Bounded classical curvature kernels and arbitrary-background potential quotients

Fifth research record in non-Archimedean Bedford–Taylor theory

Research record prepared with ChatGPT
Relative to the authorized manuscripts R, B, and A2

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Abstract

Let $K/\mathbf{Q}_p$ be finite with its prescribed absolute value. On an ordinary open in a good Hausdorff boundaryless quasi-smooth K-analytic space, we prove that the actual difference of two locally bounded classical plurisubharmonic functions with equal original CLD curvature is continuous and classically pluriharmonic. It follows that their relative curvatures agree for every fixed symmetric original-positive separately d'- and d''-closed background current. The mixed bounded Bedford–Taylor calculus therefore descends in its differentiated slots to currents with bounded classical local potentials.

The proof supplies the two gates left open in the fourth record: kernel preservation after every complete scalar extension and ordinary continuity of the actual curvewise harmonic difference. The first uses prescribed smooth rows, finite-etale traces, full analytic fiber potentials including type IV, and a simultaneous product induction. The second uses a Shilov bound proved without ambient semicontinuity, polynomial Noether coordinates, and exact logarithmic interpolation on strictly semistable alteration charts. All equalities are tested on the full original CLD test sheaf. Explicit scope tests exclude weaker closedness, arbitrary bounded upper-semicontinuous representatives, and an unrestricted passage to relative energy or capacitary completions. The previously proved seven fixed-background BT properties are retained as inputs. An exact global finite-dimensional curvature hypothesis yields local uniform compactness of normalized classical potentials. Without that hypothesis, a two-intersection example shows that the relative classical image can fail to be closed in total variation even when no boundary values are prescribed.

Reading guide

The exact main theorem and its short final proof are in Section 10. The new analytic and geometric arguments are in Sections 4–9. Section 1 declares the inputs and the point-value and test-form conventions. Section 30 compares the current status with R4. The later examples distinguish false extensions from the stated theorem.

This is a focused continuation. It contains the new proofs rather than reprinting all 304 pages of R4. The complete archive retains the exact frozen R4 archive, its original internal inputs, all modular and stand-alone R5 LaTeX sources, source-location records, independent reading records, and the continuation handoff. References beginning "R4–" are numbered results in that frozen final record. Historical open statements in R4 are superseded only in the precise scopes stated here.

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1 The question, conventions, and frozen inputs

The question left open in the fourth record was

$$D[u] = D[v] \quad \Longrightarrow \quad D((u-v)S) = 0, \qquad u, v \in \mathcal{C}_b(U), \tag{1.1}$$

for an arbitrary fixed symmetric positive separately closed background current S. The equality on the left is an equality of original CLD currents with background [1]. It must not be silently interpreted as equality after multiplication by a singular coefficient measure. The actual point values of u and v matter on the right.

The original ground field throughout the main theorem is a finite extension $K/\mathbf{Q}_p$ with its given absolute value. An *allowed space* is good, Hausdorff, boundaryless, quasi-smooth, and of pure dimension. All opens are ordinary topological opens. We set $D = d'd''$. A (p,p)-current on a space of dimension n acts on original compact smooth $(n-p, n-p)$-forms. Unless a different degree is explicitly stated, positivity is the original positive cone of R, namely the symmetric quadratic-form cone of [CLD12, Definition 5.1.1(c) and §5.4]. It is not identified with either strong or weak positivity.

The coefficient-Radon arguments use this middle cone directly. They do not invoke the strong-test order-zero assertion as printed in [CLD25, Proposition 7.2.9]. Section 21 gives an explicit closed real-tropical counterexample to that broader assertion and checks the distinct 2012 hypotheses. This does not change R's stipulated cone or any of the authorized internal inputs.

The word *actual* means that no function is replaced on a null, polar, or coefficient-null set. For a locally bounded Borel function a and a coefficient-Radon current T, multiplication means

$$\langle aT, \eta \rangle = \int a \, d\mu_{T,\eta}$$

on the actual Berkovich space. A measure specified only on one tropical image does not give this operation.

1.1 Authorized internal results

`INTERNAL_INPUT`: R.
R provides the continuous original-positive local calculus over the original field, its continuous semipositive smoothing, the original positive-cone coefficient-Radon controls, and compatibility of weak convergence of original-positive currents with a fixed continuous multiplier. Its statements use the original test category. Its local algebraic tube construction is used only over the original field, after B's algebraic localization.

`INTERNAL_INPUT`: B.
B, Definition 2.2, defines $\mathcal{C}_b(U)$ by actual local boundedness, upper semicontinuity, and subharmonic pullbacks along every analytic curve after every complete isometric field extension. Theorem 2.5 gives decreasing continuous rational semipositive PL rows on one fixed ordinary neighborhood. Proposition 4.5 and the pure-Fubini–Study smoothing used in its construction give the prescribed smooth positive rows needed here. Explicitly, if $p_j \downarrow u$ and $|s_j - p_j| \le \varepsilon_j$ with $\varepsilon_{j+1} \le \varepsilon_j/4$, then $s_j + 2\varepsilon_j \downarrow u$ because its values lie between $p_j + \varepsilon_j$ and $p_j + 3\varepsilon_j$. The original smooth curvatures stay positive. The fixed domain, rather than a separate shrinking domain for every j, is essential. B, Lemma D.4, supplies ordinary affine algebraic neighborhoods. B, Appendix A.2, supplies the positive Radon Laplacian and its original scalar-test identity on smooth algebraic curves over every complete nontrivially valued field of characteristic zero; that particular curve statement has the larger field range explicitly stated in B.

`INTERNAL_INPUT`: **A2.**

The same-model geometric adelic nefness result is already complete under its stated arithmetic and boundary assumptions. It is not used to prove the local quotient and is not counted as an open problem or a result of this round.

No comprehensive proof audit of R, B, or A2 has been undertaken. The checks concern their statements, fields, positivity cones, test categories, and dependencies.

1.2 Results proved in the earlier project records

The frozen R4 record contains the full R1–R3 sources and has 304 pages in its final stand-alone version. References of the form "R4–$a.b$" below refer to that final pagination and numbering, not to the older intermediate modular build. These results retain their original project provenance; they are not new theorems of R5.

Input	Exact content used	Location in final R4
Fixed-background BT	All seven bounded-potential properties, including weighted joint decreasing convergence, for one arbitrary original-positive separately closed S.	Theorem 3.7 and preceding construction; p. 18 onward
Common smooth rows	Prescribed smaller norm boxes admit simultaneous smooth decreasing positive rows for finitely many original-K classical potentials.	Lemma 7.1, p. 24
Compatible piecewise tests	Tangentially compatible original piecewise smooth tests admit uniform pairing approximation against mass-bounded positive closed first-order currents.	Theorem 71.4, p. 240
Original-field analytic kernel	$D[u] = D[v]$ remains true after every analytic map of allowed original-K spaces; in particular after every finite polydisc projection.	Theorem 75.3, p. 253
Exact fiber test spreading	Every compact original test on an algebraic parameter fiber is the exact restriction of a compact original test on an ambient parameter neighborhood.	Lemma 77.2, p. 256
Finite-coefficient descent	Every finite family of original tests after L/K descends exactly to $\overline{K(c_1, \dots, c_m)} \subset L$ for a finite tuple.	Lemma 78.1, p. 260
Energy and completions	The faithful form, complete-pole splitting, and bounded relative capacitary closure are used in the energy consequences and scope tests, not in the proof of (Q).	Theorems 17.6, 18.5, 84.2–84.3, 86.1–86.2 in the frozen record

The initial statement $D[u] = D[v]$ is never pulled back as an arbitrary current. Each transfer theorem proves equality for the actual composed potentials, on every original test on the new space. The distinction is indispensable.

1.3 Logical structure of the new argument

The analytic part extends the prescribed-row finite-etale kernel argument to auxiliary complete fields. A continuous potential for the *full* analytic fiber of every nonrigid point of the affine

line then closes the scalar-extension gate, including type IV. The product kernel is propagated simultaneously in every finite product dimension. The resulting analytic-map kernel gives actual harmonicity on all curve pullbacks, including their type-I and type-IV values.

The geometric part does not assume that a curvewise harmonic function is upper semicontinuous. It proves its Shilov bound by finite flat coordinates and normalized algebraic curve slices. Strictly semistable alteration charts supply continuous logarithmic interpolation of their finite Shilov data. The Shilov bound identifies the actual function with those interpolants, and proper compact descent proves continuity.

Only after that argument are both signs of $u - v$ known to be classical psh. Applying the existing first-order positive construction to the arbitrary background S makes $D((u-v)S)$ and its negative positive, which proves (1.1). The quotient is therefore downstream of all the kernel and continuity arguments.

Status convention. `NEW_PROVED` means that a complete argument is supplied in this project relative to the declared inputs. It does not claim worldwide priority, external refereeing, or formal verification. It is distinct from `DIRECT_COROLLARY`, `INTERNAL_INPUT`, and `EXTERNAL_KNOWN`. The logical statuses are `PROVED`, `CONDITIONAL`, `OPEN`, and `FALSE`; an invalid proof mechanism is labeled `FAILED_ATTEMPT`.

2 Field-independent tools for prescribed smooth positive rows

Let F be a complete non-Archimedean field with its given absolute value. Let U be a good Hausdorff boundaryless F-analytic space of pure dimension $n \geq 1$. We use only the original CLD smooth forms, their canonical integration, and $D = d'd''$. A current is positive in the original positive cone used in R, and all currents called positive below are symmetric. This is not a convention identifying the positive, strongly positive and weakly positive cones.

A *prescribed smooth positive row* for a function u on an ordinary open O means original smooth functions u_j on that same O, with positive $(1,1)$-forms Du_j, such that u_j decreases pointwise to the actual finite locally bounded function u. Whenever a continuous potential is used below, its smooth positive approximation is likewise part of the hypotheses. No regularization theorem over F is asserted. In particular, these results may be applied to rows already obtained over the original finite extension of $\mathbf{Q}_p$ and then pulled back by an analytic map or by scalar extension.

2.1 Positive coefficient measures over the given field

Lemma 2.1 (Finite positive controls and actual coefficient measures)**.** *Let T be an original positive current of bidegree (p,p) on O, and put $q = n - p$. For every original compact smooth complementary test η, there is a signed Radon measure $\mu_{T,\eta}$ on O such that*

$$\langle T, a\eta \rangle = \int a \, d\mu_{T,\eta}$$

for every original smooth scalar a. The assignment is compatible with finite localization, relations between original forms, and common original moment refinements.

There are finitely many original positive compact smooth tests Θ_ℓ and a constant C_η, depending only on a fixed presentation of η and its compact support stage, such that

$$|\mu_{T,\eta}| \leq C_\eta \sum_\ell \mu_{T,\Theta_\ell}. \tag{R}$$

The same controls work after adjoining additional scalar moments. Consequently a fixed locally bounded Borel multiplier a defines the original current aT. If $T_j \to T$ weakly and all these currents are positive, then $aT_j \to aT$ for every fixed continuous multiplier a. Weak convergence alone is not claimed to allow an arbitrary fixed discontinuous multiplier.

Status: `DIRECT_COROLLARY` of the original finite-dimensional positive-cone controls and Radon representation. The proof below records why this part has no restriction to a discretely valued or locally compact ground field.

Proof. Localize the test into finitely many ordinary moment presentations with compact support. On one such presentation write $t = (\log|g_1|, \dots, \log|g_r|)$, where the g_i are analytic units, and put

$$\beta = \sum_{\nu=1}^{r} d't_\nu \wedge d''t_\nu.$$

In the usual signed identification of symmetric (q,q)-tensors with symmetric quadratic forms on $\bigwedge^q \mathbf{R}^r$, $\beta^q/q!$ is the identity matrix. On a fixed compact real presentation stage it therefore dominates, up to a finite constant, both signs of the symmetric coefficient tensor of η. With an original compact smooth cutoff χ equal to one near that stage, this gives

$$C\chi\beta^q \pm \eta \geq 0.$$

The control $\chi\beta^q$ is strongly positive. The inequalities are in the original positive cone; they do not assume that η is strongly positive. Antisymmetric parts have zero pairing with the symmetric current.

For every nonnegative original smooth scalar b, positivity gives

$$|\langle T, b\eta\rangle| \leq C\langle T, b\chi\beta^q\rangle.$$

Original smooth scalar functions separate points on each good affinoid neighborhood: the absolute values of analytic functions separate Berkovich points, and smooth real profiles constant near zero turn these into original smooth scalars. Their compact versions are uniformly dense in continuous compactly supported functions on a fixed compact neighborhood. This is the original scalar Stone–Weierstrass argument, valid over every complete F; see also [CLD25, Propositions 15.2.6–15.2.9]. Positivity and the displayed control extend the functionals uniquely to that continuous scalar space. Riesz representation gives the positive measure of the control and the signed measure of η, with the displayed domination. The finitely many pieces give (R).

Relations between forms and changes of presentation are compatible because the corresponding original scalar-tested functionals are identical; uniqueness of Radon measures preserves every such identity. The matrix inequalities pull back under common refinements, so adding scalar coordinates does not change the controls. Applying this argument to the finitely many coefficient basis forms gives the actual coefficient-Radon description. It also proves continuity of aT in the original test topology for locally bounded Borel a: on each fixed compact presentation, the coefficient norms of a test bound the pairing by its fixed control masses times $\sup|a|$.

For the last assertion, weak convergence bounds the finitely many positive control masses. Approximate the fixed continuous scalar a uniformly on their compact support by original smooth scalars. For each smooth scalar the claimed convergence is the definition of weak convergence. Inequality (R) bounds the two approximation errors uniformly in j. Taking the approximation error to zero proves the assertion. No countable atlas or measure separability has entered the argument. □

2.2 A direct first-order estimate near an analytic zero set

Let H be analytic on O and not identically zero on any irreducible component, and put $Z = \{H = 0\}$. A fact from the original calculus used below is

$$\operatorname{supp}\omega \cap Z = \varnothing \quad \text{for every original smooth top-degree form } \omega. \tag{2.1}$$

Indeed $\dim Z < n$, and the germ of an original form of bidegree (n,n) is zero at any point whose residual transcendence dimension $d_F(x)$ is less than n; see [CLD25, Proposition 15.2.1]. In particular $\log|H|$ is bounded on the compact actual support of every such compact form. This also implies its usual local integrability against original smooth top-degree trace measures.

For smooth real functions put

$$\Gamma(a,b) = \tfrac{1}{2}(d'a \wedge d''b + d'b \wedge d''a), \qquad \Gamma(a) = d'a \wedge d''a.$$

All first-order pairings in the following proof involve smooth functions. No energy object for an unknown nonsmooth function is being assumed.

Lemma 2.2 (Uniform analytic-zero-set estimate for a prescribed row)**.** *Let $u_j \downarrow u$ be a prescribed smooth positive row on O. Fix one original moment presentation near a compact set K, put $\beta = \sum d't_\nu \wedge d''t_\nu$, and set $\theta = \beta^{n-1}$, with $\theta = 1$ when $n = 1$. Choose original smooth cutoffs $0 \le \chi, \chi_1 \le 1$ with $\chi = 1$ near K and $\chi_1 = 1$ near $\operatorname{supp}\chi$, both compactly supported in that presentation.*

After shrinking to a relatively compact ordinary neighborhood of these supports, there are smooth psh functions q_N, $N \ge N_0$, such that

$$-1 \le q_N \le 0, \qquad q_N \longrightarrow -\mathbf{1}_Z$$

pointwise, and a constant C, independent of j, N, such that

$$0 \le \int \chi(-q_N)\, Du_j \wedge \theta \le CN^{-1/2}. \tag{2.2}$$

The constant depends on the two cutoffs, the fixed moment frame, H, and one bound for the row on their compact neighborhood.

Proof. Choose a constant A so that $\varphi = \log|H| - A \le -2$ on this neighborhood. Let ψ be a smooth convex nondecreasing real function which is zero on $(-\infty, -1]$, equals the identity on $[1,\infty)$, and satisfies

$$\max(0,s) \le \psi(s) \le \max(0,s) + 1.$$

Set

$$\ell_N = -N + \psi(\varphi + N), \qquad q_N = \ell_N/N,$$

extending ℓ_N by $-N$ near every zero of H. These extensions are original smooth: the nonconstant part is supported in the unit locus of H. Convexity gives $Dq_N \ge 0$. For $N \ge 2$ the displayed bounds and pointwise limit hold.

Whenever ω is a fixed compact original smooth top form, (2.1) shows that φ is bounded on $\operatorname{supp}\omega$. Hence, for all sufficiently large N depending on ω,

$$q_N = \varphi/N \quad \text{on } \operatorname{supp}\omega. \tag{2.3}$$

Stokes against the smooth closed form θ gives

$$0 \le \int \chi Dq_N \wedge \theta = \int q_N D\chi \wedge \theta = O(N^{-1}).$$

The constant is finite by (2.3) applied to $D\chi \wedge \theta$.

Write $E_\chi(a) = \int \chi \Gamma(a) \wedge \theta$. The smooth square identity and compact Stokes yield

$$2E_\chi(q_N) = \int q_N^2 D\chi \wedge \theta - 2 \int \chi q_N Dq_N \wedge \theta.$$

Since $-1 \le q_N \le 0$, the second term is at most $2 \int \chi Dq_N \wedge \theta = O(N^{-1})$. The absolute value of the first term is $O(N^{-2})$, again by (2.3). Consequently

$$E_\chi(q_N) = O(N^{-1}), \qquad \int q_N^2 \Gamma(\chi) \wedge \theta = O(N^{-2}). \tag{2.4}$$

The second assertion uses the same support argument for the fixed top form $\Gamma(\chi) \wedge \theta$.

The row is uniformly bounded on the compact support stage. Indeed $u \le u_j \le u_1$, u is locally bounded, and u_1 is continuous. Write $|u_j| \le M$ there. For any fixed nonnegative compact smooth cutoff ξ in that stage,

$$\int \xi Du_j \wedge \theta = \int u_j D\xi \wedge \theta \le M\, |\mu_{D\xi \wedge \theta}|(O).$$

Combining this with

$$2E_\xi(u_j) = \int u_j^2 D\xi \wedge \theta - 2 \int \xi u_j Du_j \wedge \theta$$

gives a uniform bound for $E_\xi(u_j)$. Apply it to χ and χ_1.

One integration by parts, and symmetry of the remaining form, now give the exact identity

$$\int \chi(-q_N) Du_j \wedge \theta = \int \chi \Gamma(q_N, u_j) \wedge \theta + \int q_N \Gamma(\chi, u_j) \wedge \theta.$$

The positive-square Cauchy–Schwarz inequality bounds its absolute value by

$$E_\chi(q_N)^{1/2} E_\chi(u_j)^{1/2} + \left(\int q_N^2 \Gamma(\chi) \wedge \theta \right)^{1/2} E_{\chi_1}(u_j)^{1/2}.$$

Here $\chi_1 = 1$ near $\operatorname{supp} d\chi$ justifies the last energy factor. Equation (2.4) proves (2.2). All integrations occur inside one original moment presentation with compact cutoffs, so no boundary term or unproved global exhaustion is used. □

Theorem 2.3 (First-order nonconcentration for prescribed rows)**.** *Let u have a prescribed smooth positive row locally on U. Then $D[u]$ is an original positive separately closed coefficient-Radon current. If an analytic function H does not vanish identically on any local irreducible component, every coefficient measure of $D[u]$ gives zero mass to $\{H = 0\}$. In particular this holds on every proper analytic divisor. All assertions are on the full original test sheaf over F.*

Status: `NEW_PROVED` for the stated prescribed-row first-order extension. The conclusion is not an assertion of decreasing regularization for all classical functions over F.

Proof. Bounded convergence on each compact support against the fixed smooth top-form measures gives $[u_j] \to [u]$. Hence $Du_j \to D[u]$. Positivity and separate closedness pass to this limit, and Lemma 2.1 gives its coefficient measures.

In the notation of the preceding lemma, fix N. The form $\chi(-q_N)\theta$ is an original smooth compact test. Weak convergence passes (2.2) to

$$\int \chi(-q_N)\, D[u] \wedge \theta \le C N^{-1/2}.$$

Now apply bounded convergence to the ONE fixed finite positive measure $\chi D[u] \wedge \theta$. Since $-q_N \to \mathbf{1}_Z$, it follows that

$$(\chi D[u] \wedge \theta)(Z) = 0.$$

For an arbitrary complementary test on this stage, the finite positive control construction dominates the variation of its coefficient measure by a constant times $\chi D[u] \wedge \beta^{n-1}$, with a cutoff equal to one on its support. Thus every coefficient measure vanishes on Z. Finite localization handles an arbitrary compact test. First j and then N were sent to infinity; no joint discontinuous limit is used. □

2.3 One relative slot and exchange with a continuous slot

Proposition 2.4 (Relative first-order construction from a given row)**.** *Let S be any original positive separately closed (p,p)-current, $p < n$, and let u have a prescribed smooth positive row. Then*

$$D(uS) = \lim_j Du_j \wedge S$$

is an original positive separately closed coefficient-Radon current. Its definition is independent of the prescribed row.

Proof. Lemma 2.1 defines the actual Borel product uS. Since u_j is locally uniformly bounded and converges pointwise, bounded convergence for the fixed coefficient measures of S gives $u_j S \to uS$. Distributional differentiation gives the displayed convergence. Every $Du_j \wedge S$ is positive and separately closed. Apply Lemma 2.1 to its limit. The defining expression $D(uS)$ depends only on the fixed actual function u, so it is independent of the row. □

Proposition 2.5 (Exchange with a prescribed continuous potential)**.** *Let S be as above, with $p \le n-2$. Let u have a prescribed smooth positive row. Let v be finite continuous and admit locally either a uniformly convergent smooth positive approximation or a prescribed smooth positive decreasing row. Then, on every original compact complementary test,*

$$D\big(uD(vS)\big) = D\big(vD(uS)\big).$$

Status: `DIRECT_COROLLARY` for both propositions. They use no general regularization over F and no joint decreasing continuity theorem for two discontinuous slots.

Proof. The relative first-order currents exist by the preceding proposition; for a uniform approximation of v, the same proof uses uniform convergence instead of bounded convergence. For every smooth u_j, the distributional product rule and separate closedness give

$$D\big(u_j D(vS)\big) = Du_j \wedge D(vS) = D\big(v(Du_j \wedge S)\big).$$

On the left, bounded convergence uses the fixed coefficient measures of $D(vS)$ and yields $D(uD(vS))$. On the right, $Du_j \wedge S \to D(uS)$ through positive currents, and the fixed multiplier v is continuous. The last assertion of Lemma 2.1 therefore gives the limit $D(vD(uS))$. These are two justified one-parameter passages, not an interchange of discontinuous limits. □

Scope of the package. The field-independent ingredients are finite positive matrix controls, compact smooth scalar density, original Stokes, and the vanishing of top-degree form germs on proper analytic subsets. The only approximations of the potentials are the prescribed rows. These results supply the first-order analytic-divisor removal and continuous-slot exchange used in an original-test projection argument. They do not establish a bounded mixed calculus for arbitrary unknown potentials over F, the arbitrary-background potential quotient, or a new pullback operation for arbitrary currents.

3 Finite etale kernel transfer for prescribed smooth rows

Let F be a complete nontrivially valued field of characteristic zero. Its absolute value is fixed. All analytic spaces in this section are good, Hausdorff, boundaryless, and quasi-smooth of pure dimension. All forms and currents are in the original CLD sheaf. In particular, "positive" denotes the positive semidefinite cone, with no replacement by the weakly smooth or delta-form sheaf.

Definition 3.1 (A prescribed smooth row)**.** An actual locally bounded classical psh function a on an ordinary open U is said to have a prescribed smooth row if every point has one ordinary neighborhood V and original smooth functions a_j on V such that $a_j \downarrow a$ pointwise and Da_j is a positive $(1,1)$-form. The neighborhoods do not depend on j. The word "classical" retains the upper-semicontinuity and all-complete-extension curve conditions. Those conditions are not deduced from the current equation in this definition.

The functions occurring in the original problem have these rows by the authorized regularization in B, followed by smoothing its pure Fubini–Study presentations on a common smaller domain. Explicitly, if $p_j \downarrow a$ are the continuous semipositive weights, choose smooth positive weights s_j with $|s_j - p_j| \le \epsilon_j$, where $\epsilon_{j+1} \le \epsilon_j/4$. Then $s_j + 3\epsilon_j$ decrease and have limit a. The same fixed rows pull back to every complete extension and every analytic morphism: smooth pullback commutes with D, preserves positive $(1,1)$-forms, and preserves pointwise decrease. Classicality of the limit is already supplied by the original all-extension definition. This is a `DIRECT_COROLLARY` about specified rows; it is not a regularization theorem for all classical psh functions over F.

Lemma 3.2 (Piecewise tests over an arbitrary complete field)**.** *Let η be a compactly supported finite original piecewise smooth (q,q)-test: in a finite unit-moment presentation it is piecewise smooth on a finite affine hyperplane arrangement, smooth up to each closed chamber, and its tangential restrictions agree on all common faces. If symmetric positive separately closed currents T_j of complementary bidegree converge to a symmetric positive separately closed current T on all original smooth tests, then*

$$\langle T_j, \eta\rangle \longrightarrow \langle T, \eta\rangle.$$

The piecewise pairing uses the actual coefficient Radon measures and is independent of the chamber representatives chosen on the walls.

Proof. The proof is finite dimensional once the field-independent coefficient control of Lemma 2.1 has been supplied. We give the quantitative steps, so that no arithmetic hypothesis from R4 is carried over implicitly.

Write the unit logarithms as $t_1, \dots, t_m$, put $\beta = \sum_i d't_i \wedge d''t_i$, and enlarge the presentation to include the finitely many cutoffs. For an affine wall equation ℓ, the smooth convex function $g_\epsilon = (\ell^2 + \epsilon^2)^{1/2}$ satisfies

$$Dg_\epsilon \ge 2^{-3/2}\epsilon^{-1}\mathbf{1}_{\{|\ell|\le\epsilon\}} d'\ell \wedge d''\ell.$$

Compact Stokes, separate closedness, and the fixed coefficient controls imply, on each compact stage H,

$$(d'\ell \wedge d''\ell \wedge \beta^{q-1} \wedge T_j)(H \cap \{|\ell| \le \epsilon\}) \le C\epsilon M_j, \qquad M_j = \sum_b \langle T_j, \Theta_b\rangle.$$

Here the Θ_b are finitely many fixed positive compact smooth tests, and $\sup_j M_j < \infty$ by weak convergence. The same bound holds for T. Matrix Cauchy–Schwarz for the original positive

cone therefore bounds every coefficient with one primed or double-primed normal factor on that tube by $C\epsilon^{1/2}M_j$.

Convolve the ambient chamber tensors by a radius-ϵ real smooth kernel, then multiply by the fixed compact cutoff. This gives an original smooth η_ϵ. On any one wall, tangential matching writes the branch difference as

$$\ell\theta + d'\ell \wedge \xi + d''\ell \wedge \zeta$$

with uniformly bounded coefficients. For finitely many walls, subdivide a segment joining two points at distance at most ϵ by its wall crossings. The smooth variations along its chambers sum to $O(\epsilon)$; each jump is in the corresponding normal ideal, and its wall is at distance $O(\epsilon)$ from the initial point. A fixed linear right inverse for each normal ideal gives uniform coefficient bounds. Degenerate segments follow by limits. Thus

$$|\langle T_j, \eta_\epsilon - \eta\rangle| \le C\epsilon^{1/2}M_j.$$

The normal-factor estimate with decreasing tubes also makes every change of chamber representative on a wall invisible to the pairing. First pass $j \to \infty$ with the smooth η_ϵ fixed, then let $\epsilon \downarrow 0$. This proves the assertion for $q \ge 1$. For $q = 0$, matching means continuity and ordinary uniform convolution approximation suffices. A finite original partition handles all presentations. Each inequality is preserved on further original refinements, because it is an inequality of pulled-back positive tensors. No valuation-group or residue-field property was used. □

Lemma 3.3 (Finite etale trace in the original piecewise class)**.** *Let $p : Z \to U$ be finite etale between the spaces above. The pushforward current of any compact original smooth form on Z is represented locally on U by finite original compatible piecewise smooth forms. The identity holds on every original complementary test and every further original moment refinement.*

Proof. The construction of Theorem R4–73.3 in R4 uses only the following operations, all valid over the present F. We recall the construction and the calibration rather than assuming a field-extension rule for currents.

Properness and finiteness allow one to shrink an ordinary base neighborhood until the source is a finite disjoint union of pieces carrying single moment presentations. Treat one such piece of rank d, with source units $h_1, \dots, h_m$ and form $\omega(\log|h_1|, \dots, \log|h_m|)$. For each $a \in \{0, \dots, d\}^m$, take the characteristic polynomial of $h^a = \prod_i h_i^{a_i}$ in the finite locally free analytic algebra:

$$Q_a(T) = T^d + c_{a,1}T^{d-1} + \cdots + c_{a,d}.$$

The roots over an algebraic closure of each completed residue field give the d geometric sheets with their full multiplicities. Their ordered logarithmic norms are the Newton-polygon slopes of Q_a. On a fixed compact neighborhood all these norms are bounded by some B. A coefficient vanishing at the chosen base point can be omitted after shrinking until its norm is less than $\exp(-dB - 1)$: it cannot lie on the lower Newton hull. The other coefficients are original analytic units after a further shrinking. Consequently the ordered root norms are finite continuous piecewise affine functions of finitely many original base unit logarithms.

These records determine the unordered vectors $z_l = (\log|h_i|)_i$, including multiplicities. Indeed, the records determine $\prod_{l=1}^{d}(X - a\cdot z_l)$ on the grid $\{0, \dots, d\}^m$; polynomial interpolation determines this polynomial in all its variables, and its monic linear factorization determines the vectors. All possible joint listings are obtained by permuting coordinate lists. Consistency is a finite system of linear equalities and inequalities. Thus on finitely many polyhedral cells

one may list the vectors as affine maps $z_l(t)$ and set

$$\tau = \sum_{l=1}^{d} z_l^* \omega.$$

On common faces the unordered lists agree. A finite further subdivision fixes their matching permutations, so the forms have equal tangential restrictions. This construction initially gives a form on a finite closed polyhedral locus in real moment space. Triangulate this locus together with a larger box and barycentrically subdivide. On the neighborhood where the sum of hat functions of its vertices is greater than $1/2$, their normalized barycentric combination retracts piecewise smoothly onto the locus. Pullback by this retraction extends τ to a full real neighborhood of each compact moment image, preserving tangential compatibility. The resulting form is therefore an original piecewise test, not merely a form on an abstract skeleton.

