BEDFORD–TAYLOR PRODUCTS WITH A POSITIVE CLOSED BACKGROUND: CURVATURE QUOTIENTS, RELATIVE ENERGY AND SOURCE EQUATIONS

Abstract. We study Bedford–Taylor products in the original Chambert–Loir–Ducros calculus over a fixed finite extension of $\mathbf{Q}_p$. A symmetric middle-positive current, separately closed for d' and d'', is retained as the background. Locally bounded classical plurisubharmonic potentials give intrinsic mixed currents with weighted joint decreasing continuity. The kernel of curvature on bounded classical differences yields a potential-independent operation, and continuous coefficient relations extend it to a module of curvature currents. An order estimate then gives a faithful completion of that module inside the original currents. On strict Laurent domains whose closed boxes carry the background, the relative first-energy calculus admits a complete Borel realization with critical weighted continuity. We describe its variational source map and prove that a single geometric potential detects the entire original-test defect of a literal Radon datum. In remaining degree one, finite signed energy-diffuse data have a unique relative renormalized zero-boundary solution. The regularization and analytic energy estimates used in the constructions are stated with their precise hypotheses and cited to the accompanying foundational manuscripts.

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1. Introduction

The Bedford–Taylor recursion differentiates a potential after multiplying it by a positive closed current. Over a non-Archimedean field, this construction raises two distinct questions. The first is analytic: which potential classes make the recursive products positive and stable under approximation? The second concerns the objects on which the construction depends: when can a potential be replaced by another potential with the same curvature, or a finite curvature expression by another expression representing the same current?

We develop these questions with one positive closed current retained as the background. The resulting local calculus leads to an ordered module of curvature currents. On specified analytic domains it also leads to relative capacities, a complete first-energy space, and source equations. The proofs use the local regularization of classical potentials and the analytic energy estimates in the accompanying manuscripts [Cont, Bound, Order, Energy, Dual]. The arguments below give the constructions and the transitions between these theories; the cited analytic inputs are stated at the points where they enter.

1.1. **The local product.** Fix a finite extension $K/\mathbf{Q}_p$ with its prescribed absolute value. Let X be a good, Hausdorff, boundaryless, quasi-smooth K-analytic space of pure dimension n. Opens are ordinary opens in the Berkovich topology. On such an open U, write $\mathcal{C}_b(U)$ for the actual real-valued, locally bounded, upper-semicontinuous functions whose pullback to every analytic curve after every complete isometric extension of K is subharmonic. These are the locally bounded classical plurisubharmonic functions of [GR25]. Put $D = d'd''$.

Forms and currents belong to the original Chambert–Loir–Ducros calculus, with its canonical calibration. Positivity means the middle positive cone of [CLD12, Definition 5.1.1(c)]; a positive current is nonnegative on the corresponding positive complementary forms. A background of degree (p,p) is a symmetric positive current S such that $d'S = d''S = 0$. All current identities below are tested on every original compactly supported smooth complementary form. The tests retain all their actual finite moment refinements, including those on which the moment map loses rank.

Theorem 1.1 (Bounded products with a fixed background)**.** *Let S be a background of degree (p,p) on U, and put $q = n - p$. For $0 \le k \le q$ there are unique recursively defined coefficient Radon currents*

$$T_{S,0} = S, \qquad T_{S,k}(u_1, \dots, u_k) = D\big(u_1 T_{S,k-1}(u_2, \dots, u_k)\big), \qquad u_i \in \mathcal{C}_b(U). \tag{1.1}$$

The multiplication in this formula uses the given Borel functions and the coefficient measures on the analytic space. The currents are positive, symmetric as currents, and separately d'- and d''-closed. The operation is invariant under permutations of the differentiated slots, additive and positively homogeneous in each such slot, local for ordinary restriction, and independent of the regularization used to compute them.

If $u_{i,j} \downarrow u_i$ in $\mathcal{C}_b(U)$ for $0 \le i \le k$, with locally bounded limits, then

$$u_{0,j} T_{S,k}(u_{1,j}, \dots, u_{k,j}) \longrightarrow u_0 T_{S,k}(u_1, \dots, u_k) \tag{1.2}$$

on every original compact smooth complementary test. In top degree this is local weak convergence of signed Radon measures against compactly supported continuous functions. Finally, if $D[u_i] = D[v_i]$ for every differentiated slot, then

$$T_{S,k}(u_1, \dots, u_k) = T_{S,k}(v_1, \dots, v_k).$$

Proof in Section 3.

Coefficient multiplication and the ordered construction are established in Section 2. The convergence proof uses a common collar on a small norm box. The collar permits Green exchange with a smooth argument before any two discontinuous approximation parameters are passed to a limit. Positivity then gives a lower bound for a fixed trace test. A separate upper bound for decreasing weights determines the trace limit and hence all original test limits. The final curvature-independence assertion is proved in Section 4.

1.2. **Curvature relations and order completion.** Let $\mathscr{M}^1(U)$ be the sheaf of coefficient Radon $(1,1)$-currents locally admitting a finite expression

$$A = \sum_\alpha f_\alpha D[g_\alpha], \qquad f_\alpha \in C(U,\mathbf{R}), \quad g_\alpha \in \mathcal{C}_b(U). \tag{1.3}$$

Actual equality of currents is the equivalence relation. Its positive cone is inherited from the original middle-positive cone. Given $Q \in \mathscr{M}^1(U)$ positive, define

$$E_Q = \{A \in \mathscr{M}^1(U) : -cQ \le A \le cQ \text{ for some } c \ge 0\},$$
$$\|A\|_Q = \inf\{c \ge 0 : -cQ \le A \le cQ\}.$$

We denote by $\mathscr{M}^1_{\mathrm{ord}}$ the sheaf obtained from the completions of these normed spaces, using the common control $Q+R$ to compare limits originally controlled by Q and R.

Theorem 1.2 (Continuous coefficient action and its order completion)**.** *For every background S of degree (p,p) and $r \le n-p$, the formula*

$$\Phi_{S,r}(A_1,\dots,A_r) = \sum_{\alpha_1,\dots,\alpha_r} \left(\prod_{i=1}^r f_{i,\alpha_i}\right) T_{S,r}(g_{1,\alpha_1},\dots,g_{r,\alpha_r}) \tag{1.4}$$

is independent of all presentations (1.3). It defines a symmetric $C(U,\mathbf{R})$-multilinear local operation, positive when every slot is positive.

Each completion $\widehat{E}_Q$ embeds faithfully and isometrically into the actual symmetric coefficient Radon currents. These embeddings are compatible under change of control. The operation $\Phi_{S,r}$ extends uniquely to $\mathscr{M}^1_{\mathrm{ord}}$, locally and independently of the approximating sequences. If $-M_iQ_i \le A_i, B_i \le M_iQ_i$ and $-\delta_iQ_i \le A_i - B_i \le \delta_iQ_i$, then

$$-ER_S \le \Phi_{S,r}(A_1,\dots,A_r) - \Phi_{S,r}(B_1,\dots,B_r) \le ER_S, \tag{1.5}$$

where

$$R_S = \Phi_{S,r}(Q_1,\dots,Q_r), \qquad E = \sum_{i=1}^r \delta_i \prod_{j\ne i} M_j.$$

The output is separately closed whenever it is obtained as a limit of separately closed mixed approximants.

Proof in Section 5.

The bounded-difference criterion is the link between potential theory and this order structure: for $a,b \in \mathcal{C}_b$, positivity of $D[a-b]$ makes the actual difference $a-b$ classical psh. Freezing continuous coefficients on a finite cover then transfers their relations to every fixed background. The multilinear estimate (1.5) follows by a positive expansion and telescoping. The embedding and extension are proved in Section 6.

The continuous coefficient operation acts on degree-one presentations. A higher-degree analytic relation involving cancellation between several nonzero mixed terms is a separate relation problem. For example, potential independence concerns each equality $D[u] = D[v]$, whereas a quadratic relation can have the form $\sum_\nu c_\nu D_\delta h_\nu \wedge D_\delta k_\nu = 0$. The latter is the subject of the companion research program.

1.3. **Relative energy and source equations.** The global energy theory uses a domain with an explicit boundary. Let $B \subset X$ be a compact affinoid, and let

$$K_0 = \{|f_a| \le R_a,\ |g_b| \ge r_b\} \Subset \mathrm{Int}_X(B), \qquad V = \{|f_a| < R_a,\ |g_b| > r_b\} \ne \varnothing. \tag{1.6}$$

All thresholds are positive, the analytic functions are defined on B, and S is defined on an ordinary neighborhood of K_0. Put $q = n-p \ge 1$ and $d = q+1$.

For a Borel set $E \subset V$, define

$$C_S(E;V) = \sup_{\substack{h\in\mathcal{C}_b(V)\\ 0\le h\le 1}} T_{S,q}(h,\dots,h)(E), \qquad C_S^*(E;V) = \inf_{\substack{G\supset E\\ G \text{ ordinary open}}} C_S(G;V).$$

The actual class $\mathcal{E}^1_{S,\partial}(V)$ consists of negative extended classical potentials with compact negative sublevel closures, canonical truncation no loss, and finite first energy. The precise definition and the nonpolar construction occur in Section 7. For its members write

$$B(u_0,\dots,u_q)=\int_V(-u_0)\,Du_1\wedge\cdots\wedge Du_q\wedge S,\qquad e(u)=B(u^{[d]}),$$

and

$$I(u,v)=e(u)+e(v)-B(u,v^{[q]})-B(v,u^{[q]}).$$

A row is strongly Cauchy if $\sup_{j,l\ge N}I(u_j,u_l)\to 0$. Define $\mathscr{C}$ to consist of its actual Borel limits in every local truncated outer S-capacity, modulo local outer-capacity zero.

Theorem 1.3 (The complete relative first-energy space)**.** *Under (1.6), every strongly Cauchy row in $\mathcal{E}^1_{S,\partial}(V)$ has a unique limit in $\mathscr{C}$. There are intrinsic mixed curvature, first-order and energy currents, and global mixed energies on $\mathscr{C}$, independent of all defining rows and outer-capacity-null representatives. The mixed energy is symmetric and satisfies*

$$\widehat{B}(u_0,\dots,u_q)\le\prod_{i=0}^{q}\widehat{e}(u_i)^{1/d}. \tag{1.7}$$

The completed discrepancy $\widehat{I}$ separates relative classes and is complete. Moreover,

$$\widehat{I}(u_j,u)\to 0\quad\Longleftrightarrow\quad\begin{cases}u_j\to u\text{ in every local truncated outer capacity,}\\ \widehat{e}(u_j)\to\widehat{e}(u).\end{cases} \tag{1.8}$$

For a finite tuple, let F be continuous on the nonpositive orthant and satisfy $|F(x)|\le A(1-\sum_i x_i)^a$, with $a\ge 0$. Give a curvature current with k curvature slots the degree budget $b=k$; for a first-order or energy current over k curvature slots put $b=k+1$ or $b=k+2$, respectively. In the legal bidegrees, if $a+b\le q+1$, these multipliers are coefficient-integrable and their weighted currents converge on every original compact smooth test under strong convergence of the tuple. The global mixed weighted top measures also converge. The actual classical class is strongly dense.

Proof in Section 8.

The proof first extracts uniform deep tails from the Cauchy condition. These tails give the uniform integrability required at the critical degree. Compact sublevels of one fixed late member of each row also give global tightness, without a common lower barrier. Truncation then constructs the currents before unbounded multiplication is used.

Put $r(u)=\widehat{e}(u)^{1/d}$. The completed cone gives the energy-difference space $\mathbb{D}_S=\mathscr{C}-\mathscr{C}$ with norm

$$\|h\|_{\mathrm{ed}}=\inf_{h=a-b}\big(r(a)+r(b)\big),\qquad a,b\in\mathscr{C}.$$

Write $\mathbb{D}^*_{S,\mathrm{s}}$ for its continuous linear functionals whose restriction to $\mathscr{C}$ is continuous under strong convergence. A positive Radon measure μ defines a *literal* member of this class if $\ell_\mu(h)=-\int h\,d\mu$ is absolutely defined on $\mathbb{D}_S$, independent of outer-null representatives, norm continuous and strongly continuous on $\mathscr{C}$.

Theorem 1.4 (Variational sources and one geometric probe)**.** *Under (1.6), $\mathbb{D}_S$ is Banach and*

$$J(u)(h)=\widehat{B}(h,u^{[q]})=-\int h\,d\widehat{\mu}_u$$

defines an injective map with norm-closed image in $\mathbb{D}^*_S$. The metric $\|J(u)-J(v)\|_*$ is complete and induces the strong topology on $\mathscr{C}$. Every $\ell\in\mathbb{D}^*_{S,\mathrm{s}}$ has a unique variational maximizer

$$u=\Pi(\ell)=\operatorname*{argmax}_{v\in\mathscr{C}}\big(\ell(v)-\widehat{e}(v)/d\big)$$

and a unique decomposition $\ell=J(u)+n$ with $n(v)\le 0$ for every $v\in\mathscr{C}$ and $n(u)=0$.

There is one bounded continuous classical $-1<\psi_0<0$, determined by the Laurent box and having finite top mass, such that $\psi_0\pm\epsilon_\phi\phi\in\mathcal{E}^1_{S,\partial}(V)$ for every original compact smooth

real scalar ϕ and some $\epsilon_\phi > 0$. The choice of ψ_0 and ϵ_ϕ works for every extending allowed background. For a literal positive Radon datum as above, set $u = \Pi(\ell_\mu)$ and

$$\Xi_{\psi_0}(\mu) = J(u)(\psi_0) - \ell_\mu(\psi_0) \geq 0.$$

Then

$$\Xi_{\psi_0}(\mu) = 0 \quad \Longleftrightarrow \quad \widehat{\mu}_u = \mu \quad \Longleftrightarrow \quad \mu = \widehat{\mu}_v \text{ for some } v \in \mathscr{C}.$$

The solution is unique. For every such datum and every original compact smooth scalar test,

$$\left| \int \phi \, d(\mu - \widehat{\mu}_u) \right| \leq \epsilon_\phi^{-1} \Xi_{\psi_0}(\mu).$$

Proof in Section 9.

The probe is a convergent sum of exact-moment compensators. Its selected summands admit both signs of a prescribed compact smooth direction. The normal inequality on these two competitors gives the defect bound; literal integration then turns equality on all tests into equality of Radon measures.

The remaining degree $q = 1$ gives a linear equation even when the ambient dimension is larger than one. For this assertion choose one compact affinoid B, ordinary opens $V \Subset \Omega$ with $\overline{\Omega} \subset \mathrm{Int}_X(B)$, and a background S of degree $(n-1, n-1)$ on Ω. On $F = V \cap \mathrm{supp}\, S$, the original compact smooth core has a faithful, regular, strongly local, coercive Dirichlet-form realization $(\mathcal{E}, H)$ in a finite reference measure λ. This construction and its identification with the original current are stated in Section 10. Let c_V be its energy capacity and $T_k(s) = \max(-k, \min(k, s))$.

Theorem 1.5 (The finite diffuse relative Dirichlet equation)**.** *In the preceding domain, let μ be a finite signed Radon measure on F whose variation vanishes on every c_V-polar Borel set. There is a unique $u \in L^1(F, \lambda)$ with a finite energy-quasicontinuous representative, $T_k u \in H$ for every $k > 0$, the original chain identity*

$$D(\Phi(u)S) = \Phi'(u)\mu + \Phi''(u)\Gamma_S(u)$$

for every $\Phi \in C^2(\mathbf{R})$ with compactly supported first derivative, and $k^{-1}\mathcal{E}(T_{2k}u - T_k u) \to 0$. Here $\Gamma_S(u)$ is the compatible sigma-finite energy measure of its clippings. This solution satisfies $D(uS) = \mu$ on every original compact scalar test and is independent of the faithful reference measure. If $\mu_j \to \mu$ in total variation, then

$$\|u_{\mu_j} - u_\mu\|_{L^1(\lambda)} \to 0, \qquad \mathcal{E}(T_k u_{\mu_j} - T_k u_\mu) \to 0 \quad (k > 0).$$

Proof in Section 10.

1.4. **Relation to the existing calculus.** The recursive construction and continuous-potential theory originate in [CLD12, §5.6]. The revised treatment [CLD25, Lemma 8.5.5 and Propositions 8.7.2, 8.7.4] supplies positivity, local uniform stability and symmetry for continuous psh potentials on a positive closed background. Its discussion [CLD25, §8.6.1–8.6.2] isolates the question of annihilating zero-curvature potential differences against a background. The regularization manuscripts [Cont, Bound] supply the particular continuous and locally bounded classical approximation statements used here. The bounded curvature quotient and ordered difference criterion are developed in [Quot, Order].

The relative energy construction in [Energy] works with the original coefficient measures and the full local test sheaf. Its global boundary class is tied to (1.6). The dual construction in [Dual] then treats a source as a literal measure when measure equality is required. This separation of local products, relative completion and source compatibility determines the organization of the present article.

Wu [Wu26, Theorem 4.2 and Appendix A] constructs top-degree local Monge–Ampère measures for a class defined by decreasing tropical-convex PL approximation on a fixed neighborhood. The classical regularization input here supplies that neighborhood and approximation from the complete-extension curve definition; the background calculus retains all allowed bidegrees. On balanced polyhedral and tropical spaces, [BMPS26, APW26] develop related product,

energy and source theories. Their geometric categories provide useful comparison for the relation and solvability questions arising from original analytic currents.

Relative capacity with a positive closed current is classical in the complex setting [DE06]. Relative non-pluripolar products and their closedness are developed in [Vu21, Theorem 3.7], and the constant-one first-energy mixed inequality has the complex precedent [Ceg98, Theorem 3.2]. The role of these constructions here is their realization with the fixed original non-Archimedean background, coefficient measures and boundary class.

General renormalized measure-data equations for transient regular Dirichlet forms satisfying the strong sector condition are treated in [KR15]. In the linear part of this article, the analytic work is the faithful realization of the part form, its measures and its scalar chain identities in the original CLD calculus. The abstract finite-energy nest theorem used there is [KR17, Lemma 2.1].

2. Coefficient measures and the ordered construction

2.1. **Positive currents and analytic-space coefficients.** A current is of coefficient Radon type if its coefficient functionals in every finite original moment presentation are signed Radon measures on the analytic domain, compatibly with scalar multiplication, relations between forms, and common refinements. In particular, for every compact original test η there is a signed measure $\mu_{T,\eta}$ satisfying

$$\int a\, d\mu_{T,\eta} = \langle T, a\eta\rangle \quad (a \text{ original smooth}).$$

This condition permits the definition

$$\langle hT, \eta\rangle = \int h\, d\mu_{T,\eta} \tag{2.1}$$

for an actual locally bounded Borel function h.

Lemma 2.1 (Finite test control)**.** *Every positive current is of coefficient Radon type. For each compact original test η there is a nonnegative compact trace test Θ and a constant c_η such that*

$$|\langle T, h\eta\rangle| \le c_\eta \|h\|_\infty \langle T, \Theta\rangle \tag{2.2}$$

for positive T and bounded Borel h on the chosen compact stage. The positive cone is weakly closed and pointed. If positive currents T_j converge on all original tests, their coefficient variations are uniformly bounded on each compact stage. Thus $f_j T_j \to fT$ for locally uniformly convergent continuous $f_j \to f$.

Proof. Localize to a finite moment presentation containing the coefficients of η and a smooth cutoff χ equal to one near its support. A positive current annihilates the antisymmetric part of a test. For the symmetric part, put

$$\beta = \sum_{a=1}^{N} d' t_a \wedge d'' t_a, \qquad \Omega = \beta^m/m!,$$

where (m, m) is the test bidegree. The coefficient matrix of Ω is positive definite on this finite exterior space. Matrix domination therefore gives $c_\eta\|h\|_\infty \chi\Omega \pm h\eta \ge 0$ for smooth h. The inequality persists after adjoining scalar moments and pulling back to a common refinement. Positivity proves (2.2) for smooth h.

