The ordered potential quotient, delta tests, and relative Bedford–Taylor theory

Sixth research record in non-Archimedean Bedford–Taylor theory

Research record prepared with ChatGPT
Relative to the authorized manuscripts R, B, and A2

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Abstract

This record continues the arbitrary-background bounded Bedford–Taylor calculus and actual bounded-classical potential quotient proved in R4–R5. Its main further result identifies the positive cone of the actual bounded classical difference class: if $a, b \in \mathcal{C}_b$, then $D[a-b] \geq 0$ in the original CLD cone if and only if $a-b$ is itself classical psh. The proof establishes order transfer on every complete extension and analytic test map, retains actual curve endpoint values, and recovers ambient upper semicontinuity through proper-fiber maxima of finite Shilov interpolants. Curvature order and continuous coefficient relations therefore transfer to every fixed allowed positive closed background, including discontinuous potential slots. The resulting mixed order estimates give qualified total-variation stability. Arbitrary assigned bounded Borel representatives remain outside these assertions.

In the actual delta-test category, every total-degree $(1,1)$ delta-form acts over every allowed background with ordinary locality. Constant real cycle relations act in all degrees, and every real delta-form has an exact finite local generation by piecewise smooth coefficients and constant cycles. Full actual analytic delta-testing is constructed for the local weak closure of positive closed delta-form backgrounds, including all their bounded BT tails, smooth regular flags, singular effective Cartier divisors, integrable extended classical potential curvatures, and integrable diffuse barycenters. The higher-degree analytic-relation kernel for an arbitrary background remains open.

An actual bounded classical potential below the curvature of a continuous potential is automatically continuous. Normalized families with a compact continuous family of upper potentials are compact in local uniform topology. Their actual mixed images are weakly closed on the same domain, and their represented curvature quotients have joint weak stability even with a varying positive closed background. A local finite Shilov normalization also gives weighted stability for the specified representatives; it does not identify arbitrary weights by curvature alone.

In remaining degree one, sharp Hardy and strong capacitary constants are respectively one and four. The universal finite-energy threshold for concave capacity-growth moduli is determined. A compact Gauss set has zero relative BT capacity but positive energy capacity; a measure on it has a bounded continuous energy solution but no local bounded classical difference potential. Exact convex-envelope and diffuse carrier capacities give further qualified Borel regularity results. Exact continuous classical interpolation on a fast polar Cantor set gives explicit capacities for arbitrary bounded Borel radial graphs and filled tubes over that diffuse parameter set.

For Dirichlet theory, completed degree-one feasibility has an exact energy-dual criterion, while higher-degree variation criteria retain explicit feasibility hypotheses. Counterexamples rule out unrestricted relative Hessian root concavity and show that the actual zero-boundary classical Monge–Ampere image can be nonconvex in every remaining degree at least two,

even for sums of smooth analytic-subspace integration currents. All new proofs, direct consequences, failed routes and unresolved gates have separate status labels.

Reading guide

Section 1 fixes the hypotheses and frozen dependencies. The main ordered-quotient advance is in Sections 19–21; its module and stability consequences follow immediately afterward. The delta-extension conclusions are in Sections 10–16. The sharp arbitrary-background estimates are in Sections 32–33. The precise changes from R5 and the next proof gates are recorded at the end. This is a continuation with complete new proofs, rather than a replacement for the frozen primary-literature comparison or the authorized internal manuscripts. References marked R4 or R5 refer to the exact numbered results retained in the complete continuation archive.

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1 Scope, conventions, and dependencies

Fix a finite extension $K/\mathbf{Q}_p$ with its original absolute value. An allowed space is a good Hausdorff boundaryless quasi-smooth K-analytic space of pure dimension n. All opens are ordinary topological opens. Put $D = d'd''$. A background of degree (p, p) is symmetric, positive in the original middle CLD cone specified by R, and separately d'- and d''-closed. It acts on all original compact smooth CLD forms of complementary degree. The words positive, strongly positive, and weakly positive are not interchanged.

The class $\mathcal{C}_b$ consists of actual locally bounded classical psh functions as defined in B. Values on ambient polar sets are retained. Products with coefficient-Radon currents use actual Borel multiplication. Local assertions are checked on all original moments and refinements, not just on one chosen skeleton. Signed bounded classical differences are expanded into finite signed combinations of positive mixed currents whenever coefficient estimates are used.

1.1 Authorized and frozen inputs

The internal manuscripts R and B are used as authorized inputs. R supplies continuous local regularization, the original middle-cone coefficient-Radon estimates, and fixed-continuous-multiplier weak continuity. B supplies the prescribed local decreasing continuous semipositive PL rows and their original smooth positive refinements, the class definition, and the original [1] mixed calculus. Their statements are not reproved or globally audited in this record. A2 remains the completed geometric adelic nefness input under its stated number-field, integrable adelic-line, and boundary assumptions. No part of A2 is treated as a new open problem.

The following results of the preceding records are frozen inputs.

1. The complete bounded BT calculus for every fixed allowed arbitrary background: existence, positivity, separate closedness, recursive and regularization independence, differentiated-slot symmetry, multilinearity, ordinary locality and gluing, and weighted joint decreasing convergence on all original tests.
2. Relative CLN estimates with finitely many original positive control forms, bounded-classical energy measures and Green identities, and their weighted convergence. R4, Lemmas 71.1 and 71.2 and Theorem 71.4 supply normal-factor variation control and uniform piecewise smooth test approximation.
3. R5, Theorems 10.1–10.2, identify the actual bounded classical curvature kernel with continuous classical pluriharmonic functions. In particular

$$D[u] = D[v] \quad \Longrightarrow \quad D((u - v)S) = 0$$

   for every allowed fixed S. Its signed differentiated-potential calculus is also retained. This arbitrary-background potential quotient is already `PROVED`; it is not an open gate of this round.
4. The qualified comparison, maximum-locality, quasicontinuity, completed bounded calculus, and faithful relative energy-form results retain the domains and hypotheses of R4. In particular an energy solution or a completed relative potential is not automatically an actual ambient classical psh function.

The full eleven-direction classical-to-non-Archimedean comparison, with primary theorem locations, is retained in R4, Section 13, pp. 41–50. The present record gives a precise update at the gates it actually changes. A result marked `NEW_PROVED` has a proof here relative to the stated inputs; the label makes no claim of priority over every published result. `DIRECT_COROLLARY` is used conservatively for deductions already supplied by the frozen calculus. External facts and conditional hypotheses are identified where used.

1.2 Two distinct extension questions

Ordinary potential independence, preservation of represented curvature order, and testing on every delta-form are different assertions. R5 settled independence. The new bounded-difference positivity criterion settles order, and its proof uses no higher-degree delta extension. The latter extension question retains the separate scopes below. This round uses Mihatsch's actual analytic delta-form sheaf B, as defined by compatible data on all skeletons and all refinements. Its total bidegree is used unless a tridegree is displayed. The external map $B \to D$ is faithful, but faithfulness of that map does not by itself extend an arbitrary ordinary current to the larger test sheaf.

We prove the following sufficient mechanism. Let $\mathcal{W}^p(V)$ be the ordinary weak closure, on one fixed ordinary domain V, of positive separately closed actual delta-form backgrounds. Full actual delta-testing is constructed for $\mathcal{W}^p_{\mathrm{loc}}$. All bounded classical BT tails of such backgrounds stay in that class. Smooth regular flags and their integrable diffuse barycenters give additional examples. The equality of this class with all allowed backgrounds is not assumed.

The all-background ambient cycle action is stronger in a different direction: constant real tropical cycles and all their ambient relations act over every fixed allowed S. Actual analytic zero-presentations introduce an additional kernel. Explicit saturated torus and matroidal chart hypotheses remove that kernel in their fixed-chart scope. Their locality must not be upgraded to arbitrary ordinary subopens without another argument.

1.3 Capacity and Dirichlet notation

On a fixed capacity domain Ω, normalized testers are actual classical functions defined on all of Ω with $0 \le u \le 1$. Raw capacity is their supremum of masses, and the star denotes the infimum over ordinary open neighborhoods in the same domain. C1 denotes sharp compact no-gain under the relevant weak closure; C2 denotes raw Borel equality with that ordinary-open outer capacity. Both remain open in general unless a theorem below supplies an explicit additional hypothesis.

When S has degree $(n-1, n-1)$, the remaining degree is one and $\mathcal{E}_S(f) = \int \Gamma_S(f)$ for real original compact smooth f. The sharp Hardy and strong-capacity estimates below hold for every such allowed S, in every ambient dimension. Faithful closed form and Dirichlet conclusions invoke the additional frozen form setup where stated. For a background of smaller degree one may fix the other bounded BT slots to obtain a degree-one tail; this does not freeze a nonlinear family in which all slots vary.

Part I
Ordinary currents and actual delta-tests

2 What ordinary equality determines in the actual delta-form category

Throughout this section, K is the original finite extension of $\mathbf{Q}_p$, with its original absolute value. The analytic space X is good, Hausdorff, boundaryless, and purely n-dimensional. When the relative bounded-potential calculus is used, X is also quasi-smooth and has the local hypotheses of that calculus. Write

$$A_c^{a,b}(U) = \text{the original compactly supported CLD smooth forms}, \qquad B^{a,b}(U) = B^{a,b}_{\mathrm{Mih}}(U).$$

Here B is the sheaf of *actual* delta-forms of [Mih24, Definition 4.2]. Its total bidegree is used: a tridegree (a, b, c) has total bidegree $(a + c, b + c)$. We do not identify B with the algebraic sheaf B_{GK} without the hypotheses of the comparison between those two sheaves.

2.1 The external faithfulness statement

Proposition 2.1 (Faithfulness on actual delta-forms)**.** `EXTERNAL_KNOWN`. *For every ordinary open $U \subset X$, integration defines an injective map*

$$\iota_U : B^{a,b}(U) \longrightarrow D^{a,b}_{\mathrm{CLD}}(U), \qquad \langle \iota_U(\omega), \eta \rangle = \int_U \omega \wedge \eta, \quad \eta \in A_c^{n-a,n-b}(U).$$

It commutes with d' and d''. The source is defined using all original toric refinements; it is not the quotient obtained from one chosen tropicalization alone.

The precise input is [Mih24, Proposition 4.11, printed pp. 70–71]. The standing field and analytic-space assumptions are given at the start of its Section 3, printed p. 42; in particular its field is any complete non-Archimedean field, and its analytic spaces are good and Hausdorff. Section 4 assumes pure dimension and empty analytic boundary. Thus no algebraic closure of the fixed K is needed here. The original smooth-test topology is recalled in the proof of that proposition. Definition 4.2 and Lemma 4.4, printed pp. 61–62, are the relevant all-refinement definitions and vanishing criterion.

Corollary 2.2 (Ordinary equality for two actual delta-forms)**.** `DIRECT_COROLLARY`. *If $\alpha, \beta \in B^{a,b}(U)$ and $\iota_U(\alpha) = \iota_U(\beta)$ on all original smooth tests, then $\alpha = \beta$ as actual delta-forms. Consequently, for every $\gamma \in B_c^{n-a,n-b}(U)$,*

$$\int_U \alpha \wedge \gamma = \int_U \beta \wedge \gamma.$$

The same conclusion holds after multiplication by any further actual delta-form for which the total degree and support make the integral meaningful.

Proof. Injectivity gives $\alpha - \beta = 0$ in the sheaf B. Its wedge product and integration are operations on sections of that sheaf. Applying them to this equality proves the assertions. No density of smooth tests in a delta-test topology is asserted or needed. □

2.2 The precise BT comparison for delta-form backgrounds

Lemma 2.3 (Multiplication by a piecewise smooth function)**.** `DIRECT_COROLLARY`. *Let $u \in PS^{0,0}(U)$ and $\omega \in B^{a,b}(U)$. Then u is a continuous actual function, $u\omega$ is an actual delta-form, and*

$$\iota_U(u\omega) = u\, \iota_U(\omega)$$

as coefficient-Radon currents. The right side is the ordinary Borel multiplier, with its actual point values.

Proof. By [Mih24, Proposition 4.7, printed p. 66], $PS^{0,0} = B^{0,0,0}$. On a common finite skeletal presentation over the support of a test, multiplication by u multiplies each smooth coefficient of the polyhedral current by the restriction of the continuous function u; this is the product described in [Mih24, Remark 4.3(1), printed p. 61]. The coefficient measures of the resulting ordinary current are therefore u times the coefficient measures of $\iota_U(\omega)$. A finite partition of unity and the refinement compatibility in the definition of B identify the local statements. Equality holds for every original smooth test, including tests requiring a further moment refinement. □

Theorem 2.4 (Comparison on an actual delta-form background)**.** `DIRECT_COROLLARY`. *Let $\sigma \in B^{p,p}(U)$ and suppose that $S = \iota_U(\sigma)$ is symmetric, positive in the project's original middle CLD cone, and separately d'- and d''-closed. Let $u_1, \dots, u_k \in \mathcal{C}_b(U) \cap PS^{0,0}(U)$, with $0 \le k \le n-p$. Put $D_\delta = d'd''$ in B. Then*

$$T_{S,k}(u_1, \dots, u_k) = \iota_U\big(D_\delta u_1 \wedge \cdots \wedge D_\delta u_k \wedge \sigma\big). \tag{2.1}$$

In particular the delta-form on the right represents a positive, separately closed ordinary current. This includes semipositive PL potentials, whenever those potentials are actual piecewise smooth functions in the indicated original moment category.

If two such delta-form expressions represent the same ordinary BT current, then they agree as actual delta-forms and on every complementary compact delta-test. If $u_0 \in PS^{0,0}(U)$, the weighted current $u_0 T_{S,k}$ is also represented by the delta-form $u_0 D_\delta u_1 \wedge \cdots \wedge D_\delta u_k \wedge \sigma$.

Proof. The separate closedness of S and Proposition 2.1 imply $d'\sigma = d''\sigma = 0$ in B. Suppose inductively that the product with $u_2, \dots, u_k$ is represented by the separately closed delta-form

$$\tau = D_\delta u_2 \wedge \cdots \wedge D_\delta u_k \wedge \sigma.$$

The ordinary recursive definition and Lemma 2.3 give

$$\begin{aligned} T_{S,k}(u_1, \dots, u_k) &= D\big(u_1 \iota_U(\tau)\big) \\ &= D\,\iota_U(u_1\tau) = \iota_U\big(D_\delta(u_1\tau)\big) = \iota_U\big(D_\delta u_1 \wedge \tau\big). \end{aligned}$$

The last equality is the Leibniz rule with both derivatives of τ zero. This proves the induction, starting with $\tau = \sigma$. Positivity and closedness are supplied by the already established ordinary BT theorem, not by an unidentified positivity convention for arbitrary delta-forms. The full delta-test equality follows from Corollary 2.2; the weighted assertion follows from the multiplier lemma. □

Remark 2.5 (What this does and does not say for arbitrary backgrounds)*.* For every allowed fixed positive closed S, even when S is not an actual delta-form, the currents $D(uS)$ and $T_{S,k}(u_1, \dots, u_k)$ above exist by the ordinary BT theory. Their individual differentiated slots depend only on $D[u_i]$, by the R5 potential quotient theorem. This does not by itself construct a functional on all delta-tests.

For example, a formal expression $\sum_j \eta_j \wedge D_\delta u_{j1} \wedge \cdots \wedge D_\delta u_{jr}$, with admissible slots $u_{jl} \in \mathcal{C}_b(U) \cap PS^{0,0}(U)$ and compactly supported original smooth coefficients η_j of complementary degree, can be paired with S by the corresponding finite sum of ordinary BT pairings. To call this an action on the *delta-form represented by that expression*, one must additionally prove that every zero relation between such expressions annihilates S. Independence of one potential in one slot is only the linear part of this relation problem. Proposition 2.1 does not supply a product of an arbitrary current with a delta-form.

An arbitrary ordinary current is, by definition, a functional on A_c, not on B_c. A full delta-test identity involving it is therefore not a statement until the enlarged action has been defined and shown independent of all presentations and refinements. This is distinct from the already settled comparison for two *actual delta-forms*.

2.3 The algebraic GK category is a separate input

In [GK17], the standing base field is algebraically closed, complete, and nontrivially valued, and the space is the analytification of an algebraic variety; see its Notations and Terminology, printed p. 8 of arXiv:1406.7637v3. Definition 4.4 imposes vanishing after all algebraic morphisms, and

Definition 4.13 sheafifies the corresponding algebras. Remark 6.7 explicitly does not assume faithfulness of the ordinary-current map; Proposition 6.8 gives a criterion using *all algebraic pullbacks* for its generalized delta-forms P.

Mihatsch states the embedding $B_{\mathrm{GK}} \hookrightarrow B$ in [Mih24, Section 4.2, printed p. 67], under those GK hypotheses, and explains its basis in Lemma 4.4 and Theorem 4.5. He labels that comparison as given without formal proof. Once this embedding is available for the actual datum, Corollary 2.2 applies to its image, and equality there implies equality on GK delta-tests as well. None of these sentences replaces the original finite K by its completed algebraic closure or transfers an arbitrary current across that extension.

The theorem above accordingly uses Mihatsch's actual sheaf directly over the original K. It neither asserts that P_{GK} embeds into it nor that arbitrary CLD currents acquire all the nonclosed, nonpositive, and off-diagonal operations of the delta calculus.

3 The local divisor generation that is actually available

This section separates three algebras: constant-weight rational tropical cycles, smooth-coefficient expressions in those cycles, and all actual Mihatsch delta-forms. The distinction matters before one asks whether a formula involving an arbitrary background current descends through relations.

Proposition 3.1 (The fan generation input)**.** `EXTERNAL_KNOWN`. *The graded ring of rational tropical cycle fans in $\mathbf{R}^r$, with rational weights and stable intersection, is isomorphic to the rational polytope algebra. Every codimension-c such cycle is a finite rational linear combination of cth powers of tropical hypersurfaces of rational polytopes.*

The precise source is [JY16, Theorem 5.1 and Corollary 5.2], printed pp. 19–20 of arXiv:1309.7064v3. Its convention on printed p. 3 explicitly restricts Sections 3–6 to fans. Its Section 5 works over $\mathbf{Q}$, not with arbitrary non-rational polyhedral directions. The constructive paragraph after Corollary 5.2 uses a subdivision into a simple polytopal fan and products of nonnegatively weighted divisor fans. These are equality statements for cycles, not only equality of their supports or rational-equivalence classes.

Corollary 3.2 (Constant real weights and local affine cycle germs)**.** `DIRECT_COROLLARY`. *Let C be a constant-weight rational polyhedral tropical cycle of codimension c on a neighborhood of $a \in \mathbf{R}^r$. Real constant weights are allowed, and the affine offsets of its polyhedra may be real. After shrinking to a neighborhood W of a, there are finitely many real numbers b_j and convex piecewise affine functions φ_j whose gradients are integral such that*

$$[C]|_W = \sum_j b_j (D\varphi_j)^c|_W \quad \text{as actual real delta-forms on } W. \tag{3.1}$$

The functions φ_j have real additive constants; no extension of the valued base field is required when their pullbacks are used later.

Proof. Choose a finite subordinate polyhedral complex near a. On a sufficiently small neighborhood, this cycle is the restriction of the translate by a of its local cone fan. The directions of that fan are rational. For a fixed finite rational fan, its constant real weight vectors satisfying balancing form the real kernel of a matrix with rational entries: choose rational bases in the finitely many normal quotient spaces and write the balancing equations. This kernel has a rational basis. Consequently the real-weight fan is a finite real linear combination of rational-weight fans on the same complex.

Apply Proposition 3.1 to each of these finitely many rational-weight fans. If P_j is a resulting rational polytope, its support function

$$h_{P_j}(x) = \max_{m \in P_j} \langle m, x \rangle$$

is convex piecewise linear, and Dh_{P_j} is its tropical divisor. Translate the argument to $x - a$. Choose an integer $d_j > 0$ clearing the denominators of the vertices of P_j and set $\varphi_j(x) = d_j h_{P_j}(x - a)$. Then φ_j has integral gradients, and the factor d_j^{-c} is absorbed into b_j. The divisor product agrees with stable intersection by [Mih23, Theorem 4.1(4)], proving (3.1). □

Corollary 3.3 (Original moment pullbacks)**.** `DIRECT_COROLLARY`. *Let $t_f : U \to \mathbf{R}^r$ be an original toric moment map over the fixed K. On a neighborhood on which a delta-form is represented by a finite sum*

$$\omega = \sum_i t_f^*(\eta_i) \wedge t_f^\star[C_i],$$

where the η_i are smooth superforms and the C_i have constant weights and rational directions, it is locally a finite real linear combination of expressions

$$t_f^*\eta \wedge (D_\delta v_1) \wedge \cdots \wedge (D_\delta v_c), \qquad v_j = \varphi_j \circ t_f,$$

with continuous piecewise affine classical psh functions v_j. The identities hold in the actual delta-form sheaf and hence after every further original moment refinement.

Proof. Apply Corollary 3.2 in a neighborhood of $t_f(x)$ and shrink U accordingly. A convex piecewise affine function is a finite maximum of affine functions. Each affine pullback is a real linear combination of logarithms of analytic units plus a real constant. Such affine pullbacks are pluriharmonic under all complete extensions, and their finite maxima are classical psh. The delta pullback commutes with D and wedge products by [Mih24, Definition 4.2(3)]; this proves the displayed identity. Refinement compatibility is part of the same definition, not a separate assertion made on only the original chart. □

Remark 3.4 (Generation is not the relation theorem). The preceding corollary constructs representatives. It does not prove that the ordinary BT pairing of such a representative with an arbitrary fixed S is independent of the representative. The remaining relation problem is exactly that a zero delta-form expression must have zero relative BT action. The R5 potential quotient settles individual degree-one potential changes, not automatically all polynomial relations between several factors.

Example 3.5 (A false generation statement for the full Mihatsch algebra)**.** `FALSE`; direct degree check. The full algebra $B(\mathbf{R})$ is not generated by smooth forms and the curvature factors $D\varphi$ alone. Its degree-zero part contains the continuous piecewise linear function $x \mapsto \max(0, x)$, by [Mih23, Lemma 3.7]. The degree-zero part of the algebra generated by smooth forms and factors of positive bidegree $(1, 1)$ consists only of smooth functions. Thus this function is absent from that proposed generated algebra. The same example occurs over the original K on an annulus whose skeleton contains $\log|T| = 0$: the function $u = \max(0, \log|T|)$ has $D[u] = \delta_\xi$, where ξ is the unit Gauss point, by the original torus calibration. If u were original smooth near ξ, its curvature would be a smooth top-degree form there. Such a form has Lebesgue densities on finitely many local calibrated one-dimensional skeletal cells, and hence has no atom at ξ. This contradiction proves the claimed non-smoothness. A corner seen only on one fixed skeleton would not by itself prove non-smoothness after all moment refinements.

This disproves the literal generation assertion; it does not disprove a generation statement allowing piecewise smooth coefficients, first derivatives, or further operations.

Scope of the two published definitions. The generalized delta-preforms of [GK17, Definition 2.9] allow tropical cycles with *smooth* weights. Their differential subalgebra B_{GK} is defined through the closed subspace Z and the algebra AZ in §4.6 and Definitions 4.9 and 4.13. Consequently the constant-weight hypothesis of Corollary 3.3 must be retained; it is not a verified description of every section of B_{GK}.

Mihatsch's real delta-forms are all balanced polyhedral currents, equivalently those polyhedral currents whose d' and d'' derivatives remain polyhedral; see [Mih23, Definitions 3.1–3.2 and Theorem 3.3]. His theory also permits non-rational polyhedra. Neither this class of variable coefficients nor non-rational direction data is covered by the rational fan generation theorem quoted above. Their treatment cannot be replaced by an invocation of the polytope algebra without an additional proof.

A useful external structural formula, not a missing generation theorem. For real delta-forms, [Mih23, Theorem 4.1 and its uniqueness proof, printed pp. 13–14] characterizes the product by the Leibniz rules, multiplication by piecewise smooth forms, and

$$S \wedge T = p_{1,*}(\Delta \wedge (S \times T)), \qquad \Delta = \bigwedge_{i=1}^{r} D\max(x_i, y_i).$$

Divisor intersection satisfies

$$(D\varphi) \cdot T = D(\varphi T) - \varphi DT + d''\varphi \wedge d'T - d'\varphi \wedge d''T,$$

and reduces to $D(\varphi T)$ for a separately closed T; see Definition 4.6 and Remark 4.7, printed p. 15. These formulas may guide an extension to a new class of currents. They are proved in the delta-form category, however, and do not assert that the required diagonal intersections, exterior products, or presentation-independent piecewise operations exist for every ordinary CLD current.

What the additional proofs in this round now supply. Theorem 7.5 treats constant cycles with arbitrary real directions. The exact finite construction in Theorem 10.2 treats all smooth coefficient arrays, including flat ones, and proves that every real Mihatsch delta-form is locally a finite sum of piecewise smooth forms times constant real cycles. Thus the two omissions from the quoted rational fan theorem have been addressed by separate proofs in this round. These proofs do not turn the quoted JY theorem into a stronger external input. The further actual analytic relation problem is settled for the weak-approximation class in Theorem 13.1; its all-background version remains a distinct open question.

4 Closed Radon currents and piecewise smooth tests

Throughout this section K is the original finite extension of $\mathbf{Q}_p$, and U is a good Hausdorff boundaryless quasi-smooth K-analytic space of pure dimension n. All smooth forms are the original CLD forms. The coefficient-Radon property and Borel multiplication are understood on the actual Berkovich space, as in R and B. No pushforward to one selected skeleton is used.

Write $\mathrm{PS}_U^{a,b}$ for the sheaf of piecewise smooth forms defined using original toric moment presentations and their full tropical images. Its equivalence relation is the one in Mihatsch, §4.2, equation (4.9), printed page 65. On a polyhedral complex, the smooth pieces agree after restriction to the affine tangent space of every common face; see §2.1(16), printed pages 13–14. By his Proposition 4.7, this is precisely the codimension-zero part $B_U^{a,b,0}$ of his sheaf of actual delta-forms. It is not the entire sheaf B_U.

Provenance before the new argument. R4, Theorem 71.4 (printed p. 240), already proves the piecewise-test action and uniform smooth approximation for arbitrary original-positive separately closed currents. Its wall estimates even give an $O(\varepsilon^{1/2})$ bound uniform under fixed finite mass controls. That positive case is a `DIRECT_COROLLARY` here, including its identification with the codimension-zero delta-test class. The additional assertions below concern closed coefficient-Radon currents without a positivity assumption, all smooth level sets rather than only affine moment walls, and the resulting piecewise smooth module structure. They do not replace the stronger quantitative R4 estimate.

Lemma 4.1 (Tangency of the mass on a smooth level set)**.** *Let T be a coefficient-Radon current on U. If $d'T = 0$, then for every real original smooth function h and every $c \in \mathbf{R}$,*

$$d'h \wedge \mathbf{1}_{\{h=c\}}T = 0.$$

If $d''T = 0$, the corresponding identity with $d''h$ holds. These are identities on all original complementary compact smooth tests, including arbitrary further moment refinements. No positivity hypothesis is needed.

Status: `NEW_PROVED` for the stated nonpositive closed-current extension. The positive affine-wall case is already part of R4.

Proof. Put $g_\varepsilon(t) = \varepsilon \arctan((t-c)/\varepsilon)$. Its absolute value is at most $\pi\varepsilon/2$, while

$$0 \le g'_\varepsilon(t) = \frac{1}{1 + ((t-c)/\varepsilon)^2} \le 1, \qquad g'_\varepsilon(t) \longrightarrow \mathbf{1}_{\{c\}}(t).$$

Uniform convergence and the local finite coefficient variations give $g_\varepsilon(h)T \to 0$ as a current. Thus its d' tends to zero. The smooth product rule and $d'T = 0$ identify that derivative with $g'_\varepsilon(h)d'h \wedge T$. On any fixed original test, coefficientwise bounded convergence identifies its limit with $\mathbf{1}_{\{h=c\}}d'h \wedge T$. This proves the first identity. The same proof gives the second. Every argument is made after the chosen test has been presented on a common refinement; hence no restriction of the test class has occurred. □

Definition 4.2. Let $\mathscr{R}_U$ be the sheaf of coefficient-Radon currents T such that, for every local real original smooth h and every $c \in \mathbf{R}$, both identities in Lemma 4.1 hold. Thus separately closed coefficient-Radon currents belong to $\mathscr{R}_U$. This auxiliary sheaf records exactly the tangency needed to multiply by piecewise smooth forms; its elements need not be closed.

Theorem 4.3 (Canonical piecewise smooth multiplication)**.** *There is a unique local multiplication obtained from tangential restriction of the smooth pieces,*

$$\mathrm{PS}_U^{a,b} \times \mathscr{R}_U^{p,q} \longrightarrow \mathscr{R}_U^{p+a,q+b}, \qquad (\alpha, T) \longmapsto \alpha \wedge T.$$

It extends multiplication by original smooth forms and by actual continuous piecewise smooth functions. It is real bilinear, associative with the wedge product of piecewise smooth forms, and compatible with restriction and gluing on ordinary opens. It is independent of subdivisions, smooth extensions from faces, toric moment presentations, and further refinements. The resulting objects are ordinary coefficient-Radon currents. In particular every allowed arbitrary positive separately closed background S, and every frozen bounded BT product over S, acts canonically on all compact piecewise smooth complementary tests.

Status: `NEW_PROVED` for the general $\mathscr{R}_U$ module formulation; `DIRECT_COROLLARY` for the asserted action on the project's positive closed backgrounds and BT products, by R4 and the displayed external identification of the piecewise smooth sheaf.

Proof. Work first in one moment presentation $t : V \to \mathbf{R}^r$ and on a compact stage. Choose a finite polyhedral subdivision adapted to the piecewise smooth representative of α. Use all relatively open faces, so their inverse images $E_\sigma = t^{-1}(\operatorname{relint}\sigma)$ are disjoint Borel sets. Locally near the compact stage, choose a smooth ambient extension $\widetilde{\alpha}_\sigma$ of the smooth form on each closed face. Define

$$\alpha \wedge T = \sum_\sigma t^* \widetilde{\alpha}_\sigma \wedge \mathbf{1}_{E_\sigma} T. \tag{4.1}$$

There are only finitely many terms on the stage. Smooth extensions have bounded coefficients there, so this is a coefficient-Radon current. If several extension neighborhoods are needed, use a finite smooth partition on the ambient face; the same argument below proves independence of it.

Let L_σ be the affine span of σ, written as the common zero set of affine functions $\ell_1, \dots, \ell_m$. Tangency gives

$$d'(\ell_j \circ t) \wedge \mathbf{1}_{E_\sigma} T = d''(\ell_j \circ t) \wedge \mathbf{1}_{E_\sigma} T = 0.$$

Indeed $E_\sigma \subset \{\ell_j \circ t = 0\}$, and restriction of a zero coefficient-Radon current to a Borel set remains zero. If two smooth extensions have the same tangential restriction on σ, their pointwise difference there belongs to the exterior ideal generated by the conormals $d'\ell_j, d''\ell_j$. This follows by completing a basis of L_σ^* to an ambient dual basis. The coefficients of that decomposition are Borel and bounded on the compact stage. The preceding identities annihilate this difference in (4.1). Thus the formula does not depend on extension choices.

Subdivision does not change the formula. On each smaller face the old and new smooth pieces have identical tangential restrictions, and the same conormal argument applies. Any two finite subdivisions have a common subdivision.