For the current identity, use the finite-flat degree formula for calibrated skeletons: [Mih24, Lemma 3.15 and Proposition 3.19, especially equation (3.20)]. Those results are stated over an arbitrary complete non-Archimedean field. Their local weights are finite algebra degrees; they are not replaced by a product of value-group and residue-field degrees. Hence possible valued-field defect causes no omitted factor. Adjoining the source units does not increase graded transcendence degree, by their characteristic equations. Omitted coefficients are still strictly below the maximal terms of each initial equation, so the retained unit coefficients suffice for this assertion. The degree formula therefore gives, for every original complementary test γ on the base,

$$\int_Z \omega(\log|h|) \wedge p^*\gamma = \int_U \tau \wedge \gamma.$$

It remains valid after every extra base moment refinement. The extension off the consistent real locus contributes nothing on the actual calibrated cells. There is no additional wall current. Finite partitions and Lemma 3.2 complete the construction and its independence statements. □

Theorem 3.4 (Finite etale preservation for prescribed rows)**.** *Let $p : Z \to U$ be finite etale, and let u, v be actual locally bounded classical psh functions on U, each with prescribed smooth rows. If $D[u] = D[v]$ on all original tests on U, then*

$$D[u \circ p] = D[v \circ p]$$

on all original tests on Z.

Status: `NEW_PROVED` field extension of the R4 finite-etale argument, relative to the stated coefficient lemma. No regularization of new functions over F is asserted.

Proof. Let $\alpha \in A_c^{n-1,n-1}(Z)$. Choose a finite original smooth partition on a neighborhood of its projected compact support, subordinate to ordinary neighborhoods carrying rows for both u and v. Replace α in turn by $(p^*\chi)\alpha$ for the members χ of this partition. It suffices by linearity to treat each resulting test, whose projected support now lies in one common row neighborhood. For such a test, still denoted by α, put $P = p_*[\alpha]$ and $\nu = DP = p_*[D\alpha]$. The latter is one fixed signed Radon measure of compact support. By the preceding lemma, P is represented on a finite compact cover by compatible original piecewise tests. Take the given bounded smooth decreasing rows u_j, v_j on the chosen neighborhood of the projected support. Their positive separately closed curvatures converge to the same current, by fixed smooth-test dominated convergence. Lemma 3.2 gives

$$\langle P, D(u_j - v_j)\rangle \longrightarrow 0.$$

Compact smooth Stokes identifies this expression with $\int (u_j - v_j)\, d\nu$. Bounded convergence against this fixed measure identifies its limit with

$$\int (u - v)\, d\nu = \langle D[(u - v) \circ p], \alpha \rangle.$$

Sum the finitely many localized identities. The derivatives in $D\alpha$ include those of the fixed partitioned test; no derivative of a partitioned potential is taken. This proves the equality on the arbitrary original test without assuming a global row. □

Corollary 3.5 (Finite scalar extensions)**.** *If F'/F is finite, the same prescribed pair satisfies $D[u_{F'}] = D[v_{F'}]$ on all original F'-tests.*

Proof. Characteristic zero makes the scalar extension map finite etale. Apply the theorem with $U_{F'}$ viewed as an F-space. Its original analytic functions, moment tests, and their topology are the same in both interpretations. Its integration current in the F-calibration is $[F' : F]$ times its integration current in the F'-calibration. This is the finite-constant formula of CLD, also checked directly in Mihatsch, Lemma 3.16. Dividing by this single positive degree proves the assertion. The different scalar pushforward formula has factor one; it is not being substituted for the present same-space calibration. □

4 Complete scalar extensions through one-variable fiber potentials

The original finite extension $K/\mathbf{Q}_p$ keeps its given absolute value. Auxiliary complete fields below always carry extending absolute values. Tests are original CLD forms. The new-field input is the prescribed-row finite-etale kernel theorem 3.4, with the coefficient control of Lemma 2.1. Neither a regularization theorem for arbitrary psh functions over an auxiliary field nor the arbitrary-background potential quotient is used in this section.

Here a *prescribed smooth row* for u means a sequence of original smooth functions $u_j \downarrow u$, locally uniformly bounded, with positive $(1,1)$-curvature forms. When such a row is pulled back by scalar extension, the smooth forms remain positive: in an original moment presentation their pullbacks are the pullbacks of the same positive quadratic forms under linear maps of tangent spaces. The extra moment directions introduced by a refinement cause no change to this argument. Thus the rows supplied over the original K are used as specified functions over later fields; no new regularization assertion over those fields is inferred. All row arguments are local on fixed ordinary algebraic neighborhoods.

4.1 Exchange with a continuous potential needs only a prescribed row

Lemma 4.1 (Continuous-slot exchange over an auxiliary field)**.** *Let F be a complete non-Archimedean field, let W be a good Hausdorff boundaryless quasi-smooth F-analytic space, and suppose u has a prescribed smooth row on W. Let b be continuous and suppose $D[b]$ is a symmetric original-positive current. Then*

$$D(uD[b]) = D(bD[u]).$$

All products here are multiplication of coefficient-Radon currents by the displayed actual Borel or continuous function. In particular, if u, v have prescribed rows and $D[u] = D[v]$, then

$$D((u - v)D[b]) = 0.$$

Status: `DIRECT_COROLLARY` of smooth Leibniz, fixed-measure dominated convergence, and the prescribed-row coefficient control. No joint BT convergence is needed.

Proof. For a smooth row member u_j, separate closedness and smooth Leibniz give

$$D(u_j D[b]) = Du_j \wedge D[b] = D(bDu_j).$$

For a fixed compact original test, its coefficient measures against $D[b]$ are finite signed Radon measures. The row has a common bound near that support and decreases to the actual function u. Dominated convergence therefore gives $u_j D[b] \to uD[b]$ on the fixed test and on its D-derivative. On the other hand $D[u_j] \to D[u]$ distributionally. Positivity and the coefficient control permit multiplication of this convergence by the fixed continuous function b. Passing to the limit proves the first identity. Subtract the corresponding identity for v to obtain the last assertion. □

4.2 A bounded continuous potential for a nonrigid parameter point

Lemma 4.2 (Full-fiber integration is continuous in the parameter)**.** *Let F be complete, let Z be a good Hausdorff boundaryless smooth F-analytic space of pure dimension n, and let A be an ordinary open in $\mathbf{A}_F^{1,\mathrm{an}}$. For an original compact smooth (n,n)-form η on $Z \times_F A$, put*

$$I_\eta(a) = \int_{Z_{\mathscr{H}(a)}} \eta_a.$$

Then I_η is an ordinary continuous function on A. The fiber is the whole completed-residue-field analytic fiber and its integration uses that field's original calibration.

Status: `EXTERNAL_KNOWN` in the stated product setting.

Proof. The projection is separated, flat, boundaryless and purely of relative dimension n. The compact support is in particular relatively compact over A. The assertion follows from [CLD25, Theorem 16.3.3(d), pp. 447–450], whose conclusion is that I_η is G-tropically C^∞. For clarity about the ordinary topology, on an affinoid neighborhood of a base point a finite compact analytic G-cover represents such a function by continuous functions of finitely many unit norms. These representatives agree on intersections. Finite closed-set pasting on that neighborhood gives ordinary continuity. This argument concerns a smooth unweighted form; no continuity of an integral with a discontinuous psh multiplier is asserted here. □

Proposition 4.3 (A continuous full-fiber potential at every nonrigid point)**.** *Let C be an algebraically closed complete non-Archimedean field, and let $a \in \mathbf{A}_C^{1,\mathrm{an}}$ be a point other than a C-rational point. There is a finite continuous function b_a on $\mathbf{A}_C^{1,\mathrm{an}}$, locally bounded, such that for every Z as in Lemma 4.2,*

$$D[b_a \circ p] = j_{a*}[1]_{Z_{\mathscr{H}(a)}} \tag{4.1}$$

on all original tests of $Z \times_C \mathbf{A}_C^{1,\mathrm{an}}$. The right-hand side denotes the full analytic fiber, with coefficient one. In particular the left-hand side is original-positive and separately closed. The equality remains valid after multiplication by any fixed locally bounded Borel function on the product.

Status: `DIRECT_COROLLARY`**.** The type-II and type-III formula is `EXTERNAL_KNOWN`, namely the CLD Gauss-fiber formula. The type-IV passage uses a uniform potential limit and unweighted fiber-integral continuity. The limiting construction is included because using the full fiber, and only then adding a fixed Borel multiplier, is essential in the subsequent kernel argument.

Proof. Over C, the classification of points of the affine line says that a type-II or type-III point is $\zeta_{c,r}$ with $c \in C$ and $r > 0$. In that case take

$$b_a(t) = \log \max(r, |t - c|).$$

Equation (4.1) is exactly [CLD25, Corollary 18.6.2, pp. 533–534] for the analytic function $t - c$ and one parameter. The additive constant $\log r$ has no curvature.

For type IV, choose nested closed discs $D(c_j, r_j)$ with empty intersection in C representing a, and put $a_j = \zeta_{c_j,r_j}$. The radii can be chosen decreasing to $r > 0$, and $a_j \to a$ in the ordinary Berkovich topology. If their infimum were zero, the centers would be Cauchy and completeness would give a point in their intersection. These classification statements are recorded, with their proofs, in [PT20, Lemma I.2.11, Proposition I.2.12, and Remark I.2.13, pp. 8–9].

Set $b_j(t) = \log \max(r_j, |t - c_j|)$. For $\ell \geq j$, nesting gives $|c_\ell - c_j| \leq r_j$ and $r_\ell \leq r_j$. Outside $D(c_j, r_j)$, the two displayed maxima for b_j and b_ℓ agree. Inside that disc, $b_j = \log r_j$ and $\log r_\ell \leq b_\ell \leq \log r_j$. Thus on the entire analytic line,

$$0 \leq b_j - b_\ell \leq \log(r_j/r_\ell).$$

Consequently b_j decreases uniformly to a finite continuous function b_a, with

$$0 \leq b_j - b_a \leq \log(r_j/r) \longrightarrow 0. \tag{4.2}$$

Although the individual functions grow at infinity, this uniform error bound is global and their values are bounded on every compact set.

For an original compact (n, n)-test η on the product, distributional differentiation and uniform convergence give

$$\begin{aligned} \langle D[b_a \circ p], \eta \rangle &= \lim_j \langle D[b_j \circ p], \eta \rangle \\ &= \lim_j I_\eta(a_j) = I_\eta(a). \end{aligned}$$

The middle equality is the already known Gauss-fiber identity; the last one is Lemma 4.2. This proves (4.1). Positivity and separate closedness follow either from the positive closed currents $D[b_j \circ p]$ or from the displayed full-fiber equality.

To add a Borel multiplier, fix η and test the unweighted identity also on $\chi\eta$ for every original smooth scalar χ. This identifies the fixed signed coefficient Radon measure of $D[b_a \circ p]$ with the pushforward of that of η_a. Integrate the fixed locally bounded Borel function in this measure identity. In particular no limit of Borel-weighted a_j-fiber measures is used. □

4.3 Spreading and algebraic closure with prescribed tests

Lemma 4.4 (Spreading over an auxiliary complete field)**.** *Let F be complete, let X be an ordinary open in a smooth affine F-variety, and let $a \in \mathbf{A}_F^{1,\mathrm{an}}$. Every original compact test β on $X_{\mathscr{H}(a)}$ is exactly the restriction of an original compact test α on $X \times_F A$ for some ordinary neighborhood A of a. The equality commutes with D.*

Proof. The affine-case proof of Lemma R4–77.2 uses only affinoid density and original smooth cutoffs, and works over the present F. Its essential steps can be stated directly. On finitely many compact rational norm boxes, replace the analytic unit moments of β by rational moments with the same norms, using errors smaller than their positive minimum norms. Localize with norm profiles constant below a positive threshold at every possible zero; impose separate lower cutoffs on the unit moments. There are then finitely many coefficients in $\mathscr{H}(a)$.

Approximate them by values of rational functions in $F(t)$ with denominators nonzero at a. Such values are dense by the definition of the completed residue field. Uniform coefficient bounds on a fixed compact affine stage preserve exactly every active unit norm, and preserve every zero-protected profile on the whole stage. Thus the localized forms remain exactly equal on the fiber. Remove the finite denominator zero loci in an ordinary parameter neighborhood and add compact cutoffs in X and in that neighborhood. Their supports lie in a finite union of compact analytic products of affinoid domains. This gives α with the stated exact restriction. All operations were constructions of smooth test forms, with no psh regularization over F. □

Lemma 4.5 (Transfer to the completed algebraic closure)**.** *Let F be complete of characteristic zero, let X be an ordinary open in a smooth affine F-variety, and let u, v be actual locally bounded classical psh functions with prescribed rows. Suppose $D[u] = D[v]$. Under the prescribed-row finite-etale kernel theorem, the equality also holds on all original tests over $C = \widehat{F^{\mathrm{alg}}}$.*

Proof. For each finite F'/F, the scalar map $X_{F'} \to X$ is finite etale when viewed over F. Apply Theorem 3.4; then divide by the positive same-space calibration factor $[F' : F]$ to obtain the equality on original F'-tests.

Now take an original compact test β on X_C. Rationalize and localize its finitely many unit moments as in the preceding lemma. Only finitely many coefficients in C remain. Approximate all of them by elements of F^{alg}, with errors below all the fixed unit and profile thresholds. They lie in a common finite extension F'/F. The resulting compact F'-test η pulls back to exactly β: active logarithms and zero-protected profiles, hence the differential forms, are unchanged. For $f = u - v$, the factor-one scalar integration identity gives

$$\int_{X_C} f_C D\beta = \int_{X_{F'}} f_{F'} D\eta = 0.$$

This uses the fixed signed-measure form of [CLD25, Proposition 15.3.1, p. 425]. No limit of the actual values of u or v at moving points has been taken. □

4.4 One parameter and then arbitrary complete extensions

Proposition 4.6 (One-parameter transfer from a product kernel)**.** *Let F be a complete characteristic-zero field with a countable dense subfield, and let X be an ordinary open in a smooth affine F-variety. Let u, v be actual locally bounded classical psh functions with prescribed rows. Suppose both*

$$D[u] = D[v] \quad \text{on } X, \qquad D[u \circ q] = D[v \circ q] \quad \text{on } X \times_F \mathbf{A}_F^{1,\mathrm{an}},$$

where q is the projection. Then for every $a \in \mathbf{A}_F^{1,\mathrm{an}}$,

$$D[u_{\mathscr{H}(a)}] = D[v_{\mathscr{H}(a)}]$$

on all original tests. Its new-field analytic inputs are exactly the prescribed-row coefficient control and finite-etale kernel.

Proof. Write $E = \mathscr{H}(a)$ and take a completed algebraic closure Ω of E. Embed an algebraic closure of F in Ω; its closure is an isometric copy of $C = \widehat{F^{\mathrm{alg}}}$. The coordinate value $a(T) \in E \subset \Omega$ defines a point $y \in \mathbf{A}_C^{1,\mathrm{an}}$. Its completed residue field $H = \mathscr{H}(y)$ is the closure of $C(a(T))$ in Ω and contains E isometrically.

Apply Lemma 4.5 to the two assumed kernel equalities, including the product equality. Thus they hold over C. This field still has a countable dense subfield: approximate finite polynomial

coefficients in a countable dense subfield of F, and use the separable root approximation argument to see that its algebraic closure is dense in F^{alg} and then in C.

If y is C-rational, then $H = C$ and there is nothing further to prove over H. Otherwise take the continuous parameter potential b_y of Proposition 4.3. On $X_C \times_C \mathbf{A}_C^{1,\mathrm{an}}$ the product equality and Lemma 4.1 give

$$D((u_C - v_C) \circ q \; D[b_y \circ p]) = 0.$$

For an arbitrary original compact complementary test β on X_H, spread it exactly to a compact original C-test α on $X_C \times_C A$, using Lemma 4.4. Restrict the last equality to this ordinary neighborhood. The full-fiber identity, with its fixed Borel multiplier, yields

$$0 = \int_{X_H} (u_H - v_H) D\beta.$$

This proves the kernel equality over H in all cases.

Finally pull an arbitrary original E-test back to X_H. The factor-one scalar integration identity gives the same pairing on X_E as on X_H, which is zero. This descends the equality to E. None of the spaces over C, E or H was regarded as a finite-type F-analytic space. □

Theorem 4.7 (Scalar kernel functoriality for every complete extension)**.** *Let X be an allowed space over the original finite field K, let $u, v \in \mathcal{C}_b(X)$, and assume $D[u] = D[v]$ on every original K-test. Then for every complete extension L/K,*

$$D[u_L] = D[v_L]$$

on every original L-test, with the inherited absolute value and the actual pullback functions.

Status: `NEW_PROVED`, relative to the prescribed-row finite-etale theorem and coefficient control stated at the beginning of this section. The proof does not apply B's regularization theorem over an auxiliary infinite field.

Proof. The assertion is local on X. Work in an ordinary affine algebraic neighborhood over K, and choose the local prescribed smooth decreasing rows supplied by the accepted original-K inputs R and B. They and their scalar pullbacks have common bounds on each fixed compact stage. The curvature-form positivity noted at the beginning ensures that these are prescribed rows over every auxiliary field used below.

For a complete field F/K and an integer $r \geq 0$, let $\mathsf{P}(F, r)$ denote the original-current kernel equality for the pullbacks of u, v to $X_F \times_F \mathbf{A}_F^{r,\mathrm{an}}$. The R4 same-field product result proves $\mathsf{P}(K, r)$ for every r. We propagate all these statements simultaneously; this avoids presuming a new same-field product theorem at each infinite intermediate field.

Choose finitely many elements $c_1, \ldots, c_m$ in a complete extension of K, and set

$$F_i = \overline{K(c_1, \ldots, c_i)}, \qquad F_0 = K.$$

Each F_i has a countable dense subfield. It is the completion of $F_{i-1}(c_i)$ and therefore is the residue field of a point of $\mathbf{A}_{F_{i-1}}^{1,\mathrm{an}}$. Assume $\mathsf{P}(F_{i-1}, r)$ for all r. For each r, apply Proposition 4.6 to $X_{F_{i-1}} \times \mathbf{A}_{F_{i-1}}^{r}$. Its product-kernel hypothesis is precisely $\mathsf{P}(F_{i-1}, r+1)$. It gives $\mathsf{P}(F_i, r)$. Induction proves all r statements over every such finite-parameter field.

For an arbitrary complete L/K and an arbitrary original compact test β on X_L, apply the exact finite-coefficient descent of Lemma R4–78.1. It gives $E = \overline{K(c_1, \ldots, c_m)} \subset L$ and a compact original E-test η with $\beta = p^*\eta$. The just-proved $\mathsf{P}(E, 0)$ and the fixed scalar integration identity give

$$\langle D[u_L - v_L], \beta \rangle = \langle D[u_E - v_E], \eta \rangle = 0.$$

The test was arbitrary. Ordinary localization proves the assertion on the original allowed space X. □

This theorem closes the scalar-extension gate isolated in R4. It does not by itself prove continuity of the actual difference $u - v$, or its functoriality under every analytic map over an arbitrary extension. Those are separate assertions. In particular the arbitrary-background quotient is not invoked anywhere in the proof and is not claimed as a consequence without the remaining analytic argument.

5 Analytic pullback of the bounded kernel after every complete extension

The ground field of the original potentials is the fixed finite extension $K/\mathbf{Q}_p$, with its original absolute value. An allowed space over a complete extension L/K is good, Hausdorff, boundaryless, and quasi-smooth. In particular, its structural morphism is smooth in the sense of Berkovich. We use the original CLD forms and currents over the indicated field. The functions are actual locally bounded classical psh functions, with the all-complete-extension curve condition.

The result of this section is a pullback statement for differences of specified functions. No pullback of an arbitrary current is defined. Algebraic neighborhoods and tubular neighborhoods are used only over the original finite field K, where B, Lemma D.4, and R, Lemma 5.2, supply them. The new source over L is handled by genuinely etale coordinates and a field-independent normal trace.

Lemma 5.1 (Ordinary finite etale coordinates)**.** *Let L be a complete nontrivially valued field and let Z be a good, boundaryless, quasi-smooth L-analytic space. For every point $z \in Z$ of local dimension d, there are ordinary open neighborhoods Z_0 of z and B in $\mathbf{A}_L^{d,\mathrm{an}}$ and a finite etale morphism*

$$g : Z_0 \longrightarrow B.$$

The neighborhoods can be made small enough to meet any prescribed ordinary neighborhood condition at z.

Status: `EXTERNAL_KNOWN`**.** The precise geometry is the definition of smooth morphisms in Berkovich [Ber93, §3.5], together with the definition of quasifinite morphisms in [Ber93, Definition 3.1.1 and Lemma 3.1.2] and the fact that etale morphisms are quasifinite in [Ber93, 3.3.4]. These conventions and their ordinary-open meaning are also stated explicitly on printed page 9 of that paper.

Proof. Smoothness gives, after an ordinary shrinking at z, an etale map to $\mathbf{A}_L^{d,\mathrm{an}}$. In Berkovich's terminology an etale map is quasifinite. Quasifiniteness means that at the chosen source point it restricts to a finite map between ordinary open neighborhoods of that point and its image. The restriction remains etale. Starting with the prescribed smaller ordinary neighborhood proves the final assertion. This argument does not replace etale by quasi-etale: an analytic-domain immersion with relative boundary would not justify the assertion. Boundarylessness in the hypotheses is used precisely to pass from quasi-smoothness to smoothness. □

Lemma 5.2 (Normal trace over an arbitrary complete field)**.** *Let L be complete, let B be an allowed pure d-dimensional L-analytic space, and put*

$$W = B \times_L \mathbb{D}_s, \qquad \pi : W \to B, \qquad i : B \to W, \qquad \mathbb{D}_s = \{|t| < s\}.$$

For every actual locally bounded classical psh function a on W and every original $\eta \in A_c^{d,d}(B)$, one has

$$\lim_{N\to\infty} \int_W a\,\beta_N \wedge \pi^*\eta = \int_B (a\circ i)\,\eta, \tag{5.1}$$

where

$$\beta_N = D[-N + \psi(\log|t| + N)]$$

and ψ is any fixed smooth convex function equal to zero for $x \le -\epsilon$ and to x for $x \ge \epsilon$. The function in brackets is extended constantly at $t = 0$; for large N, β_N is an original smooth positive closed form supported on the compact normal annulus $e^{-N-\epsilon} \le |t| \le e^{-N+\epsilon} < s$. All coefficients are evaluated on their actual Borel representatives.

Consequently, for actual locally bounded classical psh functions a, b on W,

$$D[a] = D[b] \quad \Longrightarrow \quad D[a\circ i] = D[b\circ i]$$

on every original complementary test on B.

Status: `NEW_PROVED` extension in the ground field of the normal-trace argument from R3. The proof requires the given functions to be classical; it does not obtain classicality from a distributional inequality or assume a regularization theorem over L.

Proof. For $x \in B$ and $r > 0$, let $\sigma_r(x)$ denote the image in W of the radius-r Gauss point in the normal disc over $\mathscr{H}(x)$. Let μ_η be the signed Radon measure associated with the original smooth top form η. Original product calibration gives, for every bounded Borel coefficient c on a neighborhood of the relevant compact support,

$$\int_W c\,\beta_N \wedge \pi^*\eta = \int_B \int_{\mathbf{R}} c(\sigma_{e^t}(x))\psi''(t+N)\,dt\,d\mu_\eta(x). \tag{5.2}$$

Here the inner measure is supported below $\log s$ when N is large. To specify the calibration, take a finite original moment presentation for the base top form, retaining the individual calibrated top-dimensional cells. Adjoining the unit t on the normal annulus gives the product of each such cell with the radial interval. The relative cell measure is dt with multiplicity one; the base multiplicities are already included in μ_η. Thus smooth scalar coefficients satisfy (5.2) by integration on these product cells. The same calculation identifies the coefficient Radon measure itself, so Borel multiplication and ordinary Fubini give the displayed formula for bounded Borel c. Finite partitions and signed linearity recover all original η. This is the original product calibration; it is not an assertion about arbitrary points of a set-theoretic product or about products of arbitrary currents. The argument is unchanged over L.

The Gauss lift $(x, r) \mapsto \sigma_r(x)$ is continuous on every compact affinoid stage and compact radial interval. Indeed, the norm of a polynomial $\sum_j f_j t^j$ there is $\max_j |f_j(x)| r^j$; uniform analytic approximation proves the claim for analytic functions. The integrand in (5.2) is therefore Borel.

Fix $x \in B$. Classicality, including complete extension to $\mathscr{H}(x)$, makes the restriction of a to the normal $\mathscr{H}(x)$-disc a locally bounded subharmonic function. The maximum principle on a closed disc, whose Shilov boundary is its Gauss point, gives

$$a(i(x)) \le a(\sigma_r(x)).$$

As $r \downarrow 0$ the Gauss points tend to the zero section. Upper semicontinuity gives the reverse limiting upper bound, so

$$a(\sigma_r(x)) \longrightarrow a(i(x)). \tag{5.3}$$

This concerns a pullback of a function to a curve. It assumes no extension or pullback of a current equality.

Cover $\operatorname{supp}\eta$ by finitely many compact affinoid neighborhoods in B and choose a fixed closed normal disc of radius smaller than s. The union of their actual analytic products is compact. Local boundedness gives a uniform bound for a there. For large N all the normal annuli in (5.2) lie in this compact product, and their radial measures have mass one and radii tending uniformly to zero. Apply (5.3) and dominated convergence against the fixed finite measure $|\mu_\eta|$. This proves (5.1).

For the last assertion assume $d \geq 1$, take an arbitrary original $\alpha \in A_c^{d-1,d-1}(B)$, and test the given equality against $\beta_N \wedge \pi^*\alpha$. Separate closedness of β_N gives

$$0 = \int_W (a-b)\,\beta_N \wedge \pi^*(D\alpha).$$

Apply the trace limit to a and b separately. The limit is $\langle D[a\circ i] - D[b\circ i], \alpha\rangle$. When $d = 0$ there are no curvature tests and the assertion is immediate. Iterating the argument gives the same restriction statement for a zero section of any finite-dimensional normal polydisc, with arbitrary allowed analytic base over L. □

Theorem 5.3 (Analytic functoriality over every complete extension)**.** *Let X be an allowed K-analytic space, let $u, v \in \mathcal{C}_b(X)$ satisfy $D[u] = D[v]$ on every original compact smooth complementary test, let L/K be any complete extension with the induced absolute value, and let*

$$F : Z \longrightarrow X_L$$

be an arbitrary analytic morphism from an allowed L-analytic space. Then

$$D[u_L \circ F] = D[v_L \circ F] \tag{5.4}$$

on every original compact smooth complementary test on Z. Both functions are their actual locally bounded classical pullbacks.

Status: `NEW_PROVED`, relative to Theorem 4.7, the prescribed-row finite-etale theorem, and the accepted original-K geometric inputs. In particular the theorem includes analytic curve maps over arbitrary complete extensions, not only maps defined over a finite extension of K.

Proof. The assertion is local at a point $z \in Z$. A zero-dimensional source has no curvature tests. A zero-dimensional target is locally constant, so its pullbacks also have zero curvature. We may assume that the relevant target and source dimensions are positive.

Let x be the image in X of $F(z)$. B, Lemma D.4, gives an ordinary algebraic neighborhood of x over the original K. Choose a smooth affine K-variety Y containing this neighborhood and a closed algebraic embedding $Y \hookrightarrow \mathbf{A}_K^N$. After shrinking at x, R, Lemma 5.2, gives an ordinary open $V \subset X$ containing x, an ordinary open $W \subset \mathbf{A}_K^{N,\mathrm{an}}$, and an analytic isomorphism

$$\Phi : V \times_K \mathbb{D}_s^c \xrightarrow{\sim} W$$

whose zero section is the given embedding. Let $R_W = \mathrm{pr}_V \circ \Phi^{-1}$ and put

$$\widetilde{u} = u \circ R_W, \widetilde{v} = v \circ R_W.$$

These are actual classical functions. They have prescribed smooth positive rows by pulling back the original-K rows. The R4 polydisc kernel theorem and invariance of original forms and calibrated integrals under an analytic isomorphism show that $D[\widetilde{u}] = D[\widetilde{v}]$ on W. For every integer $d \geq 0$, the same R4 product theorem gives the kernel equality for the functions depending only on W on $W \times_K \mathbf{A}_K^{d,\mathrm{an}}$. Theorem 4.7 then gives that equality on

$$W_L \times_L \mathbf{A}_L^{d,\mathrm{an}}. \tag{5.5}$$

Only the originally defined functions were extended in this step.

Let $d = \dim_z Z$. Shrink at z until F takes values in V_L. By Lemma 5.1, a further ordinary neighborhood Z_0 admits a finite etale map $g : Z_0 \to B$, with B ordinary open in $\mathbf{A}_L^{d,\mathrm{an}}$. The morphism

$$\mathrm{id}_{W_L} \times g : W_L \times_L Z_0 \longrightarrow W_L \times_L B$$

is finite etale. Apply Theorem 3.4 to the restriction of (5.5). The needed rows are the scalar pullbacks of the original rows of $\widetilde{u}, \widetilde{v}$, followed by the projection. This proves the kernel equality for $\widetilde{u}_L(x)$ and $\widetilde{v}_L(x)$ on $Z_0 \times_L W_L$. No algebraization or regularization theorem over L is used.

Regard $F|_{Z_0}$ as an analytic map to $\mathbf{A}_L^N$ and use the analytic translation

$$A : Z_0 \times_L \mathbf{A}_L^{N,\mathrm{an}} \longrightarrow Z_0 \times_L \mathbf{A}_L^{N,\mathrm{an}}, \qquad A(c,t) = (c, F(c) + t).$$

Its inverse is $(c,x) \mapsto (c, x - F(c))$. The inverse image $O = A^{-1}(Z_0 \times_L W_L)$ is an ordinary open containing the zero section. On it the two actual classical functions

$$a(c,t) = \widetilde{u}_L(F(c) + t), \qquad b(c,t) = \widetilde{v}_L(F(c) + t)$$

have equal original curvature currents, because A is an analytic isomorphism.

The compact-tube proof of Lemma R4–75.2 works over L verbatim. Indeed, on a compact affinoid base neighborhood the nested compact analytic tubes of radii tending to zero intersect in the zero section; compactness forces a sufficiently small entire tube into O. Hence there is an ordinary neighborhood Z_1 of z and $r > 0$ such that $Z_1 \times_L \mathbb{D}_r^N \subset O$. This is an assertion about actual analytic products, not about Cartesian products of their points. Restrict the equality to this tube and apply Lemma 5.2 successively in the N normal coordinates. It gives

$$D[a|_{t=0}] = D[b|_{t=0}] \quad \text{on } Z_1.$$

The zero-section values are precisely $u_L \circ F$ and $v_L \circ F$. Locality proves (5.4) on all of Z. □

Corollary 5.4 (All complete-extension curve kernels)**.** *Under the hypotheses on X, u, v in the theorem, every complete L/K and every analytic map from a smooth boundaryless L-analytic curve C to X_L satisfy*

$$D[(u_L - v_L) \circ F] = 0$$

on every original compact smooth scalar test on C.