Compact uniform density of original smooth scalar functions and the Riesz representation theorem produce the coefficient measures. Their uniqueness preserves the original relations and refinements. Bounded Borel approximation against these measures gives (2.2) in the stated form. If $T_j \to T$, the trace masses $\langle T_j, \Theta\rangle$ are bounded, so the same coefficient estimate is uniform. Approximate f uniformly by a smooth function, pass the fixed smooth pairing, and then let the uniform error go to zero; the term $(f_j - f)T_j$ is controlled by the same bound. Closedness of the positive cone follows on each positive test. If both T and $-T$ are positive, the trace control makes every pairing zero. This is the coefficient argument of [Cont, Lemma

Continuous coefficients] in the present analytic category, using the ordinary local transport of [Bound]. □

2.2. **Regularization on a fixed neighborhood.** We use the following two precise regularization consequences. The first is [Bound, Theorem 1.5]; its construction uses the prescribed field and the original calibration.

Proposition 2.2 (Classical regularization)**.** *For every $u \in \mathcal{C}_b(U)$ and every $x \in U$ there is one ordinary neighborhood W of x and continuous semipositive rational PL functions p_j over K, defined on W, with $p_j \downarrow u$. The class $\mathcal{C}_b$ is stable under decreasing locally bounded limits. For finitely many decreasing families with locally bounded endpoints, the common projective extension construction gives one such neighborhood for all rows. Continuous members of these rows admit smooth positive uniform approximants on smaller fixed neighborhoods. Smooth majorants of each bounded member can consequently be chosen pointwise convergent and locally uniformly bounded.*

The simultaneous neighborhood assertion uses the norm-box projective extension in [Bound, Appendix D]. Smooth approximation of the continuous rows is [Cont, Lemma *Smooth approximation of FS metrics* and Proposition *Regularized maximum*]. The smooth approximating rows in the last sentence are not required to decrease. Their precise collar version is proved below.

Proposition 2.3 (First order and ordered products)**.** *If R is a background and $u \in \mathcal{C}_b$, then $D(uR)$ is positive, symmetric, separately closed and of coefficient Radon type. Iterating this operation defines (1.1). The ordered products are local and separately additive and positively homogeneous in the potentials. They are independent of the approximants used in the first-order construction.*

Proof. Work on a regularization neighborhood and choose continuous positive $p_j \downarrow u$. The continuous background theorem [Cont, Theorem *Continuous potentials on a positive closed background*], transported to this neighborhood as in [Bound], makes $D(p_j R)$ positive. Bounded convergence against the fixed coefficient measures of R gives $p_j R \to uR$ on every original test. Distributional differentiation gives $D(p_j R) \to D(uR)$. Positivity and symmetry pass to the limit. The identities $d'D = d''D = 0$ give separate closedness, and lemma 2.1 gives coefficient Radon type. The result is the derivative of the already defined actual product uR, so it is independent of the row. The same argument applies successively to every positive closed intermediate current. Linearity of multiplication and differentiation gives the algebraic properties; restriction commutes with both operations. □

Proposition 2.4 (Relative local mass estimate)**.** *Fix compact test stages $L \Subset W$ and $k \le q$. For each finite original coefficient presentation on L there is a constant C and a compact positive trace test Θ on W such that*

$$\|T_{S,k}(u_1, \dots, u_k)\|_L \le C \prod_{i=1}^{k} \|u_i\|_{L^\infty(W)} \langle S, \Theta \rangle$$

whenever the displayed bounds are finite. Here the left side denotes the sum of the coefficient variations in that finite presentation. The estimate is local and may be applied after every actual refinement.

Proof. Choose nested compactly supported cutoffs between L and W and add the finitely many moment coordinates used by their coefficients. The trace estimate in lemma 2.1 reduces a coefficient variation to a positive trace pairing. For one differentiated slot, distributional integration by parts writes it as $\langle u_1 R, D(\chi\alpha) \rangle$, where α is a fixed closed trace form. On its compact support both signs of $D(\chi\alpha)$ are bounded by a constant times the next positive trace form. The coefficient bound is therefore at most $C\|u_1\|_\infty$ times the trace mass of R on the next compact stage. Repeat with the remaining differentiated slots. The last trace is a

fixed compact test against S. This proves the estimate; finite localization supplies the general compact stage. □

3. Weighted decreasing continuity

The ordered products from proposition 2.3 are defined before symmetry or joint convergence is used. The following collar construction supplies the Green exchanges needed to prove convergence.

Lemma 3.1 (A common domain for smooth majorants)**.** *Let $x \in U$, and let $u_{i,j} \downarrow u_i$ be finitely many decreasing families in $\mathcal{C}_b(U)$ with locally bounded limits. After shrinking around x, there is one ordinary neighborhood W, independent of i, j, with the following property. Each of the functions $a = u_{i,j}$ and $a = u_i$ has smooth psh majorants a_ν on W such that*

$$a_\nu \geq a, \qquad a_\nu(x') \longrightarrow a(x') \quad (x' \in W).$$

The majorants are bounded uniformly in ν on each fixed compact subset after a finite prefix is discarded. If $a \leq M$ on a compact neighborhood $C \Subset W$, its majorants may be chosen with $a_\nu \leq M + 1$ on C. The index discarded may depend on the function a.

Proof. Use [Bound, Proposition *Transport to an ordinary analytic chart*] to work on one ordinary algebraic neighborhood, with the original field and calibration. Apply [Bound, Lemma D.2, *A common collar for decreasing families*] to the entire finite family of sequences, before choosing approximants. It gives one projective variety, one ample line, one frame on W, and bounded classical metrics whose weights on W are the given functions plus one fixed constant. By [Bound, Theorem *Projective contact and regularization*], every such metric is a decreasing limit of pure Fubini–Study metrics on this same line. Their weights p_ν are continuous, satisfy $p_\nu \geq a$, and decrease pointwise to a after the fixed constant is removed.

Smooth each pure Fubini–Study metric uniformly, using [Cont, Lemma *Smooth approximation of FS metrics* and Proposition *Regularized maximum*]. If the uniform error is at most $\epsilon_\nu \downarrow 0$, add ϵ_ν to the smoothed weight. The resulting smooth psh function a_ν satisfies

$$p_\nu \leq a_\nu \leq p_\nu + 2\epsilon_\nu.$$

This proves pointwise convergence and the majorant property. On C, the increasing open sets $\{p_\nu < M + 1/2\}$ cover C whenever $a \leq M$. Compactness gives a common index on C. Discard this prefix and require $2\epsilon_\nu < 1/2$. The same argument with any bound for a gives local uniform boundedness of the row. No monotonicity of the *smoothed* row is asserted or needed. □

Lemma 3.2 (A smooth psh collar)**.** *Under the assumptions of lemma 3.1, there are compact polynomial norm boxes*

$$x \in \operatorname{Int} C_1, \qquad C_1 \subset \operatorname{Int} C_{\mathrm{mid}}, \qquad C_{\mathrm{mid}} \subset \operatorname{Int} C_2 \Subset W,$$

a smooth psh function b on a neighborhood of C_2, and a nonnegative original smooth cutoff χ compactly supported in $\Omega = \operatorname{Int} C_2$, with the following properties. After adding one constant c to all the original functions, put

$$v_{i,j} = \max\{u_{i,j} + c, b\}, \qquad v_i = \max\{u_i + c, b\}.$$

Then:

(i) *$v_{i,j}, v_i \in \mathcal{C}_b(\Omega)$ and $v_{i,j} \downarrow v_i$;*
(ii) *$v_{i,j} = u_{i,j} + c$ and $v_i = u_i + c$ on a neighborhood of C_1;*
(iii) *$\chi = 1$ on a neighborhood of C_{mid};*
(iv) *all $v_{i,j}, v_i$ equal b on one fixed neighborhood of $\operatorname{supp} d'\chi \cup \operatorname{supp} d''\chi$;*
(v) *each patched function v has smooth psh majorants $v_\nu \geq v$ converging pointwise to v, uniformly bounded on $\operatorname{supp}\chi$, and equal to b on that same collar.*

The majorant rows in (5) need not be decreasing.

Proof. Choose the boxes by the construction in [Bound, Lemma D.2, *A common collar for decreasing families*]. In particular, lower-bound polynomials have no zero near C_2. Choose thresholds strictly between the defining thresholds for C_1 and C_{mid}. The finite maximum

$$b_0 = \max\{0, \log|P_a^{N_a}/c_a|, \log|d_l/Q_l^{M_l}|\}$$

is zero on a neighborhood of C_1 and at least some $\delta > 0$ on a neighborhood in C_2 of $C_2 \setminus \operatorname{Int} C_{\mathrm{mid}}$. All its branches other than the zero branch are logarithms of analytic units where they are active. A branch with a zero is strictly inactive near that zero.

Apply the regularized maximum of [Cont, Proposition *Regularized maximum*], with smoothing radius smaller than all the fixed gaps. The result $\widetilde{b}$ is an original smooth psh function: zero branches at polynomial zeros are deleted by the exact inactivity property. On a neighborhood of C_1, only the constant zero branch is active, so $\widetilde{b} = 0$. On the outer collar, $\widetilde{b} \geq \delta$.

The endpoint bounds give a constant c and a finite M such that

$$1 \leq u_i + c \leq u_{i,j} + c \leq M \quad \text{on a neighborhood of } C_2.$$

Multiply $\widetilde{b}$ by a positive constant so that the resulting b is greater than $M + 3$ on the outer collar. Choose χ, with $0 \leq \chi \leq 1$, equal to one near C_{mid} and compactly supported in Ω. Smooth cutoff existence and a finite localization are part of the original CLD test calculus. Its derivative support lies where the fixed outer-collar inequality holds.

Classical psh functions are stable under maxima and addition of constants. This proves (1)–(4). To prove (5), apply lemma 3.1 to the *unpatched* function $a = u_{i,j} + c$ or $u_i + c$. Discard a prefix so that its smooth psh majorants a_ν are at most $M + 1$ on C_2. Define

$$v_\nu = \operatorname{rmax}_{\epsilon_\nu}(a_\nu, b), \qquad \epsilon_\nu \downarrow 0,$$

with all radii less than the fixed collar gap. The regularized maximum majorizes the ordinary maximum and has error at most ϵ_ν. It is exactly b on the collar because the other branch is inactive there. These facts give all the assertions in (5). □

The collar lies on a chosen norm box. The closed positive form that controls a test is selected from that test's finite moment presentation.

Lemma 3.3 (Green exchange on a common collar)**.** *Let Q be a symmetric positive d'- and d''-closed current of bidegree $(n-1, n-1)$ on Ω. Let $\chi \in A_c^{0,0}(\Omega)$. Suppose that a is smooth, g is locally bounded Borel, and both equal one smooth function b on a neighborhood of* $\operatorname{supp} d'\chi \cup \operatorname{supp} d''\chi$. *Whenever $D(gQ)$ has coefficient Radon type, one has*

$$\langle aD(gQ), \chi\rangle = \langle gD(aQ), \chi\rangle.$$

In this statement positivity of a and g is not required.

Proof. Since Q is closed and a is smooth, $D(aQ) = Da \wedge Q$. The definition of the distributional derivative gives

$$\begin{aligned}\langle aD(gQ) - gD(aQ), \chi\rangle &= \langle gQ, D(\chi a) - \chi Da\rangle \\ &= \langle Q, g\{aD\chi + d'\chi \wedge d''a + d'a \wedge d''\chi\}\rangle.\end{aligned}$$

Every term in braces is supported in the collar. Substituting $a = g = b$ there makes the last expression

$$\langle bQ, D(\chi b) - \chi Db\rangle = 0.$$

The final equality follows by applying the same distributional identity to the smooth pair (b, b). The substitutions are valid identities of coefficient measures. □

Lemma 3.4 (Monotonicity of the ordered trace integral)**.** *Let S have bidegree (p, p), let $0 \leq k \leq n - p$, and let α be a smooth, positive, d'- and d''-closed form of bidegree $(n-p-k, n-p-k)$ such that wedging it with a positive current preserves positivity. Let $\chi \geq 0$ be compactly supported in Ω.*

For $0 \le i \le k$, suppose $f_i, g_i \in \mathcal{C}_b(\Omega)$, $f_i \le g_i$, and all these functions equal the same smooth psh function b near the derivative support of χ. Assume that each g_i has the smooth majorants with this exact collar supplied by lemma 3.2. Define, using the ordered products,

$$J(f_0; f_1, \dots, f_k) = \langle f_0 T_{S,k}(f_1, \dots, f_k), \chi\alpha\rangle.$$

Then

$$J(f_0; f_1, \dots, f_k) \le J(g_0; g_1, \dots, g_k).$$

Proof. The case $k = 0$ is measure monotonicity. For $k > 0$, fix smooth psh majorants $G_i \ge g_i$ with the common collar. First replace the weight f_0 by G_0, using positivity of $T_{S,k}(f_1, \dots, f_k) \wedge \alpha$. Apply lemma 3.3 with

$$Q = T_{S,k-1}(f_2, \dots, f_k) \wedge \alpha$$

to move G_0 into a differentiated position and f_1 into the weight. The resulting current $DG_0 \wedge Q$ is positive and closed. Replace the weight f_1 by G_1, and repeat. At the end one obtains

$$J(f_0; f_1, \dots, f_k) \le \langle G_k\, DG_0 \wedge \cdots \wedge DG_{k-1} \wedge S, \chi\alpha\rangle.$$

A final application of the same Green identity, together with commutation of smooth even-degree factors, makes the right-hand side $J(G_0; G_1, \dots, G_k)$.

Now let the index of G_k tend to infinity, keeping every other index fixed. Bounded convergence against the fixed coefficient measures of S implies $D(G_k S) \to D(g_k S)$. The outer factors and the weight are smooth and fixed, so this convergence can be tested directly. Next let the index of G_{k-1} tend to infinity. The background $D(g_k S)$ is now fixed, so the same argument applies. Continue in the order $k, k-1, \dots, 1$. Finally, with the ordered product fixed, let the weight G_0 tend to g_0 by bounded convergence. The result is exactly $J(g_0; g_1, \dots, g_k)$ in its original order. The order of these limits keeps the coefficient measure fixed at each discontinuous multiplication. □

Lemma 3.5 (An upper bound for decreasing weights)**.** *Let P_j, P be positive coefficient Radon currents of one bidegree, with $P_j \to P$ on all original compact smooth tests. Let $a_j \downarrow a$ be locally uniformly bounded functions. Suppose that, near the support of a fixed positive compact test Θ, every fixed a_l has bounded continuous majorants converging pointwise to a_l. Then*

$$\limsup_{j\to\infty} \langle a_j P_j, \Theta\rangle \le \langle aP, \Theta\rangle.$$

Proof. For $j \ge l$ and a continuous majorant $a_{l,\nu} \ge a_l$, positivity gives

$$\langle a_j P_j, \Theta\rangle \le \langle a_{l,\nu} P_j, \Theta\rangle.$$

Keep l, ν fixed and pass to the limit in j, using lemma 2.1. Bounded convergence against the fixed positive Radon measure $\mu_{P,\Theta}$ first allows $\nu \to \infty$ and then $l \to \infty$. □

Theorem 3.6 (Weighted decreasing continuity for a fixed arbitrary background)**.** *Let S be any symmetric positive original CLD current of bidegree (p,p) on U, separately d'- and d''-closed. Let $0 \le k \le n-p$. If $u_{i,j} \downarrow u_i$ in $\mathcal{C}_b(U)$ for $0 \le i \le k$, with locally bounded limits, then*

$$u_{0,j} T_{S,k}(u_{1,j}, \dots, u_{k,j}) \longrightarrow u_0 T_{S,k}(u_1, \dots, u_k)$$

on every original compact smooth complementary test form. The ordered unweighted products have the corresponding joint decreasing continuity.

Proof. Write W_k for the weighted assertion with k differentiated slots. Bounded convergence for the fixed coefficient measures of S proves W_0. Distributional differentiation shows that W_{k-1} implies joint unweighted convergence in degree k. We prove W_k assuming this unweighted convergence.

The statement is local on compact test supports. Use a finite original smooth partition to reduce to one test η supported in a small norm box C_1 as in lemma 3.2. Include in the chosen

ordinary neighborhood all the unit coordinates needed to present η. Let these coordinates be $t_1, \dots, t_N$, and set

$$\beta = \sum_{a=1}^{N} d't_a \wedge d''t_a, \qquad r = n - p - k, \qquad \alpha = \frac{\beta^r}{r!}.$$

For $r = 0$ set $\alpha = 1$. The form α is closed and strongly positive. In the fixed finite coefficient space it is a positive definite trace form. After replacing η by its symmetric part, there is $C > 0$ such that

$$\Theta_\pm = C\chi\alpha \pm \eta$$

are positive original compact tests. This is precisely the matrix domination used in lemma 2.1; additional scalar coordinates are handled by pullback to a common refinement.

Patch all the potentials to $v_{i,j}, v_i$ by lemma 3.2, and put

$$P_j = T_{S,k}(v_{1,j}, \dots, v_{k,j}), \qquad P = T_{S,k}(v_1, \dots, v_k).$$

The induction hypothesis gives $P_j \to P$. Applying lemma 3.4 to the ordered tuples $(v_0, \dots, v_k) \le (v_{0,j}, \dots, v_{k,j})$ yields

$$\langle v_0 P, \chi\alpha\rangle \le \langle v_{0,j} P_j, \chi\alpha\rangle.$$

lemma 3.5 gives the reverse limsup inequality. Consequently

$$\langle v_{0,j} P_j, \chi\alpha\rangle \longrightarrow \langle v_0 P, \chi\alpha\rangle. \tag{3.1}$$

Apply lemma 3.5 to Θ_+ and Θ_-. Together with (3.1), the two resulting inequalities give respectively the upper and lower bounds for $\langle v_{0,j} P_j, \eta\rangle$. Thus this pairing converges to $\langle v_0 P, \eta\rangle$.

On a neighborhood of $\operatorname{supp}\eta$, the patched potentials are $u_{i,j} + c$ and $u_i + c$. Ordinary locality of the recursion identifies the differentiated products there with the original products, since constants in differentiated slots vanish. Remove the constant c from the weight by the unweighted degree-k convergence already established. Finite summation over the test partition proves W_k.

This induction proves all W_k and all allowed unweighted assertions.

□

Corollary 3.7 (Symmetry and the first-stage calculus)**.** *Under the hypotheses of theorem 3.6, the currents $T_{S,k}$ are symmetric in the differentiated arguments, positive, symmetric as currents, and separately d'- and d''-closed. They are additive and positively homogeneous in each potential, commute with ordinary restriction and gluing, and are independent of the regularization used to compute them. In particular,*

$$D\big(uD(vS)\big) = D\big(vD(uS)\big) \qquad (u, v \in \mathcal{C}_b(U),\ p \le n-2).$$

Proof. All assertions except permutation symmetry were proved for the ordered recursion. Choose, locally on one fixed domain, continuous semipositive PL approximants decreasing to each potential, as in proposition 2.2. Continuous arbitrary-background products are symmetric, by uniform smooth approximation and fixed-continuous-multiplier convergence. theorem 3.6, with a constant weight, passes this identity to the given potentials. The recursion identifies the result independently of the approximants and on overlaps. □

Corollary 3.8 (Top-degree convergence as measures)**.** *If $p + k = n$, the weighted convergence in theorem 3.6 holds against every compactly supported continuous scalar function, locally as signed Radon measures.*

Proof. Unweighted convergence against one nonnegative smooth cutoff equal to one near a prescribed compact set bounds the masses of the positive measures there. The weights have a common actual pointwise bound on a slightly larger compact neighborhood. Thus the weighted variations are uniformly bounded. Uniform density of original smooth scalar functions, with a fixed cutoff, extends convergence to continuous scalar tests. □

4. The curvature quotient of bounded classical potentials

Write $\mathscr{DSH}_{\mathrm{lb}}(U)$ for the actual real functions locally expressible as differences of two members of $\mathcal{C}_b$. Let $\mathscr{PH}(U)$ consist of finite continuous functions whose two signs are classical psh. The following result from the ordered-potential manuscript is the regularity input for this section [Order, Theorem 21.4].

