LOCALLY BOUNDED CLASSICAL PLURISUBHARMONIC FUNCTIONS AND MIXED CURRENTS

Abstract. Let K be a finite extension of $\mathbb{Q}_p$ and let X be a good, Hausdorff, boundaryless, quasi-smooth K-analytic space. We prove that every locally bounded classically plurisubharmonic function on X is locally a decreasing limit of semipositive rational piecewise-linear functions defined over K on one fixed neighborhood. We use this regularization to construct mixed currents in the original Chambert–Loir–Ducros calculus. The unweighted currents are symmetric, closed and strongly positive. Their coefficient measures are Radon measures on the analytic space, so multiplication by a locally bounded classical weight uses its given Borel representative. These weighted products are jointly continuous along decreasing sequences with locally bounded limits, and the construction is compatible with étale pullback of the potentials. The regularization proof uses curve slicing, projective contact rigidity, and continuity of ample envelopes on normal models. A fixed vertical correction for mixed-characteristic test ideals provides the envelope estimate.

1. Introduction

Let K be a finite extension of $\mathbb{Q}_p$. A classically plurisubharmonic function on a Berkovich K-analytic space is an upper-semicontinuous function whose pullback to every analytic curve, after any complete extension of K, is subharmonic [10]. This definition tests positivity on curves and does not require the function to have a smooth local expression. The question addressed here is whether locally bounded functions in this class admit decreasing piecewise-linear approximation, and whether their curvatures have mixed products in the Chambert–Loir–Ducros calculus.

For continuous semipositive potentials, uniform approximation provides the mixed products. A locally bounded classical potential may be discontinuous. The recursive construction then requires multiplication of a current by a Borel function, and its continuity requires a limit theorem in which both that function and the current vary. We prove these assertions for the original CLD test forms. The approximation takes place over K on a fixed neighborhood of each point; all weighted products use the given pointwise representatives.

1.1. Definitions and main results.

Convention 1.1. Fix a finite extension $K/\mathbb{Q}_p$ with its prescribed absolute value. Choose a uniformizer π and put $\lambda = -\log|\pi|$. Unless another field is specified, X is a good, Hausdorff, boundaryless, quasi-smooth K-analytic space of pure dimension $n \geq 1$. All opens are ordinary topological opens. A real function is *locally bounded* if it is bounded above and below on a neighborhood of every point. No replacement on a null or polar set is made.

Definition 1.2. For an ordinary open $U \subset X$, let $\mathcal{C}_b(U)$ be the real-valued, locally bounded, upper-semicontinuous functions u such that, for every complete isometric extension F/K and every morphism $C \to U_F$ from an F-analytic curve, the pullback of u is subharmonic in the sense of [10, Definition 4.1 and §3].

Let $\mathrm{PL}_{\mathrm{sp}}(U)$ denote the continuous real piecewise-linear functions semipositive in the local model, or equivalently reduction, sense of [10, Theorem 4.3]. Define $\mathcal{R}_b(U)$ to be the locally bounded functions u such that every point has one ordinary neighborhood $V \subset U$ and a sequence

$$p_m \in \mathrm{PL}_{\mathrm{sp}}(V), \qquad p_m \downarrow u|_V.$$

The neighborhood is independent of m. On an algebraic chart, *rational* PL means rational slopes and constants in $\lambda\mathbb{Q}$.

Date: 10 November 2026.

Convention 1.3. We use the original CLD forms $A^{p,q}$ and their compactly supported test spaces $A_c^{p,q}$, with the canonical calibration. Put $D = \mathrm{d}'\mathrm{d}''$. A current of bidegree (k,k) acts on $A_c^{n-k,n-k}$. A symmetric current is called *strongly positive* if it is nonnegative on weakly positive complementary original forms. The weakly positive cone is tested by decomposable vectors; the strongly positive cone is generated by decomposable squares. These conventions concern the original test sheaf, not the larger weakly smooth sheaf of [9].

A metric is written additively in weights $-\log\|e\|$, so its curvature is $D[-\log\|e\|]$. On a projective variety a continuous *model-semipositive* metric means a uniform limit of nef rational model metrics on the same identified line bundle. A *bounded classical metric* is a metric whose local weights are classical and whose difference from a continuous reference metric is bounded. A rational *pure Fubini–Study metric*, abbreviated pure-FS, is given by a finite generating tuple $s_0, \dots, s_N$ of a tensor power mL, with local weight

$$\frac{1}{m}\max_i(\log|s_i/e^{\otimes m}| + c_i), \qquad c_i \in \lambda\mathbb{Q}.$$

After replacing m by a multiple, the constants can be absorbed by multiplying the sections by powers of π.

The recursive formula for mixed currents will involve multiplying a current by a locally bounded, possibly discontinuous function. The following condition specifies the measures used for that multiplication.

Definition 1.4 (Coefficient Radon currents)**.** A current T is of *coefficient Radon type* if each finite original moment presentation represents its coefficient functionals by signed Radon measures on the analytic domain, compatibly with all original smooth scalar functions, relations between forms, and common refinements. In particular, for every compact original test η, there is a signed Radon measure $\mu_{T,\eta}$ with

$$\int a\, d\mu_{T,\eta} = \langle T, a\eta\rangle \quad \text{for every original smooth scalar } a.$$

For a locally bounded Borel function h, multiplication is defined by

$$\langle hT, \eta\rangle = \int h\, d\mu_{T,\eta}. \tag{1.1}$$

The compatibility requirements make this definition independent of the chosen presentation. A measure only on one tropical image does not suffice for this definition.

Theorem 1.5 (Local decreasing regularization)**.** *For every ordinary open $U \subset X$,*

$$\mathcal{C}_b(U) = \mathcal{R}_b(U).$$

For $u \in \mathcal{C}_b(U)$, the local approximants may be chosen rational PL, defined over K on one fixed neighborhood. Both classes are stable under decreasing sequences whose pointwise limit is locally bounded.

Proof on p. 14.

Theorem 1.6 (Mixed currents)**.** *For $u_1, \dots, u_k \in \mathcal{C}_b(U)$, $0 \le k \le n$, there are coefficient Radon currents*

$$T_0 = [1], \qquad T_k(u_1, \dots, u_k) = D\big(u_1 T_{k-1}(u_2, \dots, u_k)\big). \tag{1.2}$$

They are symmetric, d'- and d''-closed, strongly positive, and multilinear on the cone in the differentiated arguments. They agree with the continuous CLD products whenever the potentials are continuous and semipositive. They are independent of decreasing PL regularization and commute with ordinary restriction and gluing. For $u_0 \in \mathcal{C}_b(U)$, the weighted current

$$S_k(u_0; u_1, \dots, u_k) = u_0 T_k(u_1, \dots, u_k) \tag{1.3}$$

uses the Borel multiplication of Definition 1.4.

Proof on p. 15.

Theorem 1.7 (Joint decreasing continuity)**.** *Suppose that, for $0 \le i \le k$, the functions $u_{i,j} \in \mathcal{C}_b(U)$ decrease pointwise to a locally bounded function u_i. Then*

$$S_k(u_{0,j}; u_{1,j}, \dots, u_{k,j}) \longrightarrow S_k(u_0; u_1, \dots, u_k)$$

on every form in $A_c^{n-k,n-k}(U)$. The corresponding unweighted currents also converge. In top degree these are locally weak convergences of Radon measures, against every continuous compactly supported scalar function.

Proof on p. 15.

Corollary 1.8. *The preceding results apply on every ordinary open in the analytification of a smooth quasi-projective K-variety. Local boundedness there imposes no boundedness condition at a deleted algebraic boundary.*

Remark 1.9 (The undifferentiated argument). A constant in a differentiated argument has zero curvature. A change $u_0 \mapsto u_0 + c$ in the undifferentiated argument changes S_k by cT_k. Thus S_k is symmetric in $u_1, \dots, u_k$, but not in all its arguments. The weighted current may be signed and need not be closed. The hypothesis on the limit in Theorem 1.7 is essential to the stated class; for example, the decreasing constants $-j$ have no locally bounded limit.

1.2. **Proof strategy and relation to earlier work.** The proof has two parts. We first pass from the curve condition to local decreasing regularization. We then use this regularization to construct the mixed currents and prove their continuity. The first part uses only first-order currents and products of continuous metrics, so it does not presuppose the products in Theorem 1.6.

The continuous products come from uniform approximation by semipositive model metrics. Approximation by Fubini–Study metrics and by smooth semipositive metrics is supplied by [3, Corollary 7.6 and Theorems 7.8, 7.14], in the current calculus of [5]. To treat a discontinuous classical potential v, we slice a test presentation by algebraic curves. The curve condition applies over the completed residue fields of the slicing points, even when those fields are nondiscretely valued. Thuillier's lisse approximation and Dirichlet energy [14, §§3.4–3.5] give a positive Laplacian on each curve. Integrating these Laplacians gives the first-order current $D[v]$, its coefficient measures, and the Green formulas needed below. The lisse functions are auxiliary piecewise-affine potentials on skeleta; the tests remain original CLD forms. We use the skeleton integration formulas of [6] on their original unit-moment presentations.

The passage to regularization is made on a smooth projective geometrically integral variety over a field in the finite unit-Gauss tower class $\mathcal{G}_K$ of Definition 2.1, with an ample line bundle L. The contact theorem states that a bounded classical metric r and a continuous model-semipositive metric p satisfy

$$r \ge p, \qquad \int (r-p) c_1(L,p)^n = 0 \quad \Longrightarrow \quad r = p$$

pointwise. Its proof is an induction on dimension. The first-order Green estimate controls the gap $r - p$ against smooth backgrounds. A generic hyperplane then identifies its limit against the continuous background and reduces the vanishing assertion to dimension $n - 1$. The hyperplane is defined over a unit-Gauss completion, which is why the induction treats arbitrary data over all finite unit-Gauss towers at once.

To obtain the approximants, choose continuous metrics $\rho_\ell \downarrow r$ and take their semipositive envelopes $s_\ell = P_L \rho_\ell$. Continuity and orthogonality of these envelopes give

$$\int (\rho_\ell - s_\ell) c_1(L, s_\ell)^n = 0.$$

Apply contact to $\max(r, s_\ell)$ and s_ℓ. Their nonnegative gap is bounded by $\rho_\ell - s_\ell$, so contact gives $r \le s_\ell \le \rho_\ell$ and hence $s_\ell \downarrow r$. The envelope theorem is proved in Section C on a normal model. Its algebraic input is the corrected subadditivity inequality of Theorem C.4; the vertical correction is independent of the ideals and their exponents and therefore contributes an error tending to zero after normalization by degree. To return to a local potential, Lemma D.2 realizes it, up to a constant, as a metric weight on a smaller fixed neighborhood.

For the mixed-current construction, the induction is on exterior degree. Strong positivity supplies coefficient Radon measures on the analytic space, making $u_1 T_{k-1}$ meaningful before we differentiate it. To prove weighted convergence, we use Wu's top-degree theorem for decreasing piecewise-linear potentials [16, Theorem 4.2(4)]. A complementary test is reduced to products of curvatures of auxiliary quadratic potentials. In middle degrees, this reduction is only modulo the annihilator of decomposable squares. Strong positivity kills that term in the current pairing, and the corresponding coefficient measure remains zero after Borel multiplication. Thus the reduction applies to every original complementary test.

It remains to replace decreasing PL sequences by arbitrary decreasing classical potentials. We first place the whole family on one projective variety, using the common bounds supplied by its first terms and its limits. A compact diagonal argument then chooses decreasing PL approximants whose current pairings approach the required ones. Choosing the common local domain before the approximants resolves the domain dependence noted in [16, Remark 4.6] for this class of functions.

Sections 2 and 3 establish first-order positivity and regularization; Sections 4 and 5 construct the products and prove decreasing continuity. Section 6 transports these results to ordinary analytic charts and glues the currents. Section 7 treats specified potential tuples under étale pullback, including the finite-degree formula. The curve arguments, normal-model envelope construction, and common local domain are proved in the appendices. The index of notation records the definitions and normalizations used throughout.

Contents

1. Introduction 1
1.1. Definitions and main results 1
1.2. Proof strategy and relation to earlier work 3
2. First-order currents and energy 5
3. Contact rigidity and decreasing regularization 7
4. Coefficient measures and the mixed-current construction 10
5. Decreasing families of classical potentials 12
6. Ordinary localization and the main theorems 14
7. Compatibility for specified potential tuples 16
Appendix A. Curve Laplacians and Borel slicing 17
Appendix B. Bounded curve energy and a localized Green estimate 20
Appendix C. Envelopes on a normal model 23
C.1. A correction independent of ideals and exponents 23
C.2. Weighted ideals and removal of the auxiliary line 26
Appendix D. Algebraic coordinates and a common local domain 28
Use of generative AI 31
References 31
Index of notation 33

2. First-order currents and energy

We construct $D[v]$ by integrating curve Laplacians in the directions of complementary test forms. The resulting coefficient measures allow us to form the quadratic energy expressions for bounded potentials. These first-order expressions supply the contact estimate in Theorem 3.4; no higher product of discontinuous potentials is used at this stage.

Definition 2.1. Let $\mathcal{G}_K$ be the finite unit-Gauss tower class of Lemma A.1. The induction below is carried out simultaneously over every $F \in \mathcal{G}_K$, for arbitrary data defined over F. The residue field of such an F need not be finite or perfect. Its valuation ring is nevertheless a complete excellent mixed-characteristic DVR, and its compact analytic domains used here are metrizable.

For a current of bidegree $(1,1)$, the complementary degree is $n-1$. Every $(n-1)$-vector on a face of dimension at most n is decomposable. This is the linear-algebra fact that permits the following reduction to curve-slicing directions.

Lemma 2.2 (One-curvature complementary spanning)**.** *Let $r = n-1$. On a finite original moment presentation, every symmetric compact (r,r)-test is, after finite localization, a signed sum*

$$\eta = \sum_{\ell} a_\ell \Omega_\ell, \qquad \Omega_\ell = \bigwedge_{b=1}^{r} \mathrm{d}'\ell_{\ell b} \wedge \mathrm{d}''\ell_{\ell b}, \tag{2.1}$$

where the a_ℓ are smooth compact scalars and each $\ell_{\ell b}$ is a rational linear combination of logarithmic norms of units.

Proof. Let E be the ambient moment tangent space. The orthogonal complement of the span of decomposable squares in $\mathrm{Sym}^2(\bigwedge^r E^*)$ consists of quadratic forms vanishing on decomposable r-vectors. Every r-vector in a face tangent space of dimension at most $r+1 = n$ is decomposable. Polarization therefore makes this complement zero on every face. Rational decomposable squares span the remaining finite dimensional space. Choose a finite basis from them and apply a fixed linear splitting to the smooth coefficient matrix of η. The scalar coefficients remain smooth. Clear rational denominators and use a finite original smooth partition of unity. For $n = 1$ the assertion is the scalar statement with empty wedge. □

Proposition 2.3 (First-order measures and positivity)**.** *Let $F \in \mathcal{G}_K$, let X/F be smooth algebraic of pure dimension n, and let v be locally bounded classical psh on an ordinary open $U \subset X^{\mathrm{an}}$. Then $D[v]$ is a closed positive original current of coefficient Radon type. In every algebraic-unit curve-slicing presentation it satisfies the Borel kernel identity of Theorem A.4.*

Proof. By Lemma D.1, the finitely many analytic units in a localized test presentation may be replaced, with exactly the same norms, by algebraic units on a fixed smaller norm-box interior. After clearing denominators, each direction in (2.1) is given by $n-1$ algebraic units. These units define a map $p : W \to \mathbb{G}_m^{n-1}$. When this map is dominant, its fibers over the base skeleton are the curves on which we use the subharmonicity of v. Choose a nonnegative coordinate top form on the base equal to its constant volume form near the projected test support. If p is dominant, Theorem A.4 shows that $a \mapsto \langle D[v], a\Omega_\ell \rangle$ is a positive Radon measure on W. If p is not dominant, the pairing vanishes by tropical dimension. The signed spanning lemma gives order-zero bounds in every coefficient direction. Smooth scalar relations and common refinements follow from Radon uniqueness.