Status: `DIRECT_COROLLARY`**.** The assertion includes the actual representatives of both subharmonic pullbacks. It does not by itself assert that $u - v$ or $v - u$ is upper semicontinuous on X.

Scope of the remaining quotient question. This section closes the analytic curve-test gate after all complete extensions. It does not infer continuity on a higher-dimensional space merely from the existence of those curve tests. In particular, the implication

$$D[u] = D[v] \quad \Longrightarrow \quad D((u - v)S) = 0$$

for every allowed positive closed background S still requires the separate actual-regularity or arbitrary-background argument. The all-extension functoriality established here does not define the pullback of such an S.

6 The actual harmonic representative on complete-extension curves

The distinction between equality of curve distributions and equality of actual functions matters here. Thuillier's degree-zero distributions are determined by the values at points of type II or III. An arbitrary Borel modification at a rigid point would not be detected by those values. The following argument recovers the missing values from the separate subharmonicity of the two given functions. It does not assume that their difference is semicontinuous.

Lemma 6.1 (The algebraic scope of the curve input)**.** *Over any complete nontrivially valued field F, a good Hausdorff boundaryless quasi-smooth analytic curve is, locally for the ordinary topology, isomorphic to an ordinary open in the analytification of a smooth affine algebraic F-curve. No extension of F is necessary. Consequently the explicitly complete-field statement of B, Lemma A.2, applies locally to actual locally bounded subharmonic functions on such an analytic curve, when* $\operatorname{char} F = 0$.

Status: `EXTERNAL_KNOWN` geometry and `DIRECT_COROLLARY` for the use of the internal input.

Proof. Apply Ducros [Duc12, Lemma (0.21)] with the full radius group $\Gamma = \mathbf{R}_{>0}$. The lemma is stated for an arbitrary complete non-Archimedean field. At the chosen point it gives an affinoid neighborhood V identified with an affinoid domain in a smooth affine algebraic F-curve Y. Shrink the initial neighborhood to be separated. The relative interior composition formula, with its locally separated intermediate spaces, identifies

$$\operatorname{Int}(V/Z) = \operatorname{Int}(V/F) = \operatorname{Int}(V/Y^{\mathrm{an}}),$$

because Z and Y^{an} are boundaryless. For an analytic-domain embedding the relative interior is the ordinary topological interior. The common interior contains the chosen point and is ordinary open in both ambient spaces.

This uses the external algebraization lemma in its stated field generality, not an unstated extension of B, Lemma D.4. The invoked part of B, Lemma A.2, already assumes an arbitrary complete nontrivially valued characteristic-zero field and an ordinary open in a smooth algebraic curve. Its canonical-calibration identity is

$$\int_C \alpha\, d(\Delta_C a) = \int_C a\, D\alpha, \qquad \alpha \in A_c^{0,0}(C), \tag{6.1}$$

where $\Delta_C a$ is the positive Radon Laplacian of the given subharmonic function. Thus its hypotheses hold on the obtained ordinary algebraic neighborhood. □

Lemma 6.2 (Recovering the actual curve function)**.** *Let F be complete, algebraically closed, nontrivially valued, and of characteristic zero. Let C be a good Hausdorff smooth boundaryless F-analytic curve. Let a, b be actual locally bounded subharmonic functions on C. If*

$$D[a] = D[b]$$

on every original compact smooth scalar test, then $a - b$ is an actual continuous harmonic function on C in the curve-potential sense. In particular both signs of $a - b$ are subharmonic.

Status: `DIRECT_COROLLARY` of the complete-field curve Laplacian input and the classical curve kernel theorem, with the actual-value passage proved explicitly below. The external distributional kernel is Thuillier's Lemma 3.3.12; Wanner [Wan22, Remark 7.2, p. 22, and Remark 7.12, p. 27] explicitly applies it before identifying the endpoint values. We also give its finite-graph verification in the present bounded setting.

Proof. Work on an ordinary algebraic neighborhood supplied by Lemma 6.1. B, Lemma A.2, and (6.1) show that the positive Radon measures $\Delta_C a$ and $\Delta_C b$ agree against every original compact smooth scalar function. Original smooth functions are uniformly dense on each compact stage, with a cutoff on a fixed larger stage. The measures have finite mass on that larger stage, so they agree against all compact continuous functions and hence as Radon measures. Therefore the degree-zero curve distribution defined by $a - b$ has Laplacian zero. This distribution records the actual values of $a - b$ on the subset $I(C)$ of type-II and type-III points.

Here is a local graph argument for its harmonic representative. Take a connected strictly affinoid domain Y compactly contained in the chosen curve open, with nonempty boundary. On a finite semistable skeleton, solve the finite linear Dirichlet problem with boundary values $(a - b)|_{\partial Y}$, and extend the solution by the skeleton retraction. The resulting function h is continuous on Y and harmonic on its interior. Existence and uniqueness on the finite graph follow by minimizing the strictly positive Dirichlet quadratic form with fixed boundary values; equivalently, the weighted graph Laplacian on the nonboundary vertices is invertible.

For $x \in I(\operatorname{Int} Y)$ use the curve Green function $g_{Y,x}$, extended by zero outside Y. It is lisse and compactly supported; its Laplacian is supported on $\partial Y \cup \{x\}$ and has coefficient -1 at x. These are the Green functions of Thuillier [Thu05, Proposition 3.3.7 and Corollary 3.3.9], also stated in Wanner [Wan19, Proposition 2.12]. The equation $\Delta(a - b) = 0$, evaluated on $g_{Y,x}$, and the finite-graph Green identity for the harmonic extension h, give

$$\int (a - b - h)\, d(\Delta g_{Y,x}) = 0.$$

The boundary values of $a - b - h$ vanish, so $a(x) - b(x) = h(x)$. Such interiors cover C locally, and their harmonic functions agree on overlaps by continuity and density of type-II and type-III points. They therefore give a continuous harmonic h agreeing with $a - b$ on $I(C)$.

It remains to identify actual values at points of types I and IV. Each such point x has an ordinary open-ball neighborhood; see [Wan22, Theorem 2.11]. In that ball choose nested closed discs D_j containing x, with Gauss boundary points $\xi_j \in I(C)$ tending to x. For type I their radii tend to zero about the corresponding rational center. For type IV the discs come from its nested-disc description, with empty intersection of classical points and positive limiting radius. They may be chosen sufficiently far along the system to lie in the given open-ball neighborhood. Their radii may also be chosen in $|F^\times|$, which is dense in $\mathbf{R}_{>0}$ after taking logarithms, so the disc maximum principle can be used on strictly affinoid domains.

Apply the subharmonic maximum principle separately to a and b:

$$a(x) \le a(\xi_j), \qquad b(x) \le b(\xi_j).$$

Upper semicontinuity of each given function and $\xi_j \to x$ give the reverse limiting upper bounds. Thus

$$a(\xi_j) \to a(x), \qquad b(\xi_j) \to b(x).$$

Since $a(\xi_j) - b(\xi_j) = h(\xi_j)$ and h is continuous, we conclude $a(x) - b(x) = h(x)$. This proves equality at every point, including the endpoint values invisible to the degree-zero distribution by itself. □

Theorem 6.3 (Actual harmonicity on every complete-extension curve)**.** *Let $u, v \in \mathcal{C}_b(X)$ on an allowed original K-space satisfy $D[u] = D[v]$. For every complete extension L/K and every analytic map $F : C \to X_L$ from a good Hausdorff smooth boundaryless L-analytic curve, the actual function*

$$h_C = (u_L - v_L) \circ F$$

is continuous and harmonic: both h_C and $-h_C$ are subharmonic on C.

Status: `DIRECT_COROLLARY` of Theorem 5.3, the preceding curve lemma, and scalar descent for subharmonic functions.

Proof. Let Ω be a completed algebraic closure of L. Theorem 5.3, applied to $F_\Omega : C_\Omega \to X_\Omega$, gives the original-current equality for the two actual subharmonic pullbacks on C_Ω. Lemma 6.2, applied locally, says that $h_C \circ p$ is continuous and harmonic on C_Ω, where $p : C_\Omega \to C$ is scalar projection.

Continuity descends without a countability assumption. On every compact affinoid neighborhood $A \subset C$, the map $A_\Omega \to A$ is a continuous surjection from a compact space onto a Hausdorff space, hence a quotient map. The continuous function $h_C \circ p$ is constant on its fibers, so $h_C|_A$ is continuous. Affinoid neighborhoods give continuity locally on C. Finally the scalar-extension equivalence for subharmonic functions, Gubler–Rabinoff [GR25, Proposition 3.5(7)], applied separately to the now continuous functions h_C and $-h_C$, gives their subharmonicity on C. □

A boundary of this conclusion. The theorem establishes actual continuity and harmonicity on each of the permitted source curves, after all complete extensions. It does not replace a proof of ordinary continuity of $u - v$ on the higher-dimensional target. In the definition of classical pluriharmonicity that additional topological condition still has to be established; no such inference is made from density of curve images alone.

7 A Shilov maximum principle without ambient upper semicontinuity

Throughout this section the absolute values are fixed, and scalar extensions carry the extending absolute value. Harmonicity on a smooth analytic curve means harmonicity of the actual function; in particular that function is continuous on the curve. We do not assume that a function which has harmonic curve pullbacks is upper semicontinuous on its higher-dimensional ambient space.

Lemma 7.1 (The Gauss fiber of a finite flat affinoid map)**.** *Let F be a complete non-Archimedean field, let $A = F\langle T_1, \dots, T_n\rangle$, and let B be a nonzero finite flat A-algebra. Write $V = \mathcal{M}(B)$, $\mathbb{B}_F^n = \mathcal{M}(A)$, and let $g : V \to \mathbb{B}_F^n$ be the resulting map. If ζ is the unit-polydisc Gauss point, then*

$$\Gamma(V) = g^{-1}(\zeta).$$

In particular this is a finite nonempty set. Nilpotents in B do not change the assertion.

Status: `DIRECT_COROLLARY` of the standard finite-algebra norm and Shilov-boundary facts. We give the argument to fix the exact boundary needed below.

Proof. The finite locally free A-module B has a positive constant rank d. For $b \in B$, let

$$\det(T - m_b) = T^d + c_1 T^{d-1} + \cdots + c_d \in A[T]$$

be the characteristic polynomial of multiplication by b. For each $y \in \mathbb{B}_F^n$, its roots after extension to an algebraic closure of $\mathscr{H}(y)$ are the values of b on the finite geometric fiber, with their algebraic multiplicities. The maximum of their absolute values is

$$\max_{x \in g^{-1}(y)} |b(x)| = \max_{1 \le i \le d} |c_i(y)|^{1/i}.$$

Since $|c_i(y)| \le |c_i(\zeta)|$, this maximum is at most the corresponding maximum over $g^{-1}(\zeta)$. Thus that finite fiber is a boundary for B, and the Shilov boundary is contained in it.

Put $H = \mathscr{H}(\zeta)$ and $D = B \otimes_A H$. This finite-dimensional H-algebra is a product of Artinian local algebras, one for each point of $g^{-1}(\zeta)$. Fix a point z in that fiber and let e_z be the idempotent of its factor. The field $\mathrm{Frac}(A)$ is dense in H, and consequently $B \otimes_A \mathrm{Frac}(A)$ is dense in D. Approximate e_z by an element b/a, with $b \in B$ and $0 \neq a \in A$, closely enough that

$$|(b/a)(z)| = 1, \qquad |(b/a)(z')| < 1 \quad (z' \neq z, z' \in g^{-1}(\zeta)).$$

The denominator has the same nonzero absolute value $|a(\zeta)|$ at every point of the fiber. Hence $|b|$ has a strict unique maximum at z within this finite boundary. Since the Shilov boundary is contained in the fiber and is itself a boundary, it must contain z. This proves the reverse inclusion. The proof uses the full finite algebra and therefore also applies in the presence of nilpotents or valued-field defect. □

Theorem 7.2 (Curvewise harmonic functions satisfy the affinoid Shilov bound)**.** *Let F be a complete nontrivially valued field of characteristic zero, let Y be a smooth affine F-variety of pure dimension $n \ge 1$, and let $U \subset Y^{\mathrm{an}}$ be ordinary open. Let $f : U \to \mathbf{R}$ be an actual real-valued function. Assume the following curve property:*

for every complete extension E/F, every smooth algebraic E-curve C, and every algebraic morphism $h : C \to Y_E$, the actual pullback of f is harmonic on each ordinary open subset of C^{an} mapped into U_E.

Suppose $V \subset U$ is a nonempty strict affinoid domain and $g_1, \dots, g_n$ are regular functions on Y such that

$$g = (g_1, \dots, g_n) : V \longrightarrow \mathbb{B}_F^n$$

is finite flat. Then

$$|f(x)| \le \max_{z \in \Gamma(V)} |f(z)| \qquad (x \in V). \tag{7.1}$$

No ordinary upper semicontinuity, local boundedness, or Borel measurability of f on U is assumed.

Status: `NEW_PROVED`. Only the one-dimensional maximum principle is used; higher-dimensional ambient upper semicontinuity is not inserted into its hypotheses. The finite Shilov bound proves boundedness on the displayed domain.

Proof. We first prove a one-coordinate inequality. Fix $x \in V$, write $b = g(x)$, and fix an index i. Let y be the image of b in the polydisc of the other $n-1$ coordinates, and put $E = \mathscr{H}(y)$. The fiber

$$W = V \times_{\mathbb{B}_F^{n-1}} \mathcal{M}(E)$$

is an affinoid E-space, finite flat over the unit disc with coordinate T_i. It is purely one-dimensional. The point x defines a point of W, using the canonical embedding $E \hookrightarrow \mathscr{H}(x)$.

Here the full analytic fiber is used. It must not be replaced by a selected independent section. Its points, including the chosen lift of x, retain their actual pullback values of f.

Let $a_j \in E$, $j \neq i$, be the coordinate values of y. Inside Y_E, form the algebraic slice

$$Z = \{g_j = a_j \text{ for all } j \neq i\}.$$

The space W is an analytic domain of Z^{an} lying in U_E. Local analytic dimension is unchanged by restriction to an analytic domain, including at its ordinary boundary; see [Duc18, §1.4.9]. Thus Z^{an} has local dimension one at every point of W. Dimension and irreducible components commute with algebraic analytification [Duc18, Propositions 2.7.7(2) and 2.7.16]. Remove the

finitely many algebraic components of Z of dimension different from one, all disjoint from W, and give the resulting algebraic curve its reduced structure. Its normalization C is finite and, since E has characteristic zero, is a smooth algebraic E-curve. Its algebraic map to Y_E will be denoted by h.

The inverse image W^ν of W is a compact affinoid analytic domain in C^{an}. Normalization commutes with restriction to analytic domains; equivalently one may use finite algebraic normalization followed by analytification, as in [Duc18, Lemma 2.7.15]. The map

$$W^\nu \longrightarrow \mathbb{B}^1_E$$

is finite. Each of its irreducible components dominates the disc: W is finite flat and purely one-dimensional, and finite normalization preserves that dimension. Its coordinate algebra is therefore torsion-free over the regular one-dimensional Tate algebra $E\langle T_i\rangle$, hence is finite flat. By Lemma 7.1, $\Gamma(W^\nu)$ is its fiber over the unit-disc Gauss point.

By hypothesis $f_E \circ h$ is harmonic on the ordinary open subset $h^{-1}(U_E)$ of C^{an}, which contains all of W^ν. It is consequently continuous on a genuine neighborhood of this compact domain, including its boundary. The curve maximum principle, applied to both signs, gives

$$|f(x)| \leq \max_{w\in\Gamma(W^\nu)} |f_E(h(w))|.$$

This use of continuity is solely on a smooth curve. The higher-dimensional function f has not been assumed continuous or upper semicontinuous.

Define $G_i(b)$ as follows: retain the other coordinate point y, take the unit-radius Gauss point in the T_i-disc over $\mathscr{H}(y)$, and map it into $\mathbb{B}^n_F$. Every point $h(w)$ in the last maximum projects to a point of V lying over $G_i(b)$. Since g is finite, the function

$$M(b) = \max\{|f(z)| : z \in V,\ g(z) = b\}$$

is finite and defined at every $b \in \mathbb{B}^n_F$: each fiber is nonempty and finite, and f is real-valued. No common bound over different fibers is assumed. The one-coordinate argument proves

$$M(b) \leq M(G_i(b)). \tag{7.2}$$

For completeness, these coordinate Gauss operations commute. For a polynomial $P = \sum_k P_k(T_1,\dots,\widehat{T_i},\dots,T_n)T_i^k$ one has

$$|P(G_i(b))| = \max_k |P_k(y)|.$$

Applying this formula successively to two different coordinates gives the maximum of the absolute values of the corresponding double coefficient polynomials, independently of their order. After all n coordinates, the value of every polynomial is its Gauss norm over F. By uniform density of polynomials in the Tate algebra, the resulting point is precisely the full polydisc Gauss point ζ.

Iterating (7.2) therefore gives

$$|f(x)| \leq M(g(x)) \leq M(\zeta).$$

Lemma 7.1 identifies the last maximum with $\max_{\Gamma(V)} |f|$, proving (7.1). □

Scope and use of the ambient neighborhood. The regularity of the g_i on Y ensures that every sliced affinoid curve is contained in the analytification of an algebraic curve extending across its compact boundary. Merely knowing a harmonic function on the interior of an abstract affinoid curve would not justify its boundary maximum with arbitrary assigned boundary values. The theorem states the needed ambient condition explicitly. It requires no global psh exhaustion and no strictly positive ambient form. Producing the indicated affinoids and ambient regular normalization coordinates is a separate geometric input; it is not inferred from the maximum principle itself.

8 Vertical divisors and continuous Shilov interpolation

This elementary geometric input replaces any need to identify an entire skeleton over a non-algebraically closed field. The base field F is complete and discretely valued, with valuation ring F° and uniformizer ϖ. Its given absolute value is retained.

Lemma 8.1 (Vertical components are the exact Shilov boundary)**.** *Let R be an integral, regular, flat, finite-type F°-algebra. Suppose its nonempty special fiber is reduced and connected, with irreducible components $D_0, \dots, D_r$. Write*

$$R^\wedge = \varprojlim_m R/\varpi^m R, \qquad A = R^\wedge[1/\varpi], \qquad W = \mathcal{M}(A).$$

For each i, the height-one prime of $R^\wedge$ corresponding to the generic point of D_i defines a divisorial point $\xi_i \in W$. Then

$$\Gamma(W) = \{\xi_0, \dots, \xi_r\}, \qquad \sup_{x \in W} |a(x)| = \max_i |a(\xi_i)| \quad (a \in A). \tag{8.1}$$

If, in addition, D_i is the principal effective Cartier divisor (x_i) on $\operatorname{Spec} R$ for each i, and $x_0 \cdots x_r = \varpi$, then

$$|x_i(\xi_j)| = \begin{cases} |\varpi|, & i = j, \\ 1, & i \neq j. \end{cases} \tag{8.2}$$

Status: `DIRECT_COROLLARY` of regular completion and the height-one intersection theorem for normal rings. The proof is included to specify the original-field points and norms.

Proof. First, $R^\wedge$ is a Noetherian regular ring. Here is a reduction to local completion, avoiding an unstated formal-model normality assumption. Every maximal ideal $\mathfrak{n}$ of $R^\wedge$ contains ϖ: for every $b \in R^\wedge$, the geometric series makes $1 - \varpi b$ invertible. Put $\mathfrak{m} = \mathfrak{n} \cap R$. Since $R^\wedge/\varpi R^\wedge = R/\varpi R$, this is a maximal ideal and $\mathfrak{n} = \mathfrak{m}R^\wedge$. The local map

$$R_{\mathfrak{m}} \longrightarrow (R^\wedge)_{\mathfrak{n}}$$

is flat, has the same residue field, and extends the maximal ideal to the maximal ideal. It therefore induces an isomorphism of maximal-ideal completions [Stacks, Tag 0AGX]. Regularity is equivalent to regularity of that completion [Stacks, Tag 07NY]. Thus every maximal localization of $R^\wedge$ is regular, and so is $R^\wedge$.

The ring $R^\wedge$ is normal. It is also connected: any idempotent reduces to 0 or 1 modulo ϖ because $R/\varpi R$ is connected, and that reduction determines the idempotent in the complete ring. A connected Noetherian normal ring is a domain. Flatness of completion preserves the fact that ϖ is a nonzero divisor. The height-one primes containing ϖ are exactly the generic points of the components of $R^\wedge/\varpi R^\wedge = R/\varpi R$. At each such prime the local ring is a discrete valuation ring. Reducedness of the special fiber says that its order v_i satisfies $v_i(\varpi) = 1$.

The formula $|a|_i = |\varpi|^{v_i(a)}$ on $\operatorname{Frac}(R^\wedge)$ restricts to a bounded multiplicative norm on A extending the prescribed absolute value of F. Boundedness follows because it is at most one on $R^\wedge$, which is the ring of definition of the affinoid algebra. This gives the point ξ_i.

For $a \neq 0$ in A, set $m = \min_i v_i(a) \in \mathbf{Z}$ and $b = \varpi^{-m}a$. At every height-one prime above ϖ, the order of b is nonnegative. At every other height-one prime it is also nonnegative, since $b \in R^\wedge[1/\varpi]$ and ϖ is a unit there. The height-one intersection theorem for the Noetherian normal domain $R^\wedge$ [Stacks, Tag 0AVB], applied to the module $R^\wedge$ itself, gives $b \in R^\wedge$. Hence $|b(x)| \leq 1$ for every $x \in W$, and

$$|a(x)| \leq |\varpi|^m = \max_i |a(\xi_i)|.$$

The case $a = 0$ is immediate. Thus the finite set of ξ_i is a boundary and contains $\Gamma(W)$.

To see that no ξ_i can be omitted, choose $b_i \in R^\wedge$ belonging to all the generic primes $\mathfrak{p}_j$ for $j \neq i$ and not belonging to $\mathfrak{p}_i$. Such a choice exists: for each $j \neq i$ choose an element of $\mathfrak{p}_j \setminus \mathfrak{p}_i$ and multiply them; use $b_i = 1$ when there is only one component. Then $|b_i(\xi_i)| = 1$ and $|b_i(\xi_j)| < 1$ for $j \neq i$. Since the Shilov boundary is contained in the displayed finite boundary, it must contain ξ_i. This proves (8.1).

Under the additional principal-divisor assumptions, the order of x_i along D_j is δ_{ij}. This proves (8.2). In that case the particularly simple peaking function ϖ/x_i takes absolute value 1 at ξ_i and $|\varpi|$ at the other displayed points. □

Lemma 8.2 (Small algebraic charts with logarithmic interpolation)**.** *Let $\mathcal{N}$ be a strictly semistable finite-type F°-scheme in the sense of de Jong [dJ96, §2.16]: it is regular and flat, and its special fiber is a reduced strict normal crossings divisor. Let $s \in \mathcal{N}_s$. There is an algebraic affine neighborhood $\mathcal{U} = \operatorname{Spec} R$ of s with the following properties. Its nonempty special fiber is reduced and connected, with irreducible components $D_0, \dots, D_r$ all containing s. There are regular functions $x_0, \dots, x_r$ defining these components as effective Cartier divisors and satisfying $x_0 \cdots x_r = \varpi$. Its formal generic fiber W is a strict affinoid domain in the smooth algebraic generic fiber $\mathcal{U}_F^{\mathrm{an}}$, and its Shilov boundary is the set $\{\xi_0, \dots, \xi_r\}$ in Lemma 8.1.*

For any prescribed real numbers $c_0, \dots, c_r$, the function

$$h = -\frac{1}{-\log|\varpi|} \sum_{i=0}^{r} c_i \log|x_i| \tag{8.3}$$

is continuous on the whole algebraic generic fiber, is harmonic after every pullback to a smooth analytic curve over every complete extension, and satisfies $h(\xi_i) = c_i$. Every topological affinoid generator of W can be chosen as the restriction of a regular function on $\mathcal{U}_F$.

Status: `DIRECT_COROLLARY` of the strictly semistable divisor description and the preceding lemma. No algebraic closure or splitting of all strata is needed.

Proof. Strict semistability implies regularity of $\mathcal{N}$, reducedness of its special fiber, and that each irreducible component of the special fiber is an effective Cartier divisor; these are the properties in [dJ96, §2.16]. Remove all special-fiber components not containing s. Shrink to an algebraic affine neighborhood of s on which the remaining finitely many Cartier divisors are principal, say with equations x_i. Their restrictions remain irreducible: each is a nonempty Zariski-open subset of its original irreducible component. They all meet at s, so their union is connected. The ambient regular scheme can be restricted to its component containing s, giving the integral ring R.

The effective Cartier divisor of ϖ is exactly $D_0 + \cdots + D_r$, each with coefficient one. Therefore $x_0 \cdots x_r/\varpi$ is a regular unit after a further shrinking around s. Absorb its inverse into x_0 to obtain the stated product equality. Each x_i is a unit on $\mathcal{U}_F$.

Apply Lemma 8.1. Its coordinate values give $h(\xi_i) = c_i$. The function h is a real linear combination of logarithms of analytic units on a genuine boundaryless algebraic neighborhood of W. Such logarithms are continuous and have harmonic curve pullbacks, also after arbitrary complete scalar extension. Finally, finite-type algebraic generators of R give topological generators after completion and inverting ϖ, and are regular on $\mathcal{U}_F$. □

Use in the continuity argument. For an actual curvewise harmonic function f, choose $c_i = f(\xi_i)$. If the separately proved Shilov principle applies on W with its ambient regular Noether coordinates, then it applies to $f - h$. The latter vanishes on every Shilov point, and hence $f = h$ everywhere on W. This deduction uses no ambient semicontinuity of f and no pre-existing pluriharmonicity of f.

9 Algebraic neighborhoods for the continuity argument

This section isolates the geometry needed for a Shilov maximum argument. The original field is a fixed finite extension $K/\mathbf{Q}_p$, with its given absolute value. No function in this section is assumed upper semicontinuous merely because its restrictions to curves are harmonic.

Lemma 9.1 (Polynomial Noether coordinates)**.** *Let F be a complete nontrivially valued field of characteristic zero. Let Y be a smooth affine F-variety of pure dimension n, and let $V \subset Y^{\mathrm{an}}$ be a strictly affinoid domain. Assume that the affinoid algebra $A = \mathscr{O}(V)$ has a finite Tate presentation $A = F\langle T_1, \dots, T_m\rangle / I$ whose generators $a_1, \dots, a_m$ are restrictions of regular functions on Y. Then there are polynomials $P_1, \dots, P_n \in F[T_1, \dots, T_m]$ for which*

$$g_i = P_i(a_1, \dots, a_m), \qquad F\langle Z_1, \dots, Z_n\rangle \longrightarrow A, \quad Z_i \mapsto g_i,$$

is finite and faithfully flat. Thus $g = (g_1, \dots, g_n)$ gives a finite flat surjection $V \to \mathbb{B}^n_F$, and each g_i extends regularly to the boundaryless algebraic ambient space.

Status: `DIRECT_COROLLARY` of polynomial Noether normalization and miracle flatness. The extension of the coordinates is part of the conclusion; it is not an assumed extension theorem for arbitrary affinoid functions.

Proof. Write $A = F\langle T_1, \dots, T_m\rangle / I$ using the given tuple. The proof of affinoid Noether normalization uses triangular polynomial automorphisms

$$T_i \longmapsto T_i + T_m^{N_i} \quad (i < m), \qquad T_m \longmapsto T_m,$$

their polynomial inverses, and coordinate inclusions. One chooses a nonzero element of the current ideal, makes it distinguished in the last variable, and uses Weierstrass division to obtain a finite map from the algebra in one fewer variables. Induction on m gives an injective finite map from a Tate algebra of dimension $\dim A = n$. Since every change of variables is polynomial, the images of its variables are polynomials in the original tuple. This is the construction in Bosch's April 2008 manuscript, §1.2, Lemma 7 and Corollary 10, printed pages 11 and 14 [Bosch08]. It does not replace the given tuple by arbitrary analytic functions on V.

The algebra A is regular and of pure dimension n, since V is an analytic domain in the smooth Y^{an}. At a maximal ideal of the Tate algebra, its local ring is regular of dimension n. Every maximal localization of the finite algebra above it is Cohen–Macaulay of dimension n, and its fiber is zero dimensional. Miracle flatness, in the form of [Stacks, Tag 00R4], gives flatness at these localizations. A finite module is flat if and only if its localizations at maximal ideals are flat, so the map is finite flat. Its injection and integrality make the map on spectra, hence the finite map of affinoid spaces, surjective. The regular extension of g_i is the displayed polynomial expression. □

Lemma 9.2 (Algebraic norm boxes)**.** *Let Y be a smooth affine K-variety and let U be an ordinary open in Y^{an}. Every $x \in U$ has a strictly affinoid neighborhood $V \subset U$ which is the generic fiber of the π-adic completion of an integral affine finite-type flat K°-scheme $\mathcal{M}$, after restricting to one irreducible component at x. Its algebraic generic fiber is an affine Zariski open $Y_0 \subset Y$, and $\mathscr{O}(V)$ admits a finite Tate-presentation generating tuple whose entries are restrictions of regular functions on Y_0. The point x is in the ordinary interior of V.*

Proof. Use the topology of the analytification of an affine variety: absolute values of finitely many regular functions give basic ordinary neighborhoods. Choose strict upper and, where needed, strict positive lower bounds at x, within the prescribed neighborhood. The bounds can be chosen in $|K^\times|^{\mathbf{Q}}$, since its logarithm is a dense subgroup of $\mathbf{R}$. Add sufficiently loose

upper bounds for a finite set of affine coordinates to make the closed box compact. Replacing the bounds by non-strict inequalities gives an affinoid neighborhood still contained in U; the strict inequalities at x ensure that x is an interior point.

After inverting the functions with positive lower bounds, we are on an affine Zariski open Y_0. Raising a defining inequality to a positive integral power and multiplying by an integral power of π writes it as $|a_i| \leq 1$, with a_i regular on Y_0. Add a scaled finite generating set of $K[Y_0]$ whose norms are at most one on the box. These additional inequalities do not alter the box. Put

$$M_0 = K^\circ[a_1, \ldots, a_m] \subset K[Y_0].$$

Then $M_0[1/\pi] = K[Y_0]$ and M_0 is a finite-type, torsion-free K°-algebra. It is flat over the DVR K°. Its π-adic completion has generic fiber the affinoid domain defined by the chosen inequalities: the universal affinoid presentation adjoins precisely the displayed power-bounded generators. Thus $\mathcal{M} = \operatorname{Spec} M_0$ has the required properties. Smooth components of Y are disjoint locally; restrict to the component containing x to obtain integrality. This construction uses the original absolute value and makes no normalization $|\pi| = e^{-1}$. □

Proposition 9.3 (Semistable covers with an algebraic ambient map)**.** *Let $V \subset U \subset Y^{\mathrm{an}}$ be an algebraic norm box as in Lemma 9.2. There exist a finite extension K'/K, a smooth K'-variety Z, a proper surjective K-analytic map*

$$q : W \longrightarrow V$$

from a compact analytic domain $W \subset Z^{\mathrm{an}}$, and a finite affinoid cover $W = \bigcup_\lambda W_\lambda$ with the following properties.