Theorem 4.1 (The bounded-difference criterion)**.** *For $h \in \mathscr{DSH}_{\mathrm{lb}}(U)$,*

$$D[h] \geq 0 \quad \Longleftrightarrow \quad h \in \mathcal{C}_b(U).$$

Both sides concern the given actual function.

We recall the two geometric steps behind the criterion, since its use is stronger than a choice of a good representative for a distribution. Order transfer along analytic curve maps after every complete extension preserves the actual bounded classical difference and the original curvature inequality. On a curve the positive difference of the two Laplacians gives an actual subharmonic difference: positive Green potentials identify its endpoint values along nested discs. To recover ambient upper semicontinuity, use a compact semistable cover of a norm box and its finite Shilov interpolants. If $h = a - b$ and $a < a(x) + \epsilon$ on the box, the one-sided curve maximum principle gives $h \circ q \leq Ph$ and $b \circ q \leq Pb$ on each compact chart. The fiber maximum of the continuous function Ph is upper semicontinuous and majorizes h, while its value above x is at most $h(x) + \epsilon$. Letting ϵ decrease proves ambient upper semicontinuity. The full complete-extension curve condition then gives classical psh membership. These steps and the required algebraic localization and trace arguments are proved in [Order, Sections 19–21]; the zero-curvature predecessor appears in [Quot, Section 10].

Theorem 4.2 (The actual curvature kernel)**.** *The equality*

$$\ker\bigl(D : \mathscr{DSH}_{\mathrm{lb}}(U) \longrightarrow \mathcal{D}'^{1,1}(U)\bigr) = \mathscr{PH}(U)$$

holds for actual functions. In particular, if $u, v \in \mathcal{C}_b(U)$ and $D[u] = D[v]$, the difference $u - v$ is continuous and both its signs are classical psh.

Proof. If $D[h] = 0$, apply theorem 4.1 to h and $-h$. Both are upper semicontinuous, so h is continuous; they are classical psh by the same criterion. Conversely, for $h \in \mathscr{PH}$, first-order positivity gives $D[h] \geq 0$ and $-D[h] \geq 0$. Pointedness of the cone in lemma 2.1 makes $D[h] = 0$. This reasoning is local and hence applies to every locally represented difference. □

Theorem 4.3 (Background invariance and order)**.** *Let S be a background of degree (p, p), with $p < n$. For $u, v \in \mathcal{C}_b(U)$,*

$$D[u] \geq D[v] \quad \Longrightarrow \quad D(uS) \geq D(vS).$$

Equality of the ambient curvatures gives equality of the displayed relative curvatures. The same replacement is valid in every differentiated slot of a mixed product whose other slots are fixed.

Proof. The actual difference $h = u - v$ has positive curvature, hence is classical psh by theorem 4.1. The first-order construction gives $D(hS) \geq 0$, and linearity identifies this with $D(uS) - D(vS)$. For equality, both h and $-h$ give positive relative curvatures; pointedness makes them zero. To change one mixed slot, use symmetry to put it first. The product of the remaining slots with S is one fixed positive separately closed background, to which the first-order assertion applies. □

This proves the last assertion of theorem 1.1. It also constructs products of curvature currents having bounded classical local potentials.

Corollary 4.4 (Gluing local-potential currents)**.** *Suppose each α_i is a $(1,1)$-current locally of the form $D[a_i]$ with $a_i \in \mathcal{C}_b$. For $k \le n-p$ there is a unique positive separately closed current $\alpha_1 \wedge \cdots \wedge \alpha_k \wedge S$ whose expression on such a neighborhood is $T_{S,k}(a_1, \ldots, a_k)$. It is symmetric, local, additive and positively homogeneous in these current slots.*

Proof. On an overlap, corresponding potentials have the same ambient curvature. Replace the slots one at a time using theorem 4.3. The local currents therefore agree on all original tests and glue by the sheaf property of currents. Their remaining properties follow locally from the bounded calculus. □

Corollary 4.5 (Continuous curvature domination)**.** *Let $u \in \mathcal{C}_b(U)$, let g be continuous classical psh, and suppose $0 \le D[u] \le CD[g]$ for $C \ge 0$. Then u and $Cg - u$ are continuous classical psh functions.*

Proof. The bounded classical difference $Cg - u$ has positive curvature, so it is upper semicontinuous and classical. The identity $u = Cg - (Cg - u)$ makes u lower semicontinuous. It is already upper semicontinuous, hence continuous, and the same follows for $Cg - u$. □

An undifferentiated weight retains its actual value. For example, $D[0] = D[1]$, while $0S = 0$ and $1S = S$. The quotient constructed here is therefore precisely the quotient in the differentiated slots.

5. Continuous coefficients and mixed curvature operations

Theorem 5.1 (Transfer of continuous coefficient relations)**.** *Let $g_1, \ldots, g_m \in \mathcal{C}_b(U)$ and $f_1, \ldots, f_m \in C(U, \mathbf{R})$. For a fixed background S, put*

$$A = \sum_i f_i D[g_i], \qquad A_S = \sum_i f_i D(g_i S).$$

Then $A \ge 0$ implies $A_S \ge 0$. In particular, $A = 0$ implies $A_S = 0$ on every original complementary test.

Proof. Set $T = \sum_i D(g_i S) \ge 0$. For $\epsilon > 0$, work near a point on a neighborhood where $|f_i - c_i| < \epsilon$ for suitable real constants. There,

$$D\left[\sum_i (c_i + \epsilon) g_i\right] = A + \sum_i (c_i + \epsilon - f_i) D[g_i] \ge 0.$$

The actual function inside brackets is a bounded classical difference, as is seen by separating its positive and negative real coefficients. By theorem 4.1 it is classical psh. Its curvature against S is positive, so $\sum_i c_i D(g_i S) \ge -\epsilon T$. The coefficient error gives $A_S \ge -2\epsilon T$ on this neighborhood.

Cover the support of a positive original test by finitely many such neighborhoods and use a nonnegative smooth partition. Summing gives the same inequality on that test. Its control mass against T is finite, so $\epsilon \downarrow 0$ proves $A_S \ge 0$. If $A = 0$, apply the argument to both coefficient lists (f_i) and $(-f_i)$ and use pointedness of the positive cone. □

We now prove the finite-presentation part of theorem 1.2.

Proof of the continuous coefficient operation. Each expression in (1.4) is a finite sum of continuous multiples of positive coefficient Radon currents. If a presentation of one slot changes without changing its actual current, fix a multi-index in the other slots and set

$$R = T_{S,r-1}(g_{1,\alpha_1}, \ldots, \widehat{g_{i,\alpha_i}}, \ldots, g_{r,\alpha_r}).$$

This is a fixed positive separately closed background. The zero-relation case of theorem 5.1 makes the change in that slot zero. Multiply by the remaining continuous coefficients and sum. Changing one slot at a time proves independence of all presentations. Symmetry, multilinearity and ordinary locality follow from the bounded products.

For positivity, fix a positive compact complementary test and localize to one finite presentation. Its coefficients are bounded by $M \geq 1$ on a slightly larger compact set. Subdivide the test support into ordinary neighborhoods on which $|f_{i,\alpha} - c_{i,\alpha}| < \epsilon$, with $0 < \epsilon < 1$. On one such neighborhood put

$$H_i = \sum_\alpha (c_{i,\alpha} + \epsilon) g_{i,\alpha}.$$

If $A_i \geq 0$, then

$$D[H_i] = A_i + \sum_\alpha (c_{i,\alpha} + \epsilon - f_{i,\alpha}) D[g_{i,\alpha}] \geq 0.$$

Thus $H_i \in \mathcal{C}_b$ by theorem 4.1, and $T_{S,r}(H_1, \dots, H_r) \geq 0$.

For each multi-index, telescope the product of its r scalar coefficients. The difference between the coefficient in (1.4) and that in the last positive product is at most $2r\epsilon(M+2)^{r-1}$ in absolute value. Let R_0 be the finite sum of all the positive BT tails in that presentation, with their scalar coefficients omitted. Hence

$$\Phi_{S,r}(A_1, \dots, A_r) \geq -2r\epsilon(M+2)^{r-1} R_0$$

on each member of the subdivision. A nonnegative partition and the fixed finite control mass pass this inequality to the original test. Letting ϵ decrease proves positivity. Telescoping in the slots then proves monotonicity when the remaining slots are positive. □

Example 5.2 (A Borel coefficient can destroy a relation). On the bidannulus $U = \{r < |T_1| < R,\ r < |T_2| < R\}$, where $0 < r < 1 < R$, take $Y = \{T_2 = 1\} \cap U$ and $g = \max(0, \log|T_1|)$. The first-order analytic-divisor nonconcentration theorem gives $\mathbf{1}_Y D[g] = 0$ as an ambient coefficient Radon current. The subvariety formula and the slope jump on the T_1-annulus give

$$\mathbf{1}_Y D(g[Y]) = D(g[Y]) = i_* \delta_{\zeta_{0,1}} \neq 0, \qquad i : Y \hookrightarrow U.$$

Here $\zeta_{0,1}$ is the radius-one Gauss point; the slope jump is one. A compact nonnegative original scalar test equal to one near that point has integral one. This is the obstruction to Borel coefficient transfer described in [Order, Section *Products in the continuous coefficient curvature module*].

Continuous variable coefficients also affect closedness. The operation is an operation on coefficient currents; an output serves as a new background when its separate closedness has been established.

6. Order completion inside the original currents

The continuous coefficient operation has a useful continuity stronger than distributional continuity. A fixed positive control measures the size of a signed current and, simultaneously, the error of all its background actions. We follow the intrinsic order construction of [OrdComp].

Lemma 6.1 (Signed products). *Let $Q_i \in \mathscr{M}^1$ be positive and suppose $-M_i Q_i \leq A_i \leq M_i Q_i$, where $M_i \geq 0$. Then*

$$-\Big(\prod_i M_i\Big) R_S \leq \Phi_{S,r}(A_1, \dots, A_r) \leq \Big(\prod_i M_i\Big) R_S, \qquad R_S = \Phi_{S,r}(Q_1, \dots, Q_r). \tag{6.1}$$

The difference estimate (1.5) also holds.

Proof. Set $A_i^\pm = (M_i Q_i \pm A_i)/2$. These are positive, $A_i = A_i^+ - A_i^-$, and $A_i^+ + A_i^- = M_i Q_i$. Expand the mixed product into a signed sum of positive currents. The unsigned sum is $\Phi_{S,r}(M_1 Q_1, \dots, M_r Q_r)$. This unsigned sum plus or minus the signed sum is twice a positive subsum, proving (6.1). Telescope the difference between two mixed products one slot at a time and apply this inequality to each term. Addition gives (1.5). □

For one positive $Q \in \mathscr{M}^1(U)$, the space E_Q and its gauge were defined in the introduction. Every finite presentation belongs locally to some such space: bound its coefficients and sum its positive curvature generators.

Theorem 6.2 (Faithful realization)**.** *The completion $\widehat{E}_Q(U)$ embeds isometrically into the actual symmetric coefficient Radon $(1,1)$-currents. Its image is the closure of $E_Q(U)$ for the ambient order norm defined by Q. If an actual current is obtained using controls Q and R, its two realizations agree in the completion controlled by $Q+R$.*

Proof. If $\|A\|_Q = 0$, then $-\epsilon Q \le A \le \epsilon Q$ for every $\epsilon > 0$. The finite test control makes every pairing of A zero. Thus the gauge is a norm. Addition of order bounds gives its triangle inequality. Weak closedness of the positive cone shows that the infimum in a finite gauge is attained.

Let (A_j) be order Cauchy. On each fixed original compact test its values are Cauchy by lemma 2.1. The finite coefficient bound makes their limit a distribution and a coefficient Radon current on every compact presentation. Symmetry and compatibility with all relations and refinements pass to this limit, denoted A. If $\|A_j - A_l\|_Q \le \epsilon$ for $j, l \ge N$, let l tend to infinity in both order inequalities. Weak closedness gives

$$-\epsilon Q \le A_j - A \le \epsilon Q \qquad (j \ge N). \tag{6.2}$$

The convergence is therefore convergence in the ambient order norm. If $A = 0$, the same inequality shows that the original sequence represents zero in the completion. This proves injectivity, the isometry, and the description of the image.

Since $Q, R \le Q + R$, both completion sequences are Cauchy for that larger control. Their actual limits agree; injectivity for $Q + R$ identifies the two completion classes. □

An actual symmetric coefficient current belongs to $\mathscr{M}^1_{\mathrm{ord}}$ when it belongs locally to one of these embedded completions. This is a sheaf by the local definition and the sheaf property of currents. Local addition uses $Q + R$, and multiplication by a continuous function uses its bound on a smaller neighborhood.

Theorem 6.3 (Extension of the mixed operation)**.** *The maps $\Phi_{S,r}$ extend uniquely to symmetric locally $C(U, \mathbf{R})$-multilinear operations on $\mathscr{M}^1_{\mathrm{ord}}$, continuous for every common finite family of local order controls. They are positive, local, compatible with gluing, and satisfy the signed and difference estimates. Their supports satisfy*

$$\operatorname{supp} \Phi_{S,r}(A_1, \dots, A_r) \subset \operatorname{supp} S \cap \bigcap_i \operatorname{supp} A_i. \tag{6.3}$$

If the mixed products along an order-convergent approximation are separately closed, their limit is separately closed.

Proof. Choose local controls Q_i and Cauchy approximants $A_{i,j}$. Their order norms are bounded. The difference estimate and finite test control make the mixed products Cauchy in coefficient measure on every fixed compact presentation. Their limit is therefore a coefficient Radon current. The same estimate applied to two approximating families proves independence. For different controls R_i, pass to $Q_i + R_i$ and use theorem 6.2; the resulting limits agree.

Algebraic identities pass to the limit. If $A_i \ge 0$, the inequality (6.2) gives positive approximants $A_{i,j} + \epsilon_{i,j} Q_i$, with $\epsilon_{i,j} \to 0$. These have the same limits, so positivity passes from the finite-presentation operation. The order estimates also pass by weak closedness.

Restriction is contractive for the restricted controls, so these local constructions agree on intersections and glue. On an open where one slot is zero, its completion class is zero by faithful realization, and multilinearity makes the product zero. The same holds where $S = 0$, proving (6.3). Finally, distributional continuity of d' and d'' proves the closedness assertion. □

This completes theorem 1.2. The numerical coefficient in the order estimate is independent of S; the controlling trace mass is evaluated against the particular background and test.

Corollary 6.4 (Fixed-generator approximation by curvatures)**.** *Suppose locally*

$$A = \sum_{i=1}^{N} f_i D[g_i], \qquad A_j = \sum_{i=1}^{N} f_{i,j} D[g_i] = D[v_j],$$

where g_i and v_j are finite continuous local classical psh differences, $f_i, f_{i,j}$ are continuous, and $f_{i,j} \to f_i$ locally uniformly. Then $\Phi_{S,1}(A)$ is separately closed for every allowed background S.

Proof. Write $g_i = g_i^+ - g_i^-$ and choose the fixed control $Q = \sum_i (D[g_i^+] + D[g_i^-])$. On a smaller neighborhood the error is bounded between $\pm\epsilon_j Q$, where $\epsilon_j = \max_i \|f_i - f_{i,j}\|_\infty \to 0$. Thus the actions converge on all original tests. Continuous relation transfer identifies $\Phi_{S,1}(A_j) = D(v_j S)$, a separately closed current. Both first derivatives pass to the limit. This argument uses the coefficient convergence and requires no bound or normalization on v_j. □

Proposition 6.5 (A support-conormal certificate)**.** *Let $A_i = D[g_i]$ for continuous classical psh differences, and set $T_i = D(g_i S)$ and $F_i = \operatorname{supp} A_i$. Then $\operatorname{supp} T_i \subset F_i$. If a smooth function h vanishes on F_i, then $d'h \wedge T_i = d''h \wedge T_i = 0$.*

Suppose smooth covectors admit an identity

$$\theta_i = \sum_a f_{ai}\eta_a + \sum_j b_{ij} d' h_{ij}, \qquad \sum_i f_{ai} A_i = 0, \qquad h_{ij}|_{F_i} = 0,$$

with continuous scalar coefficients and original smooth η_a, h_{ij}. Then $\sum_i \theta_i \wedge T_i = 0$. The same assertion holds after interchanging the two differential columns.

Proof. On the complement of F_i, the zero curvature relation transfers to $T_i = 0$. This proves the support assertion. Hence $hT_i = 0$; separate closedness and the smooth product rule give the conormal identities. Each scalar relation transfers to $\sum_i f_{ai} T_i = 0$. Multiply by η_a, apply the conormal identity to the remaining terms, and sum. □

7. Relative power energies on Laurent domains

7.1. **Truncation, capacity and the boundary class.** We now impose the strict Laurent-domain and background-extension hypotheses (1.6). All capacities use the entire normalized classical tester class on the indicated domain. Its ordinary open outer regularization is denoted C_S^*. Local truncated capacity convergence means that, for every ordinary $\Omega \subset V$, compact $L \Subset \Omega$, $R > 0$ and $\delta > 0$,

$$C^*_{S|\Omega}(L \cap \{|\max(u_j, -R) - \max(u, -R)| > \delta\}; \Omega) \to 0.$$

Let $u_1, \dots, u_m \le 0$ be extended classical potentials, with their actual values. Put $u_i^{(j)} = \max(u_i, -j)$ and $E_j = \bigcap_i \{u_i > -j\}$. The restrictions

$$\mathbf{1}_{E_j} T_{S,k}(u_1^{(j)}, \dots, u_k^{(j)}) \tag{7.1}$$

are consistent on their common finite-value regions. Their increasing positive coefficient limits define the nonpolar product whenever those limits are locally finite. For a single potential write $\mu_{u,j} = T_{S,q}((u^{(j)})^{[q]})$, $\nu_{u,j} = \mathbf{1}_{\{u \le -j\}}\mu_{u,j}$, and μ_u for the nonpolar top measure.

Definition 7.1 (The actual boundary power classes)**.** The boundary condition $\mathrm{BG}(u, 0)$ means $\overline{\{u < -t\}} \Subset V$ for every $t > 0$. The class $\mathcal{F}_{S,\partial}(V)$ consists of negative extended classical potentials satisfying this condition and $\nu_{u,j}(V) \to 0$. For $\sigma > 0$, set

$$\mathcal{E}^\sigma_{S,\partial}(V) = \left\{ u \in \mathcal{F}_{S,\partial}(V) : e_\sigma(u) = \int_V (-u)^\sigma \, d\mu_u < \infty \right\}.$$

The integral is over the finite-value locus, to which μ_u is carried. Write $\mathcal{E}^1 = \mathcal{E}^1_{S,\partial}(V)$.

The following analytic results will be used with their displayed domain. Their proofs in [Energy] supply the coefficient cutoffs, capacity comparison, and tail bounds needed for the later completions.

Theorem 7.2 (Analytic truncation estimates)**.** *The following statements hold.*

(i) *A locally coefficient-finite product (7.1) is separately closed on all original tests. This is an ordinary-local assertion. Under* $\mathrm{BG}(u,0)$*,* μ_u *is locally finite and*

$$u \in \mathcal{F}_{S,\partial}(V) \quad \Longleftrightarrow \quad t^q C_S(\{u < -t\}; V) \to 0.$$

(ii) *For* $\sigma > 0$ *and* u *satisfying* $\mathrm{BG}(u,0)$*, put*

$$J_\sigma(u) = \int_0^\infty t^{q+\sigma-1} C_S(\{u < -t\}; V)\, dt.$$

Then $J_\sigma(u) < \infty$ *exactly when* $u \in \mathcal{E}^\sigma_{S,\partial}(V)$*, and*

$$\frac{\sigma}{2^{q+\sigma}} J_\sigma(u) \le e_\sigma(u) \le \frac{\sigma 2^q}{1-2^{-\sigma}} J_\sigma(u). \tag{7.2}$$

Raw and outer capacities agree on these ordinary open sublevels.

