed-F_0 Lefschetz open. The members $DF_0 + aF_1$, with Chinese-remainder conditions on a, have smooth reduction at the finitely many exceptional primes, including two, and ordinary reduction elsewhere. Apply Theorem 11.4. Varying F_1 proves the asserted full-space density.

For the product profile, prepare transverse integral factors $P = \prod_i f_i$ and a smoothing section Q. Choose $q > D$ outside their finite bad set and impose $F_1 \equiv P - DF_0 + qQ \pmod{q^2}$ as well. For $a \equiv 1$ modulo q^2, the exact equation at q is $P + qQ_a$ with $\bar{Q}_a = \bar{Q}$. The ordered product-model calculation in [18] gives full local vanishing and the displayed extreme monodromy rank. Its ramified-extension theorem treats all new cycles after arbitrary finite

extension. At the other primes use the preceding ordinary argument. The complementary-degree criterion gives the heights. □

Theorem 11.10 (Successive complete-intersection sections)**.** *Let $\mathcal{Y} \subset \mathbb{P}^N_{\mathcal{O}_F}$ be a supplied integral flat projective relative complete intersection of dimension $m \geq 4$, with smooth geometrically integral generic fiber and geometrically integral special fibers smooth outside finite sets. For every $d \geq 3$, after one fixed finite extension E/F there is a dense subset of the full degree-d section system on Y_E whose smooth members have full height domains over all finite extensions of E.*

Consequently one can construct complete intersections of any specified multidegree $(d_1, \ldots, d_t)$ with all $d_i \geq 3$ and final dimension at least three, over one fixed finite extension of any prescribed number field. The first equations may be fixed so that density holds in the final section system. A split quadric may be used as the initial ambient variety.

Proof. The Koszul resolution supplies the full section space from ambient polynomials. S-mooth residue sections can be chosen to avoid the finite ambient singular loci, after finite residue preparation. The degree-d jets for $d \geq 3$ give the fixed-member Lefschetz open. Apply Lemma 11.8; the resulting integral complete-intersection model has smooth or odd-characteristic ordinary fibers and good fibers at two. Use Theorem 11.4. Its model has the same finite-singular-locus property, so one can iterate, fixing the earlier equations before varying the last one. □

There is also a product-profile final-section version on the fixed ambient of Theorem 11.10, with $m = n + 1 \geq 5$. If the ambient degree is δ, the final degree is $D = \sum_{i=1}^{s} e_i$ with $s \geq 3$, $e_i \geq 1$, the companion construction gives a dense final-section family after one fixed extension, with

$$\operatorname{rk} N^n = \delta \sum_{|I|=n+1} \prod_{i \in I} e_i - \binom{s-1}{n}.$$

For example degree-five sections of the split quadric fivefold give a dense family of type-$(2,5)$ fourfolds with $\operatorname{rk} N^4 = 1$. The local product theorem and the same congruence-pencil proof give their complete height domains.

11.6 Product ambients and positive fourfold hosts

Theorem 11.11 (Dense product-ambient families)**.** *For each fixed set of input data below, the constructed dense set has a full height domain over all finite extensions of one fixed finite extension of its stated initial number field.*

(i) *For every $D \geq 5$ and every prescribed number field F_{base}, one fixed finite F/F_{base} supports a dense subset of the full bidegree-(D, D) system on $(\mathbb{P}^1 \times \mathbb{P}^4)_F$. A distinguished product model has*

$$\operatorname{rk} N^4 = (D-1)\binom{D-1}{4} > 0.$$

(ii) *Let $\mathcal{S}/\mathcal{O}_F$ be a supplied smooth projective geometrically integral surface model and let A be relatively very ample. On $\mathcal{S} \times \mathbb{P}^3$, put $L = A \boxtimes \mathcal{O}(1)$. After one fixed finite extension E/F, the image of degree-five embedding polynomials has a dense such subset, with $\operatorname{rk} N^4 = 10 \deg(A^2) - 1$. If those polynomials span the full section space, this is density in $|L^5|$.*