1. *The map q is the restriction of an algebraic map $Z \to Y_0$ over K. The inverse image of U is therefore an ordinary boundaryless neighborhood of the whole of W.*
2. *Each W_λ is the generic fiber of the completion of a small affine open in a strictly semistable algebraic K'°-scheme. Its topological affinoid generators extend to regular functions on a smooth affine neighborhood in Z.*
3. *Each chart has regular functions $x_0, \ldots, x_r$ defining the irreducible components of its reduced special fiber, with $x_0 \cdots x_r = \varpi$ for a uniformizer ϖ of K'. Its Shilov boundary is precisely the corresponding set $\{\xi_0, \ldots, \xi_r\}$ of divisorial points. The continuous logarithmic functions*

$$\lambda_i = -\frac{\log |x_i|}{-\log |\varpi|}, \qquad \sum_{i=0}^{r} \lambda_i = 1$$

satisfy $\lambda_i(\xi_j) = \delta_{ij}$. Each x_i is a unit on the algebraic generic-fiber neighborhood.

In particular, the compact charts occur inside actual ordinary ambient neighborhoods. They are not treated as boundaryless merely because they have formal models.

Status: `DIRECT_COROLLARY` of de Jong's Theorem 6.5 [dJ96] and Lemma 8.2. Strict semistability here has de Jong's meaning: the total scheme is regular and its special fiber is a reduced strict normal-crossing divisor. No assertion about an entire skeleton is needed.

Proof. Apply [dJ96, Theorem 6.5] to $\mathcal{M}$ and its special fiber. It gives a finite extension of traits $\operatorname{Spec} K'^\circ \to \operatorname{Spec} K^\circ$, an alteration $\mathcal{N} \to \mathcal{M}$, and an open embedding of $\mathcal{N}$ in a projective strictly semistable scheme. Thus $\mathcal{N}$ itself is strictly semistable and is proper over $\mathcal{M}$. Its generic fiber Z is smooth over K' and has an algebraic proper surjection to Y_0.

Let W be the generic fiber of the completion of $\mathcal{N}$ along its special fiber. It is exactly the inverse image of V in Z^{an}. To check this at an actual point z, the condition that its image belong to V says that the corresponding map to $\mathcal{M}$ extends over $\mathscr{H}(z)^\circ$. The valuative

criterion for the proper map $\mathcal{N} \to \mathcal{M}$ uniquely extends the map to $\mathcal{N}$. Its center belongs to the special fiber, which is exactly the condition $z \in W$. The reverse inclusion follows directly from the morphism of completions. The analytic map $W \to V$ is proper and surjective and W is compact.

Apply Lemma 8.2 at every point of the special fiber of $\mathcal{N}$. Quasicompactness gives finitely many of its small algebraic affine charts. Completing them gives the asserted finite affinoid cover. Their finite-type coordinate rings supply regular topological generators on their algebraic generic fibers. The lemma gives the exact Shilov boundary and the logarithmic coordinate values over the same field K'. In particular no algebraic closure or splitting of strata is involved. Every chart maps into V, while its ordinary algebraic neighborhood maps into U after restricting to the inverse image of U. This proves the ambient assertion. □

Theorem 9.4 (Continuity from actual curvewise harmonicity)**.** *Let $f : U \to \mathbf{R}$ be an actual real-valued function on an allowed K-analytic space. Suppose that after every complete extension F/K, its pullback along every analytic map from an ordinary open in a smooth algebraic F-curve is an actual continuous harmonic function. Then f is continuous and therefore locally bounded.*

Status: `NEW_PROVED`, using Theorem 7.2. No semicontinuity, local boundedness, or Borel measurability of f is among the assumptions. Finiteness of its actual values and actual harmonicity on the indicated curves are essential.

Proof. Zero-dimensional smooth spaces are locally discrete, and their case is immediate. In positive dimension, B, Lemma D.4, allows us to work in an ordinary open of a smooth affine K-variety. Around a given point choose a norm box $V \Subset U$ with that point in its ordinary interior. Use Proposition 9.3. For a chart W_λ, denote its Shilov points by $\xi_0, \dots, \xi_r$ and put

$$h_\lambda = \sum_{i=0}^{r} (f \circ q)(\xi_i)\lambda_i.$$

This is continuous. It is a real linear combination of logarithms of analytic units, so its pullback to every smooth curve is harmonic. The same is true of $f \circ q - h_\lambda$ by the assumed curve property of f. The difference vanishes on every Shilov point of W_λ.

The affinoid generators of W_λ extend to its smooth algebraic neighborhood. Lemma 9.1 therefore gives finite flat Noether coordinates extending to that neighborhood. Apply Theorem 7.2 over K' to the difference. Its absolute value is bounded by its maximum on the Shilov boundary, which is zero. Hence

$$f \circ q = h_\lambda \quad \text{on } W_\lambda.$$

The finitely many affinoid domains W_λ are compact, hence closed in the Hausdorff compact space W. Closed-set pasting shows that $f \circ q$ is continuous on W. Finally a continuous surjection from a compact space onto the Hausdorff space V is a quotient map. Since $f \circ q$ is constant on its fibers by its definition, $f|_V$ is continuous. The ordinary interiors of these V cover U, proving the claim. □

Logical boundary. This argument has not used upper semicontinuous regularization of f, convergence of moving coefficient measures, pullback of an arbitrary current, or a strictly psh exhaustion of every analytic domain. Its indispensable analytic input is the Shilov principle for the actual curvewise harmonic function on compact slices with boundaryless ambient neighborhoods.

10 The arbitrary-background potential quotient

Fix a finite extension $K/\mathbf{Q}_p$ with its original absolute value. Let U be an ordinary open in a good Hausdorff boundaryless quasi-smooth K-analytic space of pure dimension n. Throughout this section $D = d'd''$, the test forms are all original compactly supported CLD smooth complementary forms, and positivity is the original positive cone used in R. Every positive current under consideration is symmetric. Strong and weak positivity are not substituted for that cone.

Write $\mathscr{DSH}_{\mathrm{lb}}(U)$ for the actual real functions which are locally differences of two members of $\mathcal{C}_b$. A function in $\mathscr{PH}(U)$ is a finite continuous function whose two signs are classical psh. Such a function is locally bounded. No quotient by a measure-null set occurs in either definition.

Theorem 10.1 (Regularity of the bounded classical curvature kernel)**.** *If $u, v \in \mathcal{C}_b(U)$ and $D[u] = D[v]$ on every original test, then the actual difference $u - v$ is continuous and belongs to $\mathscr{PH}(U)$. More generally,*

$$\ker\bigl(D : \mathscr{DSH}_{\mathrm{lb}}(U) \longrightarrow \mathcal{D}'^{1,1}(U)\bigr) = \mathscr{PH}(U).$$

The equality is an equality of spaces of actual functions. It does not assert regularity for the kernel on all locally integrable functions or on all abstract distributions.

Status: `NEW_PROVED` in the stated bounded-classical scope, relative to the declared internal inputs and the preceding proofs.

Proof. Let $H = u - v$. Theorem 6.3 shows that after every complete extension F/K, the pullback of H along every allowed smooth analytic curve is an actual continuous harmonic function. In particular it has the algebraic-curve property required by Theorem 9.4. That theorem applies to the actual locally bounded H and proves its ordinary continuity on U.

Both H and $-H$ are therefore upper semicontinuous. To check the full curve quantifier, let $C \to U_F$ be an arbitrary F-analytic curve map, including curves with boundary or singularities. Its two pullbacks are continuous. In the definition of subharmonicity on C, test further over any complete extension E/F by a smooth connected strict E-curve mapping to C_E. The composite maps to U_E, so the preceding harmonicity theorem makes both signs subharmonic on that test curve. Smooth curves can be localized in good Hausdorff boundaryless ordinary opens, and subharmonicity is local. Definition 3.4 and §4.1.3 of [GR25] therefore give subharmonicity on the original C. Thus the full classical psh condition holds for both signs, including its boundary and nonsmooth curve tests. They belong to $\mathcal{C}_b(U)$, and $H \in \mathscr{PH}(U)$.

For a function in $\mathscr{DSH}_{\mathrm{lb}}$ with zero curvature, apply this argument on its local difference presentations. Continuity and classical pluriharmonicity are local and hence glue. Conversely, for $H \in \mathscr{PH}$ the already established first-order positive-current theorem applied to H and $-H$ shows that $D[H]$ and $-D[H]$ are positive. A symmetric current and its negative can both be positive only if it is zero: on each finite moment presentation every symmetric smooth test is dominated in both signs by a positive control. Thus $D[H] = 0$. □

Theorem 10.2 (Potential independence for every fixed positive closed background)**.** *Let $0 \le p < n$, and let S be any symmetric original-positive (p,p)-current on U, separately d'-closed and d''-closed. For any $u, v \in \mathcal{C}_b(U)$,*

$$D[u] = D[v] \quad \Longrightarrow \quad D(uS) = D(vS). \tag{Q}$$

All products by u and v are the actual coefficient-Radon Borel products. The conclusion holds on every original compact smooth test of bidegree $(n-p-1, n-p-1)$, including all original moment refinements.

No smoothness, model origin, analytic-cycle representation, or pre-existing BT representation is required of S. For $p = n$ the corresponding differentiated statement is vacuous.

Status: `NEW_PROVED`. The final positivity step is a direct use of the previously established arbitrary-fixed-background first-order construction; the new input is Theorem 10.1.

Proof. Put $H = u - v$. Theorem 10.1 gives $H, -H \in \mathcal{C}_b(U)$ as actual functions. Apply the existing first-order BT construction, or explicitly Proposition 2.4 to the B-prescribed rows of H and $-H$. For the arbitrary fixed background S it gives two positive separately closed currents

$$D(HS), \qquad D((-H)S).$$

Borel multiplication and distributional differentiation are linear, so the second is the negative of the first. The positive-cone argument at the end of the preceding proof makes this current zero on every original complementary test. Linearity now gives $D(uS) - D(vS) = D(HS) = 0$.

Only the already defined multiplication of a fixed positive current by a locally bounded Borel function was used. In particular no pullback of S, no regularization of S, and no moving-background limit occurred. □

Corollary 10.3 (Differentiated-slot descent)**.** *Let S be as above, let $0 \le k \le n - p$, and let $u_1, \dots, u_k, v_1, \dots, v_k \in \mathcal{C}_b(U)$. If $D[u_i] = D[v_i]$ for every i, then*

$$T_{S,k}(u_1, \dots, u_k) = T_{S,k}(v_1, \dots, v_k)$$

on all original tests.

Status: `DIRECT_COROLLARY` of (Q) and the frozen fixed-background BT calculus.

Proof. Replace one differentiated slot at a time. By symmetry of the established mixed product, place the slot being changed first. The product of the remaining slots with S is one fixed positive separately closed background. Apply (Q) to this background and the two potentials in that slot. After the finitely many replacements the displayed equality follows. The case $k = 0$ is the identity $S = S$. □

Definition 10.4 (Positive currents with bounded classical local potentials)**.** An original $(1,1)$-current α on U has bounded classical local potentials if every point has an ordinary neighborhood O and $a \in \mathcal{C}_b(O)$ such that $\alpha|_O = D[a]$.

These currents are positive and separately closed by B. The definition is a local-potential condition, and it is not asserted for all positive closed $(1,1)$-currents.

Corollary 10.5 (A current-valued mixed operation with one arbitrary background)**.** *Let $\alpha_1, \dots, \alpha_k$ have bounded classical local potentials on U, with $k \le n - p$. There is a well-defined original positive separately closed current*

$$\alpha_1 \wedge \cdots \wedge \alpha_k \wedge S$$

whose expression on any ordinary open carrying local potentials $\alpha_i = D[a_i]$ is $T_{S,k}(a_1, \dots, a_k)$. It is independent of all local potentials and is symmetric in the α_i. It has the restriction, gluing, additivity, and positive homogeneity properties inherited from the bounded-potential BT calculus.

Status: `DIRECT_COROLLARY`.

Proof. Choose simultaneous local potentials on an ordinary cover. On an overlap, both local potentials of each α_i have the same ambient original curvature current. Corollary 10.3 gives equality of the two local mixed currents on every original test. The ordinary sheaf property for currents glues them uniquely. Their positivity and separate closedness are local. Symmetry and the linearity assertions follow from those properties for the chosen local potentials; potential independence removes the dependence on those choices. □

Corollary 10.6 (Pullback of bounded-potential curvature currents)**.** *Let α have bounded classical local potentials on an allowed original K-space X. Let L/K be any complete extension and let $F : Z \to X_L$ be analytic with Z allowed over L. The local rule*

$$F^*\alpha = D[a_L \circ F], \qquad \alpha = D[a] \text{ locally},$$

defines an original positive separately closed $(1,1)$-current on Z, independent of the chosen original-K potentials. Iterating the rule with the pullbacks of the original potentials agrees with further scalar extension and analytic composition. For maps between allowed original-K spaces, this is the usual intrinsic functoriality on the stated local-potential class.

Status: `DIRECT_COROLLARY` of Theorem 5.3. This is a pullback for the specified local-potential class, not for an arbitrary positive current.

Proof. On each inverse-image neighborhood, the actual function $a_L \circ F$ has a prescribed smooth positive row pulled back from its original-K row. Its original curvature is therefore positive and separately closed by the fixed-row first-order argument. Two choices of a on an original overlap have equal ambient curvature; the all-extension analytic kernel theorem gives equal displayed currents on the whole inverse-image overlap. They glue uniquely. Composition and extension commute with the actual compositions of functions and with their original smooth rows. Applying the same independence statement to the original potentials proves the asserted compatibility. When all intermediate spaces are over the original K, the theorem also applies to every alternative bounded classical potential on each intermediate space, giving intrinsic functoriality there. Over an arbitrary new field, the assertion about iteration is restricted to the induced original-potential presentations; it is not an extension of B to all new functions over that field. □

Undifferentiated weights and limits. The quotient is for differentiated slots. An undifferentiated weight is still an actual function: replacing it by another function with the same curvature need not preserve a weighted current. For example $D[0] = D[1] = 0$, but $0S = 0$ and $1S = S$ for every nonzero S. The seven established fixed-background BT properties retain their original functional hypotheses. This section does not turn arbitrary weak convergence of curvature currents or arbitrary varying backgrounds into weighted joint convergence.

Original tests and the remaining boundaries of scope. The operation above is an equality of ordinary original CLD currents. It is not promoted to equality on all delta-test forms. It does not define the wedge of two completely arbitrary positive closed currents. An arbitrary permitted S may still charge an ambient pluripolar set. Neither the regularity theorem nor the quotient proof requires that such concentration be excluded.

11 Signed currents with bounded classical difference potentials

The potential quotient also permits a real multilinear operation on a specified signed local-potential class. It does not provide local potentials for arbitrary closed currents.

Definition 11.1 (The signed local-potential class)**.** Let U be an allowed ordinary open over the original finite extension $K/\mathbf{Q}_p$. Write $\mathscr{V}_{\mathrm{cl}}^{1,1}(U)$ for the real original $(1,1)$-currents α such that every point has an ordinary neighborhood O on which

$$\alpha|_O = D[H], \qquad H = a - b, \quad a, b \in \mathcal{C}_b(O).$$

Equivalently, locally α is a difference of two positive currents admitting bounded classical psh potentials. The equivalence is local; a global positive-minus-positive decomposition on U is not required.

Every current in this class is symmetric, separately closed, and of coefficient-Radon type. These properties follow locally from $\alpha = D[a] - D[b]$. The class is a real vector space, because sums and nonnegative multiples of classical psh functions remain classical, while a negative coefficient can be absorbed by exchanging the two terms.

Theorem 11.2 (Signed differentiated slots over one arbitrary positive background)**.** *Let S be any symmetric original-positive (p,p)-current, separately d'- and d''-closed, and let $0 \le k \le n-p$. There is a well-defined coefficient-Radon separately closed current*

$$\alpha_1 \wedge \cdots \wedge \alpha_k \wedge S, \qquad \alpha_i \in \mathscr{V}_{\mathrm{cl}}^{1,1}(U),$$

which is symmetric and real multilinear in the differentiated slots. It commutes with ordinary restriction and gluing.

On a simultaneous local presentation $\alpha_i = D[a_i] - D[b_i]$, with $a_i, b_i \in \mathcal{C}_b$, it is given by the finite polarization

$$\sum_{\varepsilon \in \{0,1\}^k} (-1)^{\varepsilon_1 + \cdots + \varepsilon_k} T_{S,k}(c_{1,\varepsilon_1}, \ldots, c_{k,\varepsilon_k}), \qquad c_{i,0} = a_i, \quad c_{i,1} = b_i. \tag{11.1}$$

The result is independent of the actual difference potentials, their presentations as differences of classical functions, and the chosen positive-minus-positive current decompositions within the displayed local-potential class. The equalities hold on every original compact complementary smooth test, including every original moment refinement.

Status: `DIRECT_COROLLARY` of Theorem 10.2 and the frozen cone-additive symmetric BT calculus.

Proof. First choose one simultaneous local presentation. The finite sum (11.1) is a coefficient-Radon separately closed current, since each of its terms has these properties. Symmetry follows by permuting the slots in the known positive-potential mixed product and reindexing the finite sum.

We prove independence in one slot, holding the others fixed. Expand the other slots in their positive and negative choices. For each resulting term their mixed product with S is one fixed symmetric positive separately closed current Q. Thus it suffices to prove independence of

$$D(aQ) - D(bQ)$$

whenever $D[a] - D[b]$ is fixed. Suppose another allowed presentation gives the same current:

$$D[a] - D[b] = D[\widetilde{a}] - D[\widetilde{b}].$$

Then the two classical psh functions $a + \widetilde{b}$ and $\widetilde{a} + b$ have equal ambient original curvature. Applying the arbitrary-background quotient to this fixed positive Q gives

$$D((a + \widetilde{b})Q) = D((\widetilde{a} + b)Q).$$

Linearity of coefficient-Radon multiplication and distributional differentiation yields

$$D(aQ) - D(bQ) = D(\widetilde{a}Q) - D(\widetilde{b}Q).$$

This proves the required cancellation. In particular, it covers the case of two decompositions of the same actual difference, and the more general case of distinct difference potentials with the same current. It also covers any two positive-minus-positive decompositions whose terms admit the specified classical local potentials. Replacing one slot at a time proves full independence. The quotient theorem was applied only to positive backgrounds Q; no quotient for an arbitrary signed background was assumed.

For addition in a slot, present two signed potentials by $a - b$ and $c - d$ and present their sum by $(a + c) - (b + d)$. Cone additivity of $T_{S,k}$, followed by the displayed cancellation, gives additivity of the signed operation. Nonnegative scalar homogeneity follows in the same way. A negative scalar exchanges the positive and negative choices and contributes its sign, proving real linearity. This proves real multilinearity in all slots.

Choose such presentations on an ordinary open cover of U. On an overlap, the given currents α_i have not changed, so the independence just proved makes the two local finite sums equal on all original tests. The sheaf property for currents glues them uniquely. Separate closedness and coefficient-Radon type are local; the algebraic properties and restriction identities consequently glue as well. When $k = 0$, the operation is simply S. □

Positivity and scope. If every differentiated slot admits a classical psh local potential, the operation agrees with the already positive BT product. For the general signed class above, positivity is not part of the assertion. In particular, this section does not claim that a positive current of the form $D[H]$ with a bounded classical difference potential necessarily has a classical psh potential. It does not claim that all positive closed currents, or all signed closed currents, belong to $\mathscr{V}_{\mathrm{cl}}^{1,1}$.

There is one arbitrary positive separately closed background and finitely many differentiated local-potential slots. No product of two entirely arbitrary currents is defined. The construction is an equality of original ordinary CLD currents, with no promotion to all delta-test forms or to the general nonclosed, nonpositive, off-diagonal GK calculus.

12 Arbitrary backgrounds over extensions for induced potentials

In this section the potentials originate over the fixed finite extension $K/\mathbf{Q}_p$, but the background current need not. Every scalar extension carries the extending absolute value. The test category over each field is its full original CLD smooth test sheaf, and positivity is the original positive cone.

Corollary 12.1 (The quotient after an arbitrary extension and analytic map)**.** *Let U be an ordinary open in an allowed original-K space. Let $u, v \in \mathcal{C}_b(U)$ satisfy $D_K[u] = D_K[v]$ on all original K-tests. Let L/K be any complete extension, let Z be a good Hausdorff boundaryless quasi-smooth L-analytic space of pure dimension m, and let*

$$F : Z \longrightarrow U_L$$

be analytic. For every $0 \leq p < m$ and every symmetric original-positive (p,p)-current S on Z which is separately d'- and d''-closed, one has

$$D_L((u_L \circ F)S) = D_L((v_L \circ F)S). \tag{12.1}$$

The equality holds on every original compact smooth $(m-p-1, m-p-1)$-test on Z. Multiplication uses the actual coefficient-Radon Borel functions. The current S is arbitrary over L; it need not descend to K or be the pullback of any current.

Status: `DIRECT_COROLLARY` of Theorem 10.1 and the field-independent prescribed-row first-order construction.

Proof. The original-field regularity theorem gives an actual finite continuous function $H = u - v$ with $H, -H \in \mathcal{C}_b(U)$. The authorized original-K regularization supplies prescribed smooth positive rows for both signs, locally on fixed ordinary neighborhoods. Pull each such row first to U_L and then by F. Its members remain original smooth with positive $(1,1)$-curvature forms and decrease to the corresponding actual function $h = H_L \circ F$ or $-h$. The rows are locally uniformly bounded. This uses specified rows from K, not a regularization theorem for arbitrary new functions over L.

Proposition 2.4 applies over L to the fixed background S and each of these two rows. Thus $D_L(hS)$ and $D_L((-h)S)$ are positive symmetric currents. They are opposites by linearity of the actual Borel product and distributional differentiation. Finite positive controls on all original moment refinements therefore make them zero. Since $h = (u_L \circ F) - (v_L \circ F)$, this proves (12.1). Ordinary locality glues the conclusions from the row neighborhoods. No operation on S other than its fixed coefficient-Radon multiplication was used. □

Corollary 12.2 (Ordered products of induced local potentials)**.** *Keep L, Z, F, S as above, allowing $0 \le p \le m$, and let $0 \le k \le m-p$. For $u_1, \dots, u_k \in \mathcal{C}_b(U)$ put $a_i = u_{i,L} \circ F$. The ordered recursion*

$$T^{\mathrm{ord}}_{S,0} = S,$$
$$T^{\mathrm{ord}}_{S,k}(a_1, \dots, a_k) = D_L(a_1 T^{\mathrm{ord}}_{S,k-1}(a_2, \dots, a_k))$$

defines a symmetric original-positive separately closed current on all original tests and is independent of the chosen prescribed rows. Here symmetric *refers to the CLD involution of the resulting current, not to permutation of the ordered slots.*

If also $v_1, \dots, v_k \in \mathcal{C}_b(U)$ and $D_K[u_i] = D_K[v_i]$ for every i, then

$$T^{\mathrm{ord}}_{S,k}(u_{1,L} \circ F, \dots, u_{k,L} \circ F) = T^{\mathrm{ord}}_{S,k}(v_{1,L} \circ F, \dots, v_{k,L} \circ F). \tag{12.2}$$

Consequently an ordered tuple of original-K curvature currents with bounded classical local potentials gives a well-defined ordered operation with the arbitrary background S on Z, by choosing those local potentials on inverse-image ordinary opens. The operation respects ordinary restriction and gluing.

Status: `DIRECT_COROLLARY`. No permutation symmetry or weighted joint convergence theorem over an arbitrary new field is included in this assertion.

Proof. Each a_i has a prescribed smooth positive row induced from its original-K row. Starting at the innermost slot, apply Proposition 2.4. At each step its background is one already constructed fixed positive separately closed current. The actual recursion is defined using its coefficient-Radon measures, and gives the claimed positivity, current symmetry, separate closedness, and independence of row choices. The same argument works locally and commutes with ordinary restriction.

To replace slot i, keep all other slots fixed and put

$$R_i = T^{\mathrm{ord}}_{S,k-i}(a_{i+1}, \dots, a_k).$$

This is one fixed positive separately closed background. The preceding corollary gives

$$D_L((u_{i,L} \circ F)R_i) = D_L((v_{i,L} \circ F)R_i).$$

Apply successively the unchanged outer operations $R \mapsto D_L(a_{i-1}R), \dots, R \mapsto D_L(a_1 R)$ to this equality. Both inputs at every stage are the same positive current, so their actual Borel products and derivatives are equal. No exchange of adjacent slots is used. Replacing the finitely many slots proves (12.2); for $k = 0$ the identity is $S = S$.

On an overlap of original local-potential charts, the two original-K potentials of each curvature current have the same ambient original curvature. Apply the proved slot descent on the inverse-image overlap. The local ordered currents therefore agree on all original tests and glue uniquely. □

Exact boundary of the conclusion. The background over L is unrestricted within the stated positive separately closed class. The potentials, however, are induced from bounded classical potentials over the original K. The descent hypothesis is equality of the original K-curvatures; no assertion is made for additional equalities arising only after pullback over L. Neither result asserts regularization, kernel regularity, full slot symmetry, or joint monotone BT continuity for all unknown bounded classical functions over L. No pullback of S, or of a mixed current with arbitrary background, is defined. All equalities remain ordinary original-current equalities rather than equalities on every delta-test form.

13 Finite evaluation and stability inside the exact kernel

Write

$$\mathscr{H}_{\mathrm{cl}}(U) = \{H \in \mathscr{DSH}_{\mathrm{lb}}(U) : D[H] = 0\}$$

for the space of actual bounded-classical local differences in the original-current kernel. Theorem 10.1 identifies this space with $\mathscr{PH}(U)$. The consequences below use that identification and known properties of pluriharmonic functions. Their status is `DIRECT_COROLLARY`; finite-dimensionality and the pluriharmonic Shilov bound are not new claims of this project.

Corollary 13.1 (Finite evaluations control an exact kernel on a compact stage)**.** *Let U be an allowed original-K space and let $K_0 \subset U$ be a nonempty compact subset. There are finitely many compact affinoid domains $V_1, \dots, V_m \subset U$ whose ordinary interiors cover K_0. Put*

$$V = \bigcup_{a=1}^{m} V_a, \qquad \Gamma = \bigcup_{a=1}^{m} \Gamma(V_a).$$

Then Γ is a finite subset of U, and every $H \in \mathscr{H}_{\mathrm{cl}}(U)$ satisfies

$$\sup_{x \in V} |H(x)| = \max_{z \in \Gamma} |H(z)|, \qquad \sup_{x \in K_0} |H(x)| \le \max_{z \in \Gamma} |H(z)|. \tag{13.1}$$

The domains and the finite set can be fixed using only U and K_0; they do not depend on H, on a presentation $H = u - v$, or on a background current.

The space of restrictions $\mathscr{H}_{\mathrm{cl}}(U)|_V$ is a finite-dimensional subspace of $C(V)$, of dimension at most $|\Gamma|$, and is closed for the uniform norm.

Proof. Goodness supplies an affinoid neighborhood at every point of K_0, chosen inside U. Compactness supplies finitely many whose ordinary interiors cover K_0. Each Shilov boundary

is finite. By Theorem 10.1, the restriction of H to every V_a is pluriharmonic. The known Shilov maximum and minimum principle [GR25, Proposition 7.3] gives

$$\sup_{V_a} |H| = \max_{\Gamma(V_a)} |H|.$$

Taking the maximum over the finite cover gives (13.1). Evaluation on Γ is therefore an injective linear map from the space of restrictions into $\mathbf{R}^{\Gamma}$, and is an isometry for the displayed supremum norms. Its image is a linear subspace of a finite-dimensional vector space, hence is finite-dimensional and closed. This proves the last assertion. All domain choices preceded the choice of H. □

Corollary 13.2 (Finite pointwise limits in the exact kernel)**.** *Let $(H_i)_{i\in I}$ be a net in $\mathscr{H}_{\mathrm{cl}}(U)$ which converges at every actual point of U to a finite real value $H(x)$. Then the convergence is uniform on every compact subset of U, and*

$$H \in \mathscr{H}_{\mathrm{cl}}(U) = \mathscr{P}\mathscr{H}(U).$$

Neither continuity of the pointwise limit nor common bounds on the individual psh presentations of H_i are hypotheses.

In particular, on any quasicompact allowed U the kernel space itself is finite-dimensional, and the convergence is uniform on U.

Proof. Fix a compact $K_0 \subset U$ and the finite set Γ of Corollary 13.1. Apply its inequality to $H_i - H_j$, which is again in the exact kernel. For fixed i, pass to the pointwise limit in j. The maximum is over a finite set, so for every $x \in K_0$,

$$|H_i(x) - H(x)| \le \max_{z\in\Gamma} |H_i(z) - H(z)|.$$

The right-hand side tends to zero with i. This proves uniform convergence on K_0. In particular H is continuous locally. Local uniform limits of pluriharmonic functions are pluriharmonic by [GR25, Proposition 7.2(4)]; for a net, one may choose a uniformly convergent sequence on each of the fixed compact neighborhoods just constructed. Hence $H \in \mathscr{P}\mathscr{H}(U)$, and the kernel identification gives the displayed conclusion.

If U is quasicompact, choose a finite affinoid cover of U itself. The same finite-evaluation argument applies to the whole space. This is the argument underlying [GR25, Theorem 7.4]. □

Provenance and limits of the consequence. The preceding argument is the finite-dimensional Shilov argument of [GR25, Proposition 7.3 and Theorems 7.4–7.6], applied after the new identification of the actual original-current kernel. The proof of its Theorem 7.6 already contains the finite-evaluation mechanism; here it is written for nets and with continuity of the finite pointwise limit derived explicitly. No novelty is claimed for that elementary consequence. The finite dimension is asserted on a fixed quasicompact domain or for restrictions to a fixed compact affinoid cover. It does not assert a uniform bound for dimensions as neighborhoods shrink, or a finite-dimensional stalk of the pluriharmonic sheaf.