The slicing argument gives positivity in constant decomposable directions. To obtain positivity on every complementary test, we must also show that coefficient mass on a face has tangent indices on that face. Closedness supplies this tangency. Multiply the coefficient measures by a nonnegative compact smooth cutoff and push them to moment space. Write their symmetric matrix as M_{IJ}, and view each column as an r-vector measure F_J. Closedness of $D[v]$ implies that its distributional boundary is a measure; localization adds only the derivative of the cutoff times the order-zero current. For an affine linear equation ℓ, choose $\ell_\epsilon = \ell\theta(\ell/\epsilon)$,

where $\theta = 1$ near zero and has compact support. This cutoff tends uniformly to zero, while its differential detects precisely the mass on $\{\ell = 0\}$. Insert ℓ_ϵ times a compact scalar and a constant $(r-1)$-covector into the boundary equation. Terms containing ℓ_ϵ tend to zero. Its differential is bounded and converges pointwise to $\mathbf{1}_{\{\ell=0\}}\, d\ell$. Dominated convergence for the finite coefficient measures gives

$$(\iota_{d\ell} F_J)|_{\{\ell=0\}} = 0.$$

The equations of the affine span of each face therefore make both indices tangent to that face. On its tangent space every r-vector is decomposable. Positivity on rational decomposable directions, followed by density, makes the coefficient matrix positive semidefinite there. It pairs nonnegatively with every positive original test. For $n = 1$ positivity is the curve statement.

The coefficient identities are identities of Radon measures and therefore remain valid after multiplication by a bounded Borel scalar, as in Theorem A.4. □

Proposition 2.4 (Smooth-background energy)**.** *Let X/F be smooth projective, $F \in \mathcal{G}_K$, and L ample. Let r be a bounded classical metric and let p, q be continuous model-semipositive metrics on the same line. Set $g = r - p$, $b = q - p$ and $V = \deg_F(L^n)$. If Ω is a product of $n-1$ smooth semipositive regularized-FS curvatures on L, put*

$$\Gamma(g) = \tfrac{1}{2} D[g^2] - gD[g], \tag{2.2}$$

$$\Gamma(g,h) = \tfrac{1}{2}(D[gh] - gD[h] - hD[g]), \qquad E_\Omega(g,h) = \int_{X^{\mathrm{an}}} \Gamma(g,h) \wedge \Omega.$$

Then

$$E_\Omega(g) \geq 0, \qquad |E_\Omega(g,b)|^2 \leq E_\Omega(g) E_\Omega(b), \tag{2.3}$$

$$E_\Omega(g) = -\int gD[g] \wedge \Omega, \qquad E_\Omega(g,b) = -\int gD[b] \wedge \Omega, \tag{2.4}$$

$$0 \leq E_\Omega(b) \leq 2V \|q - p\|_\infty. \tag{2.5}$$

Proof. We first check that the energy expressions are defined using the first-order measures already constructed. Locally translate the bounded classical summands to be nonnegative. Their squares are then classical by convex increasing composition on curves. Polarization expresses squares and products of their differences as finite linear combinations of such squares. Thus Proposition 2.3 defines the differentiated terms in (2.2), and its coefficient measures permit multiplication by the bounded gaps.

The curve chain rule of Lemma B.1, transported through Theorem A.4, makes the quadratic expression nonnegative on each positive decomposable background. A smooth regularized-FS curvature has a positive semidefinite ambient Hessian, and their product is strongly positive. With one dominating coefficient measure, positivity on countably many rational directions passes by continuity to its variable decomposable directions. Polarization proves (2.3).

Stokes against the smooth closed Ω gives the first equality of (2.4). For the second, approximate p, q uniformly by smooth regularized-FS metrics, keeping Ω fixed. The localized variation estimate of Lemma B.2 supplies an integrable factor multiplying the uniform approximation error on the curve slices. A finite partition of unity and the kernel formula therefore give Green symmetry with the continuous b. The positive continuous measures $c_1(L,p) \wedge \Omega$ and $c_1(L,q) \wedge \Omega$ each have mass V. Their difference gives (2.5). □

Remark 2.5 (The energy current). The current $\Gamma(g)$ need not be closed. The face-tangency argument above is applied only to $D[v]$. Energy positivity is used on the strongly positive smooth backgrounds of Proposition 2.4. Its construction uses only first-order currents, independently of the higher mixed products.

3. Contact rigidity and decreasing regularization

In this section X/F is smooth projective geometrically integral, $F \in \mathcal{G}_K$, and L is ample. The continuous products used here are constructed from model-semipositive metrics. Uniform pure-FS approximation and smoothing are valid over these fields [3, Corollary 7.6 and Theorems 7.8, 7.14]. Their model measure is

$$c_1(L_1,\phi_1)\cdots c_1(L_n,\phi_n) = \sum_B b_B \deg_B(\mathcal{L}_1\cdots\mathcal{L}_n)\delta_{x_B} \tag{3.1}$$

on a common normal model, where b_B is the multiplicity of the special-fiber component B and all intersection degrees are taken over the residue field. This convention retains residual degrees and the prescribed absolute value.

Definition 3.1. For a continuous metric ρ on L, let

$$P_L\rho = \sup\{s : s \text{ continuous model-semipositive on } L,\ s \le \rho\},$$

pointwise in a common local frame. Nef rational model minorants give the same envelope: approximate a continuous competitor uniformly and shift its approximants down by rational valuation constants. Continuity and orthogonality of this envelope are proved in Theorem C.6.

The next lemma replaces one Fubini–Study curvature by integration over a generic hyperplane. It is used as an identity of measures: we will integrate the bounded, possibly discontinuous gap $r - p$ against it in the proof of contact rigidity.

Lemma 3.2 (A single generic hyperplane)**.** *Suppose $n \ge 2$. Let q be a pure-FS metric on L, given by generating sections $s_0,\dots,s_N$ of mL, with rational constants absorbed in a tensor power. Put*

$$E = F(\widehat{Z_0,\dots,Z_N})_{\mathrm{Gauss}}, \qquad H = \left\{\sum_i Z_i s_i = 0\right\} \subset X_E.$$

Then H/E is smooth projective geometrically integral of dimension $n-1$. For the natural map $\iota : H^{\mathrm{an}} \to X^{\mathrm{an}}$ and continuous model-semipositive metrics $a_1,\dots,a_{n-1}$ on L,

$$c_1(q)\prod_{i=1}^{n-1} c_1(a_i) = \frac{1}{m}\,\iota_*\prod_{i=1}^{n-1} c_1(a_i|_H), \tag{3.2}$$

$$\deg_E((L_E|_H)^{n-1}) = m\deg_F(L^n). \tag{3.3}$$

The first equality is an equality of Radon measures.

Proof. The associated morphism $X \to \mathbb{P}^N_F$ is finite onto its image: the pullback of $\mathcal{O}(1)$ is ample, so a positive-dimensional proper fiber is impossible. Generic Bertini in characteristic zero gives smoothness and geometric irreducibility of the hyperplane pullback; for irreducibility see [12, Lemma 37.32.3]. Completion of the Gauss parameter field preserves these geometric properties.

First take all a_i to be nef rational model metrics, and choose a common normal projective model carrying them and the generated model line of q. Work over $R[Z_0,\dots,Z_N]_{(\pi)}$ and then its completion. For an irreducible special component B, the incidence in affine coefficient space is the kernel vector bundle of the generating evaluation map. Its projectivization is a projective bundle over B. If the image of B under the generated linear system is a point, the generic hyperplane misses it, and its intersection degree with that generated line is zero. Otherwise the incidence dominates the parameter space; its generic fiber is integral. It need not be geometrically integral.

The multiplicity of the induced component is the original vertical multiplicity b_B. This can be read at its generic point, away from the singular loci of the model and of B: the hyperplane cuts a parameter on B, whereas the uniformizer has order b_B. Such generic points avoid every fixed proper closed subset of B in the universal incidence. Normalization retains the associated residual degrees. The divisorial points of the hyperplane model restrict to x_B, and

the projection formula for their intersection degrees gives exactly the coefficient of δ_{x_B} in (3.1). The root presentation of mL supplies the factor $1/m$. Completion preserves normality because it is a regular morphism.

This proves (3.2) for model metrics. Uniform approximation passes it to the stated continuous metrics, on both sides as Radon measures. Finally the algebraic intersection number of $H \sim mL_E$ gives (3.3). There is no finite-degree factor for the transcendental Gauss extension. □

Lemma 3.3 (Weighted uniform stability in lower dimension)**.** *Suppose that bounded classical metrics in dimension d admit decreasing continuous semipositive regularization. On a smooth projective d-fold, let r be a bounded classical metric, p continuous model-semipositive on the same ample line, and $h = r - p$. For continuous semipositive a_i, b_i on that line, with W its top degree,*

$$\left| \int h \left(\prod_{i=1}^{d} c_1(a_i) - \prod_{i=1}^{d} c_1(b_i) \right) \right| \leq 2W \sum_{i=1}^{d} \|a_i - b_i\|_\infty. \tag{3.4}$$

Proof. Choose continuous semipositive $r_\ell \downarrow r$. At a fixed ℓ, telescope the product and use the continuous Green formula to move the difference $D[a_i - b_i]$ onto $r_\ell - p$. The two resulting positive mixed measures each have mass W. The corresponding term is bounded by $2W\|a_i - b_i\|_\infty$. Sum the terms and let $\ell \to \infty$. On the two fixed measures in the left side, the functions $r_\ell - p$ are uniformly bounded and converge pointwise to h, so bounded convergence applies. □

Theorem 3.4 (Projective contact and regularization)**.** *For every $F \in \mathcal{G}_K$ and every positive dimension, the following statements hold.*

(1) *If r is a bounded classical metric on L, p is continuous model-semipositive, $r \geq p$, and*

$$\int (r - p) c_1(L, p)^n = 0,$$

then $r = p$ at every point.

(2) *Every bounded classical metric r on L is the decreasing pointwise limit of continuous model-semipositive metrics, and also of rational pure-FS metrics, on the same identified line over F.*

Proof. We prove contact in dimension n using both assertions in smaller dimensions, and then deduce regularization in dimension n from contact. The induction is simultaneous for all $F \in \mathcal{G}_K$, so the lower-dimensional assertions remain available over the unit-Gauss field of a generic hyperplane. In dimension one, contact is Proposition B.4; the envelope argument at the end of the proof then gives regularization.

Assume $n \geq 2$, and set $g = r - p \geq 0$, $V = \deg_F(L^n)$. Choose smooth semipositive regularized-FS metrics $p_j \to p$ uniformly and put $\Omega_j = c_1(p_j)^{n-1}$. The function g is bounded and upper semicontinuous. Continuous mixed convergence and the upper Portmanteau inequality give

$$A_j = \int g\, c_1(p) \wedge \Omega_j \longrightarrow 0.$$

Since $D[g] = c_1(r) - c_1(p)$, the Green identity gives $E_{\Omega_j}(g) = A_j - \int g\, c_1(r) \wedge \Omega_j$. The second term is nonnegative by first-order positivity and $g \geq 0$. Together with Proposition 2.4, this yields

$$0 \leq E_{\Omega_j}(g) \leq A_j.$$

For a fixed pure-FS metric q, set $I_j(q) = \int g\, c_1(q) \wedge \Omega_j$. Green symmetry identifies $I_j(q) - A_j$ with $-E_{\Omega_j}(g, q - p)$. Cauchy–Schwarz and $E_{\Omega_j}(q - p) \leq 2V\|q - p\|_\infty$ therefore yield

$$0 \leq I_j(q) \leq A_j + \left(2V\|q - p\|_\infty A_j\right)^{1/2} \longrightarrow 0. \tag{3.5}$$

The convergence of the background measures alone does not identify their integrals against the discontinuous function g. Apply Lemma 3.2 to q. On its hyperplane H/E, the dimension-$(n-1)$ induction gives decreasing regularization of $r|_H$, which is the hypothesis needed for the

weighted stability estimate. Applying Lemma 3.3 there and using (3.3), the factor m cancels the $1/m$ in (3.2). Consequently

$$I_j(q) \longrightarrow \int g\, c_1(q)c_1(p)^{n-1}.$$

The measure identity and (3.5) now give

$$\int_{H^{\mathrm{an}}} g|_H\, c_1(p|_H)^{n-1} = 0.$$

The lower-dimensional contact theorem shows $g|_H = 0$.

We now pass from vanishing on the generic hyperplane to pointwise vanishing on X. The intermediate points are the Shilov boundary points of a model associated with a pure-FS metric q close to p. Normalize the projective closure of its image in the finite function-field extension $F(X)$. This gives a normal model finite over that image, with the generated model line of mL relatively ample. Every special component has dimension n and meets the generic hyperplane. The divisorial points of these intersections restrict to all its component points x_B, so the equality $g|_H = 0$ implies $g(x_B) = 0$.

On a formal generating-frame domain, the model weight of mq is zero. The local weight $m(r-q)$ is classical psh there. The maximum principle on its Shilov boundary [10, Theorem 6.4 and Remark 7.5] therefore gives

$$r - q \le \max_B (p-q)(x_B) \le \|p-q\|_\infty.$$

These domains cover X^{an}. Hence

$$0 \le r - p \le 2\|p-q\|_\infty.$$

Pure-FS approximation to p proves the contact assertion.

It remains to deduce regularization from contact. Compact metrizability gives continuous metrics $\rho_\ell \downarrow r$. Set $s_\ell = P_L\rho_\ell$. By Theorem C.6, each s_ℓ is continuous model-semipositive and satisfies envelope orthogonality. Although r is not yet known to be a semipositive competitor, the classical metric $\max(r, s_\ell)$ lies above s_ℓ and satisfies

$$0 \le \max(r, s_\ell) - s_\ell \le \rho_\ell - s_\ell.$$

Its contact integral is therefore zero. The contact assertion just proved implies $\max(r, s_\ell) = s_\ell$, or $r \le s_\ell$. Since $s_\ell \le \rho_\ell$ and envelopes preserve order,

$$s_\ell \downarrow r.$$

Uniform pure-FS approximation need not preserve the order of the sequence. To retain monotonicity, choose rational pure-FS a_ℓ with $\|a_\ell - s_\ell\|_\infty \le \epsilon_\ell$, where $\epsilon_\ell = \lambda 4^{-\ell}$. Then $q_\ell = a_\ell + 3\epsilon_\ell$ is decreasing: its next term is at most $s_\ell + \epsilon_\ell$, whereas its current term is at least $s_\ell + 2\epsilon_\ell$. It converges pointwise to r. This proves the second assertion and completes the simultaneous induction. □

Proposition 3.5 (Local algebraic regularization)**.** *Let $F \in \mathcal{G}_K$, and let u be locally bounded classical psh on an ordinary open of a smooth geometrically integral algebraic F-variety. Near each point, u is the decreasing limit of rational semipositive PL functions over F on one fixed ordinary domain. The approximants may be taken in Wu's algebraic tropically convex PL class on that domain.*

Proof. Choose an affine algebraic neighborhood of the prescribed point. Compactify it projectively over F, and resolve the compactification while preserving the smooth affine open [2, Theorem 1.1]. The resulting variety is smooth projective and geometrically integral over F, and contains the ordinary open on which u is given. The projective theorem applies to a global metric, whereas u is given only locally. Apply Lemma D.2 to the single function u. On a smaller fixed neighborhood it realizes $u + c$ as the exact weight of a bounded classical metric on an ample line; outside a larger neighborhood the metric is supplied by a fixed Fubini–Study barrier. Theorem 3.4 provides decreasing pure-FS metrics there. Restrict to the fixed frame

and subtract the same constant c. The inactive floor of Lemma D.3 gives the stated finite algebraic tropically convex representations, including at zeros of individual sections. □

4. Coefficient measures and the mixed-current construction

We construct the products by induction using $T_k = D(u_1 T_{k-1})$. Two points require attention: the Borel product $u_1 T_{k-1}$ must be independent of a moment presentation, and top-degree PL convergence must determine the pairing with an arbitrary complementary test. The coefficient-measure lemma addresses the first point and supplies the linear-algebra reduction for the second.