(iii) *For a finite tuple* $\mathbf{u}$ *in this power class and a multi-index* γ *with* $k = |\gamma| \le q$*, its unbounded mixed curvature* N_γ *is defined, positive and separately closed. For* $k \le q-1$*,*

$$D(u_i N_\gamma) = N_{\gamma+e_i}, \qquad \mathcal{J}'_{\gamma,i} = d'(u_i N_\gamma), \quad \mathcal{J}''_{\gamma,i} = d''(u_i N_\gamma). \tag{7.3}$$

The first-order currents are of coefficient Radon type. When $k+2 \le q+\sigma$*, the energy matrix is also of coefficient Radon type and is given by*

$$G^\gamma_{ih} = \tfrac{1}{2}\left[D(u_i u_h N_\gamma) - u_i N_{\gamma+e_h} - u_h N_{\gamma+e_i}\right]. \tag{7.4}$$

It is positive semidefinite in the middle cone.

(iv) *Put* $\kappa = q+\sigma$*,* $w = \sum_i u_i$*,* $W = 1-w$*, and* $J_w(R) = \int_R^\infty t^{\kappa-1} C_S(\{w < -t\}; V)\, dt$*. For a curvature, first-order or energy coefficient current* P*, assign the degree budget* $b = k, k+1, k+2$*, respectively. If* $a+b \le \kappa$*, the actual and canonically truncated coefficients have the uniform tail bound*

$$\int_{\{w \le -R\}} W^a\, |d\mu_{P,\eta}| \le C_\eta R^{a+b-\kappa} J_w(R/4), \qquad R \ge 1, \tag{7.5}$$

with the corresponding bound for every bounded truncation. Canonical truncations converge in local coefficient total variation, including continuous tuple multipliers of growth at most W^a*.*

(v) *Bounded actual Borel capacity limits of classical functions in* $[-R,0]$ *have an intrinsic bounded calculus, including first-order and energy currents. Its degree-b coefficients satisfy*

$$|\mu_{P_R,\eta}|(E) \le C_\eta R^b C^*_S(E; V). \tag{7.6}$$

These currents are independent of defining rows and outer-null changes, and bounded continuous tuple weights pass under capacity convergence.

(vi) *There is a bounded continuous classical barrier*

$$\rho = \max\left(-1, \max_a \log\frac{|f_a|}{R_a}, \max_b \log\frac{r_b}{|g_b|}\right) \tag{7.7}$$

on a neighborhood of K_0*, with* $-1 \le \rho < 0$ *on* V*, compact negative sublevels, and* $\mu_\rho(V) < \infty$*. For* $u \in \mathcal{E}^\sigma$*, the bounded finite-mass potentials* $\max(u, j\rho)$ *decrease to* u*. Fixed lower-barrier families have critical weighted convergence, including global mixed-energy convergence in first energy.*

The individual restricted stages in (7.1) may have boundary derivatives; the closedness theorem concerns their locally finite limit. The exact sources of theorem 7.2 are [Energy, Sections 8, 12, 15–20, 31]. In particular, the first-order estimate is (FO8), the curvature and energy bounds are (PG3) and (PG12), and the bounded Borel construction is the lemma *Analytic bounded capacitary calculus.* The boundary comparison and all-degree capacity estimates also give the following consequences used below: equality of two no-loss top measures implies equality of their potentials modulo local outer capacity; the actual power classes are closed under finite sums and maxima; and capacity convergence of uniformly energy-bounded

tuples passes bounded weighted products below the critical degree. These are proved in [Energy, Sections 12, 14–18, 28].

7.2. **Global Green symmetry and the mixed energy inequality.** For bounded classical a, b and a closed background Q of degree $(n-1, n-1)$, write $\Gamma_Q(a,b) = \frac{1}{2}[D(abQ) - aD(bQ) - bD(aQ)]$. The bounded energy identity and its positive energy matrix are those of the relative first-order calculus [Rel]; the same definition is justified for the unbounded products by (7.4).

Lemma 7.3 (Global bounded Green exchange)**.** *Let $a, b \le 0$ be bounded classical potentials with compact negative sublevel closures. If $D(aQ)$ and $D(bQ)$ have finite total mass, then*

$$\int_V (-a) D(bQ) = \int_V (-b) D(aQ) = \int_V \Gamma_Q(a,b).$$

The diagonal energy is finite, and the cross energy has finite total variation.

Proof. Put $a^\epsilon = \max(a, -\epsilon)$. The difference $a - a^\epsilon$ is supported in $K_\epsilon = \overline{\{a < -\epsilon\}} \Subset V$. Compact Green exchange gives

$$\int (a - a^\epsilon) D(bQ) = \int b D(aQ) - \int b D(a^\epsilon Q).$$

The measures $D(a^\epsilon Q)$ and $D(aQ)$ have the same finite total mass: their difference is the derivative of a compactly supported current, which pairs to zero with a cutoff equal to one near that support. For fixed $\delta > 0$, relative CLN on $L_\delta = \overline{\{b < -\delta\}}$ gives $D(a^\epsilon Q)(L_\delta) \le C_{\delta,Q}\epsilon$. Therefore

$$\left| \int b D(a^\epsilon Q) \right| \le \|b\|_\infty C_{\delta,Q}\epsilon + \delta D(aQ)(V).$$

Let $\epsilon \downarrow 0$ and then $\delta \downarrow 0$ to obtain the first equality.

The bounded clipping rule and the zero energy of an exact scalar level give

$$\int_{\{a<-\epsilon\}} \Gamma_Q(a) = \int (a^\epsilon - a) D(aQ).$$

Monotone convergence identifies its limit with $\int (-a) D(aQ) < \infty$. Apply the diagonal identity also to b and $a + b$ and polarize. Measure Cauchy–Schwarz bounds the total variation of the cross energy by the square root of the two diagonal masses. □

Theorem 7.4 (First mixed energy)**.** *For $u_0, \dots, u_q \in \mathcal{E}^1$, the finite functional*

$$B(u_0, \dots, u_q) = \int_V (-u_0) N_q(u_1, \dots, u_q)$$

is symmetric and separately additive and positively homogeneous. For $Q = N_{q-1}(c_2, \dots, c_q)$,

$$\int_V \Gamma_Q(a,b) = B(a, b, c_2, \dots, c_q), \qquad |B(a,b,\mathbf{c})|^2 \le B(a,a,\mathbf{c}) B(b,b,\mathbf{c}). \tag{7.8}$$

The energy matrix stays positive on finite real linear combinations of the potentials, and all cross energies have finite total variation. Moreover,

$$B(u_0, \dots, u_q) \le \prod_{i=0}^{q} e(u_i)^{1/d}, \qquad e(u+v)^{1/d} \le e(u)^{1/d} + e(v)^{1/d}. \tag{7.9}$$

The constant one is optimal whenever the class contains a positive energy potential.

Proof. **Step 1: exchange the undifferentiated slot.** Use $b_{i,j} = \max(u_i, j\rho)$. For fixed j these potentials are bounded and their sum is bounded below by $(q+1)j\rho$. Boundary comparison with $c(q+1)j\rho$, followed by $c \downarrow 1$, gives

$$\mu_{\sum_i b_{i,j}}(V) \le [(q+1)j]^q \mu_\rho(V) < \infty.$$

The positive mixed expansion makes each top mixed measure finite. Apply lemma 7.3 after fixing the last $q-1$ slots. Together with differentiated-slot symmetry this proves full bounded mixed symmetry. The fixed lower-barrier convergence in theorem 7.2 passes the identity globally

to the u_i. Finiteness is the mixed power-energy estimate in [Energy, (PE11)]. Additivity follows from integration and the unbounded recursion.

Step 2: identify the global energy measures. For $a^\epsilon = \max(a, -\epsilon)$, apply compact Green to the canonical bounded difference $a^{(j)} - a^\epsilon$ and to the bounded background Q_j. The clipping rule gives

$$\int_{\{a<-\epsilon\}} \Gamma_{Q_j}(a^{(j)}) = \int (a^\epsilon - a^{(j)}) D(a^{(j)} Q_j).$$

The support lies in a fixed compact set. Critical local total-variation convergence from theorem 7.2 passes the energy side; the degree-one weighted curvature convergence passes the other side. Thus the same identity holds for a, Q. Let $\epsilon \downarrow 0$. The energy is zero on the pole and the scalar level $a = 0$, and the right side tends to its finite mixed energy. This proves the diagonal identity. Polarization and positive energy matrices give (7.8) and the variation bound.

Step 3: prove the sharp integral inequality. The zero-factor case follows from the mixed power-energy estimate. Otherwise normalize $e(u_i) = 1$. Among the finitely many mixed values $B_\alpha = B(u_0^{[\alpha_0]}, \dots, u_q^{[\alpha_q]})$, $\sum \alpha_i = d$, choose a largest value M, and among its indices maximize $\sum \alpha_i^2$. If two entries α_i, α_l are positive, energy Cauchy–Schwarz gives

$$M^2 \le B_{\alpha + e_i - e_l} B_{\alpha - e_i + e_l} \le M^2.$$

Since a diagonal value is one, $M > 0$, and both neighboring values equal M. At least one neighboring index has larger sum of squares, a contradiction. Thus a maximizing index is pure and $M = 1$. Rescaling proves the first inequality in (7.9). Expand $e(u + v)$ and use the binomial theorem for the second. Equality with all slots equal proves optimality. □

7.3. **Discrepancy and quantitative tests.** Set $\mathscr{A}(u) = -e(u)/d$. Symmetric multilinearity and global Green give, for $h = v - u$ and $u_t = (1 - t)u + tv$,

$$\frac{d}{dt}\mathscr{A}(u_t) = \int h \, d\mu_{u_t}, \qquad \frac{d^2}{dt^2}\mathscr{A}(u_t) = -q \int \Gamma_{N(u_t^{[q-1]})}(h) \le 0. \tag{7.10}$$

In particular

$$I(u, v) = e(u) + e(v) - B(u, v^{[q]}) - B(v, u^{[q]}) = \sum_{l=0}^{q-1} \int \Gamma_{N(u^{[l]}, v^{[q-1-l]})}(u - v). \tag{7.11}$$

It is nonnegative and vanishes precisely on equal relative capacity classes. To see the latter assertion, zero discrepancy makes each diagonal energy in the sum zero. Compact Green and Cauchy–Schwarz, passed through canonical truncations, then give $D((u-v)N(u^{[l]}, v^{[q-1-l]})) = 0$ on every test. Summing the telescoping identity gives $\mu_u = \mu_v$. The boundary comparison consequence of theorem 7.2 identifies their relative classes. The reverse follows from representative independence.

Theorem 7.5 (Uniform energy tests)**.** *Put $\alpha = 2^{-q}$. For $u, v, \psi \in \mathcal{E}^1$ and $M = \max\{e(u + v), e(\psi)\}$,*

$$\int_V |u - v| \, d\mu_\psi \le 2^{(q-1)\alpha} I(u, v)^\alpha M^{1-\alpha}. \tag{7.12}$$

For a mixed top measure, replace ψ by the sum of its q potentials and divide the bound by $q!$. At $M = 0$ the integral is zero.

Proof. Let $h = u - v$, $w = u + v$, and $J_k = \int \Gamma_{N(\psi^{[k]}, w^{[q-1-k]})}(h)$ for $0 \le k \le q - 1$. Positive expansion gives $J_0 \le 2^{q-1} I(u, v)$. Exchange one ψ slot with one h slot by global symmetry and expand the latter as $u - v$. Cauchy–Schwarz in the two positive backgrounds, then in their two scalar terms, gives

$$J_{k+1} \le \{J_k B(\psi^{[k+2]}, w^{[q-1-k]})\}^{1/2} \le (J_k M)^{1/2}.$$

Hence $J_{q-1} \le J_0^{2^{-(q-1)}} M^{1-2^{-(q-1)}}$. The bounded absolute-value chain, the zero energy of an exact scalar level, and critical total-variation passage give $\Gamma_Q(|h|) = \Gamma_Q(h)$ in these backgrounds. Global Green gives

$$\int |h|\, d\mu_\psi = -\int \Gamma_{N(\psi^{[q-1]})}(|h|, \psi) \le J_{q-1}^{1/2} e(\psi)^{1/2}.$$

Combine the inequalities. The zero case follows from mixed energy finiteness and (7.9). Positive expansion $q!N(\psi_1, \dots, \psi_q) \le \mu_{\sum \psi_i}$ proves the last assertion. □

We record the remaining quantitative consequences needed for the completion. They are proved in [Energy, Section *Quantitative first-energy convergence in every local capacity*]. For an energy bound $e(u), e(v) \le M_0$ and every compact/domain pair $L \Subset \Omega \subset V$,

$$C^*_{S|\Omega}(L \cap \{|u - v| > \delta\}; \Omega) \le C_{L,\Omega,M_0,S} \delta^{-1} I(u, v)^\alpha, \tag{7.13}$$

$$|e(u)^{1/d} - e(v)^{1/d}|^d \le I(u, v), \tag{7.14}$$

$$I(u, u^{(R)}) \le \int (u^{(R)} - u)\, d\mu_u \longrightarrow 0, \tag{7.15}$$

$$\int_R^\infty t^q C_S(\{u < -t\}; V)\, dt \le 2^d \int (-u - R/2)^+\, d\mu_u. \tag{7.16}$$

The capacity estimate uses a normalized tester a replaced near L by $\max(a-2, A_L\rho)$. Its energy is uniformly bounded by the finite mass of ρ, so (7.12) applies. Outer quasicontinuity and the analytic domain comparison give all local outer capacities. For (7.14), write $a = e(u)^{1/d}$, $b = e(v)^{1/d}$ and use mixed Hölder to bound $I(u, v)$ below by $(a - b)(a^{d-1} - b^{d-1}) \ge |a - b|^d$. The clipping bound follows from the definition of I and positivity; dominated convergence passes its right side. The tail estimate is the tail form of (7.2).

Proposition 7.6 (Sharp midpoint bound)**.** *For $u, v \in \mathcal{E}^1$,*

$$\mathscr{A}\left(\frac{u+v}{2}\right) - \frac{\mathscr{A}(u) + \mathscr{A}(v)}{2} \ge c_q I(u, v), \qquad c_q = \frac{1 - 2^{-q}}{2(q+1)}. \tag{7.17}$$

This constant is optimal if a positive-energy potential exists.

Proof. The scalar Green formula on $[0, 1]$ and (7.10) express the left side as $\sum_{l=0}^{q-1} a_{q,l}\gamma_l$, where

$$\gamma_l = \int \Gamma_{N(u^{[l]}, v^{[q-1-l]})}(u - v) \ge 0, \quad a_{q,l} = \frac{q}{2}\binom{q-1}{l} \int_0^1 \min(t, 1-t) t^l (1-t)^{q-1-l}\, dt.$$

Their sum $\sum_l \gamma_l$ is $I(u, v)$. We verify $a_{q,l} \ge c_q$. Put $p_i(s) = \binom{q}{i} s^i (1-s)^{q-i}$, $B_k(s) = \sum_{i=k}^q p_i(s)$, and $b_k(t) = q\binom{q-1}{k-1} t^{k-1}(1-t)^{q-k}$. Differentiation gives $B_k' = b_k$. For $0 \le s \le 1/2$,

$$P_k(s) := \int_s^{1-s} b_k(t)\, dt = 1 - B_k(s) - B_{q-k+1}(s).$$

If $k \le (q+1)/2$, then $P_k(s) - P_1(s) = \sum_{i=1}^{k-1} (p_i(s) - p_{q-i}(s)) \ge 0$, since $(s/(1-s))^{2i-q} \ge 1$; the remaining cases follow from $P_k = P_{q+1-k}$. Layer cake for the tent function gives $a_{q,k-1} = \frac{1}{2}\int_0^{1/2} P_k(s)\, ds \ge a_{q,0}$. The endpoint integral is $a_{q,0} = c_q$, proving the bound. For $v = 0$ and $e(u) > 0$, homogeneity gives $I(u, 0) = e(u)$ and equality in (7.17). □

8. The complete relative first-energy space

Retain the domain and background of (1.6). Put $d = q + 1$, $\alpha = 2^{-q}$ and $\mathcal{E}^1 = \mathcal{E}^1_{S,\partial}(V)$. We construct actual Borel classes from the strong Cauchy condition and then define their currents through bounded truncations.

8.1. **Uniform tails and the boundary of an energy-Cauchy row.** Call an actual row $(u_j) \subset \mathcal{E}^1$ strongly Cauchy if

$$\lim_{N\to\infty} \sup_{j,l\geq N} I(u_j, u_l) = 0. \tag{8.1}$$

The Cauchy condition is stated directly in terms of this discrepancy.

Lemma 8.1 (Uniform deep tails and global weighted tightness)**.** *Every strongly Cauchy actual row has bounded first energies and*

$$\begin{aligned} \lim_{R\to\infty} \sup_j \int_{\{u_j<-R\}} (-u_j)\, d\mu_{u_j} = 0, \\ \lim_{R\to\infty} \sup_j \int_R^\infty t^q C_S(\{u_j < -t\}; V)\, dt = 0. \end{aligned} \tag{8.2}$$

For finitely many strongly Cauchy actual rows $u_{i,j}$, $0 \leq i \leq q$, the finite positive Radon measures

$$(-u_{0,j}) M(u_{1,j}, \dots, u_{q,j}) \tag{8.3}$$

are uniformly tight on V. The same conclusions hold if the indices of the finitely many rows tend independently to infinity.

Proof. The energy-root estimate (7.14) bounds the energies by some M. The case $M = 0$ is immediate from the constant-one mixed Hölder inequality; otherwise enlarge M to be positive. On $\{-u_j > R\}$ split according to whether $-u_N \leq (-u_j)/2$. The pointwise splitting used below gives

$$\int_{\{u_j<-R\}} (-u_j)\, d\mu_{u_j} \leq 2 \int |u_j - u_N|\, d\mu_{u_j} + 4 \int |u_N - u_N^{(R/4)}|\, d\mu_{u_j}, \qquad u^{(L)} = \max(u, -L). \tag{8.4}$$

By (7.12), the first integral is at most $C_{q,M} I(u_j, u_N)^\alpha$. For fixed N, the second is at most $C_{q,M} I(u_N, u_N^{(R/4)})^\alpha$, uniformly in j. The latter discrepancy tends to zero by (7.15). Choose N large using (8.1), then let R grow, and finally make the Cauchy error tend to zero. The finitely many indices $j < N$ have vanishing tails because their first energies are finite. This proves the first line of (8.2). The tail form of (7.16),

$$\int_R^\infty t^q C_S(\{u_j < -t\}; V)\, dt \leq 2^{q+1} \int (-u_j - R/2)^+\, d\mu_{u_j}, \tag{8.5}$$

proves the second. For a finite sum of rows, the sublevel union bound gives the same uniform tail conclusion, with the factor $(\#\text{slots})^{q+1}$ and a fixed rescaling of R.

For (8.3), all slot energies are uniformly bounded. Write $M_j = M(u_{1,j}, \dots, u_{q,j})$. Positivity of the expansion of $\mu_{\sum_{i=1}^q u_{i,j}}$ and (7.12) give

$$\int |u_{0,j} - u_{0,N}|\, dM_j \leq C I(u_{0,j}, u_{0,N})^\alpha, \tag{8.6}$$

where C depends on the uniform energies and q, but not on the derivative-row indices. Fix a sufficiently late N. For $\epsilon > 0$, the set $K_\epsilon = \overline{\{u_{0,N} < -\epsilon\}}$ is compact. On its complement $u_{0,N} = u_{0,N}^{(\epsilon)}$, and hence

$$\begin{aligned} \int_{V\setminus K_\epsilon} (-u_{0,j})\, dM_j &\leq \int |u_{0,j} - u_{0,N}|\, dM_j + \int (-u_{0,N}^{(\epsilon)})\, dM_j \\ &\leq C I(u_{0,j}, u_{0,N})^\alpha + e_1(u_{0,N}^{(\epsilon)})^{1/d} \prod_{i=1}^q e_1(u_{i,j})^{1/d}. \end{aligned} \tag{8.7}$$

The last line is the global constant-one Hölder inequality. The small-value energy vanishes:

$$e_1(u_{0,N}^{(\epsilon)}) \leq C_q \int_0^\epsilon t^q C_S(\{u_{0,N} < -t\}; V)\, dt \longrightarrow 0, \tag{8.8}$$

by the power-energy/capacity equivalence. First make the Cauchy error small, then choose ϵ small. The finitely many omitted weights $u_{0,j}$, with derivative rows still varying, are handled

by the same argument with zero first term: outside their own compact negative sublevel, the Hölder bound with $u_{0,j}^{(\epsilon)}$ tends uniformly to zero. A finite union of these compacts proves uniform tightness. This observation also proves the assertion for independent row indices, without taking a diagonal assumption on any derivative slot. □

8.2. **The Borel class and its intrinsic original currents.**

Definition 8.2 (Strong relative first-energy completion)**.** Let $\widehat{\mathcal{E}}^1_{\mathrm{str}}(V)$ consist of actual Borel functions $u : V \to [-\infty, 0]$ which are limits of an actual strongly Cauchy row in every local truncated outer S-capacity. Equality means local outer-capacity equality. The capacities may equivalently be taken on all compact cores in the fixed domain V, by the analytic capacity-domain comparison in theorem 7.2. A choice of actual defining row is not part of the element of this quotient.