Exact vanishing $D[H_i] = 0$ is essential to this statement. Weak convergence $D[H_i] \to 0$ alone does not imply it, even for uniformly bounded continuous classical difference presentations: Example 22.3 has a nonzero actual polar-carried limit and a persistent relative curvature defect. Thus this corollary does not replace any capacitary or weighted convergence hypothesis by weak curvature convergence.

14 Cutoff energy and strong stability inside the exact kernel

Let Ω be a norm-box domain on which the faithful degree-one energy construction of Section R4–84 applies. Fix a symmetric original-positive, separately d'- and d''-closed current S of bidegree $(n-1,n-1)$. Write $(\mathcal{E}_S,\mathscr{D}_S)$ for its closed form and λ_S for the chosen faithful reference measure. It is locally finite; no finite total mass on Ω is assumed.

Corollary 14.1 (Exact cutoff energy identity)**.** *Let $H\in\mathscr{DSH}_{\mathrm{lb}}(\Omega)$ be actual, with $D[H]=0$ on all original compact tests. For every real original smooth compact function χ on Ω, one has $\chi H\in\mathscr{D}_S$ and*

$$\mathcal{E}_S(\chi H,\chi H)=\int_\Omega H^2\,\Gamma_S(\chi). \tag{14.1}$$

In particular the right-hand side is a fixed coefficient integral, not an integral against a varying curvature measure.

Status: `DIRECT_COROLLARY` of the quotient theorem and the frozen bounded BT–energy Green and product identities.

Proof. Theorem 10.1 makes H continuous and pluriharmonic. The arbitrary-background quotient theorem gives $D(HS)=0$. Proposition R4–86.1, applied to a local bounded classical difference representing H, gives local form membership and the bounded product identities.

Choose an ordinary relatively compact neighborhood W of $\operatorname{supp}\chi$ and an original smooth compact θ equal to one on a neighborhood of $\overline{W}$. For every original smooth compact φ supported in W, the Green identity gives

$$\mathcal{E}_S(\theta H,\varphi)=-\langle D(HS),\varphi\rangle=0.$$

The vector χ^2H belongs to the part space $\mathscr{D}_{S,0}(W)$: the support-preserving approximations in Proposition R4–86.1, multiplied by χ^2, are original core functions supported in W and converge in form norm. Hence the displayed equality extends to this test vector. Since $\theta=1$ near its support, the local Leibniz identity yields

$$0=\mathcal{E}_S(\theta H,\chi^2H)=\int\chi^2\Gamma_S(H)+2\int\chi H\,\Gamma_S(H,\chi).$$

All terms are finite by local boundedness and energy Cauchy–Schwarz. The product square identity gives

$$\mathcal{E}_S(\chi H,\chi H)=\int\chi^2\Gamma_S(H)+2\int\chi H\,\Gamma_S(H,\chi)+\int H^2\Gamma_S(\chi).$$

Subtracting the preceding equality proves (14.1). Thus no nonsmooth test was inserted directly into a distributional equation: it was reached through the closed form and compact core density. □

Corollary 14.2 (Finite evaluations control the local form norm)**.** *Fix χ as above. There is a finite set $\Gamma\subset\Omega$, depending on Ω and $\operatorname{supp}\chi$ but not on H or on its classical-difference presentation, such that*

$$\begin{aligned}\mathcal{E}_S(\chi H,\chi H)&\le\Big(\int\Gamma_S(\chi)\Big)\max_{z\in\Gamma}|H(z)|^2,\\ \|\chi H\|^2_{\mathscr{D}_S}&\le\Big(\int\Gamma_S(\chi)+\int\chi^2\,d\lambda_S\Big)\max_{z\in\Gamma}|H(z)|^2,\end{aligned} \tag{14.2}$$

where $\|f\|^2_{\mathscr{D}_S}=\mathcal{E}_S(f,f)+\|f\|^2_{L^2(\lambda_S)}$. Both constants are finite for the fixed current and cutoff.

Consequently, if a net (H_i) in the actual exact kernel converges at every point of Ω to a finite real function H, then H is pluriharmonic and

$$\chi H_i \longrightarrow \chi H \quad \textit{strongly in } \mathscr{D}_S$$

for every fixed S and every such χ.

Proof. If $\chi = 0$ there is nothing to prove. Otherwise apply Corollary 13.1 to the compact support of χ. Its finite set controls the supremum of $|H|$ on that support. The positive measure $\Gamma_S(\chi)$ is carried by this compact set, so (14.1) gives the first bound. The fixed locally finite reference measure gives the second bound.

For the net, Corollary 13.2 first proves that H is pluriharmonic. Apply (14.2) to $H_i - H$. The maximum over the fixed finite set tends to zero, proving the asserted strong convergence. This requires no common bound on the separate psh presentations of the net. □

Remaining degree and scope. The scalar closed form above has a degree-one background. Starting with an arbitrary allowed (p,p) background S_0 and $q = n - p \geq 1$, one may freeze any $q-1$ bounded classical slots and use

$$S = T_{S_0,q-1}(u_1, \ldots, u_{q-1})$$

as that fixed positive closed background. The same identities then apply to its faithful form. They do not assert an unfrozen all-degree Hilbert form, uniform constants as S varies, or continuity for $S_i \to S$. Exact membership in the actual kernel is essential: weakly vanishing ambient curvatures and uniformly bounded classical decompositions do not give these conclusions for their assigned limits, as Example 22.3 shows.

15 A fixed-background energy criterion for curvature convergence

Fix the original field K, an allowed ordinary open U of pure dimension n, and a symmetric original-positive separately d'- and d''-closed current S of bidegree (p,p). Put $q = n - p \geq 1$. All tests below are original compact smooth CLD forms. For a degree-one background R on an algebraic norm-box domain Ω, let $(\mathcal{E}_R, \mathscr{D}_R)$ denote the faithful closed form of Theorem R4–84.3. Its values on bounded classical differences agree with the bounded BT energy by Proposition R4–86.1.

Proposition 15.1 (Finite-test energy control)**.** *For every original compact $(q-1,q-1)$-test η on U, there are finitely many ordinary open algebraic norm-box domains $\Omega_b \Subset U$, original smooth compact real functions f_{ba} on Ω_b, original smooth compact cutoffs χ_b equal to one near all the supports of f_{ba}, and constant original-positive closed forms θ_{ba} of degree $(q-1,q-1)$ in fixed unit-moment coordinates on Ω_b, such that the following statements hold.*

The currents

$$R_{ba} = (S|_{\Omega_b}) \wedge \theta_{ba}$$

are symmetric original-positive separately closed currents of bidegree $(n-1,n-1)$. For every actual $H \in \mathscr{DSH}_{\mathrm{lb}}(U)$, $\chi_b H \in \mathscr{D}_{R_{ba}}$ and

$$\langle D(HS), \eta \rangle = -\sum_{b,a} \mathcal{E}_{R_{ba}}(\chi_b H, f_{ba}). \tag{15.1}$$

Consequently

$$\left|\langle D(HS),\eta\rangle\right| \le C_{S,\eta}\left(\sum_{b,a}\mathcal{E}_{R_{ba}}(\chi_b H,\chi_b H)\right)^{1/2}, \qquad C_{S,\eta}^2=\sum_{b,a}\mathcal{E}_{R_{ba}}(f_{ba},f_{ba})<\infty. \tag{15.2}$$

The boxes, cutoffs, forms, and constant are fixed by S and the original test; none depends on H or on a regularization of its classical-difference presentations.

Status: `DIRECT_COROLLARY` of the original positive-square test decomposition, the faithful closed-form construction, and the bounded Green identity.

Proof. The current $D(HS)$ is symmetric, so replace η by its symmetric part. A finite original smooth partition localizes it in moment presentations. Refine that finite localization to relatively compact ordinary open algebraic norm boxes inside the presentations. The local algebraic neighborhoods and these boxes are available over the original K by B, Lemma D.4, and Lemma 9.2. Each localized test has compact support in one such box. Use strict slack in that construction: Ω_b is defined by strict norm inequalities and its closed norm box is compactly contained in the moment presentation. The faithful energy form is taken on the boundaryless ordinary open Ω_b; the closed affinoid box supplies only its compact containment.

Fix one box and write its moment coordinates as $t_1,\dots,t_r$. The symmetric coefficient matrix of the localized test is a smooth real matrix indexed by $(q-1)$-multi-indices. Constant positive rank-one matrices span all such symmetric matrices: use the squares of the basis vectors and the squares of sums of two basis vectors, and absorb the signed scalar coefficients into f_{ba}. Thus its exact expression is

$$\sum_a f_{ba}\theta_{ba}.$$

The scalar coefficients can be multiplied by one compact cutoff equal to one near the localized support, so they belong to $A_c^{0,0}(\Omega_b)$. Each θ_{ba} is the positive signed square of a constant $(q-1,0)$-form in the $d't_i$. This form need not be decomposable. The construction is the finite square span of (22.2); it uses the full original positive cone, not just strongly positive squares. For $q=1$ the square is simply the constant scalar 1.

The forms θ_{ba} are separately closed. To check positivity of R_{ba}, test it by an original positive $(1,1)$-form on any common refinement. Such a form is a sum of positive squares of one-covectors. Its wedge with θ_{ba} is a sum of positive squares of the exterior products of those one-covectors with the defining $(q-1)$-covector. It is therefore an original positive test for S. This proves positivity of R_{ba}, including all original refinements. Symmetry and separate closedness follow from the same properties of S and the constant closed form.

By Proposition R4–86.1, a cutoff times a bounded classical difference belongs to $\mathscr{D}_{R_{ba}}$. For a locally presented difference, use a finite partition on the compact cutoff support and the bounded smooth multiplier property of the closed form. This proves the asserted membership for $\chi_b H$.

The smooth closed product rule and the scalar Green identity now give, with precisely the displayed sign,

$$\begin{aligned}\langle D(HS), f_{ba}\theta_{ba}\rangle &= \langle D(HR_{ba}), f_{ba}\rangle\\ &= \int_{\Omega_b} H\, D(f_{ba}R_{ba}) = -\mathcal{E}_{R_{ba}}(\chi_b H, f_{ba}).\end{aligned}$$

The cutoff is one near the support of f_{ba} and its derivatives. The last identity is the strong-form extension of compact smooth Stokes, not an insertion of a nonsmooth function as an original smooth test. Summing proves (15.1). Apply Cauchy–Schwarz in each fixed closed form and

then in the finite index set to obtain (15.2). Every energy of a fixed smooth compact coefficient f_{ba} is finite by local coefficient-Radon control. This also establishes the claimed independence of the constants. □

Corollary 15.2 (Convergence modulo actual pluriharmonic corrections)**.** *Let $H_j \in \mathscr{DSH}_{\mathrm{lb}}(U)$ be a sequence or net. Suppose that for every original test η and the finite family supplied by Proposition 15.1, there are $h_{b,j} \in \mathscr{PH}(\Omega_b)$ such that*

$$\sum_{b,a} \mathcal{E}_{R_{ba}}\big(\chi_b(H_j - h_{b,j}), \chi_b(H_j - h_{b,j})\big) \longrightarrow 0. \tag{15.3}$$

Then $D(H_jS) \longrightarrow 0$ on every original compact complementary test on U.

In particular, if $u_j, u \in \mathcal{C}_b(U)$ and the hypothesis holds with $H_j = u_j - u$, then

$$D(u_jS) \longrightarrow D(uS).$$

No local uniform bound on the separate psh presentations is required beyond what is needed for each displayed individual energy; the convergence hypothesis itself is substantive.

Status: `DIRECT_COROLLARY` of the preceding estimate and the arbitrary-background quotient.

Proof. For each fixed box, actual pluriharmonicity and Theorem 10.2 give $D(h_{b,j}S) = 0$ there. Equivalently they give $D(h_{b,j}R_{ba}) = 0$ for every member of its finite family. Apply (15.1) to each localized term, subtract that zero current, and then apply (15.2) with the local difference $H_j - h_{b,j}$. Its right-hand side tends to zero by (15.3), while the left-hand side is the pairing of $D(H_jS)$ with the original test. This proves the statement for every test. Linearity gives the last assertion. □

What the criterion does and does not require. All the degree-one backgrounds are fixed smooth wedges of the one given S; there is no sequence of backgrounds. The positive square spanning family accounts for every original complementary test, including tests with new moment coordinates. The criterion requires energy of the error, after an actual pluriharmonic correction, to tend to zero. Strong energy convergence to a different assigned Borel representative does not satisfy this requirement. In particular it does not rescue the weak-current or completed-polar-selector failures recorded earlier. No assertion about varying mixed BT tails or joint higher mixed-current convergence follows from this first-order estimate alone.

Corollary 15.3 (No pluriharmonic correction removes the smooth relative defect)**.** *In Example 24.1, one can choose an ordinary algebraic norm box $\Omega_0 \Subset \Omega$, a nonnegative original smooth compact scalar φ on Ω_0, and an original smooth compact cutoff χ equal to one near $\operatorname{supp}\varphi$, with the following property. Put $R = [Y]|_{\Omega_0}$. There is a constant $c_0 > 0$, independent of N and of $h \in \mathscr{PH}(\Omega_0)$, such that*

$$\mathcal{E}_R\big(\chi(v_N - w - h), \chi(v_N - w - h)\big) \geq c_0.$$

Status: `DIRECT_COROLLARY`.

Proof. The example supplies the fixed nonzero positive measure $D(q[Y])$. Choose a nonnegative compact annular scalar profile detecting it and multiply by a compact original cutoff in the z-direction equal to one near $z = 0$. Its support can be placed inside a relatively compact algebraic norm box Ω_0 in the bidisc. This gives φ with

$$c = \langle D(qR), \varphi\rangle > 0.$$

Choose χ as stated. For every N, the example gives $D((v_N - w)R) = -D(qR)$, whereas the quotient theorem gives $D(hR) = 0$. Green and Cauchy–Schwarz in the one fixed closed form therefore imply

$$c = |\mathcal{E}_R(\chi(v_N - w - h), \varphi)| \le \mathcal{E}_R(\chi(v_N - w - h), \chi(v_N - w - h))^{1/2} \mathcal{E}_R(\varphi, \varphi)^{1/2}.$$

The last fixed energy is finite. It is positive, since a zero value would contradict $c > 0$ in this inequality. Thus $c_0 = c^2/\mathcal{E}_R(\varphi, \varphi)$ proves the claim. The conclusion concerns this fixed relative form; it does not define an intrinsic all-degree quotient norm. □

16 A finite-field check for normal Gauss coarea

This section checks a possible failure mode in the normal trace: could a finite residue-field extension split after adjoining the normal Gauss parameter, or create extra correlated points on the calibrated support? The answer for this particular product support is no. The following finite-dimensional argument requires neither stability of the valued field nor a ramification–residue-degree formula.

Lemma 16.1 (Finite extensions commute with Gauss completion)**.** *Let F be a complete non-Archimedean field, let E/F be a finite valued extension with its extending absolute value, and let $r > 0$ be any real radius. Denote by F_r the completion of $F(T)$ for the radius-r Gauss norm and by E_r the analogous completion of $E(T)$. Then the natural homomorphism*

$$E \otimes_F F_r \longrightarrow E_r \tag{16.1}$$

is an isomorphism of finite F_r-algebras. In particular, its source is a field and

$$[E_r : F_r] = [E : F].$$

There is only one point in the Berkovich spectrum of this finite F_r-algebra, with the full displayed algebraic degree. No rationality assumption on r is imposed.

Status: `DIRECT_COROLLARY` of finite-dimensional norm equivalence and the Gauss norm. The proof is given because it is exactly the field assertion needed in the product calibration, including in the presence of defect.

Proof. Choose an F-basis $e_1, \dots, e_d$ of E. There are constants $0 < c \le C < \infty$ such that

$$c \max_i |a_i| \le \left| \sum_i e_i a_i \right| \le C \max_i |a_i| \qquad (a_i \in F). \tag{16.2}$$

For completeness, the lower estimate needs only completeness of F. Induct on the number of basis vectors. The span of the first $d-1$ vectors is complete by the induction hypothesis, hence closed in E. Its distance from e_d is positive. Consequently the norm of $\sum_i e_i a_i$ controls $|a_d|$; the ultrametric triangle inequality then controls the norm of the first $d-1$ summands, and induction controls their coefficients. The upper estimate is the triangle inequality. This proves (16.2) without an orthogonal-basis assertion.

For polynomials $P_i(T) \in F[T]$, apply (16.2) separately to every coefficient and then take the weighted maximum over the powers of T. The result is

$$c \max_i \|P_i\|_r \le \left\| \sum_i e_i P_i \right\|_r \le C \max_i \|P_i\|_r. \tag{16.3}$$

The same inequalities hold for $P_i \in F(T)$ after clearing one common denominator in $F[T]$ and dividing by its nonzero Gauss norm.

Algebraically $E \otimes_F F(T) = E(T)$, and the same e_i form an $F(T)$-basis. To see explicitly that a denominator in $E[T]$ causes no problem, multiplication by such a nonzero polynomial is injective on the free $F[T]$-module $E[T]$. After localizing $F[T]$, it is an injective endomorphism of a finite-dimensional $F(T)$-space, hence is invertible. Its adjugate divided by its nonzero determinant gives its inverse with coefficients in $F(T)$. Thus every element of $E(T)$ has the required basis expression.

The Gauss norm on $E(T)$ is multiplicative. Its completion E_r is a field: if a Cauchy sequence converges to a nonzero element in that completion, the norms of its terms eventually have a fixed positive value, and their inverses are Cauchy by the identity $x^{-1} - y^{-1} = (y - x)/(xy)$. Completing (16.3) identifies this completion, as a normed vector space, with F_r^d in the basis e_i. The lower bound proves injectivity of (16.1); the same estimate and density prove its surjectivity. Multiplication commutes with completion, so it is the claimed algebra isomorphism.

Finally a finite field extension of the complete field F_r has its unique extending absolute value. Its Berkovich spectrum therefore consists of one point. The degree is the vector-space dimension d retained by the displayed basis; it has not been computed from residue or ramification indices. □

Corollary 16.2 (The special product top-form coarea)**.** *Let F be complete, nontrivially valued, and of characteristic zero, and let B be a good Hausdorff boundaryless smooth F-analytic space of pure dimension d, and let $\eta \in A_c^{d,d}(B)$ be an original form, with its signed coefficient Radon measure μ_η. Let κ be a smooth real function of compact support in a radial interval and put, on the corresponding normal annulus,*

$$\beta_\kappa = \kappa(\log|T|)\, d' \log|T| \wedge d'' \log|T|.$$

Write $\sigma_r(x)$ for the image in $B \times_F \mathbf{A}_F^{1,\mathrm{an}}$ of the radius-r Gauss point over $\mathscr{H}(x)$. For every bounded Borel function c on a neighborhood of the indicated compact product support,

$$\int c\, \beta_\kappa \wedge \pi^* \eta = \int_B \int_{\mathbf{R}} c(\sigma_{e^t}(x))\kappa(t)\, dt\, d\mu_\eta(x). \tag{16.4}$$

Thus this product coefficient measure is carried by the canonical normal Gauss lifts of the base top-form measure.

Status: `DIRECT_COROLLARY` of the original product calibration, with the finite-field step made explicit above. This checks the particular formula used in Lemma 5.2; it is not an independent-section formula for an arbitrary full analytic fiber.

Proof. Use finitely many original moment presentations for η. On a calibrated top-dimensional cell, choose a d-coordinate monomial projection of full rank. At each of its relevant Gauss points the finite source fibers have completed residue fields E finite over the moment Gauss field F_0. Their full finite algebra degrees are exactly the degrees retained by original calibration. Adjoining the normal coordinate replaces F_0 by its radius-r Gauss completion. By Lemma 16.1, each finite field E yields precisely its canonical Gauss completion E_r, with the same full degree. Consequently there is one normal Gauss lift for each such base point, and there is no additional split factor or lost multiplicity on this calibrated support.

The product cell measure in the added coordinate is dt with multiplicity one. The preceding field calculation therefore gives (16.4) for original smooth scalar coefficients. Refining moments for such a coefficient does not change the calibrated support or its degree. Signed linearity and the finite partition recover an arbitrary original top form η.

The normal Gauss lift is continuous on each compact affinoid stage and compact radial interval: for a polynomial $\sum_j a_j T^j$ its norm is $\max_j |a_j(x)| r^j$, and uniform analytic approximation gives the general assertion. Hence the right side defines a finite signed Radon measure on the compact support. Equality on original smooth scalar tests, which are uniformly dense on compact stages, identifies that measure with the left-hand coefficient measure. Integration of a fixed bounded Borel function proves the stated identity. □

The excluded inference. The full fiber of the projection to the normal parameter may contain correlated points, such as those on which $|x - T|$ is smaller than both $|x|$ and $|T|$. It has not been replaced by the image of σ_r. The conclusion concerns the coefficient measure of a *pulled-back base top form* wedged with a radial normal top form. An arbitrary form on the full fiber, involving mixed moments such as $x - T$, need not have this support. This precise distinction is what makes the normal trace compatible with the correlated-point counterexample recorded in R4.

17 A finite-etale collision in the original piecewise test class

The trace construction must retain joint unit norms, allow genuine coefficient jumps in normal directions, and retain full finite degrees. The following computations check these three features. They are illustrations of the proved construction, not additional kernel or regularization theorems.

Example 17.1 (Two joint norm vectors which collide)**.** Let F be a complete nontrivially valued field of characteristic zero, with its given absolute value. Choose $0 < \epsilon < 1$ and put

$$U = \{\epsilon < |T| < \epsilon^{-1},\ 4T + 1 \neq 0\} \subset \mathbf{A}^{1,\mathrm{an}}_F.$$

The inverse image Z under the polynomial map

$$p : Y \longmapsto T = Y(Y - 1)$$

is finite etale of degree two over U. Both Y and $Y - 1$ are analytic units on Z. Write

$$s = \log |T|, \qquad z_1 = \log |Y|, \qquad z_2 = \log |Y - 1|.$$

The unordered joint norm vectors, counted with their full geometric multiplicities over each completed residue field, are

$$\{(z_1, z_2)\} = \begin{cases} \{(s, 0), (0, s)\}, & s < 0, \\ \{(0, 0), (0, 0)\}, & s = 0, \\ \{(s/2, s/2), (s/2, s/2)\}, & s > 0. \end{cases} \tag{17.1}$$

For the original smooth source form

$$\alpha = d' z_1 \wedge d'' z_2,$$

the ordinary pushforward current is represented by the finite compatible piecewise form

$$p_*[\alpha] = [\tau], \qquad \tau = \begin{cases} 0, & s < 0, \\ \frac{1}{2}\, d's \wedge d''s, & s > 0. \end{cases} \tag{17.2}$$

The representative on the wall $s = 0$ is immaterial. The current identity is on every compact original complementary test on U; no all-delta-test assertion is being made.

Status: `DIRECT_COROLLARY` of Lemma 3.3 and the elementary Newton polygon calculation below.

Proof. The map is finite free of degree two, since its equation is monic in Y. Its derivative $2Y-1$ vanishes only over $T=-1/4$, which was removed. It is therefore finite etale. The two source functions are units because their product is the base unit T.

If $|T|<1$, the polynomial Y^2-Y-T has root norms $|T|$ and 1. For the small root $|Y-1|=1$; for the other root the identity $Y(Y-1)=T$ gives $|Y-1|=|T|$. If $|T|>1$, both roots have norm $|T|^{1/2}>1$, and $|Y-1|=|Y|$. If $|T|=1$, both Y and $Y-1$ have norm one. These computations prove (17.1), over algebraic closures of the completed residue fields and hence with exactly the finite local multiplicities used in the trace theorem.

On $s<0$, each of the two vector maps has one constant coordinate, so both pullbacks of $d'z_1\wedge d''z_2$ vanish. On $s>0$, each vector map is $(s/2,s/2)$ and contributes $\frac{1}{4}d's\wedge d''s$. Their sum is the second line of (17.2). This is precisely the joint-norm trace formula of the cited lemma. Local compact cutoffs make its ordinary-current identity applicable to every compact test. A cutoff may also use the original unit $4T+1$; no radial cutoff which remains nonzero arbitrarily close to the removed critical value is being declared compact on U.

Both tensor restrictions to the wall tangent space vanish, so the piecewise form is tangentially compatible even though its ambient normal coefficient jumps. The unit-radius Gauss point belongs to U: it is not the removed rigid point $T=-1/4$. Thus this is a genuine collision within an allowed ordinary neighborhood, not a limiting event excluded from the space. □

Why marginal records are insufficient. For $s<0$, the two individual coordinate records are both $\{s,0\}$. Matching them independently as $\{(s,s),(0,0)\}$ would incorrectly give $d's\wedge d''s$ instead of zero. The joint monomial record for $Y(Y-1)=T$ rules out this matching: its two log norms are s,s, whereas the false matching would give $2s,0$. The full finite-grid tomography in the trace proof is the general version of this elementary correction.

Why the coefficient jump is allowed. The same example may be multiplied by any additional smooth analytic factor and the finite map replaced by $p\times\mathrm{id}$. Then τ is an intermediate-degree piecewise form on the base. For a symmetric original-positive separately closed current R of complementary bidegree, the wall estimate in Lemma 3.2 gives, on each fixed compact stage H,

$$(d's\wedge d''s\wedge R)(H\cap\{|s|\le\delta\})\le C\delta\sum_b\langle R,\Theta_b\rangle.$$

In particular this coefficient measure gives the wall zero mass. The value assigned to the discontinuous coefficient in (17.2) therefore does not affect the pairing. For a weakly convergent positive closed family, the fixed-control masses are uniformly bounded, so smoothing this particular jump has a uniform $O(\delta)$ error. General compatible jumps with only one normal factor use the proved $O(\delta^{1/2})$ matrix Cauchy–Schwarz estimate. There is no claim that the trace is an original smooth form.

A degree check with a power map. For $T=Y^d$ on a torus annulus, let $s=\log|T|$ and $z=\log|Y|$. There are d geometric root-norm records $z=s/d$, even when they describe a single Berkovich fiber point. Thus the trace of a scalar $\varphi(z)$ is $d\varphi(s/d)$, while the trace of $\varphi(z)d'z\wedge d''z$ is

$$d\left(\frac{1}{d}\right)^2\varphi(s/d)d's\wedge d''s=\frac{1}{d}\varphi(s/d)d's\wedge d''s.$$

The latter agrees with the direct radial integration change of variables. Counting only Berkovich fiber points would lose this factor. The construction uses the full finite algebra degree; it never replaces that degree by a ramification–residue product.

18 An exact closedness test for the potential quotient

The following elementary example checks a hypothesis of the arbitrary-background theorem. It is not a counterexample under that theorem's stated assumptions.

Example 18.1 (Merely D-closed backgrounds do not suffice)**.** Let π be a uniformizer of the original finite extension $K/\mathbf{Q}_p$, put $R = |\pi|^{-1}$ and $\ell = \log R > 0$, and let

$$A = \{1 < |T| < R\} \subset \mathbf{A}_K^{1,\mathrm{an}}, \qquad t = \ell^{-1} \log |T|.$$

Thus $0 < t < 1$ and t is an original smooth classical pluriharmonic function on the ordinary annulus A. Set

$$u = t, \qquad v = 0, \qquad S = (1+t)[1]_A.$$

The symmetric $(0,0)$-current S is original-positive and $DS = 0$. Nevertheless

$$D[u] = D[v] = 0, \qquad D(uS) - D(vS) = 2[d't \wedge d''t] \neq 0. \tag{18.1}$$

In particular, the separate d'- and d''-closedness in Theorem 10.2 cannot be replaced by D-closedness alone.

Status: `FALSE` for the weakened statement; `DIRECT_COROLLARY` of the original smooth product rule for the displayed construction.

Proof. The analytic function T is a unit on A, so $Dt = 0$ and both signs of t are classical psh. Since $1 + t > 0$, the displayed coefficient multiplication makes S positive. Smooth Leibniz and closedness of $[1]_A$ give

$$DS = D[1+t] = 0, \qquad D(uS) = D[t + t^2] = 2[d't \wedge d''t].$$

The last current is not zero. Indeed, for a nonnegative nonzero $\varphi \in C_c^\infty((0,1))$, original annulus calibration with the coordinate $x = \log |T| = \ell t$ gives

$$\langle 2[d't \wedge d''t], \varphi(t) \rangle = \frac{2}{\ell} \int_0^1 \varphi(s)\, ds > 0.$$

The support of $\varphi(t)$ is a compact closed subannulus, so this is an allowed original test. The same formulas show that $d'S = [d't]$ and $d''S = [d''t]$ are not zero. Hence the example violates exactly the separate-closedness assumption, while retaining positivity, symmetry, local boundedness, the original field, and its prescribed absolute value. □

Remark 18.2 (A nonconstant kernel is permitted). **Status:** `DIRECT_COROLLARY`**.** On the same annulus, let S instead be any background satisfying the actual theorem's separate-closedness assumptions, on A or on its product with another allowed space. The smooth product rule then gives $D(tS) = Dt \wedge S = 0$. Thus the theorem recovers the nonconstant logarithmic kernel; it does not imply that a curvature-kernel function is locally constant. It also allows backgrounds carried by analytic subvarieties of that product, since a restriction of the unit T is still a unit there.

Actual type-IV values. The endpoint argument has a separate check. On a smooth curve over an algebraically closed complete extension, let x be of type IV and choose nested closed discs with Gauss points $\xi_j \to x$. For either of the original bounded subharmonic pullbacks w separately, the disc maximum principle and upper semicontinuity give

$$w(x) \le w(\xi_j), \qquad \limsup_j w(\xi_j) \le w(x).$$

Thus both original functions have their correct actual endpoint values, and so does their difference. This checks the passage in Lemma 6.2; it does not treat a type-IV point as a Gauss point or infer a Borel-weighted limit from weak convergence of fiber measures.

No faithfulness condition on the background. The final argument for S uses that a symmetric original current and its negative cannot both be positive unless they are zero. It does not require the coefficient measures of S to detect every point or every ordinary open set. Thus $S = 0$, backgrounds with smaller support, and integration currents carried by ambient pluripolar subvarieties all remain within scope. At $p = n$ there is no further differentiated degree; the statement is vacuous rather than an additional analytic assertion.

Finite extensions retain the original calibration. For a finite extension F/K of degree d, the integration current of an F-space viewed as a K-space is d times its F-calibrated current on common original tests. By contrast, scalar pushforward of the F-calibrated top-form measure has factor one. These are respectively the conventions of [CLD25, Remark 14.4.3, p. 410, and Proposition 15.3.1, p. 425]. They are compatible: the finite morphism viewed over K has degree d. The kernel arguments divide a zero pairing by this single positive factor when necessary; they neither change the absolute value nor introduce a power of d for differentiated slots. No ramification–residue-degree identity is required.