It remains to check the quotient defining the analytic sheaf, not just a fixed polyhedral presentation. Suppose a presented piecewise smooth form is zero in that quotient. After passing to a common moment presentation, its restriction to the full tropical image $T'(A, t)$ is zero for every affinoid domain A in the presentation. Cover the support of a given original test by finitely many interiors of such affinoids, and use an original smooth partition subordinate to those interiors. On one member subdivide $T'(A, t)$ and the piecewise smooth presentation simultaneously. For each resulting face, the representative has zero tangential restriction. The conormal argument therefore makes its product with the corresponding Borel part of T zero. Summing proves zero on the chosen original test. This also proves invariance under the equivalence of two moment presentations, by taking their joint moments and subtracting the two representatives. The argument uses the full $T'(A, t)$ and works for every affinoid A; it does not replace the defining quotient by a selected skeleton.

If h is any further original smooth function, apply the same formula to $\mathbf{1}_{\{h=c\}} d'h \wedge (\alpha \wedge T)$. Each summand is a smooth multiplier times a Borel restriction of $\mathbf{1}_{\{h=c\}} d'h \wedge T$, hence zero. The d'' identity is identical. Thus the product belongs to $\mathscr{R}_U$. Using one common subdivision for two piecewise smooth forms proves associativity directly in (4.1). Bilinearity follows in the same way. Restriction is manifest, and the independence already proved glues the local constructions uniquely. For a continuous piecewise smooth function the face values are its actual values, so the formula is ordinary continuous coefficient multiplication. □

Proposition 4.4 (Coefficient control and fixed-current smoothing)**.** *Fix $T \in \mathscr{R}_U$ and one compact stage with a finite original moment presentation. Let α be a piecewise smooth form there. There exist smooth forms α_ε on a slightly smaller stage such that, for every fixed original smooth test η on that stage,*

$$\left|\mu_{(\alpha_\varepsilon - \alpha)\wedge T, \eta}\right|(K) \longrightarrow 0$$

for each compact K in the smaller stage. All these coefficient variations admit a common finite bound by the variations of finitely many coefficients of T times a fixed bound for the

piecewise smooth coefficients and η. The assertion is for fixed T; it does not assert uniformity over an arbitrary varying family of currents.

Proof. Use a piecewise smooth representative on an open subset of $\mathbf{R}^r$ containing the relevant tropical image. In each full-dimensional polyhedron choose its smooth coefficient array. Assign arbitrary bounded values to the array on lower-dimensional faces and convolve locally with a nonnegative compactly supported smooth mollifier of radius ε. Work on a relatively compact smaller real open, so no boundary extension issue occurs. The arrays remain uniformly bounded there and define α_ε.

For a fixed point x in the relative interior of a face σ, all adjacent pieces have the same restriction to the tangent space of σ. Consequently, after restricting all covectors to that tangent space, the mollified array at x tends to the common tangential array at x. To verify this, discard the finitely many polyhedra whose closures do not contain x, then use continuity of the smooth arrays on each remaining closed polyhedron. The mollifier averages only these finitely many arrays for sufficiently small ε. Their common tangential limits agree.

On E_σ, all components in the conormal exterior ideal are annihilated by T, as in the preceding theorem. Thus the coefficients of $(\alpha_\varepsilon - \alpha) \wedge T$ on that set converge pointwise, after contraction, to zero, with a uniform bound by finitely many fixed coefficient variations of T. Bounded convergence gives convergence in total variation on $K \cap E_\sigma$. Sum the finitely many faces. For an additional original test, make a common refinement first; the argument and the finite coefficient domination are unchanged. This proves the assertion on all original tests. A partition of unity gives the same statement for finitely many local presentations. □

Corollary 4.5 (First derivatives of continuous piecewise smooth potentials)**.** *Let T be separately closed and coefficient-Radon. For every real continuous piecewise smooth function f,*

$$d'(fT) = d'_P f \wedge T, \qquad d''(fT) = d''_P f \wedge T.$$

In particular both first derivatives are coefficient-Radon. These statements do not assert that $d'\alpha = d'_P\alpha$ for an arbitrary piecewise smooth form of positive degree; corner terms can occur in that situation.

Proof. The local ambient representative of f is continuous and piecewise smooth on a finite polyhedral subdivision. It is locally Lipschitz: the gradients on each smooth piece are bounded on compact subsets and the pieces agree on common faces. Its mollifications converge locally uniformly to f. Their first derivatives are the mollifications of its almost-everywhere piecewise derivatives, because continuity eliminates a distributional jump term in the first derivative. The tangential convergence proof of Proposition 4.4 therefore applies to these derivatives. The smooth closed product rule gives $d'(f_\varepsilon T) = d' f_\varepsilon \wedge T$. The left side converges distributionally to $d'(fT)$ by local uniform convergence; the right side converges as a coefficient-Radon current to $d'_P f \wedge T$. Repeat for d''. Local identities glue by the preceding theorem. □

Exact scope. The construction solves the codimension-zero delta-test extension for every fixed allowed background and its BT products. A positive-codimension delta-form is not merely a piecewise smooth coefficient array. Pairing with such a form can require an additional intersection. No arbitrary delta-test extension, no product of two arbitrary currents, and no varying-background monotone theorem is asserted here.

5 A lattice valuation identity over an arbitrary background

Fix the original K, an allowed ordinary open U, and one arbitrary symmetric original-positive separately closed (p,p)-current S. Put $q = n - p$. The signed differentiated calculus of R5 is a

frozen input: locally bounded differences of actual classical psh functions give coefficient-Radon mixed currents by finite polarization, with symmetry, recursion, and dependence only on their differentiated currents. All equalities below are ordinary current equalities on every original compact complementary smooth test and every further refinement.

Let $\mathscr{H}_c(U)$ denote the continuous actual functions which are locally differences of two members of $\mathcal{C}_b$. No continuity of those two individual presentations is required. Their pointwise maximum is again classical psh. Consequently $\mathscr{H}_c$ is a real vector lattice: if $h = a - b$, then

$$h^+ = \max(a, b) - b, \qquad h^- = \max(a, b) - a,$$

and the maximum and minimum of two local differences can be reduced to this identity. These formulas are only used on a common local presentation; the resulting functions are their actual pointwise positive and negative parts.

Lemma 5.1 (Orthogonality of positive and negative parts)**.** *Let $h \in \mathscr{H}_c(U)$ and suppose $q \geq 2$. Then*

$$D[h^+] \wedge D[h^-] \wedge S = 0. \tag{5.1}$$

The same conclusion holds with S replaced by any signed mixed tail generated from S by the frozen bounded classical-difference calculus, provided the total degree is allowed.

Status: `DIRECT_COROLLARY` of the frozen signed calculus and ordinary locality. Its proof is recorded because it supplies the contact-set identity needed for the new cycle-relation argument below.

Proof. First let Q be any one positive separately closed mixed tail over S, including $Q = S$. The signed current $C_- = D(h^- Q)$ is coefficient-Radon by finite polarization. The set $G = \{h > 0\}$ is an ordinary open because h is continuous. On G the actual function h^- is zero, so the recursive definition, or ordinary restriction of $D(h^- Q)$, gives $C_-|_G = 0$. Every coefficient Radon measure of C_- therefore vanishes on G. On $U \setminus G$ the actual multiplier h^+ is zero. The two assertions imply

$$h^+ C_- = 0$$

as a coefficient-Radon current on all of U. Applying D and using the signed recursive calculus proves (5.1) with Q in place of S. A signed mixed tail is a finite linear combination of such positive tails on each local presentation. Apply the preceding argument to its terms and use ordinary restriction and gluing. No assertion about the ordinary openness of a gap set between discontinuous psh functions is involved, and no simultaneous limit of two singular measures has been exchanged. □

Theorem 5.2 (Max–min valuation for every mixed power)**.** *Let $u, v \in \mathscr{H}_c(U)$ and set $a = \max(u, v)$, $b = \min(u, v)$. For every $0 \leq k \leq q$,*

$$(D[a])^k \wedge S + (D[b])^k \wedge S = (D[u])^k \wedge S + (D[v])^k \wedge S. \tag{5.2}$$

The degree-zero convention is $2S = 2S$. A further fixed signed mixed tail may be inserted in every term whenever the degree permits it. The functions a and b are continuous classical differences; b need not be psh. Thus the formula uses the signed calculus in its stated scope rather than asserting positivity of every term.

Status: `DIRECT_COROLLARY` of the preceding orthogonality identity and finite polynomial algebra. The ensuing application to ambient cycle relations is a separate theorem, rather than a new regularization theorem.

Proof. On the module of signed mixed tails over S, write $A = D[a]$, $B = D[b]$, $U_1 = D[u]$, and $V_1 = D[v]$ for their commuting differentiated-slot operators. This notation means composition of the defined recursive operations; it does not multiply two arbitrary currents. Linearity gives $A + B = U_1 + V_1$. With $h = v - u$ one has $a - u = h^+$ and $a - v = h^-$. Lemma 5.1 gives

$$(A - U_1)(A - V_1) = 0$$

on every allowed tail. It follows that $AB = A(U_1 + V_1 - A) = U_1V_1$.

For two commuting symbols x, y, their power sums obey

$$P_0 = 2, \quad P_1 = x + y, \quad P_k = (x + y)P_{k-1} - xyP_{k-2} \quad (k \geq 2).$$

This is checked by expanding the right side and cancelling the two cross terms. The pairs (A, B) and (U_1, V_1) have the same sum and product, hence have the same power sums by induction. Apply that polynomial identity to S or the fixed tail. All operators commute by the frozen signed BT symmetry, and all intermediate terms have the specified degree. Local potential presentations give the identity locally; the original current sheaf glues it. □

Corollary 5.3 (A finite exponential valuation)**.** *For $h \in \mathscr{H}_c(U)$ let L_h act on the module of signed mixed tails generated over S by $L_hT = D(hT)$. The exponential*

$$E_h = \sum_{j=0}^{q} \frac{L_h^j}{j!}$$

is finite, with terms beyond the available degree zero. It satisfies

$$E_{h+k} = E_hE_k, \qquad E_0 = I, \qquad E_{\max(h,k)} + E_{\min(h,k)} = E_h + E_k.$$

If h is actual pluriharmonic, then $L_h = 0$ and $E_h = I$.

Status: `DIRECT_COROLLARY` of Theorem 5.2 and the frozen R5 quotient.

Proof. The product hT and its derivative depend on the actual current T, so the operator is independent of a chosen signed-tail presentation. Expanding that presentation shows that it stays within the stated module and agrees with insertion of the differentiated slot. Those operators commute and satisfy $L_{h+k} = L_h + L_k$. The ordinary finite binomial formula gives the first assertion. The valuation identity follows by summing the power identities with coefficients $1/j!$. For a pluriharmonic h, R5 makes its slot zero on every positive tail; finite polarization gives the same conclusion on the entire module. □

Continuity is used at one precise place. The proof uses that $\{h > 0\}$ is an ordinary open. For a general discontinuous classical difference this need not hold. The preceding argument therefore supplies no new fine locality theorem for that larger class. Continuous PL support functions, which are the application below, satisfy the hypothesis exactly.

6 Descent through ambient tropical cycle relations

Fix the same original field and allowed space as above, one arbitrary allowed background S, and original toric moments $t = (\log|f_1|, \ldots, \log|f_r|)$ on an ordinary open U. Changing all signs to additive valuations merely changes the chosen moment convention. We use the maximum convention for support functions and tropical divisors, consistently with [JY16, Notations and conventions, p. 3].

Let $\mathscr{M}_S$ be the graded real module of ordinary currents obtained locally as finite linear combinations of the signed bounded-potential mixed tails over S. Its grading records the number of differentiated slots. An operator of degree larger than the remaining dimension acts as zero. A continuous local classical difference h defines $L_h : T \mapsto D(hT)$ on this module. This is a well-defined operator on actual currents: Borel multiplication and D do not depend on a presentation of T, while the frozen signed calculus shows that its value is another allowed mixed tail. The operators L_h commute.

Lemma 6.1 (Support functions under a convex union)**.** *Let P, Q be nonempty compact convex polytopes such that $P \cup Q$ is convex. Then*

$$h_{P\cup Q} = \max(h_P, h_Q), \qquad h_{P\cap Q} = \min(h_P, h_Q).$$

In particular $P \cap Q$ is nonempty.

Proof. The maximum formula is immediate from taking the supremum over a union. Fix a linear functional ℓ. If $h_P(\ell) < h_Q(\ell)$, choose maximizers $p \in P$, $q \in Q$. For every $s \in (0, 1]$, the point $(1 - s)p + sq$ has ℓ-value strictly greater than $h_P(\ell)$. The whole segment lies in $P \cup Q$, so all these points belong to Q. By closedness $p \in Q$, and the maximum on $P \cap Q$ is at least $h_P(\ell)$. The reverse inequality is automatic. The case of the opposite strict inequality is symmetric. If the two support values agree, the segment of maximizers lies in that common supporting hyperplane. Its intersections with the closed sets P and Q cover the segment and contain its respective endpoints. Connectedness forces them to intersect. The intersection point has the common support value. This proves the second formula and nonemptiness. □

Theorem 6.2 (Action of the rational fan cycle ring)**.** *Let $\mathcal{T}_r$ be the graded ring of constant-weight rational tropical cycle fans in $\mathbf{R}^r$, allowing real weights. There is a canonical graded real algebra homomorphism*

$$\mathcal{A}_{t,S} : \mathcal{T}_r \longrightarrow \mathrm{End}(\mathscr{M}_S)$$

characterized by

$$\mathcal{A}_{t,S}(Dh_P)T = D\big((h_P \circ t)T\big) \tag{6.1}$$

for every rational polytope P. Thus every polynomial relation between constant rational tropical cycle fans annihilates every allowed mixed tail over the arbitrary S. The output for a homogeneous cycle of codimension c is a separately closed coefficient-Radon current of degree c above that tail. No positivity claim is made for a general positive tropical cycle presented as a signed combination of hypersurface products.

Status: `NEW_PROVED`: the relation descent uses the elementary frozen-calculus consequence proved above and the external polytope algebra isomorphism. It is not a new construction of the ordinary BT product.

Proof. First work over $\mathbf{Q}$. The rational polytope algebra Π_r is generated additively by symbols $[P]$, with translation invariance and the relation

$$[P \cup Q] + [P \cap Q] = [P] + [Q]$$

when $P \cup Q$ is convex. Its multiplication is Minkowski sum, $[P][Q] = [P + Q]$, with $[\varnothing] = 0$ and the point class as unit; see [JY16, Section 5, p. 19].

For $P \neq \varnothing$ assign

$$[P] \longmapsto \exp L_{h_P \circ t}, \qquad [\varnothing] \longmapsto 0.$$

The support function pulled back by t is a finite maximum of original pluriharmonic functions, hence a continuous classical psh function. All its operators are therefore defined. Translating

P adds an affine function of t. That function is original smooth pluriharmonic, so its operator is zero. Thus translation invariance holds. The support function of a Minkowski sum is the sum of the support functions. The finite exponential identity in Corollary 5.3 proves multiplicativity. The preceding lemma and the max–min exponential valuation prove the additive polytope relation. If one polytope is empty the relation is immediate. All defining relations are consequently respected, and the assignment gives a unital algebra homomorphism $\rho : \Pi_r \to \mathrm{End}(\mathscr{M}_S)$.

By [JY16, Theorem 5.1, pp. 19–20], the map

$$[P] \longmapsto \sum_{j=0}^{r} \frac{(Dh_P)^j}{j!}$$

is an isomorphism $\Phi : \Pi_r \to \mathcal{T}_{r,\mathbf{Q}}$. Its logarithm sends $\log[P]$ to Dh_P. In the nilpotent operator algebra, $\log \exp L_{h_P \circ t} = L_{h_P \circ t}$. Therefore $\rho\Phi^{-1}$ has property (6.1). The graded component of degree c of the polytope algebra is spanned by $(\log[P])^c$, so this action is graded and its homogeneous values are finite linear combinations of the signed BT currents asserted in the theorem. Separate closedness and coefficient-Radon type follow term by term.

For real weights, use scalar extension. On any fixed finite rational fan, the real balanced weight space is the real scalar extension of its rational balanced weight space, because balancing is the kernel of a rational matrix. Every finite collection of cycles has a common finite rational subdivision. Hence $\mathcal{T}_r = \mathcal{T}_{r,\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{R}$, with its products preserved, and the action extends uniquely by real linearity. Hypersurface generation gives the claimed characterization. This proof never changes the valued field and never replaces S by a geometric or smooth background. □

Corollary 6.3 (Local affine cycles, support, and ambient refinements)**.** *Let C be a constant-weight rational polyhedral tropical cycle on a real open $W \subset \mathbf{R}^r$, with real affine offsets and real weights. There is a local action $\mathcal{A}_t(C)S$ on $t^{-1}(W)$. It is linear in C, separately closed, and coefficient-Radon, and it has support contained in $t^{-1}(|C|)$. It is independent of local fan representations and compatible with restriction, gluing, and affine rational refinements of ambient moment maps. Every equality of such ambient cycle germs gives an equality of their actions on all original compact smooth tests, including tests with additional moment coordinates.*

Proof. We first verify locality of the fan action, rather than assuming it from its global algebraic definition. Near $a \in \mathbf{R}^r$, a polytope support function satisfies

$$h_P(x) = h_P(a) + h_{F_P(a)}(x - a),$$

where $F_P(a)$ is the exposed maximizing face. Indeed the finitely many other vertices have a strict support gap at a, which persists in a sufficiently small neighborhood. The equality is exact on that neighborhood. It remains exact after moment pullback, so its differentiated operators agree there. A finite expression for a cycle as powers of hypersurfaces therefore localizes to the corresponding expression for its translated local fan. If the cycle germ is zero, that fan relation is zero, and Theorem 6.2 annihilates it. This proves locality and the support assertion.

Every affine polyhedral cycle is, near a given point, the translate of its local fan. Apply the fan theorem to that translate, using $t - a$ in place of t. On the overlap of two such neighborhoods their cycle germs agree; localizing both expressions once more at any point of the overlap reduces their equality to the same fan relation. Hence the currents agree on the overlap and glue uniquely. The same argument proves linearity and compatibility with restriction.

For an affine rational map F of real moment spaces, pullback of a hypersurface potential is literal composition with F. On a sufficiently small neighborhood it is again a convex piecewise affine function with rational gradients, with real constants allowed. The delta pullback is compatible with D and products. Hypersurface generation and the identity of the actual composed potentials prove compatibility of the actions. Refinement never restricts the final test class: the current identity can be paired with any original test after its own common refinement. □

The remaining analytic kernel is separate. The preceding results kill relations in the ambient real cycle algebra and its germs. They do not yet prove

$$t^\star C = 0 \text{ in } B(U) \quad \Longrightarrow \quad \mathcal{A}_t(C)S = 0$$

for an arbitrary original moment map on an arbitrary analytic space. A nonzero ambient cycle may have zero actual pullback. Annihilation of that additional kernel is required before the construction descends to all actual analytic delta-tests. The R5 quotient settles the corresponding differentiated potential relation in degree one; it does not automatically settle this higher-degree analytic relation. The fixed-coordinate torus case below has an independent faithfulness check.

7 Real cycle relations and pointwise polyhedral relations

The preceding rational construction has a stronger real version. The additional argument uses the presentation of piecewise polynomial functions on a simplicial fan. It also identifies a class of analytic relations that can be annihilated without approximating the background.

Lemma 7.1 (A vanishing product of continuous potentials)**.** *Let $f_1, \dots, f_m$ be continuous local bounded classical differences on an ordinary open and suppose $f_1 \cdots f_m = 0$ at every point. Then, over every allowed positive separately closed background Q,*

$$D[f_1] \wedge \cdots \wedge D[f_m] \wedge Q = 0$$

whenever the degree is allowed. The conclusion also holds on every signed mixed tail over a fixed allowed background.

Status: `DIRECT_COROLLARY` of the frozen signed recursion and ordinary locality.

Proof. Induct on m. The case $m = 1$ is immediate. For $m > 1$ let $R = D[f_2] \wedge \cdots \wedge D[f_m] \wedge Q$. It is a coefficient-Radon current. On the ordinary open $G = \{f_1 \neq 0\}$ the remaining product of actual functions is zero. Induction on that open gives $R|_G = 0$. On its complement the multiplier f_1 is zero. Thus $f_1 R = 0$ as an actual coefficient-Radon current, and its D is zero. Signed tails are handled by their finite positive-tail expansions. All of these arguments are local on ordinary opens. □

Lemma 7.2 (Finite piecewise linear functions give allowed signed slots)**.** *Every continuous piecewise linear function on a finite complete real fan is a difference of two finite convex piecewise linear functions. After composition with original unit moments it is therefore a continuous local bounded classical difference, over the original field.*

Proof. Refine the fan by the finite arrangement of all hyperplanes containing its facets. Write their linear equations as $\ell_j = 0$ and set $\rho = \sum_j |\ell_j|$. Continuity of the given function f implies that, across each adjacent pair of full-dimensional chambers, its gradient jump is a scalar

multiple of the corresponding normal. There are finitely many such jumps. For sufficiently large M, every oriented normal jump of $f + M\rho$ is nonnegative. Along a generic line, its successive slopes are then nondecreasing. By continuity, the same convexity inequality holds on every line, including lines in walls. Thus $f + M\rho$ and $M\rho$ are convex piecewise linear functions. Each is a finite maximum of linear functions. Their moment pullbacks are finite maxima of original pluriharmonic functions, hence classical psh under the full complete-extension curve definition. The actual difference is the required function. □

Lemma 7.3 (The piecewise polynomial presentation)**.** *Let Δ be a finite complete simplicial fan in $\mathbf{R}^r$. Choose one nonzero generator v_ρ of each ray and let λ_ρ be its Courant function: it is linear on each cone and takes value $\delta_{\rho\tau}$ at v_τ. Then*

$$\mathrm{PP}(\Delta) = \mathbf{R}[x_\rho : \rho \in \Delta(1)]/I_\Delta, \qquad x_\rho \longmapsto \lambda_\rho,$$

where I_Δ is generated by the squarefree monomials whose rays are not all in one cone. Here $\mathrm{PP}(\Delta)$ consists of continuous functions polynomial on every cone, with the homogeneous grading.

If $C \subset \Delta$ is a subfan, the kernel of restriction $\mathrm{PP}(\Delta) \to \mathrm{PP}(C)$ is generated by the monomials whose ray supports lie in no cone of C.

Proof. A nonface monomial vanishes pointwise, since on any cone only its own ray variables can be nonzero. Thus the displayed map factors through the stated quotient. In that quotient every polynomial is a linear combination of monomials whose support is a face. If it restricts to zero on a maximal cone σ, all monomials supported in σ have zero coefficients: the corresponding $\lambda_\rho|_\sigma$ are independent linear coordinates and a polynomial vanishing on the relative interior is the zero polynomial. Every allowed monomial occurs in such a cone. This proves injectivity, and the same argument on maximal cones of a subfan proves the asserted description of the restriction kernel.

For surjectivity take a homogeneous piecewise polynomial F and construct its expression in increasing order of face dimension. After matching all proper faces of a cone σ, its residual polynomial on σ vanishes on every facet. In its independent ray coordinates it is therefore divisible by $\prod_{\rho\subset\sigma} \lambda_\rho$. Write the quotient as a polynomial in those coordinates and use the identical polynomial in the global Courant functions, multiplied by that product. This corrects the residual on σ and vanishes on every other face of dimension at most $\dim \sigma$. Processing all faces of that dimension consequently preserves earlier corrections. At the maximal cones the residual is zero everywhere. The initial constant on the zero cone is handled by a constant polynomial. This finite induction gives surjectivity degree by degree. □

Theorem 7.4 (Piecewise polynomial functional calculus)**.** *For original moments $t : U \to \mathbf{R}^r$ and an arbitrary allowed background S, there is a canonical graded algebra homomorphism*

$$\Theta_{t,S} : \mathrm{PP}(\mathbf{R}^r) \longrightarrow \mathrm{End}(\mathscr{M}_S)$$

which sends a degree-one piecewise linear function f to $L_{f\circ t}$. The source is the direct limit over finite complete real fans and their refinements. Every global linear function lies in its kernel, and hence so does the ideal generated by such functions.

More strongly, if C is a subfan of a finite complete simplicial fan Δ, if $t(U) \subset |C|$, and if $F \in \mathrm{PP}(\Delta)$ vanishes on $|C|$, then $\Theta_{t,S}(F) = 0$.

Status: `NEW_PROVED` for the operator descent and the stated pointwise-image ideal.

Proof. On one simplicial fan assign $x_\rho \mapsto L_{\lambda_\rho\circ t}$. The operators are defined by Lemma 7.2 and commute by signed BT symmetry. A nonface monomial has zero product of the actual Courant

functions. Lemma 7.1 therefore annihilates its operator. Lemma 7.3 gives a homomorphism from $\mathrm{PP}(\Delta)$.

On a simplicial refinement, a coarse Courant function is the linear combination of the fine Courant functions given by its values at the fine ray generators. Linearity of $h \mapsto L_h$ shows that the two homomorphisms agree. Every two finite complete real fans have a common finite simplicial refinement, obtained by common subdivision and compatible triangulation of the cones. Thus these homomorphisms give the asserted direct-limit map, with no dependence on the ray normalizations. A global linear function pulled back by unit moments is original smooth pluriharmonic. Its operator on a separately closed tail is zero, so its generated ideal is killed.

Finally the kernel of restriction to C is generated by its nonface monomials, by the preceding lemma. Their actual products vanish on U because $t(U) \subset |C|$. The same zero-product lemma annihilates them. This proves the stronger assertion. Notice that it uses actual pointwise vanishing, not vanishing almost everywhere for the ambient integration current. □

Theorem 7.5 (Constant real tropical cycles act over every fixed background)**.** *The ambient cycle action and its local affine version extend from rational directions to all constant-weight real tropical cycle fans and germs. It remains linear, graded, local in the real moment domain, and compatible with real affine changes of moments. It agrees with the previous action on rational cycles. Every output is an ordinary separately closed coefficient-Radon current, and all equalities hold on all original tests.*

Status: `NEW_PROVED` using Theorem 7.4 and the following external isomorphism.

Proof. Esterov's Theorem 3.1 identifies the graded ring of continuous conewise-polynomial functions modulo the ideal generated by global linear functions with the ring of constant-weight real tropical cycle fans; see [Est12, Section 3, printed p. 12]. The identification sends a piecewise linear function f to its corner divisor. With the standard ambient calibration, that divisor is Df: on the elementary hinge $\max(0, \ell)$ both conventions assign the normal weight $d\ell$. The ring products are the same transverse, hence stable, intersection products. No rationality or lattice hypothesis appears in this real isomorphism.

The preceding theorem kills its entire defining kernel. Composing with the inverse isomorphism gives the required cycle action. It sends Df to $L_{f\circ t}$, including signed PL functions. This also proves agreement with the rational action, because both are generated by these divisors. Homogeneous degree c piecewise polynomials are finite linear combinations of products of c Courant functions. Their values are consequently finite signed BT currents, giving the asserted closedness and Radon properties.

Locality and affine-germ gluing follow as in the rational case: localize each PL generator to its exact piecewise linear germ at the point in question and discard its constant value. The localized polynomial is the local cycle germ under the corner-locus isomorphism. A zero germ is annihilated by the same ring relation. Composition of potentials proves compatibility with real affine moment changes. Again, this is an ambient relation theorem; it does not discard the remaining actual analytic pullback kernel described above. □

An exact additional criterion for analytic relations. On a local moment image contained in a subfan C, every piecewise polynomial representative in

$$\ker(\mathrm{PP}(\Delta) \to \mathrm{PP}(C)) + (\text{global linear functions})$$

has zero relative action over every S. The first ideal is killed by actual zero products; the second by pluriharmonicity. It remains a separate question whether every actual zero cycle-presentation on a given analytic chart lies in this sum of ideals. The statement here is an

independently proved sufficient criterion, not an assumption that the local tropical image has a Euclidean or matroidal intersection ring.

8 An arbitrary-background action for fixed-coordinate tropical cycle tests

The ambient cycle action proved in Theorem 6.2 and Corollary 6.3 will be denoted $C \mapsto \mathcal{A}_C(S)$. We use its stated linearity, locality, separate closedness, coefficient-Radon property, support property, and invariance under equality of ambient constant-weight rational cycles. Its hypersurface action is $\mathcal{A}_{D\varphi}(S) = D(\varphi S)$. For signed cycles the action is interpreted by the proved finite polarization of positive classical-potential products.

8.1 Coefficients and relations on all polyhedral strata

Theorem 8.1 (Piecewise smooth coefficient relations)**.** `NEW_PROVED`, relative to the ambient cycle action. *Let $t_f : U \to W \subset \mathbf{R}^r$ be an original moment map and let S be any allowed fixed positive separately closed background. Let $C_1, \dots, C_N$ be constant-weight rational cycles of the same codimension c on W, and let $\eta_i \in PS^{a,b}(W)$. If*

$$\sum_{i=1}^{N} \eta_i \wedge [C_i] = 0 \quad \text{as real delta-forms on } W, \tag{8.1}$$

then the well-defined piecewise coefficient products satisfy

$$\sum_{i=1}^{N} t_f^* \eta_i \wedge \mathcal{A}_{C_i}(S) = 0 \quad \text{as original CLD currents on } U. \tag{8.2}$$

The equality is tested against all original complementary compact smooth forms, including tests requiring further moment refinements. Only degrees within the dimension of U are asserted.

Proof. Put $T_i = \mathcal{A}_{C_i}(S)$. These are separately closed coefficient-Radon currents, not necessarily positive. Their piecewise smooth coefficient products are defined by Theorem 4.3.

The assertion is local over a compact moment stage. Choose a common finite polyhedral complex subordinate to all the C_i and η_i on a neighborhood of that stage. All arguments below can be localized to slightly smaller stages, so artificial outer faces of the chosen complex play no role. Write $w_{i\tau}$ for the constant weight of C_i on a cell τ of dimension $r - c$, with zero weight when that cell is absent from C_i.

For a face F of this complex let

$$E_F = t_f^{-1}(\operatorname{relint} F).$$

These are Borel sets. If L_F is the affine span of F, choose affine equations $\ell_j = 0$ for L_F. The original smooth functions $\ell_j \circ t_f$ vanish on E_F. Lemma 4.1, applied separately to each T_i, gives

$$d'(\ell_j \circ t_f) \wedge \mathbf{1}_{E_F} T_i = d''(\ell_j \circ t_f) \wedge \mathbf{1}_{E_F} T_i = 0.$$

Thus, on this stratum, the multiplier $t_f^* \eta_i$ depends only on the restriction of η_i to L_F. In particular all terms vanish if $\dim F < \max(a, b)$.

Otherwise choose constant tangent coordinate forms on L_F and write, on F,

$$\eta_i|_F = \sum_{I,J} c_{i,IJ}\, \vartheta_{IJ}.$$

These scalar coefficients are smooth on F, locally bounded, and Borel after pullback. For every incident cell τ of dimension $r-c$, relation (8.1) implies

$$\sum_i w_{i\tau}\eta_i|_\tau = 0.$$

Restrict this equality to $F \subset \tau$. For each fixed tangent tensor component, the vector $c_{IJ}(x) = (c_{1,IJ}(x), \dots, c_{N,IJ}(x))$ lies in the kernel of the fixed real matrix

$$M_F = (w_{i\tau})_{\tau \supset F,\ i}.$$

The case in which no such τ occurs causes no problem: all T_i vanish on a neighborhood of the stratum by their support property.