Proposition 8.3 (Borel limits of all strongly Cauchy actual rows)**.** *Every row satisfying (8.1) has a limit in definition 8.2, unique modulo local outer-capacity zero. Every member of that space is finite and continuous locally outside ordinary open sets of arbitrarily small outer capacity. Its pole set is locally outer-capacity zero.*

Proof. Uniform energy and (7.13) make each fixed bounded truncation of the row Cauchy in every local outer capacity. Apply the countable compact-core and integer-level diagonal construction from the bounded completion. A subsequence has compatible pointwise limits of all bounded truncations off a locally outer-null set. Its actual pointwise limsup defines a Borel $u \le 0$ on all of V, and the Cauchy property gives the required capacity convergence for the full row.

There is a uniform deep sublevel estimate. The no-loss sublevel inequality and the first-energy bound imply

$$C_S(\{u_j < -R\}; V) \le C_q M R^{-d}. \tag{8.9}$$

These sets are ordinary open, so the same bound holds for their outer capacities. On a fixed compact core, the set $\{u < -2R\}$ is contained in $\{u_j < -R\}$ together with a positive gap of the fixed truncations at $3R$. Pass $j \to \infty$ in the outer-capacity subadditivity bound. It follows that the pole of u has outer capacity zero on that core. Outside an ordinary open cover of a sufficiently deep sublevel, u agrees with one bounded truncation. Its bounded-completion quasicontinuity then proves the stated finite local quasicontinuity. The level-set triangle inequality proves uniqueness. □

Theorem 8.4 (Canonical critical calculus in the strong completion)**.** *For a finite tuple* $\mathbf{u}$ *in* $\widehat{\mathcal{E}}^1_{\mathrm{str}}(V)$*, with actual strongly Cauchy defining rows* $\mathbf{u}_j$*, there are intrinsic coefficient-Radon original currents*

$$\widehat{N}_\gamma(\mathbf{u}), \qquad \widehat{\mathcal{J}}'_{\gamma,i}, \quad \widehat{\mathcal{J}}''_{\gamma,i}, \qquad \widehat{G}^\gamma_{ih}. \tag{8.10}$$

Curvatures have $k = |\gamma| \le q$; first-order and energy currents have $k \le q-1$. All energy degrees are included, since $k+2 \le q+1 = d$. Curvatures are separately closed, symmetric and middle-positive; energy matrices are positive. First-order and energy currents are not asserted to be closed.

For any continuous F on the nonpositive orthant satisfying $|F(x)| \le A(1 - \sum_i x_i)^a$, the actual multiplier limit

$$F(\mathbf{u}_j)P_j \longrightarrow F(\mathbf{u})\widehat{P} \qquad (a+b \le d) \tag{8.11}$$

holds on every original compact smooth complementary test and every actual moment refinement. Here the degree budget b is k, $k+1$, or $k+2$ for curvature, first-order, or energy. All these multipliers are coefficient-integrable. The currents and weighted limits are independent of

defining rows and of outer-capacity-null representatives, and they annihilate the joint pole set. They satisfy all original recursive identities, including

$$\begin{aligned} &D(u_i\widehat{N}_\gamma) = \widehat{N}_{\gamma+e_i}, \\ &d'(u_i\widehat{N}_\gamma) = \widehat{\mathcal{J}}'_{\gamma,i}, \qquad d''(u_i\widehat{N}_\gamma) = \widehat{\mathcal{J}}''_{\gamma,i} \qquad (k \le q-1), \end{aligned} \tag{8.12}$$

and the actual three-term energy identity obtained by placing hats on (7.4).

For $q+1$ completed slots, the global mixed energy is finite and

$$B(u_{0,j},\dots,u_{q,j}) \longrightarrow \widehat{B}(u_0,\dots,u_q) := \int_V (-u_0)\widehat{N}(u_1,\dots,u_q). \tag{8.13}$$

The corresponding positive weighted measures converge against every bounded original smooth scalar test, including those without compact support. The assertion also permits independent indices tending to infinity in the defining rows.

Proof. For a finite tuple define

$$\Phi(R) = \sup_j \int_R^\infty t^q C_S\left(\left\{\sum_i u_{i,j} < -t\right\}; V\right) dt. \tag{8.14}$$

By lemma 8.1, $\Phi(0) < \infty$ and $\Phi(R) \to 0$; the same is true for the supremum over independently large row indices, with the finitely many earlier indices included if needed. At every original compact test stage, the actual truncation estimate (7.5) give

$$\|\mu_{P_j - P_j^{(L)},\eta}\|_{\mathrm{TV}} \le C_\eta L^{b-d}\Phi(L/4). \tag{8.15}$$

The bounded analytic completion supplies intrinsic currents $P^{(L)}$ of the completed clipped tuple. Passing j to infinity in the difference of two bounded stages shows that $P^{(L)}$ are Cauchy in local coefficient total variation, by comparing two truncation levels with the same tail bound. Their limit defines $\widehat{P}$. Coefficient/refinement identities, positivity and separate closedness pass on all original tests.

The limiting coefficients have the common modulus

$$\inf_{L\ge 1}\{L^b s + L^{b-d}\Phi(L/4)\} \longrightarrow 0 \quad (s \downarrow 0). \tag{8.16}$$

Its weak-limit proof keeps L fixed until after ordinary-open variation lower semicontinuity and Radon outer regularity. Thus it controls both the defining and limiting coefficients, including the critical case $b = d$. It annihilates every locally outer-null set and, in particular, the joint pole set. The bounded quasicontinuous multiplier argument therefore passes every bounded continuous function of a fixed clipped tuple.

For endpoint integrability apply that argument to $\theta_R(z)\min((1+z)^a, H)$, where $z = -\sum_i u_i$ and θ_R is zero below R and one above $2R$. For fixed R, H this is a continuous function of one sufficiently large clipped tuple; once a slot exceeds that clipping level its value is the constant H. When $a = 0$, use θ_R itself. The actual row estimates bound the integral by $CR^{a+b-d}\Phi(R/8)$. Passing to the bounded multiplier limit and then $H \uparrow \infty$ proves the same tail bound for the limiting positive traces. Variation lower semicontinuity handles signed first-order coefficients, and positive diagonal controls handle cross energies. Adding the bounded-value part gives actual coefficient integrability. Clipping a general F and then letting its clipping level tend to infinity proves (8.11). Consequently all multipliers in (8.12) and (7.4) are defined before distributional differentiation, and the original row identities pass to the limit.

For two different defining tuples with the same Borel capacity limits, their bounded completed stages are identical. The union of their finitely many row families has the same vanishing-tail property. The preceding construction therefore gives identical unbounded currents and actual multipliers. This proves approximation independence.

Finally, (8.11) gives local convergence of (8.3). The uniform global tightness in lemma 8.1, together with the uniform finite mixed-energy bound (7.9), identifies a finite global limiting measure and gives its total-mass convergence. This is precisely (8.13). Tightness also passes

every bounded original scalar test. The independent-index assertions follow from the same bounds, or by applying the argument to every sequence of independently diverging indices. □

8.3. **Intrinsic discrepancy and completeness.** Define on this Borel space

$$\begin{aligned} \widehat{e}(u) &= \widehat{B}(u^{[q+1]}), \\ \widehat{I}(u,v) &= \widehat{e}(u) + \widehat{e}(v) - \widehat{B}(u, v^{[q]}) - \widehat{B}(v, u^{[q]}). \end{aligned} \tag{8.17}$$

Lemma 8.5 (Uniform discrepancy continuity on energy balls)**.** *The completed space is closed under finite sums and nonnegative scalar multiples. Its mixed energy is symmetric, positive and multilinear on that cone, and satisfies the constant-one Hölder inequality. The formulas (7.12), (7.13), (7.14) and the sharp midpoint inequality (7.17) hold in the completion, with their actual Borel multipliers and local capacities. In particular, $\widehat{I} \geq 0$, and it vanishes exactly on equal relative classes.*

If $\widehat{e}(a), \widehat{e}(a'), \widehat{e}(b) \leq M$, then

$$|\widehat{I}(a,b) - \widehat{I}(a',b)| \leq L_q M^{1-\alpha} \widehat{I}(a,a')^{\alpha}, \qquad L_q = 2(q+1)2^{(q-1)\alpha}3^{d(1-\alpha)}. \tag{8.18}$$

The assertion at $M = 0$ means that both sides are zero.

Proof. For two strongly Cauchy defining rows, the sum rows have the required Borel capacity limit. Expand the discrepancy of two independently late sum rows into finitely many global mixed energies. Equation (8.13) makes each term converge to the corresponding completed mixed energy; the total limiting discrepancy is zero. Thus the sum row is strongly Cauchy. Positive scaling is immediate, and zero scaling means the zero function even at poles. Passing the actual finite mixed identities and (7.9) through (8.13) now proves all asserted cone identities.

The actual inequalities (7.12), (7.14), and (7.17) pass by global mixed-energy convergence. For (7.12), its left side is the actual weight $|u-v|$, which has degree-one growth and is bounded by $(-u)+(-v)$; (8.11) and the tightness of these two critical positive mixed measures pass its global integral. For (7.13), use defining rows, their convergent energies, and the level-set triangle inequality for clipped gaps; then remove deep sublevels using (8.9) and decrease the intermediate gap threshold. This also proves that $\widehat{I} = 0$ gives equality in all local outer capacities. The converse follows from intrinsic current and multiplier independence.

It remains to prove (8.18). It suffices first to consider actual potentials. Write $h = a - a'$. Telescoping the three terms of $I(a,b)$ that depend on a, and using full mixed symmetry, expresses their difference as $2(q+1)$ signed integrals of $-h$ against positive mixed top products whose slots lie among a, a', b. Every such product is bounded by μ_ψ, where $\psi = a + a' + b$, by the positive expansion. Thus

$$|I(a,b) - I(a',b)| \leq 2(q+1) \int |a - a'| \, d\mu_\psi. \tag{8.19}$$

Minkowski gives $e_1(\psi) \leq 3^d M$ and $e_1(a+a') \leq 2^d M$. Apply (7.12) to (8.19) to obtain (8.18). For completed potentials pass through actual defining rows whose energies tend to the completed energies; first allow $M + \epsilon$ and then let $\epsilon \downarrow 0$. The zero-energy case follows separately from Hölder. □

Theorem 8.6 (Completeness and the intrinsic strong topology)**.** *Every $\widehat{I}$-Cauchy sequence in $\widehat{\mathcal{E}}^1_{\mathrm{str}}(V)$ converges to a unique relative class in that space, with discrepancy tending to zero. The actual classical class is strongly dense.*

For members u_j, u of this space, one has

$$\widehat{I}(u_j, u) \longrightarrow 0 \quad \Longleftrightarrow \quad \begin{cases} u_j \longrightarrow u \text{ in all local truncated outer capacities}, \\ \widehat{e}(u_j) \longrightarrow \widehat{e}(u). \end{cases} \tag{8.20}$$

Strongly convergent finite tuples have every original local weighted-current limit in (8.11), including all its critical endpoints, and the global mixed-energy and weighted-measure limits (8.13). The weighted limits here are original-current limits.

Proof. If (v_l) is an actual defining row for a completed u, then for fixed l, (8.13) gives $\widehat{I}(v_l, u) = \lim_{r\to\infty} I(v_l, v_r)$. Its Cauchy property therefore implies $\widehat{I}(v_l, u) \to 0$. This proves strong density.

Now let (u_j) be completed and $\widehat{I}$-Cauchy. The energy-root estimate bounds its energies. Choose actual v_j with $\widehat{I}(v_j, u_j) < 2^{-j}$ and uniformly bounded energies. Applying (8.18) in its two slots shows

$$I(v_j, v_l) \le \widehat{I}(u_j, u_l) + C\big(2^{-j\alpha} + 2^{-l\alpha}\big). \tag{8.21}$$

Hence (v_j) is strongly Cauchy and has a completed Borel limit u. Its defining property and strong density imply $\widehat{I}(v_j, u) \to 0$. A further use of (8.18) gives $\widehat{I}(u_j, u) \to 0$. Separation proves uniqueness. This proves completeness using the uniform discrepancy modulus.

The forward implication of (8.20) follows from the completed energy-root and capacity estimates. For the reverse, the energies are uniformly bounded. Bounded completed stages and the strict subcritical truncation bound for top unweighted currents give local convergence of their top measures. More explicitly, the stage error has order at most CM/L for degree $q < d$; first fix L, use bounded-completion capacity continuity, and then let L grow. The same uniform coefficient modulus permits bounded quasicontinuous weight passage. Consequently, by first truncating the weight and inserting a compact nonnegative cutoff, then increasing both,

$$\liminf_j \widehat{B}(u_j, u^{[q]}) \ge \widehat{e}(u), \qquad \liminf_j \widehat{B}(u, u_j^{[q]}) \ge \widehat{e}(u). \tag{8.22}$$

Hölder gives upper bounds tending to $\widehat{e}(u)$ for both cross energies. Expanding (8.17) proves the converse. This argument uses no critical weighted continuity to prove that very continuity.

For a strongly convergent completed tuple choose, at the jth row, classical approximants with discrepancy at most 2^{-j}, and sufficiently accurate values on the first j compact test stages and bounded clipped weights. The defining-row convergence allows these finitely many requirements simultaneously. By (8.18), these actual rows are strongly Cauchy, with the same Borel limits and energies. The preceding construction and (8.11)–(8.13) give the asserted limits. Equivalently, any failure on a fixed test would furnish a subsequence and actual approximants contradicting those already proved actual-row limits. This last argument also covers bounded original scalar tests in the global statement.

□

8.4. **Monotone closure and the capacity integral.**

Proposition 8.7 (Monotone closure)**.** *For completed potentials $a \le b$ relatively quasi-everywhere,*

$$0 \le \widehat{I}(a, b) \le \widehat{e}(a) - \widehat{e}(b).$$

An increasing sequence in the strong cone converges strongly to its pointwise monotone Borel limit. The same holds for a decreasing sequence with uniformly bounded energies. Equalities and monotonicity are understood outside one locally outer-null set.

Proof. The mixed energy is decreasing in each potential for pointwise order: this follows from positivity in the weight slot and symmetry in every other slot. Both cross terms in the discrepancy are consequently at least $\widehat{e}(b)$, proving the inequality. Along an increasing sequence the energies decrease and are nonnegative; along a decreasing sequence they increase and are bounded by hypothesis. In either case they converge, so the displayed inequality makes the row strongly Cauchy. Completeness supplies a limit. A summable-capacity diagonal subsequence of its bounded truncations converges pointwise off a locally outer-null set; monotonicity identifies this limit with the specified Borel monotone limit. Thus the full sequence converges strongly. □

We also use the following precise capacity characterization from [Energy, Theorem 37.2]. A Borel function $u \le 0$ represents a member of the strong cone if and only if each $u^{(m)}$ is a

bounded capacitary limit of actual classical functions on the same domain V with values in $[-m, 0]$, and

$$J_1^*(u) := \int_0^\infty t^q C_S^*(\{u < -t\}; V)\, dt < \infty. \qquad 2^{-d} J_1^*(u) \le \widehat{e}(u) \le 2^d J_1^*(u). \tag{8.23}$$

The admissibility of the bounded stages is part of this criterion. The completed sublevel theorem in [Energy, Section 36] proves the capacity inequalities and the forward direction. For the reverse, form $v_j = \max(u, j\rho) = \max(u^{(j)}, j\rho)$. Bounded classical approximants of $u^{(j)}$, followed by this maximum, are bounded below by the actual first-energy barrier $j\rho$. The fixed-barrier critical convergence of theorem 7.2 makes their limit a strong member v_j. The sublevel bounds give $\widehat{e}(v_j) \le 2^d J_1^*(u)$. The v_j decrease, hence converge strongly by proposition 8.7. On a compact core and for fixed R, strict negativity of ρ gives $v_j^{(R)} = u^{(R)}$ for all large j. The completed limit is therefore the given capacity class of u. The sublevel theorem then gives both inequalities in (8.23).

9. The energy dual and the source equation

In this section $\mathscr{C} = \widehat{\mathcal{E}}^1_{\mathrm{str}}(V)$, and B, e, I denote the completed mixed energy, energy and discrepancy. Write $r(u) = e(u)^{1/d}$. A difference is formed on the common finite locus of its two Borel potentials and is assigned any finite value on the outer-null exceptional set.

9.1. The Banach space of energy differences.

Definition 9.1 (Energy-difference space and its norm)**.** Let $\mathbb{D}_S = \mathscr{C} - \mathscr{C}$ be the real vector space of such Borel differences modulo local outer-capacity equality. Define

$$\|h\|_{\mathrm{ed}} = \inf\{r(a) + r(b) : a, b \in \mathscr{C},\ h = a - b\}. \tag{9.1}$$

Lemma 9.2 (Continuous extension of the mixed energy)**.** *There is a unique symmetric real d-linear form on $\mathbb{D}_S$ extending B, and it satisfies*

$$|B(h_0, \dots, h_q)| \le \prod_{i=0}^{q} \|h_i\|_{\mathrm{ed}}. \tag{9.2}$$

Moreover $\|u\|_{\mathrm{ed}} = r(u)$ for every $u \in \mathscr{C}$.

Proof. Expand each difference into its two cone terms and extend by the finite multilinear expansion. If $a - b = a' - b'$, then $a + b' = a' + b$ in the relative cone. Separate additivity of the completed B therefore makes the expansion independent of that presentation, one slot at a time. Positive real homogeneity and changing the signs of differences give real multilinearity. For specified decompositions $h_i = a_i - b_i$, the triangle inequality and cone Hölder give the bound $\prod_i (r(a_i) + r(b_i))$. Taking independent infima proves (9.2).

The definition gives $\|u\|_{\mathrm{ed}} \le r(u)$. Applying (9.2) to the diagonal tuple gives $e(u) \le \|u\|_{\mathrm{ed}}^d$, so the reverse inequality follows. This argument also covers zero energy. □

Theorem 9.3 (Actual Borel realization and Banach completeness)**.** *Formula (9.1) is a norm, $\mathbb{D}_S$ is complete for that norm, and $\mathscr{C}$ is a closed convex cone in this Banach space. Norm convergence implies convergence in the full local outer S-capacity topology. More precisely, with $C_0 = d2^d$, one has*

$$C_S^*(\{|h| > t\}; V) \le C_0 t^{-d} \|h\|_{\mathrm{ed}}^d \qquad (h \in \mathbb{D}_S,\ t > 0). \tag{9.3}$$

For $u, v \in \mathscr{C}$,

$$I(u, v) \le (r(u)^q + r(v)^q)\, \|u - v\|_{\mathrm{ed}}. \tag{9.4}$$

Proof. The infimum in (9.1) is finite. Minkowski on $\mathscr{C}$ proves the triangle inequality; switching the two cone terms proves symmetry, and homogeneity follows by rescaling decompositions. The capacity integral bound (8.23) and monotonicity of the sublevels give

$$\frac{t^d}{d}C_S^*(\{u<-t\};V)\le\int_0^t s^{d-1}C_S^*(\{u<-s\};V)\,ds\le 2^d e(u).$$

For $h=a-b$, the set $\{|h|>t\}$ is contained, modulo an outer-null set, in $\{a<-t\}\cup\{b<-t\}$. Hence its outer capacity is at most $C_0t^{-d}(r(a)^d+r(b)^d)$, bounded by $C_0t^{-d}(r(a)+r(b))^d$. Infimizing proves (9.3), and shows that a zero seminorm has the zero relative Borel class. Local outer-null sets here also have zero outer capacity in the fixed domain: use a countable compact exhaustion and the analytic domain comparison on its cores. The same comparison transfers norm convergence to every smaller-domain original capacity.