19 Why same-field curve tests do not suffice

Example 19.1 (A point invisible to every same-field analytic curve)**.** Let $K/\mathbf{Q}_p$ carry its given absolute value, choose a uniformizer π, and put $r = |\pi| < 1$. In $U = (\mathbf{A}^2_K)^{\mathrm{an}}$, let x be the two-variable Gauss point of polyradius (r, r) and set

$$f = \mathbf{1}_{\{x\}}.$$

The function is bounded, Borel and upper semicontinuous. Its pullback along every analytic map from every K-analytic curve to U is identically zero. Nevertheless f is not continuous and fails the Shilov bound on the unit bidisc. Thus replacing all complete extensions by same-field curve tests in Theorems 7.2 and 9.4 is false, even with these extra boundedness and semicontinuity hypotheses.

Status: `FALSE` for the weakened testing assertion; `DIRECT_COROLLARY` for this explicit construction from the dimension invariant of analytic points.

Proof. Write the affine coordinates as T_1, T_2. The norm defining x is

$$\left|\sum_{i,j} c_{ij} T_1^i T_2^j\right|(x) = \max_{i,j} |c_{ij}| r^{i+j}.$$

The residue field of $\mathscr{H}(x)$ contains the two algebraically independent residues of T_1/π and T_2/π. In fact it is $\widetilde{K}(\overline{T_1/\pi}, \overline{T_2/\pi})$, and the value group has no new rational rank. Hence $d_K(x) = 2$.

Every point c of a K-analytic curve has $d_K(c) \leq 1$; this is the pointwise dimension bound [Duc18, §1.4.6]. An analytic map induces an isometric embedding of completed residue fields from its image point to its source point. Such an embedding injects residue fields and value groups, so it cannot increase d_K at the image. Consequently no point of a K-analytic curve can map to x. Every indicated pullback of f is therefore the zero harmonic function. This argument includes analytic maps whose images are not contained in an algebraic curve.

The singleton $\{x\}$ is closed in the Hausdorff space U, so its indicator is upper semicontinuous. As $\rho \uparrow r$ with $\rho < r$, the Gauss points of polyradius (ρ, r) converge to x and are distinct from it. This proves noncontinuity. The unit bidisc has the single Shilov point $\zeta_{(1,1)}$, different from x, while x lies in its ordinary interior. Therefore

$$f(x) = 1 > 0 = \max_{\Gamma(\mathbb{B}^2_K)} |f|,$$

which is the asserted failure of the Shilov bound. □

An explicit detecting extension and curve. The all-extension hypothesis detects this example directly. Let $L = \mathscr{H}(x)$ and let $b \in L$ be the value of T_2 at x. On the smooth algebraic L-line consider

$$h(s) = (s, b) : \mathbf{A}^1_L \longrightarrow \mathbf{A}^2_L.$$

For its radius-r Gauss point $\zeta_{0,r}$ and every polynomial $P = \sum_{i,j} c_{ij} T_1^i T_2^j \in K[T_1, T_2]$,

$$\begin{aligned} |P(h(\zeta_{0,r}))| &= \max_i \left| \sum_j c_{ij} b^j \right| r^i \\ &= \max_{i,j} |c_{ij}| r^{i+j} = |P(x)|. \end{aligned}$$

Thus its image in the original U is exactly x. For $0 < \rho < r$, the image of $\zeta_{0,\rho}$ has $|T_1| = \rho$ and hence is not x. The actual pullback $f_L \circ h$ takes value one at $\zeta_{0,r}$ and zero along these approaching points. It is not continuous, and therefore not harmonic, on the ordinary open disc $|s| < 1$ which contains them all. This is a calculation of actual points, not a replacement of a fiber integration current by a selected section.

The same obstruction remains if one tests over all finite extensions of K: those extensions have $d_K = 0$, and an analytic curve over any one of them still has $d_K \leq 1$ at each point.

The assigned spike also has zero ambient distribution. For every original compact smooth $(2,2)$-form η, its signed integration measure has no atom at x. Indeed, the calibrated integration construction represents its total variation locally by densities with respect to two-dimensional Lebesgue measure on skeletal cells; see [CLD25, §§5.7.1, 5.8.4–5.8.5 and 16.8.1]. A point outside the carrying skeleton has zero mass, and a point on it is Lebesgue-null on each local cell, including their lower-dimensional faces. Finite compact localization therefore gives $|\mu_\eta|(\{x\}) = 0$. Consequently

$$[f] = 0, \qquad D[f] = 0.$$

This argument concerns the measures defined by original smooth top forms. It does not assert that arbitrary positive currents are non-atomic, or that $\{x\}$ is pluripolar or has zero Bedford–Taylor capacity.

The example concerns the testing hypotheses for arbitrary actual functions. It is not a counterexample to the quotient theorem for actual bounded classical differences, whose proof establishes the requisite complete extension functoriality before applying the curve criterion.

20 The weak-positive current cone is not pointed

We use the terminology of CLD exactly. A symmetric current is *weakly positive* when it is nonnegative on strongly positive compact tests; it is *positive* when it is nonnegative on positive compact tests; it is *strongly positive* when it is nonnegative on weakly positive compact tests. These are [CLD12, Definition 5.4.1] and [CLD25, Definition 7.2.1]. R, Convention 2.2, chooses the middle, original positive cone. The distinction is essential to the two-sign conclusion used in the quotient theorem.

Example 20.1 (A nonzero smooth closed current in the weak lineality space)**.** On $X = (\mathbf{G}^4_{m,K})^{\mathrm{an}}$ there is a nonzero symmetric original smooth $(2,2)$-form γ such that its integration current $S = [\gamma]$ is separately d'- and d''-closed and both S and $-S$ are weakly positive on *all* original tests. Neither is positive in R's cone. In particular

$$S \text{ and } -S \text{ weakly positive} \quad \Longrightarrow \quad S = 0.$$

Status: `EXTERNAL_KNOWN` for the linear-algebra phenomenon, explicitly recorded in [CLD25, Proposition 1.2.13 and the following paragraph]; `DIRECT_COROLLARY` for the following signed calculation and original-torus realization. No new positivity convention is introduced.

Proof. Write $t_i = \log|T_i|$, let $e_i = d't_i$, and for $i < j$ put $e_{ij} = e_i \wedge e_j$. The CLD involution sends e_i to $d''t_i$. The positive $(4,4)$-volume is

$$\Omega = e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge Je_1 \wedge Je_2 \wedge Je_3 \wedge Je_4.$$

Indeed the sign $\sigma_4 = (-1)^{4\cdot 3/2}$ equals one. For a two-form $a = \sum_{i<j} z_{ij} e_{ij}$, the positive square is $\theta_a = -a \wedge Ja$, since $\sigma_2 = -1$. It is strongly positive when a is decomposable. Define

$$\begin{aligned}\gamma = -\tfrac{1}{2}\big(&e_{12} \wedge Je_{34} + e_{34} \wedge Je_{12} - e_{13} \wedge Je_{24} - e_{24} \wedge Je_{13} \\ &+ e_{14} \wedge Je_{23} + e_{23} \wedge Je_{14}\big).\end{aligned}$$

This form is symmetric. Direct exterior multiplication gives

$$\gamma \wedge \theta_a = (z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{23})\,\Omega. \tag{20.1}$$

For example, the complementary pairings $e_{12} \wedge e_{34}$, $e_{13} \wedge e_{24}$, $e_{14} \wedge e_{23}$ have signs $+,-,+$ against e_{1234}. Moving a two-form past a two-form has sign $+1$, so the two minus signs in γ and θ_a give exactly the displayed coefficient. Equivalently, the complement matrix C on the ordered basis $(12,13,14,23,24,34)$ interchanges the pairs $(12,34),(13,24),(14,23)$ with signs $+,-,+$; it satisfies $C^2 = 1$, and the coefficient is $\frac{1}{2}z^{\mathsf{T}}Cz$.

If a is decomposable then $a \wedge a = 0$, while in general

$$a \wedge a = 2(z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{23})e_{1234}.$$

Thus γ and $-\gamma$ pair to zero with every strongly positive $(2,2)$-form on $\mathbf{R}^4$. They are both weakly positive there by the exact definition [CLD12, Definition 5.1.1(b)], equivalently [CLD25, Definition 1.2.3(a)].

Pull this constant form to the analytic torus by its original moment coordinates. Weak positivity is preserved by pullback to every common refinement. Its wedge with a strongly positive test is weakly positive; the same holds with its negative. On every top-dimensional calibrated cell of the four-dimensional analytic space, the resulting top form is therefore both nonnegative and nonpositive. Its integral is zero. A finite localization of any original compact strongly positive test proves

$$\langle S, \eta \rangle = 0.$$

This verifies both weak positivity assertions for the entire original test sheaf, not only for tests expressed in the four displayed coordinates. The pullback and wedge facts used here are [CLD25, Lemma 1.2.11 and Proposition 1.2.12].

To see nonvanishing, take $a_+ = e_{12} + e_{34}$ and $a_- = e_{12} - e_{34}$. These are nondecomposable, and their squares θ_{a_+}, θ_{a_-} are positive in the middle cone. Formula (20.1) gives respectively Ω and $-\Omega$. Choose a nonzero nonnegative $\rho \in C_c^\infty(\mathbf{R}^4)$. The function $\rho(t_1, \dots, t_4)$ has compact support on X, since the inverse image of a compact tropical box is a compact polyannulus. The original torus calibration, of degree one, gives

$$\langle S, \rho\theta_{a_+} \rangle = \int_{\mathbf{R}^4} \rho(t)\, dt > 0, \qquad \langle S, \rho\theta_{a_-} \rangle = -\int_{\mathbf{R}^4} \rho(t)\, dt < 0.$$

Hence $S \neq 0$, S is not positive, and $-S$ is not positive. All coefficients of γ are constant in unit-log coordinates, so $d'\gamma = d''\gamma = 0$. Compact Stokes on the boundaryless torus proves separate closedness of S. It is a smooth-form current and therefore of coefficient-Radon type; no general order-zero claim for weakly positive currents is needed. □

Consequence for the quotient argument. The original positive cone used by R is pointed on symmetric currents: positive forms span the full symmetric test space, and two opposite positive currents consequently vanish [CLD25, Remark 7.2.3]. Strongly positive tests do not span that space in the example above. Thus replacing "positive" by "weakly positive" in the last two-sign step would be invalid. The construction is not a counterexample to potential independence for weak backgrounds: it supplies neither a nonzero quotient defect nor any pair of potentials with such a defect. It only disproves that proposed proof step and fixes the positivity scope of the present result.

21 A closed weakly positive tropical current need not have order zero

This section concerns the standard real tropical space $\mathbf{R}^5$, with its ordinary smooth superform test sheaf. No distribution is pushed from a skeleton into a Berkovich space. We use the convention of [CLD25, §1.1.2]: $Jd't_i = d''t_i$, $Jd''t_i = -d't_i$, and a symmetric form is fixed by J. A symmetric current is weakly positive if it is nonnegative on all strongly positive tests, exactly as in [CLD25, Definition 7.2.1].

Example 21.1 (Separate closedness does not repair strong-test control)**.** There is a symmetric current T of bidegree $(3,3)$ on $\mathbf{R}^5$ such that

$$d'T = d''T = 0, \qquad T \neq 0, \qquad T \text{ and } -T \text{ are weakly positive,}$$

but T does not have order zero. In particular it is not a coefficient-Radon current.

Status: `DIRECT_COROLLARY` for the construction from the explicit Plücker form in Example 20.1; `FALSE` for the assertion that nonnegativity on all strongly positive tests implies order zero, even with separate closedness.

Proof. Write $t = (t_0, t')$, where $t' = (t_1, t_2, t_3, t_4)$, and $e_i = d't_i$. Let γ_4 be the constant symmetric $(2,2)$-form from Example 20.1, in the four variables t'. Put

$$\Omega_4 = e_{1234} \wedge Je_{1234}, \qquad \gamma_5 = e_0 \wedge Je_0 \wedge \gamma_4, \qquad \Omega_5 = e_0 \wedge Je_0 \wedge \Omega_4.$$

The last form is the positive standard volume on $\mathbf{R}^5$: moving Je_0 past the four e_i has sign $+1$, and $\sigma_5 = (-1)^{5\cdot 4/2} = 1$. All three displayed forms are symmetric in the stated convention. For $\eta \in A_c^{2,2}(\mathbf{R}^5)$ define the smooth scalar coefficient c_η by

$$\gamma_5 \wedge \eta = c_\eta \Omega_5$$

and set

$$\langle T, \eta \rangle = -\int_{\mathbf{R}^4} \frac{\partial c_\eta}{\partial t_0}(0, t')\, dt'. \tag{21.1}$$

This is a current of order at most one, since each c_η is a fixed linear combination of the coefficients of η and the integral is over a fixed compact set when the test supports are fixed. Algebraically it is the constant form γ_5 with coefficient distribution $\delta_0'(t_0) \otimes dt'$.

Because $J\gamma_5 = \gamma_5$ and $J\Omega_5 = \Omega_5$, we have $c_{J\eta} = c_\eta$; hence $T(J\eta) = T(\eta)$. Suppose that η is strongly positive. In $\gamma_5 \wedge \eta$, every term of η involving e_0 or Je_0 disappears. Its remaining part is the restriction of η to the four-dimensional slice with t_0 fixed. Restriction preserves strong positivity, and γ_4 annihilates every strongly positive $(2,2)$-form on that slice. Consequently c_η is identically zero, not merely zero at $t_0 = 0$. Formula (21.1) therefore gives $T(\eta) = 0$. Both T and $-T$ are weakly positive.

For separate closedness, differentiation in any t_i with $i > 0$ of $\delta_0' \otimes dt'$ vanishes, by ordinary compactly supported integration by parts. Its only possibly nonzero coefficient derivatives are in t_0. In $d'T$ these acquire the factor e_0, already present in γ_5, and in $d''T$ they acquire Je_0, also already present. Both products are zero. This proves $d'T = d''T = 0$ as distributional supercurrents; the convention-dependent overall differentiation sign does not affect either zero identity.

Finally, take

$$a_+ = e_{12} + e_{34}, \qquad \theta_+ = -a_+ \wedge Ja_+.$$

The signed calculation in (20.1) gives $\gamma_4 \wedge \theta_+ = \Omega_4$, and therefore $\gamma_5 \wedge \theta_+ = \Omega_5$. Choose $\chi \in C_c^\infty(\mathbf{R})$ identically one near zero and nonzero $\rho \in C_c^\infty(\mathbf{R}^4)$ with $\rho \geq 0$. For positive integers m put

$$\eta_m = m^{-1} \sin(m^2 t_0) \chi(t_0) \rho(t') \theta_+.$$

These tests have a common compact support, and all their coefficients converge uniformly to zero. Nevertheless,

$$\langle T, \eta_m \rangle = -m \int_{\mathbf{R}^4} \rho(t')\, dt'.$$

Thus $T \neq 0$ and is not continuous for the coefficient C^0 topology on a fixed compact support. It does not have order zero, and its coefficients cannot all be Radon measures. □

Corollary 21.2 (Nonunique decreasing smooth multiplier limits)**.** *For the fixed separately closed weakly positive current T above, the limit of $u_j T$ need not be independent of the smooth convex decreasing approximation to a continuous convex function u, even when the convergence of the functions is uniform. Here all products $u_j T$ are the ordinary products of a smooth function and a distribution.*

Status: `DIRECT_COROLLARY`; `FALSE` for the proposed approximation-independent continuous-multiplier construction under only weak positivity.

Proof. Let $u(t) = \max(0, t_0)$. For $c \geq 0$ and $\varepsilon > 0$ set

$$u_{\varepsilon,c}(t) = \varepsilon \log(1 + \exp(t_0/\varepsilon + c)).$$

These are smooth nonnegative convex functions; their only nonzero Hessian entry is nonnegative in the t_0 direction. Moreover

$$0 \le u_{\varepsilon,c} - u \le \varepsilon(c + \log 2) \quad \text{on all of } \mathbf{R}^5.$$

For fixed t_0, put $x = t_0/\varepsilon$ and $s = \exp(x+c)/(1+\exp(x+c))$. Direct differentiation gives

$$\frac{\partial u_{\varepsilon,c}}{\partial \varepsilon} = -s\log s - (1-s)\log(1-s) + cs \ge 0.$$

Thus every sequence $\varepsilon_j \downarrow 0$ gives smooth convex majorants decreasing uniformly to the same u.

Let T_0 be defined by $\langle T_0, \eta\rangle = \int c_\eta(0,t')\,dt'$. It is the nonzero order-zero current with coefficient $\delta_0 \otimes dt'$ and form factor γ_5. The distributional identity $a\delta_0' = a(0)\delta_0' - a'(0)\delta_0$ gives

$$u_{\varepsilon,c}T = \varepsilon\log(1+e^c)\,T - \frac{e^c}{1+e^c}\,T_0 \longrightarrow -\frac{e^c}{1+e^c}\,T_0.$$

For instance $c = 0$ and $c = \log 3$ give the distinct limits $-T_0/2$ and $-3T_0/4$. This proves the multiplier assertion. Every $u_{\varepsilon,c}T$ is separately closed: the only derivative of the multiplier has the already present factor e_0 or Je_0. In particular

$$D(u_{\varepsilon,c}T) = 0 \quad \text{for every } \varepsilon, c.$$

The example therefore does not give a nonunique BT curvature or a quotient defect. It isolates the failure of a multiplier limit on a non-Radon weakly positive background, solely in the standard real tropical calculus. □

Precise comparison with the printed statements. As written, [CLD25, Proposition 7.2.9, p. 162] assumes symmetry and nonnegativity on strongly positive tests, and concludes order zero on a locally differentially separated tropical space. The standard space $\mathbf{R}^5$ has that separation property, so Example 21.1 contradicts the order-zero conclusion under its printed hypotheses. The displayed domination step in that proof would require an arbitrary symmetric form α to satisfy

$$\alpha + C\|\alpha\|\beta^p \quad \text{strongly positive}$$

for a fixed positive definite $(1,1)$-form β. The relative-interior statement in [CLD25, Proposition 1.2.13, pp. 27–28] does not give this: strongly positive forms span only a proper subspace in the middle degrees used here. The same proposition explicitly records this proper-span phenomenon.

There is also a distinct printed strong-span assertion in [CLD12, Lemma 5.3.3, p. 81]: every smooth (p,p)-form with strongly paracompact support is asserted to be a difference of strongly positive forms. Example 20.1 disproves that assertion even for symmetric compactly supported *original Berkovich* forms. Indeed, on the four-dimensional analytic torus take $\omega = \rho(t)\theta_{a_+}$ as in that example. Its compact support meets the stated support hypothesis, and $\langle[\gamma_4], \omega\rangle > 0$. If $\omega = \omega_1 - \omega_2$ with both ω_i strongly positive, choose an original nonnegative smooth compact cutoff ξ equal to one near $\operatorname{supp}\omega$. Then $\omega = \xi\omega_1 - \xi\omega_2$, whereas $[\gamma_4]$ annihilates both compact strongly positive terms. This is a contradiction. The replacement by the middle positive span is the one stated in [CLD25, Proposition 6.5.2]. This analytic strong-span counterexample uses only the smooth original torus current; it does not transport the derivative distribution constructed above.

The earlier [CLD12, Lemma 5.4.4, p. 82] has a different hypothesis: its linear functional is nonnegative on every *weakly positive* test. Our current does not satisfy that stronger test requirement, so this construction does not refute that lemma. Explicitly, choose the preceding

χ nonnegative with support in $(-1/2, 1/2)$. The test $(1+t_0)\chi(t_0)\rho(t')\theta_+$ is middle positive, hence weakly positive, whereas its pairing with T is $-\int \rho < 0$. In particular one must not silently identify the two printed hypotheses.

For the standard real test space, nonnegativity on the *middle positive* cone is already a sufficient replacement for the order-zero conclusion. Indeed β^p lies in the full interior of that cone. On a fixed compact support, finite-dimensional cone domination bounds a symmetric test above and below by $C\|\eta\|_{C^0}\zeta\beta^p$, with a fixed nonnegative smooth compact cutoff ζ equal to one on the support. Positivity then gives a coefficient C^0 bound; symmetry annihilates the antisymmetric part. This is precisely the distinction relevant to R, Convention 2.2, which chooses the middle cone of [CLD12, Definition 5.1.1(c)]. Neither this example nor its proof changes that internal input.

Scope of the obstruction. The derivative distribution above is defined on an ordinary real tropical space. It has not been transported to a Berkovich skeleton. Such a transport would require control of all original refined tests, whose skeletal restrictions need not be smooth in these fixed coordinates. The original analytic counterexample in Example 20.1 remains a smooth, order-zero closed current. The present stronger non-Radon statement is confined to the real tropical setting, and neither example asserts that the quotient theorem is false for every possible weak-positive background class.

22 The degree-one energy reduction and its precise obstruction

Fix the original finite extension $K/\mathbf{Q}_p$ and its absolute value. All forms, currents and positivity cones in this section are the original CLD objects. Write $D = d'd''$. An allowed space and the class $\mathcal{C}_b$ have their frozen meanings. In particular the functions in $\mathcal{C}_b$ have specified actual values.

We study the implication

$$D[u] = D[v] \quad \Longrightarrow \quad D((u-v)S) = 0 \tag{22.1}$$

for every fixed symmetric original-positive, separately closed background S. The full implication is not proved here.

22.1 An exact reduction to remaining degree one

Proposition 22.1 (Degree-one sufficiency for the arbitrary-background quotient)**.** *Suppose that, on every allowed ordinary open set of pure dimension n, implication (22.1) holds for every original-positive separately closed background of bidegree $(n-1, n-1)$. Then it holds for every such background of bidegree (p,p), $0 \le p < n$, on the same class of spaces and over the same original field.*

Status: `DIRECT_COROLLARY` of the original positive-cone algebra and localization. This reduces the remaining problem; it is not a proof of its degree-one hypothesis.

Proof. Put $q = n - p$ and $H = u - v$. The current $D(HS)$ is symmetric. It therefore suffices to test its symmetric complementary forms. Use a finite original smooth partition subordinate to moment presentations of an arbitrary compact test. On one such presentation let $t_1, \dots, t_m$ be the real moment coordinates. Every symmetric $(q-1, q-1)$-test is a finite sum

$$\sum_{\ell} f_\ell \theta_\ell, \qquad f_\ell \in A_c^{0,0}, \tag{22.2}$$

where each θ_ℓ is a constant positive $(q-1,q-1)$-form in the $d't_i, d''t_i$. Indeed the symmetric coefficient matrices are spanned by positive rank-one matrices: the diagonal squares and the polarization squares of the finite basis $d't_I$ give an explicit finite spanning family. Here a rank-one form is the positive signed square of an arbitrary constant $(q-1,0)$-form. That form need not be decomposable. Thus this argument uses the full original positive cone, not only the strongly positive decomposable squares.

Every θ_ℓ is separately closed. The current $Q_\ell = S \wedge \theta_\ell$, defined on this ordinary moment neighborhood, has bidegree $(n-1,n-1)$ and is separately closed. It is original-positive: wedging a positive $(1,1)$-test with the positive square θ_ℓ gives a positive test for S. This last assertion follows by expanding that $(1,1)$-test as a sum of positive covector squares; each product is the positive square of the exterior product of the two covectors/forms. The assertion survives pullback to a common moment refinement.

The assumed degree-one statement gives $D(HQ_\ell) = 0$. The smooth closed product rule, valid also for the coefficient Radon current HS, gives

$$\langle D(HS), f_\ell \theta_\ell \rangle = \langle D(HQ_\ell), f_\ell \rangle = 0.$$

Sum (22.2) and the finite partition. Antisymmetric tests vanish by symmetry, so every original compact complementary test has zero pairing. □

Proposition 22.2 (The exact form-theoretic meaning of the degree-one gate)**.** *Let S have bidegree $(n-1,n-1)$ on a norm box Ω to which the faithful form construction applies. If $H = u - v$ with $u, v \in \mathcal{C}_b(\Omega)$, then $H \in \mathscr{D}_{S,\mathrm{loc}}$. For every original compact smooth scalar ϕ and every original smooth compact cutoff χ equal to one near* $\operatorname{supp}\phi$*,*

$$\langle D(HS), \phi \rangle = -\mathcal{E}_S(\chi H, \phi). \tag{22.3}$$

In particular, (22.1) *in remaining degree one is precisely the assertion that the given actual difference is locally harmonic for every such faithful closed form.*

Status: `DIRECT_COROLLARY` of Proposition R4–86.1.

Proof. Apply that proposition to u and v and subtract. Its strong local form convergence and Green identity prove (22.3). The cutoff disappears from the integral because it equals one near the support of the test and its derivatives. Original compact scalar tests are exactly the tests for a top-degree current, proving the last assertion. Neither H nor $-H$ is declared psh in this argument. □

The ambient equality $D[H] = 0$ only tests H against $D\eta$ for original smooth complementary forms η. The right side of (22.3) instead pairs it with the coefficient measure of $D(\phi S)$. That measure-valued test has not been replaced by an original smooth test. The faithful measure's domination of fixed smooth coefficients does not make this replacement: in particular it does not assert $D(uS) \ll \lambda_S$.

22.2 A failure of the unrestricted energy-completion route

Example 22.3 (Uniform bounded psh decompositions and universal strong energy do not suffice)**.** Choose a uniformizer π of K, put $R = |\pi|^{-1}$ and $\ell = \log R > 0$, and work on the ordinary open bidisc

$$\Omega = \{|z| < R,\ |t| < R\} \subset (\mathbf{A}^2_K)^{\mathrm{an}}.$$

Let $Y = \{z = 0\} \cap \Omega$ and set

$$q(t) = \ell^{-1} \max\{0, \log|t|\}, \qquad b_N(z) = \max\{0, 1 + N^{-1}\log(|z|/R)\}.$$

The values at zeros are given by the constant branches. Thus $0 \le q < 1$, $0 \le b_N < 1$, and b_N increases pointwise to $\mathbf{1}_{Y^c}$. Define

$$u_N = q + \tfrac{1}{2}(q^2 + b_N^2), \qquad v_N = \tfrac{1}{2}(q + b_N)^2,$$
$$f_N = u_N - v_N = q(1 - b_N), \qquad f = \mathbf{1}_Y q.$$

Then the following assertions hold.

1. The two rows u_N, v_N consist of actual continuous classical psh functions on the whole bidisc, increase pointwise, and lie between zero and two. The functions f_N are continuous bounded differences with $0 \le f_N \le 1$ and $f_N \downarrow f$.
2. On all original compact tests, $[f_N] \to 0$ and $D[f_N] \to 0$ as ambient currents.
3. For the fixed background $S_0 = [Y]$,
$$D(f_N S_0) = D(qS_0) = \ell^{-1}\delta_{(0,\zeta_{0,1})} \neq 0 \qquad \text{for every } N.$$
Here $\zeta_{0,1}$ is the unit Gauss point of the t-disc, and the measure has the original K calibration.
4. For every fixed original-positive separately closed background S of bidegree $(1,1)$ and every original smooth compact cutoff χ, the vectors χf_N converge strongly in the faithful closed form of S to the vector represented by $\chi \mathbf{1}_Y q$. If $\mathbf{1}_Y S = 0$, this limit is zero. For $S = S_0$ the relative local energy need not tend to zero.
5. The limit f is not a difference of two actual locally bounded classical psh functions.

Status: `DIRECT_COROLLARY` for this sharpened scope test, from the frozen model formula, actual faithfulness, and complete-pole energy splitting. It proves `FALSE` an unrestricted passage from weak ambient curvature vanishing to arbitrary-background curvature vanishing, even with the displayed uniform decompositions and strong energy convergence. It does not refute (22.1) for a fixed actual difference.

Proof. The functions q and b_N are continuous maxima of constants and positive scalar multiples of logarithms of analytic functions. Their curve pullbacks are subharmonic over every complete extension, so they are classical. No rationality of the real factor ℓ^{-1} or of the additive constants is required. The functions $s \mapsto s^2$ on $[0, \infty)$ are convex and nondecreasing. Their compositions with the displayed nonnegative classical potentials are classical. This proves all psh, continuity and monotonicity assertions. The numerical bounds follow from $0 \le q, b_N \le 1$, and the stated identity for f_N is an algebraic expansion.

The proper analytic locus Y has zero mass for every original smooth top-degree coefficient measure on the bidisc. Consequently $[f] = 0$. Fixed-coefficient dominated convergence, using the common bound of f_N, proves $[f_N] \to 0$. Distributional differentiation proves $D[f_N] \to 0$; equivalently apply the same bounded convergence to the fixed top form $D\eta$ for each original compact complementary test η.

At every actual point of Y one has $b_N = 0$, so $f_N S_0 = qS_0$ as coefficient-Radon currents for every N. The one-variable original model formula $D\max(0, \log|t|) = \delta_{\zeta_{0,1}}$, followed by the closed immersion, proves assertion 3.

For assertion 4 use the complete pole witness $\log(|z|/R)$ and the splitting of Theorem R4–86.2. In particular, for every fixed compact nonnegative smooth cutoff ρ,

$$\int \rho\, \Gamma_S(b_N) \longrightarrow 0. \tag{22.4}$$

Put $g = \chi q$, which is a bounded compactly supported form vector by Proposition R4–86.1 and the product rule. The measure-valued product square inequality gives, on a compact

neighborhood of its support,

$$\mathcal{E}_S(g(b_N - b_M)) \le 2\int (b_N - b_M)^2 \Gamma_S(g) + 2\|g\|_\infty^2 \int \rho\, \Gamma_S(b_N - b_M).$$

In the second term choose $\rho = 1$ on that neighborhood and use $\Gamma_S(b_N - b_M) \le 2\Gamma_S(b_N) + 2\Gamma_S(b_M)$. Equation (22.4) makes it tend to zero. The first term tends to zero by bounded convergence against the fixed finite measure $\Gamma_S(g)$. The $L^2(\lambda_S)$ limit is $g\mathbf{1}_{Y^c}$, so closedness identifies the strong form limit. Subtract from g to obtain $\chi f_N \to \chi \mathbf{1}_Y q$. This proof applies separately to every fixed S, with its own faithful form and controls.

If $\mathbf{1}_Y S = 0$, faithfulness of λ_S makes the limit vector zero. In contrast $f_N = q$ on Y, so the local energy for $[Y]$ is constant in N. A nonzero cutoff supported on a radial segment $1 < |t| < R$ has positive energy pairing with $\Gamma_{[Y]}(q)$; the slope there is $1/\ell$. Thus this relative energy does not vanish.