Choose a fixed basis $b^1, \dots, b^m$ of $\ker M_F$. There are locally bounded Borel scalar functions $q_{IJ,l}$ on F such that

$$c_{IJ}(x) = \sum_l q_{IJ,l}(x) b^l.$$

For each l, the cycle $\sum_i b_i^l C_i$ vanishes on an ambient ordinary neighborhood W_F of relint F: remove the finitely many nonincident closed top cells and use the zero weights on all incident cells. Local ambient relation invariance therefore gives

$$\sum_i b_i^l T_i = 0 \quad \text{on } t_f^{-1}(W_F).$$

This is an equality of coefficient-Radon currents. Restricting it to E_F, multiplying by the Borel scalar $q_{IJ,l} \circ t_f$, and then by the constant tangent form $t_f^* \vartheta_{IJ}$ are legitimate coefficient-measure operations. Their finite sum proves

$$\mathbf{1}_{E_F} \sum_i t_f^* \eta_i \wedge T_i = 0.$$

Finally sum over the finitely many relevant strata. Since every compact original test is covered by such local stages, this proves (8.2) on every original test. No E_F was treated as an ordinary open set, and no cycle was inserted into a nonclosed coefficient-weighted background. □

8.2 A full-coordinate torus domain removes the pullback kernel

Let $\Omega \subset \mathbf{R}^n$ be ordinary open and set

$$U = t^{-1}(\Omega) \subset \mathbf{G}_{m,K}^{n,\mathrm{an}}, \qquad t = (\log|T_1|, \dots, \log|T_n|).$$

We keep this single coordinate moment presentation. The use of log instead of $-\log$ is a simultaneous real affine reflection; all weights and differential operations are the original ones. Define $\mathscr{B}_t(U)$ as the image under $t^\star$ of those real delta-forms on Ω which are locally finite sums $\sum_i \eta_i \wedge [C_i]$ of the indicated kind. Its elements are therefore locally presented over coordinate open subsets by

$$\sum_i t^* \eta_i \wedge t^\star [C_i],$$

where η_i are piecewise smooth superforms and C_i are constant-weight rational cycles. Different tropical codimensions are allowed and are treated in their separate trigraded summands. This is an explicitly specified subalgebra of actual $B(U)$. It is not claimed to be every actual delta-form on U.

Theorem 8.2 (A fixed-coordinate delta-test action)**.** `NEW_PROVED`, with the stated test scope. *For every allowed fixed background S of bidegree (p,p) on U, the prescription*

$$\mathcal{A}_S\left(\sum_i t^*\eta_i \wedge t^\star[C_i]\right) := \sum_i t^*\eta_i \wedge \mathcal{A}_{C_i}(S) \tag{8.3}$$

is independent of the displayed presentation and defines a linear coefficient-Radon current of the expected total bidegree. It is compatible with restriction from Ω to its real open subsets and thus from U to the corresponding saturated torus domains. The independence is equality of original currents on all original tests, not merely equality in the chosen tropical coordinates.

In complementary total degree $(n-p,n-p)$ it gives a well-defined pairing with every compactly supported test in $\mathscr{B}_t$. If S is represented by an actual delta-form σ, this pairing is the usual $\int_U \sigma\wedge\beta$.

For continuous convex rational PL functions $u_j=\varphi_j\circ t$, cycle insertion recovers the relative BT products. More precisely, whenever degrees permit and $\beta\in\mathscr{B}_t$,

$$\mathcal{A}_{T_{S,k}(u_1,\dots,u_k)}(\beta) = \mathcal{A}_S\big((D_\delta u_1)\wedge\cdots\wedge(D_\delta u_k)\wedge\beta\big).$$

Thus identities between these actual fixed-coordinate delta-tests remain valid after relative BT insertion over an arbitrary S.

Proof. The standard skeleton of U is canonically identified with Ω, with calibration one. Restricting a presented pullback to this skeleton is the original real delta-form on Ω. Consequently, if two displayed expressions agree as actual delta-forms on U, their real expressions agree on Ω. Apply Theorem 8.1 to their difference, separately in each codimension. This proves presentation independence. The restriction and Radon assertions follow from those of the cycle action and piecewise multiplication.

When the test has compact support, its real expression has compact support in Ω: a point of the support of the real expression gives a support point on the standard skeleton, lying in the compact support of the actual test. Choose a smooth real compact cutoff equal to one near that real support. Ambient locality and the already proved coefficient relations imply that the output is supported over the support of this cutoff. The inverse image of a compact subset of Ω under the torus tropicalization is compact, being a closed subset of a compact polyradius annulus. Thus in complementary degree the output is a compactly supported top-degree signed Radon measure, whose mass defines the pairing. If $S=[\sigma]$, the recursion comparison in Theorem 2.4, finite polarization, and multiplication by piecewise coefficients identify each term with its actual delta product. This proves the comparison with delta integration.

Finally $D_\delta u_j$ is the pullback of the rational tropical hypersurface $D\varphi_j$. The ambient polytope action is multiplicative on cycle factors and its hypersurface insertion is $D(u_j\,\cdot)$. The displayed compatibility follows by applying these identities before multiplying by the piecewise coefficient of each summand of β. All inserted intermediate backgrounds are separately closed signed mixed currents for which that cycle action was constructed; no operation on an arbitrary nonclosed current is used. □

The remaining actual-pullback gate. Even on the torus, the saturated-domain statement above does not by itself prove ordinary sheaf locality relative to the *actual* support of a delta-test. An ordinary open subset of U can miss the standard skeleton and need not be a saturated coordinate domain. For example $D\max(0,\log|T|)$ on a curve annulus has actual support consisting only of its Gauss point, whereas $t^{-1}\{0\}$ also contains residue branches. Its relative action does vanish on those branches, because its potential is locally constant there; this special observation is not a general pullback-kernel theorem. We have proved compact

saturated support and presentation independence, not the missing ordinary sheaf locality assertion in general.

For a general moment map $t_f : U \to \mathbf{R}^r$, especially with $r > \dim U$, an actual delta-form can vanish on U although its ambient expression is nonzero. Theorem 8.1 kills *ambient* relations. To descend through this additional pullback kernel one needs a further argument. The standard full-coordinate torus domain above has no such kernel because of its standard skeleton. A delta-form involving additional units, such as $T_i - 1$, is not automatically in $\mathscr{B}_t$; enlarging the moment tuple returns to the additional pullback problem. No extension to all actual Mihatsch tests, all GK generalized preforms, or arbitrary nonclosed backgrounds is claimed here.

9 Actual cycle relations on saturated matroidal tropical charts

This section supplies a geometric condition under which an actual analytic pullback relation is stronger than an ambient cycle relation, but is still annihilated over every allowed background. The condition concerns one fixed moment chart and its calibrated tropical image. No existence of such a chart on an arbitrary Berkovich space is asserted.

Definition 9.1 (A saturated matroidal chart)**.** Let U be an allowed pure n-dimensional space over the original K, and let $t = t_f : U \to \Omega \subset \mathbf{R}^r$ be original unit moments. The following data will be called a saturated matroidal chart.

1. t is proper onto its image $C \subset \Omega$, and C, with its induced integral affine structure, is a boundaryless rational tropical manifold of dimension n.
2. There is a positive constant m such that the following calibrated affinoid condition holds locally over every point of C. One can choose arbitrarily small compact polyhedral neighborhoods $P \Subset \Omega$, with the point in P°, for which $A_P = t^{-1}(P)$ is affinoid,

$$t\big(\Sigma(A_P, f) \cap \partial A_P\big) \subset \partial P, \qquad T(A_P, f) = m[C] \quad \text{on } P^\circ.$$

   Here $T(A_P, f)$ and its weights are exactly the original calibrated tropicalization of Mihatsch, Definition 3.4 and equation (3.8).

The constant m can equally be allowed to be a locally constant positive function on C, by making the assertions local over Ω. Restriction to a smaller real open subset of Ω always means restriction of U to its full inverse image.

The hypotheses explicitly retain the proper fibers. They do not say that U is its skeleton, or that an arbitrary ordinary subopen of U has the same calibrated tropical image.

Proposition 9.2 (The required tropical duality)**.** `EXTERNAL_KNOWN`. *If L is a rational tropical linear space, then*

$$\mathrm{PP}(L)/\mathrm{LPP}(L) \longrightarrow Z_*^{\mathrm{fan}}(L), \qquad P \longmapsto P \cdot [L]$$

is an isomorphism. Here the source is the direct limit over rational fan subdivisions, and LPP *is the ideal generated by global linear functions. The local affine version is an isomorphism of sheaves on every boundaryless rational tropical manifold. These statements remain true after tensoring the integral groups with* $\mathbf{R}$*.*

Source and normalization. These are Gross–Shokrieh, Lemma 4.4 and Theorem 4.5, [GS21, printed pp. 13–14]. Their Section 4.3, printed pp. 12–13, defines the local affine piecewise polynomial sheaf and quotients by the ideal generated by affine functions; Remark 4.2 explains why the affine convention is needed at points with nonintegral coordinates. Their Lemma 4.1

identifies the degree-one action with tropical divisor intersection. In our maximum convention this is $D\varphi$; a simultaneous sign reflection in the real moment coordinates changes neither the assertion nor its calibration. Tensoring is exact over $\mathbf{Z}$ with the torsion-free, hence flat, module $\mathbf{R}$. Equivalently, on each fixed finite rational fan the relevant polynomial and balanced-weight spaces are obtained from the integral ones by scalar extension. This proves the stated real-weight version. □

Lemma 9.3 (From an actual zero form to a tropical zero cycle)**.** `DIRECT_COROLLARY` *of the actual delta-form definition. On a saturated matroidal chart, let γ be a real delta-form on Ω. If $t^\star\gamma = 0$ as an actual analytic delta-form on U, then*

$$\gamma \wedge [C] = 0$$

locally on C. This assertion applies to forms with piecewise smooth coefficients as well as to constant-weight cycles.

Proof. Mihatsch's Lemma 4.4(5), [Mih24, printed p. 62], applied with the identity refinement of f, gives

$$T(A_P, f) \wedge \gamma = 0$$

away from $t(\Sigma(A_P, f) \cap \partial A_P)$. On P° the two chart assumptions identify this equality with $m[C] \wedge \gamma = 0$. Since $m > 0$, it gives the claimed equality. These interiors cover C. This is a consequence of the full actual delta-form vanishing; it does not test only one chosen skeleton. □

Lemma 9.4 (Annihilation of relative cycle relations)**.** `NEW_PROVED`*, using the external tropical duality. Let B be a constant-weight rational cycle of codimension c on Ω, with real weights. If $B \wedge [C] = 0$ on a relative open subset $C \cap W$, then*

$$\mathcal{A}_B(S) = 0 \quad \text{on } t^{-1}(W)$$

for every allowed fixed positive separately closed background S. The same assertion holds for finite real linear relations between such cycles, and on signed mixed tails over S.

Proof. Work near a point $b \in C \cap W$. Translate b to zero. The local cone of C is integrally linearly isomorphic, on its ambient span, to a rational tropical linear space L: the chart uses the induced affine sheaf, so both directions of this identification are locally restrictions of ambient affine maps. We make this identification; the germ of B is a rational fan cycle B_b. By the ambient cycle presentation, choose a homogeneous degree-c piecewise polynomial P on a finite complete rational fan whose corner-cycle class is B_b. Refine to a simplicial fan adapted to L. Compatibility of divisor intersection gives

$$(P|_L) \cdot [L] = B_b \wedge [L] = 0.$$

Proposition 9.2 implies, after a finite rational subdivision, that

$$P|_L = \sum_{j=1}^{N} \ell_j Q_j,$$

where ℓ_j are restrictions of ambient real linear functions and Q_j are homogeneous piecewise polynomials of degree $c-1$ on L. For $c = 0$ the assertion instead says directly that $P = 0$ on L and no sum is needed.

Extend the chosen finite subdivision of L to a finite complete simplicial ambient subdivision. This can be done by first taking a finite hyperplane arrangement refining its cones and then a

compatible simplicial subdivision. Each Q_j extends to an ambient piecewise polynomial $\widetilde{Q}_j$: in the Courant presentation, keep its face-supported monomials and assign zero coefficients to the remaining monomials. Thus

$$P - \sum_j \ell_j \widetilde{Q}_j$$

vanishes pointwise on L. The pointwise-image ideal in Theorem 7.4 annihilates its operator on the inverse image of a sufficiently small neighborhood of b. The operators of the terms $\ell_j \widetilde{Q}_j$ are zero because the original moment pullback of ℓ_j is pluriharmonic. This proves $\mathcal{A}_B(S) = 0$ there. Local ambient cycle invariance permits these identities to glue over $t^{-1}(W)$. The argument takes place in the module of signed mixed tails, so the last assertion and real linearity follow at the same time. □

Theorem 9.5 (A fixed-chart actual delta-test action)**.** `NEW_PROVED`*, on the stated chart and test class. On a saturated matroidal chart let* $\mathscr{B}_f^{\mathrm{rat}}(U)$ *be the image of those real delta-forms on* Ω *which locally are finite sums*

$$\gamma = \sum_i \eta_i \wedge [B_i],$$

where η_i *are piecewise smooth forms and* B_i *are constant-weight rational cycles with real weights. For every allowed fixed background* S *the prescription*

$$\mathcal{A}_S(t^\star \gamma) = \sum_i t^* \eta_i \wedge \mathcal{A}_{B_i}(S)$$

is independent of the displayed presentation as an actual *delta-form on* U*. It is linear and coefficient-Radon, and all its identities hold on every original complementary compact smooth test. It is compatible with restriction to saturated subdomains. In complementary total degree it gives a pairing with every compactly supported test in* $\mathscr{B}_f^{\mathrm{rat}}(U)$*.*

For a background represented by an actual delta-form this pairing agrees with the full actual delta product and its integration. Insertion of continuous convex rational PL potentials in the fixed moments agrees with relative BT insertion. No assertion identifies $\mathscr{B}_f^{\mathrm{rat}}(U)$ *with all of* $B(U)$*.*

Proof. It suffices to annihilate a displayed actual zero relation, one tropical codimension at a time. By Lemma 9.3, that relation gives

$$\sum_i \eta_i \wedge Z_i = 0, \qquad Z_i = B_i \wedge [C].$$

These Z_i are constant-weight rational cycles of dimension $n - c$ on C. Put $R_i = \mathcal{A}_{B_i}(S)$. Lemma 9.4 shows that R_i vanishes over every real open subset on which Z_i vanishes. In particular its support is contained in $t^{-1}(|Z_i|)$. It also shows that every constant local relation among the Z_i gives the identical relation among the R_i.

Choose one finite common polyhedral subdivision on a compact moment stage, subordinate to the Z_i and all coefficients. For every face F, the Borel current $\mathbf{1}_{t^{-1}(\operatorname{relint} F)} R_i$ is tangent to the affine span of F, by Lemma 4.1. Consequently only the tangent restrictions $\eta_i|_F$ matter. Expand these restrictions in constant tangent tensor components. For each component its scalar coefficient vector is in the kernel of the fixed matrix of weights of the Z_i on the incident $(n - c)$-dimensional cells. This follows by restricting the displayed zero relation from those cells to F.

Choose a fixed basis of that finite-dimensional kernel. Each basis vector defines a constant cycle relation among the Z_i on a real neighborhood of $\operatorname{relint} F$, after discarding the finitely

many nonincident closed top cells. The preceding lemma annihilates the identical combination of the R_i. Restrict that current equality to the Borel stratum and multiply by the bounded Borel scalar coordinates of the coefficient vector in the chosen kernel basis. Summing tangent components, then faces, proves

$$\sum_i t^* \eta_i \wedge R_i = 0.$$

This is the same coefficient-measure argument as in Theorem 8.1, with the relative cycles Z_i in place of the ambient cycles. It does not insert a cycle into a nonclosed coefficient-weighted background.

For compactly supported $\beta = t^\star \gamma$, the support of $\gamma \wedge [C]$ is contained in the compact set $t(\operatorname{supp} \beta)$. Indeed, outside that set the actual form vanishes on a saturated open, and the detector gives the claimed tropical vanishing. The relation argument just proved then shows that $\mathcal{A}_S(\beta)$ has support in $t^{-1}(t(\operatorname{supp} \beta))$, which is compact by properness. Its top-degree mass therefore defines the stated pairing. The remaining comparison and insertion assertions follow termwise from the ambient cycle action, Theorem 2.4, and canonical piecewise coefficient multiplication. □

Example 9.6 (Products of three-punctured lines)**.** For every $n \geq 1$, take

$$X = (\mathbf{A}^1_K \setminus \{0, 1\})^n, \qquad f = (z_1, 1 - z_1, \ldots, z_n, 1 - z_n),$$

and a real open $\Omega \subset \mathbf{R}^{2n}$. The saturated domain $U = t_f^{-1}(\Omega)$, if nonempty, has the chart data above with $m = 1$. In one factor the image is the three-ray fan

$$\{(s, 0) : s \leq 0\} \ \cup \ \{(0, s) : s \leq 0\} \ \cup \ \{(s, s) : s \geq 0\}.$$

On its first two rays the parameters z and $1 - z$, respectively, give degree-one annulus coordinates; on the third ray use z. The primitive slopes are $(1, 0)$, $(0, 1)$ and $(1, 1)$ up to the outward signs, so the original calibrated weights are one. All radii are ordinary Berkovich Gauss radii; no extension or rescaling of K is used. Taking products gives the product fan with product calibration one. It is a tropical linear space, as is also explicit in [GS21, Lemma 3.1].

The map is the closed immersion into the torus defined by $T_{2i-1} + T_{2i} = 1$. Inverse images of compact coordinate boxes are therefore affinoid closed subspaces of compact polyannuli, and the moment map is proper. Boxes with a given point in their interior can be chosen inside Ω. Their analytic boundary maps to the box boundary: the closed immersion is boundaryless, and the polyannulus boundary lies over that boundary. The calibrated tropicalization on the box interior is the indicated product fan. Thus the theorem applies to arbitrary allowed positive separately closed S on these n-dimensional domains. In dimensions greater than one this allows genuinely non-smooth backgrounds; it is not a theorem only about currents coming from the displayed curves.

The remaining scope. The theorem is about actual relations on each fixed saturated chart. Its proof uses the whole calibrated tropical image in every compact stage. An arbitrary ordinary subopen may omit part of a tropical fiber and fail these hypotheses. Ordinary sheaf locality for all such subopens, arbitrary analytic moment images, and tests outside the indicated constant-cycle-generated class remain separate questions. The external tropical Poincare duality does not assert that a general analytic space admits matroidal charts or that its actual delta-form sheaf is generated by this one presentation.

10 Every real delta-form is locally generated over piecewise smooth forms

This section proves an exact finite generation statement. Its proof uses a decomposition of smooth coefficients by their restrictions to coordinate faces. It uses no polynomial density, formal Taylor series, or convergence against singular tests. In particular a smooth flat remainder is retained exactly.

Lemma 10.1 (Anchored face decomposition)**.** *Let J be a finite set and let F be smooth on a neighborhood of the closed coordinate cube $[0,\varepsilon)^J$. For $A \subset J$ put*

$$F_A(x_A) = \sum_{B \subset A} (-1)^{|A|-|B|} F(x_B, 0_{J\setminus B}).$$

Then

$$F(x) = \sum_{A \subset J} F_A(x_A).$$

Every F_A is smooth and vanishes whenever $x_a = 0$ for one $a \in A$. These formulas commute with every fixed real linear relation between the functions involved. If one first restricts F to a coordinate face containing A, its anchored A-component remains F_A.

Proof. In the sum over A, the coefficient of the evaluation $F(x_B, 0_{J\setminus B})$ is $\sum_{A: B \subset A \subset J} (-1)^{|A|-|B|}$, which is zero unless $B = J$, when it is one. This proves the identity. If $x_a = 0$, pair the subsets B not containing a with $B \cup \{a\}$; their evaluations agree and their signs are opposite. Smoothness, linearity, and compatibility with the indicated restrictions follow directly from the finite formula. □

Theorem 10.2 (Local piecewise smooth module generation)**.** `NEW_PROVED` *by the finite construction below; no claim of external priority is made. Let T be a real delta-form of tridegree (a,b,c) on an open subset of $\mathbf{R}^r$, in Mihatsch's balanced-polyhedral-current sense. Every point has a neighborhood on which*

$$T = \sum_{j=1}^{N} \eta_j \wedge C_j, \tag{10.1}$$

where η_j is a piecewise smooth (a,b)-form and C_j is a constant-weight real tropical cycle of codimension c. The sum is finite. If the supporting polyhedra have rational directions, the C_j can be chosen with rational directions. All weights and coefficients may be real.

The equality holds in the real delta algebra itself and hence survives its pullbacks and refinements. No positivity, closedness, or diagonal-degree hypothesis is imposed on T.

Proof. Fix the point, translate it to zero, and take a finite polyhedral presentation near it. Discard the closed cells not incident to zero by shrinking the neighborhood. Each remaining cell agrees there with its local cone. Refine these cones to a finite complete simplicial fan Δ; cones off the original support have coefficient zero. This is done only in the real moment space and changes neither the current nor its weights. Put $d = r - c$ and choose weights μ_σ for the d-dimensional cones, absorbing any scalar factors into their coefficient forms.

Choose one nonzero generator v_ρ for each ray and let λ_ρ be the corresponding Courant function. On a cone σ, its own ray functions are independent linear coordinates. Choose one ordering of all rays. There are unique smooth coefficient functions $F_{\sigma,I,J}$ such that

$$\alpha_\sigma = \sum_{\substack{I,J \subset \sigma(1) \\ |I|=a,\ |J|=b}} F_{\sigma,I,J}(\lambda_{\sigma(1)})\, d'_P\lambda_I \wedge d''_P\lambda_J\big|_\sigma.$$

Their smoothness up to all coordinate faces follows from the definition of a smooth form on a closed polyhedron. There are only finitely many functions. Choose a common small $\varepsilon > 0$ such that all positive coordinate cubes $0 \leq \lambda_\rho < \varepsilon$ under consideration, together with every point obtained by setting any coordinates to zero, remain inside the original presentation neighborhood. Each coefficient then has a smooth extension to an open neighborhood of its coordinate cube.

Apply Lemma 10.1 to every $F_{\sigma,I,J}$. Denote its anchored A-component by $F_{\sigma,I,J,A}(x_A)$. Fix a triple (A, I, J) which occurs in at least one cone, and let τ be the face spanned by $A \cup I \cup J$. For each d-cone $\sigma \supset \tau$ put

$$f_\sigma(x_A) = F_{\sigma,I,J,A}(x_A),$$

using zero when the coefficient is absent. If A is empty these are simply constants.

We claim that, for every $x_A \in [0, \varepsilon)^A$, the vector $(f_\sigma(x_A))_{\sigma \supset \tau}$ is a balanced constant weight vector on the star of τ. Let κ be a $(d-1)$-cone containing τ. The balancing criterion for T, [Mih23, Definition 3.2 and Theorem 3.3], says

$$\sum_{\sigma \supset \kappa} \alpha_\sigma|_\kappa \otimes n_{\sigma,\kappa} \in A^{a,b}(\kappa) \otimes N_\kappa.$$

Compare the coefficients of $d'_P\lambda_I \wedge d''_P\lambda_J$ in the independent coordinate basis on κ. Indices outside κ restrict to zero and contribute nothing. The remaining identity is a fixed linear relation between smooth functions of the coordinates on κ, with values in $\mathbf{R}^r/N_\kappa$. Apply its anchored A-projector. Because $A \subset \kappa$, compatibility with restriction in Lemma 10.1 gives exactly

$$\sum_{\sigma \supset \kappa} f_\sigma(x_A) n_{\sigma,\kappa} \in N_\kappa. \tag{10.2}$$

This holds throughout the indicated cube, including its faces. It proves the claim.

Let K_τ be the finite-dimensional real kernel defined by all equations (10.2). Choose a fixed basis $b^1, \ldots, b^s$. There are smooth functions $g_1, \ldots, g_s$ on the A-cube such that

$$(f_\sigma(x_A))_\sigma = \sum_{l=1}^{s} g_l(x_A) b^l.$$

Indeed choose a fixed linear coordinate map on K_τ and apply it to the vector of smooth functions. Its coordinates extend smoothly to a neighborhood of the cube. Since every f_σ vanishes when any A-coordinate is zero, the same is true of every g_l on those coordinate faces.

Define the constant cycle

$$C_{\tau,l} = \sum_{\sigma \supset \tau} b^l_\sigma [\sigma + N_\tau, \mu_\sigma].$$

The projected cones σ/N_τ form the star fan. The lifted cones $\sigma + N_\tau$ have the same tangent spaces and the same weights as the original σ. Their facet normal equations are precisely those defining K_τ, so $C_{\tau,l}$ is balanced. This includes the case $\dim \tau = d$, when the quotient is zero-dimensional. There is no lattice index or change of calibration: choosing a weight on N_τ, dividing by it in the quotient, and multiplying by that same weight in the lift leaves μ_σ unchanged.

On a smaller neighborhood of zero put

$$\eta_{A,I,J,l} = g_l(\lambda_A) d'_P\lambda_I \wedge d''_P\lambda_J.$$

It is piecewise smooth. The Courant functions are continuous PL, their piecewise differentials agree tangentially on all faces, and composition with the displayed smooth functions has the same compatibility. Its tangent restriction to any cone missing a ray of τ is zero: a missing ray

in A sets one coordinate of g_l to zero, whereas a missing ray in I or J makes the corresponding differential zero.

We now verify the exact recovery. On the open star of τ, the lifted cone $\sigma + N_\tau$ agrees with σ wherever the coefficient form can contribute. More explicitly, if x is in the relative interior of a cone σ' containing τ and its quotient lies in σ/N_τ, that quotient lies in the relative interior of σ'/N_τ. The star fan intersection property implies $\sigma' \subset \sigma$. Thus on the top-dimensional pieces one obtains exactly the original cone σ and its weight. On every other cone the tangent coefficient is zero by the preceding paragraph. Lower-dimensional intersections do not change a polyhedral current. It follows that

$$\sum_l \eta_{A,I,J,l} \wedge C_{\tau,l} = \sum_{\sigma \supset \tau} f_\sigma(\lambda_A) d'_P \lambda_I \wedge d''_P \lambda_J \wedge [\sigma, \mu_\sigma].$$

Summing over the finitely many triples (A, I, J) and applying the anchored decomposition recovers every α_σ, hence T. All constructions took place in the chosen small neighborhood, so no extension of the smooth coefficients to an entire fan was assumed.

If the original support directions were rational, complete and simplicial subdivisions can be chosen rational, and the star lifts preserve rational directions. Finally equality of the resulting polyhedral currents is equality in the real delta algebra, by its definition and the refinement equivalence of [Mih23, Remark 2.3]. This proves all the assertions, including arbitrary tridegrees. □

Corollary 10.3 (Exact local generation of actual analytic delta-forms)**.** `DIRECT_COROLLARY` *of Theorem 10.2 and the definition of Mihatsch's actual analytic delta-forms. On every original analytic moment neighborhood, each actual delta-form is locally a finite sum of a piecewise smooth form times pullbacks of constant real cycles. By the real cycle presentation, the latter factors can in turn be expressed as finite real linear combinations of products of $D_\delta\varphi$, where φ is a continuous PL difference of classical psh potentials in the same original moments.*

Consequently, on any class of backgrounds for which the constant-cycle-generated action has been proved to annihilate all actual analytic relations*, that action covers the whole actual delta-test sheaf. The generation result does not itself prove that relation-annihilation property for an arbitrary background or an arbitrary analytic moment image.*

Relation to the fixed-chart theorems. For a full-coordinate saturated torus domain, the ambient real cycle action and its coefficient-relation proof apply to every delta-form in the fixed coordinate presentation. For the saturated matroidal theorem, the present generation argument applies to its rational-direction test class. The matroidal duality input used there is a rational fan theorem; no additional nonrational relative duality is inferred here. Both chart statements retain their explicit restriction to saturated domains until ordinary sheaf locality is established by a separate argument.

11 Degree-one delta relations over every allowed background

The weak approximation hypothesis is unnecessary in total degree $(1, 1)$. The reason is an order property specific to curvature currents of continuous scalar potentials. Freezing a continuous coefficient produces an arbitrarily small *order error*; it need not produce an exact constant relation on a neighborhood.

All domains in this section are ordinary opens in the analytic spaces allowed by B, over the original finite $K/\mathbf{Q}_p$. The positive cone and tests are the original ones. We use R, Theorem 2.9, through B's authorized ordinary local algebraic localization. Thus a finite continuous potential

with positive original curvature has locally uniform original smooth positive approximants. No extension of this input to a new base field is being made.

Lemma 11.1 (Continuous curvature order transfers to a fixed background)**.** *Let S be an arbitrary allowed symmetric positive separately closed (p,p)-current, with $p<n$. If h is finite continuous and $D[h]\geq 0$ on an ordinary open, then*

$$D(hS)\geq 0.$$

In particular, if $g_1,\dots,g_m$ are continuous classical psh, $G=\sum_i g_i$, and $c_i\in\mathbf{R}$ satisfy

$$-\varepsilon D[G]\leq D\left[\sum_i c_i g_i\right]\leq \varepsilon D[G],$$

then the same inequalities hold after replacing $D[g]$ by $D(gS)$.

Status: `DIRECT_COROLLARY` of the authorized continuous regularization theorem R and fixed coefficient multiplication.

Proof. Locally choose $h_j\to h$ uniformly with Dh_j original-positive smooth forms. Since S is coefficient-Radon, $h_jS\to hS$ on every original test. Separate closedness gives $D(h_jS)=Dh_j\wedge S\geq 0$. Applying the continuous distributional operator D and passing to the limit gives $D(hS)\geq 0$. The assertion is local and hence glues. Apply it to the two continuous potentials $\varepsilon G\pm\sum_i c_i g_i$ for the last statement. This argument uses uniform smooth approximation directly and does not require a new classicality equivalence. □

Theorem 11.2 (Continuous-coefficient degree-one relation descent)**.** *Let $g_1,\dots,g_m$ be continuous classical psh functions and let $f_1,\dots,f_m$ be finite continuous real functions on the same ordinary domain. Set*

$$A=\sum_i f_i D[g_i],\qquad A_S=\sum_i f_i D(g_iS).$$

These are actual coefficient-Radon currents. Then

$$A\geq 0\quad\Longrightarrow\quad A_S\geq 0.$$

Consequently $A=0$ implies $A_S=0$ on all original tests. The last implication also holds when each g_i is a continuous local difference of two continuous classical psh functions.

Status: `NEW_PROVED` by the local order argument below. The background is arbitrary and fixed; no membership in $\mathcal{W}$ is assumed.

Proof. Put $M_i=D[g_i]$, $T_i=D(g_iS)$, and $T=\sum_i T_i$. All M_i,T_i are positive and separately closed. Fix $\varepsilon>0$ and a point x. There is an ordinary neighborhood O_x on which $|f_i-f_i(x)|<\varepsilon$ for every i. Write $c_i=f_i(x)$. If $A\geq 0$, positivity of the coefficient measures gives on O_x

$$\sum_i c_i M_i\geq -\varepsilon\sum_i M_i.$$

The preceding lemma transfers this order inequality to $\sum_i c_i T_i\geq -\varepsilon T$. Continuous scalar multiplication of the positive T_i also gives

$$A_S=\sum_i c_i T_i+\sum_i (f_i-c_i)T_i\geq -2\varepsilon T\quad\text{on } O_x.$$

For a fixed compact original positive test choose a finite subcover of its support by such O_x and a nonnegative original smooth partition subordinate to that cover. Applying the local

inequality to the partitioned tests and summing gives $\langle A_S, \eta\rangle \geq -2\varepsilon\langle T, \eta\rangle$. Let $\varepsilon \downarrow 0$. This proves positivity on every original positive test, with the same cone convention as for S. If $A = 0$, apply the implication both to A and to $-A$. The positive test cone spans the symmetric test space; all these currents are symmetric. Thus $A_S = 0$ on every original test.