We prove completeness inside the space of actual Borel differences, rather than introducing an abstract completion with unidentified elements. For a norm-Cauchy sequence choose a subsequence (h_j) whose successive increments have decompositions

$$h_{j+1}-h_j=a_j-b_j,\qquad \sum_{j\ge1}(r(a_j)+r(b_j))<\infty. \tag{9.5}$$

Write $h_1=a_0-b_0$. The partial sums $A_N=a_0+\sum_{j=1}^N a_j$ and $B_N=b_0+\sum_{j=1}^N b_j$ decrease as Borel potentials, outside one countable union of outer-null sets. Their energy roots are bounded by the corresponding sums of roots, so the proposition 8.7 gives strong limits $A,B\in\mathscr{C}$. Apply the same theorem to each tail series. It yields $A-A_N,B-B_N\in\mathscr{C}$ and

$$r(A-A_N)+r(B-B_N)\le\sum_{j>N}(r(a_j)+r(b_j))\longrightarrow 0. \tag{9.6}$$

The identities with the tails hold on the common finite locus, hence in the relative Borel quotient. Thus $h=A-B\in\mathbb{D}_S$ is the norm limit of the subsequence and of the original Cauchy sequence. No abstract vector invisible to Borel capacity has been adjoined.

For $u,v\in\mathscr{C}$, multilinearity gives

$$I(u,v)=B(u-v,u^{[q]})-B(u-v,v^{[q]}).$$

Apply (9.2) to obtain (9.4). If a sequence in $\mathscr{C}$ converges in norm to $h\in\mathbb{D}_S$, its energy roots are bounded by the norm triangle inequality and the preceding lemma. Equation (9.4) makes it strongly Cauchy. theorem 8.6 supplies a limit $u\in\mathscr{C}$. The norm limit and the strong limit have the same local capacity class by (9.3) and (8.20). Thus $h=u$, proving norm closedness. Convexity is the completed cone addition property. □

Definition 9.4 (Curvature as an energy-dual functional)**.** For $u\in\mathscr{C}$ put

$$J(u)(h)=B(h,u^{[q]})=-\int_V h\,d\widehat{\mu}_u\qquad(h\in\mathbb{D}_S). \tag{9.7}$$

The displayed integral is literal and absolutely convergent: for $h=a-b$, $|h|\le(-a)+(-b)$ off a common null set, and the two mixed first energies are finite. The current measure ignores that exceptional set. Thus $J(u)$ is a genuine functional in $\mathbb{D}_S^*$, with its original current interpretation.

Theorem 9.5 (Exact dual norm and strong continuity of curvature)**.** *For every continuous linear functional ℓ on $\mathbb{D}_S$,*

$$\|\ell\|_*=\sup_{\substack{a\in\mathscr{C}\\ r(a)\le1}}|\ell(a)|. \tag{9.8}$$

For $u\in\mathscr{C}$ one has $\|J(u)\|_*=r(u)^q$. For $u,v\in\mathscr{C}$ and $R=\max(r(u),r(v))$,

$$\begin{aligned}\|J(u)-J(v)\|_*&\le qR^{q-1}\|u-v\|_{\mathrm{ed}},\\ \|J(u)-J(v)\|_*&\le\sqrt{q}\,R^{(q-1)/2}I(u,v)^{1/2}.\end{aligned} \tag{9.9}$$

At $R = 0$ both sides mean zero; for $q = 1$ the power R^0 is one. In particular strong convergence of potentials gives norm convergence of their source functionals, in every remaining degree.

Proof. If the supremum in (9.8) is L, a decomposition $h = a - b$ gives $|\ell(h)| \le L(r(a) + r(b))$. Taking the infimum yields $\|\ell\|_* \le L$; the reverse follows from $\|a\|_{\mathrm{ed}} = r(a)$. Equation (9.2) bounds the norm of $J(u)$ by $r(u)^q$. For $r(u) > 0$, evaluation on $u/r(u)$ gives equality; if $r(u) = 0$, the class and functional are zero.

The finite telescoping expansion gives

$$(J(u) - J(v))(h) = \sum_{l=0}^{q-1} B(h, u - v, u^{[l]}, v^{[q-1-l]}). \tag{9.10}$$

Equation (9.2) proves the first bound in (9.9). For the sharper strong-topology bound define

$$H(f, g) = \sum_{l=0}^{q-1} B(f, g, u^{[l]}, v^{[q-1-l]}).$$

This is a positive semidefinite bilinear form on $\mathbb{D}_S$. Indeed each summand is the integrated energy form with the fixed completed positive closed mixed background. Positivity on finite differences follows from (7.8), passed through the global completed-energy convergence (8.13). We have

$$H(u - v, u - v) = I(u, v), \qquad H(a, a) \le q r(a)^2 R^{q-1} \quad (a \in \mathscr{C}).$$

Cauchy–Schwarz for this positive semidefinite form and (9.10) bound the functional on every a with $r(a) \le 1$. Equation (9.8) proves the second bound. Strongly convergent rows have bounded energy roots by (7.14), so the final assertion follows. □

9.2. **Closed sources and variational normals.**

Theorem 9.6 (A complete curvature-dual metric and closed source image)**.** *The image $J(\mathscr{C})$ is norm closed in $\mathbb{D}_S^*$. The formula*

$$d_J(u, v) = \|J(u) - J(v)\|_* \tag{9.11}$$

defines a complete metric on $\mathscr{C}$, inducing precisely the strong discrepancy topology. For $u, v \in \mathscr{C}$,

$$\begin{aligned} I(u, v) &\le (r(u) + r(v)) d_J(u, v), \\ d_J(u, v) &\le \sqrt{q}\, \max(r(u), r(v))^{(q-1)/2} I(u, v)^{1/2}. \end{aligned} \tag{9.12}$$

Proof. Multilinearity gives the exact identity $I(u, v) = (J(u) - J(v))(u - v)$. The dual inequality and $\|u - v\|_{\mathrm{ed}} \le r(u) + r(v)$ prove the first estimate. The second is (9.9). Thus $d_J = 0$ gives $I = 0$ and equal relative classes. Symmetry and the triangle inequality come from the dual norm, not from an asserted triangle inequality for I.

For completeness, mixed Hölder also gives the explicit root bound

$$I(u, v) \ge \big(r(u) - r(v)\big)\big(r(u)^q - r(v)^q\big) \ge |r(u) - r(v)|^{q+1}.$$

The first inequality follows by bounding the two mixed terms in $e(u) + e(v) - B(u, v^{[q]}) - B(v, u^{[q]})$ separately. The second is the elementary inequality $x^q - y^q \ge (x - y)^q$ for $x \ge y \ge 0$. Hence convergence to a fixed potential in I automatically bounds the energy roots.

Suppose $J(u_j)$ is norm Cauchy. Its norms are bounded, so $r(u_j)^q = \|J(u_j)\|_*$ bounds the energy roots. The first estimate in (9.12) makes (u_j) strongly Cauchy. theorem 8.6 gives $u \in \mathscr{C}$ with $I(u_j, u) \to 0$. Its energy roots remain bounded, and the second estimate gives $J(u_j) \to J(u)$ in norm. Every norm limit of image elements is consequently in the image, proving closedness and metric completeness. The same inequalities prove equivalence of the two convergences: strong convergence bounds the roots by the (7.14), whereas d_J convergence bounds them by the exact dual norm identity. These arguments also identify their Cauchy sequences. □

Definition 9.7 (The strong-continuous energy dual)**.** Let $\mathbb{D}^*_{S,\mathrm{s}}$ consist of those $\ell \in \mathbb{D}^*_S$ for which

$$u_j, u \in \mathscr{C},\ I(u_j,u) \to 0 \quad \Longrightarrow \quad \ell(u_j) \to \ell(u). \tag{9.13}$$

Lemma 9.8 (A closed dual class containing every curvature)**.** *The class $\mathbb{D}^*_{S,\mathrm{s}}$ is a norm-closed linear subspace of $\mathbb{D}^*_S$, and $J(\mathscr{C})$ is contained in it. Consequently the norm-closed linear span of all curvature functionals is contained in this class.*

Proof. Linearity is immediate. A strongly convergent row has bounded $r(u_j)$, hence is bounded in $\mathbb{D}_S$. If $\ell_m \to \ell$ in norm, the error $|(\ell-\ell_m)(u_j-u)|$ is uniformly small for large m. Fixing m then invokes (9.13), proving closure. For fixed $v \in \mathscr{C}$, the completed tester estimate (7.12) bounds $|J(v)(u_j-u)|$ by $\int |u_j-u|\,d\widehat{\mu}_v \to 0$. Its constants are finite on the bounded-energy family. This proves the curvature assertion. □

Let

$$\mathscr{C}^\circ = \{n \in \mathbb{D}^*_S : n(v) \le 0 \text{ for every } v \in \mathscr{C}\}. \tag{9.14}$$

Theorem 9.9 (All-degree variational normal decomposition)**.** *For every $\ell \in \mathbb{D}^*_{S,\mathrm{s}}$, the functional*

$$F_\ell(v) = \ell(v) - e(v)/d \qquad (v \in \mathscr{C}) \tag{9.15}$$

has a unique maximizer $u = \Pi(\ell)$. Every maximizing sequence converges strongly to u. Equivalently, there is a unique pair (u,n) with

$$\ell = J(u) + n, \qquad u \in \mathscr{C}, \quad n \in \mathscr{C}^\circ, \quad n(u) = 0. \tag{9.16}$$

The residual also lies in $\mathbb{D}^*_{S,\mathrm{s}}$. Define the nonnegative support gauge

$$p(\ell) = \sup_{\substack{v\in\mathscr{C}\\ r(v)\le 1}} \ell(v). \tag{9.17}$$

Then

$$\begin{aligned} e(u) = \ell(u), \qquad & r(u)^q = p(\ell) \le \|\ell\|_*, \\ \sup F_\ell = \frac{q}{d}\,p(\ell)^{d/q}. & \end{aligned} \tag{9.18}$$

If ℓ is nonnegative on $\mathscr{C}$, then $p(\ell) = \|\ell\|_*$.

Proof. Since $|\ell(v)| \le \|\ell\|_* r(v)$, $F_\ell(v) \le \|\ell\|_* r(v) - r(v)^d/d$. Thus its supremum $L \ge F_\ell(0) = 0$ is finite, and every maximizing sequence has bounded energies. In the midpoint gap the linear term cancels. (7.17) gives

$$c_q I(v_j,v_k) \le L - \tfrac{1}{2}(F_\ell(v_j)+F_\ell(v_k)) \longrightarrow 0, \quad c_q = \frac{1-2^{-q}}{2d}.$$

Strong completeness provides $u \in \mathscr{C}$. Its energy converges, and (9.13) passes the datum term, so $F_\ell(u) = L$. The same midpoint estimate proves uniqueness and convergence of every maximizing sequence. Actual classical strong density also gives the same supremum over the actual classical class.

The continuous d-linear extension B makes e/d a polynomial on $\mathbb{D}_S$ whose differential at u is $J(u)$. Only segments inside $\mathscr{C}$ are used for maximization. Their first variation gives

$$(\ell - J(u))(v-u) \le 0 \qquad (v \in \mathscr{C}). \tag{9.19}$$

Conversely convexity of e on the cone makes this sufficient. Taking the competitors $0, 2u$ gives $(\ell - J(u))(u) = 0$. Then (9.19) is precisely the polar condition in (9.16). Conversely (9.16) implies (9.19). The pair is unique because the maximizer is unique and $n = \ell - J(u)$. Its strong continuity follows from the preceding lemma.

Scaling along tu gives $e(u) = \ell(u)$, also when $u = 0$. For each $v \in \mathscr{C}$, the polar inequality gives $\ell(v) \le J(u)(v) \le r(v)r(u)^q$. Thus $p(\ell) \le r(u)^q$. If $u \ne 0$, evaluation on $u/r(u)$ gives the reverse inequality; if $u = 0$, the polar condition and the allowed zero tester give $p(\ell) = 0$. This proves (9.18). Finally (9.8) says $\|\ell\|_* = \max(p(\ell), p(-\ell))$; if $\ell \ge 0$ on the cone, the latter second term is zero. □

Theorem 9.10 (A quantitative nonlinear retraction of source functionals)**.** *Put $\mathcal{P}(\ell) = J(\Pi(\ell))$ on $\mathbb{D}^*_{S,\mathrm{s}}$. This is a retraction onto the norm-closed image $J(\mathscr{C})$, and is positively homogeneous of degree one. For $\|\ell\|_*, \|m\|_* \le R$, with $R > 0$, one has*

$$\begin{aligned} I(\Pi(\ell), \Pi(m)) &\le 2R^{1/q}\|\ell - m\|_*, \\ \|\mathcal{P}(\ell) - \mathcal{P}(m)\|_* &\le \sqrt{2qR}\,\|\ell - m\|_*^{1/2}. \end{aligned} \tag{9.20}$$

The first inequality has the more precise right side $(\ell - m)(\Pi(\ell) - \Pi(m))$. If $n = \ell - \mathcal{P}(\ell)$ and $R = \|\ell\|_* > 0$, then

$$\left(\sqrt{qR + \|n\|_*} - \sqrt{qR}\right)^2 \le \operatorname{dist}_*(\ell, J(\mathscr{C})) \le \|n\|_*. \tag{9.21}$$

In particular a nonzero residual is a quantitative obstruction to source-image approximation, not merely a failure of one chosen maximization algorithm.

Proof. If $\ell = J(a)$, the pair $(a, 0)$ satisfies (9.16), so uniqueness gives the retraction. Homogeneity of J gives $\Pi(t\ell) = t^{1/q}\Pi(\ell)$ and $\mathcal{P}(t\ell) = t\mathcal{P}(\ell)$ for $t \ge 0$; the case $t = 0$ is the zero class.

Write $u = \Pi(\ell)$ and $v = \Pi(m)$. Use v, u as the competitors in their two inequalities (9.19), and add. It yields $I(u, v) \le (\ell - m)(u - v)$. Duality bounds this by $\|\ell - m\|_*(r(u) + r(v))$; (9.18) bounds the roots by $R^{1/q}$. The second line of (9.20) now follows from (9.9). If $R = 0$ both projections are zero.

Let δ be the distance in (9.21). Since $J(0) = 0$, $\delta \le R$. Choose $\eta_j \in J(\mathscr{C})$ with $\delta_j = \|\ell - \eta_j\|_* \to \delta$. Their norms are at most $R + \delta_j$. Retraction and (9.20) imply

$$\|n\|_* \le \delta_j + \sqrt{2q(R + \delta_j)\delta_j}.$$

Passing to the limit and using $\delta \le R$ gives $\|n\|_* \le \delta + 2\sqrt{qR\delta}$. Solving this quadratic inequality in $\sqrt{\delta}$ gives the lower bound. The upper bound uses the particular image element $\mathcal{P}(\ell)$. □

9.3. **One geometric potential and all compact smooth directions.** We next construct a potential that permits both signs of each original compact smooth scalar direction. The construction is made from the geometry of V before choosing S.

Lemma 9.11 (Countable exact moments and compensators)**.** *There are compact moment cores $K_l \Subset V$ and smooth classical psh functions a_l on V, with $0 \le a_l \le 1$, such that every original compact smooth scalar test can be localized to finitely many of these cores. On each selected core, Da_l dominates a positive multiple of the identity form in all the moment directions of the localized test and its smooth partition. The same assertion holds after every actual finite refinement.*

Proof. We use the exact unit replacement of [Energy, Lemma 2.1]. For clarity, its countability comes from a countable dense subring A_0 of the separable affinoid algebra $\mathcal{O}(B)$. Laurent inequalities in A_0 with rational positive thresholds give a countable neighborhood system. On a suitable such Laurent domain, a unit u is approximated by a quotient b/c from A_0 so closely that $\|u - b/c\|\|u^{-1}\| < 1$. Thus $|u| = |b/c|$ at every point of the domain. The logarithmic moment and all its CLD differentials are exactly preserved. Close this countable collection under finite refinements and select countable cofinal compact cores.

For a finite moment list $t_i = \log|b_i/c_i|$ on a core, choose a smooth nonnegative function κ of integral one supported in $(-1, 0)$ and set

$$\zeta(s) = \int_{-\infty}^{s} (s - r)\kappa(r)\,dr, \qquad F(s) = \zeta(s)^2.$$

The function F is convex and nondecreasing, vanishes for $s \le -1$, and has $F'' = 2$ for $s \ge 0$. Choose A so that $\log|b_i| + A$ and $\log|c_i| + A$ are positive near the core. The sum of $F(\log|b_i| + A)$ and $F(\log|c_i| + A)$ is smooth and classical on the affinoid interior, with the zero branches extended by zero near analytic zeros. Its Hessian dominates $\sum_i d't_i \wedge d''t_i$ near

the core, by $(\xi-\eta)^2 \le 2\xi^2+2\eta^2$. Normalize this bounded sum on $\overline{V}$ to obtain a_l. Finite localization of any actual test uses finitely many exact moment cores and includes the partition moments. This proves the assertion with the full original test sheaf. □

Proposition 9.12 (A simultaneous geometric probe)**.** *There is a bounded continuous classical potential ψ_0 determined by the box alone such that, for every extending allowed S,*

$$-1 < \psi_0 < 0, \qquad \rho \le \psi_0, \qquad \mu_{\psi_0}(V) \le \mu_\rho(V) < \infty.$$

It has compact negative sublevels and ordinary zero boundary value. For every original real compact smooth scalar ϕ, there is $\epsilon_\phi > 0$, independent of S, such that

$$\psi_0 + t\phi \in \mathcal{E}^1_{S,\partial}(V), \qquad |t| \le \epsilon_\phi. \tag{9.22}$$

These functions are bounded continuous classical psh, lie between -1 and 0, and have finite total top mass. On a fixed finite presentation one may choose

$$\epsilon_\phi \ge \frac{c_{\mathfrak{a}}}{\max(1, \|\phi\|_{2,\mathfrak{a}})}, \tag{9.23}$$

where $\|\cdot\|_{2,\mathfrak{a}}$ is the sum of the real C^2 norms of its representatives on that presentation.

Proof. **Step 1: construct the potential.** Choose $A_l \ge 1$ with $A_l\rho < -2$ near K_l, and set

$$\psi_l = \max(a_l - 2, A_l\rho), \qquad c_l = \frac{2^{-l}}{1+A_l}, \qquad \psi_0 = \sum_{l=1}^{\infty} c_l\psi_l.$$

Each ψ_l is bounded between -2 and 0, and $\psi_l \ge A_l\rho$. The series converges uniformly. Its partial sums decrease and are classical, so the finite continuous limit is classical by bounded decreasing closure. Moreover,

$$T := \sum_l c_l A_l = 1 - \sum_l c_l < 1, \qquad \psi_0 \ge T\rho > -1.$$

Strict negativity follows from any summand. The lower barrier gives compact negative sublevels and zero boundary value. Boundary comparison with $c\rho$, $c > 1$, on the increasing compact-gap sets $\{c\rho < \psi_0 - \epsilon\}$, followed by $\epsilon \downarrow 0$ and $c \downarrow 1$, gives the mass bound. Thus all finite positive power energies of ψ_0 are finite. Omitting finitely many summands gives the same properties for the remainder.