Lemma 4.1 (Coefficient measures of strongly positive currents)**.** *Every symmetric strongly positive original current is of coefficient Radon type. On a fixed finite moment presentation its coefficient variations are bounded by finitely many positive cutoff evaluations. Consequently these variations are locally uniformly bounded in a weakly convergent family of such currents.*

Proof. Put $r = n - k$, and let E be the ambient moment tangent space. Identify symmetric (r, r)-forms with symmetric bilinear tensors using the usual Lagerberg sign. Choose a rational basis e_I of $\bigwedge^r E^*$, and put $\sigma = \sum_I e_I \otimes e_I$. For any constant symmetric tensor A, choose C_A so that $C_A\sigma \pm A$ are positive semidefinite, hence weakly positive. If ϕ is an arbitrary original smooth scalar supported where $\kappa = 1$, strong positivity gives

$$|\langle T, \phi A\rangle| \le C_A \|\phi\|_\infty \langle T, \kappa\sigma\rangle. \tag{4.1}$$

The estimate applies to every original smooth scalar ϕ, including those that do not factor through the chosen moment map: pass to a common refinement, where pullback preserves weak positivity. Riesz representation therefore gives coefficient measures on the analytic domain itself.

Let W_r be the span of squares of decomposable r-vectors in $\mathrm{Sym}^2(\bigwedge^r E)$, and let $A_r = W_r^\perp$ in the dual symmetric tensor space. Both signs of a tensor in A_r are weakly positive. Hence T annihilates every smooth coefficient times such a tensor. Its coefficient measure is zero, and remains zero after multiplication by any bounded Borel function.

We use decomposable directions after passing to the quotient by A_r. Choose finitely many rational decomposable squares and a fixed linear splitting in that quotient. Every symmetric test then has a decomposition

$$\eta = \sum_\ell a_\ell \Omega_\ell + B, \qquad B(t) \in A_r, \tag{4.2}$$

where the a_ℓ are original smooth compact scalars and the Ω_ℓ are constant rational decomposable forms. The annihilator term is killed by T and by every Borel multiple of T.

All form relations and changes of presentation can be checked after multiplying by every original smooth scalar. Uniqueness of Radon measures then proves those relations as measure identities. This also proves compatibility with Borel multiplication. Finally, in a weakly convergent family, the finitely many positive evaluations on $\kappa\sigma$ converge and are bounded. Equation (4.1) gives the last assertion. □

Remark 4.2 (The middle-degree annihilator). The term B in (4.2) is necessary. For $\dim E = 4$ and $r = 2$, the Plücker quadratic

$$z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{23}$$

vanishes on all decomposable bivectors. The decomposable-square span has dimension 20, whereas the symmetric tensor space has dimension 21. Strong positivity kills this annihilator in the current pairing; it does not remove it from the original form sheaf.

To apply a top-degree convergence theorem to a test in complementary degree r, we will fill its r directions with curvature factors. The following quadratic potentials realize exactly those factors and admit PL approximants on the same domain.

Lemma 4.3 (Auxiliary quadratic potentials)**.** *After fixed finite localization, every decomposable direction in* (4.2) *is a product*

$$\Omega = \prod_{a=1}^{r} Dh_a, \qquad h_a = \tfrac{1}{2}\ell_a^2,$$

where the ℓ_a are rational linear combinations of logarithmic norms of algebraic units. Each h_a admits a decreasing sequence of algebraic tropically convex rational PL approximants, uniformly on the relevant compact neighborhood.

Proof. Use Lemma D.1 and clear rational denominators. The unit logarithms are pluriharmonic, so $D(\ell_a^2/2) = d'\ell_a \wedge d''\ell_a$. On one fixed bounded interval containing the range of ℓ_a, approximate $t^2/2$ uniformly by finite maxima of affine functions with rational slopes and constants in $\lambda\mathbb{Q}$. Such maxima are dense: approximate finitely many supporting slopes and their intercepts, using the density of $\mathbb{Q}$ and of $\lambda\mathbb{Q}$. Choose errors $\epsilon_m = \lambda 4^{-m}$ and add $3\epsilon_m$. The same margin calculation as in Theorem 3.4 makes the sequence decreasing to $t^2/2$. Clearing denominators realizes the affine branches by algebraic monomials in the units and powers of π. Lemma D.3 gives the required algebraic PL presentation. The interval, units and analytic domain remain fixed. □

Proposition 4.4 (Top-degree PL convergence and normalization)**.** *On an ordinary algebraic analytic open of dimension n over K, decreasing sequences of algebraic tropically convex PL functions $p_{i,j} \downarrow p_i$, $0 \le i \le n$, with locally bounded limits satisfy*

$$p_{0,j}\ \mathrm{MA}_{\mathrm{PL}}(p_{1,j}, \dots, p_{n,j}) \longrightarrow p_0\ \mathrm{MA}_{\mathrm{PL}}(p_1, \dots, p_n) \tag{4.3}$$

locally weakly as Radon measures. The continuous PL stages agree with the original CLD products normalized by (3.1)*.*

Proof. This is [16, Theorem 4.2(4)], applied on compact supports, with every sequence, including the undifferentiated one, in the algebraic tropically convex PL class. The local class is [16, Definition 3.8]; the continuous PL comparison is [16, Proposition 4.1(3)].

We make the normalization explicit. Wu's preliminary convention may be achieved by replacing the absolute value temporarily by $|\cdot|_* = |\cdot|^{1/\lambda}$. The canonical homeomorphism $j : X^{\mathrm{an}} \to X_*^{\mathrm{an}}$ sends a seminorm to its $1/\lambda$-power. Put $p_{i,*} \circ j = p_i/\lambda$. In logarithmic coordinates $t_* = t/\lambda$, pullback of a top Lagerberg form satisfies

$$\int_{X^{\mathrm{an}}} j^*\alpha = \lambda^{-n} \int_{X_*^{\mathrm{an}}} \alpha. \tag{4.4}$$

There are two differentials in each of the n directions, while the calibrated real volume has dimension n, which gives this factor. Also $j^* D_* p_{i,*} = \lambda^{-1} Dp_i$. The factors therefore cancel in the calibrated top product:

$$j^*\,\mathrm{MA}_*(p_{1,*}, \dots, p_{n,*}) = \mathrm{MA}_{\mathrm{PL}}(p_1, \dots, p_n)$$

as transported measures. Equivalently, both sides assign the same structural degree to every model component in (3.1). Inserting the weight gives the factor λ^{-1}, since $p_{0,*} \circ j = p_0/\lambda$; multiplication by λ returns (4.3). Thus the formula holds for the prescribed absolute value and the given functions, with the model masses in (3.1).

The constructed PL functions satisfy the stated local hypothesis by Lemmas D.3 and 4.3. In particular rational valuation constants are encoded by constant algebraic coordinates, or become rational under the normalization above. All uses of (4.3) are on compact tests; no global integrability assertion on a noncompact open is needed. □

Proposition 4.5 (Construction and decreasing PL convergence)**.** *On an ordinary open in the analytification of a smooth geometrically integral K-variety, the currents in Theorem 1.6 exist. If each potential varies in a decreasing sequence of algebraic tropically convex PL functions on one fixed open, with locally bounded classical limit, then the weighted currents converge on every original complementary test.*

Proof. We prove existence in degree k from weighted convergence in degree $k-1$, and then prove weighted PL convergence in degree k. This order is required by the recursion: differentiation of u_1T_{k-1} uses the undifferentiated argument at the preceding degree. For $k=0$, $T_0=[1]$, and the weighted assertion is bounded convergence against each fixed original top-form measure.

Suppose the statement holds in degree $k-1$. The coefficient measures of T_{k-1} allow multiplication by u_1. Distributional differentiation of this Borel product defines T_k by (1.2). Choose decreasing rational pure-FS approximants to its potentials on one fixed open. By weighted convergence in degree $k-1$, the Borel products formed from these approximants converge to u_1T_{k-1} on every original test. Applying distributional differentiation shows that their continuous mixed currents converge to the T_k just defined. Finite convex-maximum smoothing makes each approximating current strongly positive. The limit is strongly positive, and Lemma 4.1 supplies its coefficient measures. Permuting the continuous approximants proves symmetry. Closedness and multilinearity follow from the recursion and the continuous identities. The recursion also makes the candidate independent of the chosen approximants.

We still need weighted convergence in degree k; it does not follow by multiplying the unweighted current limit by a discontinuous function. Use (4.2). Its annihilator term has zero coefficient measure, so both the current and its Borel multiple kill that term. It suffices to treat a test $a\Omega$, where $\Omega = Dh_1\cdots Dh_r$, $r=n-k$, and the h_α are as in Lemma 4.3. First we prove the unweighted identity of Radon measures on the analytic domain

$$T_k(u_1,\dots,u_k)\wedge\Omega = \mathrm{MA}_{\mathrm{PL}}(u_1,\dots,u_k,h_1,\dots,h_r). \tag{4.5}$$

Let $u_{i,j}\downarrow u_i$ be PL sequences and $h_{\alpha,l}\downarrow h_\alpha$ their uniform PL approximants. At fixed j, continuous mixed convergence lets one choose an auxiliary index $l(j)\geq j$, increasing in j, so that the error on the given smooth scalar test is less than $1/j$. Apply Proposition 4.4 to the resulting n decreasing PL sequences. Its unweighted version follows by taking a fixed nonzero weight in $\lambda\mathbb{Q}$. The other limit is $T_k(u_1,\dots,u_k)\wedge\Omega$, by the already established current convergence. Equality on every smooth scalar, and Radon uniqueness, prove (4.5).

Because (4.5) is now an identity of Radon measures on the analytic domain, we may multiply it by the given Borel function u_0. For the varying PL weight $u_{0,j}$, at each fixed j use the continuous compact scalar $au_{0,j}$ to choose the auxiliary index with error less than $1/j$. Increase the indices to preserve all decreasing auxiliary sequences. The weighted assertion (4.3) yields

$$\langle u_{0,j}T_k(u_{1,j},\dots,u_{k,j}),a\Omega\rangle \longrightarrow \int au_0\,d\,\mathrm{MA}_{\mathrm{PL}}(u_1,\dots,u_k,h_1,\dots,h_r).$$

By (4.5), the right side is precisely $\langle u_0T_k(u_1,\dots,u_k),a\Omega\rangle$. For each test, this diagonal proves convergence to the current defined by the recursion. Finite summation finishes the induction. □

5. Decreasing families of classical potentials

Each classical potential has a decreasing PL regularization, but independently regularizing the members of a decreasing family need not preserve their order. We use a compact diagonal argument to choose one approximant from each row. The choice must both preserve decrease and satisfy prescribed bounds on current pairings.

Lemma 5.1 (A compact triangular diagonal)**.** *Let Z be compact, and suppose, for finitely many i, that bounded metrics $R_{i,j}\downarrow R_i$ on one identified line have continuous decreasing rows*

$P_{i,j,m} \downarrow R_{i,j}$. Let $\delta_j > 0$ decrease strictly to zero. Given arbitrary thresholds for m, one can choose $m_j \geq j$ beyond them so that

$$Q_{i,j} = P_{i,j,m_j} + \delta_j$$

decreases in j and converges pointwise to R_i.

Proof. Proceed inductively. The desired inequality $P_{i,j,m} + \delta_j < Q_{i,j-1}$ holds eventually at each point: its limit on the left is $R_{i,j} + \delta_j$, whereas the right is at least $R_{i,j-1} + \delta_{j-1}$, with a strict positive margin. The sets on which it holds are increasing open sets in m. Compactness gives one index for all points and the finite set of indices i.

Monotonicity alone would only give a limit above R_i. To force the correct limit, compare the chosen term with the first j fixed rows by imposing the finite family of caps

$$P_{i,j,m} + \delta_j \leq P_{i,a,j} + 2\delta_j, \qquad 1 \leq a \leq j.$$

The limiting left side is at most $R_{i,a} + \delta_j$, so the same strict-margin compactness argument permits all these caps. Include the prescribed thresholds and $m_j \geq j$. The decreasing limit Q_i is at least R_i. For fixed a, the cap gives $Q_i \leq R_{i,a}$ as $j \to \infty$. Letting $a \to \infty$ gives $Q_i = R_i$ pointwise. □

Proposition 5.2 (Convergence for decreasing classical potentials)**.** *The conclusion of Theorem 1.7 holds on ordinary opens in the analytification of a smooth geometrically integral K-variety.*

Proof. The decreasing-limit property on curves shows that each locally bounded limit u_i is classical [10, Proposition 4.2]. The endpoint inequalities $u_i \leq u_{i,j} \leq u_{i,1}$ give common upper and lower bounds near a prescribed compact local support. Apply Lemma D.2 on a fixed smooth projective compactification of the affine algebraic chart, as in the proof of Proposition 3.5, before choosing any regularizing sequences. It gives one smooth projective variety Z, one ample line, one frame on a relatively compact ordinary neighborhood V, and one constant c, with bounded classical metrics

$$R_{i,j} \downarrow R_i$$

whose local weights are exactly $u_{i,j} + c$ and $u_i + c$. For each i, j, choose decreasing pure-FS metrics $P_{i,j,m} \downarrow R_{i,j}$ by Theorem 3.4. One inactive floor works for all these sequences on V.

Fix a complementary test η supported in V. Write

$$F_j = \langle S_k(u_{0,j}; u_{1,j}, \dots, u_{k,j}), \eta \rangle, \qquad B_j = |\langle T_k(u_{1,j}, \dots, u_{k,j}), \eta \rangle|.$$

Choose strictly decreasing $\delta_j \in \lambda \mathbb{Q}_{>0}$ with $\delta_j \leq 2^{-j}$ and $\delta_j B_j \leq 1/j$. The small shift enforces strict decrease in the compact diagonal. It also changes the undifferentiated argument, so we choose it to make that change in the pairing at most $1/j$. This choice uses only the finite value of B_j, without a bound uniform in j. For each fixed j, decreasing PL convergence from Proposition 4.5 shows that, as $m \to \infty$, the weighted-current pairing of the shifted local weights $P_{i,j,m} + \delta_j - c$ converges to

$$F_j + \delta_j \langle T_k(u_{1,j}, \dots, u_{k,j}), \eta \rangle.$$

The shifts in the differentiated arguments have zero curvature. Choose a threshold for m making this error less than $1/j$.

Apply Lemma 5.1 with those thresholds. The caps in the diagonal lemma ensure that the local weights $Q_{i,j} - c$ converge pointwise to u_i. They are decreasing algebraic tropically convex PL functions on the same V. Their weighted-current pairing G_j converges to the desired limit by Proposition 4.5. The chosen threshold and the bound on δ_j give

$$|F_j - G_j| \leq 2/j.$$

This proves convergence on η. A finite original smooth partition of the support proves the assertion for an arbitrary compact test. The unweighted assertion follows by taking a constant weight. □

Proposition 5.3 (Continuous compatibility and top degree)**.** *The constructed currents agree with the continuous CLD products on their common continuous domain. Their top-degree weighted convergence holds against continuous compact scalar tests.*

Proof. For continuous limits, decreasing continuous PL regularization is uniform on compact subsets by Dini's argument. Continuous CLD mixed convergence therefore identifies the products. Conversely the recursive formula and Proposition 4.5 determine the current uniquely from any decreasing PL regularization.

In top degree choose a nonnegative original smooth cutoff equal to one on the test support. Its evaluations on the positive measures $T_{n,j}$ converge by Proposition 5.2, giving a common mass bound there. The weights have a common bound M on a slightly larger compact neighborhood, so $|S_{n,j}| \leq MT_{n,j}$ there. Uniform approximation of a continuous compact scalar by original smooth scalars then extends convergence to that scalar. □

6. Ordinary localization and the main theorems

The preceding constructions were made on algebraic charts. We now transfer them to the analytic spaces of Convention 1.1. The ordinary neighborhood must be fixed before regularization, and a finite change in the constant field must retain the original integration calibration. Once these points are settled, the recursive formula identifies the locally constructed currents on overlaps.