(iii) *Let $\mathcal{M}/\mathcal{O}_{F_0}$ be a supplied smooth projective scheme of relative dimension four over the whole number ring, with geometrically integral fibers. Fix an odd rational prime p unramified in F_0. At each $v \mid p$ and every finite residue extension $k'/k(v)$ assume*

$$b^-(N^2(\mathcal{M}_{k'})_{\mathbb{R}}, \deg xy) = \rho(\mathcal{M}_{k'}) - 1. \tag{11.2}$$

Choose H sufficiently positive for relative very ampleness, normal generation and section base change. Put $L_0 = H \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$ and $c = \deg H^4$. One fixed finite F/F_0, unramified above p, supports a dense subset of the full $|L_0^5|_F$ with

$$\operatorname{rk} N^4 = 4(5c-4) > 0, \qquad b_1(X) = b_1(M), \quad b_2(X) = b_2(M) + 1.$$

Proof. For (i), use D transverse $\mathcal{O}(1,1)$ factors at a distinguished prime. The ambient double-stratum comparison retains both hyperplane classes. The ordered model and its uniform finite-extension comparison, proved in [18], give full local vanishing and $5\binom{D}{5} - \binom{D-1}{4} = (D-1)\binom{D-1}{4}$.

For (ii), the local surface-ambient comparison uses the actual blowups of the double strata. Its extreme incidence complex gives $10\deg(A^2)-1$. The free module of embedding polynomials supplies the coefficient space in Lemma 11.8, proving density in its image. The stated spanning hypothesis identifies that image with the full section system.

For (iii), the positive-host suspension calculation in [20] applies to five divisors on M and five distinct point factors on $\mathbb{P}^1$. Residue Bertini prepares smooth geometrically integral intersections of one, two and three divisors, split finite-étale fourfold intersections of degree c, and empty fivefold intersection. The smoothing section cuts each positive-dimensional stratum transversely, with geometrically integral positive-dimensional cuts and finite-étale zero-dimensional cuts, and avoids the zero-dimensional strata. Condition (11.2) kills the primitive numerical residual. Its uniform form over finite residue extensions gives the all-new-cycle ramified conclusion. The incidence complex has $\dim H_3 = 5c-4$ before joining the five point vertices, hence $\operatorname{rk} N^4 = 4(5c-4)$ afterward. Weak Lefschetz gives the two Betti numbers. Relative Serre vanishing and a sufficiently high Veronese supply the full coefficient module and the required jets. Apply Lemma 11.8 to all three cases, use ordinary reduction at nondistinguished bad primes, and then Lemma 4.2. □

The signature hypothesis in (iii) is supplied by everywhere-good abelian fourfolds, by the projective-bundle host types listed after Proposition 11.2, and by $\mathbb{P}^4$. For $\mathbb{P}^4$, $H = \mathcal{O}(1)$ and $p = 5$, the result gives bidegree-$(5,5)$ fourfolds with $\operatorname{rk} N^4 = 4$, $b_1 = 0$, $b_2 = 2$.

11.7 Explicit elliptic-power families

Let $E_0/\mathbb{Q}$ be $y^2 = x^3 - x$ and put $F_{\mathrm{tor}} = \mathbb{Q}(E_0[3])$, adjoining all geometric 3-torsion. The elliptic curve has an everywhere-good model $\mathcal{E}$ over this field. At two, potential good reduction follows from complex multiplication; after adjoining 3-torsion its finite inertia image lies in the torsion-free principal congruence subgroup of $\mathrm{GL}_2(\mathbb{Z}_3)$, so it is trivial. The curve is already good at the other primes. This field is unramified above five.

Put $A = \mathcal{O}_E(3[0]) \boxtimes \mathcal{O}_E(3[0])$ on E^2, and $H = \boxtimes_{i=1}^4 \mathcal{O}_E(3[0])$ on E^4. The plane-cubic embeddings and the Segre construction prove full section spanning. The following dense subsets of full linear systems are constructed over fixed finite extensions of F_{tor} unramified above five, and have full heights after every further finite extension:

Ambient and system	Distinguished profile	Monodromy	(b_1, b_2)
$E^2 \times \mathbb{P}^3$, $\lvert(A \boxtimes \mathcal{O}(1))^5\rvert$	five section factors	$\operatorname{rk} N^4 = 179$	$(4, 7)$
same system	curve and five hyperplanes	$N^4 = 0$	$(4, 7)$
$E^4 \times \mathbb{P}^1$, $\lvert(H \boxtimes \mathcal{O}(1))^5\rvert$	two curves and five points	$\operatorname{rk} N^2 = 802816$, $N^3 = 0$	$(8, 29)$
same system	five divisors and five points	$\operatorname{rk} N^4 = 38864$	$(8, 29)$

The first row has $\deg A^2 = 18$. The second uses the mixed surface–hyperplane local theorem of [19]; its weight-two contribution contains a nonzero eight-dimensional summand. In the third row the double-stratum model is the actual blowup of the curve product along the smoothing cuts. The local signature condition holds for the abelian fourfold product, and the exact rank formula is

$$\operatorname{rk} N^2 = (2g(C) - b_1(S))(2g(D) - b_1(T))(d-1), \qquad N^3 = 0.$$

Here $g(C) = g(D) = 226$, $b_1(S) = b_1(T) = 4$, and $d = 5$. The final row is Theorem 11.11(iii), with $c = 4!\,3^4 = 1944$. The arithmetic argument is again Lemma 11.8, now with the full Segre coefficient system. The different monodromy profiles retain the distinctions among these constructed subsets even when the ambient system is the same.

12 Cycle subgroups and explicit representatives

12.1 Algebraically trivial cycles

Write $\mathrm{CH}^r(X)_{\mathbb{Q},\mathrm{alg}}$ for the rational Chow classes whose base change to an algebraic closure is algebraically equivalent to zero.

Proposition 12.1. *On every smooth projective X/K and in every codimension,*

$$\mathrm{CH}^r(X)_{\mathbb{Q},\mathrm{alg}} \subseteq \mathcal{K}^r_K(X).$$

Over a number field, the companion normalized height is therefore defined on

$$\mathrm{CH}^r(X)_{\mathbb{Q},\mathrm{alg}} \times \mathrm{CH}^{n+1-r}(X)_{\mathbb{Q},\mathrm{alg}},$$

and more generally on either algebraically trivial factor paired with the admitted complementary group $\mathcal{H}$.

Proof. Algebraic equivalence is generated by connected smooth projective curve families. After a finite extension, an algebraically trivial class is a finite sum of

$$\Gamma_*([a] - [b]), \qquad \Gamma \in \mathrm{CH}^r(C \times X)_{\mathbb{Q}},$$

with a, b rational over that finite extension on the smooth projective parameter curve. Enlarge the extension to retain the finite rational-equivalence witnesses as well. The degree-zero divisor $[a]-[b]$ is in its endpoint kernel. Kernel covariance and full-degree finite descent prove the local inclusion. The constant-field interpretation of algebraic triviality is equivalently obtained by norming the finite family certificate. Apply this inclusion at every completion and then the companion height theorem. □

Together with the endpoint results, this also gives the full divisor–zero-cycle height on an arbitrary n-fold. The complete one-cycle group lies in $\mathcal{H}^{n-1}(X)$ and pairs with $\mathcal{H}^2(X)$. Whenever one of the geometric or model criteria above establishes the entire codimension-two group, this becomes the full complementary pairing.

12.2 Finite obstruction certificates

The finite-detector theorem of [22] provides, for a fixed admissible model calculation, finitely many rational period coordinates for the actual obstruction. Their vanishing is a certificate that a specified homological Chow class belongs to $\mathcal{K}^r_K(X)$. The normalized tests use

$$\eta_j = \frac{1}{[L:K]c_L}\,\mathrm{Tr}^K_{X_L/X}(b_j), \qquad c_L = -\log|\varpi_L| = c_K/e(L/K).$$

Thus all model families, isotypic sectors and individually certified classes may be combined inside the same admitted group. For a number-field class one checks this membership at its finitely many relevant bad places; the good places are handled by the good-model construction. The output Green identity still has the complete actual test space in (1.1).