For local differences expand each $g_i = g_i^+ - g_i^-$ and replace the corresponding coefficient by f_i and $-f_i$. On any compact test support only finitely many local expansions and a partition are needed. The already proved statement applies. No freezing step asserted that $\sum_i c_i D[g_i]$ vanishes on a neighborhood. Only its two-sided small order bound was needed. □

Lemma 11.3 (Local continuous piecewise smooth potentials are differences)**.** *Every finite continuous piecewise smooth scalar in a finite original moment presentation is locally a difference of two continuous piecewise smooth classical psh functions. The same holds for a finite product of such scalars.*

Proof. It suffices to prove a real local statement before the moment pullback. Work on a convex ball with compact closure in the presentation domain, and refine the finite subdivision by all its facet hyperplanes. A continuous piecewise smooth function F has uniformly bounded Hessians on the finitely many closed pieces. Across a facet its tangential derivatives agree, so its gradient jump is normal to the facet and is uniformly bounded there. Choose affine defining functions ℓ_H for the finitely many hyperplanes, with normalized gradients. For sufficiently large positive constants A, B, the function

$$Q(x) = A|x|^2 + B\sum_H |\ell_H(x)|$$

is convex, and $F + Q$ has positive semidefinite Hessian in every open piece and nonnegative normal derivative jumps across every facet. Restriction to a generic segment therefore has nonnegative second derivative on each subinterval and nonnegative first-derivative jumps. It is convex. Continuity and approximation of segments give convexity for all segments in a smaller ball. Hence $F = (F + Q) - Q$ is a difference of convex continuous piecewise smooth functions there.

Convex real functions of original logarithmic unit moments are classical psh: one may mollify the convex functions on a smaller real ball, obtaining locally uniform original smooth positive approximants, or apply convexity to the harmonic coordinate pullbacks on every test curve. This proves the asserted local decomposition over the original field. Products of continuous piecewise smooth scalars are again of that same class, so the same real argument applies to them. □

For $(1,1)$-forms, "symmetric" below means the usual CLD prime–double-prime symmetry; in a moment coefficient matrix it is ordinary matrix symmetry. Write Sym for that projection. Constant codimension-one cycle factors are symmetric.

Lemma 11.4 (A continuous Hessian presentation in total degree one)**.** *Every symmetric actual Mihatsch delta-form of total degree* $(1,1)$ *locally has a finite presentation*

$$\alpha = \sum_i f_i D_\delta g_i,$$

where f_i are continuous piecewise smooth scalars and g_i are continuous piecewise smooth local differences of continuous classical psh functions. Each g_i may therefore be expanded into two admissible positive potentials in that formula.

For a general $(1,1)$*-delta-form its antisymmetric part is piecewise smooth, so it is covered by the already canonical PS multiplication.*

Proof. Module generation splits a local total-degree-one presentation into a PS $(1,1)$-form and continuous PS scalars times constant codimension-one cycles. The real cycle theorem writes the latter cycles as finite sums of D_δ of continuous PL functions, which have the required difference decompositions.

For the symmetric PS part use the anchored construction of Theorem 10.2 with codimension zero. Every constant balanced codimension-zero cycle on a connected real neighborhood is a constant multiple of the ambient integration current. The construction consequently writes the PS form as a finite sum

$$h\ \mathrm{Sym}(d'_P\lambda_i \wedge d''_P\lambda_j),$$

with continuous PS h and Courant PL scalars λ_i, λ_j. The actual delta Leibniz rule gives

$$\mathrm{Sym}(d'_P\lambda_i \wedge d''_P\lambda_j) = \tfrac{1}{2}\big(D_\delta(\lambda_i\lambda_j) - \lambda_i D_\delta\lambda_j - \lambda_j D_\delta\lambda_i\big).$$

Lemma 11.3 applies to every displayed potential and coefficient product. This proves the required presentation. The positive-codimension part has only scalar coefficients and symmetric cycle factors, so the antisymmetric part of a general form comes entirely from its PS part. □

Lemma 11.5 (Piecewise smooth product rule with a Radon derivative)**.** *Suppose R belongs to the tangential coefficient-Radon module of Theorem 4.3, and $d'R$ is coefficient-Radon. For a continuous PS scalar f,*

$$d'(fR) = d'_P f \wedge R + f\, d'R.$$

The corresponding d'' identity also holds. In particular, for continuous PS local differences f, g and the fixed S,

$$\begin{aligned} D(fgS) = {} & fD(gS) + gD(fS) \\ & + d'_P f \wedge d''_P g \wedge S + d'_P g \wedge d''_P f \wedge S. \end{aligned}$$

Proof. Mollify a continuous PS real representative of f on a smaller compact moment stage. The functions converge uniformly, and their first derivatives converge tangentially on each face with uniformly bounded arrays. The conormal annihilation and fixed coefficient dominated convergence in Proposition 4.4 give $d'f_\varepsilon \wedge R \to d'_P f \wedge R$ in local coefficient variation. Uniform convergence also gives $f_\varepsilon d'R \to f\, d'R$. Pass to the limit in the ordinary smooth product rule. This proves the first assertion and its double-prime counterpart.

For the second assertion, Corollary 4.5 gives $d''(gS) = d''_P g \wedge S$, which belongs to the same tangential module by its PS module property. Its d' derivative $D(gS)$ is coefficient-Radon because g is a local difference. Apply the first assertion to this current, and similarly with f, g exchanged. The first-derivative product rule for the scalar fg then yields the displayed formula. This passage uses only fixed Radon measures; it does not require a moving-measure chain rule. □

Theorem 11.6 (Canonical degree-one action for arbitrary fixed S)**.** *For every allowed positive separately closed (p,p)-current S, $p < n$, every actual delta-form α of total degree $(1,1)$ has a canonical ordinary coefficient-Radon current $\alpha \wedge S$. It is linear, independent of all presentations and refinements, and compatible with restriction and gluing on ordinary opens. It agrees with canonical PS multiplication, with $D_\delta g \wedge S = D(gS)$ for the continuous local differences above, and with the actual delta product when S is a delta-form. Its support is contained in* $\operatorname{supp}\alpha \cap \operatorname{supp} S$.

If the ordinary current $[\alpha]$ is symmetric and original-positive, then $\alpha \wedge S$ is original-positive. On each fixed finite moment/polyhedral presentation it has the finite coefficient C^0 bounds supplied by its Hessian presentation. No hypothesis $S \in \mathcal{W}_{\mathrm{loc}}$ occurs.

Status: `NEW_PROVED` by continuous order descent. This is a total-degree-one assertion, not an all-codimension analytic relation theorem.

Proof. For symmetric $\alpha = \sum_i f_i D_\delta g_i$ define $\alpha \wedge S = \sum_i f_i D(g_i S)$. If a presentation represents zero, ordinary delta integration and continuous scalar compatibility give $\sum_i f_i D[g_i] = 0$. Theorem 11.2 annihilates it after relative insertion. Subtracting presentations gives independence. The same theorem proves the stated positivity for every positive symmetric ordinary $[\alpha]$.

For a general form add the canonical action of its antisymmetric PS part. Prime–double-prime symmetry commutes with the real delta operations and their actual pullbacks, so this decomposition is intrinsic. To verify agreement with PS multiplication, use the Hessian presentation of its symmetric part. The last identity of Lemma 11.5 identifies each symmetric Hessian expression with its already defined PS wedge product. The antisymmetric part agrees by definition. Ordinary smooth multiplication is a special case.

All formulas restrict ordinarily. Independence proves their agreement on overlaps and under all original refinements. If α vanishes on an ordinary open its presentation there is a zero relation, so its action vanishes; if S vanishes there the formula also vanishes. This proves the support assertion. On an actual delta-form background the ordinary BT–delta comparison and continuous multiplication identify the formula with the actual product. If only the ordinary current $[\alpha]$ is initially known to be symmetric, actual delta faithfulness (Proposition 2.1) gives $\alpha = \operatorname{Sym}\alpha$, so the antisymmetric PS action is zero; this justifies the theorem's stated positivity hypothesis. Finally, on a fixed finite presentation the scalar coefficients in the anchored Hessian expression are fixed finite linear combinations of coordinate-face evaluations of the original arrays. The potentials and their positive/negative decompositions are fixed. Their coefficient variations therefore give the claimed C^0 bounds on smaller fixed compact stages. □

Corollary 11.7 (Full complementary delta-tests in remaining degree one)**.** *Let S be an arbitrary allowed $(n-1,n-1)$-current. Then*

$$\Lambda_S(\alpha) = \int_U \alpha \wedge S, \qquad \alpha \in B_c^{1,1}(U),$$

is a canonical full actual delta-test extension. It is separately closed, agrees with S on all original tests, and is nonnegative whenever the ordinary current $[\alpha]$ is original-positive. It is determined by the ordinary current S, not by an auxiliary weak approximation or a choice of representatives.

Proof. Compact actual support and the preceding theorem give the finite pairing and its claimed independence and positivity. To prove closedness, every actual form of total degree $(0,1)$ is PS. The codimension-zero anchored construction writes it locally as $\sum_j f_j d''_P g_j$, where f_j, g_j are continuous PS scalars. For a compact test, an original smooth partition makes each f_j compactly supported in its moment neighborhood. The actual delta Leibniz rule, agreement with PS multiplication, and Lemma 11.5 give, term by term,

$$(d'(f_j d''_P g_j)) \wedge S = d'(f_j d''_P g_j \wedge S).$$

The right side is the derivative of a compactly supported ordinary current, so its integral is zero by an original smooth cutoff equal to one near its support. Summing proves $\Lambda_S(d'\beta) = 0$. The prime–double-prime exchanged argument proves the other identity. No higher-codimension delta product is used in this argument. □

Corollary 11.8 (Uniqueness in the stated positive closed class)**.** *For $p = n-1$, the functional in Corollary 11.7 is the unique extension satisfying Definition 17.1. Thus existence and uniqueness on the whole complementary actual delta-test space hold for every allowed background*

of remaining degree one, under that explicit symmetry, separate closedness, and generator-positivity convention.

Status: `DIRECT_COROLLARY` of the forced-value theorem and the finite Hessian presentation.

Proof. Our functional is symmetric and separately closed. Each generator in Definition 17.1, now with $k = 0$ or 1, has positive ordinary current: it is a nonnegative continuous scalar times an original positive smooth form or a positive curvature current. The positivity assertion of Theorem 11.6 therefore verifies the required generator cone.

Conversely, take any extension with the stated properties. Localize a compact test by a finite original smooth partition. Each symmetric localized test has the Hessian presentation of Lemma 11.4; expand its differentiated potentials into admissible positive differences. The forced-value theorem applies with the partition function as the compact original scalar test and with the remaining continuous PS coefficient as its weight. It determines every term to be its relative BT value. The antisymmetric part is killed by symmetry of both functionals. Summing proves equality on the whole actual complementary test space. This is finite generation, not a density argument for singular tests. □

Corollary 11.9 (Weighted decreasing convergence on degree-one delta-tests)**.** *Let S be any fixed allowed (p,p)-background, let $k \geq 0$ satisfy $p + k = n - 1$, and let $u_{i,j} \downarrow u_i$ for $0 \leq i \leq k$ be actual locally bounded classical psh rows with locally bounded limits. For every compact actual $\alpha \in B_c^{1,1}(U)$,*

$$\int_U u_{0,j}\, \alpha \wedge T_{S,k}(u_{1,j}, \ldots, u_{k,j}) \longrightarrow \int_U u_0\, \alpha \wedge T_{S,k}(u_1, \ldots, u_k).$$

All weights are the actual Borel functions. The background is fixed and arbitrary; no weak-approximation hypothesis is imposed.

Status: `DIRECT_COROLLARY` of the frozen ordinary weighted BT theorem and the new finite Hessian presentation.

Proof. The antisymmetric PS part of α pairs to zero with each scalar multiple of a symmetric $(n-1, n-1)$-current. For its symmetric part, take a finite local Hessian presentation and a compact original smooth partition. After expanding each fixed signed potential, every term has the form

$$\int \chi f\, u_{0,j}\, T_{S,k+1}(u_{1,j}, \ldots, u_{k,j}, g),$$

where g is fixed continuous classical psh, f is fixed continuous PS, and χ is an original smooth compact scalar cutoff. The extra differentiated slot is a constant decreasing row, so the ordinary weighted joint theorem applies after replacing χf by an original smooth compact scalar approximant. Such approximants converge uniformly on the one fixed compact stage. Relative CLN and the common local bounds of all decreasing rows bound the unweighted positive measures uniformly there; the weights $u_{0,j}$ are uniformly bounded there as well. The scalar approximation errors therefore tend to zero uniformly in j. Pass first in j and then in the scalar approximation, and sum the finite terms. No simultaneously varying discontinuous coefficient measure has been passed without that estimate. □

Remaining distinction. The degree-one result applies to every allowed background, including those carried by ambient complete-polar sets. In larger remaining degree it supplies an action of all total-degree-one delta-forms, but it does not yet annihilate every actual relation among several positive-codimension factors. The full-test theorem in arbitrary remaining degree continues to use $\mathcal{W}_{\mathrm{loc}}$ or another explicitly proved relation mechanism.

12 A weak-approximation bridge to an actual delta-test class

This section states a sufficient condition on the background; it does not impose that condition on the frozen ordinary BT theory. Its purpose is to isolate an independent analytic condition which kills the additional pullback relations. All fields, spaces, positivity cones, and original test classes remain those fixed at the start of the round.

Definition 12.1 (The local weak closure of delta-form backgrounds)**.** For an ordinary open V, let $\mathcal{W}^p(V)$ be the weak closure, in the topology of all original compact smooth tests, of the cone of currents $[\sigma]$, where $\sigma \in B^{p,p}(V)$ and $[\sigma]$ is symmetric, original-positive, and separately closed. A current belongs to $\mathcal{W}^p_{\mathrm{loc}}$ if each point has one ordinary neighborhood V on which it belongs to $\mathcal{W}^p(V)$. The closure is allowed to use nets. The same fixed V is used throughout each approximating net; it is not a varying-domain notion.

Let $\mathscr{B}_{\mathrm{cc}}$ be the subsheaf of the actual Mihatsch delta-form sheaf whose sections locally admit finite presentations

$$\alpha = \sum_i t^* \eta_i \wedge t^\star C_i,$$

with original moments t, piecewise smooth coefficients η_i, and constant-weight real polyhedral tropical cycles C_i. Nonrational directions, real weights, and real affine offsets are allowed by the real cycle theorem. This is an explicitly specified test class, not an asserted description of all Mihatsch or GK forms.

Lemma 12.2 (Fixed continuous slots and varying positive backgrounds)**.** *Suppose $S_\lambda \to S$ weakly on all original tests, where all these currents are symmetric, original-positive, and separately closed of the same degree. Fix continuous classical psh potentials $v_1, \dots, v_c$ on their common ordinary domain. Then*

$$T_{S_\lambda, c}(v_1, \dots, v_c) \longrightarrow T_{S,c}(v_1, \dots, v_c)$$

on every original test. The same holds after wedging with one fixed piecewise smooth form and testing the resulting ordinary current on any original complementary smooth form.

Status: `DIRECT_COROLLARY` of R's fixed-continuous-multiplier lemma, the frozen positive BT recursion, and R4's piecewise-test continuity. The fixed continuity assumption on the inserted potentials is essential to this particular argument.

Proof. For one slot, R's lemma gives $v_1 S_\lambda \to v_1 S$; applying D proves the assertion. Those derivatives are again positive separately closed currents. Induct on the number of fixed continuous slots, applying the same lemma to the already convergent positive tails. For the last assertion, wedge the fixed piecewise smooth coefficient with the chosen original test on a common finite presentation. It is a compatible piecewise smooth test of complementary diagonal degree; replace it by its symmetric part if needed. R4, Theorem 71.4, gives convergence on that test, since weak convergence bounds its finitely many positive mass controls. This proof allows nets and does not invoke any monotone theorem for simultaneously varying discontinuous potentials and backgrounds. □

Theorem 12.3 (Actual relation descent under local weak delta approximation)**.** *Let $S \in \mathcal{W}^p_{\mathrm{loc}}$. For every section $\alpha \in \mathscr{B}^{a,b}_{\mathrm{cc}}(U)$ there is a canonical ordinary coefficient-Radon current $\alpha \wedge S$. On a local hypersurface presentation of the constant cycles it is computed by the corresponding finite sum of piecewise smooth coefficients wedged with the relative BT products over S.*

This operation annihilates every actual analytic zero-presentation of α, commutes with ordinary restriction and gluing, and has support contained in $\operatorname{supp}\alpha \cap \operatorname{supp} S$. *If $a = b = n - p$ and α is compactly supported, it therefore defines the finite canonical pairing $\langle S, \alpha \rangle = \int_U \alpha \wedge$*

S. When S itself is an actual delta-form background this agrees with its delta product and integration.

Status: `NEW_PROVED` under the explicit weak-approximation hypothesis. Its application to every arbitrary positive closed background is `CONDITIONAL` on the separate density assertion $\mathcal{W}^p_{\mathrm{loc}}$ = all such backgrounds. That density assertion remains `OPEN` in this round.

Proof. The real cycle theorem and its piecewise polynomial presentation express each constant cycle as a finite real linear combination of products of PL divisors. Decompose each signed PL potential into two convex PL functions by Lemma 7.2, and expand. Thus one obtains a finite real linear combination of mixed products of continuous convex PL potentials pulled back by the original moments. Thus the proposed expression is a well-defined finite sum of ordinary coefficient-Radon currents. The ambient relation theorem above already removes ambient cycle relations, but the following argument also handles the additional analytic kernel.

Work on one fixed ordinary neighborhood on which $S_\lambda = [\sigma_\lambda] \to S$ as in Definition 12.1. On each S_λ, the BT–delta comparison and piecewise smooth multiplication identify the proposed expression with $[\alpha \wedge \sigma_\lambda]$. In particular it is zero whenever the represented actual delta-form α is zero. Lemma 12.2 passes every term of the finite expression to its corresponding value over S. Hence every actual zero-presentation also has zero value over S. Subtracting two presentations proves independence.

Restriction is immediate from the formula and the fact that restriction sends $\mathcal{W}^p(V)$ into $\mathcal{W}^p(V')$ for an ordinary subopen V'. Independence on overlaps glues the local expressions. If α or S is zero on an ordinary open, the local formula and actual zero-presentation independence make the resulting current zero there. This proves the support assertion. In top degree the result is a signed Radon measure; compact support makes its total integral finite. The delta-form-background comparison was already used term by term, and yields the final assertion as well. □

Proposition 12.4 (Stability of the approximation class under all bounded slots)**.** *If $S \in \mathcal{W}^p_{\mathrm{loc}}$ and $u_1, \dots, u_k \in \mathcal{C}_b$, then*

$$T_{S,k}(u_1, \dots, u_k) \in \mathcal{W}^{p+k}_{\mathrm{loc}}.$$

In particular the class contains every bounded classical BT product over [1]*, and over every positive separately closed actual delta-form background. These differentiated potentials may be discontinuous.*

Status: `DIRECT_COROLLARY` of the authorized local smooth rows, ordinary BT continuity, and the definition as a weakly closed cone.

Proof. It suffices to insert one slot and work on one smaller ordinary neighborhood where the prescribed decreasing original smooth positive row $u_j \downarrow u$ exists. The restrictions of S remain in the corresponding weak closure on that fixed neighborhood. For fixed j, the map $R \mapsto Du_j \wedge R$ is weakly continuous. It sends each approximating actual delta-form background to an actual delta-form background whose ordinary current is positive and separately closed. Consequently $D(u_j S)$ belongs to the weakly closed cone $\mathcal{W}^{p+1}$ there. The frozen monotone theorem gives $D(u_j S) \to D(uS)$, so the latter is in the same closure. Repeat on a common smaller neighborhood for finitely many slots. The current [1] is itself an actual delta-form background. This proves the stated examples. □

Lemma 12.5 (Piecewise-test approximation with a bounded Borel weight)**.** *For a fixed compact compatible piecewise smooth test η, the smooth approximants and finite positive controls in R4*

can be chosen so that, for every allowed positive separately closed T and every bounded Borel w on that compact stage,

$$|\langle wT, \eta_\varepsilon - \eta\rangle| \le C\|w\|_\infty \varepsilon^{1/2} \sum_b \langle T, \Theta_b\rangle.$$

In complementary degree zero use the uniform scalar approximation error in place of $\varepsilon^{1/2}$.

Status: `DIRECT_COROLLARY` of the total-variation version of R4's normal-factor estimate.

Proof. R4 decomposes the error into a uniformly $O(\varepsilon)$ coefficient array and finitely many bounded normal-factor arrays supported in tubes of width $O(\varepsilon)$. Its Lemma 71.2 bounds the variations of the normal-factor coefficient measures, not merely their signed total pairings. Multiplication by w increases each such variation by at most $\|w\|_\infty$. The same statement for the regular error follows from fixed coefficient domination. Sum the finitely many terms. The proof uses closedness only for T, not for the weighted current. □

Theorem 12.6 (Weighted decreasing convergence on the specified delta-tests)**.** *Fix $S \in \mathcal{W}^p_{\mathrm{loc}}$ and $0 \le k \le n-p$. Suppose every row $u_{i,j} \downarrow u_i$, $0 \le i \le k$, consists of actual locally bounded classical psh functions and has a locally bounded limit. For every compact $\alpha \in \mathscr{B}^{n-p-k,n-p-k}_{\mathrm{cc}}(U)$,*

$$\int_U u_{0,j}\, \alpha \wedge T_{S,k}(u_{1,j}, \dots, u_{k,j}) \longrightarrow \int_U u_0\, \alpha \wedge T_{S,k}(u_1, \dots, u_k).$$

The multiplier is the actual Borel function. No identification of weights modulo their ambient curvature is made.

Status: `NEW_PROVED` for this actual delta-test class and these backgrounds. The ordinary weighted BT convergence itself is frozen input.

Proof. The preceding stability proposition and actual relation descent define all the currents in the integrals independently of the chosen test presentation. Localize the compact support by a finite original smooth partition and choose local constant-cycle hypersurface presentations. Each term is the pairing of a fixed piecewise smooth coefficient η with

$$u_{0,j}\, T_{S,k+c}(u_{1,j}, \dots, u_{k,j}, v_1, \dots, v_c),$$

where the v_i are fixed continuous convex PL classical potentials. All these unweighted tails are positive separately closed. Their finitely many mass controls are bounded by relative CLN and the common local bounds of the decreasing rows and fixed v_i. The weights $u_{0,j}$ have one local bound as well.

Replace η by its fixed smooth approximant η_ε. The frozen weighted joint decreasing theorem gives convergence for this original smooth test, with the fixed v_i treated as constant decreasing rows. Lemma 12.5 bounds both approximation errors uniformly in j by a quantity tending to zero with ε. First pass to the limit in j, then in ε. Sum the finite presentations. Compact support and actual relation descent make the resulting integrals independent of all local choices. There is no varying background S_j in this monotone statement. □

13 Full actual delta-testing over the weak approximation class

The exact module generation theorem removes the test-generation restriction from the weak-approximation bridge. The hypothesis on the background remains explicit and independent of that algebraic generation statement.

Theorem 13.1 (A full actual delta-test extension)**.** *Let $S \in \mathcal{W}^p_{\mathrm{loc}}$ be as in Definition 12.1. For every actual Mihatsch delta-form $\alpha \in B^{a,b}(U)$ there is a canonical ordinary coefficient-Radon current $\alpha \wedge S$ of bidegree $(a+p, b+p)$. It has the following properties.*

1. *It is linear in α, independent of all actual presentations and refinements, and commutes with restriction and gluing on ordinary opens.*
2. *Its support is contained in $\operatorname{supp}\alpha \cap \operatorname{supp} S$. Thus complementary compact delta-forms have finite pairings $\Lambda_S(\alpha) = \int_U \alpha \wedge S$.*
3. *It agrees with ordinary smooth multiplication and with the actual delta product on delta-form backgrounds. Curvature factors of continuous PL classical potentials act by their ordinary relative BT insertion.*
4. *It satisfies the distributional identities*

$$d'(\alpha \wedge S) = (d'\alpha) \wedge S, \qquad d''(\alpha \wedge S) = (d''\alpha) \wedge S.$$

   In particular Λ_S is separately closed on compact actual delta-tests.
5. *On every fixed finite moment, polyhedral, and support presentation it is continuous for smooth coefficient convergence. More precisely, on smaller fixed compact stages each original coefficient pairing is bounded by a constant times finitely many C^0 norms of the presented coefficient arrays. Derivative convergence is not needed for this individual pairing bound.*

Positivity is asserted on the explicit generator cone: finite sums of nonnegative continuous piecewise smooth scalars times original middle-positive smooth forms wedged with convex PL classical curvature factors. No identification with an unspecified stronger or weaker delta-positivity convention is made.

Status: `NEW_PROVED` under the stated background hypothesis. It is a full delta-test extension for this class, not a wedge product of two arbitrary currents.

Proof. Theorem 10.2 and its analytic corollary identify the actual sheaf B with the locally constant-cycle-generated sheaf $\mathscr{B}_{\mathrm{cc}}$ of the bridge. Theorem 12.3 therefore gives the construction and assertions (1)–(3), on the whole actual delta-test sheaf rather than just on a subalgebra.

For (4), work on one fixed ordinary neighborhood with $S_\lambda = [\sigma_\lambda] \to S$ as in the definition. Both α and $d'\alpha$ are actual delta-forms and have finite presentations supplied by module generation. The fixed-continuous-slot lemma and the piecewise-test estimate consequently give

$$[\alpha \wedge \sigma_\lambda] \longrightarrow \alpha \wedge S, \qquad [(d'\alpha) \wedge \sigma_\lambda] \longrightarrow (d'\alpha) \wedge S$$

on every original compact smooth test. Faithfulness of actual delta-forms makes $d'\sigma_\lambda = d''\sigma_\lambda = 0$. Their delta Leibniz identity gives

$$d'[\alpha \wedge \sigma_\lambda] = [(d'\alpha) \wedge \sigma_\lambda].$$

Distributional differentiation is weakly continuous; passing to the limit proves the first claimed identity. The second is identical. These local identities glue. For a compact test in the degree preceding top degree, its resulting ordinary current has compact support by (2). Pair its derivative with an original smooth scalar cutoff equal to one near that support. The definition of the derivative and the vanishing of the derivative of that cutoff there give zero. This proves separate closedness of the full delta-test functional.

For (5), inspect the finite anchored construction on a fixed small Courant cube. Each anchored coefficient is a finite signed sum of evaluations of one smooth coefficient on coordinate faces. Taking its coordinates in a fixed star-balancing kernel is another fixed finite linear map.

The resulting piecewise smooth coefficient arrays have C^0 norms bounded by a fixed constant times those of the original arrays on finitely many enlarged compact coordinate-face stages. All Courant covectors and cycle factors are fixed. The output is therefore a finite sum of these bounded coefficient multipliers applied to fixed coefficient-Radon cycle actions over S. Their local finite variation bounds give the asserted estimate on every original test, after its common refinement. Finite partitions handle arbitrary compact support presentations.

Finally each generator-positive term is exactly a nonnegative actual scalar times a positive BT current wedged with an original middle-positive smooth form. Its top pairing is nonnegative by the frozen cone convention and coefficient-Radon multiplication. Summing proves the stated positivity without changing that cone. □

Corollary 13.2 (Full delta-tests for bounded BT products and weights)**.** *Fix $S \in \mathcal{W}^p_{\mathrm{loc}}$. Every bounded classical mixed BT current over S has the full extension of Theorem 13.1. The weighted joint decreasing convergence in Theorem 12.6 holds for every compact actual Mihatsch delta-test of complementary degree.*

If two such unweighted currents are equal on all original smooth tests, their extensions agree on all actual compact delta-tests. They also agree after the same actual locally bounded Borel multiplier wherever the displayed pairings are defined. In particular this applies to all the bounded classical products over [1] *from B and to the ordinary potential-quotient identities over backgrounds in this class.*

Status: `DIRECT_COROLLARY` of the full extension, its ordinary-current construction, the class stability proposition, and the bridge's weighted convergence theorem.

Proof. Proposition 12.4 puts each unweighted BT current in the same kind of weak approximation class. Module generation makes every actual test admissible in the already proved weighted convergence argument. The extension is determined by the given ordinary current through fixed-potential recursion and piecewise coefficient multiplication, not by a choice of approximating net. Ordinary equality therefore gives equality of the resulting coefficient currents for each delta-form. Multiplying those identical coefficient currents by the same actual Borel function preserves equality. The final applications follow because [1] belongs to the approximation class and the R5 quotient gives the relevant ordinary equalities. □

Corollary 13.3 (Weak continuity on fixed delta-tests)**.** *Suppose $S_j, S \in \mathcal{W}^p_{\mathrm{loc}}$ are positive separately closed and $S_j \to S$ on all original compact smooth tests. Then for every fixed actual delta-form α, the ordinary currents $\alpha \wedge S_j$ converge to $\alpha \wedge S$. For compact complementary α their full delta-test pairings converge as well. This is a statement for a fixed delta-test. It does not assert simultaneous stability with varying discontinuous potential slots and arbitrary varying backgrounds.*

Proof. Choose one finite local module presentation of the fixed α over each compact test support, expand its PL slots into convex positive and negative choices, and apply Lemma 12.2 to each term. This proof needs no common approximating delta-form net for the S_j. For a compact complementary test, all the resulting measures are supported in its one fixed actual compact support. An original smooth cutoff equal to one near that compact converts ordinary current convergence into convergence of their integrals. □

The precise remaining all-background question. Every fixed allowed S already has the full ordinary bounded BT calculus and the R5 potential quotient. The unsolved assertion here is that every actual zero delta-presentation has zero relative action for that arbitrary S. Local weak approximation by positive closed actual delta-forms is one sufficient mechanism,

now proved to cover all actual delta-tests. Its availability for every S has not been established. The saturated matroidal theorem provides a different, explicitly geometric mechanism in its fixed-chart scope.

14 Integration currents obtained from smooth regular flags

The weak-approximation hypothesis has geometric examples beyond bounded products over [1]. The approximation below is an iterated weak-closure argument on one fixed ordinary domain. It does not assert a joint limit of singular logarithmic potentials.

Lemma 14.1 (Exact eventual equality on one original top test)**.** *Let Z be a good Hausdorff pure d-dimensional analytic space over the original K, and let $g \in \mathcal{O}(Z)$ be a non-zero-divisor. Put*

$$v_N = \max\{\log|g|, -N\}, \qquad N \in \mathbf{N},$$

with $v_N = -N$ on $g = 0$. For every original compactly supported smooth (d,d)-form θ there is N_θ such that

$$\int_Z v_N \theta = \int_{Z\setminus\{g=0\}} \log|g|\,\theta \qquad (N \geq N_\theta).$$

In particular $[v_N] \to [\log|g|]$ as original currents. No integrability statement against an arbitrary Radon measure is part of this assertion.