Proposition 6.1 (Transport to an ordinary analytic chart)**.** *Every point of X has one ordinary neighborhood on which Propositions 3.5, 4.5 and 5.2 apply with the original K-calibration. For a finite decreasing family of locally bounded classical functions, the neighborhood and the projective variety can be chosen independently of the sequence index.*

Proof. By Lemma D.4, a sufficiently small ordinary neighborhood is an ordinary open of a smooth affine K-variety. Its integral components are disjoint; restrict to the component containing the point. Let L be its finite constant field as in Lemma D.5. The same variety is smooth geometrically integral over the finite p-adic field L. Take its projective closure and resolve that closure while preserving the smooth affine open [2, Theorem 1.1]. This gives a smooth projective geometrically integral L-variety containing the original chart. The preceding theorems apply with L as the initial field and $\mathcal{G}_L$ as its auxiliary tower class. All local weights are functions on the identical original analytic chart. The algebraic L-functions used in their PL expressions belong to its original K-coordinate ring. If π_L is a uniformizer of L, then $-\log|\pi_L|\mathbb{Q} = \lambda\mathbb{Q}$. Thus these are rational PL functions defined over K, and the classical and model-semipositive conditions are preserved. For the currents, multiply the L-calibration by the single factor $[L : K]$ in (D.4).

For a decreasing family $u_{i,j} \downarrow u_i$, choose this chart and a slightly larger compact neighborhood first. The endpoint inequalities

$$u_i \leq u_{i,j} \leq u_{i,1}$$

give common two-sided bounds there for the finite family. The common collar of Lemma D.2 therefore fixes the projective variety, line, frame and local domain for all indices. The potentials themselves are not required to extend to the compactification. The analytic isomorphism transfers every original test presentation, so it also preserves the current identities on every original test form. □

Proof of Theorem 1.5. The inclusion $\mathcal{C}_b(U) \subset \mathcal{R}_b(U)$ follows from Proposition 6.1. Conversely, semipositive PL functions are classical by [10, Theorem 4.3]. This result is applied on the strict ordinary algebraic charts just constructed; rational tensor denominators can be cleared. The decreasing-limit property of classical subharmonic functions on curves, with the all-extension quantifier, makes the pointwise limit classical [10, Proposition 4.2]. That limit is upper semicontinuous as an infimum of continuous functions, and it is locally bounded by the definition of $\mathcal{R}_b$. This proves equality on every ordinary open.

Now let u_j decrease in either class with a locally bounded pointwise limit u. The equality just proved makes every u_j classical. Since $\{u < a\} = \bigcup_j \{u_j < a\}$, the limit is upper semicontinuous. The curve decreasing-limit property again makes u classical. Applying the regularization assertion to this function gives membership in $\mathcal{R}_b$ on one neighborhood at a time. No intersection of the separate approximation neighborhoods of the u_j is used. □

Lemma 6.2 (Equality and Borel multiplication)**.** *If two coefficient Radon currents T, T' agree on all original smooth compact tests on an ordinary open, then $hT = hT'$ for every locally bounded Borel function h.*

Proof. For a fixed compact original test η, the scalar functionals

$$a \longmapsto \langle T, a\eta \rangle, \qquad a \longmapsto \langle T', a\eta \rangle$$

agree on every original smooth scalar. Smooth scalar density on a compact neighborhood and uniqueness of Radon measures give $\mu_{T,\eta} = \mu_{T',\eta}$. Integrating the given function h against this equality proves the assertion. The coefficient relations and common refinements are measure identities by Lemma 4.1; finite localization therefore removes the choice of presentation. □

Proof of Theorems 1.6 and 1.7. Construct the currents on the ordinary charts of Proposition 6.1. At degree zero their normalization is the intrinsic integration current $[1]_K$, by Lemma D.5. Suppose the constructions agree in degree $k-1$ on an overlap. By Lemma 6.2, multiplication by the same Borel function u_1 agrees there. Applying D identifies the two degree-k currents in (1.2). A second application of the lemma with u_0 identifies the weighted currents. Thus the local currents glue uniquely.

Closedness, strong positivity, symmetry and additivity in the differentiated arguments are local properties, proved in Proposition 4.5. Positive homogeneity follows from the same recursion and approximation. A constant in a differentiated argument contributes zero because the preceding current is closed. Linearity of Borel multiplication gives the identity for the undifferentiated argument in Remark 1.9. The recurrence and the continuous compatibility of Proposition 5.3 give independence of PL regularization.

For joint convergence, Theorem 1.5 first makes each bounded limit u_i classical. Cover the compact support of a test by finitely many ordinary charts and split the test into compact summands by an original smooth partition of unity. The compact-support partition theorem [6, Corollary 15.2.9] is sufficient; no global paracompactness assumption is needed. On each slightly larger compact subchart the bounded endpoints give the common bounds used in Proposition 5.2. Apply that proposition, restore the single constant-field factor, and sum. Only the test forms are partitioned. Taking a constant weight proves the unweighted convergence. In top degree, the cutoff mass argument in Proposition 5.3 extends convergence to all continuous compact scalar tests. □

Proof of Corollary 1.8. The analytification of a smooth separated variety over K is good, Hausdorff, quasi-smooth and without relative analytic boundary [13, Exercise 5.1.1(iii), Fact 5.1.5]. These properties are preserved on ordinary opens. Every compact test uses only finitely many relatively compact charts inside the given open, to which the main theorems apply. □

Remark 6.3 (Two different boundaries). If $Y \subset \overline{Y}$ is a projective compactification, the deleted algebraic locus $\overline{Y} \setminus Y$ differs from the relative analytic boundary of $Y^{\mathrm{an}} \to \mathcal{M}(K)$, which is empty for smooth Y. A potential can be unbounded near the deleted algebraic locus and remain locally bounded at every point of Y^{an}. The present theorems apply on that ordinary open and make no claim about extension of the potential to $\overline{Y}^{\mathrm{an}}$. For a singular variety they apply on its smooth locus.

7. Compatibility for specified potential tuples

For a specified tuple of potentials, composition with an étale map gives another tuple of locally bounded classical potentials. The next theorem identifies its mixed current with the limit of the continuous-stage pullbacks. Thus the operation is defined through the potentials and their decreasing regularizations; no pullback of an arbitrary distributional current is required.

Theorem 7.1 (Étale compatibility)**.** *Let $f : Y \to X$ be étale between spaces satisfying Convention 1.1. Let $W \subset X$ be an ordinary open on which a finite tuple $u_0, \dots, u_k \in \mathcal{C}_b(W)$ has decreasing continuous semipositive regularizations $a_{i,m} \downarrow u_i$. Then, on every original compact test in $f^{-1}(W)$,*

$$S_{k,Y}(a_{0,m} \circ f; a_{1,m} \circ f, \dots, a_{k,m} \circ f) \longrightarrow S_{k,Y}(u_0 \circ f; u_1 \circ f, \dots, u_k \circ f). \tag{7.1}$$

This extension of the continuous-stage operation is independent of the regularizations and commutes with ordinary restriction and composition of étale maps.

Proof. Composition of curve tests preserves classicality, and continuity of f preserves local boundedness. At a continuous stage, local uniform smooth semipositive approximation and the pullback identity for smooth forms identify the pulled-back operation with the continuous mixed product of the pulled-back weights. To obtain the required smooth approximations, apply Theorem 1.5 to a continuous weight. Its decreasing PL approximation is uniform on a smaller compact neighborhood by Dini's theorem. In the pure-FS presentations furnished by the proof, smoothing the convex maximum gives original smooth semipositive weights with the same uniform limit; see [3, Theorem 7.14]. The argument is local on both spaces. Theorem 1.7 on Y gives (7.1) with the Borel product on its right, for every choice of base regularization. For an étale map $g : Z \to Y$, pull the same continuous approximants twice and apply the theorem on Z. Both operations give the weighted current of $(u_i \circ f \circ g)_i$. Restriction follows from locality. □

Proposition 7.2 (The finite-étale degree formula)**.** *Let $f : Y \to X$ be finite étale of constant degree d. For every tuple $u_0, \dots, u_k \in \mathcal{C}_b(X)$,*

$$f_* S_{k,Y}(u_0 \circ f; u_1 \circ f, \dots, u_k \circ f) = d S_{k,X}(u_0; u_1, \dots, u_k), \tag{7.2}$$

$$f_* T_{k,Y}(u_1 \circ f, \dots, u_k \circ f) = d T_{k,X}(u_1, \dots, u_k). \tag{7.3}$$

For locally constant degree, the formulas hold componentwise.

Proof. For smooth inputs, apply the calibrated finite-map integration formula to the pullback of $u_0 Du_1 \cdots Du_k \wedge \eta$; see [6, Proposition 15.4.3]. Locally uniform smooth approximation gives the same formula for continuous semipositive inputs. For the bounded tuple, split a compact base test into finitely many compact summands supported on ordinary opens W_a where simultaneous decreasing continuous regularizations exist. Properness of f makes the pulled-back test compact in $f^{-1}(W_a)$. Apply Theorem 1.7 on both spaces to pass each continuous-stage identity to its limit, and sum. A constant weight gives (7.3). □

Proposition 7.3 (Descent of the bounded class)**.** *Let $f : Y \to X$ be a finite étale surjection and u a given real function on X. If $u \circ f \in \mathcal{C}_b(Y)$, then $u \in \mathcal{C}_b(X) = \mathcal{R}_b(X)$. Membership of a given function in this class is also local for étale covers.*

Proof. Classical subharmonicity descends under the finite surjection by [10, Proposition 4.2(6)]. A finite surjection is proper and closed, so upper semicontinuity and local boundedness descend as follows. Since $\{u \geq a\} = f(\{u \circ f \geq a\})$, upper semicontinuity descends. Cover the finite fiber above x by finitely many open neighborhoods on which $u \circ f$ is bounded, let V be their union, and choose a common bound M. The ordinary open $X \setminus f(Y \setminus V)$ contains x, and its entire inverse image lies in V. Surjectivity gives $|u| \leq M$ there.

For an étale cover, choose a point above x and a local finite-étale representative on compact analytic neighborhoods [6, §11.6.3 and Lemma 11.6.5(a)]. Restrict the target to an ordinary open in its relative interior. The locally separated interior formula used in Lemma D.4 makes its inverse image ordinary open in the source. The positive-degree component supplies a finite-étale surjection near x. The first assertion and ordinary locality apply. Finally Theorem 1.5 gives the local regularization on X. □

Remark 7.4 (Descent data). Proposition 7.3 concerns a function already given on the base. Functions independently chosen on different sheets need not descend: constants 0 and 1 on two disjoint copies of a ball provide an example.

Appendix A. Curve Laplacians and Borel slicing

The slicing formula in this appendix expresses the first-order current of Proposition 2.3 as an integral of positive curve Laplacians. Its measure-theoretic form is needed when the test coefficient is only Borel. The measure identities in this section are identities on analytic spaces. In particular, a Borel coefficient is not replaced by a function on the image of a moment map. Put $D = \mathrm{d}'\mathrm{d}''$. On a smooth analytic curve we use the sign for which the Laplacian of a convex function on an oriented annulus has nonnegative second derivative. With weights $-\log\|e\|$, this is the sign for which a semipositive model metric has positive curvature of mass equal to the degree of its line bundle. Original unit-moment forms are used throughout. The integration theory of [6, §15.1 and §15.3] restricts on these presentations to the original one: both integrate their Lagerberg representatives with the canonical finite-degree calibration. Thus the use below of the more general skeleton formulation in [6] changes neither the test forms nor their integrals; compare the original calibration in [5, §3.5].

Lemma A.1 (Countability of the auxiliary fields)**.** *Let $K/\mathbb{Q}_p$ be finite. Let $\mathcal{G}_K$ consist of the fields obtained from K by finitely many operations*

$$E \longmapsto E(\widehat{Z_1, \dots, Z_s})_{\mathrm{Gauss}}, \qquad |Z_i| = 1.$$

Every $F \in \mathcal{G}_K$ has a countable dense subfield. Its valuation ring is a complete excellent discrete valuation ring of mixed characteristic, with the same value group as K. Every compact analytic domain admitting a finite affinoid cover in a good Hausdorff F-analytic space is metrizable. Locally in such a space, ordinary analytic opens admit countable compact exhaustions and countable families of original smooth scalar functions that are uniformly dense on each fixed compact support.

Proof. A $\mathbb{Q}$-span of a $\mathbb{Q}_p$-basis of K is countable and dense; the subfield it generates is still countable. If $E_0 \subset E$ is a countable dense subfield, then $E_0(Z_1, \dots, Z_s)$ is dense for the unit Gauss valuation in $E(Z_1, \dots, Z_s)$ and hence in its completion. The valuation ring after one step is the completion of $E^\circ[Z_1, \dots, Z_s]_{(\pi)}$. This is a discrete valuation ring with uniformizer π and residue field $\widetilde{E}(Z_1, \dots, Z_s)$. Its completion is excellent.

An affinoid algebra over a field with a countable dense subfield has a countable dense subset. Evaluation of the absolute values of these elements embeds its Berkovich spectrum in a countable product of compact intervals: uniform approximation determines the value of every other element. The spectrum is therefore compact metrizable. A finite affinoid cover gives the assertion for compact domains. An ordinary open in such a compact neighborhood is locally compact and second countable, and admits a countable compact exhaustion. Original smooth partitions of unity and uniform smooth density [6, Theorem 15.2.6 and Corollary 15.2.7] give the last assertion. For a fixed compact set one first chooses a larger compact neighborhood and a smooth cutoff equal to one on the smaller set, and then takes a countable dense subset of the resulting separable space of continuous functions. □

Lemma A.2 (Compatibility of curve Laplacians)**.** *Let E be a complete nontrivially valued field of characteristic zero, and let C be an ordinary open in the analytification of a smooth algebraic E-curve. A locally bounded subharmonic function b has a positive Radon Laplacian $\Delta_C b$, and*

$$\int_C a\, d(\Delta_C b) = \int_C b\, Da \qquad (a \in A_c^{0,0}(C)). \tag{A.1}$$

Both sides use the canonical calibration over E. The assertion does not require the residue field of E to be algebraic over a finite field.

Proof. The open C is strictly analytic. Thuillier's Theorem 3.4.12 gives its positive Radon Laplacian, and Theorem 3.4.2 gives, on a relatively compact larger open, a decreasing net b_ν of lisse subharmonic functions with limit b [14]. Here *lisse* means piecewise affine on a finite semistable skeleton after a finite extension; it does not mean original CLD smooth. The relation between this subharmonicity and the curve convention used here is [10, §3.3.1–§3.3.2].

For a lisse function, (A.1) follows from integration by parts on a common finite skeleton for the function and the original moment presentation of a. Along an annular edge the formula is the one-variable integration-by-parts formula; the endpoint terms are the weighted outgoing slopes. These are precisely the atoms of the lisse Laplacian. The edge weights retain the residue degrees. A finite extension used to obtain the skeleton can be removed by the calibrated base-extension integration formula [6, §15.3, Proposition 14.3.6]; both sides have the same normalization. This is also the lisse comparison underlying [15, Proposition 4.2].