Step 2: perturb each selected moment core. Localize $\phi = \sum_{l\in F}\phi_l$ on finitely many cores from lemma 9.11, combining repeated indices. Near $\operatorname{supp}\phi_l$, one has $\psi_l = a_l - 2$. The Hessian of the actual smooth representative of ϕ_l is bounded in the selected moment directions. The strict positive Hessian contributed by the corresponding convex-flat numerator and denominator terms of a_l dominates both signs of this perturbation. Hence there is $\delta_l > 0$ such that $\psi_l + \tau\phi_l$ is classical for $|\tau| \le \delta_l$. Outside the support it equals ψ_l, so ordinary gluing applies. Decrease δ_l to retain strict negativity on its compact support.

Write $\psi_0 = r_F + \sum_{l\in F} c_l\psi_l$ and choose $\epsilon_\phi/c_l \le \delta_l$ for every selected l. Then

$$\psi_0 + t\phi = r_F + \sum_{l\in F} c_l(\psi_l + (t/c_l)\phi_l)$$

is classical and negative. Decrease ϵ_ϕ again so that $\epsilon_\phi\|\phi\|_\infty \le (1-T)/2$. This preserves the lower bound -1. The perturbation agrees with ψ_0 off one compact set, so its boundary condition is unchanged. Its top source agrees there by locality; relative CLN gives finite mass on a larger compact neighborhood. It therefore has finite total mass and belongs to every finite positive power class.

For a fixed finite presentation, the Hessian bounds, the negativity margins and the finitely many coefficients c_l are uniform on the unit ball of its C^2 seminorm. Scaling this estimate gives (9.23). □

In particular every compact original smooth scalar belongs to $\mathbb{D}_S$: it is the difference of the two cone members $\epsilon_\phi^{-1}(\psi_0 + \epsilon_\phi\phi)$ and $\epsilon_\phi^{-1}\psi_0$.

9.4. **The scalar source defect.** A Radon measure μ defines a *literal energy functional* if $\ell_\mu(h) = -\int_V h\,d\mu$ is absolutely defined on $\mathbb{D}_S$, independent of relative representatives, and continuous for its norm. We impose strong continuity when writing $\ell_\mu \in \mathbb{D}^*_{S,\mathrm{s}}$.

Theorem 9.13 (One probe detects the literal source equation)**.** *Suppose $\ell = J(u) + n$ is a normal pair, with $u \in \mathscr{C}$, $n(v) \le 0$ for every $v \in \mathscr{C}$, and $n(u) = 0$. Set*

$$\Xi_{\psi_0}(\ell, u) = J(u)(\psi_0) - \ell(\psi_0) = -n(\psi_0) \ge 0.$$

For every original real compact smooth scalar ϕ,

$$|n(\phi)| \le \epsilon_\phi^{-1}\Xi_{\psi_0}(\ell, u). \tag{9.24}$$

Now let μ be positive Radon with $\ell_\mu \in \mathbb{D}^*_{S,\mathrm{s}}$, and let $u = \Pi(\ell_\mu)$. Then the following are equivalent:

(i) $\Xi_{\psi_0}(\mu) := \displaystyle\int_V (-\psi_0)\,d\widehat{\mu}_u - \int_V (-\psi_0)\,d\mu = 0$;
(ii) $\widehat{\mu}_u = \mu$ *on every original compact smooth scalar test;*
(iii) *there exists $v \in \mathscr{C}$ with $\widehat{\mu}_v = \mu$.*

The solution in (iii) *is unique and equals u. For all data in this class,*

$$\left|\int_V \phi\,d(\mu - \widehat{\mu}_u)\right| \le \epsilon_\phi^{-1}\Xi_{\psi_0}(\mu). \tag{9.25}$$

The same ψ_0 works for every extending background, every legal remaining degree, and every actual refinement of the original tests.

Proof. Apply the normal inequality to the two cone members $\psi_0 \pm \epsilon_\phi\phi$ from (9.22). It gives $0 \ge n(\psi_0) \pm \epsilon_\phi n(\phi)$, which is (9.24).

For a literal datum both probe integrals are finite by the energy-dual bound and mixed Hölder. The normal on a smooth test is $-\int \phi\,d(\mu - \widehat{\mu}_u)$, proving (9.25). If $\Xi = 0$, all these test integrals vanish. Original scalar density and Radon uniqueness give equality of the measures. Conversely, measure equality gives equality of the two finite probe integrals. Finally, a source potential v has $J(v) = \ell_\mu$ by literal integration. The normal pair $(v, 0)$ and the variational uniqueness theorem give $v = u$. This proves all three equivalences. □

For a nonliteral functional the first part detects its compact-test shadow. The literal hypothesis in the equivalence identifies that shadow with the specified Radon measure.

Corollary 9.14 (Additive strictness and dense admissible sources)**.** *Under the literal hypotheses of theorem 9.13, if $u = v + \varepsilon\psi_0$ for some $v \in \mathscr{C}$ and $\varepsilon > 0$, then $\widehat{\mu}_u = \mu$. For any $a \in \mathscr{C}$, the potentials $a + \varepsilon\psi_0$ admit both signs of every compact smooth direction, tend to a in the energy-difference norm, and satisfy*

$$\|J(a + \varepsilon\psi_0) - J(a)\|_* \le (r(a) + \varepsilon r(\psi_0))^q - r(a)^q.$$

Their sources are therefore dense in $J(\mathscr{C})$.

Proof. The normal satisfies $0 = n(u) = n(v) + \varepsilon n(\psi_0)$, and both terms are nonpositive. Hence $\Xi = 0$, so theorem 9.13 applies. For the second assertion use $a + \varepsilon(\psi_0 + t\phi) \in \mathscr{C}$ and $\|\varepsilon\psi_0\|_{\mathrm{ed}} = \varepsilon r(\psi_0)$. Expanding the q source slots and bounding each mixed functional by the product of its energy roots gives the displayed binomial estimate. □

10. The relative linear renormalized Dirichlet problem

Here the remaining degree is $q = 1$, so S has degree $(n-1, n-1)$. The ambient dimension is arbitrary. This section uses the following slightly more general domain: choose one compact original affinoid B, ordinary opens $V \Subset \Omega$ with $\overline{\Omega} \subset \mathrm{Int}_X(B)$, and an allowed background S on Ω. The Laurent domains used above satisfy these conditions after an intermediate neighborhood is chosen.

10.1. **The original energy form and its diffuse data.** The faithful-form construction [Energy, Theorem *A faithful coercive original form without algebraizability* and its part-space corollary] gives a bounded continuous classical $0 \le h \le 1$ on Ω and a positive reference measure $\lambda = D(hS)$. It is faithful to the original coefficients: $\lambda(E) = 0$ if and only if $\mathbf{1}_E S = 0$ for Borel E. Put $F = V \cap \operatorname{supp} S$. The measure $\lambda|_F$ is finite. The energy closure H of the original real smooth compact scalar core on V, modulo zero energy, is a regular strongly local symmetric Dirichlet form on F, with

$$\mathcal{E}(f,g) = \int_V \tfrac{1}{2}(d'f \wedge d''g + d'g \wedge d''f) \wedge S, \qquad \mathcal{E}(f,f) \ge \tfrac{1}{2}\|f\|^2_{L^2(\lambda)}. \tag{10.1}$$

For an original compact smooth ϕ,

$$D(\phi S) = b_\phi \lambda, \quad b_\phi \in L^\infty_c(\lambda), \qquad \mathcal{E}(v,\phi) = -\langle D(vS), \phi\rangle = -\int_F v b_\phi \, d\lambda \quad (v \in H). \tag{10.2}$$

Every original compact coefficient of S is bounded by a constant multiple of λ. Multiplication by a fixed core cutoff obeys

$$\mathcal{E}(\chi v, \chi v) \le 2\|\chi\|^2_\infty \mathcal{E}(v,v) + 2C_\chi \|v\|^2_2. \tag{10.3}$$

The faithful measure is obtained from a summable sequence of the exact moment compensators of lemma 9.11. Thus the form and its current identities retain every original refinement.

Let c_V be the transient zero-capacity of this part form; for quasi-open sets it is the infimum of $\mathcal{E}(v,v)$ over $v \in H$ with $\widetilde{v} \ge 1$ quasi-everywhere. A finite signed Radon datum μ on F is *energy diffuse* if $|\mu|$ vanishes on every c_V-polar Borel set. The tilde denotes the canonical energy-quasicontinuous representative. This energy capacity and the relative Bedford–Taylor capacity play different roles.

For $f \in L^2(F,\lambda)$, let $Gf \in H$ be the Riesz potential, $\mathcal{E}(Gf, z) = \int f z \, d\lambda$. The original barrier gives $|Gf| \le (2-h)\|f\|_\infty \le 2\|f\|_\infty$ for bounded f. Set $T_k(s) = \max(-k, \min(k, s))$.

Theorem 10.1 (Transposition with the full original current)**.** *Every finite signed energy-diffuse Radon measure μ has a unique $u_\mu \in L^1(F,\lambda)$ satisfying*

$$\int_F u_\mu f \, d\lambda = -\int_F \widetilde{Gf} \, d\mu \qquad (f \in L^\infty(F,\lambda)). \tag{10.4}$$

Its canonical representative is finite and energy-quasicontinuous outside a polar set, $T_k u_\mu \in H$, and $D(u_\mu S) = \mu$ on every original compact scalar test. The map is linear, order reversing on positive data, and

$$\|u_\mu - u_\nu\|_{L^1(\lambda)} \le 2\|\mu - \nu\|_{\mathrm{TV}}, \tag{10.5}$$

$$\mathcal{E}(T_k(u_\mu - u_\nu)) \le k\|\mu - \nu\|_{\mathrm{TV}}, \tag{10.6}$$

$$c_V(\{|\widetilde{u}_\mu - \widetilde{u}_\nu| > k\}) \le k^{-1}\|\mu - \nu\|_{\mathrm{TV}}. \tag{10.7}$$

Proof. **Step 1: finite-energy data.** A finite diffuse measure is a smooth measure for the regular transient form. The finite-energy nest theorem [KR17, Lemma 2.1] gives increasing Borel sets F_j with $|\mu|(F \setminus \bigcup F_j) = 0$ such that $\mathbf{1}_{F_j}|\mu|$ has finite energy integral. Put $\mu_j = \mathbf{1}_{F_j}\mu$ and $u_j = -G\mu_j$.

For a signed datum σ with finite-energy variation, order gives $|G\sigma| \le G|\sigma|$. Pair with the torsion vector $G1 \le 2$ to obtain $\|G\sigma\|_1 \le 2\|\sigma\|_{\mathrm{TV}}$. The normal contraction $s - 2T_k(s)$ gives $\mathcal{E}(T_k G\sigma) \le \mathcal{E}(G\sigma, T_k G\sigma) \le k\|\sigma\|_{\mathrm{TV}}$. Testing capacity with $|T_k G\sigma|/k$ gives the analogous capacity estimate. Apply these estimates to $\mu_j - \mu_l$.

Step 2: take the canonical limit. The u_j are Cauchy in $L^1(\lambda)$ and in energy capacity. A summable-capacity subsequence, together with the quasicontinuity of its terms, has a finite energy-quasicontinuous limit off a polar set. For fixed k, the $T_k u_j$ are bounded in H, and $\|T_k u_j - T_k u\|^2_2 \le 2k\|u_j - u\|_1$. Weak compactness and coercivity give $T_k u \in H$ and the clipping bound. The canonical representatives agree by capacity convergence. Finite-energy symmetry passes to (10.4), using total-variation convergence of μ_j and boundedness of Gf. Bounded

measurable functions separate L^1 classes, so the transposition identity gives uniqueness and linearity.

Step 3: identify every original test. Coefficient domination makes uS defined on all compact original coefficients. For a compact original smooth ϕ, (10.2) gives $G(b_\phi) = -\phi$. Use $f = b_\phi$ in (10.4) to obtain

$$\langle D(uS), \phi\rangle = \int_F u b_\phi \, d\lambda = \int_F \phi \, d\mu.$$

A common finite-energy exhaustion of $|\mu|+|\nu|$ passes the three finite-energy difference estimates to the limit. This proves the bounds and the order assertion. □

10.2. **Closed-form identities used for signed data.** For $v \in H$, its original core gradients close to a Hilbert gradient module, and its energy measure $\Gamma_S(v)$ is the total-variation limit of the core energy measures. In particular,

$$\|\Gamma_S(v) - \Gamma_S(w)\|_{\mathrm{TV}} \le \mathcal{E}(v-w)^{1/2}(\mathcal{E}(v)^{1/2} + \mathcal{E}(w)^{1/2}). \tag{10.8}$$

The closed chain and product rules give

$$dT_k v = \mathbf{1}_{\{|\widetilde{v}|<k\}}\, dv, \qquad \Gamma_S(v)\{\widetilde{v} = t\} = 0. \tag{10.9}$$

For $h_t(s) = (s-t)^+ - (-t)^+$, the finite-energy level measure is

$$L_t(v; S) = D(h_t(v)S) - \mathbf{1}_{\{\widetilde{v}>t\}} D(vS). \tag{10.10}$$

When $|D(vS)|$ has finite energy integral, this is positive, diffuse and carried by $\{\widetilde{v} = t\}$. These original-current facts are proved in [Energy, Sections 41–42]. The scalar chain is obtained by closing the original identity against tests $\phi\Phi'(v)$; one-sided smooth hinges prove (10.9) and the level assertion.

For a positive diffuse datum τ, the potential $w = G\tau$ has strongly convergent bounded clippings of its increasing finite-energy nest potentials. Their energy identities give

$$\mathcal{E}(T_{2M}w - T_M w) \le M\tau\{w > M\}. \tag{10.11}$$

The compatible energy measures of the clippings satisfy

$$\Gamma_S(T_k u) = \mathbf{1}_{\{|\widetilde{u}|<k\}} \Gamma_S(T_l u), \qquad l > k. \tag{10.12}$$

For any diffuse datum the Kato inequality, obtained from smooth one-sided hinges in the transposition approximation, defines a positive Radon measure $L_t(u; S) = D(h_t(u)S) - \mathbf{1}_{\{\widetilde{u}>t\}}\mu$. Its exact level carrier for signed data is proved below.

We use one elementary continuity fact. If ν is finite and energy diffuse, then $c_V(E_j) \to 0$ implies $|\nu|(E_j) \to 0$. Otherwise a subsequence with summable capacities and variations bounded below would have a polar limsup with positive variation. Consequently, capacity convergence passes bounded continuous scalar compositions in $L^1(|\nu|)$.

10.3. **Signed clippings, chain rules and level measures.** Fix a finite signed energy-diffuse datum μ and write $B_\mu = \|\mu\|_{\mathrm{TV}}$. Every occurrence of u below means its canonical representative when used as a measure multiplier.

Lemma 10.2 (Bounded energy tests and signed Radon functionals)**.** *Every bounded $z \in H$ has original compact smooth core approximations converging strongly in H, uniformly bounded in absolute value, and along a subsequence quasi-everywhere. If a finite signed Radon measure ν on F has a continuous core functional for the energy norm, then $|\nu|$ charges no energy-polar set. If $h \in H$ and $D(hS) = \nu$ on every original compact smooth test, then*

$$\mathcal{E}(h, z) = -\int \widetilde{z}\, d\nu \qquad (z \in H \cap L^\infty), \qquad h = u_\nu. \tag{10.13}$$

Proof. For bounded z, start with any original core sequence $z_j \to z$ in H and compose it with one fixed smooth normal clamp that fixes a neighborhood of the range of z and is bounded globally. The gradient-module chain rule proves strong convergence of these compositions to z. They are still original compact smooth functions because the clamp fixes zero. A summable-capacity subsequence converges quasi-everywhere.

Let K be an energy-polar compact set and choose an ordinary relatively open neighborhood G with compact closure such that $|\nu|(G \setminus K) < \epsilon$. The regular compact-core capacity formula, coercivity, a smooth scalar clamp, and a fixed original cutoff supported in G give core functions $0 \le a_j \le 1$, equal to one near K, supported in G, with $\mathcal{E}(a_j) \to 0$. Multiplication by that cutoff preserves the energy convergence by (10.3). Hence $\int a_j \, d\nu \to 0$, while

$$\left|\nu(K) - \int a_j \, d\nu\right| \le |\nu|(G \setminus K) < \epsilon.$$

Let $\epsilon \downarrow 0$. The same argument applies to every polar compact subset of the positive and negative Hahn sets. Radon inner regularity of the two variations therefore proves that $|\nu|$ vanishes on every polar Borel set.

For bounded $z \in H$ use the uniformly bounded core approximation just constructed. Energy continuity passes its left side in the original identity; diffuseness and dominated convergence pass the right side. This proves (10.13)'s first assertion. For $f \in L^\infty(\lambda)$ take $z = Gf$, which is bounded by the torsion estimate. Symmetry gives

$$\int h f \, d\lambda = \mathcal{E}(h, Gf) = -\int \widetilde{Gf} \, d\nu.$$

Since $h \in L^2(\lambda) \subset L^1(\lambda)$, transposition uniqueness gives $h = u_\nu$. □

The usual product rule extends to bounded members of H: $d(z_1 z_2) = z_1 dz_2 + z_2 dz_1$. Indeed the preceding uniformly bounded core approximations and the original gradient module pass this identity by dominated convergence against the fixed limiting energy measures. This fact will be used for a product of two bounded energy functions, not only a smooth cutoff.

Theorem 10.3 (Strong clipping convergence for signed nest data)**.** *Let $\mu_j = \mathbf{1}_{F_j}\mu$ be the finite-energy nest restrictions from the finite-energy nest in the proof of theorem 10.1, and put $u_j = -G\mu_j$. For every $k > 0$,*

$$T_k u_j \longrightarrow T_k u_\mu \quad \textit{strongly in } H. \tag{10.14}$$

Proof. Let $w = -u_{|\mu|} = G|\mu| \ge 0$ be the positive-data Green potential, with its canonical representative. Positivity and order of the transposition map give $|u_j|, |u_\mu| \le w$ quasi-everywhere for all j. Put $a_M = |\mu|\{w > M\}$, so $a_M \to 0$, and define

$$\zeta_M = \frac{T_{2M} w - T_M w}{M} \in H, \qquad 0 \le \zeta_M \le 1.$$

It equals one on $\{w \ge 2M\}$. The positive-data band estimate (10.11) and the closed clipping rule give

$$\mathcal{E}(\zeta_M) \le \frac{a_M}{M}, \qquad d\zeta_M = \frac{1}{M}\mathbf{1}_{\{M<w<2M\}} \, dw. \tag{10.15}$$

Here dw on the printed bounded band means the compatible clipped gradient $d(T_{2M+1}w)$ there; no globally Radon first-order current of the unbounded w is required. Test the finite-energy equation for u_j with $\zeta_M^2 T_k u_j \in H \cap L^\infty$. The original closed gradient product and clipping rules give

$$\int \zeta_M^2 \, d\Gamma_S(T_k u_j) = -\int \zeta_M^2 T_k \widetilde{u}_j \, d\mu_j - 2\int \zeta_M T_k \widetilde{u}_j \, d\Gamma_S(u_j, \zeta_M).$$

The first term in absolute value is at most $k a_M$. The cross energy is carried where $M < w < 2M$, and there $|u_j| < 2M$, so its u_j factor may be clipped at $2M$. Cauchy–Schwarz and (10.6)

therefore prove

$$\sup_j \Gamma_S(T_k u_j)\{w \geq 2M\} \leq k a_M + 2k\sqrt{\mathcal{E}(\zeta_M)\sup_j \mathcal{E}(T_{2M}u_j)} \leq k a_M + 2k\sqrt{2B_\mu a_M} \longrightarrow 0. \tag{10.16}$$

This is the required uniform control of energy in the region where positive and negative potentials could be large and cancel.