Status: `DIRECT_COROLLARY` of the carrier argument in [CLD12, Lemma 4.6.1 and its proof, printed pp. 72–73].

Proof. Set $Z^* = Z \setminus \{g = 0\}$. It is a dense Zariski open because g is a non-zero-divisor. Apply a finite original smooth partition to the compact support of θ. The cited proof supplies, for each resulting moment presentation $\lambda f^*\omega$, a compact set C contained in the interior of its affinoid stage and disjoint from $Z \setminus Z^*$. It contains the support of $\lambda f^*\omega$ for every top-degree coefficient form in that fixed presentation. Explicitly, C is the intersection of the support of λ with the union of the inverse images of the standard d-dimensional torus skeleta under the standard coordinate projections of f. The original top-degree carrier has maximal residual dimension, so it cannot meet the proper analytic zero locus of g; this is exactly the disjointness proved there.

The function $\log|g|$ is continuous and finite on each such compact C, hence has a finite lower bound there. Choose N_θ larger than the negatives of the finitely many lower bounds. Then $v_N = \log|g|$ on all these carriers for $N \geq N_\theta$. The displayed integrals are consequently equal, term by term. The logarithmic integral defines a current by the same cited lemma; thus equality eventually on each fixed test proves the claimed weak convergence in the topology of all original tests. This argument does not require a uniform N_θ over varying tests. □

Proposition 14.2 (A smooth regular flag belongs to the approximation class)**.** *Let V be one fixed ordinary open in an allowed smooth analytic space of pure dimension n. Suppose that there are analytic functions $g_1, \dots, g_p$ on V and a flag of closed analytic subspaces*

$$V = Y_0 \supset Y_1 \supset \cdots \supset Y_p, \qquad Y_i = \{g_1 = \cdots = g_i = 0\},$$

such that every Y_i is good, Hausdorff, boundaryless, and quasi-smooth of pure dimension $n - i$, and $g_i|_{Y_{i-1}}$ cuts out the reduced effective Cartier divisor Y_i in Y_{i-1}. Let $[Y_i]$ denote its integration current on V, with the original CLD calibration. Then

$$[Y_i] \in \mathcal{W}^i(V) \qquad (0 \leq i \leq p).$$

More precisely, if $v_{i,N} = \max\{\log|g_i|, -N\}$ *on* V*, then*

$$D(v_{i,N}[Y_{i-1}]) \longrightarrow [Y_i]$$

on every original compact complementary smooth test, and every current on the left belongs to $\mathcal{W}^i(V)$*. All approximations use this same* V *and the original absolute value.*

Status: `DIRECT_COROLLARY` of the preceding lemma, original Poincaré–Lelong, and the fixed-continuous-slot weak-closure mechanism.

Proof. The initial current $[Y_0] = [1]$ is an actual delta-form background. Assume that $[Y_{i-1}] \in \mathcal{W}^{i-1}(V)$. Each $v_{i,N}$ is an actual continuous locally bounded classical psh function on V. Indeed it is constant on the ordinary neighborhood $\{|g_i| < e^{-N}\}$ of the zero locus, and elsewhere is the maximum of a constant and the logarithm of the norm of an analytic unit. This also proves $v_{i,N} \in PS^{0,0}(V)$ in the original moment category. The same description after every complete field extension proves the required classicality.

Choose a net $[\sigma_\lambda] \to [Y_{i-1}]$ on the fixed V as in Definition 12.1. For fixed N, the current

$$D(v_{i,N}[\sigma_\lambda]) = [D_\delta v_{i,N} \wedge \sigma_\lambda]$$

is an actual delta-form background. Its ordinary current is positive, symmetric, and separately closed by the ordinary relative BT theorem. The fixed-continuous-slot lemma gives its weak limit $D(v_{i,N}[Y_{i-1}])$. Hence this limit belongs to $\mathcal{W}^i(V)$, with no shrinking of V depending on N.

Let $j : Y_{i-1} \hookrightarrow V$. Pullback of every original compact smooth test by this closed immersion is again an original compact smooth test. Integration and distributional differentiation give

$$D(v_{i,N}[Y_{i-1}]) = j_* D[v_{i,N}|_{Y_{i-1}}].$$

Lemma 14.1, applied intrinsically to Y_{i-1}, passes its right side to $j_* D[\log|g_i|_{Y_{i-1}}]$. Original Poincaré–Lelong [CLD12, Theorem 4.6.5, printed p. 74] identifies this with $j_* \delta_{\mathrm{div}(g_i|_{Y_{i-1}})} = [Y_i]$. There is no sign or degree factor: the cited convention is $d'd''[\log|g|] = \delta_{\mathrm{div}(g)}$, and the divisor is reduced with multiplicity one by hypothesis.

The cone $\mathcal{W}^i(V)$ is weakly closed by definition, so it contains $[Y_i]$. This completes the induction. Only a single logarithmic slot is sent to its singular limit at each induction step; no simultaneous singular-product continuity has been used. □

Corollary 14.3 (Full delta-testing for the flag and its bounded products)**.** *Every integration current which locally admits the smooth regular flags of Proposition 14.2 lies in* $\mathcal{W}^p_{\mathrm{loc}}$*. This includes a smooth algebraic closed subvariety of a smooth algebraic variety after restricting to neighborhoods on which its regular immersion has an adapted smooth flag. All its bounded classical mixed BT products belong to the same approximation class in the appropriate degree. They therefore have the canonical full actual delta-test extension of Theorem 13.1, including its weighted joint decreasing convergence.*

Proof. The proposition proves the local membership. For the stated algebraic example, choose local generators of the conormal bundle whose independent differentials give smooth successive zero loci, and shrink the algebraic neighborhood. Analytification supplies the required flag on the corresponding ordinary open. The bounded product assertion follows from Proposition 12.4; the full-test assertions then follow from the cited extension theorem and its corollary. □

Scope. The proof covers the explicit local smooth-flag hypothesis. It neither asserts that arbitrary singular analytic cycles admit such flags nor that every allowed positive closed current belongs to $\mathcal{W}_{\mathrm{loc}}$. In particular it adds examples to the sufficient approximation class without replacing the project's arbitrary-background ordinary theory by a geometric subclass.

15 Integrable classical potentials and effective Cartier backgrounds

All currents in this section remain currents on the allowed original K-space. An extended classical psh function takes values in $\mathbf{R} \cup \{-\infty\}$. Absolute integrability below refers to the total variations of the measures of all original compact smooth top-degree forms. It is not an assertion of integrability against every positive background measure.

Lemma 15.1 (A fixed approximation domain for individually bounded functions)**.** *In an ordinary algebraic neighborhood of a point, fix nested polynomial norm boxes*

$$C_1 \subset \operatorname{Int} C_{\mathrm{mid}}, \qquad C_{\mathrm{mid}} \subset \operatorname{Int} C_2 \Subset O, \qquad V = \operatorname{Int} C_1.$$

They can be chosen so that every classical psh function w on O which is bounded on a neighborhood of C_2 has, on this same V, a decreasing row of continuous semipositive PL functions. The bound for w and the row can depend on w; the domain V does not. Consequently $D[w]|_V \in \mathcal{W}^1(V)$.

Status: `DIRECT_COROLLARY` of the explicit fixed-box collar in B, Lemma D.2, and its bounded projective regularization theorem. The fixed-domain point is spelled out because it is needed for an unbounded limiting potential.

Proof. Use the ordinary algebraic localization and smooth projective compactification of B, Proposition 4.5. If its constant field is a finite extension of K, retain the original extending absolute value; the analytic functions and their moment forms are still original K-analytic data on that neighborhood. The single positive integration factor of B, Lemma D.5, appears on both sides of each current limit and does not change weak closure or positivity.

Choose the boxes once. The constructive part of B, Lemma D.2 (printed pp. 27–28), supplies from these boxes a continuous PL collar b, a pure Fubini–Study metric q, and $\delta > 0$, with

$$b = 0 \text{ on a neighborhood of } C_1, \qquad b \geq \delta \text{ on } C_2 \setminus \operatorname{Int} C_{\mathrm{mid}},$$

and with b the weight of q in a frame near C_2. These objects are chosen before the upper and lower bounds of the functions enter that construction.

Normalize the present w by a positive constant and an additive constant to a classical function v with $0 \leq v \leq 1$ on a neighborhood of C_2. Fix $c > 0$ in the permitted value group and an integer m with $m\delta > c + 2$. The metric with local weight

$$\max\{v + c, mb\}$$

on $\operatorname{Int} C_2$, and equal to mq outside C_{mid}, is a bounded classical metric on the same ample line used by that collar. On the overlap the second branch dominates, so this is the identical gluing from B. Near C_1 its framed weight is exactly $v + c$. The choices of boxes, frame, c and m work for every normalized v. No monotonicity of a family of such v is needed to construct each metric; in B it was needed only for the additional assertion that the resulting metric family decreases.

Apply B, Theorem 4.4, separately to this bounded metric. Its decreasing pure-FS row restricts on V to a continuous semipositive PL row for $v + c$. Undo the fixed additive shift and the positive normalization to obtain the desired row for w. Real shifts and positive real scalings are permitted for actual PS and delta-form data. Each row curvature is an actual delta-form with positive separately closed ordinary current. The bounded original monotone theorem, or fixed top-test bounded convergence, gives its weak limit $D[w]$ on this same V. This proves membership in $\mathcal{W}^1(V)$. □

Theorem 15.2 (An integrable extended classical potential gives a weak delta background)**.** *Let $u : U \to \mathbf{R} \cup \{-\infty\}$ be actual extended classical psh on an allowed original-K space of dimension $n \geq 1$. Assume that u is locally absolutely integrable against the total variation of every original compact smooth (n, n)-form. Then $[u]$ is a degree-zero current, $D[u]$ is symmetric positive and separately closed, and*

$$D[u] \in \mathcal{W}^1_{\mathrm{loc}}.$$

For $u_N = \max\{u, -N\}$ one has $D[u_N] \to D[u]$ on every original compact smooth complementary test. Near each point all these truncation curvatures belong to $\mathcal{W}^1(V)$ on one common ordinary V.

Status: `DIRECT_COROLLARY` of the fixed-box lemma, original integrability and weak closure. No local potential is supplied for an arbitrary given positive current.

Proof. First the integrability hypothesis defines a current, rather than only individual finite integrals. On a fixed finite compact moment stage the coefficient controls for [1] bound the variation of every top test by its coefficient C^0 seminorm times finitely many fixed positive compact top-form measures. Their integrals against $|u|$ are finite. This gives a continuous order-zero functional on that test stage. Finite localization proves continuity in the original test topology. Thus $[u]$ is well defined, with arbitrary values on the sets on which the controlling measures vanish.

Each actual truncation u_N is locally bounded classical. For $N \geq 1$, $|u_N| \leq |u|$ wherever u is finite, and the integrability hypothesis makes its infinite-value set null for every fixed top-test variation. Fixed-measure dominated convergence therefore gives $[u_N] \to [u]$, hence $D[u_N] \to D[u]$ on every original test. Since each $D[u_N]$ is symmetric positive and separately closed, its limit has the same properties.

Choose the boxes in Lemma 15.1 once near a prescribed point. Upper semicontinuity gives a common finite upper bound M for u on a neighborhood of C_2. Each u_N has the finite lower bound $-N$ there. The lemma consequently puts every $D[u_N]|_V$ in the same weakly closed cone $\mathcal{W}^1(V)$. Its weak limit $D[u]|_V$ also lies there. The geometric domain does not shrink with N; only the rows and their harmless normalizations vary. This proves the local claim. □

Corollary 15.3 (Effective Cartier divisors, with singularities and multiplicities)**.** *Let V be an allowed ordinary domain and let $f \in \mathcal{O}(V)$ be a non-zero-divisor. Let $Z = \mathrm{div}(f)$ be its effective Cartier divisor, with its original generic multiplicities. No smoothness or reducedness of Z is assumed. Then its ordinary integration current satisfies*

$$[Z] = D[\log |f|] \in \mathcal{W}^1(V).$$

More precisely, on this same whole V,

$$[D_\delta \max\{\log |f|, -N\}] \longrightarrow [Z]$$

on every original compact complementary smooth test. Hence every locally effective Cartier divisor lies in $\mathcal{W}^1_{\mathrm{loc}}$, and has the canonical full actual delta-test action already proved for that approximation class. Its bounded classical mixed BT tails have that action as well.

Status: `DIRECT_COROLLARY` of [CLD12, Lemma 4.6.1, 4.6.3 and Theorem 4.6.5, printed pp. 72–74]. The cited statements require pure dimension, topological separatedness and a non-zero-divisor; they do not require the divisor to be smooth.

Proof. Put $v_N = \max\{\log|f|, -N\}$. It is an actual continuous classical PS function on V: near a zero of f the constant branch is active, and off the zero locus it is a maximum of affine functions of the original unit moment $\log|f|$. These assertions hold after every complete extension. Thus $D_\delta v_N$ is an actual delta-form, and its ordinary current is symmetric positive and separately closed.

For every fixed original compact top test, the compact carrier in the proof of CLD12, Lemma 4.6.1, avoids the divisor. The logarithm is finite and bounded below on that carrier. Accordingly v_N and $\log|f|$ have exactly the same pairing with that test for all sufficiently large N. This is also Lemma 14.1; it assumes only a pure-dimensional space and a non-zero-divisor. Apply it to the top test $D\eta$ for any original complementary compact smooth η. It yields

$$\langle D[v_N], \eta\rangle \longrightarrow \langle D[\log|f|], \eta\rangle.$$

Original Poincare–Lelong identifies the latter current with the integration current of Z, including the generic Cartier multiplicities. In the original convention its coefficient is one: $D[\log|f|] = \delta_{\mathrm{div}(f)}$. In particular repeated factors contribute their actual multiplicities; no smoothness or reduced-divisor hypothesis has been substituted.

All approximants are actual delta-form backgrounds on V itself, so the fixed-domain weak closure gives $[Z] \in \mathcal{W}^1(V)$. For an effective Cartier divisor with only local equations, apply this on the equation neighborhoods; multiplication of an equation by a unit changes its logarithm by a smooth pluriharmonic function. The ordinary currents and the already proved full delta actions therefore glue. Bounded BT stability of $\mathcal{W}_{\mathrm{loc}}$ gives the final assertion. □

Limits of this enlargement. The extended-potential theorem assumes original-test absolute integrability. It does not assert that every extended classical function has that property, nor that every positive closed current has such a potential. The Cartier corollary is independently unconditional because its original logarithmic carrier calculation is exact. No assertion about general singular higher-codimension intersections or arbitrary analytic cycles is made here.

16 Barycenters of backgrounds in the approximation cone

The weak approximation class is not restricted to a single geometric background or to countable positive sums. The following elementary closure mechanism is useful for diffuse currents supported on ambient pluripolar sets. Its same-domain hypothesis is explicit.

Proposition 16.1 (Integrable barycenters in one fixed weak cone)**.** *Let V be one fixed ordinary open and let (A, λ) be a positive measure space. For almost every $a \in A$ let $S_a \in \mathcal{W}^p(V)$. Assume the following two conditions.*

1. *For every original compact smooth complementary test η, the scalar $a \mapsto \langle S_a, \eta\rangle$ is measurable.*
2. *On each compact test presentation the finite positive mass controls for the coefficient-Radon estimate are integrable in a.*

Then

$$\langle S, \eta\rangle = \int_A \langle S_a, \eta\rangle \, d\lambda(a)$$

defines a symmetric original-positive separately closed current, and $S \in \mathcal{W}^p(V)$. In particular it admits the full actual delta-test extension of Theorem 13.1. For each fixed actual delta-form α, whenever the corresponding finite inserted mass controls are locally integrable, one also has

$$\alpha \wedge S = \int_A \alpha \wedge S_a \, d\lambda(a)$$

on all original compact smooth tests.

Status: `DIRECT_COROLLARY` of convex weak closure and the finite-presentation construction. The same-domain membership $S_a \in \mathcal{W}^p(V)$ is not replaced by an a-dependent local membership assertion.

Proof. The positive coefficient controls imply absolute integrability of every displayed scalar pairing and continuity on each compact original test presentation. They therefore define a current. Symmetry, positivity and the two closedness identities pass under the integral.

We check membership without assuming any countable collection of tests generates the topology. Fix finitely many real original tests $\eta_1, \dots, \eta_N$, and consider

$$v(a) = (\langle S_a, \eta_1 \rangle, \dots, \langle S_a, \eta_N \rangle) \in \mathbf{R}^N.$$

Its components are integrable. Its integral belongs to the closed convex cone generated by its essential values: otherwise finite dimensional separation supplies a real linear functional nonnegative on that cone but negative on the integral, contradicting integration of that same nonnegative measurable scalar function. Hence the pairings of S on this finite list are approximated arbitrarily closely by positive finite combinations of the pairings of S_a. Every such combination belongs to the convex cone $\mathcal{W}^p(V)$. This proves that every basic weak neighborhood of S meets that weakly closed cone, and consequently S belongs to it. The argument permits infinite total parameter mass when the stated local controls are integrable.

For the last assertion take a finite module presentation of α and express its constant cycles as finite signed products of continuous PL curvature slots. Start with one continuous slot v. Uniform approximation of a fixed continuous scalar by original smooth scalars, together with the integrable control bounds, gives $vS = \int_A vS_a \, d\lambda(a)$. Applying D commutes with the integral on original tests. Iterate this argument for the finitely many fixed continuous slots; relative CLN gives the required integrable controls from the stipulated inserted controls, or from the original controls when the fixed potentials have common compact bounds. Finally use the uniform piecewise-test approximation estimate. Its errors are bounded by an integrable finite sum of the same positive control pairings, and the smooth approximants commute with the integral. This passes piecewise smooth coefficient multiplication through the integral without assuming a Borel disintegration on every individual stratum. Summing the finite expression proves the identity. Actual relation independence holds on both sides by the full extension theorem. □

Corollary 16.2 (Diffuse polar-supported fiber backgrounds)**.** *In the product-polydisc setting of Proposition 46.1, the diffuse fiber current*

$$S = \int_P [Y \times \{a\}] \, d\lambda(a)$$

belongs to $\mathcal{W}^p(\Omega)$. All its bounded classical BT products admit full actual delta-testing, separately closed extensions, and weighted joint decreasing convergence on these tests. This includes the nonatomic exact complete-pole parameter examples.

Proof. For every fixed $a \in P$, the equations $z_i - a_i = 0$ give a smooth regular flag on the same domain Ω. Proposition 14.2 therefore puts each fiber integration current in $\mathcal{W}^p(\Omega)$. The fiber-integration proof gives scalar measurability and uniform compact control bounds on the compact parameter space P. Apply Proposition 16.1, followed by the stability of $\mathcal{W}_{\mathrm{loc}}$ under bounded classical slots and Corollary 13.2. No assertion that ambient pluripolar sets are null for these currents is used. □

What remains outside this closure argument. An arbitrary positive separately closed current has not been shown to be an integrable barycenter of the displayed geometric examples, nor to be a weak limit of positive separately closed delta-forms. The proposition enlarges an established class; it does not solve the all-background analytic relation problem by definition.

17 Values forced by a positive closed delta-test extension

An arbitrary ordinary current is not, by definition, a functional on singular delta-tests. The following result is a uniqueness statement for extensions satisfying explicit positivity and Stokes requirements. It does not assume that such an extension exists.

Fix an allowed positive symmetric separately closed current S of bidegree (p,p) on U, and put $q = n-p$. An admissible potential below means a continuous function $u \in \mathcal{C}_b(U) \cap PS^{0,0}(U)$; local versions on neighborhoods of the relevant compact supports are equally allowed. Write $D_\delta u = d'd''u$ in the actual delta algebra.

Definition 17.1 (The positivity required of an extension)**.** Consider a real symmetric linear functional Λ on the actual compact delta-tests of complementary total bidegree to S. Assume the following three properties.

1. Its restriction to all original compact smooth complementary tests is S.
2. It is separately d'- and d''-closed, in the dual sense on the actual delta-test complex.
3. Whenever $0 \le k \le q$, $u_1, \dots, u_k$ are admissible, $a \ge 0$ is a continuous piecewise smooth function near the support of a compact original smooth form θ, and θ is positive in the original middle/PSD cone of bidegree $(q-k, q-k)$, one has

$$\Lambda(a\, D_\delta u_1 \wedge \cdots \wedge D_\delta u_k \wedge \theta) \ge 0.$$

This is a stated generator-positivity condition. It is not a claim that every possible convention for positivity of a delta-current automatically has these properties.

Theorem 17.2 (Forced BT values)**.** `DIRECT_COROLLARY` *of the frozen ordinary calculus, under the explicitly assumed existence of* Λ *above. For every* $0 \le k \le q$*, every admissible* $u_1, \dots, u_k$*, every continuous piecewise smooth scalar* u_0*, and every original compact smooth complementary test* η*, one necessarily has*

$$\Lambda(u_0\, D_\delta u_1 \wedge \cdots \wedge D_\delta u_k \wedge \eta) = \langle u_0 T_{S,k}(u_1, \dots, u_k), \eta \rangle.$$

All quantities use the actual original field, points, positivity cone, and full original test class. In particular two extensions satisfying Definition 17.1 agree on the linear span of these tests. Finite signed polarization gives the identical conclusion for differentiated slots which are differences of admissible potentials.

Proof. First consider $k = 0$. If θ is a positive original compact test and h is a real continuous piecewise smooth scalar, then the generator-positivity assumption applied to $\|h\|_\infty \pm h$ gives

$$|\Lambda(h\theta)| \le \|h\|_\infty \langle S, \theta \rangle,$$

where the supremum is over a fixed compact neighborhood of the support. For an arbitrary symmetric original test, the finite positive-form domination in the original PSD cone gives the same bound with a finite sum of fixed positive controls and a fixed constant. Symmetry reduces a general test to its symmetric part. This is the usual finite-frame coefficient domination; no strong cone spanning assertion is being used.

On this compact stage, a continuous piecewise smooth scalar h admits original smooth approximants h_j converging uniformly. In a finite moment presentation they are ordinary mollifications of its continuous piecewise smooth representative on a slightly larger real open; a finite original smooth partition handles several presentations. Positivity gives

$$\Lambda(h\eta) = \lim_j \Lambda(h_j\eta).$$

The original restriction of Λ and the coefficient-Radon property of S identify the right side with

$$\lim_j \langle h_j S, \eta\rangle = \langle hS, \eta\rangle.$$

Thus multiplication by any such scalar agrees with actual continuous multiplication on the ordinary restriction.

For an admissible u, define insertion into the functional by

$$\Lambda_u(\beta) = \Lambda(D_\delta u \wedge \beta).$$

It is symmetric and separately closed: $D_\delta u$ is separately closed and the delta algebra satisfies the two graded Leibniz rules. The generator-positivity assumption is inherited by Λ_u, with one fewer available differentiated slot. The dual Leibniz identity, using separate closedness of Λ, is

$$\Lambda_u = D(u\Lambda).$$

It is an algebraic identity of functionals on the delta-test complex, not a convergence statement. Restrict it to original tests. The preceding scalar argument identifies the ordinary restriction of $u\Lambda$ with uS. Differentiation commutes with restriction because original smooth tests form a subcomplex. Hence

$$\Lambda_u\big|_{\text{original}} = D(uS) = T_{S,1}(u).$$

The right side is a positive symmetric separately closed coefficient-Radon current by the frozen ordinary BT theorem.

Repeat the argument for Λ_u. Induction identifies the ordinary restriction of insertion of k factors with the corresponding relative BT product. Apply the initial scalar argument once more to that inserted functional and u_0 to obtain the displayed weighted equality. Finally expand any signed slots by finite real multilinearity. Every intermediate unsigned insertion has the proved positivity and closedness, so no quotient theorem for an arbitrary signed background is silently assumed. □

What this does and does not decide. The theorem determines the extension on a specified test span; it is not a density argument and it is not an existence theorem. It also shows that, if a formal relation between tests in this span fails after the ordinary relative BT prescription, then no extension with the three stated properties can exist. Conversely, annihilating those relations constructs an action only on that span. Generation of the whole actual delta algebra, and the positivity or continuity needed for an extension beyond it, remain separate questions. In particular the unrestricted functional counterexamples from the preceding section do not contradict this uniqueness statement.

18 Ordinary restriction does not determine an arbitrary delta-test functional

This section concerns *functionals on tests*, not equality between two actual delta-forms. Proposition 2.1 continues to apply to actual delta-forms. The examples below leave its domain and do not contradict it.

Lemma 18.1 (Top delta-forms have canonical coefficient measures)**.** `DIRECT_COROLLARY`. *On the original good Hausdorff boundaryless pure n-dimensional K-space U, each $\eta \in B_c^{n,n}(U)$ has a canonical finite signed Radon measure μ_η, supported in* $\operatorname{supp}\eta$*, characterized by*

$$\int_U a\, d\mu_\eta = \int_U a\eta \qquad (a \in A_c^{0,0}(U)).$$

For each locally bounded Borel function f on U,

$$\Lambda_f : B_c^{n,n}(U) \longrightarrow \mathbf{R}, \qquad \Lambda_f(\eta) = \int_U f\, d\mu_\eta$$

is a well-defined linear functional. Its restriction to original smooth tests is the ordinary current $[f]$.

It is continuous on every fixed finite polyhedral presentation stage with a common compact support and coefficient C^0 convergence. In particular it is continuous for the stronger coefficient smooth convergence on that stage. This statement specifies the continuity being used; it does not attribute an unmentioned delta-current topology to Mihatsch.

Proof. Choose a finite family of local presentations of η over its compact support. By [Mih24, Corollary 4.9 and Definition 4.10], its support is carried by a finite union of calibrated skeletons. On a common refinement, its restriction is a finite sum of smooth top-degree coefficient forms on weighted polyhedra. Their absolute integrals on the compact stage are finite. They therefore define a finite signed Radon measure on that stage and then on U.

Different skeletal presentations give the same integrals against every original smooth scalar test, since both compute $\int_U a\eta$. Original smooth scalar functions are uniformly dense among continuous functions on a fixed compact neighborhood, with a common compact cutoff. Uniqueness of Radon measures then identifies these constructions. This also proves linearity in η. The support assertion follows by ordinary locality.

On a fixed stage the sum of the absolute polyhedral integrals is bounded by a fixed constant times the maximum of the coefficient uniform norms. Hence

$$|\Lambda_f(\eta)| \le \sup_C |f|\, |\mu_\eta|(C)$$

has the asserted stage bound. The Borel function f is multiplied only after the fixed signed Radon measure has been identified; no tropical measurability condition on f is needed. For smooth η, the measure just constructed is its original CLD integration measure, proving the restriction statement. □

The algebraic GK counterpart of the measure construction is [GK17, Corollary 6.15]. Proposition 6.16 there states the functional result for continuous f. Its displayed bound (6.16.1), once the signed Radon measure is fixed, gives the same continuity estimate for locally bounded Borel f. This observation does not extend the base-field hypotheses of that paper; the proof above uses Mihatsch's integration directly over the original K.

Example 18.2 (A positive functional invisible to all ordinary tests)**.** `FALSE` for unrestricted uniqueness; `DIRECT_COROLLARY` construction. Choose a uniformizer π of the original K, put $\ell = -\log|\pi| > 0$, and let

$$U = \{\, |\pi| < |T| < |\pi|^{-1} \,\} \subset \mathbf{G}^{\mathrm{an}}_{m,K}, \quad t = \log|T|, \quad \xi = \zeta_{0,1}.$$

The point ξ is an actual K-Berkovich point; no algebraic closure or rescaling of the absolute value is used. Let $f = \mathbf{1}_{\{\xi\}}$. Every original smooth compact top-degree form has zero mass at ξ: locally its integration measure has one-dimensional Lebesgue densities on finitely many calibrated skeletal cells. Thus

$$[f]_{\mathrm{ordinary}} = 0, \qquad \Lambda_f|_{A^{1,1}_c(U)} = 0.$$

On the other hand,

$$b = \max(0,t) \in PS^{0,0}(U), \qquad \eta_0 = D_\delta b \in B^{1,1}_c(U)$$

has $\mu_{\eta_0} = \delta_\xi$, with coefficient one by the original one-dimensional torus calibration. Consequently

$$\Lambda_f(\eta_0) = 1.$$

The zero functional and Λ_f therefore have the same ordinary restriction and different values on actual delta-tests. Moreover $\Lambda_f(\eta) \geq 0$ whenever μ_η is a positive measure. This is positivity for the explicitly specified top-degree test cone. The function f is not an actual piecewise smooth function, and Λ_f is not induced by an actual delta-form.

Example 18.3 (Separate closedness alone does not restore uniqueness)**.** `FALSE` for closed unrestricted uniqueness. On the same annulus, define the dual derivative on all compact delta-functions by

$$\Psi(g) = \Lambda_f(D_\delta g), \qquad g \in B^{0,0}_c(U) = PS^{0,0}_c(U).$$

It is a top-degree functional and is separately closed, since its d' and d'' derivatives have degrees exceeding the dimension. Equivalently, the dual differential identities give $d'D_\delta\Lambda_f = d''D_\delta\Lambda_f = 0$. Its ordinary restriction is zero because Dg is a smooth top-degree form for original smooth g.

Choose $0 \leq \chi \in C^\infty_c((-\ell,\ell))$ equal to one near zero and put $g = \chi(t)\max(0,t)$. This is a compactly supported nonnegative actual piecewise smooth function. Its delta-curvature has coefficient one at ξ, so $\Psi(g) = 1$. Therefore Ψ is nonzero despite its zero ordinary restriction.

This example does *not* preserve positivity. Indeed choose a nonnegative original smooth compact cutoff a equal to one on $\operatorname{supp} g$, and $C \geq \sup g$. Then $Ca - g \geq 0$ but

$$\Psi(Ca - g) = -1.$$

It consequently does not disprove uniqueness of an extension that is required to be both separately closed and positive on all appropriate full Mihatsch delta-tests.

Proposition 18.4 (The closed-cycle test subalgebra)**.** `DIRECT_COROLLARY`. *On the same original-K annulus, let $\mathscr{G}$ be the differential subalgebra of B generated by original smooth forms and pullbacks of constant-weight tropical cycle forms. Then Λ_f restricted to $\mathscr{G}^{1,1}_c$ is nonzero, is positive for the Radon top-degree test cone, is separately closed as a functional on $\mathscr{G}_c$, and has zero ordinary restriction.*

Proof. A constant-weight tropical cycle generator is separately closed. The derivatives of a product of such generators and a smooth coefficient differentiate only that coefficient. Thus d'

and d'' preserve its tropical codimension. A form of total degree $(0,1)$ or $(1,0)$ necessarily has codimension zero. Its derivative of degree $(1,1)$ still has codimension zero and therefore has a piecewise smooth Lebesgue density on one-dimensional skeletal cells, without an atom at ξ. The functional Λ_f kills these derivatives, proving separate closedness in this test algebra. It is nonzero because η_0 is the pullback of the constant cycle consisting of the origin in $\mathbf{R}$ and belongs to $\mathscr{G}$. The remaining assertions were proved above. □

The last proposition is not a closed-positive nonuniqueness theorem for the *full* Mihatsch test algebra. In that larger algebra, the piecewise smooth function g from the preceding example is an admissible test and detects the failure of closedness of Λ_f. This is one concrete reason to state the test sheaf before comparing delta-currents. We have defined $\mathscr{G}$ over the original K; we have not identified it with the full algebraic B_{GK}.