On a compact neighborhood of $\operatorname{supp} a$, a tail of b_ν is bounded above, while b bounds it below. Integration of a decreasing net of continuous functions against a finite Radon measure commutes with its infimum. To see this, a continuous function strictly above the infimum eventually dominates the net on the compact set: the sets on which this domination holds form an increasing open cover. Compactness gives one index. Taking the infimum of the integrals of continuous majorants proves the assertion by Radon regularity. Apply this to the positive and negative parts of the fixed measure Da. The Laplacians converge as distributions by Thuillier's construction; their masses on a smaller compact set are bounded by testing against one nonnegative lisse cutoff. Uniform density then permits every continuous compact test, including a. Passing to the limit gives (A.1). □

Lemma A.3 (Borel coarea for a smooth relative form)**.** *Let $F \in \mathcal{G}_K$, let $n \geq 2$, let W be an ordinary algebraic analytic open of a smooth n-fold, and let $p : W \to (\mathbb{G}_m^{n-1})^{\mathrm{an}}$ be induced by a dominant algebraic morphism. In the specified torus coordinates $z_1, \dots, z_{n-1}$ put $t_i = -\log|z_i|$, and let*

$$\Omega = \omega(t) \bigwedge_{i=1}^{n-1} \mathrm{d}'t_i \wedge \mathrm{d}''t_i, \qquad 0 \leq \omega \in C_c^\infty(\mathbb{R}^{n-1}).$$

Thus Ω is presented by these coordinates and its measure is supported on their standard Gauss skeleton. Let $\beta \in A_c^{1,1}(W)$, and let h be a bounded Borel function on a neighborhood of its support. Writing $\nu_\Omega = \Omega^{\mathrm{Rad}}$ and $C_y = p^{-1}(y)$, one has

$$\int_W h\, \beta \wedge p^*\Omega = \int_{\mathrm{Sk}(\mathbb{G}_m^{n-1})} \left(\int_{C_y} h|_{C_y}\, \beta|_{C_y} \right) d\nu_\Omega(y). \tag{A.2}$$

The inner integral is measurable after completion of ν_Ω, and its absolute value is integrable. The fibers over this skeleton are smooth curves over $\mathscr{H}(y)$.

Proof. Every torus skeleton point has generic algebraic support. Generic smoothness in characteristic zero therefore makes C_y a smooth $\mathscr{H}(y)$-curve. Since the support of β is compact, the calculation is confined to finitely many compact moment presentations. The flat locus of p is dense; the hypotheses used in the smooth coarea theorem [6, Theorem 16.5.4] are thus satisfied.

For the Borel coefficient in (A.2), refine the moment presentations by the base coordinates and choose the convenient skeleton cells of [6, Lemma 14.1.5]. On each cell $D \subset W$, the refined

tropicalization $t_D : D \to \overline{D}$ is a homeomorphism onto a polyhedral cell. The Borel coefficient used on that cell is $h \circ t_D^{-1}$. Distinct cells having the same tropical image are kept separate, with their respective values of h. Using relative interiors, or a disjoint Borel subdivision, avoids double counting their boundaries.

The integration formula is now a finite sum of ordinary Lebesgue integrals on these calibrated cells. The base and fiber skeletons are compatible by [6, Lemma 14.2.10]. If a full-dimensional source cell D projects onto a full-dimensional base cell B, let d_D, d_B be their calibrated finite degrees. Its fiber calibration is d_D/d_B; this is the degree calculation in the proof of [6, Theorem 16.5.4]. Consequently ordinary Borel Fubini on $\overline{D} \to \overline{B}$ gives exactly the corresponding summand of (A.2), with coefficient d_D on the source and $d_B(d_D/d_B)$ after integration over the base. Cells of insufficient base or fiber dimension contribute zero, outside a finite union of lower-dimensional base cells. That union is ν_Ω-null. The same computation applied to absolute values gives absolute integrability, since h is bounded and the smooth form has compact support. This proves the formula and its measurability assertion. The construction evaluates h on D, not on the tropical image without a chosen inverse, and is compatible with further subdivision and common moment refinements. When using this formula on a prescribed compact source support, a coordinate cutoff ω may be chosen equal to one on a neighborhood of its compact image under $(-\log|z_1|, \dots, -\log|z_{n-1}|) \circ p$. The resulting form Ω on the base torus has compact support in a closed polyannulus, so the compactness requirement of the coarea theorem is retained. □

Theorem A.4 (The curve Laplacian kernel)**.** *In the setting of Lemma A.3, let v be locally bounded and classically plurisubharmonic. After restricting to a relatively compact neighborhood in W, there is a measurable family of positive Radon measures*

$$\mu_y = \Delta_{C_y}(v|_{C_y}) \quad \text{for } \nu_\Omega\text{-almost every } y$$

whose integral is a positive Radon measure $\mu_{v,p,\Omega}$ on the analytic domain W. It satisfies

$$\langle D[v], a\, p^*\Omega\rangle = \int_W a\, d\mu_{v,p,\Omega} = \int \left(\int_{C_y} a|_{C_y}\, d\mu_y \right) d\nu_\Omega(y) \quad (a \in A_c^{0,0}(W)). \tag{A.3}$$

For every bounded Borel h on the relevant compact support, the same measure identity gives

$$\int_W ha\, d\mu_{v,p,\Omega} = \int \left(\int_{C_y} (ha)|_{C_y}\, d\mu_y \right) d\nu_\Omega(y). \tag{A.4}$$

Scalar multiplication and relations between original test forms are preserved on common refinements.

Proof. The all-extension quantifier in classical plurisubharmonicity implies that $v|_{C_y}$ is subharmonic over $\mathscr{H}(y)$. These fields may be nondiscrete. Lemma A.2 gives their positive Laplacians. Since Ω has top degree on the base, it is closed, and distributional differentiation gives

$$L(a) := \langle D[v], a\, p^*\Omega\rangle = \int_W v\, Da \wedge p^*\Omega.$$

Apply Lemma A.3 with $h = v$ and $\beta = Da$, followed by (A.1) on each fiber. This proves that $y \mapsto \int a|_{C_y}\, d\mu_y$ is measurable modulo a null set and integrable, and that its integral is $L(a)$. In particular $L(a) \geq 0$ when $a \geq 0$.

Positivity of L gives an order-zero estimate. Fix a compact $Q \Subset W$ and an original smooth cutoff $\kappa \geq 0$ equal to one near Q. For every original smooth a supported in Q,

$$|L(a)| \leq \|a\|_\infty L(\kappa) < \infty. \tag{A.5}$$

The last value is finite because v is bounded on the compact support of $D\kappa$ and $D\kappa \wedge p^*\Omega$ is a smooth top-degree form of finite total variation. Positivity and (A.5), together with original smooth density, therefore give a positive Radon measure $\mu_{v,p,\Omega}$ by the Riesz representation theorem.

The preceding argument constructs a measure from its smooth pairings. To integrate arbitrary Borel functions fiberwise, we also need a measurable kernel defined outside one common null set. Put, on $Q' = \operatorname{supp}\kappa$, $\nu_y = \kappa|_{C_y}\mu_y$, viewed as a finite measure on the compact metric space Q' and extended by zero off the fiber. Choose a countable family of original smooth functions uniformly dense in $C(Q')$. Their products with κ have measurable integrals by the preceding coarea calculation. Together with the test κ, this involves only countably many exceptional null sets; remove one Borel null set containing their union and set $\nu_y = 0$ there. For the remaining parameters the mass $\nu_y(Q')$ is finite. Uniform approximation consequently proves measurability of $y \mapsto \int f\, d\nu_y$ for every $f \in C(Q')$.

For an open subset A of Q', continuous functions $0 \le f_j \uparrow \mathbf{1}_A$ show that $y \mapsto \nu_y(A)$ is measurable. The monotone-class theorem then gives this assertion for every Borel subset of Q', and gives measurable integration of every bounded Borel function on Q'. Moreover,

$$\int \nu_y(Q')\, d\nu_\Omega(y) = L(\kappa) < \infty.$$

Thus $\nu(A) = \int \nu_y(A)\, d\nu_\Omega(y)$ is a finite Borel measure on the compact metric space Q', hence a Radon measure. On original smooth tests it agrees with $\kappa\mu_{v,p,\Omega}$; uniform density identifies these measures. Taking a countable compact exhaustion and a countable collection of such cutoffs produces a single conull parameter set and compatible kernels. Equality of the measures on W proves (A.4).

Finally, if two presentations define the same original complementary form η, then for every original smooth scalar f their coefficient measures have integral $\langle D[v], f\eta\rangle$. They are equal by density and Radon uniqueness. The same argument applies to every finite relation among test forms and to multiplication by an original smooth scalar. These relations hold as measure identities before multiplication by h and remain true afterwards. This proves compatibility with common refinements and with Borel coefficients. □

Remark A.5. Only the complete-field curve theory is used over $\mathscr{H}(y)$ in Theorem A.4; the envelope theorem is used over $F \in \mathcal{G}_K$. The countability needed to select one conull set is that of a dense family of tests on W, not of the residue fields. The coordinate-presentation hypothesis on Ω is essential: an arbitrary original smooth form on a torus may have a refined skeleton. In the applications below, p is formed from the units in a decomposable complementary direction; a coordinate cutoff $\omega = 1$ near the projected test support gives exactly the displayed class of base forms. When $n = 1$, the base is a point and the statement is Lemma A.2. If p is not dominant, its top-degree base form pulls back to zero by tropical dimension, and the corresponding first-order pairing is zero.

Return to first-order positivity.

Appendix B. Bounded curve energy and a localized Green estimate

The local estimate below transfers differentiation from a continuous potential to a compactly supported bounded weight. Its total variation bound then controls the error when the continuous potential varies uniformly, even though the weight may be discontinuous. It is used in Proposition 2.4; the proper-curve identity supplies the initial case of Theorem 3.4.

We retain the curve convention of Lemma A.2. The field in this appendix is complete and nontrivially valued of characteristic zero. The curves used below are ordinary opens in smooth algebraic curves; in particular they are strictly analytic. The operator D on a lisse function is its weighted graph Laplacian. Original smooth functions on a common skeleton are piecewise smooth graph functions, and the same integration-by-parts formula applies to them.

Lemma B.1 (Bounded curve energy)**.** *Locally bounded subharmonic functions have locally finite Dirichlet energy. For a finite real linear combination b of such functions, the expression*

$$\Gamma_C(b) = \tfrac{1}{2}D[b^2] - bD[b]$$

is a nonnegative Radon measure. Its polarization is bilinear and satisfies Cauchy–Schwarz on compact supports. The product and chain rules are

$$D[ab] = aD[b] + bD[a] + 2\Gamma_C(a,b), \tag{B.1}$$

$$D[\Phi(b)] = \Phi'(b)D[b] + \Phi''(b)\Gamma_C(b) \tag{B.2}$$

whenever the functions are locally bounded and Φ is C^2 on a neighborhood of their relevant range.

On a smooth proper connected curve, if r, p are bounded subharmonic metrics on the same line, then their gap $g = r - p$ satisfies

$$E_C(g) = -\int gD[g] \geq 0. \tag{B.3}$$

The associated Green pairing is symmetric. If $E_C(g) = 0$, the function g is constant pointwise.

Proof. We recall how boundedness puts these statements within the complete-field curve theory. On a compact domain inside a larger curve open, Thuillier's harmonic interpolants give lisse subharmonic functions $b_S \downarrow b$, indexed by finite sets of nonrigid points containing the boundary [14, Theorem 3.4.2]. Their values on each such set are the values of the given function b. At an additional fixed inner boundary these values can be kept fixed by including its finitely many points in every S.

Take original smooth cutoffs $0 \leq \rho, \kappa \leq 1$, with $\kappa = 1$ near $\operatorname{supp}\rho$, on the larger domain. A tail of the interpolants is bounded by $M + \epsilon$ on these compact supports if $|b| \leq M$; this follows from the increasing open cover $\{b_S < M + \epsilon\}$. On a common finite skeleton, the graph product formula gives, with $x_S^2 = \int \rho^2 \Gamma_C(b_S)$,

$$x_S^2 \leq (M+\epsilon)^2 \|D\kappa\|_{\mathrm{TV}} + 2(M+\epsilon)x_S \left(\int \Gamma_C(\rho)\right)^{1/2}. \tag{B.4}$$

Indeed $\int_{\operatorname{supp}\rho} Db_S \leq (M+\epsilon)\|D\kappa\|_{\mathrm{TV}}$, by testing against κ, and integration by parts in $\int \rho^2\, d'b_S\, d''b_S$ gives the displayed mixed term. Thus the local Dirichlet energies are bounded.

Subtracting the fixed harmonic extension of the inner boundary values places the interpolants in the Dirichlet space with zero boundary values. The finite-skeleton harmonic interpolants are the orthogonal projections to the corresponding finite-dimensional Dirichlet subspaces. Their energies increase under refinement; the bound above puts their limit in the Hilbert completion. For nested point sets $S \subset S'$, orthogonality gives

$$\|b_{S'} - b_S\|_D^2 = \|b_{S'}\|_D^2 - \|b_S\|_D^2$$

after the fixed harmonic extension has been subtracted. The bounded increasing energies therefore make this net Cauchy. Its distributional limit is b, so its Dirichlet limit represents the same potential. This is the construction and projection argument in [14, §3.5, especially Lemma 3.5.3 and the discussion following it]. It applies simultaneously to a finite list of bounded subharmonic functions.

Products of bounded Dirichlet functions and C^2 compositions satisfy the graph product and chain rules. These rules pass from the interpolants to the Dirichlet limit: boundedness controls the multipliers, strong Dirichlet convergence controls the derivative terms, and the interpolants converge to the given functions at every point. Green-kernel symmetry [14, Remark 3.4.15] and the energy-integrability identity [14, Proposition 3.5.7 and its local variant, Proposition 3.5.10] identify the resulting pairings with integration of these bounded potentials against their Laplacians. Thus the measure in the statement is $\frac{1}{2}D[b^2] - bD[b]$, with the Borel product of the original representative, and not an independently chosen representative. The finite-skeleton rules give (B.1)–(B.2); positivity and Cauchy–Schwarz pass through the same Dirichlet limit.

For the proper assertion, choose a lisse semipositive reference metric on the line after replacing it by a positive tensor power if necessary. More generally a fixed difference of such reference metrics suffices. Its curvature is a finite combination of atoms at nonrigid points.

The gap of a bounded subharmonic metric from this reference is bounded. Its Green potential therefore has finite energy by [14, Proposition 3.5.7]. Polarization gives the symmetric Green pairing and (B.3), with the mass of each Chern measure equal to the degree of its line.

Finally, if $E_C(g) = 0$, Cauchy–Schwarz with the elementary Green function $g_{x,y}$ gives $g(x) = g(y)$ for all nonrigid x, y. In each local frame the subharmonic weights of r and of $p + g(x)$ therefore agree at all such points. They agree everywhere by [14, Lemma 3.4.1]. Thus the gap is constant at every point. □

Lemma B.2 (Localized variation and Green formula)**.** *Let b be locally bounded subharmonic on a smooth curve open, and let $a \in A_c^{0,0}$. Choose original smooth compact cutoffs $0 \le \rho, \kappa \le 1$ with $\rho = 1$ near $\operatorname{supp} a$ and $\kappa = 1$ near $\operatorname{supp} \rho$. Suppose $|b| \le M$ on a neighborhood of these supports. Set*

$$E(f) = \int d'f \wedge d''f, \qquad A_\kappa = \|D\kappa\|_{\mathrm{TV}}.$$

Then $D[ab]$ is a signed Radon measure and

$$\begin{aligned} \|D[ab]\|_{\mathrm{TV}} \le M\|a\|_\infty A_\kappa + M\|Da\|_{\mathrm{TV}} \\ + 2M\sqrt{E(a)}\,(\sqrt{A_\kappa} + 2\sqrt{E(\rho)}). \end{aligned} \tag{B.5}$$

For continuous subharmonic w,

$$\int ab\, D[w] = \int w\, D[ab]. \tag{B.6}$$

If the cutoffs are restrictions of fixed original source functions in Theorem A.4, the bound in (B.5) is integrable over the slicing base.

Proof. Use a decreasing lisse subharmonic regularization b_ν on a larger open. On the relevant compact set, take a tail with $|b_\nu| \le M + \epsilon$. The estimate (B.4) gives

$$x_\nu \le (M + \epsilon)(\sqrt{A_\kappa} + 2\sqrt{E(\rho)}), \qquad x_\nu^2 = \int \rho^2\, d'b_\nu \wedge d''b_\nu.$$

The local mass of Db_ν on $\operatorname{supp} a$ is at most $(M + \epsilon)A_\kappa$. Apply the graph product formula

$$D(ab_\nu) = aDb_\nu + b_\nu Da + d'a \wedge d''b_\nu + d'b_\nu \wedge d''a.$$

The first two terms have the first two bounds of (B.5), with $M + \epsilon$; Cauchy–Schwarz bounds the last two by $2\sqrt{E(a)}x_\nu$. The distributional limit is $D[ab]$, so the uniform variation bound makes it a measure with the same estimate. Let $\epsilon \downarrow 0$.