12.3 Exact-support cancellation for complete intersections

Let $\bar{L}$ be a line bundle with one fixed piecewise-smooth or smooth metric allowed by the complete-intersection construction on a smooth projective X/K, and let s_0, s_1 be regular rank-r tuples of sections of L, with $n \geq r \geq 3$. Write $Y_i = V(s_i)$ with its scheme-theoretic cycle multiplicities, and put

$$Z = [Y_1] - [Y_0], \quad S = |Y_0|^{\mathrm{an}} \cup |Y_1|^{\mathrm{an}}, \quad U = X^{\mathrm{an}} \setminus |Z|^{\mathrm{an}}, \quad W = X^{\mathrm{an}} \setminus S.$$

The complete-intersection kernel of the tropical companion [23], for a common allowed metric on L, is

$$\phi_s = \min_i(-\log\|s_i\|), \quad R = c_1(\bar{L}), \quad A_s = R - D\phi_s, \quad G_s = \phi_s \sum_{j=0}^{r-1} A_s^j R^{r-1-j}. \tag{12.1}$$

It satisfies $D[G_s] + \delta_{Y_s} = [R^r]$ against all actual tests. The raw difference is defined on W; cancelled components lie in $C_U = S \cap U$.

Proposition 12.2 (Supplied cancellation certificates)**.** *Each of the following hypotheses supplies an explicit actual Green form for Z on U, satisfying (1.1).*

(i) *On an ordinary analytic neighborhood of C_U in U, $s_1 = Ms_0$ exactly, where M is an invertible three-row block plus the identity in the other rows. Two specified entries a, d in its first column have nonvanishing opens covering C_U.*

(ii) *On a Zariski neighborhood of C_U, the two regular sequences define the same ideal. Their comparison matrix on the reduced conormal coefficient ring has a supplied finite ordered factorization into elementary shears, one-entry units, invertible two-row blocks, and invertible three-row blocks with the two-pivot cover in* (i)*. All row labels, coefficient data and inverses are part of the factorization. At each fixed word position use the same cap offsets and inner and outer partitions for every local coefficient lift.*

Proof. For (i), the relative three-row face identities in [23] give, for an ordinary neighborhood N of C_U in U, the identity on $N \cap W$

$$G_{s_1} - G_{s_0} = d'\alpha + d''\beta + E,$$

where α, β are actual on $N \cap W$ and E is actual on N. Their explicit two-pivot partition uses $v(d) - v(a)$ and includes the differentiated partition and mixed-derivative terms. Choose the

companion relative-support cutoff ζ on U, with support closed in U inside this neighborhood and value one near C_U. On W the corrected form

$$G_{s_1} - G_{s_0} - d' \operatorname{ext}_0(\zeta\alpha) - d'' \operatorname{ext}_0(\zeta\beta)$$

equals E near C_U, so it glues to an actual form on U. The relative-support Stokes calculation of the companion proves the all-actual test identity.

For (ii), lift the fixed finite word locally and execute its factors at the actual intermediate tuples. The final discrepancy is $1 + E_0$ with entries of E_0 vanishing on the reduced common zero scheme. Near C_U its entries have norm less than one, so this discrepancy preserves the tuple supremum norm and the complete expression (12.1). The coefficient-uniform germ theorem in [23] identifies the complete primitive germs for different local lifts. They therefore glue with a locally finite partition; its derivative discrepancy extends by zero. Apply the same relative-support cutoff argument. The calculation retains the original nonreduced schemes and their cycle multiplicities throughout. □

These regular sections have $[Y_i] = c_r(L^{\oplus r}) \cap [X] = c_1(L)^r \cap [X]$ in Chow. Consequently $[Z] = 0$. The constructions make the exact-support Green representative explicit. For such globally defined cycles, the Chow-descended height is

$$h_X([Z], w) = 0 \qquad (w \in \mathcal{H}^{n+1-r}(X)),$$

even though individual local Green integrals may be nonzero.