The conclusion for arbitrary backgrounds. The ordinary coefficient-Radon data of a current alone do not specify its values on singular delta-tests. A proposed extension must include its test sheaf, continuity requirements, differential compatibility, positivity requirements, and an independence proof. The examples establish the failures precisely stated above. They do not rule out an additional canonical extension mechanism for separately closed positive currents, and they do not affect the actual delta-form comparison of Theorem 2.4.

Part II
The ordered potential quotient and compactness

19 Functoriality of curvature order for actual bounded potentials

The zero-kernel transfer in R5 does not by itself transfer an order inequality. The additional points proved here are positivity of the piecewise traces, positivity on a carried polyhedral set, and positive finite-coefficient descent. These preserve the original middle CLD cone throughout. No pullback operation on arbitrary currents is defined.

Over an auxiliary complete field, a prescribed row means the fixed ordinary-neighborhood smooth positive decreasing row of R5, Definition 3.1. Only pullbacks of rows supplied over the original K are used in the final theorem. The field-independent coefficient lemma and piecewise continuity in R5, Lemmas 2.1 and 3.2, remain applicable. All complete fields carry their given extending absolute values.

Lemma 19.1 (Positivity on the polyhedral set carrying the mass)**.** *Let R be a symmetric positive separately closed coefficient-Radon current over any complete field in the preceding scope. On a fixed compact original moment stage $t : V \to \mathbf{R}^m$, suppose its coefficient measures are carried by a Borel set A whose moment image lies in a finite polyhedral complex P. Let η be a smooth or compatible piecewise smooth complementary form. If its restriction to every face of P is middle-positive in that face's affine tangent space, then*

$$\langle R, t^*\eta \rangle \geq 0$$

whenever the displayed test has the stated compact localization. The same conclusion holds after a nonnegative Borel restriction or scalar multiplier supported on this stage. No positivity of an ambient extension away from P is required.

Status: `DIRECT_COROLLARY` of the coefficient matrix description and level-set tangency, with the carried-set scope made explicit.

Proof. Use the disjoint inverse images of the relative interiors of all faces of P, restricted to A. The arctangent proof of Lemma 4.1 uses only smooth differentiation and fixed Radon dominated convergence, so works over the present complete field. On the Borel part over a face F, it annihilates every primed and double-primed conormal of its affine span. This remains true after Borel restriction. Thus only the tangential tensor of η can contribute on this part.

For completeness, choose a constant basis of the appropriate exterior power of the tangent covectors of F and extend it to the ambient moment space by one fixed linear projection. The coefficient measures of this Borel part, relative to one finite sum of diagonal control measures, give a symmetric positive semidefinite matrix M_F almost everywhere. Indeed every constant rank-one square in the middle cone has a positive measure; rational coefficient vectors and continuity give positivity for all vectors. Nondecomposable squares are included. The tangent matrix $A_F(x)$ of η is positive semidefinite. The pairing density is $\operatorname{tr}(A_F M_F) \geq 0$ in the usual signed identification of superforms. If the face dimension is smaller than the form degree, that tangent form is zero. Sum over the finite faces. Nonnegative multipliers preserve the matrix positivity, proving the final assertion. The definition of this cone and its restriction property are exactly [CLD12, 5.1.1(c), 5.1.5, 5.2.1 and 5.3.1]. □

Lemma 19.2 (The finite and split projected tests preserve positivity)**.** *The piecewise trace of an original positive compact test under a finite etale map, as constructed in R5, Lemma 3.3, is tangentially middle-positive on each actual base tropical image. The same holds for the split-disc projected complementary tests of R4, Theorem 72.1. These statements retain every further original moment refinement.*

Proof. Localize the original positive form by a nonnegative smooth partition, using positive presentations from [CLD12, Definition 5.3.1]. For a finite etale piece, the fixed trace formula is

$$P(t) = \sum_{l=1}^{d} z_l^* \omega,$$

where the joint vectors $z_l(t)$ are listed by the characteristic polynomials of finitely many monomials and are affine on a finite subdivision. Each vector over an actual base point is realized by a source point, with the full positive algebraic multiplicity. Refine the base cells by inverse images of the finitely many source faces in a positive presentation. On each resulting base face, the affine map z_l maps its tangent space into the tangent space of an actual source face. Pullback preserves the middle-positive cone by [CLD12, Remark 5.1.5]; hence every summand and their sum are positive on that face. Matching on smaller faces preserves positivity there. The extension off the consistent moment locus in the trace construction is immaterial by Lemma 19.1. This argument also works after adjoining extra base moments; the same actual sheets and positive calibrated degrees are retained.

For the split disc, use the exact formula of R4, Theorem 72.1:

$$P(t) = \sum_i \int_{-\infty}^{L_i(t)} \mathcal{I}_s(F_i^* \omega)(t, s)\, ds, \qquad F_i(t, s) = (t, \max(s, b_{i1}), \dots, \max(s, b_{im})).$$

After the finite chamber subdivision, these maps are affine and their values on actual base points and their assigned radial intervals are actual source Gauss-ray points. Restricting further by source faces makes the tangent maps land in their actual tangent spaces. The form $F_i^* \omega$ is consequently middle-positive there. Its vertical top contraction $\mathcal{I}_s$ is positive: on a positive semidefinite tensor it is the compression obtained by wedging a base exterior vector

with the one vertical vector, on both sides of the corresponding quadratic form. Its sign is the positive product-calibration sign. Integrating this positive tensor against ds, summing the assigned rays once, and multiplying by the nonnegative localization all preserve positivity. Collapsed intervals contribute zero. Tangential matching on base faces, proved in the frozen formula, gives the same conclusion on their intersections. There are no additional wall currents in that formula. □

Proposition 19.3 (Finite etale and original product order transfer)**.** *Let a, b be actual locally bounded classical functions with prescribed rows over a complete characteristic-zero field F. If $D[a-b] \geq 0$, then for a finite etale map p,*

$$D[(a-b)\circ p] \geq 0.$$

Over the original K, the same implication holds for projection from an ordinary polydisc product of any finite dimension.

Proof. Put $f = a - b$ and $T = D[f]$. In the finite etale case take a positive original compact test α upstairs and let $P = p_*[\alpha]$. Localize by a nonnegative base partition so that both prescribed rows exist on a common neighborhood of the projected support. The frozen finite trace is a finite compatible piecewise form, and $\nu = DP = p_*[D\alpha]$ is one fixed signed compact Radon measure. For the smooth rows a_j, b_j, compact Stokes gives

$$\langle P, D(a_j - b_j)\rangle = \int (a_j - b_j)\, d\nu.$$

Piecewise continuity, applied separately to the positive closed currents Da_j and Db_j, and fixed-measure bounded convergence give the exact nonzero identity

$$\langle D[f\circ p], \alpha\rangle = \langle T, P\rangle \geq 0. \tag{19.1}$$

The inequality follows from the preceding two lemmas. This does not assume that $a_j - b_j$ has positive curvature.

For an original-K disc product, follow the finite rational localization of R4, Theorem 74.3, with nonnegative partitions. For one positive localized test α, take the same fixed finite polynomial list, remove its base discriminant by the finite etale splitting map, and set $\alpha_j = \theta_j(\log|H|)\alpha$, where $0 \leq \theta_j \leq 1$ vanishes near $H = 0$ and increases pointwise to one off that fixed analytic divisor. These remain positive tests. Finite etale order transfer gives positivity of the pulled-back base curvature. For the split test upstairs, the same smooth-row calculation as in (19.1), now with the split projected test, gives a nonnegative pairing. The original finite-flat integration formula multiplies the downstairs pairing by the positive finite degree, so the downstairs pairing with α_j is nonnegative too.

The signed current $D[f\circ\pi]$ is the difference of the two positive ambient prescribed-row curvature currents. Every coefficient variation of these two currents vanishes on the removed analytic divisor, by the frozen first-order nonconcentration theorem (R5, Theorem 2.3, or its original-K predecessor). Dominated convergence against these fixed coefficient measures gives

$$\langle D[f\circ\pi], \alpha\rangle = \lim_j \langle D[f\circ\pi], \alpha_j\rangle \geq 0.$$

No uniform differentiability bound on α_j is needed. Iteration gives all finite-dimensional polydisc products, always over the original K and with the pulled-back prescribed rows. □

Lemma 19.4 (Positive finite-coefficient descent and positive fiber extraction)**.** *The following two refinements of the frozen test-spreading arguments hold.*

1. *In exact finite-coefficient descent from X_L to X_E, $E = \overline{K(c_1, \dots, c_m)}$, a positive original test can be descended to a positive original test. The analogous exact descent from a completed algebraic closure to a finite extension has the same property.*
2. *Let $j : X_H \hookrightarrow X_F \times_F A$ be the full fiber over a parameter point, with H its completed residue field. Suppose a symmetric positive separately closed current J on the product is carried by this fiber and satisfies*

$$\langle J, \alpha \rangle = \langle T_H, j^* \alpha \rangle$$

for all original compact complementary product tests, where T_H is a coefficient-Radon current on the fiber. If exact spreading of its original fiber tests is available as in R5, Lemma 4.4, then T_H is positive.

The second statement does not assert that a positive fiber form has a positive extension to the whole ambient product.

Proof. For (1), start with positive presentations from [CLD12, Definition 5.3.1] and use nonnegative scalar localizations. Perform the frozen rational-unit replacement on slightly larger compact norm boxes. It preserves the actual unit logarithms on each box, so preserves the positive tensor on its tropical image. Include all coefficients of the rational units *and of the domains and cutoff profiles* in the finite field. The resulting domain over L is the scalar extension of the corresponding domain over E. Scalar projection of an affinoid base change is surjective; their images under these common unit moments are therefore identical. The same real coefficient form is positive on exactly this common polyhedral image, which is the definition of original positivity. The descended localized tests are consequently positive.

For descent from a completed algebraic closure, approximate the finite coefficients by one finite algebraic extension, with errors strictly below all unit lower bounds and all positive profile thresholds. Closed upper and lower norm inequalities are unchanged under these ultrametric error bounds, and profiles near possible zeros are constant. Thus the same compact boxes and active norms are recovered exactly after scalar extension. The identical-image argument again proves positivity. This is stronger information than equality of the signed integrated forms, and follows from retaining the positive presentation itself.

For (2), take a positive compact fiber test and partition it into nonnegative localizations of positive moment presentations on compact fiber norm domains. Spread these particular presentations by the frozen exact construction: it retains all unit logarithms, norm-domain cutoffs, and zero-protected scalar profiles on the fiber. A local spread can be written $\chi\, \widetilde{t}^* \eta$, where χ is nonnegative, its restriction is the original localizing scalar, and η is positive on the finite tropical image P of the corresponding fiber domain.

The current χJ is positive, and its coefficient mass on the fiber is supported inside that domain. Every level tangency of J survives multiplication by χ and Borel restriction. Stratify this mass by the faces of P, whether or not those faces occur in the original ambient tropical complex. Their affine conormals are original smooth affine combinations of the spread unit logarithms. Lemma 19.1 therefore gives $\langle J, \chi \widetilde{t}^* \eta \rangle \geq 0$. Add the finite localizations and use the displayed restriction identity. The arbitrary positive fiber test has a nonnegative pairing with T_H, as required. Only the tangential tensor on the mass-carrying fiber was required to be positive. □

Theorem 19.5 (Order transfer through every complete scalar extension)**.** *Let $a, b \in \mathcal{C}_b(X)$ on an allowed original-K space and suppose $D[a - b] \geq 0$ on all original tests. Then for every complete L/K,*

$$D[a_L - b_L] \geq 0$$

on every original L-test.

Status: `NEW_PROVED`, using the positive trace and carried-fiber lemmas. The corresponding zero equality was already proved in R5.

Proof. Work locally in one original algebraic neighborhood with prescribed rows for a, b. Put $f = a - b$. Finite scalar extensions preserve positivity by Proposition 19.3; the same-space calibration factor is the positive number $[F' : F]$. Part (1) of Lemma 19.4, and the factor-one fixed signed-measure scalar integration formula [CLD25, Proposition 15.3.1], then preserve positivity on passing from an auxiliary F to its completed algebraic closure.

Here is the one-parameter step, specifying the new point where order differs from the zero-kernel argument. Suppose $D[f_F] \geq 0$ on X_F and the same holds for its pullback to $X_F \times \mathbf{A}^{1,\mathrm{an}}_F$. Given a point of the parameter line, embed its residue field in a completed algebraic closure and then take $C = \widehat{F^{\mathrm{alg}}}$ there, as in R5, Proposition 4.6. The induced point $y \in \mathbf{A}^{1,\mathrm{an}}_C$ has a residue field H containing the original parameter residue field. Both assumed positive currents remain positive over C.

For a nonrigid y, R5, Proposition 4.3, gives the full-fiber potential b_y: for types II and III it is $\log \max(r, |z - c|)$, and for type IV it is the uniform limit of the corresponding nested-disc potentials. Each is locally uniformly approximable by original smooth positive functions: smooth the real convex maximum, and in type IV also use the proved uniform error $0 \leq b_j - b_y \leq \log(r_j/r)$. These are explicit functions, not an invocation of R over C.

On the product let $T = D[f_C \circ q] \geq 0$ and put

$$J = D(b_y T) = D((f_C \circ q) D[b_y]).$$

The equality is R5, Lemma 4.1, applied separately to the prescribed rows for a and b. Smooth positive approximation of b_y and fixed coefficient-Radon multiplication show that J is positive and separately closed. The second expression shows that it is carried by the full parameter fiber. Its exact restriction identity is

$$\langle J, \alpha \rangle = \int_{X_H} f_H D(j^* \alpha),$$

by the full-fiber formula and its fixed Borel version. Part (2) of Lemma 19.4 now proves $D[f_H] \geq 0$ on every original fiber test. A rational parameter point requires no new extension. Finally, pull a positive test over the original parameter residue field to H; positive smooth pullback and factor-one scalar integration descend the inequality. This proves the one-parameter step without a positive ambient extension of an arbitrary fiber test.

Propagate these statements simultaneously with every finite number r of affine-space factors. The initial assertions over K are Proposition 19.3. If they hold over F for all r, the assertion with $r + 1$ supplies the product hypothesis for the one-parameter step applied to the assertion with r. Hence they hold over the residue field of every point of $\mathbf{A}^1_F$. Iterating through $F_i = \overline{K(c_1, \ldots, c_i)}$ proves them over every finite parameter field. These fields have the countable dense subfields used in the frozen R5 construction; their algebraic closures and all the prescribed rows are exactly those in that construction.

For an arbitrary complete L/K, descend any positive original compact test exactly to one such finite parameter field by part (1) of the preceding lemma. The fixed Borel scalar integration identity identifies its pairing with the already nonnegative pairing over that field. Nonnegative original partitions handle the initial algebraic neighborhoods. This proves the theorem on all original tests, without extending B's regularization hypothesis to any new complete field. □

Theorem 19.6 (Analytic functoriality of the represented order)**.** *Under the hypotheses of the preceding theorem, let L/K be any complete extension and let $F : Z \to X_L$ be an analytic*

morphism from an allowed L-analytic space. Then

$$D[(a_L - b_L) \circ F] \geq 0$$

on every original compact complementary test of Z. This is an order statement for the specified actual bounded classical functions, not a pullback of an arbitrary positive current.

Proof. First observe that the normal trace in R5, Lemma 5.2, preserves order. If $D[c - d] \geq 0$ on $B \times \mathbb{D}_s$, test it with $\beta_N \wedge \pi^*\eta$ for an original positive compact complementary η on B. The normal form β_N is smooth positive of degree $(1,1)$, hence strongly positive; its wedge with $\pi^*\eta$ is positive. The frozen trace identity, applied to c and d separately, gives

$$\langle D[(c-d)|_{t=0}], \eta\rangle = \lim_N \langle D[c-d], \beta_N \wedge \pi^*\eta\rangle \geq 0.$$

The trace uses actual Gauss radial limits of the two classical functions and the already checked normal coarea, including its full finite-field degree. It applies over any complete field and iterates in finitely many normal coordinates.

Use the precise geometry of R5, Theorem 5.3. Over the original K, B's ordinary algebraic localization and R's tubular neighborhood embed the target in an affine space and extend a, b constantly in normal polydisc directions. Proposition 19.3 gives the order on this ordinary ambient neighborhood and on all its finite affine products. Theorem 19.5 gives those same inequalities after extension to L.

At a source point use the genuine ordinary finite etale coordinates of R5, Lemma 5.1. Their product with the ambient target is finite etale; Proposition 19.3 transfers the inequality to that product. Translation by the analytic map F makes its graph the zero section. Analytic isomorphisms preserve the original forms, calibration, and their positive cone. The frozen compact-tube argument supplies a whole normal polydisc tube around the zero section on a smaller ordinary source neighborhood. Apply the positive normal trace just proved in each coordinate. The restriction is exactly the actual difference $(a_L - b_L) \circ F$. Zero-dimensional source or target cases are immediate. Locality proves the assertion globally on Z.

Every algebraic or tubular neighborhood used in this step is over the original K. The new source over L is handled by genuinely etale ordinary-open coordinates and normal trace. All new-field rows are the pullbacks of the original prescribed rows. □

Scope of the advance. The represented current order is now functorial for actual bounded classical differences. A bounded Borel function with distributionally positive curvature is not covered by this result. Nor has an arbitrary positive current acquired a pullback, a potential, or an action on all higher-codimension delta-tests. The carried-fiber argument specifically avoids claiming that positivity of an arbitrary fiber test extends to a positive ambient test.

20 Continuity on a curve under a continuous Laplacian dominator

This section uses an order inequality of actual curve Laplacian measures. It does not pull back an ambient current inequality. That distinct operation is isolated in the order-transfer argument.

Lemma 20.1 (Positive potential splitting)**.** *Let Y be a topological space, let $P_1, P_2 : Y \to [0, \infty]$ be lower semicontinuous, and suppose $P_1 + P_2$ is finite and continuous. Then P_1 and P_2 are finite and continuous. Consequently, for a nonnegative potential kernel G, if positive measures satisfy $0 \leq \mu \leq \tau$, their kernel potentials are lower semicontinuous, and P_τ is finite continuous on Y, then P_μ is finite continuous there.*

Status: `DIRECT_COROLLARY` of semicontinuity and additivity of nonnegative integrals.

Proof. Finiteness follows from the nonnegative sum. Each P_i is also upper semicontinuous, since it is the continuous sum minus the other lower semicontinuous function. For the potential assertion, use $P_\tau = P_\mu + P_{\tau-\mu}$ and Tonelli's theorem. □

Proposition 20.2 (Actual continuity under domination on a curve)**.** *Let F be complete, algebraically closed, nontrivially valued and of characteristic zero, with its fixed absolute value. Let C be a good Hausdorff boundaryless smooth F-analytic curve. Suppose a is a finite continuous subharmonic function and b is an actual locally bounded subharmonic function on an ordinary open $O \subset C$. If*

$$0 \le \Delta b \le \Delta a$$

as Radon measures on O, then b and $a - b$ are continuous on O, and $a - b$ is subharmonic there. The values at points of all four Berkovich types are their given actual values.

Status: `DIRECT_COROLLARY` of the curve potential kernel and distributional harmonic kernel, with the actual-value passage specified in the proof. This assertion is not a higher-dimensional distributional positivity converse.

Proof. Local algebraization, in the precise complete-field scope of R5, Lemma R5–6.1, places a neighborhood of the point under consideration in a smooth projective algebraic curve $\overline{C}$ over F. Choose a compact affinoid neighborhood Y of that point contained in O, and choose a type-II point $\zeta \in \overline{C}^{\mathrm{an}} \setminus Y$. Shrinking Y if necessary makes this possible. Set

$$\tau = \mathbf{1}_Y \Delta a, \qquad \mu = \mathbf{1}_Y \Delta b.$$

They are finite positive Radon measures and $\mu \le \tau$.

Use the potential kernel $g_\zeta(x, y) \ge 0$ of [Wan22, Definition 4.4, Lemma 4.7 and Proposition 4.8]. For a finite positive measure ρ supported in Y, write

$$P_\rho(x) = \int_Y g_\zeta(x, y)\, d\rho(y).$$

Its lower semicontinuity and its limits along paths to endpoints are the kernel-potential properties in [Wan22, Proposition 6.5]. Its distributional Laplacian is $\rho(Y)\delta_\zeta - \rho$, as follows from the fixed-kernel Laplacian identity and integration against a compact lisse test. These facts concern the original absolute value. A positive constant change in the convention for logarithmic lengths changes both sides by the same factor and is not performed here.

For type-II or type-III x, the function $y \mapsto g_\zeta(x, y)$ is finite continuous on the compact curve, so $P_\rho(x)$ is finite. Thus $a + P_\tau$ defines a degree-zero curve distribution on $\operatorname{Int} Y$ with Laplacian zero. The distributional harmonic kernel theorem (Thuillier, Lemma 3.3.12, as explicitly used in [Wan22, Remarks 7.2 and 7.12] and in R5) gives a continuous harmonic function H_a agreeing with $a + P_\tau$ on the type-II and type-III points of that interior.

This equality holds at the remaining actual points as well. If x has type I or IV, choose an open-ball neighborhood inside $\operatorname{Int} Y$ and nested closed discs with boundary Gauss points $\xi_j \to x$ on the segment toward x. The subharmonic maximum principle and upper semicontinuity of a give $a(\xi_j) \to a(x)$; here continuity of a would already suffice. The path-limit property of the positive kernel potential gives $P_\tau(\xi_j) \to P_\tau(x)$, including a possible infinite limit before the equality is used. Since $H_a(\xi_j) \to H_a(x)$ is finite, we obtain

$$P_\tau(x) = H_a(x) - a(x) < \infty.$$

The same equality was already known on the other two types. It follows that $P_\tau = H_a - a$ is finite continuous throughout $\operatorname{Int} Y$.

Apply Lemma 20.1 to $P_\mu + P_{\tau-\mu} = P_\tau$. It shows that P_μ and $P_{\tau-\mu}$ are continuous on this interior. Similarly, $b + P_\mu$ has harmonic degree-zero distribution, so it agrees on type-II and type-III points with a continuous harmonic H_b. At a type-I or type-IV point, the same nested-disc argument now uses the given subharmonicity of b:

$$b(x) \le b(\xi_j), \qquad \limsup_j b(\xi_j) \le b(x).$$

Thus $b(\xi_j) \to b(x)$. Continuity of P_μ and H_b gives $b = H_b - P_\mu$ at that point too. This proves continuity of the actual b on $\operatorname{Int} Y$.

Finally $a - b$ is continuous and its curve distribution has Laplacian $\Delta a - \Delta b \ge 0$. The continuous distributional criterion for curve subharmonicity gives that it is subharmonic; see [Wan22, Definition 3.11 and Remark 3.12]. The interiors of such Y cover O, proving all assertions locally and hence on the entire open. □

How this enters the higher-dimensional argument. For a classical u and a continuous classical g, a separately proved order-transfer statement would give $0 \le \Delta(u \circ F) \le C\Delta(g \circ F)$ on every smooth curve after every complete scalar extension. The proposition then makes $u \circ F$ and $(Cg - u) \circ F$ continuous and the latter subharmonic. This step uses actual curve functions. Ordinary continuity on the higher-dimensional space still requires the finite Shilov interpolation argument; it is not inferred from continuity along a dense collection of curves alone.

21 Positive curvature in the actual bounded classical difference class

The difference class in this section consists of actual functions, not arbitrary representatives of degree-zero currents. Its two positive parts belong to $\mathcal{C}_b$ on the same ordinary neighborhood. This restriction is essential to the curve endpoint arguments and to the ambient semicontinuity proof.

Lemma 21.1 (A positive difference of actual curve Laplacians)**.** *Let F be algebraically closed, complete, nontrivially valued and of characteristic zero. Let a, b be actual locally bounded subharmonic functions on an ordinary open in a smooth boundaryless analytic F-curve. If $\Delta a - \Delta b \ge 0$, then the actual function $a - b$ is locally bounded subharmonic. Neither a nor b is assumed continuous.*

Status: `DIRECT_COROLLARY` of the positive curve kernel, with actual endpoint values retained.

Proof. Use the local projective algebraization and compact affinoid Y of Proposition 20.2. Choose its type-II kernel base point ζ outside Y, and put $\tau = \mathbf{1}_Y \Delta a$, $\mu = \mathbf{1}_Y \Delta b$ and $\rho = \tau - \mu \ge 0$. For any of these finite positive measures write $P_\nu(x) = \int g_\zeta(x, y)\, d\nu(y)$. These are nonnegative lower semicontinuous potentials with the path-limit and Laplacian properties of [Wan22, Propositions 6.5–6.6]. They are finite at type-II and type-III points.

On $\operatorname{Int} Y$ the distributions $a + P_\tau$ and $b + P_\mu$ have Laplacian zero. The curve distributional kernel theorem gives continuous harmonic functions H_a, H_b agreeing with them on those two types. At a type-I or type-IV point x, use nested closed discs inside an open-ball neighborhood, with boundary points $\xi_j \to x$ along its incoming path. The actual subharmonic maximum principle and upper semicontinuity give $a(\xi_j) \to a(x)$ and $b(\xi_j) \to b(x)$ separately. The kernel path limits then identify

$$P_\tau = H_a - a, \qquad P_\mu = H_b - b$$

at x as well, and show that both are finite there. No continuity of a or b has been used for this actual-value identification. The same identities already hold on types II and III.

Additivity of nonnegative integrals now yields $P_\tau = P_\mu + P_\rho$, with all terms finite on the interior. Consequently

$$a - b = H_a - H_b - P_\rho$$

is upper semicontinuous there. We verify harmonic comparison with these actual values, rather than use an unrestricted representative criterion. Put $f = a - b$. Let $Z \Subset \operatorname{Int} Y$ be a connected strictly affinoid domain and let h be harmonic on Z with $f \le h$ on ∂Z. For a type-II or type-III point $x \in \operatorname{Int} Z$, the compact nonnegative lisse Green function $g_{Z,x}$ of [Wan19, Proposition 2.12] satisfies

$$\Delta g_{Z,x} = \omega_x - \delta_x, \qquad \operatorname{supp} \omega_x \subset \partial Z, \qquad h(x) = \int_{\partial Z} h \, d\omega_x.$$

The boundary measure ω_x is nonnegative: the lisse Green function is nonnegative, vanishes on the boundary and outside Z, and its outward slope sum at a boundary point is therefore nonnegative. Its mass is one, as follows by taking $h = 1$. Evaluation of the positive distribution Δf on this nonnegative lisse test gives

$$0 \le \langle \Delta f, g_{Z,x} \rangle = \int_{\partial Z} f \, d\omega_x - f(x) \le h(x) - f(x).$$

At a type-I or type-IV interior point, the earlier nested-disc argument gives the actual path limits of a and b separately. Pass the last inequality along type-II/III points on that path, using continuity of h, to obtain $f \le h$ there as well. Thus f satisfies the original affinoid harmonic comparison principle, so it is subharmonic. Local boundedness is inherited from the original difference. Cover by such interiors. □

Lemma 21.2 (One-sided Shilov bounds without ambient semicontinuity)**.** *In the geometric setting of R5, Theorem* R5–7.2*, replace harmonic curve pullbacks by actual subharmonic curve pullbacks. Then an actual finite function f satisfies*

$$f(x) \le \max_{\Gamma(V)} f \quad (x \in V).$$

No upper semicontinuity of f on the ambient space is assumed. The same bound applies on each of the frozen semistable charts.

Status: `DIRECT_COROLLARY` of the one-sided argument in the frozen curve-slicing proof.

Proof. Use the finite flat map $V \to \mathbb{B}^n$ whose coordinates extend regularly to the algebraic ambient space. Fixing all but one coordinate and passing to its completed residue field gives the full affinoid curve fiber. Its algebraic normalization is a smooth curve defined across the whole compact fiber. Subharmonicity of the actual pullback and the curve maximum principle bound its value at every point by its maximum on that fiber's finite Shilov boundary. The boundary is the preimage of the corresponding one-variable Gauss point, by the frozen finite flat Gauss-fiber lemma.

If $M(t)$ is the maximum of f on the finite nonempty fiber over $t \in \mathbb{B}^n$, this proves $M(t) \le M(G_i(t))$ for each coordinate Gauss operation G_i. These operations commute and their composition is the full Gauss point. Iterating proves the displayed inequality. All maxima of finite fibers are actual finite numbers. The argument uses only one sign and no ambient semicontinuity. For a semistable chart, its regular topological generators give the required finite flat algebraic Noether coordinates by R5, Lemma R5–9.1. □

Theorem 21.3 (Upper semicontinuity of a curvewise subharmonic difference)**.** *Let $a, b \in \mathcal{C}_b(\Omega)$ on an allowed original-K space, and put $f = a - b$ as an actual function. Suppose every pullback of f to a smooth analytic curve after every complete extension is an actual subharmonic function. Then f is upper semicontinuous on Ω. In particular $f \in \mathcal{C}_b(\Omega)$.*

Status: `NEW_PROVED`. The essential step is an upper semicontinuous majorant obtained from a proper finite-chart cover; no ambient regularity is assumed for the difference.

Proof. Fix $x \in \Omega$ and $\epsilon > 0$. Since a is upper semicontinuous, choose an ordinary neighborhood O of x with $a(z) < a(x) + \epsilon$ for every $z \in O$. Use the frozen algebraic localization and norm-box construction to obtain $x \in \operatorname{Int} V$ with $V \subset O$. Let $q : W \to V$ and its finitely many compact semistable charts W_λ be the cover of R5, Proposition R5–9.3.

For each chart, use its nonnegative unit-logarithm coordinates λ_i, summing to one, to define the continuous pluriharmonic interpolants

$$P_\lambda a = \sum_i a(q(\xi_i))\lambda_i, \qquad P_\lambda b = \sum_i b(q(\xi_i))\lambda_i, \qquad P_\lambda f = P_\lambda a - P_\lambda b.$$

All functions are defined on genuine ordinary neighborhoods of the charts. The Shilov values of $f \circ q - P_\lambda f$ vanish, and this difference has actual subharmonic curve pullbacks. Lemma 21.2 therefore gives

$$f \circ q \le P_\lambda f.$$

The ordinary classical maximum principle applied to $b \circ q - P_\lambda b$ gives $b \circ q \le P_\lambda b$. Since every $q(\xi_i)$ lies in $V \subset O$ and the coefficients are nonnegative and sum to one, $P_\lambda a < a(x) + \epsilon$ on the entire chart. In particular, for every chart point y above x,

$$P_\lambda f(y) \le a(x) + \epsilon - b(x) = f(x) + \epsilon.$$

Take the disjoint union of these finitely many compact charts and let $Q : \coprod_\lambda W_\lambda \to V$ be its continuous surjection. The function which equals $P_\lambda f$ on each component is continuous. Define

$$e(z) = \max\{P_\lambda f(y) : y \in W_\lambda,\ q(y) = z\}.$$

Every fiber is nonempty and compact. For any real c, the set $\{e \ge c\}$ is the image under Q of the compact set on which this continuous componentwise function is at least c; it is closed in the Hausdorff space V. Hence e is upper semicontinuous. The preceding inequalities give $f \le e$ on V and $e(x) \le f(x) + \epsilon$. Since x is in the ordinary interior,

$$\limsup_{z \to x} f(z) \le e(x) \le f(x) + \epsilon.$$

Let $\epsilon \downarrow 0$. This proves upper semicontinuity at x. The difference is locally bounded. Once this ambient upper semicontinuity has been established, the smooth-curve testing reduction in [GR25, 4.1.3 and Definition 3.4] applies to the assumed complete-extension curve property and gives membership in the actual classical class. In particular it covers test curves with boundary or singularities by the compositions appearing in that definition. Zero-dimensional components are locally discrete and immediate. □

Theorem 21.4 (Positivity determines the bounded classical difference cone)**.** *On an allowed original-K ordinary open Ω, let $f = a - b$ with $a, b \in \mathcal{C}_b(\Omega)$. Then*

$$D[f] \ge 0 \text{ in the original CLD cone} \quad \Longleftrightarrow \quad f \in \mathcal{C}_b(\Omega).$$

The same assertion holds for a locally represented difference, with possibly different decompositions on different ordinary neighborhoods. No assertion is made for an arbitrary locally bounded Borel function whose associated degree-zero distribution has positive curvature.