On $A_M = \{w \leq M\}$ one has $|u_j - u_l| \leq 2M$. The first-order clipping rule and (10.6), applied to the finite-energy datum $\mu_j - \mu_l$, give

$$\|\mathbf{1}_{A_M}(du_j - du_l)\|^2 \leq \mathcal{E}(T_{2M+1}(u_j - u_l)) \leq (2M+1)\|\mu_j - \mu_l\|_{\mathrm{TV}} \longrightarrow 0. \tag{10.17}$$

Thus the localized gradients have a fixed limit in the original Hilbert gradient module. Its energy measure is absolutely continuous with respect to λ. For a smooth scalar P with $P(0) = 0$ and compactly supported first derivative, bounded convergence against this fixed limiting module measure proves that $\mathbf{1}_{A_M} d(P(u_j))$ converges strongly. Here $u_j \to u$ in $L^1(\lambda)$, and hence in measure; continuity and boundedness of P' supply the required convergence of its scalar multipliers. The energy of $P(u_j)$ outside A_M is bounded by a fixed multiple of the tail of a clipping $T_L u_j$, where L contains the support of P'. Formula (10.16) makes it small uniformly in j as $M \to \infty$. It follows that $P(u_j)$ is Cauchy in H; its L^2 limit is $P(u)$, by boundedness and the L^1 convergence of u_j.

Finally approximate T_k by smooth normal clamps P_ϵ with uniform scalar error at most ϵ, whose derivatives differ from that of T_k only where $k - \epsilon < |s| < k + \epsilon$. For $0 < \epsilon < k/2$, the finite-energy band formula and the source equation give

$$\begin{aligned} \mathcal{E}(T_k u_j - P_\epsilon(u_j)) &\leq \mathcal{E}(T_{k+\epsilon}u_j - T_{k-\epsilon}u_j) \\ &= -\int (T_{k+\epsilon}\widetilde{u}_j - T_{k-\epsilon}\widetilde{u}_j)\, d\mu_j \leq 2\epsilon B_\mu. \end{aligned} \tag{10.18}$$

The smooth compositions are strongly convergent by the preceding paragraph. First let $j, l \to \infty$ and then $\epsilon \downarrow 0$ in the triangle inequality. This makes $T_k u_j$ Cauchy in H. Its already known L^2 limit is $T_k u$, proving (10.14). □

Theorem 10.4 (Full signed renormalized chain and energy distribution)**.** *For $u = u_\mu$, the increasing compatible energy measure $\Gamma_S(u) = \lim_k \Gamma_S(T_k u)$ is sigma-finite, finite on every bounded value strip, and zero on every exact scalar level. If $\Phi \in C^2(\mathbf{R})$ has bounded first derivative and compactly supported second derivative, then*

$$D(\Phi(u)S) = \Phi'(\widetilde{u})\mu + \Phi''(\widetilde{u})\Gamma_S(u) \tag{10.19}$$

on all original compact smooth tests. Every term is coefficient- defined, and the right side is a finite signed Radon measure. If in addition Φ' has compact support, then $\Phi(u) - \Phi(0) \in H$. For every $0 \leq a < b$,

$$\mathcal{E}(T_b u - T_a u) = -\int (T_b\widetilde{u} - T_a\widetilde{u})\, d\mu. \tag{10.20}$$

The scalar energy image has density

$$\widetilde{u}_*\Gamma_S(u) = k_\mu(t)\, dt, \qquad k_\mu(t) = \begin{cases} \mu\{\widetilde{u} < t\}, & t < 0, \\ -\mu\{\widetilde{u} > t\}, & t > 0. \end{cases} \tag{10.21}$$

This density is nonnegative and bounded by B_μ almost everywhere; its value at zero is immaterial. In particular all bounded-value weighted energy measures used here are finite Radon measures.

Proof. Compatibility and level zero follow from the H clipping identities exactly as in (10.12) and (10.9), without any sign hypothesis on u. Choose M containing the support of Φ'' in its interior. The finite-energy chain for u_j reads

$$D(\Phi(u_j)S) = \Phi'(\widetilde{u}_j)\mu_j + \Phi''(\widetilde{u}_j)\Gamma_S(T_M u_j).$$

The left side passes because Φ is Lipschitz and $u_j \to u$ in $L^1(\lambda)$. The source term passes in variation by the capacity convergence of u_j to u, diffuseness of $|\mu|$, bounded continuity of Φ',

and total-variation convergence of μ_j to μ. Strong convergence in (10.14), the energy variation estimate (10.8), and bounded convergence against the fixed limiting energy measure pass the last term. This proves (10.19). For merely C^2 scalar functions, ordinary smoothing gives the same limit, uniformly in the first two derivatives on the bounded active range and uniformly in scalar values up to the original Lipschitz error. If Φ' is compactly supported, its normalized composition is a smooth function of one bounded clipping and lies in H.

The same strong clipping convergence passes the finite-energy band identity, proving (10.20). Apply the argument separately to the bounded one-sided bands

$$B^+_{a,b}(s) = \min((s-a)^+, b-a), \qquad B^-_{a,b}(s) = -B^+_{a,b}(-s).$$

Their derivatives are the indicators of (a,b) and $(-b,-a)$, respectively. Composing a strongly convergent clipping with these finite hinge combinations is strongly continuous: the closed gradient formula applies, and the limiting H energy is zero on their finitely many corner levels by (10.9). Hence

$$\Gamma_S(u)\{a<u<b\} = -\int_a^b \mu\{u>t\}\,dt,$$
$$\Gamma_S(u)\{-b<u<-a\} = \int_a^b \mu\{u<-t\}\,dt.$$

There is no energy mass at zero or at the interval endpoints. These formulas prove (10.21). Nonnegativity follows because it is the density of a positive measure; the bound follows from total variation of μ. □

Proposition 10.5 (Finite global level measures and clipping curvatures)**.** *The Kato measures $L_t(u;S)$ are finite for every real t, with*

$$\begin{aligned} L_t(u;S)(F) &\le |\mu|\{u\le t\} \quad (t<0),\\ L_t(u;S)(F) &\le |\mu|\{u>t\} \quad (t>0),\\ L_0(u;S)(F) &\le B_\mu. \end{aligned} \tag{10.22}$$

For $M>0$, the actual signed Radon curvature $\nu_M = D(T_M u\, S)$ satisfies

$$\begin{aligned} \nu_M &= \mathbf{1}_{\{-M<u\le M\}}\mu + L_{-M}(u;S) - L_M(u;S),\\ \|\nu_M\|_{\mathrm{TV}} &\le B_\mu, \qquad \|\nu_M-\mu\|_{\mathrm{TV}} \le 2|\mu|\{|u|\ge M\} \longrightarrow 0. \end{aligned} \tag{10.23}$$

Both ν_M and every $L_t(u;S)$ are energy diffuse.

Proof. Apply (10.19) to the smooth one-sided hinges. Their positive defects are $\Phi''_\epsilon(u-t)\Gamma_S(u)$, supported in the scalar strip $t<u<t+\epsilon$. Formula (10.21), the nonnegative unit-integral second derivative, and its bounded density show that their masses are at most B_μ. For $t>0$ the sharper bound is $|\mu|\{u>t\}$; for $t<0$ and sufficiently small ϵ it is $|\mu|\{u<t+\epsilon\}$. They converge on original compact tests to $L_t(u;S)$. Positivity, compact smooth exhaustion and then $\epsilon\downarrow 0$ prove (10.22).

The scalar identity $T_M s = h_{-M}(s) - h_M(s)$ gives the first formula of (10.23). The three source regions $\{u\le -M\}$, $\{-M<u\le M\}$ and $\{u>M\}$ partition the finite-value locus. Triangle inequalities and (10.22) therefore give both variation estimates in (10.23). Since $T_M u \in H$, its finite Radon curvature is diffuse by lemma 10.2.

For any fixed t and $M>|t|$, the exact scalar identity

$$h_t(T_M u) = h_t(u) - h_M(u) \tag{10.24}$$

holds on the finite-value locus and hence in the coefficient-defined L^1 sense. Its left side belongs to H. Its curvature is finite Radon by (10.22) and the Kato decompositions, hence diffuse by the same lemma. Thus

$$L_t(u;S) - L_M(u;S) = D(h_t(T_M u)S) - \mathbf{1}_{\{t<u\le M\}}\mu$$

is a diffuse finite signed measure. On any polar Borel set, the mass of L_t is therefore at most $L_M(F)$, which tends to zero by (10.22). This proves diffuseness of L_t without assuming it passes through an arbitrary weak measure limit. □

Lemma 10.6 (Finite signed Radon data in the H level theorem)**.** *Let $v \in H$ and suppose $D(vS) = \sigma$ is a finite signed Radon measure on the original tests. Then the conclusion (10.10) and its exact carrier assertion holds, with a finite diffuse positive level measure, without assuming finite energy integral of $|\sigma|$.*

Proof. The datum is diffuse and the current equation extends to all bounded H tests by Lemma 10.2. Consequently the finite-energy scalar chain proof applies: its test $\psi\Phi'(v)$ is bounded and lies in H. The one-sided band tests are also bounded H vectors and give (10.21) with v, σ in place of u, μ. Thus the smoothed hinge defects have uniformly bounded total mass, and their positive limit $L_t(v; S)$ is finite. The signed measure $D(h_t(v)S) = \mathbf{1}_{\{v>t\}}\sigma + L_t(v; S)$ is finite and comes from an H vector. It is therefore diffuse by the preceding lemma, as is L_t.

The original current identity for this hinge source now extends to every bounded H test. Take $z = \psi g(v)$ with ψ an original compact nonnegative smooth function and g smooth bounded, with bounded derivative, nonnegative and zero near t. Strong convergence of the smooth hinges in H and bounded convergence against the finite measure $|\sigma|$ pass

$$\int z\, dL_{t,\epsilon} = -\mathcal{E}(h_{t,\epsilon}(v), z) - \int z\Phi'_\epsilon(v - t)\, d\sigma$$

to $\int z\, dL_t$. Its left side is zero for small ϵ because $g\Phi''_\epsilon = 0$. The same countable family of nonnegative tests as in the finite-energy level-carrier proof cited above shows that L_t is carried by $\{\widetilde{v} = t\}$. □

Theorem 10.7 (Exact level carriers and coarea for all signed diffuse data)**.** *For every finite signed energy-diffuse datum and every real t,*

$$\mathbf{1}_{\{\widetilde{u}\neq t\}} L_t(u; S) = 0. \tag{10.25}$$

For every bounded compactly supported Borel scalar θ,

$$\int \theta(t) L_t(u; S)\, dt = \theta(\widetilde{u})\Gamma_S(u) \tag{10.26}$$

as finite signed Radon measures. The hard-curvature variation bounds (10.23) hold on the complete original space and every actual moment refinement.

Proof. First identify $\nu_M = \mu$ on $A_M = \{|u| < M\}$. Choose a smooth Φ with Φ' zero outside $[-M, M]$ and strictly positive on $(-M, M)$. The identities $\Phi(T_M u) = \Phi(u)$ and (10.12) make their weighted energies agree. Compare (10.19) with the bounded H chain for $T_M u$ and its finite signed datum ν_M. The latter chain is valid by the bounded-test extension, so $\Phi'(u)(\nu_M - \mu) = 0$, proving the assertion on A_M.

Fix t, choose $M > |t|$, and apply the preceding H level lemma to $v = T_M u$. The level measure $L_t(v; S)$ is carried by $\{v = t\} = \{u = t\}$. Compare the curvature of the two sides of (10.24), and then restrict their actual Radon measures to A_M. The just proved identity $\nu_M = \mu$ on A_M cancels the source terms and gives

$$\mathbf{1}_{A_M}(L_t(u; S) - L_M(u; S)) = \mathbf{1}_{A_M} L_t(T_M u; S).$$

It follows that $L_t(u; S)(A_M \cap \{u \neq t\}) \leq L_M(u; S)(F) \to 0$. The sets A_M exhaust the finite-value locus, and L_t is diffuse, so it gives no mass to the omitted polar infinite- value set. This proves (10.25) by monotone measure convergence.

For smooth compactly supported θ, take $\Phi(s) = \int \theta(t)(s - t)^+ dt$. It has bounded first derivative and compactly supported second derivative θ, so (10.19) and Fubini prove (10.26) on every original test. The positive kernels L_t, their global bound by B_μ, and their Borel original-test pairings extend the equality to nonnegative Borel θ; subtract positive and negative parts. The right side is finite by (10.21). □

Definition 10.8 (The signed renormalized zero-boundary class)**.** For the specified finite signed diffuse datum, a candidate is $v \in L^1(F,\lambda)$ with a finite energy-quasicontinuous representative and $T_k v \in H$ for every $k > 0$. Use the compatible sigma-finite energy measure of its clippings. Require the original current identity

$$D(\Phi(v)S) = \Phi'(\widetilde{v})\mu + \Phi''(\widetilde{v})\Gamma_S(v) \tag{10.27}$$

for every $\Phi \in C^2(\mathbf{R})$ with compactly supported first derivative, and the radiation condition $k^{-1}\mathcal{E}(T_{2k}v - T_k v) \to 0$. These conditions specify the boundary and solution class without using the transposition identity in their definition.

Theorem 10.9 (The complete finite-diffuse relative renormalized Dirichlet problem)**.** *Every finite signed energy-diffuse Radon datum has exactly one candidate in definition 10.8. It is the transposition solution u_μ, and all its original current, exact-level, coarea and global curvature-variation conclusions above hold. The construction and solution class are independent of the chosen faithful reference measure.*

Proof. Existence follows from (10.19), the truncation membership, quasicontinuity, and the band identity (10.20), which gives $k^{-1}\mathcal{E}(T_{2k}u - T_k u) \le |\mu|\{|u| > k\} \to 0$. For uniqueness choose smooth odd scalar clamps Φ_k fixing the interval $[-k,k]$, constant outside $[-2k,2k]$, with $\Phi_k(0) = 0$, $0 \le \Phi_k' \le 1$ and $|\Phi_k''| \le C/k$ on the two transition bands. Then $|\Phi_k(s)| \le |s|$ and $\Phi_k(s) \to s$. For a candidate v, its prescribed chain gives

$$\begin{aligned} \nu_k &:= D(\Phi_k(v)S) = \Phi_k'(v)\mu + \Phi_k''(v)\Gamma_S(v), \\ \|\nu_k - \mu\|_{\mathrm{TV}} &\le |\mu|\{|v| > k\} + \frac{C}{k}\mathcal{E}(T_{2k}v - T_k v) \to 0. \end{aligned} \tag{10.28}$$

The measures ν_k are finite signed Radon measures. The potential $\Phi_k(v) = \Phi_k(T_{2k}v)$ belongs to H, so lemma 10.2 proves both diffuseness of ν_k and $\Phi_k(v) = u_{\nu_k}$. The established total-variation stability gives $\|\Phi_k(v) - u_\mu\|_1 \le 2\|\nu_k - \mu\|_{\mathrm{TV}} \to 0$. Dominated convergence also gives $\Phi_k(v) \to v$ in $L^1(\lambda)$. Thus $v = u_\mu$ almost everywhere. Equality of all clipped H vectors identifies the canonical representatives quasi-everywhere. Reference independence follows because the original core energy $\mathcal{E}(f,g) = \int d'f \wedge d''g \wedge S$ is independent of the faithful measure. Coercivity identifies each realization with the same Hilbert completion. Its compact-core zero-capacity, polar sets and canonical representatives coincide. A common finite-energy nest then gives the same capacity limit of the same Riesz potentials. □

10.4. **Stability under signed measure convergence.**

Theorem 10.10 (Strong clipping and smooth-curvature stability)**.** *Suppose $\|\mu_j - \mu\|_{\mathrm{TV}} \to 0$, and write $u_j = u_{\mu_j}$, $u = u_\mu$ for the unique intrinsic renormalized solutions of $D(u_j S) = \mu_j$, $D(uS) = \mu$. Then, for every fixed $k > 0$,*

$$\|T_k u_j - T_k u\|_H \longrightarrow 0. \tag{10.29}$$

In particular, for any fixed $k, l > 0$, the finite original energy measures converge in global total variation:

$$\Gamma_S(T_k u_j, T_l u_j) \longrightarrow \Gamma_S(T_k u, T_l u). \tag{10.30}$$

If $\Phi \in C^2(\mathbf{R})$ has bounded first derivative and compactly supported second derivative, then

$$D(\Phi(u_j)S) \longrightarrow D(\Phi(u)S) \quad \text{in global total variation.} \tag{10.31}$$

If, more restrictively, Φ' has compact support, then $\Phi(u_j) - \Phi(0) \to \Phi(u) - \Phi(0)$ strongly in H. For this latter class both original first-order currents $d'(\Phi(u_j)S)$ and $d''(\Phi(u_j)S)$ converge in local coefficient total variation, against every original compact complementary smooth test and every actual moment refinement. The zero-order multipliers $\Phi(u_j)S$ converge in that same local coefficient sense under the hypotheses of (10.31).

Proof. The finite-datum estimates give $u_j \to u$ in energy capacity and in $L^1(\lambda)$. The clipped functions converge in $L^2(\lambda)$: they are bounded by k, and $\lambda(F) < \infty$ because the part domain is relatively compact in the neighborhood where λ is Radon. Also $\mathcal{E}(T_k u_j) \le k\|\mu_j\|_{\mathrm{TV}}$, so these vectors are bounded in H. Every weakly convergent subsequence therefore has the unique L^2 representative $T_k u$. Consequently the whole sequence converges weakly in H.

Take $a = 0, b = k$ in (10.20). The exact norm identity is

$$\mathcal{E}(T_k u_j) = -\int_F T_k \widetilde{u}_j \, d\mu_j, \qquad \mathcal{E}(T_k u) = -\int_F T_k \widetilde{u} \, d\mu. \tag{10.32}$$

The difference between the first datum integral and the second is bounded in absolute value by

$$k\|\mu_j - \mu\|_{\mathrm{TV}} + \int_F |T_k \widetilde{u}_j - T_k \widetilde{u}| \, d|\mu|. \tag{10.33}$$

The second term tends to zero by energy-capacity convergence, finite diffuseness of $|\mu|$, and boundedness of the clippings. Thus the norms in (10.32) converge. Weak convergence and convergence of norms in the Hilbert space prove (10.29). The exact band identity is what turns weak clipping convergence into norm convergence.

The original closed gradient module gives, for $f_j, g_j, f, g \in H$,

$$\|\Gamma_S(f_j, g_j) - \Gamma_S(f, g)\|_{\mathrm{TV}} \le \|f_j - f\|_H \|g_j\|_H + \|f\|_H \|g_j - g\|_H. \tag{10.34}$$

This is measure Cauchy–Schwarz after adding and subtracting $\Gamma_S(f, g_j)$. Applying (10.29) proves (10.30).

For (10.31), choose M larger than the support radius of Φ''. The all-original chain identity reads

$$D(\Phi(u_j)S) = \Phi'(\widetilde{u}_j)\mu_j + \Phi''(\widetilde{u}_j)\Gamma_S(T_M u_j). \tag{10.35}$$

The first term converges in variation: its changing-datum error is at most $\|\Phi'\|_\infty \|\mu_j - \mu\|_{\mathrm{TV}}$, and its fixed-datum multiplier error tends to zero by finite diffuseness and capacity convergence. For the second term use (10.30) and then bounded convergence in measure against the fixed finite diffuse measure $\Gamma_S(T_M u)$. This gives its variation convergence as well and proves (10.31). The zero-order multiplier claim follows from the Lipschitz bound for Φ, $L^1(\lambda)$ stability and the faithful original coefficient domination.

If Φ' has compact support, its normalized composition is a smooth function of one sufficiently large clipping. The closed gradient formula and (10.29) give strong convergence: the changing-gradient term is bounded by $\|\Phi'\|_\infty \|T_M u_j - T_M u\|_H$, and the changing scalar factor tends to zero in the fixed finite energy measure. Its L^2 representative converges by the same bounded composition argument. Finally, on a fixed original compact stage, write any complementary one-form test as a finite sum of bounded smooth coefficients times differentials of original local moment functions. Multiply those moment functions by original compact cutoffs equal to one on the test support. The coefficient pairing of a first-order derivative with each such term is an original mixed gradient measure; (10.34), or its one-variable Cauchy–Schwarz bound, controls its variation by a fixed constant times the H-norm of the changing potential. This proves local coefficient-TV convergence of both first-order currents. The argument starts with an arbitrary original test and is valid after any further actual moment refinement.

□

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[Dual] *Closed bands, energy duals, and nonlinear source gaps*, internal research manuscript, twelfth record, 12 November 2026. See Sections 12–13, 25 and 31 for the dual and probe constructions.

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