At fixed ν, approximate w by lisse subharmonic functions on a larger domain. Since w is continuous, the decreasing net converges uniformly on the compact support by the finite-cover form of Dini's argument. The finite-skeleton Green formula therefore passes to w. Now let ν decrease to its limit. The left side integrates ab_ν against the fixed positive measure $D[w]$. Decreasing continuous-net integration, applied after decomposing the sign of a, gives its limit ab. On the right, distributional convergence and the common total variation bound imply convergence on continuous compact tests: approximate such a test uniformly by original smooth functions and use the variation bound for the error. A cutoff equal to one on $\operatorname{supp} a$ permits applying this observation to w. This proves (B.6).

For the base integral, finite moment presentations and compact coefficient bounds give positive original smooth $(1,1)$-forms majorizing both signs of $D\kappa$ and Da. Their restrictions to the fibers majorize $A_\kappa(y)$ and $\|Da_y\|_{\mathrm{TV}}$. Their integrals over the base are finite by Lemma A.3. The same is true of $E(a_y)$ and $E(\rho_y)$, since these are fiber integrals of fixed positive smooth forms. Cauchy–Schwarz on the base bounds the products of square roots in (B.5). This proves integrability without a uniform degree assertion for the fibers. □

Corollary B.3. *If continuous subharmonic $w_j \to w$ uniformly on the relevant support, then*

$$\left| \int ab\,(D[w_j] - D[w]) \right| \le \|w_j - w\|_\infty \|D[ab]\|_{\mathrm{TV}}.$$

The same conclusion holds for finite real linear combinations of the bounded subharmonic weight b.

Proof. Subtract (B.6) for w_j and w, and use the variation norm. Then use linearity. □

Proposition B.4 (The curve contact theorem)**.** *Let C/F be smooth projective geometrically integral, and L ample. Let p be a continuous model-semipositive metric on L and r a bounded classical metric on L. If*

$$g = r - p \ge 0, \qquad \int g\, c_1(L,p) = 0,$$

then $g = 0$ at every point.

Proof. Both local weights are subharmonic. By Lemma B.1,

$$0 \le E_C(g) = -\int g D[g] = \int g\, c_1(L,p) - \int g\, c_1(L,r) \le 0.$$

Hence g is constant. The measure $c_1(L,p)$ has positive mass $\deg_F L$, so the constant is zero. □

Return to smooth-background energy.
Return to projective contact and regularization.

Appendix C. Envelopes on a normal model

We prove continuity of the ample envelopes used in Theorem 3.4. Weighted base ideals encode the semipositive minorants of a model obstacle. We replace them by test ideals to obtain global generation after one fixed ample twist. The required comparison comes from subadditivity with a vertical correction independent of the ideals and their exponents. Dividing logarithmic norms by degree makes the resulting error tend to zero.

Throughout this appendix R is a complete discrete valuation ring of mixed characteristic, $F = \mathrm{Frac}(R)$ has characteristic zero, π is a uniformizer, and $\lambda = -\log|\pi| > 0$. No finiteness or perfection assumption is imposed on the residue field. All test ideals are those of [1, Definition 8.37]; after completion we use the stable complete-local ideals of [1, Definition 8.13].

C.1. **A correction independent of ideals and exponents.** Let $(S, \mathfrak{m})$ be a complete Noetherian normal local domain of mixed characteristic $(0,p)$ and dimension d, and let $\Delta \ge 0$ be an integral Weil divisor such that $K_S + \Delta$ is Cartier. Choose a local trivialization of this Cartier divisor and a compatible rational canonical section. The canonical module is then represented by

$$W = \mathcal{O}_S(-\Delta) \subset S, \qquad K_S = -\Delta.$$

Choose $c \ne 0$ with $cS \subset W$. Put

$$E = H^d_{\mathfrak{m}}(W), \quad F_S = H^d_{\mathfrak{m}}(S), \quad \alpha : E \longrightarrow F_S, \quad \beta : F_S \longrightarrow E,$$

where α is induced by inclusion and β by multiplication by c. Both composites are multiplication by c. Top local duality identifies the Matlis dual of E with $\mathrm{Hom}_S(W, W) = S$, so E is the injective hull of the residue field. This does not require S to be Cohen–Macaulay, and α need not be injective.

The Noether different will supply the local correction factor. Its role is to factor multiplication on a big Cohen–Macaulay algebra through a flat module, where annihilators of finitely generated ideals commute with base change.

Lemma C.1 (A flat factorization)**.** *Let $A \subset S$ be a finite extension of complete local domains, with A regular of dimension d. Write*

$$\mathfrak{N}_{S/A} = \mu(\mathrm{Ann}_{S\otimes_A S}(\ker \mu)), \qquad \mu : S \otimes_A S \longrightarrow S.$$

If B is a balanced big Cohen–Macaulay S^+-algebra, multiplication by every $z \in \mathfrak{N}_{S/A}$ on B factors S-linearly as

$$B \xrightarrow{j_z} S \otimes_A B \xrightarrow{\mu_B} B, \qquad \mu_B j_z = z\,\mathrm{id}_B,$$

and $S \otimes_A B$ is S-flat. For every finitely generated ideal $I \subset S$, it follows that

$$z \operatorname{Ann}_{H^d_{\mathfrak{m}}(B)}(I) \subset \mathrm{im}\big((S \otimes_A B) \otimes_S \operatorname{Ann}_{F_S}(I) \longrightarrow H^d_{\mathfrak{m}}(B)\big). \tag{C.1}$$

Proof. Every system of parameters of A is a system of parameters of S. Thus B is balanced big Cohen–Macaulay over A, and is A-flat by [11, Setup 2.1 and Proposition 4.9]. Choose $\gamma = \sum u_i \otimes v_i$ annihilating $\ker \mu$ and satisfying $\mu(\gamma) = z$. Set $j_z(b) = \sum u_i \otimes v_i b$. The relations $(s \otimes 1 - 1 \otimes s)\gamma = 0$ prove S-linearity; multiplication proves the asserted composite. Flat base change gives the flatness of the middle module $C = S \otimes_A B$. A top Čech cokernel gives $H^d_{\mathfrak{m}}(C) = C \otimes_S F_S$. Since I is finitely generated and C is flat, $\operatorname{Ann}_{C\otimes F_S}(I) = C \otimes \operatorname{Ann}_{F_S}(I)$. Apply local cohomology to the factorization and then apply μ_B. □

Proposition C.2 (Complete-local corrected subadditivity)**.** *With the preceding notation, for nonzero ideals $\mathfrak{a}, \mathfrak{b}$ and rational $s, t \geq 0$,*

$$c\mathfrak{N}_{S/A}\, \tau(S, \Delta, \mathfrak{a}^s\mathfrak{b}^t) \subset \tau(S, \Delta, \mathfrak{a}^s)\tau(S, \Delta, \mathfrak{b}^t). \tag{C.2}$$

Proof. Choose a single sufficiently large perfectoid balanced BCM S^+-algebra B for the three stable test ideals. Such a choice, and the mixed element-wise description below, are supplied by [1, Definitions 8.2 and 8.13, Theorem 8.11]. For an ideal datum $\mathfrak{a}^s$, let $\mathcal{R}_{\mathfrak{a},s}$ be the allowed root multipliers: powers f^q in B, where $0 \neq f \in \mathfrak{a}^m$ and $q \geq s/m$ is rational. Put $T_{\mathfrak{a}} = \tau(S, \Delta, \mathfrak{a}^s)$. Then

$$\operatorname{Ann}_E(T_{\mathfrak{a}}) = \bigcap_{g \in \mathcal{R}_{\mathfrak{a},s}} \ker\big(E \xrightarrow{\alpha} F_S \longrightarrow H^d_{\mathfrak{m}}(B) \xrightarrow{g} H^d_{\mathfrak{m}}(B)\big). \tag{C.3}$$

Indeed, the root maps defining the parameter test modules start from F_S, whose Matlis dual is W. The dual of α is the chosen inclusion $W \hookrightarrow S = \mathrm{Hom}_S(W, W)$. Precomposition therefore gives precisely the pair ideals as ideals of S. The equality follows from Matlis duality. The pair factor is one because $K_S + \Delta = 0$ in the chosen trivialization.

Here stability is required simultaneously for the three ideal data, rather than for every ideal on S at once. The existence and comparison theorem in [1, Theorem 8.11] permits replacing their three sufficiently large algebras by a common one. The mixed version of [1, Definition 8.2] is a sum of the element-wise parameter test modules for products of root multipliers. Matlis duality changes that sum into the intersection of kernels in (C.3), and gives the corresponding intersection over pairs (g, h) for the mixed datum.

Take $\eta \in \operatorname{Ann}_E(T_{\mathfrak{a}}T_{\mathfrak{b}})$ and $h \in \mathcal{R}_{\mathfrak{b},t}$. For every $a \in T_{\mathfrak{a}}$, the element $a\eta$ is annihilated by $T_{\mathfrak{b}}$; its image under $h\alpha$ is therefore zero by (C.3) for $\mathfrak{b}^t$. Hence $h\alpha(\eta)$ is annihilated by $T_{\mathfrak{a}}$. For $z \in \mathfrak{N}_{S/A}$, (C.1) expresses

$$zh\alpha(\eta) = \sum_i b_i\zeta_i, \qquad \zeta_i \in \operatorname{Ann}_{F_S}(T_{\mathfrak{a}}), \quad b_i \in B.$$

Now $\beta(\zeta_i) \in \operatorname{Ann}_E(T_{\mathfrak{a}})$. Equation (C.3) gives $gc\zeta_i = g\alpha\beta(\zeta_i) = 0$ for every $g \in \mathcal{R}_{\mathfrak{a},s}$. Hence $czgh\alpha(\eta) = 0$. The mixed element-wise definition tests exactly the products gh. Thus

$$cz \operatorname{Ann}_E(T_{\mathfrak{a}}T_{\mathfrak{b}}) \subset \operatorname{Ann}_E \tau(S, \Delta, \mathfrak{a}^s\mathfrak{b}^t).$$

Taking annihilators yields (C.2). No factorization of an arbitrary root of an element of $(\mathfrak{a}\mathfrak{b})^m$ is used. □

To globalize the correction, we need finitely many local differents whose sum is the unit ideal on the generic fiber. The next lemma obtains them from finite projections that are étale on a covering of that fiber. Their common failure is then supported vertically and is killed by one power of π.

Lemma C.3 (Finite projections and a vertical different)**.** *Let $\mathcal{X}$ be normal, integral, projective and flat over R, with smooth generic fiber X of dimension n. There are finitely many finite morphisms $f_j : \mathcal{X} \to \mathbb{P}^n_R$ whose generic étale loci cover X. The sum $\mathfrak{J}$ of their coherent Noether-different ideals contains $\pi^{C_0}\mathcal{O}_\mathcal{X}$ for some C_0.*

Proof. Choose a relatively very ample line $\mathcal{H}$. On the special fiber, choose $n+1$ homogeneous parameters without a common zero. This works over an arbitrary residue field: at each stage, Serre interpolation provides a high-degree section nonzero at a chosen closed point of each component of the preceding zero locus. The dimension drops at each step. Taking powers makes the degrees equal; taking further powers permits lifting the sections to $\mathcal{X}$ by Serre vanishing. Every tuple congruent to these lifts modulo π has no base point on $\mathcal{X}$: a nonempty proper closed base locus would meet the special fiber. It defines a finite morphism to $\mathbb{P}^n_R$, because the pullback of $\mathcal{O}(1)$ is relatively ample and so no fiber can have positive dimension.

Fix a closed point $x \in X$. Increase the degree until sections surject onto its first infinitesimal neighborhood. Since $\kappa(x)/F$ is separable, one can prescribe one nonzero section and n ratios with independent differentials at x. Being étale at x is therefore a nonempty Zariski-open condition on the tuple; the determinant condition can be expressed over F by its field norm. The congruence coset of lifts modulo π is Zariski dense in the F-vector space of tuples, since F is infinite and the section module is a full R-lattice. Choose a tuple in this open set. The resulting generic étale loci cover every closed point of X, hence all of the Jacobson scheme X. Quasi-compactness selects finitely many morphisms.

On affine target charts their Noether-different ideals glue, since the diagonal ideal is finitely generated and annihilators commute with localization. On the smooth generic fiber the different is a unit at every étale point; see [12, §49.12]. Thus $\mathfrak{J}|_X = \mathcal{O}_X$. The coherent quotient $\mathcal{O}_\mathcal{X}/\mathfrak{J}$ is supported on the special fiber. A finite affine cover gives one exponent C_0. □

Theorem C.4 (Global fixed correction)**.** *Let $\mathcal{X}/R$ be as in Lemma C.3. Fix an effective integral vertical Weil divisor Δ_0 such that $K_\mathcal{X} + \Delta_0$ is Cartier. There is an integer $C \geq 0$, independent of the nonzero coherent ideals $\mathfrak{a}, \mathfrak{b}$ and rational exponents $s, t \geq 0$, such that*

$$\pi^C \tau(\mathcal{X}, \Delta_0, \mathfrak{a}^s\mathfrak{b}^t) \subset \tau(\mathcal{X}, \Delta_0, \mathfrak{a}^s)\tau(\mathcal{X}, \Delta_0, \mathfrak{b}^t). \tag{C.4}$$

Proof. Choose T globally with $\pi^T\mathcal{O}_\mathcal{X} \subset \mathcal{O}_\mathcal{X}(-\Delta_0)$. It exists because the latter coherent ideal is the unit ideal on X. It gives $c = \pi^T$ in every local canonical embedding above.

Fix a closed point y of the special fiber and a finite projection from Lemma C.3. Complete the regular local ring of $\mathbb{P}^n_R$ at the image of y. Its finite algebra splits into complete local factors; the factor at y is $S = \widehat{\mathcal{O}}_{\mathcal{X},y}$. It is a normal domain by excellence. Both rings have dimension $n+1$, so the map from the complete regular domain A to S is injective. An original Noether-different tensor witness base changes to the finite completed algebra and projects in both tensor factors to a witness for S/A. Thus the completed image of the global different is contained in $\mathfrak{N}_{S/A}$.

Apply Proposition C.2. Corollary 8.40 of [1] identifies the completed global test ideals with the complete-local ones, and faithful flatness descends the containment. Sum over the finitely many projections and use $\pi^{C_0}\mathcal{O} \subset \mathfrak{J}$. This gives (C.4) with $C = T + C_0$ at every closed special-fiber point. A nonzero coherent failure quotient on the proper R-scheme would have such a point in its support. The containment therefore holds everywhere, with a constant chosen before the ideals and exponents. □

Remark C.5. The uncorrected subadditivity in [1, Theorem 8.41(h)] applies on a regular scheme without a boundary in its two factors. In Theorem C.4, the scheme is normal and the boundary Δ_0 occurs in both factors. The factor π^C accounts for these differences. Its independence of the ideals and the exponents will give an error of order m^{-1} in the envelope construction. Fang–Gubler–Künnemann prove continuity for model obstacles on regular models carrying an ample extension, and for arbitrary continuous obstacles under resolution of projective models [8, Theorem 1.1]. The proof below instead uses Theorem C.4 on one normal model and retains a uniform error term throughout the construction.