Status: `NEW_PROVED`, relative to the order-functoriality theorem and the preceding semicontinuity mechanism.

Proof. The reverse implication is the positive-current assertion already contained in B. Suppose $D[a-b] \geq 0$. For every complete extension and every smooth analytic curve map, Theorem 19.6 gives positive original curvature for the actual pullback difference. Over an algebraically closed complete field, the frozen curve algebraization and B's explicitly complete-field Lemma A.2, in the scope checked in R5, Lemma R5–6.1, identify this as $\Delta(a \circ F) - \Delta(b \circ F) \geq 0$ as Radon measures. Indeed the inequality holds on all nonnegative original smooth scalar tests; their compact uniform density extends it to nonnegative continuous tests. Lemma 21.1 makes the actual difference subharmonic on the curve.

For a curve over a non-algebraically closed complete field, perform the argument after completed algebraic closure. Upper semicontinuity descends through scalar projection: on a compact affinoid neighborhood it is a continuous surjection from a compact space onto a Hausdorff space, and the upper level sets of the fiber-constant pullback are closed; their images give the upper level sets downstairs. Subharmonicity then descends by [GR25, Proposition 3.5(7)]. We have proved the actual subharmonicity assertion on every smooth complete-extension test curve. Theorem 21.3 now supplies the missing ambient upper semicontinuity and then applies the full classical smooth-curve reduction. Local boundedness is already known. This proves the forward implication. Ordinary restriction and locality prove the locally represented version. □

Corollary 21.5 (Order of represented curvatures transfers to every background)**.** *For any allowed fixed positive separately closed (p,p)-background S, with $p < n$, and $a, b \in \mathcal{C}_b(\Omega)$,*

$$D[a] \geq D[b] \quad \Longrightarrow \quad D(aS) \geq D(bS).$$

All inequalities and equalities use every original complementary compact smooth test. The same order replacement is valid in any one differentiated slot of a mixed BT product, with its other slots fixed in $\mathcal{C}_b$.

Status: `DIRECT_COROLLARY`. Equality of the represented curvatures was already handled by the R5 quotient; the new conclusion preserves their order, including discontinuous actual potentials.

Proof. The difference $a - b$ is actual classical by the theorem. Its product $D((a-b)S)$ is positive by the frozen arbitrary-background calculus, and equals $D(aS) - D(bS)$ by linearity of Borel multiplication and D. For another differentiated slot, use symmetry and replace S by the positive separately closed tail containing the other fixed potentials. No arbitrary-current wedge is introduced. □

Theorem 21.6 (Continuous coefficient relations for bounded potential slots)**.** *Let $g_1, \dots, g_m \in \mathcal{C}_b(\Omega)$ be actual locally bounded classical potentials, and let $f_1, \dots, f_m$ be finite continuous real functions. Define the coefficient-Radon currents*

$$A = \sum_{i=1}^{m} f_i D[g_i], \qquad A_S = \sum_{i=1}^{m} f_i D(g_i S).$$

For every fixed allowed background S, $A \geq 0$ implies $A_S \geq 0$. In particular $A = 0$ implies $A_S = 0$ on all original tests. Finite continuous-coefficient presentations by curvatures of locally bounded classical differences therefore have a well-defined positive linear action over S, independently of their presentation. The differentiated potentials need not be continuous. The scalar coefficients in this statement are continuous.

Status: `NEW_PROVED` extension of the continuous-potential module theorem to all actual bounded classical slots. The coefficient-bearing currents are not asserted separately closed merely because their individual curvature factors are closed.

Proof. Put $T = \sum_i D(g_i S) \geq 0$. Fix $\epsilon > 0$. Near any point choose constants c_i with $|f_i - c_i| < \epsilon$. On this ordinary neighborhood,

$$D\left(\sum_i c_i g_i + \epsilon \sum_i g_i\right) = A + \sum_i (c_i + \epsilon - f_i) D[g_i] \geq 0.$$

The actual function inside parentheses is a bounded classical difference, by separating its finitely many positive and negative real coefficients. Theorem 21.4 therefore makes it classical, and its curvature over S is positive. Thus $\sum_i c_i D(g_i S) \geq -\epsilon T$. The coefficient error gives

$$A_S \geq -2\epsilon T$$

on this neighborhood. Cover the support of a positive original test by finitely many such neighborhoods and apply a nonnegative original smooth partition. Positivity is local, so the same inequality holds on that test globally. Let $\epsilon \downarrow 0$, using its fixed finite control mass against T. This proves $A_S \geq 0$.

If $A = 0$, apply the implication both to the coefficients f_i and to $-f_i$. The original middle positive cone of currents is pointed, so $A_S = 0$. To compare two signed-potential presentations, expand each classical difference into its positive classical parts, subtract the presentations, and use the proved zero relation. Addition and continuous scalar multiplication are then well defined; the positive-order assertion is exactly the first part of the theorem. □

Corollary 21.7 (No a priori continuity is needed in a curvature order interval)**.** *Let $u \in \mathcal{C}_b(\Omega)$ and let g be continuous classical. If $0 \leq D[u] \leq CD[g]$ for some $C \geq 0$, then u and $Cg - u$ are continuous classical functions. Consequently all compactness, represented-quotient, actual-image and joint-continuity conclusions above for normalized continuous curvature order families remain true when their defining family initially allows all actual $u \in \mathcal{C}_b$. This also applies to the family with $D[u] \leq D[g]$ for some g in a fixed compact continuous upper-potential family $\mathscr{G}$.*

Status: `DIRECT_COROLLARY` of the new bounded-difference positivity criterion.

Proof. The actual difference $Cg - u$ has positive curvature, so it is classical and upper semicontinuous. Since Cg is continuous, this says that $u = Cg - (Cg - u)$ is lower semicontinuous. It is already upper semicontinuous, hence continuous; the complement is then continuous as well. Thus the enlarged defining family is exactly the previously proved continuous family. In the compact-$\mathscr{G}$ case apply this argument separately with the upper potential attached to each member. All prior conclusions apply without a new approximation step. □

Dependency and remaining limits. The proof uses the order-transfer construction for the prescribed functions, the curve kernel and actual endpoint values, the frozen semistable geometric covers, and the one-sided Shilov bound. It does not infer ambient semicontinuity from curve continuity alone, nor invoke an unproved general distributional regularization theorem. The bounded-difference criterion does not supply a potential for an arbitrary positive closed current. The continuous-coefficient theorem has one curvature factor in each summand; it does not annihilate every higher-degree analytic delta relation or permit arbitrary Borel coefficient replacements.

22 An actual upper-semicontinuous spike is outside the DSH criterion

Status: `DIRECT_COROLLARY` of original smooth top-form atomlessness and the curve maximum principle. The proposed replacement of the actual bounded classical difference hypothesis by bounded upper semicontinuity is `FALSE`. This is the one-dimensional version of the already recorded R5 assigned-spike obstruction, not a new counterexample mechanism.

Fix a uniformizer π of the original K, and let

$$\Omega = \{|\pi|^2 < |T| < |\pi|^{-2}\} \subset \mathbf{G}^{\mathrm{an}}_{m,K}, \qquad x = \zeta_{0,1}, \qquad f = \mathbf{1}_{\{x\}}.$$

The actual function f is bounded, Borel and upper semicontinuous, since its nonzero level set is a closed singleton. Nevertheless

$$[f] = 0, \qquad D[f] = 0,$$

while f is not classical psh and is not continuous.

Proof. Every original compact smooth $(1,1)$-form has an atomless signed integration measure. The finite-moment proof is the one given in Section 62: positive smooth top-form controls push forward to calibrated one-dimensional smooth densities, which have no atom at a tropical point; positivity then gives no mass to its entire inverse image. Rank-zero presentations give the zero top form, and finite localization covers every test. Thus the total variation of every such test measure gives mass zero to $\{x\}$. Integrating the bounded assigned function f yields zero on every original test, proving both current equalities. This does not say that arbitrary positive currents have no atoms.

If f were classical on Ω, it would be subharmonic on this same original analytic curve. On the compact subannulus

$$A = \{|\pi| \le |T| \le |\pi|^{-1}\} \Subset \Omega$$

the two Shilov boundary points are different from x and have value zero. The subharmonic maximum principle would give $f \le 0$ throughout A, contradicting $f(x) = 1$. The distinct Gauss points $\zeta_{0,r}$ with $r \to 1$ also show directly that f is not continuous at x. □

In particular f cannot be an actual difference of two locally bounded classical psh functions on any ordinary neighborhood of x. Such a presentation would have equal original curvatures for its two terms. The proved R5 kernel theorem would make their actual difference pluriharmonic and continuous there, contrary to this spike. Equivalently, the R6 ordered-DSH criterion applied to both signs would give the same contradiction.

This check preserves the exact representative scope of Theorem 21.4. It does not disprove existence of a continuous classical representative of the zero distribution: the constant zero function is such a representative. The failure lies in prescribing the spike's actual point value and then treating it as if it were already in the classical-difference domain of that theorem.

23 Products in the continuous coefficient curvature module

This section records a consequence of the represented-order theorem. The input currents have a prescribed local finite potential presentation. No product of completely arbitrary positive closed currents is defined.

Definition 23.1. On an allowed ordinary open U, let $\mathscr{M}^1(U)$ be the $C(U,\mathbf{R})$-module of original coefficient-Radon $(1,1)$-currents locally admitting a finite expression

$$A = \sum_\alpha f_\alpha D[g_\alpha], \qquad f_\alpha \in C(U,\mathbf{R}), \quad g_\alpha \in \mathcal{C}_b(U).$$

The expression is local; actual equality of currents is the equivalence relation. Signed bounded classical potentials can be used after expanding them into two classical parts. Its positive cone consists of those represented currents which are positive in the original middle CLD sense. Variable continuous coefficients need not preserve separate closedness.

Theorem 23.2 (Positive products of continuously weighted curvatures)**.** *Let S be any fixed allowed (p,p)-background and $r \le n-p$. There is a well-defined symmetric $C(U,\mathbf{R})$-multilinear map*

$$\Phi_{S,r} : (\mathscr{M}^1)^r \longrightarrow \{\textit{original coefficient-Radon } (p+r,p+r)\textit{-currents}\}$$

characterized on finite local presentations by

$$\Phi_{S,r}(A_1,\dots,A_r) = \sum_{\alpha_1,\dots,\alpha_r} \left(\prod_{i=1}^r f_{i,\alpha_i}\right) T_{S,r}(g_{1,\alpha_1},\dots,g_{r,\alpha_r}). \tag{23.1}$$

It commutes with ordinary restriction and gluing. If every A_i is positive, its image is positive. In particular the map is increasing in each slot when all the other slots are positive. When every slot is $D[g_i]$, it is the frozen bounded BT product.

No separate closedness of the output is asserted for general continuous coefficients. The map does not factor through every higher-degree ordinary-current relation merely from its construction.

Status: `DIRECT_COROLLARY` of the new bounded-difference positivity criterion and continuous-coefficient relation theorem, with the positivity argument written out.

Proof. Each expression in (23.1) is a finite sum of continuous multiples of positive coefficient-Radon currents. Suppose the presentation of one slot is changed without changing its original current. Fix one multi-index in the remaining slots and put

$$R = T_{S,r-1}(g_{1,\alpha_1},\dots,\widehat{g_{i,\alpha_i}},\dots,g_{r,\alpha_r}).$$

This is an allowed positive separately closed background. The zero-relation assertion of Theorem 21.6, applied to R, makes the change in that slot zero. Multiplication by the remaining continuous coefficient product and finite summation preserve zero. Changing the slots one at a time proves presentation independence. Finite linearity, scalar multiplication, symmetry of the frozen BT tails, ordinary locality and gluing give the stated module and sheaf properties.

For positivity fix a positive original compact complementary test η. Use finitely many local presentations near its support and nonnegative smooth localization; it suffices to treat one presentation. All its coefficients are bounded by a fixed number $M \ge 1$ on a slightly larger compact neighborhood. For $0 < \epsilon < 1$, subdivide the test support into finitely many ordinary neighborhoods on which

$$|f_{i,\alpha} - c_{i,\alpha}| < \epsilon$$

for suitable constants $c_{i,\alpha}$. On one such neighborhood put

$$H_i = \sum_\alpha (c_{i,\alpha} + \epsilon) g_{i,\alpha}.$$

It is an actual bounded classical difference, and

$$D[H_i] = A_i + \sum_{\alpha} (c_{i,\alpha} + \epsilon - f_{i,\alpha}) D[g_{i,\alpha}] \geq 0.$$

Theorem 21.4 therefore gives $H_i \in \mathcal{C}_b$ on that neighborhood. Hence $T_{S,r}(H_1, \dots, H_r) \geq 0$ there.

For every multi-index the scalar difference between its coefficient in (23.1) and its coefficient in $T_{S,r}(H_1, \dots, H_r)$ has absolute value at most $2r\epsilon(M+2)^{r-1}$. This follows by telescoping the product of r scalars, since each paired scalar differs by at most 2ϵ. Let R_0 be the finite sum of all the positive BT tails in (23.1), without coefficients. On this neighborhood the positive test cone therefore gives

$$\Phi_{S,r}(A_1, \dots, A_r) \geq -2r\epsilon(M+2)^{r-1} R_0.$$

The same constant and the same R_0 work for every member of the finite subdivision. Apply a nonnegative original smooth partition to η, sum the inequalities, and let $\epsilon \downarrow 0$. The fixed control mass $\langle R_0, \eta \rangle$ is finite. The pairing of $\Phi_{S,r}$ with η is nonnegative. This proves positivity on all original tests in the specified cone.

For monotonicity subtract the two products and telescope their slots. Each term has one positive difference slot and all other slots positive, so the proved positivity applies. The BT specialization is its defining one-term presentation. □

Example 23.3 (Borel coefficients do not preserve current relations)**.** The word continuous in the preceding module cannot be replaced by bounded Borel, even when all potential functions are continuous. Take the ordinary bidannulus

$$U = \{r < |T_1| < R,\ r < |T_2| < R\} \subset (\mathbf{G}^2_{m,K})^{\mathrm{an}}, \qquad 0 < r < 1 < R,$$

and let $Y = \{T_2 = 1\} \cap U$, $S = [Y]$, and $g = \max\{0, \log|T_1|\}$. This is a continuous locally bounded classical psh function. The frozen first-order analytic-divisor nonconcentration theorem, applied to $D[g]$, gives

$$\mathbf{1}_Y D[g] = 0$$

as an ambient coefficient-Radon current. On the other hand, the restriction and projection formulas for the subvariety background, and the one-variable slope computation, give

$$\mathbf{1}_Y D(g[Y]) = D(g[Y]) = i_* \delta_{\zeta_{0,1}} \neq 0, \qquad i : Y \hookrightarrow U.$$

Here $\zeta_{0,1}$ is the radius-one Gauss point in the T_1 annulus; the slope jump of $\max(0, \log|T_1|)$ is exactly one. An original nonnegative compact smooth scalar test equal to one near this point pairs with the displayed measure to give one. Thus the failure is visible on original tests, not only on an enlarged test sheaf.

Status: `FALSE` for the proposed Borel extension; the example is a `DIRECT_COROLLARY` of the frozen polar-carrier obstruction and first-order nonconcentration. It is not counted as a new counterexample mechanism. R5, Corollary R5–23.2 and Remark R5–22.4, already show why arbitrary assigned bounded upper semicontinuous values must also be excluded from the bounded-classical-difference positivity criterion.

What this construction does not settle. Independence of the degree-one presentations means that relations in $\mathscr{M}^1$ may be substituted inside (23.1). It does not say that every ordinary zero current formed from several curvature factors is such a relation. The higher-degree analytic delta kernel remains a separate problem. Nor is the output of a continuous coefficient product an admissible new background unless its required positivity and separate closedness are independently available.

24 Quantitative stability in the represented curvature order

The ordered quotient gives a quantitative convergence statement which is stronger than weak convergence but has correspondingly stronger, explicit hypotheses. The upper potentials here may be discontinuous. The background is fixed throughout.

Proposition 24.1 (A mixed order bound)**.** *Let S be any fixed allowed (p,p)-background and $k \le n-p$. Let $u_i, v_i, w_i \in \mathcal{C}_b(U)$ and let $M_i, \epsilon_i \ge 0$ be finite constants such that, on all original ambient tests,*

$$0 \le D[u_i], D[v_i] \le M_i D[w_i], \qquad -\epsilon_i D[w_i] \le D[u_i - v_i] \le \epsilon_i D[w_i] \quad (1 \le i \le k).$$

Set

$$R = T_{S,k}(w_1, \dots, w_k), \qquad c = \sum_{i=1}^{k} \epsilon_i \prod_{\ell \ne i} M_\ell.$$

Then

$$-cR \le T_{S,k}(u_1, \dots, u_k) - T_{S,k}(v_1, \dots, v_k) \le cR \tag{24.1}$$

in the original positive cone. If $k = n-p$, these are Radon measures and their signed difference ν satisfies $|\nu| \le cR$ on all Borel sets. In particular this controls total variation on every compact subset, without continuity of any w_i.

Status: `DIRECT_COROLLARY` of the new order-preserving quotient and the frozen mixed calculus.

Proof. Telescope the mixed difference, replacing the slots one at a time. The ith signed term has differentiated potential $u_i - v_i$, with all earlier slots chosen from the v list and all later slots from the u list. Their product with S is a positive separately closed tail. Corollary 21.5, applied to the two inequalities for $D[u_i - v_i]$, bounds this term above and below by ϵ_i times the same tail with slot w_i. Replace each other slot in that positive tail by $M_\ell w_\ell$, using the same order corollary and mixed symmetry. Its bound is $\epsilon_i(\prod_{\ell \ne i} M_\ell)R$. Summation proves (24.1). No division by M_i is used, so vanishing constants cause no difficulty.

In top degree the inequalities say $-cR \le \nu \le cR$ as signed Radon measures. Radon–Nikodym differentiation with respect to R gives $\nu = hR$ with $|h| \le c$ almost everywhere, hence $|\nu| \le cR$. This proves the Borel and total-variation assertions. □

Corollary 24.2 (Weighted total-variation convergence)**.** *Let $k = n-p$ and let $u_{i,j}, u_i, w_i \in \mathcal{C}_b(U)$ be* sequences *with common finite constants M_i such that*

$$0 \le D[u_{i,j}], D[u_i] \le M_i D[w_i], \qquad -\epsilon_{i,j} D[w_i] \le D[u_{i,j} - u_i] \le \epsilon_{i,j} D[w_i], \qquad \epsilon_{i,j} \longrightarrow 0.$$

Put $R = T_{S,k}(w_1, \dots, w_k)$, $T_j = T_{S,k}(u_{1,j}, \dots, u_{k,j})$ and $T = T_{S,k}(u_1, \dots, u_k)$. Let a_j, a be actual Borel functions which are uniformly bounded on every compact subset and satisfy $a_j \to a$ R-almost everywhere. Then

$$|a_j T_j - aT|(K) \longrightarrow 0$$

for every compact $K \Subset U$. The weights need not be continuous or psh under these hypotheses.

Status: `DIRECT_COROLLARY`. This is not a consequence of bare weak convergence with arbitrary Borel weights.

Proof. Let $c_j = \sum_i \epsilon_{i,j} \prod_{\ell \neq i} M_\ell$, and put $M = \prod_i M_i$. The proposition gives $|T_j - T| \leq c_j R$ and order replacement gives $T \leq MR$. Therefore, if B_K bounds $|a_j|$ on K,

$$|a_j T_j - aT|(K) \leq B_K c_j R(K) + M \int_K |a_j - a| \, dR.$$

The first term tends to zero and the second tends to zero by fixed finite-measure dominated convergence. The sequence hypothesis is intentional: arbitrary pointwise convergent nets do not satisfy this version of dominated convergence. □

Scope. The propositions concern finite locally represented curvature order intervals and their explicit order errors. They neither turn ordinary weak convergence into order convergence nor enlarge the continuous coefficient relation module to arbitrary Borel coefficients. The Borel weights in the last corollary are multiplied only after the relative measures and a common relative dominating measure have been fixed. No varying-background theorem is inferred from these estimates.

25 A qualified varying-background continuity theorem

The original fixed-background monotone theorem must not be replaced by an arbitrary varying-background assertion. Local uniform convergence to continuous potentials supplies a sufficient additional hypothesis. It also gives weak continuity of the new degree-one delta action on the full cone of allowed backgrounds.

Proposition 25.1 (Continuous potentials and weakly varying backgrounds)**.** *On one common ordinary domain, let $S_j \to S$ be a net of allowed positive separately closed currents of degree (p,p), converging on every original compact smooth test. Suppose that, for $1 \leq i \leq k \leq n-p$, the actual functions $u_{i,j}, u_i$ are continuous classical psh and $u_{i,j} \to u_i$ locally uniformly. Then*

$$T_{S_j,k}(u_{1,j}, \ldots, u_{k,j}) \longrightarrow T_{S,k}(u_1, \ldots, u_k)$$

on all original compact smooth tests. If a_j, a are finite continuous real functions with $a_j \to a$ locally uniformly, the same assertion holds after multiplying the two sides by a_j and a respectively. Finite signed expansions of the continuous potential slots are allowed when their positive parts satisfy these same hypotheses.

Status: `DIRECT_COROLLARY` of R's fixed continuous multiplier lemma, coefficient control and the frozen BT recursion. No monotonicity is required under these stronger uniform and continuity hypotheses.

Proof. First record a multiplier fact. If $R_j \to R$ are positive currents of one degree and $b_j \to b$ locally uniformly with b continuous, then for each original compact test η,

$$\langle b_j R_j - bR, \eta \rangle = \langle (b_j - b) R_j, \eta \rangle + \langle b(R_j - R), \eta \rangle \longrightarrow 0.$$

The second term converges by R. The first is bounded by $\|b_j - b\|_\infty$ on a fixed compact stage times finitely many positive control pairings of R_j. Those controls are eventually bounded by weak convergence. This proves the fact for nets, without a global mass bound or a topology determined by countably many tests.

Use induction on k. The positive separately closed tails with $k-1$ slots converge by induction. Apply the multiplier fact with $b_j = u_{1,j}$ and then the continuous distributional operator D. The frozen recursion identifies the two derivatives with the stated mixed currents. The weighted assertion is one further application of the same multiplier fact to those positive currents. Finite signed expansions are handled term by term. □

Corollary 25.2 (All-background weak continuity on a fixed degree-one delta-test)**.** *Let S_j, S be as above and let α be a fixed actual delta-form of total degree $(1,1)$. Then the canonical ordinary currents*

$$\alpha \wedge S_j \longrightarrow \alpha \wedge S$$

converge on every original compact smooth test. If $p = n-1$ and α has compact actual support, the full delta pairings converge. Thus, on the positive separately closed cone in remaining degree one, the ordinary weak topology already controls every fixed compact actual delta-test.

Status: `DIRECT_COROLLARY` of the proved degree-one relation theorem and continuous multiplier stability.

Proof. For the symmetric part use a fixed local Hessian presentation $\alpha = \sum_i f_i D_\delta g_i$, with continuous PS coefficients and continuous PS classical-difference potentials. Expand each g_i into its fixed continuous positive parts. The proposition, first with one constant potential row and then with the fixed continuous coefficient f_i, proves convergence of each term on every original test. A finite partition covers a compact test support. The antisymmetric part is PS; the frozen piecewise-test convergence theorem applied to S_j handles it on any original complementary test. Actual relation independence makes these local formulas the canonical action for all backgrounds in the statement.

For a compact complementary α, the ordinary supports of all outputs are contained in the same actual compact support of α. An original smooth cutoff equal to one near that compact converts the current convergence into convergence of their total integrals. Original smooth tests are themselves actual delta-tests, so the assertion about the two weak topologies follows in this degree. □

Excluded extension. Neither assertion allows a simultaneously varying discontinuous Borel weight under mere weak background convergence. Likewise locally bounded psh rows converging only pointwise cannot be substituted for the local uniform continuous rows in the proposition. The frozen weak-current counterexamples and the main fixed-S monotone hypothesis retain their force.

26 Compactness of continuous curvature order intervals

The order bound in this section is on the original ambient curvature. It holds on the whole domain on which the functions are normalized. No finite-dimensional assumption on these curvatures is made. Fix an allowed ordinary open Ω, a continuous $g \in \mathcal{C}_b(\Omega)$, and a real constant $C \geq 0$. Put

$$\mathscr{P}(g, C; \Omega) = \{u \in C(\Omega) \cap \mathcal{C}_b(\Omega) : 0 \leq u \leq 1, \quad 0 \leq D[u] \leq CD[g]\}. \tag{26.1}$$

The inequalities between currents use every original positive complementary test. All scalar extensions below retain the given absolute value.

Lemma 26.1 (The continuous curvature converse used here)**.** *Let h be a finite continuous function on an allowed ordinary open. If $D[h]$ is original CLD-positive, then h is locally bounded classical psh. In particular, both u and $Cg - u$ are continuous classical psh for every $u \in \mathscr{P}(g, C; \Omega)$.*

Status: `DIRECT_COROLLARY` of the authorized continuous regularization in R and the external smooth curve criterion. This is not a converse for discontinuous distributional potentials.

Proof. After B's ordinary algebraic localization, R, Theorem 2.9, supplies on a fixed neighborhood smooth original functions h_j with positive curvature forms and $h_j \to h$ locally uniformly. An original smooth function with positive curvature form is classical psh: original smooth forms pull back along analytic maps and complete scalar extension, and positive $(1,1)$ forms remain positive. On a smooth curve the resulting smooth function is subharmonic by the smooth case of Wanner's Theorem 4.7 [Wan19]. The passage to all analytic curves and all complete extensions is exactly the smooth-function argument recorded in [GR25, §1.2, p. 3; §4.1.3]. It concerns pullback of smooth forms, not pullback of an arbitrary current. Wanner's auxiliary normalization of the logarithmic value group is harmless here: multiplying all logarithmic lengths by a positive constant preserves positivity and the subharmonic comparison inequality, and the absolute value in the present construction is never changed.

The continuous pointwise limit h is classical psh by [GR25, Theorem 6.5]. This assertion can also be checked directly by passing a local uniform limit through the curve comparison inequalities. Thus no regularization theorem over a new complete field has been used. Continuity gives local boundedness. Apply this to $h = Cg - u$ for the last assertion. □

Lemma 26.2 (Positive Shilov interpolation on the fixed charts)**.** *Let V be an original algebraic norm box as in R5, Lemma 9.2, contained in an ordinary algebraic neighborhood. Use the compact surjective semistable cover $q : W \to V$ and one of its finitely many affinoid charts W_λ from R5, Proposition 9.3. Write its Shilov boundary as $\{\xi_0, \dots, \xi_r\}$ and its logarithmic coordinates as*

$$\lambda_i = -\frac{\log|x_i|}{-\log|\varpi|}, \qquad P_\lambda a = \sum_{i=0}^{r} a(q(\xi_i))\lambda_i.$$

Then $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$ on W_λ. If a is continuous classical psh on an ordinary neighborhood of V, then

$$a \circ q \leq P_\lambda a \quad \text{on } W_\lambda.$$

For every $u \in \mathscr{P}(g, C; \Omega)$ and every such chart over $V \subset \Omega$,

$$0 \leq P_\lambda u - u \circ q \leq C(P_\lambda g - g \circ q) \leq C \operatorname{osc}_V(g). \tag{26.2}$$

Status: `DIRECT_COROLLARY` of the frozen semistable charts and the affinoid maximum principle.

Proof. The regular functions x_i are power bounded on their formal generic fiber, and $x_0 \cdots x_r = \varpi$. Hence the logarithmic coordinates are nonnegative and sum to one. R5, Lemma 8.2, identifies their values at the Shilov points as $\lambda_i(\xi_j) = \delta_{ij}$. Each x_i is an analytic unit on a genuine ordinary algebraic neighborhood of the chart. Therefore $P_\lambda a$ is a real linear combination of logarithms of units and is pluriharmonic there.

The function $a \circ q - P_\lambda a$ is classical psh near the entire chart and vanishes at each of its Shilov points. The affinoid maximum principle [GR25, Theorem 6.4] proves the asserted inequality. Apply it first to $a = u$ and then to $a = Cg - u$, which is classical by Lemma 26.1. Linearity of P_λ gives the middle inequality in (26.2). Since $P_\lambda g$ is a convex combination of values of g in V, the final inequality follows. No arbitrary vertex interpolant is assumed harmonic: these are precisely the unit-logarithm interpolants on the frozen charts. □

Theorem 26.3 (Finite evaluation and local uniform compactness)**.** *For each compact $K_0 \Subset \Omega$ and each $\epsilon > 0$ there is a finite set $F_\epsilon \subset \Omega$, depending on g, K_0, ϵ and the chosen geometric neighborhoods, such that*

$$\|u - v\|_{K_0} \leq \max_{z \in F_\epsilon} |u(z) - v(z)| + 2C\epsilon \qquad (u, v \in \mathscr{P}(g, C; \Omega)). \tag{26.3}$$

The family $\mathscr{P}(g, C; \Omega)$ is compact in the topology of uniform convergence on compact subsets of Ω. In particular, every net in this family has a subnet converging locally uniformly to an actual member of the same family.

The sets F_ϵ need not be independent of ϵ. Neither finite-dimensionality of the family nor convergence in total variation of its curvature measures is asserted.

Status: `NEW_PROVED`, using R's continuous regularization, the external affinoid maximum principle, and the frozen R5 geometric construction. This label makes no claim about priority over other compactness theorems.

Proof. By continuity of g, cover K_0 by finitely many ordinary neighborhoods on each of which $\operatorname{osc}(g) < \epsilon$. Use B's ordinary local algebraization and R5, Lemma 9.2, to choose finitely many norm boxes V_a within these neighborhoods whose ordinary interiors cover K_0. Apply the frozen semistable cover to each V_a. It gives finitely many charts and finitely many Shilov points on each chart. Define F_ϵ to be all their images in Ω. These points and covers depend on g and the accuracy, but not on u or v.

For $x \in K_0$, choose a box containing it and a preimage $y \in W_\lambda$ in one of its charts. The preceding lemma and the nonnegative coordinates give

$$\begin{aligned} |u(x) - v(x)| &\le |P_\lambda u(y) - P_\lambda v(y)| + 2C\epsilon \\ &\le \max_{z \in F_\epsilon} |u(z) - v(z)| + 2C\epsilon. \end{aligned}$$

This proves (26.3). In particular, a finite grid in $[0,1]^{F_\epsilon}$ gives finite uniform nets on K_0: choose one actual family member for each occupied grid cell and apply the estimate to any two members in that cell. No interpolant has to descend through the alteration.

For full compactness, take any net (u_j) in the family. Tychonoff compactness of $[0,1]^\Omega$ gives a pointwise convergent subnet; write its actual limit as u. Given a compact K_0 and a positive tolerance, choose ϵ so that $2C\epsilon$ is smaller than half the tolerance (when $C = 0$ choose any ϵ). The subnet is Cauchy on the finite set F_ϵ, so the displayed estimate makes it uniformly Cauchy on K_0. Completeness of $C(K_0)$ gives uniform convergence there, and that uniform limit is the already chosen pointwise function u.