12.4 Comparison with supplied arithmetic models

Suppose an admitted pair z, w has regular numerically admissible arithmetic lifts $\widehat{\Theta}, \widehat{\Phi}$ on a regular flat projective model over $\mathcal{O}_F$, with zero archimedean curvature in the comparison normalization. The companion model-comparison theorem [34] identifies its normalized height with

$$h_X(z, w) = [F : \mathbb{Q}]^{-1} \widehat{\deg}(\widehat{\Theta} \cdot \widehat{\Phi}).$$

If $f : Y \to X_E$ is a degree-d alteration with Y smooth projective, and the supplied admissible lifts of $f^* z_E$, $f^* w_E$ lie on a regular flat projective $\mathcal{O}_E$-model of Y, with the same zero-curvature archimedean normalization, the corresponding formula is

$$h_X(z, w) = \frac{1}{d[E : \mathbb{Q}]} \widehat{\deg}(\widehat{\Theta} \cdot \widehat{\Phi}).$$

The generic degree and the absolute number-field degree are separate factors. These formulas and the abelian comparison in the next section describe the same normalized pairing on their common domains.

13 Height pairings and recursive formulas

13.1 Proof of the normalized height theorem

Let X/F satisfy the hypothesis of Theorem 1.4. At each finite place v, complete a field of definition of the Chow retraction at a place above v. The resulting extension of F_v is finite. Theorem 1.2, applied with the original absolute value of F_v, shows that every geometric homological class has zero local obstruction. Both groups in (1.6) are therefore in the full domain of the companion height theorem.

Normalized finite-extension invariance, the Chow identity and (2.4) give

$$\begin{aligned} h_X^{\mathrm{an}}(z,w) &= h_{X_E}^{\mathrm{an}}(z_E, w_E) \\ &= \sum_j h_{X_E}^{\mathrm{an}}(v_{j*}u_{j*}z_E, w_E) \\ &= \sum_j h_{A_j}^{\mathrm{an}}(u_{j*}z_E, {}^t v_{j*} w_E). \end{aligned}$$

The image classes are homologically trivial and have codimensions $r+a_j$ and $\dim A_j+1-r-a_j$. An image in codimension zero is zero because its geometric cycle class detects its rank. Out-of-range images also vanish. Apply (2.5) to each remaining term to obtain (1.7). Every permitted choice computes the same canonical left side, proving independence. This completes the proof of Theorem 1.4.

13.2 The signed join recursion

Consider a join $g : W = \mathrm{Bl}_C V \to H$ of degree d over a number field, with the potential abelian Chow retractions constructed above for V, C, W, H. For geometrically homologically trivial rational Chow classes z, w on H in complementary codimensions, put

$$a_z = q_* g^* z, \quad a_w = q_* g^* w, \qquad b_z = p_* i^* g^* z, \quad b_w = p_* i^* g^* w.$$

Corollary 13.1. *The normalized heights satisfy*

$$h_H^{\mathrm{an}}(z,w) = \frac{1}{d}\left(h_V^{\mathrm{an}}(a_z, a_w) - h_C^{\mathrm{an}}(b_z, b_w)\right). \tag{13.1}$$

The center height is summed over its components after a finite splitting field. Its two degrees are $r-1$ and $s-1$, where $r+s = \dim H + 1$; zero or impossible homological terms are omitted.

Proof. Adjunction and $g_* g^* = d$ yield $h_W^{\mathrm{an}}(g^*z, g^*w) = d\, h_H^{\mathrm{an}}(z,w)$. Apply (7.1) to the first entry on W. The transposes of q^* and $i_* p^*$ act by q_* and $p_* i^*$, respectively. A second application of adjunction gives the right side of (13.1), with the negative sign from the normal bundle of the exceptional divisor. The center codimensions sum to $\dim C + 1$. □

13.3 The cyclic-cover scalar

For the product cover $f : Y_E \to X_E$ of rank M and the factorization of Y_E used in (10.3), the same adjunction gives

$$\begin{aligned} h_X^{\mathrm{an}}(z,w) &= \frac{1}{M} h_{Y_E}^{\mathrm{an}}(f^* z_E, f^* w_E) \\ &= \frac{1}{M[E:\mathbb{Q}]} \sum_J \left\langle u_{J*} f^* z_E, {}^t v_{J*} f^* w_E \right\rangle_{A_J, \mathrm{Kun}}. \end{aligned} \tag{13.2}$$

Indeed, $f_* f^* z_E = M z_E$. The global pairing is already absolutely normalized, while the factor M^{-1} records the finite flat rank in (10.2). In particular, using the maps (U_J, V_J) instead absorbs this factor into V_J and gives the same formula.

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