C.2. **Weighted ideals and removal of the auxiliary line.**

Theorem C.6 (Continuity and orthogonality of ample envelopes)**.** *Let X/F be smooth projective geometrically integral and L ample. For every continuous metric ρ on L, its pointwise envelope*

$$P_L\rho = \sup\{q : q \text{ is continuous model-semipositive on } L, q \le \rho\}$$

is continuous and model-semipositive on the same identified line. It is uniformly approximable by nef rational model metrics over F, and

$$\int_{X^{\mathrm{an}}} (\rho - P_L\rho) c_1(L, P_L\rho)^n = 0. \tag{C.5}$$

Proof. Take a normal projective flat model $\mathcal{X}$ carrying an extension $\mathcal{L}$ of L. Replace it by the normalization of the blowup of its rank-one canonical fractional sheaf. This modification is an isomorphism on the smooth generic fiber and gives a Cartier extension G of its canonical line. Compatible rational sections make $G - K_{\mathcal{X}}$ vertical. For sufficiently large N_0,

$$\Delta_0 = G + N_0 \operatorname{div}(\pi) - K_{\mathcal{X}} \ge 0, \qquad K_{\mathcal{X}} + \Delta_0 \text{ Cartier}.$$

First suppose that the obstacle is an integral model metric r; rational denominators will be cleared and divided out afterwards. Let θ be the metric of $\mathcal{L}$. On a dominating normal model $f : \mathcal{W} \to \mathcal{X}$, a constant translation permits writing

$$r = \theta + u, \qquad u = \log|\mathcal{E}|, \qquad (\pi^N)\mathcal{O}_{\mathcal{W}} \subset \mathcal{E} \subset \mathcal{O}_{\mathcal{W}},$$

with $\mathcal{E}$ an invertible vertical ideal. The section lattices below encode the condition that a Fubini–Study weight lie below r. We form their base ideals $\mathfrak{c}_l$ and the associated asymptotic test ideals β_m:

$$\begin{aligned} V_l &= H^0(\mathcal{W}, (f^*\mathcal{L} \otimes \mathcal{E})^l), \\ \mathfrak{c}_l &= \operatorname{im}(V_l \otimes \mathcal{L}^{-l} \longrightarrow \mathcal{O}_{\mathcal{X}}), \\ \beta_m &= \sum_{l \gg 0} \tau(\mathcal{X}, \Delta_0, \mathfrak{c}_l^{m/l}), \quad \psi_m = m^{-1}\log|\beta_m|, \quad P = P_L r - \theta. \end{aligned} \tag{C.6}$$

Only sufficiently large degrees are used, so these ideals are vertical. The sum is directed under divisibility, since $\mathfrak{c}_l^q \subset \mathfrak{c}_{lq}$ and test ideals commute with powers. It stabilizes by Noetherianity.

Normality identifies V_l with the unit lattice for the sup norm of lr. Pure-FS density on an ample line [3, Corollary 7.6 and Theorem 7.8] gives

$$P = \sup_{l \gg 0} l^{-1}\log|\mathfrak{c}_l|, \qquad (mq)^{-1}\log|\mathfrak{c}_{mq}| \longrightarrow P \quad (q \longrightarrow \infty) \tag{C.7}$$

for fixed m. For the latter limit, products of unit sections give superadditivity of the degree-weighted functions. Divide large degrees by any fixed sufficiently large l, leaving remainders in a finite interval of globally generated degrees. Their weights are bounded on the compact analytic space, so their contribution divided by the total degree tends to zero. Taking the supremum over l proves the stated pointwise limit.

Let $\tau_0 = \tau(\mathcal{X}, \Delta_0)$. It contains a power of π, and Cartier translation and monotonicity give $\mathfrak{c}_{mq}\tau_0 \subset \beta_{mq}$. Iterating (C.4) and using repeated-factor compatibility [1, Theorem 8.41(c)], choose a degree computing β_{mq} to get

$$\pi^{C(q-1)}\beta_{mq} \subset \beta_m^q. \tag{C.8}$$

Indeed, (C.8) gives

$$\frac{1}{mq}\log|\beta_{mq}| - \frac{C(q-1)\lambda}{mq} \le \psi_m.$$

Use $\mathfrak{c}_{mq}\tau_0 \subset \beta_{mq}$ and (C.7), then let $q \to \infty$. The contribution of the fixed ideal τ_0 tends to zero, leaving $P - C\lambda/m \le \psi_m$.

There is also a fixed upper bound

$$\psi_m \le u + B\lambda/m. \tag{C.9}$$

Indeed, put $\Delta_{\mathcal{W}} = f^*(K_{\mathcal{X}} + \Delta_0) - K_{\mathcal{W}}$. For a noneffective $\Delta_{\mathcal{W}}$, choose an effective vertical Cartier divisor A with $\Delta_{\mathcal{W}} + A \ge 0$ and interpret its test ideal as $\mathcal{O}(A)\tau(\mathcal{W}, \Delta_{\mathcal{W}} + A)$. Cartier translation makes this independent of A. Birational inclusion, monotonicity, powers and Cartier translation [1, Proposition 7.16(b), Theorem 8.41(b),(c),(j)] imply

$$\tau(\mathcal{X}, \Delta_0, \mathfrak{c}_l^{m/l})\mathcal{O}_{\mathcal{W}} \subset \mathcal{E}^m\tau(\mathcal{W}, \Delta_{\mathcal{W}}).$$

Indeed, the pushforward inclusion of [1, Theorem 8.41(j)] gives the first pullback containment by adjunction. The further containment follows from $\mathfrak{c}_l\mathcal{O}_{\mathcal{W}} \subset \mathcal{E}^l$ and Cartier translation for the invertible ideal $\mathcal{E}$. The last test ideal is a fixed coherent fractional ideal, equal to $\mathcal{O}_X$ on the generic fiber. One power π^B clears its vertical poles, proving (C.9).

The bounds on ψ_m compare functions, but do not yet give semipositive metrics. For that purpose we show that one fixed ample twist makes every $\beta_m\mathcal{L}^m$ globally generated. Fix l and choose R-module generators $\sigma_1, \dots, \sigma_a$ of V_l, regarded as sections of $\mathcal{L}^l$ on $\mathcal{X}$. Discard any zero generator and let $D_b = \operatorname{div}(\sigma_b)$, an effective Cartier divisor linearly equivalent to $l\mathcal{L}$. Locally trivializing $\mathcal{L}^l$ expresses $\mathfrak{c}_l$ as the sum of the principal ideals defined by these sections. Repeated application of the summation formula [1, Theorem 8.41(g)], together with Cartier translation, gives

$$\tau(\mathcal{X}, \Delta_0, \mathfrak{c}_l^{m/l}) = \sum_{\substack{t_b \in \mathbb{Q}_{\ge 0} \\ \sum_b t_b = m/l}} \tau\left(\mathcal{X}, \Delta_0 + \sum_b t_b D_b\right).$$

The local expressions agree on overlaps because they use the Cartier divisors of global sections. Each divisor $D = \sum_b t_b D_b$ is effective and satisfies $D \sim_{\mathbb{Q}} m\mathcal{L}$. By Noetherianity, finitely many terms suffice for this sum and for the sum defining β_m; compare [1, Remark 8.42].

Choose a generated ample Cartier divisor H and write $G_0 = K_{\mathcal{X}} + \Delta_0$. This is Cartier by construction. For each such D, apply [1, Corollary 7.24] with $M = m\mathcal{L} + G_0 + H$. Its required difference is

$$M - K_{\mathcal{X}} - (\Delta_0 + D) \sim_{\mathbb{Q}} H,$$

which is big and nef. Since the special fiber has dimension n, the sheaf

$$\tau(\mathcal{X}, \Delta_0 + D) \otimes \mathcal{L}^m \otimes \mathcal{O}(G_0 + (n+1)H)$$

is generated. Take a generated ample model line $\mathcal{A}$ and choose m_0, independently of m, l, D, so that $\mathcal{A}^{m_0} \otimes \mathcal{O}(-G_0 - (n+1)H)$ is generated. Tensoring with this fixed generated line and then taking the sum proves that $\beta_m\mathcal{L}^m\mathcal{A}^{m_0}$ is generated for every m.

Let $A = \mathcal{A}|_X$ with its model metric q_A. Generation gives a pure-FS metric t_m on $mL + m_0A$ with exact weight $m\theta + m\psi_m + m_0q_A$. This metric is on $mL + m_0A$; to compare it with the envelope, we must return to L. Choose r_0 so that $r_0L - A$ is generated, and a pure-FS metric ν on that line. Adding $m_0\nu$ cancels the auxiliary A and produces a metric on $(m + r_0m_0)L$. Set

$$M_m = m + r_0m_0, \qquad h_m = (t_m + m_0\nu)/M_m, \qquad d = q_A + \nu - r_0\theta, \quad D_0 = \|d\|_\infty.$$

Every h_m is a nef rational model metric on L. Since $-N\lambda \le u \le 0$ and $P \le u$, the lower estimate and (C.9) give, with a constant C_1 independent of m,

$$h_m - P_L r \ge -\frac{C\lambda + m_0 D_0}{M_m}, \qquad h_m - r \le \frac{C_1}{M_m}.$$

Shift h_m down by its upper error C_1/M_m. The result is a continuous semipositive minorant of r, hence lies below $P_L r$ by the definition of the envelope. Combining this with the lower estimate gives

$$-\frac{C\lambda + m_0 D_0}{M_m} \le h_m - P_L r \le \frac{C_1}{M_m}. \tag{C.10}$$

This proves uniform convergence for a model obstacle on the prescribed line. Real constant translations are allowed in the uniform closure of rational model metrics; rational downward approximations give the same pointwise envelope. Undo the preparatory translation.

For a rational model obstacle clear its tensor denominator, use $P_{dL}(dr) = dP_L r$, and divide back. For a continuous obstacle, uniform model density and the estimate $\|P_L r - P_L r'\|_\infty \le \|r - r'\|_\infty$ give continuity and same-line semipositive approximation by a diagonal. Finally [4, Theorem 6.7], applicable because X is geometrically reduced, gives (C.5) after this continuity has been established. □

Remark C.7. Every field in $\mathcal{G}_K$ satisfies the hypotheses of this appendix by Lemma A.1. The proof applies to arbitrary projective varieties, lines and obstacles defined over that field; it does not require their descent to K. This is the field scope needed for the induction through generic hyperplanes.

Return to projective contact and regularization.

Appendix D. Algebraic coordinates and a common local domain

The projective approximation theorem must be applied to local potentials without changing them near the test support. The collar in Lemma D.2 achieves this for an entire decreasing family at once: its first terms and limits determine a common bound, and that bound determines one barrier and one local domain. The last two lemmas transfer these constructions to the analytic spaces of Convention 1.1, including their integration normalizations.

In the first three lemmas, $F \in \mathcal{G}_K$, π is the chosen uniformizer, and $\lambda = -\log|\pi|$. All constructions take place over F on the indicated ordinary neighborhoods.

Lemma D.1 (Exact replacement of analytic units)**.** *Let Y/F be smooth integral affine, let $O \subset Y^{\mathrm{an}}$ be ordinary open, and let $f_1, \dots, f_a$ be analytic units on O. Every $x \in O$ has a compact affinoid neighborhood $C \subset O$ and rational functions $g_1, \dots, g_a \in F(Y)$ such that*

$$\|f_i - g_i\|_C < \min_C |f_i|, \qquad |g_i(y)| = |f_i(y)| \quad (y \in C). \tag{D.1}$$

The g_i are units on one algebraic open containing C. Consequently the logarithms and their original smooth differentials agree on the ordinary interior of C. A compactly supported smooth test can be divided into finitely many tests on such neighborhoods, with one simultaneous replacement for each finite moment presentation.

Proof. Embed Y in affine space with coordinates $T_1, \dots, T_N$. Polynomial evaluations generate the topology of Y^{an}. Since $|F^\times|^{\mathbb{Q}}$ is dense in $\mathbb{R}_{>0}$, there are finitely many polynomial inequalities, with strict slack at x, whose closed solution set C contains an ordinary neighborhood of x and lies in O. Include upper bounds for every T_j. Thus C is compact, and it has a presentation

$$|T_j| \le R_j, \qquad |P_i| \le S_i, \qquad |Q_l| \ge r_l > 0, \qquad R_j, S_i, r_l \in |F^\times|^{\mathbb{Q}}.$$

It is obtained from the closed polydisc by imposing the equations of Y, followed by Weierstrass and Laurent localizations. Its affinoid algebra is a quotient of the generalized Tate algebra with coordinates T_j, U_i, V_l and relations

$$U_i = P_i(T), \qquad Q_l(T)V_l = 1.$$

The radii for U_i and V_l are S_i and r_l^{-1}. Polynomials in these coordinates are dense in that Tate algebra, and hence their images are uniformly dense in $\mathcal{O}(C)$. Those images belong to $F[Y][Q_1^{-1},\dots,Q_b^{-1}]$. Compactness and invertibility give $\min_C |f_i| > 0$. Approximate the finitely many $f_i|_C$ by elements of the displayed localization with errors smaller than these minima. The ultrametric inequality gives the pointwise equalities in (D.1). Their numerators and denominators do not vanish anywhere on C; removing their algebraic zero loci gives one algebraic open on which all g_i are units. The logarithms are equal as smooth functions on $\operatorname{Int} C$, so applying $\mathrm{d}', \mathrm{d}''$ preserves the equality. For the last assertion, cover the compact test support by finitely many such interiors and use a smooth partition subordinate to this finite cover. Apply the construction to the whole finite unit list on each summand. □

Lemma D.2 (A common collar for decreasing families)**.** *Let Y/F be smooth projective and let $O \subset Y^{\mathrm{an}}$ be ordinary open. For finitely many indices i, let $u_{i,j} \downarrow u_i$ be classical plurisubharmonic functions whose first terms and limits are locally bounded. Near every $x \in O$, there are one ordinary neighborhood V with compact closure in O, an ample line bundle L on Y, an algebraic frame e near $\overline{V}$, a constant $c \in \lambda\mathbb{Q}$, and bounded classical metrics $R_{i,j} \downarrow R_i$ on this same L such that*

$$\mathrm{wt}_e(R_{i,j}) = u_{i,j} + c, \qquad \mathrm{wt}_e(R_i) = u_i + c \quad \text{on } V. \tag{D.2}$$

All geometric choices are independent of i, j.

Proof. Choose a projective embedding by a sufficiently ample line A and an affine coordinate chart containing x. As in the proof of Lemma D.1, choose nested polynomial norm boxes in this chart,

$$x \in \operatorname{Int} C_1, \qquad C_1 \subset \operatorname{Int} C_{\mathrm{mid}}, \qquad C_{\mathrm{mid}} \subset \operatorname{Int} C_2, \qquad C_2 \subset O,$$

using the same finite polynomial list and strictly nested thresholds. All lower-bound polynomials are nonzero on C_2. Choose additional thresholds strictly between those for C_1 and C_{mid}. Clear rational powers in the thresholds to obtain a function

$$b = \max\left\{0, \log|P_a^{N_a}/c_a|, \log|d_l/Q_l^{M_l}|\right\}, \qquad c_a, d_l \in F^\times. \tag{D.3}$$

The barrier must vanish on the inner box, where we want to retain the given potentials, and be uniformly positive on the outer collar, where the extension must agree with a global metric. The function b is finite continuous on C_2, zero on C_1, and satisfies

$$b \geq \delta > 0 \quad \text{on } C_2 \setminus \operatorname{Int} C_{\mathrm{mid}}$$

for one δ. Indeed, outside the strict intermediate box at least one defining bound fails; the ratio between that threshold and its chosen inner threshold is strictly larger than one. The minimum of the finitely many positive logarithmic gaps gives δ.

Clear the denominators in (D.3) and homogenize. After increasing the common degree by a power of the affine-chart denominator, the branches become ratios s_a/s_0 of sections of A^{m_0}, where s_0 is nowhere zero on C_2; include the branch $s_0/s_0 = 1$. This denominator can be chosen as a product of the homogenized Q_l and the affine-chart denominator, so its nonvanishing on C_2 is preserved. Choose m_0 large enough that A^{m_0} is generated. Append small F-scalar multiples of a global generating tuple. Since C_2 is compact and s_0 has no zero there, the appended ratios have absolute value less than one on C_2. Their ratios stay below the branch 1 on C_2, so they do not change b there. They ensure that the enlarged tuple generates the line everywhere. Its pure Fubini–Study metric q has weight b in the frame s_0 on C_2.