Finally, if $f = a - b$ for $a, b \in \mathcal{C}_b(\Omega)$, the ambient identity $[f] = 0$ would imply $[a] = [b]$. Frozen actual current-faithfulness would give $a = b$ everywhere, contrary to $f = q > 0$ on the part of Y with $|t| > 1$. This proves assertion 5. □

The excluded estimate. The example rules out any general estimate whose hypotheses are only one uniform bound on the two classical decompositions, weak vanishing of their ambient curvature difference, and strong convergence in all fixed relative energy spaces, and whose conclusion is vanishing of the relative curvature differences. It also rules out obtaining the desired quotient by simply replacing actual classical differences with the full linear local energy domains. The function $\mathbf{1}_Y q$ is an energy vector for every such background and annihilates all ambient smooth current tests, but it does not annihilate $D(\phi[Y])$. Therefore an approximation of those latter functionals in the dual of the unrestricted linear energy space cannot hold in general. Any successful energy argument must preserve an additional property of the actual classical differences; normal trace or a genuine regularity conclusion cannot be discarded in completion.

Remark 22.4 (Bounded upper semicontinuity does not repair the scope). **Status:** `DIRECT_COROLLARY.` The limit $f = \mathbf{1}_Y q$ in the example is bounded and upper semicontinuous. Indeed, outside the closed set Y it is locally zero. At a point of Y, the inequalities $0 \le f \le q$ and continuity of q give $\limsup_{x\to y} f(x) \le q(y) = f(y)$. Nevertheless

$$[f] = 0, \qquad D[f] = 0, \qquad D(f[Y]) = \ell^{-1}\delta_{(0,\zeta_{0,1})} \neq 0.$$

Thus the actual bounded-classical-difference hypothesis in the kernel theorem cannot be replaced even by arbitrary bounded upper semicontinuous functions with zero original curvature. In particular, merely calling such a representative original CLD-positive does not place it in $\mathcal{C}_b$. This is not a counterexample to the quotient for actual $\mathcal{C}_b - \mathcal{C}_b$ differences. Nor does it disprove existence of a continuous representative for a zero distribution: the distribution $[f] = 0$ here already has the continuous representative zero. The failure is precisely equality with the chosen upper semicontinuous values on the analytic locus.

Remark 22.5 (A quantitative fixed-test ambient error). **Status:** `DIRECT_COROLLARY.` For every fixed original compact smooth top form η on Ω, there is a constant $C_\eta < \infty$ such that

$$|\langle [f_N], \eta\rangle| \le C_\eta / N.$$

Indeed the fixed original top-form integration is carried by a compact set $K_\eta \subset \Omega \setminus Y$. This is the finite moment-presentation support property used in the proof of [CLD12, Lemma 4.6.1,

pp. 72–73]: the calibrated top-dimensional supports avoid every proper analytic locus. Let μ_η be its signed coefficient measure and set

$$A_\eta = \sup_{K_\eta} \log(R/|z|) < \infty.$$

The pointwise formula

$$0 \le f_N = q \min\{1, N^{-1} \log(R/|z|)\} \le A_\eta/N \qquad \text{on } K_\eta$$

proves the claim with $C_\eta = A_\eta |\mu_\eta|(K_\eta)$. For every fixed original compact $(1,1)$-test α, apply the same statement to $D\alpha$ to obtain

$$|\langle D[f_N], \alpha\rangle| \le C_{D\alpha}/N.$$

These constants depend on the fixed tests. They are not uniform over an unbounded class of tests or over tests approaching Y. The nonzero relative defect in the example is unchanged.

Remark 22.6 (Failure on relative capacitary completions). **Status: `DIRECT_COROLLARY`; the proposed completed-class extension of (Q) is `FALSE`.** The two increasing classical rows in the example have actual limits

$$a = q + \tfrac{1}{2}(q^2 + \mathbf{1}_{Y^c}), \qquad b = \tfrac{1}{2}(q + \mathbf{1}_{Y^c})^2.$$

They are uniformly bounded by two on the entire bidisc. Thus Corollary R4–18.5 places both limits in $\mathcal{A}_{S,b}(\Omega)$ for every permitted fixed S of positive remaining degree. The convergence is the all-domain local outer-capacity convergence of that corollary; no ambient upper regularization is substituted for the displayed limits.

In particular take $S_0 = [Y]$. The completed first-order recursion of Theorem R4–17.6 is still $\mathcal{T}_{S_0,1}(a) = D(aS_0)$, with the actual Borel multiplier, and similarly for b. Since $a - b = \mathbf{1}_Y q$, one has

$$[a] = [b], \qquad D[a] = D[b], \qquad \mathcal{T}_{S_0,1}(a) - \mathcal{T}_{S_0,1}(b) = \ell^{-1}\delta_{(0,\zeta_{0,1})} \neq 0.$$

Thus even a common global defining bound and increasing classical defining rows do not extend the ambient-curvature quotient to these relative completed classes. This does not contradict the completed calculus's invariance under equality outside an S_0-capacity-zero set: the two displayed functions are not equal in that relative sense. It also leaves the quotient for actual $\mathcal{C}_b$ potentials untouched.

23 The universal relative Borel kernel is larger than the classical kernel

The quotient theorem concerns actual locally bounded differences of classical psh functions. Even vanishing against every allowed positive closed background does not impose the same regularity on arbitrary Borel functions.

Let U be an allowed pure n-dimensional ordinary open, with $n \ge 1$. Define

$$\mathscr{K}_{\mathrm{Bor}}(U) = \left\{ H : U \to \mathbf{R} : \begin{array}{l} H \text{ is locally bounded and Borel, and } D(HS) = 0 \\ \text{for every symmetric original-positive, separately} \\ d'\text{- and } d''\text{-closed } (p,p)\text{-current } S, \quad 0 \le p < n \end{array} \right\}.$$

The product HS means actual Borel multiplication of the coefficient-Radon current. The asserted distributional equality is on all original compact smooth complementary forms. No positive sign is required of H or of the product HS.

Proposition 23.1 (Finite complete-pole selectors)**.** *Let $P_1, \dots, P_m$ be pairwise disjoint Borel subsets of U. Assume each P_a is locally the exact pole set of an extended classical psh function. Put $P_0 = U \setminus \bigcup_{a=1}^m P_a$. If $h_0, \dots, h_m \in \mathscr{PH}(U)$, then*

$$H = \sum_{a=0}^{m} \mathbf{1}_{P_a} h_a \quad \text{belongs to } \mathscr{K}_{\mathrm{Bor}}(U).$$

In particular both $\mathbf{1}_P$ and $\mathbf{1}_{P^c}$ belong to this kernel for every locally complete classical pole set P.

Status: `DIRECT_COROLLARY` of the already proved complete-pole restriction theorem and first-order positivity over a fixed background.

Proof. Fix one of the currents S in the definition. The R2 restriction theorem, Theorem R4–24.3, applied locally to the exact pole witness and then glued, makes

$$S_a = \mathbf{1}_{P_a} S \qquad (1 \le a \le m)$$

symmetric, original-positive and separately closed. The complement

$$S_0 = S - \sum_{a=1}^{m} S_a = \mathbf{1}_{P_0} S$$

is separately closed by finite linearity and is positive because the sets are disjoint. This argument is a restriction of a current already defined on U. It does not assert an extension theorem for a current initially defined off a pole set.

For each a, both h_a and $-h_a$ are continuous locally bounded classical psh functions. First-order positivity over the fixed positive closed background S_a therefore makes both $D(h_a S_a)$ and its negative positive. The original positive cone is pointed, so $D(h_a S_a) = 0$. Actual Borel multiplication and finite linearity give

$$D(HS) = \sum_{a=0}^{m} D(h_a S_a) = 0.$$

The finite sum is locally bounded and Borel. This proves the claim for every S and every original test. □

Corollary 23.2 (Strict enlargement and its precise boundary)**.** *On a bidisc with coordinates (z,t) in $(\mathbf{A}^2_K)^{\mathrm{an}}$, with $Y = \{z = 0\}$, the function $\mathbf{1}_Y$ is bounded and upper semicontinuous, and*

$$[\mathbf{1}_Y] = 0, \qquad D(\mathbf{1}_Y S) = 0 \quad \text{for every allowed } S.$$

It is not continuous and hence is not pluriharmonic. Thus $\mathscr{PH}(U) \subset \mathscr{K}_{\mathrm{Bor}}(U)$ can be a strict inclusion. On every allowed U, however,

$$\mathscr{K}_{\mathrm{Bor}}(U) \cap \mathscr{DSH}_{\mathrm{lb}}(U) = \mathscr{PH}(U).$$

Proof. The divisor Y is the exact pole set of $\log|z|$, so the proposition applies. The indicator of a closed set is upper semicontinuous. It is discontinuous, for example at the rigid origin, which is approached by the points $(\pi^j, 0)$ off Y. Original smooth top forms have support disjoint from the lower-dimensional analytic locus Y [CLD25, Proposition 15.2.1]; consequently $[\mathbf{1}_Y] = 0$. Its noncontinuity and Theorem 10.1 therefore exclude $\mathbf{1}_Y$ from $\mathscr{DSH}_{\mathrm{lb}}$.

Every pluriharmonic function lies in the universal kernel by the two signs of first-order positivity. Conversely, if $H \in \mathscr{K}_{\mathrm{Bor}}(U) \cap \mathscr{DSH}_{\mathrm{lb}}(U)$, the permitted choice $S = [1]$ gives $D[H] = 0$. Theorem 10.1 then makes H pluriharmonic, proving the equality. □

A necessary distinction. Being carried by a complete pole set does not by itself imply membership in the universal kernel. With the notation of Example 22.3, $f = \mathbf{1}_Y q$ still satisfies $[f] = 0$, but

$$D(f[Y]) = D(q[Y]) = \ell^{-1}\delta_{(0,\zeta_{0,1})} \neq 0.$$

The finite selector proposition requires the displayed ambient pluriharmonic weights; it does not replace them by arbitrary bounded Borel or classical psh weights. Nor does it assert a classification of the whole Borel kernel, countable selector closure, or invariance under arbitrary changes on ambient null sets. In particular the discontinuous selector does not contradict existence of a continuous representative of the zero distribution: its ambient current already has the representative zero. It shows that universal relative curvature does not recover that continuous representative from arbitrary assigned actual Borel values.

24 The descended operation is not weakly continuous

Potential independence is a statement about exact equality of curvature currents. It gives no continuity assertion for the operation when those currents vary weakly. The following example keeps the background fixed and makes both the approximating curvatures and their common ambient limit original smooth.

Example 24.1 (Smooth weak convergence with a fixed relative defect)**.** Let $K/\mathbf{Q}_p$ and its absolute value be the prescribed ones. Fix a uniformizer π, put $R = |\pi|^{-1}$ and $\ell = \log R$, and let

$$\Omega = \{|z| < R,\ |t| < R\} \subset (\mathbf{A}_K^2)^{\mathrm{an}}, \qquad Y = \{z = 0\} \cap \Omega, \qquad S = [Y].$$

There exist original smooth classical psh functions v_N, w on Ω, all between zero and two, such that

$$D[v_N] \longrightarrow D[w]$$

on every original compact test, whereas

$$D(v_N S) \not\longrightarrow D(wS).$$

More precisely, $D(v_N S)$ is independent of N, and $D(wS) - D(v_N S)$ is a fixed nonzero positive current. The ambient error is $O(N^{-1})$ on each fixed original test, with a test-dependent constant.

Status: `FALSE` for weak-current continuity of the descended operation; the construction and proof are `DIRECT_COROLLARY` of smooth chain rules, fixed-coefficient dominated convergence, and the original top-form support property. This is not a counterexample to exact potential independence.

Proof. Choose a nonnegative even $\kappa \in C_c^\infty((-1,1))$ with integral one and not identically zero. Put

$$\psi(s) = \int_{-\infty}^{s} (s-a)\kappa(a)\, da.$$

Then ψ is smooth and convex, $0 \le \psi' \le 1$, $\psi'' = \kappa$, it is zero on $(-\infty, -1]$, and it equals s on $[1, \infty)$. Its derivative bounds give

$$0 \le 1 - \psi(1 - A/N) \le A/N \qquad (A \ge 0). \tag{24.1}$$

Define

$$q(t) = \psi\Big(\frac{\log|t|}{\ell}\Big), \qquad b_N(z) = \psi\Big(1 + \frac{\log(|z|/R)}{N}\Big),$$

with value zero at the corresponding coordinate zero. These are original smooth functions: the profiles are identically zero on an ordinary neighborhood of each zero locus, and elsewhere they are smooth functions of unit logarithms. They are classical psh after every complete extension, by the curve chain rule for convex nondecreasing profiles of log norms. On the open bidisc,

$$0 \le q, b_N \le 1, \qquad b_N \uparrow \mathbf{1}_{Y^c}.$$

Now set

$$v_N = \tfrac{1}{2}(q + b_N)^2, \qquad w = \tfrac{1}{2}(q+1)^2.$$

The square is convex and nondecreasing on the nonnegative range. Thus v_N, w are original smooth classical psh and are bounded by two. Their original curvatures are positive smooth forms, by the smooth chain rule. The row v_N increases, but its actual limit on Y is $q^2/2$, not $w|_Y$. No decreasing-row convergence theorem applies to this non-upper-semicontinuous actual limit.

Every fixed original compact smooth top form η has compact actual support disjoint from Y, by [CLD25, Proposition 15.2.1]. On that support the number

$$A_\eta = \sup \log(R/|z|)$$

is finite. The displayed formulas and (24.1) give

$$\begin{aligned} 0 \le w - v_N &= \tfrac{1}{2}(1 - b_N)(2q + 1 + b_N) \\ &\le 2(1 - b_N) \le 2A_\eta/N. \end{aligned}$$

Integration against the fixed signed measure of η proves

$$|\langle [v_N] - [w], \eta \rangle| \le 2A_\eta |\mu_\eta|(\Omega)/N.$$

For a fixed original compact $(1,1)$-test α, apply this estimate to $D\alpha$. This proves the stated ambient curvature convergence and its fixed-test rate.

At every actual point of Y, $b_N = 0$. Consequently

$$D(v_N S) = D\Big(\tfrac{1}{2} q^2 [Y]\Big) \quad \text{for every } N,$$

while separate closedness of $[Y]$ gives

$$D(wS) = D\Big(\tfrac{1}{2} q^2 [Y]\Big) + D(q[Y]).$$

The last summand is positive and nonzero. Indeed, on the t-annulus in Y, with $s = \log|t|/\ell$,

$$Dq = \kappa(s) d's \wedge d''s.$$

For any nonnegative smooth compact scalar profile ρ with $\int \rho(s)\kappa(s)\, ds > 0$, the original annulus calibration gives

$$\int_Y \rho(s) Dq = \ell^{-1} \int \rho(s)\kappa(s)\, ds > 0.$$

Multiply the scalar by an original compact cutoff in the z-direction equal to one near $z = 0$ to obtain a genuine compact test on Ω. Hence $D(q[Y]) \neq 0$, proving the persistent relative defect. □

Consequences for the current-valued formulation. Let $\alpha_N = D[v_N]$ and $\alpha = D[w]$. By Corollary 10.5, $\alpha_N \wedge [Y]$ and $\alpha \wedge [Y]$ are independent of their bounded classical potentials. The example therefore proves an intrinsic failure of weak continuity on this class of positive currents, not an ambiguity caused by the choice of potentials. In fact all the displayed currents already come from smooth positive forms. The failure occurs when a fixed singular background detects the unchanged trace on Y.

The example extends to every ambient dimension $n \geq 2$ by taking a product with an open polydisc of dimension $n-2$ and pulling back the displayed functions. The same hyperplane integration current remains the background; a compact positive top form in the extra directions detects the unchanged defect. No claim about arbitrary varying backgrounds is needed.

25 A finite-curvature sufficient condition for compact no-gain

This section isolates one exact consequence of the new potential quotient for the unresolved compact-capacity gate C1. It does not claim that an arbitrary normalized family satisfies the additional finite-dimensional hypothesis, or that approximation by such families suffices.

Fix the original finite extension $K/\mathbf{Q}_p$, an allowed ordinary open Ω, and a symmetric original-positive separately closed (p,p)-current S. Put $q = n - p \geq 1$, and write

$$\mathcal{P}(\Omega) = \{u \in \mathcal{C}_b(\Omega) : 0 \leq u \leq 1 \text{ everywhere on } \Omega\}.$$

The diagonal raw capacity is

$$C_S(E;\Omega) = \sup_{u \in \mathcal{P}(\Omega)} T_{S,q}(u,\dots,u)(E) \qquad (E \text{ Borel}).$$

The capacity domain and this normalization are fixed throughout.

Proposition 25.1 (Exact finite-dimensional curvature images)**.** *Let $W \subset \Omega$ be an ordinary open, and let $\mathcal{F} \subset \mathcal{P}(\Omega)^q$ be a family of tuples. Suppose that for each $i = 1,\dots,q$ there is a fixed finite-dimensional real vector space E_i of original $(1,1)$-currents on W such that*

$$D[u_i]|_W \in E_i \qquad ((u_1,\dots,u_q) \in \mathcal{F}).$$

These are exact equalities in the ambient original current space on the whole same open W.

Then the restrictions to W of

$$\mu_{\mathbf{u}} = T_{S,q}(u_1,\dots,u_q) \qquad (\mathbf{u} \in \mathcal{F})$$

lie in a fixed finite-dimensional vector space of signed Radon measures, of dimension at most $\prod_i \dim E_i$. If all slots come from one space E of dimension m, the bound can be replaced by $\binom{m+q-1}{q}$, with the zero-space case interpreted as the zero image.

For every net $(\mathbf{u}_a)$ in this family, local weak convergence $\mu_{\mathbf{u}_a}|_W \to \nu$ on all original compact smooth scalar tests implies

$$|\mu_{\mathbf{u}_a} - \nu|(L) \longrightarrow 0 \qquad (L \Subset W \text{ compact}). \tag{25.1}$$

No finite total mass on W is required.

Status: `DIRECT_COROLLARY` of the arbitrary-background quotient, signed potential-current polarization, and finite-dimensional linear algebra.

Proof. Replace E_i by the span of the curvatures actually occurring in its ith slot. Choose a basis

$$\alpha_{i,l} = D[b_{i,l}]|_W, \qquad 1 \le l \le m_i,$$

from those occurring curvatures, retaining their original potentials $b_{i,l} \in \mathcal{P}(\Omega)$. Write $D[u_i]|_W = \sum_l c_{i,l}\alpha_{i,l}$. The current-valued operation and its real multilinear signed extension from Section 11 give, on W,

$$\mu_{\mathbf{u}} = \sum_{l_1,\dots,l_q} \left(\prod_{i=1}^{q} c_{i,l_i} \right) T_{S,q}(b_{1,l_1}, \dots, b_{q,l_q}). \tag{25.2}$$

All the currents on the right are fixed positive Radon measures on W. To make the dependence on the quotient explicit, a linear relation among potentials with zero ambient curvature is a locally bounded classical difference; the quotient theorem makes its insertion against each fixed positive expanded tail vanish. Finite polarization therefore respects precisely the displayed ambient current relation. There is no assumption that a signed linear combination of the potentials is itself psh.

The expansion proves the product bound on dimension. If all slots use one E, choose one basis from the union of the occurring curvatures. Symmetry identifies the products whose indices differ only by permutation; there are $\binom{m+q-1}{q}$ possible multisets of q indices. If one occurring curvature space is zero, that slot has the same curvature as the zero potential and the product vanishes by the quotient theorem.

Let M be the resulting finite-dimensional measure space, and choose a basis $\tau_1, \dots, \tau_d$ of M. Original smooth compact scalar tests separate these signed Radon measures. Consequently there exist $\phi_1, \dots, \phi_d \in A_c^{0,0}(W)$ such that the matrix

$$A_{st} = \langle \tau_t, \phi_s \rangle$$

is invertible. Indeed, the common annihilator in M of all original tests is zero, and a finite subcollection of those linear functionals already has zero annihilator. Writing $\mu_{\mathbf{u}_a} = \sum_t x_{a,t}\tau_t$, weak convergence on these particular tests gives

$$(x_{a,1}, \dots, x_{a,d})^{\mathsf{T}} = A^{-1}\big(\langle \mu_{\mathbf{u}_a}, \phi_s \rangle\big)_s \longrightarrow (x_1, \dots, x_d)^{\mathsf{T}}.$$

The measure $\sum_t x_t\tau_t$ has the same original scalar pairings as ν, so it equals ν. For every compact $L \Subset W$,

$$|\mu_{\mathbf{u}_a} - \nu|(L) \le \sum_{t=1}^{d} |x_{a,t} - x_t|\, |\tau_t|(L) \longrightarrow 0.$$

This proves the assertion, including its local-mass scope. □

Corollary 25.2 (Sharp diagonal compact no-gain under the exact hypothesis)**.** *Let $K_0 \Subset W \subset \Omega$, with K_0 compact and W ordinary open. Let $u_a \in \mathcal{P}(\Omega)$, and assume that all $D[u_a]|_W$ lie in one fixed finite-dimensional space of original ambient currents. If $T_{S,q}(u_a, \dots, u_a)|_W \to \nu$ locally weakly, then*

$$\nu(K_0) = \lim_a T_{S,q}(u_a, \dots, u_a)(K_0) \le C_S(K_0; \Omega).$$

In particular, in remaining degree one, a normalized completed potential with a global normalized defining sequence satisfying this exact hypothesis has no completed compact-mass gain on K_0.

Status: `DIRECT_COROLLARY`.

Proof. The proposition gives convergence in total variation on K_0. Every term before passage to the limit is a permitted diagonal tester on the original whole domain Ω; hence its mass is at most $C_S(K_0;\Omega)$. This proves the exact constant one. For a normalized completed defining sequence, the existing completed calculus supplies the required weak convergence. This last invocation does not assert a completed representation for every higher-degree weak limit. □

The same measure-space argument applies to mixed families, but its constant-one capacity bound is then with respect to that mixed family's own raw supremum. It does not assert a constant-one mixed-to-diagonal capacity inequality. For the cutoff version of C1, choose W inside the ordinary neighborhood on which the fixed cutoff equals one. The restricted weak limit is unchanged there, so the corollary applies without changing the capacity domain.

Example 25.3 (Strong approximation still permits compact-mass gain)**.** Let $\lambda = -\log|\pi| > 0$, and on the original ordinary annulus

$$\Omega = \{|\pi| < |T| < |\pi|^{-1}\}$$

take $S = [1]$. Write $t = \log|T|$, let ζ_s be its radius-e^s Gauss point, and put

$$s_j = \frac{\lambda}{j+2}, \qquad u_j = \frac{\max(0, t - s_j)}{\lambda}, \qquad u = \frac{\max(0,t)}{\lambda}.$$

These are actual continuous classical psh functions on the same Ω, with $0 \le u_j, u \le 1$, and

$$u_j \uparrow u, \qquad \|u_j - u\|_{L^\infty(\Omega)} \le s_j/\lambda.$$

The original curve Laplacian and degree-one torus calibration give the equalities of Radon measures, hence of currents on all original tests,

$$D[u_j] = \lambda^{-1}\delta_{\zeta_{s_j}}, \qquad D[u] = \lambda^{-1}\delta_{\zeta_0}.$$

Thus for the fixed compact $K_0 = \{\zeta_0\}$,

$$D[u_j](K_0) = 0, \qquad D[u](K_0) = \lambda^{-1}. \tag{25.3}$$

Even the normalized psh-dual distance tends to zero. Indeed, for every $a \in \mathcal{P}(\Omega)$ the function $s \mapsto a(\zeta_s)$ is convex on $(-\lambda, \lambda)$. This follows by the one-dimensional annulus maximum principle, or by Lemma R4–51.1. Its range lies in $[0,1]$, so its Lipschitz constant on $[-\lambda/2, \lambda/2]$ is at most $2/\lambda$, by the two endpoint secant inequalities. Therefore

$$\sup_{a\in\mathcal{P}(\Omega)} \left| \int a\,(D[u_j] - D[u]) \right| \le 2s_j/\lambda^2 \longrightarrow 0.$$

By contrast the total variation of $D[u_j] - D[u]$ on the fixed compact closed subannulus $\{|\pi|^{1/2} \le |T| \le |\pi|^{-1/2}\}$ is $2/\lambda$ for every j.

There is also local strong convergence for each fixed faithful degree-one energy form. Set $f_j = u_j - u$, and take any original compact smooth χ. The bounded BT Green and product identities, valid in the closed form by Proposition R4–86.1, give

$$\begin{aligned}
\mathcal{E}(\chi f_j, \chi f_j) &= \int f_j^2\,\Gamma_{[1]}(\chi) - \int \chi^2 f_j\, D[f_j] \\
&= \int f_j^2\,\Gamma_{[1]}(\chi) + \chi(\zeta_{s_j})^2 s_j/\lambda^2 \\
&\le \frac{s_j^2}{\lambda^2}\Gamma_{[1]}(\chi)(\Omega) + \frac{s_j}{\lambda^2}\|\chi\|_\infty^2 \longrightarrow 0.
\end{aligned}$$

Here $f_j(\zeta_0) = 0$ and $f_j(\zeta_{s_j}) = -s_j/\lambda$ fix the sign of the atomic term. The accompanying local L^2 convergence follows from the uniform bound and local finiteness of the reference measure. Uniform convergence also implies convergence in every local outer-capacity pair, since each fixed positive-gap set is eventually empty.

Status: `DIRECT_COROLLARY` of the explicit annulus calculation; `FALSE` for an inference from these strong potential or psh-dual convergences to continuity of mass on an arbitrary fixed compact. This is not a counterexample to C1: the limit potential u itself is a permitted tester and realizes the limiting mass. The point masses at the distinct ζ_{s_j} are linearly independent on every fixed neighborhood of ζ_0 after discarding finitely many terms, so the exact finite-dimensional curvature hypothesis does not hold in this example.

The remaining gate. For arbitrary S in remaining degree one, the normalized completed class still supplies only the outer compact value

$$C_{\mathcal{A}}(K_0;\Omega) = C_S^*(K_0;\Omega).$$

The required general assertion is still that for every normalized completed a and every $\varepsilon > 0$ there exists $v \in \mathcal{P}(\Omega)$ with

$$D(vS)(K_0) \geq D(aS)(K_0) - \varepsilon.$$

The proposition proves this for exact finite-curvature defining families. No such hypothesis or approximation theorem is known here for an arbitrary family. The new potential quotient controls exact relations, and finite Shilov evaluations control the exact harmonic kernel. Neither statement turns approximate curvature relations into exact ones. The annulus example shows why uniform, outer-capacity, local form, or psh-dual density alone does not fill that logical gap. No compact or Borel outer regularity for the full arbitrary-background capacity is claimed by this section.

26 Global finite-curvature families retain normalized classical limits

Status: `DIRECT_COROLLARY` of the actual kernel theorem, finite Shilov control, and the existing bounded BT calculus. No countability hypothesis on the domain is imposed. The global finite-dimensional hypothesis is stronger than the neighborhood hypothesis in Proposition 25.1.

Proposition 26.1. *Let Ω be an ordinary open in an allowed original-K space of pure dimension $n \geq 1$, where $K/\mathbf{Q}_p$ is finite with its original absolute value. Let E be a finite-dimensional real vector space of original $(1,1)$-currents on the* whole *Ω, and put*

$$\mathcal{P}_E = \{u \in \mathcal{C}_b(\Omega) : 0 \leq u \leq 1,\ D[u] \in E\}.$$

Then $\mathcal{P}_E$ is compact for local uniform convergence of the actual functions.

For every fixed symmetric original-positive separately closed (p,p)-current S, with $q = n - p \geq 1$, the image

$$\{T_{S,q}(u,\dots,u) : u \in \mathcal{P}_E\}$$

is locally weakly compact and closed. Every one of its weak closure points has the same-domain normalized classical representative displayed in this formula. On this image weak convergence of nets implies convergence in total variation on each compact subset.

The corresponding assertions hold for the mixed image of $\mathcal{P}_{E_1} \times \cdots \times \mathcal{P}_{E_q}$ when each E_i is finite-dimensional on the whole Ω.

Proof. We first record a compactness property of the exact kernel. For $M \geq 0$, the set

$$\mathscr{H}_M = \{H \in \mathscr{PH}(\Omega) : |H(x)| \leq M \text{ for every } x \in \Omega\}$$

is a closed subset of the pointwise product $[-M, M]^\Omega$. Indeed a pointwise limit of a net in this set is finite at every actual point, and Corollary 13.2 makes it pluriharmonic and the convergence locally uniform. Equivalently, the finite-evaluation estimate on every compact stage shows directly that pointwise and local uniform convergence have the same nets on $\mathscr{H}_M$. Tychonoff's theorem therefore makes $\mathscr{H}_M$ compact also for local uniform convergence. This argument uses no countable exhaustion or sequential compactness assertion.

Replace E by the span of the curvatures actually occurring in $\mathcal{P}_E$, and choose a basis $\alpha_l = D[b_l]$, $1 \leq l \leq m$, with $b_l \in \mathcal{P}_E$. For $u \in \mathcal{P}_E$ write

$$D[u] = \sum_l c_l(u)\alpha_l.$$

Finitely many original complementary compact smooth tests extract the coordinates $c_l(u)$, because such tests separate the independent currents α_l. Each coordinate-extracting test η has a fixed compactly supported top form $D\eta$. Its integration measure is a finite signed Radon measure, and $0 \leq u \leq 1$ gives directly

$$|\langle D[u], \eta\rangle| = \left|\int_\Omega u\, D\eta\right| \leq |\mu_{D\eta}|(\Omega).$$

Thus these finitely many pairings are uniformly bounded; no CLN estimate is needed for this particular step. Consequently all coefficient vectors lie in a fixed compact box $B \subset \mathbf{R}^m$.

The actual function

$$H_u = u - \sum_l c_l(u)b_l$$

is locally a difference of locally bounded classical psh functions and has zero ambient curvature on all of Ω. Theorem 10.1 gives $H_u \in \mathscr{PH}(\Omega)$. Because the basis potentials also satisfy $0 \leq b_l \leq 1$ on the whole domain,

$$|H_u| \leq 1 + \sum_l |c_l(u)| \leq M \quad \text{everywhere on } \Omega$$

for one finite M. Thus $\mathcal{P}_E$ is contained in the image of the compact space $B \times \mathscr{H}_M$ under the map

$$(c, H) \longmapsto H + \sum_l c_l b_l.$$

This map is continuous for local uniform convergence, since the finitely many b_l are bounded. Its image is therefore a compact set of actual locally bounded functions.

The subset $\mathcal{P}_E$ is closed in that compact image. To check this for nets, a local uniform limit of its members still has values in $[0, 1]$ and is upper semicontinuous. After every complete extension and analytic curve pullback, the convergence is uniform on each compact curve neighborhood: its image in Ω is compact. The subharmonic class is closed under such uniform limits, so the limit is classical psh. If desired, this last local assertion for a net reduces on each compact neighborhood to a uniformly convergent sequence chosen from the net. Distributional differentiation is continuous for local uniform convergence, and E is a finite-dimensional, hence closed, subspace of the Hausdorff original-current space. The limit curvature therefore remains in E. This proves compactness of $\mathcal{P}_E$. When $m = 0$ the sum is empty and the same argument simply uses the compact kernel ball.