The inequalities $u_i \leq u_{i,j} \leq u_{i,1}$ and compactness give one two-sided bound for all the functions on C_2. Choose $c \in \lambda\mathbb{Q}$ and M with $0 < u_{i,j} + c \leq M$ and $0 < u_i + c \leq M$ there. Choose an integer $m > 0$ with $m\delta > M + 1$, and put $L = A^{mm_0}$, $e = s_0^m$. On $\operatorname{Int} C_2$, give $R_{i,j}$ the e-weight

$$\max\{u_{i,j} + c, mb\}.$$

On $Y^{\mathrm{an}} \setminus C_{\mathrm{mid}}$, set $R_{i,j} = mq$. These are an ordinary open cover, and their prescriptions agree on the overlap because $mb > M$ there. The maximum and locality properties of classical psh

functions [10, Proposition 4.2] prove classicality. The metric difference $R_{i,j} - mq$ lies in $[0, M]$; thus it is bounded globally. Use the same prescription with u_i to define R_i. For a decreasing sequence of real numbers, the maximum with a fixed number commutes with its limit. It follows that $R_{i,j} \downarrow R_i$ at every point. Finally take $V = \operatorname{Int} C_1$. Here $b = 0$ and the first branch is positive, proving (D.2). The closure of V lies in $C_1 \subset O$. □

Lemma D.3 (One inactive floor)**.** *In Lemma D.2, suppose that $P_{i,j,m} \downarrow R_{i,j}$ are pure Fubini–Study metrics on L. Their weights on the same V, and all their nonnegative translates in $\lambda\mathbb{Q}$, have finite tropically convex rational PL presentations in algebraic coordinates. The algebraic chart and V are independent of i, j, m.*

Proof. After clearing a tensor denominator, each weight in the fixed frame has the form

$$P = \frac{1}{N} \max_a \log |s_a / e^{\otimes N}|,$$

where the finitely many s_a generate NL. Rational constants in $\lambda\mathbb{Q}$ can be absorbed by raising to one common tensor power and multiplying sections by powers of π. All these weights lie above their limits as $m \to \infty$, and those limits lie above the finitely many bounded functions $u_i + c$ on V. Choose a single $B \in \lambda\mathbb{Q}$ strictly below all of them. This floor will keep the tropical expression finite at zeros of individual sections, while remaining inactive in every approximating weight. Then $P = \max\{B, P\}$ on V, for every i, j, m.

Write $B = a\lambda/N_0 = (1/N_0) \log |\pi^{-a}|$. Appending the nonzero constant regular function π^{-a} to the algebraic moment coordinates represents the floor by a rational linear expression with zero constant term. In the valuation coordinates $t_f = -\log |f|$, the resulting function is a maximum of rational linear functions $-t_f/N$. The floor remains finite even when some section coordinates vanish. Near such a point, those coordinates have value $+\infty$ and their branches are inactive; deleting them leaves a finite rational PL expression on a neighborhood of that point. Restrict the ambient extended tropical domain to the locus where the appended constant coordinate is finite. The finitely many possible subsets of inactive coordinates give the finite PL cover in Wu's definition. The expression is continuous on this domain and convex on its finite-coordinate part, hence tropically convex by [16, Lemma 3.4(1), Definitions 3.1 and 3.7]. The appended constant coordinate is algebraic; if an affine embedding is desired, include generators of the fixed affine coordinate ring as additional coordinates which the displayed function ignores.

A nonnegative translate still lies above B, and a translate in $\lambda\mathbb{Q}$ is handled by the same section-power operation. Thus the floor and the underlying domain are independent of the indices and of the translate. Subtracting the fixed c in (D.2) only replaces the floor by $B - c$. This construction does not require setting $\lambda = 1$. □

Lemma D.4 (Ordinary algebraic neighborhoods)**.** *Let X be good, Hausdorff, boundaryless and quasi-smooth over K. For every $x \in X$ and ordinary neighborhood N of x, there is a smaller ordinary neighborhood $W \subset N$ which is K-analytically isomorphic to an ordinary open of a smooth affine K-variety.*

Proof. Shrink N inside the interior of an affinoid neighborhood of x; it is then separated and remains boundaryless. Ducros's algebraization lemma [7, Lemma (0.21)], applied with the full group of positive radii, gives an affinoid neighborhood V of x in N, identified with an affinoid domain in the analytification of a smooth affine K-variety Y. No extension of K is made.

The relative-interior composition formula, with locally separated intermediate spaces [10, §2.11(3)–(4)], gives

$$\operatorname{Int}(V/N) = \operatorname{Int}(V/K) = \operatorname{Int}(V/Y^{\mathrm{an}}),$$

because N and Y^{an} are boundaryless. Relative interiors of analytic-domain embeddings are their ordinary topological interiors; see [13, Fact 3.4.11 and Fact 4.2.15(i)]. The common interior is therefore ordinary open in both N and Y^{an}, and contains x since V was a neighborhood.

It is the required W. One may shrink it further to have compact closure in a larger such chart. □

Lemma D.5 (Finite constants and integration)**.** *Let Y be a smooth integral affine K-variety, and let L be the algebraic closure of K in $K(Y)$. Then L/K is finite separable, Y is naturally an L-variety, and Y/L is smooth and geometrically integral. Viewing this same analytic space over K or over L does not change its functions, absolute values or classical psh functions. If $d = [L : K]$, the calibrated integration currents satisfy*

$$[1]_K = d[1]_L. \tag{D.4}$$

Consequently the mixed currents and weighted currents, whenever constructed by the recursion in the main theorem, satisfy

$$T_{k,K} = dT_{k,L}, \qquad S_{k,K} = dS_{k,L}.$$

Proof. The extension L/K is finite because $K(Y)/K$ is finitely generated, and it is separable in characteristic zero. Every element of L is integral over K, hence over the normal affine coordinate ring of Y; it therefore belongs to that ring. This gives the L-structure. The diagonal factor of $Y \times_K L$, singled out by $L \otimes_K L \to L$, is Y itself as an L-scheme. It is open and closed, so it is smooth over L. The function-field extension $K(Y)/L$ is regular, giving geometric integrality.

There is a unique extension of the absolute value of the complete field K to L. Thus the two analytifications have the same points, topology and analytic functions. An E/L-curve test maps to the diagonal open-and-closed factor $Y_L \times_L E$ of $Y_K \times_K E$, and is therefore a permitted K-curve test. Conversely, for an E/K-curve test, pass to a completed algebraic closure Ω/E. Then $L \otimes_K \Omega = \prod_{\sigma: L \to \Omega} \Omega$. The inverse images of these factors are open-and-closed curves. Classicality over L gives subharmonicity on each of them. Sheafiness and descent under analytic field extension [10, Proposition 3.5(7)] give subharmonicity on the original E-curve. Upper semicontinuity and local boundedness are unchanged.

The moment skeletons are the same in the two interpretations. Forgetting L multiplies each finite moment degree by d; its calibration is therefore multiplied by d, as in [6, Remark 14.4.3]. This proves (D.4). Differentiation and multiplication by a fixed Borel function are linear. Induction through the recursive definition gives the same single factor d at every exterior degree, not d^k. □

Return to transport to ordinary analytic charts.
Return to local algebraic regularization.

Use of generative AI

Generative AI was used in developing and drafting mathematical arguments and in revising the exposition of this manuscript.

References

[1] B. Bhatt, L. Ma, Z. Patakfalvi, K. Schwede, K. Tucker, J. Waldron, and J. Witaszek, with an appendix by R. Datta, *Test ideals in mixed characteristic: a unified theory up to perturbation*, arXiv:2401.00615v2, 2025.
[2] E. Bierstone and P. D. Milman, *Functoriality in resolution of singularities*, Max-Planck-Institut für Mathematik preprint 2007-30, 2007. Author manuscript.
[3] S. Boucksom and D. Eriksson, *Spaces of norms, determinant of cohomology and Fekete points in non-Archimedean geometry*, arXiv:1805.01016v3, 2021.
[4] S. Boucksom, W. Gubler, and F. Martin, *Non-Archimedean volumes of metrized nef line bundles*, Épijournal de Géométrie Algébrique **5** (2021), Article 13; arXiv:2011.06986v3.
[5] A. Chambert–Loir and A. Ducros, *Formes différentielles réelles et courants sur les espaces de Berkovich*, arXiv:1204.6277v1, 2012. This edition fixes the original test sheaf used here.
[6] A. Chambert–Loir and A. Ducros, *Formes différentielles réelles et courants sur les espaces de Berkovich*, arXiv:1204.6277v2, 2025.

[7] A. Ducros, *Espaces de Berkovich, polytopes, squelettes et théorie des modèles*, Confluentes Mathematici **4** (2012), no. 4, Article 1250007, 57 pp. The numbering used here is that of the 53-page author manuscript, Lemma (0.21).
[8] Y. Fang, W. Gubler, and K. Künnemann, *On the non-Archimedean Monge–Ampère equation in mixed characteristic*, Nagoya Mathematical Journal **259** (2025), 548–561; arXiv:2203.12282v3.
[9] W. Gubler, P. Jell, and J. Rabinoff, *Forms on Berkovich spaces based on harmonic tropicalizations*, arXiv:2111.05741v3.
[10] W. Gubler and J. Rabinoff, *Classically psh and pluriharmonic functions on Berkovich spaces*, arXiv:2506.13548v2, 2025.
[11] H. Holm, *The structure of balanced big Cohen–Macaulay modules over Cohen–Macaulay rings*, Glasgow Mathematical Journal **59** (2017), 549–561. Author manuscript.
[12] The Stacks Project Authors, *The Stacks Project*, 2026. Section 49.12: The Noether different; Section 37.32: Bertini theorems.
[13] M. Temkin, *Introduction to Berkovich analytic spaces*, preliminary lecture notes, 2010. Lecture notes.
[14] A. Thuillier, *Théorie du potentiel sur les courbes en géométrie analytique non archimédienne. Applications à la théorie d'Arakelov*, doctoral thesis, Université de Rennes 1, 2005. Thesis.
[15] V. Wanner, *Comparison of two notions of subharmonicity on non-archimedean curves*, arXiv:1801.04713v3, 2018.
[16] L. Wu, *On the local theory of non-archimedean psh functions*, arXiv:2609.27610v1, 2026.

Index of notation

The entries are grouped by mathematical role. Each locator links to the definition or principal use and gives its page.

Symbol	Meaning and scope	Definition or first use
Fields, domains and input classes		
K, π, λ	Finite extension of $\mathbb{Q}_p$, chosen uniformizer, and $\lambda = -\log\lvert\pi\rvert > 0$ for the prescribed absolute value.	Convention 1.1, p. 1
X, U, n	Good Hausdorff boundaryless quasi-smooth analytic space, ordinary open, and pure dimension.	Convention 1.1, p. 1
$\mathcal{C}_b(U)$	Locally bounded upper-semicontinuous classical psh functions; curve tests after every complete isometric field extension.	Definition 1.2, p. 1
$\mathrm{PL}_{\mathrm{sp}}(U)$	Continuous model-semipositive PL functions. Rational slopes and constants in $\lambda\mathbb{Q}$ are used in the constructions.	Definition 1.2, p. 1
$\mathcal{R}_b(U)$	Locally bounded functions locally obtained as decreasing PL limits on one fixed ordinary domain.	Definition 1.2, p. 1
$\mathcal{G}_K$	Fields obtained from K by finitely many unit-Gauss completions; the simultaneous dimension induction uses arbitrary data over them.	Definition 2.1 and lemma A.1, p. 17
$L/K,\ d$	Finite algebraic constant field of a smooth integral affine K-variety, and its degree; forgetting constants multiplies each calibrated current by the single factor d.	Lemma D.5, p. 31
Forms, coefficient measures and mixed currents		
$A^{p,q}, A_c^{p,q}$	Original CLD smooth forms and compact test forms, with their canonical calibration.	Convention 1.3, p. 2
$D, \mathrm{d}', \mathrm{d}''$	The original differentials and curvature operator $D = \mathrm{d}'\mathrm{d}''$.	Convention 1.3, p. 2
$[1]$	Integration current in the ground-field calibration.	(1.2), p. 3
$\mu_{T,\eta}, hT$	Radon measure on the analytic domain representing the scalar functional $a \mapsto \langle T, a\eta\rangle$, and multiplication by a locally bounded Borel function.	Definition 1.4, p. 2
$T_k(u_1, \dots, u_k)$	Symmetric closed strongly positive mixed curvature current of bidegree (k, k).	Theorem 1.6, p. 3
$S_k(u_0; u_1, \dots, u_k)$	Weighted current $u_0 T_k(u_1, \dots, u_k)$, with undifferentiated argument u_0.	(1.3), p. 3
$W_r,\ A_r$	Span of decomposable squares in the symmetric tensor space of r-vectors, and its annihilator in the dual tensor space.	Lemma 4.1, p. 10
$\mathrm{MA}_{\mathrm{PL}}$	Top-degree measure in the local PL theory, compared with the continuous CLD product.	Proposition 4.4, p. 11
$j,\ \lvert\cdot\rvert_*$	Transport under $\lvert\cdot\rvert_* = \lvert\cdot\rvert^{1/\lambda}$, used only to compare valuation conventions with the PL input.	(4.4), p. 11

Continued on the next page

Symbol	Meaning and scope	Definition or first use
Metrics, energy and slicing		
$c_1(L,\phi)$	Curvature of the additive metric ϕ, with model mass equal to structural degree.	Convention 1.3, (3.1), p. 7
$\Gamma(g), \Gamma(g,h)$	First-order quadratic energy current and its symmetric polarization.	(2.2), p. 6
$E_\Omega(g,h)$	Energy paired with a smooth strongly positive closed complementary background.	Proposition 2.4, p. 6
$P_L\rho$	Pointwise envelope of continuous model-semipositive minorants of ρ on the same ample line L.	Definition 3.1, p. 7
$E,\ H,\ \iota$	Unit-Gauss parameter field, its generic hyperplane, and the map from that hyperplane to the original projective analytic space.	Lemma 3.2, p. 7
$\Delta_C b, \Gamma_C$	Positive curve Laplacian of a subharmonic function, and the curve energy measure.	Lemmas A.2 and B.1, p. 20
$p : W \to \mathbb{G}_m^{n-1},\ \Omega$	Algebraic curve-slicing map and a nonnegative top form in the specified standard torus coordinates.	Lemma A.3, p. 18
$\nu_\Omega,\ \mu_y$	Base skeletal measure and the curve Laplacian kernel, defined off one common null parameter set.	Theorem A.4, p. 19
Normal-model envelopes and local geometry		
R, F	In Appendix C, a complete mixed-characteristic DVR and its characteristic-zero fraction field; no residue-field finiteness is assumed.	Section C, p. 23
S, W, E, F_S	Complete local normal domain, its chosen canonical module $W = \mathcal{O}_S(-\Delta)$, and their top local cohomology modules.	Section C, p. 23
$\mathfrak{N}_{S/A}$	Noether different of a finite extension from a regular local subring; its witnesses give the flat factorization.	Lemma C.1, p. 24
$\tau(\mathcal{O}_\mathcal{X}, \Delta, \mathfrak{a}^s)$	Mixed-characteristic test ideal; the global fixed correction is independent of the ideals and rational exponents.	Theorem C.4, p. 25
$\mathfrak{c}_\ell,\ \beta_m,\ \psi_m$	Weighted base ideals, stable asymptotic test ideals, and their normalized logarithmic norms in the envelope construction.	(C.6), p. 26
$C_1, C_{\mathrm{mid}}, C_2,\ b$	Nested compact norm boxes and their logarithmic collar barrier.	Lemma D.2, p. 29
$R_{i,j}, P_{i,j,m}, Q_{i,j}$	Bounded projective metrics on a common line bundle, decreasing pure-FS rows, and the compact triangular diagonal.	Lemmas D.2 and 5.1, p. 12