Fix S. On tuples of normalized bounded classical potentials the mixed top operator is continuous for local uniform convergence, including for nets. One may use the established

capacity continuity, or verify the assertion directly by telescoping the slots. Against any fixed original compact smooth scalar test ϕ, each telescoping term has the form

$$\langle (u_i - v_i)T_i, D\phi \rangle,$$

where T_i is a positive mixed tail with the fixed background S. Coefficient-Radon control and relative CLN bound its absolute value by a fixed constant times $\|u_i - v_i\|$ on a fixed compact neighborhood, uniformly over the other normalized slots. This tends to zero under local uniform convergence. The same CLN mass bounds and uniform approximation of compact continuous scalar tests give continuity for local vague convergence of Radon measures.

The diagonal image is therefore a compact subset of that Hausdorff measure space, hence is closed, and every closure point has a representative in $\mathcal{P}_E$. For mixed images apply the same argument to the finite product of the compact spaces $\mathcal{P}_{E_i}$. Finally Proposition 25.1, with $W = \Omega$, gives the asserted local total-variation convergence inside these images. □

This conclusion requires exact finite-dimensional control on the entire capacity domain and retains the original pointwise normalization there. Finite-dimensional curvature span only near one compact core gives the earlier local-TV result, but does not supply this global normalized representative. No finite dimension of the pluriharmonic stalk, compactness of the whole unrestricted normalized family, or finite-dimensional approximation theorem for general C1 is asserted.

27 The classical relative image need not be TV-closed without boundary conditions

The obstruction in this section concerns exact classical representatives of relative curvature measures. No boundary value is prescribed. The limiting measure nevertheless satisfies the desired sharp capacity bound, so this is not a counterexample to compact no-gain C1.

Lemma 27.1 (A necessary carrier-incidence condition)**.** *Let Ω be an allowed original-K space of pure dimension n, and let $\iota_i : \Delta_i \hookrightarrow \Omega$, $1 \le i \le m$, be closed analytic embeddings, where each Δ_i is an ordinary open unit disc. Suppose that the images C_i have finite pairwise intersections, and let Z be the union of these intersection sets. Take $h_i \in \mathcal{C}_b(\Delta_i)$ such that $D[h_i]$ gives no mass to $\iota_i^{-1}(Z)$, and set*

$$S = \sum_i [C_i], \qquad \nu = \sum_i (\iota_i)_* D[h_i].$$

If $u \in \mathcal{C}_b(\Omega)$ satisfies $D(uS) = \nu$, then there are constants $c_i \in \mathbf{R}$ with

$$u \circ \iota_i = h_i + c_i \quad \text{on } \Delta_i.$$

In particular every intersection point $P \in C_i \cap C_l$ imposes the actual-value relation

$$c_l - c_i = h_i(\iota_i^{-1}P) - h_l(\iota_l^{-1}P). \tag{27.1}$$

The sum of the right sides along every cycle of the incidence multigraph must be zero. This is only a necessary condition; no ambient extension theorem is asserted.

Status: `DIRECT_COROLLARY` of the actual curve kernel, positivity, and integration-current compatibility.

Proof. Write $\mu_i = D[u \circ \iota_i]$. These are positive measures, and first-order integration compatibility gives $D(uS) = \sum_i (\iota_i)_* \mu_i$. On $C_i \setminus Z$ no other carrier contributes. Thus μ_i agrees there

with $D[h_i]$. The target measure gives zero mass to Z, so positivity of all the summands forces each μ_i to give zero mass to its preimage as well. Consequently $\mu_i = D[h_i]$ on the whole disc.

The actual curve kernel theorem makes $u \circ \iota_i - h_i$ pluriharmonic. Every finite pluriharmonic function on an ordinary open unit disc is constant: exhaust it by nested closed discs, apply the maximum and minimum principles on each disc with its singleton Shilov boundary, and note that the constants agree on the overlaps. This proves the displayed representation. The single actual value $u(P)$ at an intersection gives (27.1). Summing it along a cycle makes the constants telescope to zero. □

Proposition 27.2 (A normalized boundary-free image obstruction in every $n \geq 2$)**.** *For every $n \geq 2$ there are an ordinary unit polydisc Ω over the original finite extension $K/\mathbf{Q}_p$, a symmetric original-positive separately closed current S of bidegree $(n-1, n-1)$, normalized continuous classical potentials a_j on Ω, and a finite positive measure ν such that*

$$0 \leq a_j \leq 1, \qquad a_j \uparrow a, \qquad \|D(a_j S) - \nu\|_{\mathrm{TV}} \longrightarrow 0,$$

but there is no actual $u \in \mathcal{C}_b(\Omega)$ with $D(uS) = \nu$, even without any normalization or boundary condition on u.

The actual limit a belongs to the normalized relative completion and satisfies $D(aS) = \nu$. Moreover the approximating measures can be chosen to dominate ν:

$$D(a_j S) \geq \nu \quad \textit{for every sufficiently large } j, \qquad \nu(E) \leq C_S(E; \Omega) \quad (E \textit{ Borel}). \tag{27.2}$$

Here C_S is the diagonal raw capacity in remaining degree one on the same whole Ω.

Status: `NEW_PROVED` for this boundary-free strengthening. The assertion that the actual classical relative image is always TV-closed even without boundary conditions is `FALSE`. The proof uses previously established curve and completed-calculus results; it does not modify them.

Provenance of the strengthening. Corollary R4–65.2 already gives two coordinate carriers, a fixed zero-boundary condition, an exact TV error, and domination of the limiting measure by classical approximating measures. The completed carrier construction and the distinction between capacity regularity and exact representability are therefore prior inputs, not new claims here. The new point is the use of *two* actual intersections: their incidence cycle eliminates the otherwise free additive constants and proves nonexistence after all boundary conditions have been removed.

Proof. Put $\lambda = -\log|\pi| > 0$ and fix $0 < \delta < 1/2$. Write the coordinates of the ordinary unit polydisc as $(x, y, z_3, \ldots, z_n)$, omitting the last coordinates for $n = 2$. Define two closed analytic embedded unit discs by

$$\iota_1(t) = (t, 0, 0, \ldots, 0), \qquad \iota_2(t) = (t, t(t-\pi), 0, \ldots, 0).$$

They meet precisely at the two original rigid points with $t = 0$ and $t = \pi$. Set $S = [C_1] + [C_2]$ for their images. These integration currents have exactly the stated positivity, closedness, and bidegree on the boundaryless ordinary domain.

On the open unit disc put

$$h(t) = \delta + (1 - 2\delta)\max\{0, 1 + (2\lambda)^{-1}\log|t|\}, \qquad c = \frac{1-2\delta}{2\lambda} > 0.$$

This is continuous classical psh, with $\delta \leq h \leq 1-\delta$, and the original curve Laplacian calibration gives

$$D[h] = c\delta_{\zeta_{0,|\pi|^2}}, \qquad h(0) = \delta, \qquad h(\pi) = 1/2.$$

Define $z_* = \iota_2(\zeta_{0,|\pi|^2})$ and $\nu = c\delta_{z_*}$. Its support is disjoint from C_1: at that Gauss point the norm of $t(t-\pi)$ is $|\pi|^3 > 0$. In particular the target gives zero mass to both intersection points.

Define actual potentials on the whole polydisc by

$$a_j(x,y,z) = \max\{0, h(x) + j^{-1}\log|y|\}, \qquad \log|0| = -\infty. \tag{27.3}$$

They are continuous classical psh and satisfy $0 \le a_j \le 1-\delta$. For completeness, away from the zero coordinates the expression is a convex coordinatewise nondecreasing function of the logarithmic unit norms. Near $x = 0$ the inner maximum in h is constant, and near $y = 0$ the outer maximum in (27.3) is identically zero, using the common upper bound for h. Thus these expressions extend continuously across both zero loci. Truncating the analytic logarithms and taking the same finite maxima verifies classicality after every complete extension. Since $\log|y| \le 0$, the sequence increases pointwise to

$$a(x,y,z) = \begin{cases} h(x), & y \neq 0, \\ 0, & y = 0. \end{cases}$$

Corollary R4–18.5 puts this actual Borel limit in the relative completion, using the same globally normalized defining row on Ω.

We compute every relative measure for $j > \max\{3\lambda/\delta, 4\lambda\}$. The restriction of a_j to C_1 is zero. On C_2 it is

$$b_j(t) = \max\{0, h(t) + j^{-1}\log|t(t-\pi)|\}.$$

Put

$$r_{0,j} = \exp(\lambda - j\delta), \qquad r_{\pi,j} = \exp(\lambda - j/2).$$

The chosen inequalities give $r_{0,j} < |\pi|^2$ and $r_{\pi,j} < |\pi|$. On the disc $|t| < |\pi|^2$ one has $h(t) = \delta$ and $|t-\pi| = |\pi|$, so

$$b_j(t) = \max\{0, \delta + j^{-1}(\log|t| - \lambda)\}.$$

It is zero on the disc of radius $r_{0,j}$ and gains outward logarithmic slope $1/j$ at its Gauss boundary. On $|t-\pi| < |\pi|$ one has $|t| = |\pi|$ and $h(t) = 1/2$, so likewise

$$b_j(t) = \max\{0, 1/2 + j^{-1}(\log|t-\pi| - \lambda)\},$$

with the same slope jump at radius $r_{\pi,j}$.

Outside these two smaller discs the nonzero branch is nonnegative. To check that no additional contact locus was omitted, for $|\pi|^2 \le |t| < |\pi|$ it is at least $\delta - 3\lambda/j > 0$; for $|t| > |\pi|$ it is at least $1/2 - 2\lambda/j > 0$; and the remaining case $|t| = |\pi|$ is exactly the second displayed radial expression or has $|t-\pi| = |\pi|$, where the same positive bound applies. At the circle $|t| = |\pi|^2$ the branch is strictly active. Its curvature there is therefore $D[h]$. On the complement of the two zeros, the additional polynomial logarithm is harmonic. The one-dimensional slope formula, or original Poincaré–Lelong and ordinary maximum locality on the displayed annuli, now gives on all original curve tests

$$D[b_j] = D[h] + j^{-1}\delta_{\zeta_{0,r_{0,j}}} + j^{-1}\delta_{\zeta_{\pi,r_{\pi,j}}}.$$

Both centers are K-rational, so each annular slope has degree-one calibration. The radii need not belong to $|K^\times|$; the Gauss points are points of the original K-analytic disc and the same slope formula applies.

First-order integration compatibility and linearity in S therefore give the exact original-current identity

$$D(a_jS) = \nu + j^{-1}\delta_{\iota_2(\zeta_{0,r_{0,j}})} + j^{-1}\delta_{\iota_2(\zeta_{\pi,r_{\pi,j}})}. \tag{27.4}$$

In particular its total-variation distance to ν on the whole Ω is exactly $2/j$. The completed calculus identifies this limit with $D(aS)$, by Corollary R4–18.6.

Suppose that some actual $u \in \mathcal{C}_b(\Omega)$ had $D(uS) = \nu$. The two intrinsic curve Laplacians of its restrictions are positive. Their pushforwards sum to ν, whose support is disjoint from C_1. The first Laplacian is therefore zero, and the second is $D[h]$. As in Lemma 27.1, the actual restrictions have the forms

$$u \circ \iota_1 = c_1, \qquad u \circ \iota_2 = h + c_2.$$

At the two common actual points this would give simultaneously $c_1 = \delta + c_2$ and $c_1 = 1/2 + c_2$, contrary to $\delta < 1/2$. This proves nonexistence without imposing boundary values or any global bound on u beyond its membership in the locally bounded classical class.

Finally (27.4) gives $\nu \le D(a_j S)$ for every sufficiently large j. Since each a_j is an admissible normalized tester on the same whole Ω, for every Borel E one has

$$\nu(E) \le D(a_j S)(E) \le C_S(E; \Omega).$$

This proves (27.2) with the exact constant one. □

What is, and is not, excluded. The convergence in (27.4) concerns the relative measures $D(a_j S)$, not the ambient curvatures $D[a_j]$. The counterexample shows that finding an exact actual classical representative of every completed relative measure is a strictly stronger task than the C1 mass bound: for this specific limit no such representative exists, while every sufficiently late classical approximant already dominates it. It makes no assertion that general C1 fails for this background or for arbitrary backgrounds. The arbitrary-background quotient remains valid; its hypothesis concerns two actual classical potentials, and the completed a here cannot be replaced by any such potential with the same relative curvature. Finite pluriharmonic adjustments cannot remove the two incompatible actual intersection values.

28 A wall obstruction to a naive Gauss-current regularization

The following calculation rules out one proposed construction of a regularized background. It is not a counterexample to potential independence for actual classical psh functions.

Proposition 28.1 (The piecewise Jacobian rule does not preserve closedness)**.** *Let K retain its original absolute value, and let*

$$A = \{a < \log|T| < b\} \subset \mathbf{G}^{\mathrm{an}}_{m,K}, \qquad a < c < b.$$

Write $s = \log|T|$ and $r = e^c$. The radius-r Gauss inflation of a point, formed by adjoining a Gauss variable of radius r to its completed residue field and replacing T by $T + Z$, satisfies

$$s \circ \tau_r = \max(s, c).$$

Consider the proposed operation on $[1]_A$ which pulls back a radial top test by the ordinary Jacobian of this piecewise affine map, with no wall term. On every $\phi \in C^\infty_c((a,b))$ this rule is

$$\langle R_r[1], \phi(s) d's \wedge d''s \rangle = \int_c^b \phi(t)\, dt. \tag{28.1}$$

No separately closed original current can satisfy (28.1) on all these tests.

Status: `FALSE` for this proposed closed-current regularization; the following elementary obstruction is `NEW_PROVED`. The ordinary annular integration calibration used in the proof is `EXTERNAL_KNOWN`.

Proof. For the Gauss seminorm over $\mathscr{H}(x)$ one has $|T(x)+Z| = \max(|T(x)|, r)$, which proves the displayed formula for $s \circ \tau_r$. The derivative of $t \mapsto \max(t, c)$ is zero on $t < c$ and one on $t > c$. Pulling back both differential factors therefore gives zero on the first interval and the original top form on the second. The ordinary annulus has tropical weight one, so integration against $[1]_A$ gives (28.1). The value assigned to the derivative at $t = c$ does not change this integral.

Choose $\psi \in C_c^\infty((a,b))$ with $\psi(c) = 1$, and set $\alpha = \psi(s)d''s$. This is an original compact smooth $(0,1)$-test, and

$$d'\alpha = \psi'(s)d's \wedge d''s.$$

Consequently the proposed rule gives

$$\langle R_r[1], d'\alpha\rangle = \int_c^b \psi'(t)\,dt = -1.$$

This is incompatible with $d'R_r[1] = 0$. Interchanging primes gives the corresponding obstruction to separate d''-closedness. □

What the calculation excludes. The finite-wall pairing theorem permits compatible piecewise smooth tests to be evaluated against closed positive currents. It does not state that piecewise differentiation commutes with D or that its wall terms vanish. Indeed $Ds = 0$, whereas $D\max(s, c)$ is the unit Dirac curvature at the radius-e^c Gauss point. Thus compatible piecewise test approximation alone cannot turn Gauss inflation into a closed-current smoothing operator. A construction using integration over the full analytic product fiber would require its own pullback, calibration, and closedness argument; it is not supplied by this failed Jacobian rule.

29 A failed Gauss-perturbation shortcut

One possible route to scalar kernel functoriality was to perturb a finite parameter tuple by arbitrarily small independent Gauss variables, hoping that the resulting parameter point would always be Abhyankar. Exact norm-profile stability would then avoid any limit with moving coefficient measures. The premise is false, even for one parameter over the original finite field K.

Example 29.1 (A non-Abhyankar point fixed by every sufficiently small perturbation)**.** Let π be a uniformizer of K, and normalize its additive valuation by $v(\pi) = 1$. There are a complete extension M/K, an element $a \in M$, and a constant $r_0 > 0$ such that the point $x \in \mathbf{A}^{1,\mathrm{an}}_K$ represented by a is Zariski generic and non-Abhyankar, but for every $0 < \epsilon < r_0$ the point represented by $a + U$ in the radius-ϵ Gauss extension M_ϵ is exactly x.

Status: `FALSE` for the proposed universal Abhyankar-perturbation premise. The following explicit construction also records why the obstruction occurs over the original K, not only over a changed ground field.

Proof. Let $C = \widehat{\overline{K}}$. Choose increasing primes ℓ_j so rapidly that

$$\ell_j > j \prod_{i<j} \ell_i.$$

Put $q_j = 1 - 1/\ell_j$, choose $\theta_j \in \overline{K}$ with $\theta_j^{\ell_j} = \pi^{\ell_j - 1}$, and set

$$c_j = \sum_{i=1}^{j} \theta_i, \qquad r_j = |\pi|^{q_{j+1}}, \qquad r_0 = |\pi|.$$

The closed discs $D_j = D(c_j, r_j)$ in C are nested, since $|c_{j+1} - c_j| = r_j$, and $r_j \downarrow r_0 > 0$.

They have empty intersection in C. To check this directly, first fix $c \in \overline{K}$ and put $d = [K(c) : K]$. For $j > d$, the finite extension

$$H_j = K(c, \theta_1, \dots, \theta_{j-1})$$

has degree at most $d \prod_{i<j} \ell_i < \ell_j$. Its ramification index is therefore smaller than ℓ_j, so its value group does not contain $q_j = (\ell_j - 1)/\ell_j$. Since $c - c_{j-1} \in H_j$, the two summands in $c - c_j = (c - c_{j-1}) - \theta_j$ cannot have equal finite valuation. Thus

$$v(c - c_j) = \min\{v(c - c_{j-1}), q_j\} \le q_j < q_{j+1},$$

also when $c = c_{j-1}$. Hence $c \notin D_j$. If $b \in C$ belonged to every D_j, approximate it by $c \in \overline{K}$ with $|b - c| < r_0$. Since every $r_j > r_0$, the ultrametric inequality would put c in every D_j, a contradiction.

Let y be the multiplicative seminorm on $C[T]$ obtained as the decreasing limit of the Gauss norms of these discs. It is the type-IV point represented by the nested discs. Put $M = \mathscr{H}(y)$ and let a be the image of T in M. For every $c \in C$, some D_j omits c, and nesting makes $|c_j - c|$ constant from that stage onward. In particular

$$|a - c| > r_0 \qquad (c \in C). \tag{29.1}$$

Every nonzero polynomial over C splits into such linear factors; its value at a is therefore nonzero. This also shows that the restricted point x on $\mathbf{A}_K^{1,\mathrm{an}}$ is Zariski generic.

For completeness, M/C is immediate. Given a nonzero rational function $R \in C(T)$, choose j so large that D_j contains none of its finitely many zeros or poles. Its norm on every later disc equals $|R(c_j)|$. Moreover all its linear factors satisfy

$$\left| \frac{a - c}{c_j - c} - 1 \right| \le \frac{r_j}{|c_j - c|} < 1.$$

Taking products and quotients gives $|R(a)/R(c_j) - 1| < 1$. Thus both the value group and every initial residue of $C(a)$ are already supplied by C. Completion does not change the value group or residue field, so the same is true of M. The value group of C is $|K^\times|^{\mathbf{Q}}$ and its residue field is algebraic over $\widetilde{K}$. Any intermediate completed residue field, in particular $\mathscr{H}(x) \subset M$, therefore has rational value-group rank zero and residue transcendence degree zero over K. Consequently $d_K(x) = 0$, whereas its Zariski closure has dimension one. It is not Abhyankar.

Finally take $0 < \epsilon < r_0$ and give $M(U)$ the radius-ϵ Gauss norm. By (29.1), every $c \in C$ satisfies

$$|a + U - c| = \max\{|a - c|, \epsilon\} = |a - c|.$$

Factor any polynomial in $K[T]$ over C and use multiplicativity. Its norm at $a + U$ is exactly its norm at a. The two points of $\mathbf{A}_K^{1,\mathrm{an}}$ are therefore identical for every such ϵ. Arbitrarily small independent Gauss perturbations do not remove this transcendence defect. □

This is a failure of the proposed perturbation mechanism, not of scalar kernel functoriality. In the one-variable proof, the positive limiting radius supplies a different tool: the continuous potential obtained from the uniformly Cauchy truncated logarithms of the nested discs. Its full-fiber curvature is treated in Proposition 4.3.

30 Status change from R4 and remaining proof boundaries

The general question (Q) in the last subsection of R4 was exactly $D[u] = D[v] \Rightarrow D((u-v)S) = 0$ for every allowed fixed S. Its two stated unresolved gates were all-complete-extension curve testing and continuity of the actual bounded classical difference. Sections 4– 9 prove those gates; Theorem 10.2 therefore changes that precise question from `OPEN` to `PROVED`, relative to the declared inputs.

30.1 Focused theorem comparison

Statement	R4	R5
Seven bounded BT properties for fixed arbitrary S	`PROVED` in the earlier records	Unchanged input. Existence, positivity, separate closedness, symmetry, potential multilinearity, ordinary restriction/gluing, and weighted joint decreasing convergence are not recounted as new results.
Original-K projection, finite-etale and analytic kernel	`PROVED`	Unchanged starting point for the simultaneous product induction.
Every complete scalar extension L/K	Only finite-over-Gauss extensions proved; general parameter residue fields remained `OPEN`	`PROVED`, Theorem 4.7. Every finite tuple of coefficients is treated by successive one-parameter fields, including non-Gauss type-IV parameters.
Every analytic map over every such L	Only the original-K analytic theorem; no unrestricted new-field assertion	`PROVED`, Theorem 5.3, for the actual induced original potentials.
Actual curve harmonicity with all extensions	The full extension quantifier remained `OPEN`	`PROVED`, Theorem 6.3. The type-I and type-IV values are recovered separately for each of the two original subharmonic functions.
Curvewise harmonicity without ambient semicontinuity	No continuity mechanism established	`PROVED` in the stated original-field scope by the new Shilov bound and semistable logarithmic interpolation. Even local boundedness and Borel measurability need not be assumed for the finite real-valued curvewise harmonic function.
$\ker D$ on actual $\mathcal{C}_b - \mathcal{C}_b$	Regularity `OPEN`	`PROVED`: it is exactly the sheaf of continuous classical pluriharmonic functions.
Potential quotient for every fixed S	`OPEN`	`PROVED`, Theorem 10.2. There is no new restriction on the background.
Mixed operation on bounded-potential curvature currents	The general quotient obstruction remained	`DIRECT_COROLLARY`: local potentials and differentiated slots descend and glue by Corollaries 10.3 and 10.5.
Signed differentiated local-potential slots	No general potential-independent current operation	`DIRECT_COROLLARY`: real multilinearity on the specified bounded classical difference-potential class, Theorem 11.2.

Statement	R4	R5
Arbitrary new-field backgrounds and original induced potentials	No unrestricted complete-extension transfer	`DIRECT_COROLLARY`: the background over L is arbitrary, while relations between local potentials must originate over K; Section 12.
Finite pointwise limits of exact kernel functions	Known PH finite-evaluation theory, but no general identification of the actual current kernel	`DIRECT_COROLLARY`: locally uniform convergence, including nets, with the finite-evaluation bound of Section 13.
Energy stability inside the actual exact kernel	The faithful form was available	`DIRECT_COROLLARY`: the exact cutoff identity and strong local form convergence for pointwise finite kernel nets, Section 14.
Relative curvature convergence modulo PH corrections	Green and closed-form tools available	`DIRECT_COROLLARY`: a finite-test fixed-background energy bound supplies a sufficient convergence condition; Section 15.
Weak continuity of the descended current operation	Not a consequence of the potential calculus	`FALSE`: Example 24.1 has smooth positive curvatures, a smooth positive limit, uniformly bounded smooth potentials, and fixed $S = [Y]$.
Q for arbitrary bounded USC representatives or the full relative capacitary completion	No justified extension	`FALSE`, by the polar trace example and Remarks 22.4 and 22.6.
Only same-field or finite-extension curve tests	Insufficient to close the complete-extension gate	`FALSE` as a substitute: the full Gauss-point spike passes every such curve test and has zero ambient distribution, but is not continuous; Section 19.
Regularity from universal vanishing on arbitrary Borel functions	Not an asserted replacement for classical potentials	`FALSE`: complete-pole selectors can vanish against every allowed relative curvature and still be discontinuous, Section 23.
Replacing the middle cone by weak positivity	Not permitted by the original input convention	The two-sign vanishing step is `FALSE` in the weak cone. The printed general strong-test order-zero inference also has a closed real-tropical counterexample; Sections 20–21.
Exact finite-dimensional ambient curvature families	The general current quotient was unavailable	`DIRECT_COROLLARY`: all-degree mixed images are finite-dimensional, weak convergence is locally total variation, and diagonal compact capacity control has constant one; Section 25. The additional exact hypothesis is not automatic.

Statement	R4	R5
Normalized families with exact finite curvature on the whole domain	No general actual-current kernel identification	`DIRECT_COROLLARY`: local uniform compactness and retention of normalized classical representatives for every fixed-S mixed weak limit in the specified family; Proposition 26.1. The proof uses nets and imposes no countability assumption.
Classical relative image without prescribed boundary values	Failure of TV-closedness for a prescribed zero-boundary problem was already proved	`FALSE` even with all boundary requirements removed: two intersections impose incompatible actual values. The boundary-free strengthening in Proposition 27.2 is `NEW_PROVED`. Its limit still obeys the sharp capacity bound.

30.2 What the proof adds

The main new arguments are the all-extension kernel propagation and the passage from actual curvewise harmonicity to ambient continuity without assuming either sign semicontinuous. The former preserves all original tests through full analytic fibers. The latter supplies the regularity statement needed by the final arbitrary-background positivity argument. The coefficient controls, standard finite-flat Shilov facts, polynomial normalization, vertical-divisor interpolation, and the subsequent current linearizations retain their stated direct or external provenance.

The complete-field steps have deliberately narrower function hypotheses than an unrestricted new-field regularization theorem: the functions come with prescribed positive smooth rows, and the rows used in the main proof are pullbacks of original-K rows. No statement of B over an arbitrary new field is inferred from the existence of those particular rows.

30.3 The eleven-direction project remains larger than Q

The exact original-literature theorem comparison remains in the frozen R4 record, Section 13, pp. 41–50. This continuation changes its positive-current/mixed-product row and adds the bounded classical kernel theorem. It does not replace the individual hypotheses of the remaining capacity, comparison, or Dirichlet results.

Direction	Current boundary
1. Locally bounded psh	B's class and regularization remain inputs. The actual bounded-difference curvature kernel now has a continuous PH representative equal to its actual values.
2. Positive closed currents and mixed products	The seven fixed-background properties and the general differentiated potential quotient are now both proved.
3. CLN inequalities	The earlier finite-control relative CLN theorem is unchanged. The prescribed-row auxiliary-field estimate is used in this proof.
4. Symmetry and weighted monotone convergence	The fixed-S bounded theorem is unchanged. Weak current convergence is strictly weaker than its hypotheses, as the new smooth counterexample demonstrates.

Direction	Current boundary
5. Locality and maxima	The earlier ordinary and Borel maximum-locality results are unchanged. A strict gap set of two discontinuous psh functions is still not automatically an ordinary open.
6. BT capacities	The qualified earlier results remain. General sharp compact control C1 and Borel outer regularity C2 remain `OPEN`.
7. Quasicontinuity and convergence in capacity	The earlier relative bounded and energy-representative conclusions retain their stated domains and capacities. Q adds no automatic compactness theorem for the normalized psh family.
8. Pluripolar and negligible sets	Relative measures can still charge ambient pluripolar sets. The examples here use that possibility and do not remove it by changing representatives.
9. Comparison and domination	The earlier compact-gap, relative-q.e., and qualified ambient principles remain. No new unrestricted pointwise uniqueness statement follows from the quotient.
10. Relative extremal functions	The earlier constrained variational and qualified ambient results remain. The false unrestricted relative obstacle-contact formula is not repaired by potential independence.
11. Monge–Ampere and Dirichlet problems	The qualified existence results remain unchanged. The new two-intersection example strengthens the earlier relative nonexistence result by removing every boundary condition. A useful general relative solvability criterion remains `OPEN`.

30.4 Minimal next gates

There is no remaining unproved lemma in the stated proof of Q. For the broader project the next gates remain those isolated in R4:

1. **C1, sharp compact approximation.** On one fixed normalized capacity domain, a measure ν in the local vague closure of the classical BT measure family should satisfy

$$\nu(K_0) \le C_S(K_0;\Omega)$$

for every compact core K_0. In the remaining-degree-one completed image, the precise missing step is lower approximation of the completed compact mass by a normalized classical tester on that same domain. Neither weak density nor Q proves this lower compact-mass approximation. Section 25 proves the sharp bound for defining families whose ambient curvatures lie exactly in one fixed finite-dimensional space near the compact set. If this exact hypothesis holds on the whole domain, Proposition 26.1 also supplies normalized actual classical representatives of limits in that specified family. Its local hypothesis alone does not supply this global conclusion. The annulus example shows that uniform potential, strong energy, and psh-dual convergence by themselves do not give convergence of compact masses. The example does not refute C1, since its normalized classical limit is admissible. Moreover, Section 27 rules out the stronger strategy of always representing a completed relative measure by one exact actual classical potential: its explicit measure satisfies the sharp capacity bound and is dominated by normalized classical approximants, but has no such representative even without boundary conditions.
2. **C2, ordinary-open outer regularity.** Even sharp compact control would still need the appropriate increasing-set/outer-capacity argument. The proved finite tester-pasting

lemma supplies the final combination step; the required simultaneous compact extraction with vanishing capacity and pasting losses is not supplied by this round.

3. **D, a verifiable relative solvability criterion.** The completed constrained images and their relative-q.e. uniqueness are available. A useful independent criterion for a datum to lie in the actual image remains to be proved. The unrestricted bounded-subsolution version in the global bounded-row completion is already false and must not be reinstated as a conjectural automatic principle. For actual classical representatives, the new carrier-incidence condition is an additional necessary constraint: harmonic adjustments on unit-disc carriers are constants, and their values must match at every actual intersection point.

Outside the main field/function scope, a general regularity theorem for all locally integrable original-current potentials, and a new-field theorem for all new classical functions without prescribed rows, are not established here. The two questions raised in CLD2025, Remark 7.8.3, have a broader tropical or distributional scope. The present actual bounded-classical difference theorem must not be advertised as resolving all of those questions.

Comparison with GK and arithmetic applications. The conclusions are original CLD-current identities. Equality on every GK δ-test needs a separate comparison theorem. No wedge of two arbitrary positive currents and no nonclosed, off-diagonal GK calculus is asserted. A2 remains a completed arithmetic input. These boundaries are unchanged by the new potential quotient.

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