The space Ω is locally compact Hausdorff and has compact neighborhoods, so these compact restrictions show that u is continuous and that convergence is locally uniform. The inequalities $0 \le u \le 1$ persist. Every fixed original compact test pairs continuously with $D[u_j]$ under local uniform convergence: apply the fixed signed top-form measure to $u_j - u$. Consequently $0 \le D[u] \le CD[g]$. Finally, u is classical by [GR25, Theorem 6.5], or by Lemma 26.1. Hence u belongs to the same family. The net characterization of compactness now proves the assertion without a countability assumption on Ω. □

Lemma 26.4 (Uniform continuity of the fixed-background BT map)**.** *Fix an allowed positive separately closed background S. Suppose $u_{i,j}, u_i$ are continuous classical psh, for $1 \le i \le k$, and $u_{i,j} \to u_i$ locally uniformly. Then, on every original compact complementary smooth test,*

$$T_{S,k}(u_{1,j}, \dots, u_{k,j}) \longrightarrow T_{S,k}(u_1, \dots, u_k).$$

The assertion remains true after multiplication by continuous $w_j \to w$ locally uniformly, with locally bounded w_j.

Status: `DIRECT_COROLLARY` of the fixed-continuous-multiplier result in R and relative CLN.

Proof. Proceed by induction through the ordered recursion. If the positive tails R_j converge to R, relative CLN gives uniform coefficient variation bounds on each fixed compact test stage.

Thus $(u_{1,j} - u_1)R_j \to 0$ by local uniform convergence. R's fixed continuous multiplication gives $u_1 R_j \to u_1 R$, so applying D proves the next induction step. The same two-term decomposition with $w_j - w$ proves the weighted assertion. This proof works for nets, uses all original test forms, and never varies the initial background S. □

Corollary 26.5 (Actual representatives of weak limits on the same domain)**.** *Fix $k \le n - p$ and, for each i, continuous classical g_i on the whole Ω and constants $C_i \ge 0$. For every fixed allowed background S, the set*

$$\{T_{S,k}(u_1, \dots, u_k) : u_i \in \mathscr{P}(g_i, C_i; \Omega)\}$$

is compact and closed in the ordinary weak current topology. Every member of its weak closure therefore has a representation with actual normalized continuous classical potentials on the same Ω, satisfying the same ambient curvature bounds. The corresponding diagonal image is also compact and closed.

For $k = n - p$, the restricted diagonal raw capacity has no compact mass gain under weak closure. This conclusion concerns the displayed family; no common continuous curvature dominator for the unrestricted normalized family is asserted.

Status: `DIRECT_COROLLARY` of Theorem 26.3.

Proof. The finite product of the potential families is compact, and the BT map is continuous by Lemma 26.4. Its image is compact, and the weak current topology is Hausdorff because all original tests are retained. Thus the image is closed. Restrict the same argument to the diagonal when all slots use one family. A measure in that diagonal image is itself one of the normalized test measures defining the restricted raw capacity, which proves the final compact-mass assertion with constant one. This argument does not infer compact-mass continuity from weak convergence. □

Corollary 26.6 (Ambient weak curvature stability within an order interval)**.** *For each i, let $u_{i,j} \in \mathscr{P}(g_i, C_i; \Omega)$ and suppose that the ambient currents $D[u_{i,j}]$ converge weakly on all original tests to A_i. There are actual $u_i \in \mathscr{P}(g_i, C_i; \Omega)$ with $D[u_i] = A_i$, and for every fixed allowed background S,*

$$T_{S,k}(u_{1,j}, \dots, u_{k,j}) \longrightarrow T_{S,k}(u_1, \dots, u_k).$$

The right side is independent of the chosen cluster potentials. No statement for varying backgrounds or for an undominated family is made.

Status: `DIRECT_COROLLARY` using the frozen R5 arbitrary-background potential quotient, Theorem 10.2.

Proof. Compactness gives a simultaneous locally uniform cluster tuple, and pairing with original tests identifies its curvatures with the A_i. Every other cluster tuple has those same curvatures. The R5 quotient and differentiated-slot symmetry identify their mixed products over S, slot by slot with a positive closed tail. Lemma 26.4 identifies every cluster of the mixed-current net with this same current. Since its image lies in a compact Hausdorff set, a unique cluster point implies convergence of the whole net. No convergence of a chosen pluriharmonic gauge is needed. □

Limits of the conclusion. The local uniform compactness is a property of continuous actual potentials under a fixed continuous ambient order bound. It does not follow from a relative energy bound alone. In particular the sine example in Section 44 is compatible with this theorem: its potentials converge uniformly, while the curvature measures fail to converge

in total variation. If the order bound is known only on a smaller $W \subset \Omega$, the argument gives compactness on compact subsets of W; it does not produce a normalized representative on all of Ω.

27 The weak topology on represented curvature order intervals

This section states the topological consequence on ambient curvature currents themselves. Its compactness input concerns actual normalized continuous potentials on one common ordinary domain. It is not a compactness assertion for the unrestricted bounded psh class.

For continuous classical g on Ω and $C \geq 0$, define

$$\mathscr{A}(g, C; \Omega) = \{D[u] : u \in \mathscr{P}(g, C; \Omega)\},$$

where $\mathscr{P}$ is defined in (26.1). Give $\mathscr{A}$ the subspace topology of ordinary weak current convergence on every original test. Write $\mathscr{S}^p_+(\Omega)$ for the cone of allowed symmetric original-positive separately closed (p, p) currents, with the same weak topology.

Proposition 27.1 (The curvature quotient has its ordinary weak topology)**.** *The set $\mathscr{A}(g, C; \Omega)$ is compact, convex, and Hausdorff. The map*

$$\mathscr{P}(g, C; \Omega) \longrightarrow \mathscr{A}(g, C; \Omega), \qquad u \longmapsto D[u],$$

is a closed quotient map. Its fibers are exactly the equivalence classes obtained by allowing pluriharmonic differences while retaining membership in the normalized potential family.

Status: `DIRECT_COROLLARY` of the compactness theorem and the frozen R5 actual potential quotient.

Proof. The potential family is compact by Theorem 26.3. Its map to currents is continuous, since a fixed original pairing with $D[u]$ is an integral of u against a fixed signed top-form measure. The target weak current space is Hausdorff. A continuous surjection from a compact space to a Hausdorff space is closed and is a quotient map. Both the potential family and its curvature image are convex, by the defining linear order inequalities. R5, Theorems 10.1–10.2, identify equality of the actual bounded-classical curvatures with a continuous pluriharmonic difference. This identifies the fibers without selecting a gauge or asserting the existence of a continuous section. □

Theorem 27.2 (Joint weak continuity in background and ambient curvatures)**.** *Fix $k \leq n - p$. For each i, fix a continuous classical g_i on the whole Ω and $C_i \geq 0$, and put $\mathscr{A}_i = \mathscr{A}(g_i, C_i; \Omega)$. There is a well-defined continuous map*

$$\begin{aligned} &\mathscr{S}^p_+(\Omega) \times \prod_{i=1}^{k} \mathscr{A}_i \longrightarrow \mathscr{S}^{p+k}_+(\Omega), \\ &(S, A_1, \ldots, A_k) \longmapsto \mathsf{T}(S; A_1, \ldots, A_k) := T_{S,k}(u_1, \ldots, u_k), \qquad D[u_i] = A_i. \end{aligned} \tag{27.1}$$

All topologies in this display are the ordinary weak topologies on all original compact smooth tests. In particular, for nets $S_j \to S$ and $A_{i,j} \to A_i$ in these specified spaces,

$$\mathsf{T}(S_j; A_{1,j}, \ldots, A_{k,j}) \longrightarrow \mathsf{T}(S; A_1, \ldots, A_k).$$

For each fixed actual delta-form α of total degree $(1, 1)$, with $p + k + 1 \leq n$, the same map followed by the canonical operation $R \mapsto \alpha \wedge R$ is continuous on original tests. If $p + k = n - 1$, this includes convergence against every fixed compact complementary actual delta-test.

Status: `DIRECT_COROLLARY` of Theorem 26.3, Proposition 25.1, and the frozen arbitrary-background potential quotient. The hypotheses on the represented order intervals are essential to the deduction.

Proof. Choose representatives u_i in the normalized continuous potential families. The R5 quotient, with positive separately closed mixed tails, shows that replacing any representative by another with the same ambient curvature leaves the product unchanged. This proves that (27.1) is well defined.

To check continuity, consider a convergent net of input tuples and choose arbitrary normalized representatives $u_{i,j}$ for its curvature slots. Every subnet of the resulting tuple of representatives has a further subnet converging locally uniformly, simultaneously in its finitely many slots, by compactness of their finite product. Let $(u_1, \dots, u_k)$ be such a limit. Its ambient curvatures are the specified limits A_i, by local uniform convergence on original tests. The backgrounds still converge to S along this further subnet. Proposition 25.1 gives

$$T_{S_j,k}(u_{1,j}, \dots, u_{k,j}) \longrightarrow T_{S,k}(u_1, \dots, u_k).$$

The right side is the value prescribed in (27.1), independently of this choice of cluster representatives. Thus every subnet admits a further subnet converging to the same output. This is precisely convergence of the original output net: if an output neighborhood failed to contain a tail, one could take a subnet lying in its complement and obtain a contradiction. This proves joint continuity without metrizability, a gauge, or a continuous choice of representatives.

The outputs are positive separately closed currents. Corollary 25.2 therefore applies to them and proves the final assertion. Its actual delta-test is fixed; no convergence of varying delta-test presentations is asserted. □

Weights are not curvature classes. The theorem concerns differentiated slots. A weight cannot be passed to the same quotient: the constants 0 and 1 have the same ambient curvature, while their products with any nonzero background are different. If continuous actual weights converge locally uniformly, the weighted conclusion follows instead from Proposition 25.1. Likewise, the theorem does not concern arbitrary weakly convergent curvature currents outside the displayed compact represented intervals, nor does it assert total variation convergence.

28 Compact families of continuous upper potentials

A curvature upper bound supplied by one continuous potential is a useful sufficient condition, but it can be replaced by compactness of the upper potentials themselves. All families and normalizations in this section live on one common ordinary domain Ω.

Let $\mathscr{G}$ be a nonempty compact subset of $C(\Omega) \cap \mathcal{C}_b(\Omega)$ for the topology of uniform convergence on compact subsets. Define

$$\mathscr{P}^c_{\mathscr{G}} = \{u \in C(\Omega) \cap \mathcal{C}_b(\Omega) : 0 \le u \le 1, \quad D[u] \le D[g] \text{ for some } g \in \mathscr{G}\},$$
$$\mathscr{A}_{\mathscr{G}} = \{D[u] : u \in \mathscr{P}^c_{\mathscr{G}}\}.$$

The curvatures $D[u]$ are already nonnegative by membership in the classical class. No convexity of $\mathscr{G}$ or of the displayed unions is assumed.

Proposition 28.1 (Compactness with varying upper potentials)**.** *The family $\mathscr{P}^c_{\mathscr{G}}$ is compact in local uniform topology. Its curvature image $\mathscr{A}_{\mathscr{G}}$ is compact Hausdorff in the ordinary weak topology, and the curvature map is a closed quotient map with the frozen pluriharmonic fibers. For every compact $K_0 \Subset \Omega$ and $\epsilon > 0$, there is a finite set $F_\epsilon \subset \Omega$ such that*

$$\|u - v\|_{K_0} \le \max_{z \in F_\epsilon} |u(z) - v(z)| + 2\epsilon \quad (u, v \in \mathscr{P}^c_{\mathscr{G}}).$$

Status: `DIRECT_COROLLARY` of positive Shilov interpolation, with a larger compactness scope than one fixed curvature dominator.

Proof. First, compactness of $\mathscr{G}$ gives uniform small oscillation near each point. Choose a compact neighborhood L of a given point x. The restrictions of $\mathscr{G}$ to L form a compact subset of $C(L)$, so choose finitely many functions giving a uniform $\epsilon/3$ net. Continuity of these finitely many functions gives an ordinary neighborhood of x inside L on which all their oscillations are less than $\epsilon/3$. Every $g \in \mathscr{G}$ then has oscillation less than ϵ on that neighborhood.

Cover K_0 by finitely many such neighborhoods and choose the frozen norm boxes and finite semistable Shilov covers inside them. For any u choose an upper potential $g \in \mathscr{G}$. Both u and $g - u$ are continuous classical by Lemma 26.1, so positive Shilov interpolation gives on each of the fixed charts

$$0 \le P_\lambda u - u \circ q \le P_\lambda g - g \circ q < \epsilon.$$

The covers and their finite set of Shilov image points depend only on $\mathscr{G}, K_0, \epsilon$, not on either chosen potential. Subtract the inequalities for u, v and use the positive barycentric coordinates. This proves the finite-evaluation estimate.

For a net u_j choose an associated $g_j \in \mathscr{G}$. Compactness of $[0,1]^\Omega \times \mathscr{G}$ gives a simultaneous subnet with pointwise $u_j \to u$ and locally uniform $g_j \to g$. The finite-evaluation estimate makes the same subnet uniformly Cauchy on each compact set, hence locally uniformly convergent to u. The actual limit is continuous classical, lies between zero and one, and satisfies $0 \le D[u] \le D[g]$ by original smooth testing. Thus it belongs to the displayed family. This proves compactness without a countable exhaustion. The quotient and fiber assertions follow exactly as in Proposition 27.1 from the continuous curvature map, Hausdorff target, and frozen R5 quotient. □

Corollary 28.2 (Joint stability and actual weak-image closure)**.** *For finitely many compact upper-potential families $\mathscr{G}_i$, the operation*

$$(S, A_1, \dots, A_k) \longmapsto T_{S,k}(u_1, \dots, u_k), \qquad D[u_i] = A_i \in \mathscr{A}_{\mathscr{G}_i},$$

is well defined and jointly continuous for ordinary weak convergence of the positive separately closed background S and all ambient curvature slots. For fixed S its image is compact and closed, with actual normalized representatives on the whole Ω. The same is true for the diagonal image of one such family. Consequently, in top degree this diagonal family has sharp compact no-gain under weak closure. These conclusions do not assert general Borel outer regularity, setwise convergence, or a common measure dominator.

Status: `DIRECT_COROLLARY`.

Proof. For every subnet of representatives choose a further subnet converging locally uniformly in the compact finite product of their potential families. The qualified varying-background theorem identifies its mixed-current limit, and the frozen quotient makes the answer independent of the cluster representatives. The every-subnet argument of Theorem 27.2 proves convergence of the whole net. For fixed S the image of the compact family is compact in the Hausdorff current space, hence closed. In top degree a weak-limit diagonal measure is itself one of the actual normalized testers on Ω, so its mass on any compact set is bounded by the raw capacity. This uses actual representatives, not an invalid compact-set Portmanteau inequality in the opposite direction. □

Example 28.3 (Compact upper potentials without a common measure upper bound)**.** On the original annulus with $t = -\log|T| \in (-2, 2)$, put

$$g_r(t) = \frac{1}{4}|t - r|, \qquad -\frac{1}{2} \le r \le \frac{1}{2}.$$

These are continuous classical functions, lie in $[0,1]$, and depend uniformly continuously on r. Their family is compact even in the uniform norm, and

$$Dg_r = \frac{1}{2}\delta_{\zeta_{e-r}}$$

with the original logarithmic calibration. Thus they are included in the preceding compactness statement by choosing each potential as its own upper potential.

There is no single locally finite positive Radon measure ρ on this annulus for which every Dg_r is at most $C_r\rho$ with a finite constant C_r. Otherwise ρ would have a positive atom at every one of the uncountably many displayed Gauss points in a fixed compact subannulus. A finite measure has only countably many positive atoms: for each integer m only finitely many can have mass at least $1/m$. In particular no single continuous psh potential has curvature dominating this whole family, even with member-dependent finite multipliers.

Also, if $r_j \to 0$ with $r_j \neq 0$, then $g_{r_j} \to g_0$ uniformly and their measures converge weakly, whereas

$$Dg_{r_j}(\{\zeta_1\}) = 0, \qquad Dg_0(\{\zeta_1\}) = \tfrac{1}{2}.$$

So the compact upper-potential theorem cannot be upgraded to setwise convergence. It genuinely differs from the common-Radon-dominator mechanism in Section 44.

Status of the example: `DIRECT_COROLLARY` of the original annular hinge calibration and elementary measure countability. It limits this new sufficient criterion; it does not supply a counterexample to general compact C1.

29 Compact continuous dominators do not give Borel outer regularity

This example concerns the full normalized family with a curvature dominated by one member of a specified compact continuous family. It does not replace that family by just the displayed extremizers. It realizes the frozen abstract vertical-fiber obstruction on an actual smooth Berkovich torus, with its original tests and calibration.

Theorem 29.1 (A restricted capacity with different Borel and outer values)**.** *Work over the original K, with its original absolute value. Put*

$$\Omega = \{(T_1,T_2) \in (\mathbf{G}^2_{m,K})^{\mathrm{an}} : -1 < t_1, t_2 < 2\}, \qquad t_i = \log|T_i|, \qquad S = 3[d't_2 \wedge d''t_2].$$

Let

$$A = \{0\} \cup \{1/j : j \geq 1\}, \qquad g_s = \tfrac{1}{3}\max\{t_1 - s, 0\} \quad (s \in A), \qquad \mathscr{G} = \{g_s : s \in A\}.$$

The family $\mathscr{G}$ is compact in $C_{\mathrm{loc}}(\Omega)$ and consists of continuous classical psh functions with $0 \leq g_s \leq 1$. For the entire family

$$\mathscr{P}(\mathscr{G}) = \{u \in \mathcal{C}_b(\Omega) : 0 \leq u \leq 1,\ D[u] \leq D[g_s] \text{ for some } s \in A\},$$

define

$$C_{\mathscr{G}}(E) = \sup_{u \in \mathscr{P}(\mathscr{G})} D(uS)(E), \qquad C^*_{\mathscr{G}}(E) = \inf_{\substack{G \supset E \\ G \text{ ordinary open in } \Omega}} C_{\mathscr{G}}(G).$$

There is an F_σ subset E of a fixed compact subset of Ω such that

$$C_{\mathscr{G}}(E) = \tfrac{1}{2}, \qquad C^*_{\mathscr{G}}(E) = 1.$$

Thus Borel outer regularity fails for this restricted capacity even though the whole potential family $\mathscr{P}(\mathscr{G})$ is compact in the local uniform topology and its represented measure image is weakly compact.

Status: `NEW_PROVED` realization in the actual normalized curvature-order family, using the frozen abstract vertical-fiber example. No new general abstract capacity counterexample or failure of unrestricted BT capacity is claimed.

Proof. The background S is an original smooth, symmetric, positive, separately closed $(1,1)$-current. The affine unit moment $t_1 - s$ is pluriharmonic after every complete extension, so its maximum with zero is classical psh. Since $t_1 < 2$ and $s \geq 0$, one has $0 \leq g_s < 2/3$. Also

$$\|g_s - g_r\|_{\infty,\Omega} \leq \tfrac{1}{3}|s - r|.$$

The parameter space A is compact, hence $\mathscr{G}$ is compact even for the global uniform norm. By Proposition 28.1 and Corollary 21.7, the full displayed family $\mathscr{P}(\mathscr{G})$ is compact in C_{loc}; it is not necessary to assume continuity in its definition.

Let $\sigma : (-1,2)^2 \to \Omega$ be the original two-variable Gauss section, with radii (e^{t_1}, e^{t_2}). Original torus calibration gives the equality of Radon measures

$$D(g_s S) = \nu_s := \sigma_*(\delta_s \otimes dy), \qquad -1 < y < 2. \tag{29.1}$$

The slope jump of g_s is $1/3$, and the factor 3 in S cancels it. There is no change of the field or its absolute value. This is an identity on every original smooth compact scalar test: on a common moment refinement, torus top integration is supported on the original Gauss section and has degree one. Equivalently, smooth convex approximations of the one-variable hinge give the same product-calibrated measure in the limit. The identity is therefore not a prescription of pairings only against functions which factor through the two radius coordinates.

If $u \in \mathscr{P}(\mathscr{G})$, choose its parameter s. The arbitrary-background order theorem gives

$$0 \leq D(uS) \leq D(g_s S) = \nu_s.$$

Conversely $g_s \in \mathscr{P}(\mathscr{G})$ for every s. It follows for every Borel $B \subset \Omega$ that

$$C_{\mathscr{G}}(B) = \sup_{s \in A} \nu_s(B). \tag{29.2}$$

This equality establishes the upper bound for all normalized competitors, not only for the hinges. Local uniform BT continuity also shows that the full represented measure image is weakly compact.

In the real radius box set

$$E_0 = (\{0\} \times [0, \tfrac{1}{2}]) \ \cup \ \bigcup_{j \geq 1} (\{1/j\} \times (\tfrac{1}{2}, 1]), \qquad E = \sigma(E_0).$$

The Gauss map is a topological embedding, and $\sigma(A \times [0,1])$ is compact in Ω. Every half-open vertical segment in this formula is a countable union of compact segments. Thus E is F_σ in the actual Berkovich space. Each ν_s assigns it mass $1/2$, so (29.2) gives $C_{\mathscr{G}}(E) = 1/2$.

Let G be any ordinary open set containing E. The inverse image $O = \sigma^{-1}(G)$ is an ordinary real open set containing E_0. Compactness of $\{0\} \times [0, 1/2]$ gives a $\delta > 0$ such that

$$(-\delta, \delta) \times [0, 1/2] \subset O.$$

For every sufficiently large j, this includes the lower half of the vertical segment at $1/j$, while its upper half already belongs to E_0. Hence $\{1/j\} \times [0,1] \subset O$ and $\nu_{1/j}(G) \geq 1$. The full-family identity proves $C_{\mathscr{G}}(G) \geq 1$ for every such ordinary open G.

For the opposite bound, take $0 < \epsilon < 1$ and the ordinary open set

$$G_\epsilon = \{-\epsilon < t_2 < 1 + \epsilon\} \cap \Omega.$$

It contains E, and every ν_s gives it mass $1 + 2\epsilon$. Thus $C_{\mathscr{G}}(G_\epsilon) = 1 + 2\epsilon$ by the same identity. Taking the infimum and then $\epsilon \downarrow 0$ proves $C^*_{\mathscr{G}}(E) = 1$. □

Corollary 29.2 (The restricted outer capacity fails continuity from below)**.** *There are increasing compact sets $K_m \uparrow E$ in the preceding example such that*

$$C^*_{\mathscr{G}}(K_m) = C_{\mathscr{G}}(K_m) = \tfrac{1}{2} \quad \text{for every } m, \qquad C^*_{\mathscr{G}}(E) = 1.$$

Proof. For $m \geq 3$ let

$$K_m = \sigma\left((\{0\} \times [0, \tfrac{1}{2}]) \cup \bigcup_{j=1}^{m} (\{1/j\} \times [\tfrac{1}{2} + 1/m, 1]) \right).$$

These are compact, increase, and have union E. Their raw capacity is $1/2$ by (29.2). To see the outer value directly, separate the finitely many parameters $1, \dots, 1/m$ by pairwise disjoint small real open intervals, each containing no other point of A. Around 0 use one disjoint interval containing only 0 and a tail of A. Take their products with slightly enlarged versions of the respective vertical segments, and pull back the resulting real open union by (t_1, t_2). For any $\epsilon > 0$ the enlargements can be chosen so that every ν_s gives this ordinary open neighborhood of K_m mass at most $1/2 + \epsilon$. The outer value is consequently at most $1/2$, and the reverse inequality follows from monotonicity. The theorem supplies the strict increase at the union. □

What the example excludes. Compactness of an actual normalized potential family, compactness of its weak measure image, and sharp compact-set outer regularity do not imply Borel outer regularity for its upper-envelope capacity. This example does not contradict the fixed-Radon domination theorem: the measures ν_s charge distinct vertical fibers, and no locally finite Radon measure can dominate all of them with one fixed density bound on this compact rectangle. Nor is this a counterexample for the unrestricted capacity formed from every normalized classical psh function on Ω. Additional unrestricted competitors may change both displayed capacity values; no value for that larger capacity is asserted here.

30 The restricted outer-regularity failure in every positive relative degree

Status: `DIRECT_COROLLARY` of Theorem 29.1, mixed order replacement, and original product calibration. This is a dimensional propagation of the same restricted-family obstruction.

Proposition 30.1. *For every pair $p, q \geq 1$ there is an allowed smooth background of bidegree (p, p) on a smooth $(p+q)$-dimensional original-K torus domain, and a compact continuous normalized upper-potential family, whose restricted relative degree-q capacity satisfies*

$$C(E) = \tfrac{1}{2}, \qquad C^*(E) = 1$$

on one F_σ set E contained in a fixed compact subset. The family defining C includes every normalized actual classical potential whose ambient curvature is bounded by a member of the specified upper family.

Proof. Use coordinates $T_1, \dots, T_q, Z_1, \dots, Z_p$ and put $x_i = \log|T_i|$, $y_a = \log|Z_a|$. Let

$$\Omega_{p,q} = \{-1 < x_i, y_a < 2\} \subset (\mathbf{G}^{p+q}_{m,K})^{\mathrm{an}}, \qquad A = \{0\} \cup \{1/j : j \geq 1\}.$$

Take

$$g_s = \frac{(x_1 - s)_+ + \sum_{i=2}^{q} (x_i)_+}{3q}, \qquad S = \frac{(3q)^q}{q!} \left[\bigwedge_{a=1}^{p} d' y_a \wedge d'' y_a \right].$$

An empty hinge sum for $q = 1$ is zero. The background is smooth, symmetric, strongly positive and separately closed. Every g_s is continuous classical psh, $0 \le g_s < 2/3$, and

$$\|g_s - g_r\|_\infty \le \frac{|s-r|}{3q}.$$

Thus $\mathscr{G} = \{g_s : s \in A\}$ is a compact normalized continuous family on this same whole domain.

Let σ be the original $(p+q)$-variable Gauss section. The exact ordinary top measure is

$$T_{S,q}(g_s, \dots, g_s) = \nu_s := \sigma_*(\delta_s \otimes \delta_0^{\otimes(q-1)} \otimes dy_1 \cdots dy_p). \tag{30.1}$$

To check the coefficient, each one-variable hinge curvature has calibration one and squares to zero. In the qth power of their sum, only terms using all q distinct coordinate curvatures survive; there are $q!$ such terms. The remaining scalar is

$$\frac{(3q)^q}{q!}\frac{q!}{(3q)^q} = 1.$$

Wedging with the p smooth passive-coordinate top factors gives their product Lebesgue measure. This verifies the entire product calibration, rather than multiplying p or q independent field degrees. As in the two-dimensional case, smooth convex approximants of each hinge and the original torus top-form measure identify (30.1) on all original compact scalar tests, including further moment refinements.

Define the whole competitor family by

$$\mathscr{P}(\mathscr{G}) = \{u \in \mathcal{C}_b(\Omega_{p,q}) : 0 \le u \le 1,\ D[u] \le D[g_s] \text{ for some } s \in A\}.$$

For a competitor with parameter s, replace the differentiated slots one at a time using Corollary 21.5. Each remaining mixed tail is a fixed positive separately closed background, so this yields

$$0 \le T_{S,q}(u, \dots, u) \le T_{S,q}(g_s, \dots, g_s) = \nu_s.$$

Each g_s is itself a competitor. Consequently the restricted capacity is exactly $C(B) = \sup_{s\in A} \nu_s(B)$ for every Borel B, with no restriction to just the displayed hinge testers.

Write $Q_0 = [0,1]^{p-1}$, interpreted as a point if $p = 1$, and set

$$\begin{aligned} E_0 &= \{(0,0,\dots,0)\} \times ([0,\tfrac12] \times Q_0) \\ &\quad \cup \bigcup_{j\ge1} \{(1/j,0,\dots,0)\} \times ((\tfrac12,1] \times Q_0), \qquad E = \sigma(E_0). \end{aligned}$$

It is F_σ and lies in the compact Gauss image of $A \times \{0\}^{q-1} \times [0,1]^p$. Every ν_s assigns E mass $1/2$, proving the raw value.

For any ordinary open $G \supset E$, its inverse image under σ contains the compact lower plate $\{0\}^q \times [0,1/2] \times Q_0$. A uniform small displacement in the x_1-direction stays in this real open set. Therefore, for all sufficiently large j, it contains the lower plate over $x_1 = 1/j$. The upper plate there already belongs to E_0. The entire passive cube $[0,1]^p$ at this parameter is contained in $\sigma^{-1}(G)$, so $\nu_{1/j}(G) \ge 1$ and $C^*(E) \ge 1$. Conversely the ordinary open sets

$$G_\epsilon = \{-\epsilon < y_a < 1+\epsilon \text{ for } 1 \le a \le p\} \cap \Omega_{p,q}, \qquad 0 < \epsilon < 1,$$

contain E and have ν_s-mass $(1+2\epsilon)^p$ for every s. Let $\epsilon \downarrow 0$ to obtain $C^*(E) \le 1$. This proves the assertion for all $p, q \ge 1$, including the requested $p = 1$, $n = q+1$ case. □

The ambient background in this construction is smooth and belongs to the full delta approximation class. The failure is caused by the restricted normalized family, not by a lack of delta tests or by singularity of the background. It neither disproves compact-set outer regularity for that compact represented family nor determines the unrestricted relative Bedford–Taylor capacity of the same set.

31 A local finite-point normalization of the potential quotient

A differentiated potential quotient does not itself determine a multiplier. On the compact represented families constructed above, one can nevertheless make an explicit local choice of representatives. The choice depends on a fixed affinoid and finitely many normalization points; it is not a canonical global gauge for all psh functions.

Fix a compact affinoid norm box $V \Subset \Omega$ with nonempty ordinary interior, and let $\Gamma(V)$ be its finite Shilov boundary. Let $\mathscr{H}_V$ be the vector space of restrictions to V of continuous pluriharmonic functions defined on some ordinary neighborhood of V inside Ω. The neighborhood may depend on the function. Finite sums are defined on intersections of these neighborhoods, so this is a vector space of actual continuous functions on V.

Lemma 31.1 (Finite Shilov normalization)**.** *The space $\mathscr{H}_V$ has dimension $d \le \#\Gamma(V)$. There are points $\xi_1, \dots, \xi_d \in \Gamma(V)$, a basis $h_1, \dots, h_d$ of $\mathscr{H}_V$ represented by pluriharmonic functions on one fixed ordinary neighborhood W of V, and a bounded linear projection*

$$L_V : C(V) \longrightarrow \mathscr{H}_V, \qquad (L_V f)(\xi_j) = f(\xi_j) \quad (1 \le j \le d).$$

The projection is the identity on $\mathscr{H}_V$. Its construction uses actual values at the displayed points.

Status: `DIRECT_COROLLARY` of the classical Shilov maximum principle and finite-dimensional linear algebra.

Proof. Apply the affinoid maximum principle to both signs of any representing pluriharmonic function, as in [GR25, Theorem 6.4]. Its restriction to V is determined by its values on $\Gamma(V)$: zero values there force the function to vanish on all of V. Evaluation therefore embeds $\mathscr{H}_V$ in $\mathbf{R}^{\Gamma(V)}$, proving the dimension bound. Choose a basis, intersect its finitely many defining ordinary neighborhoods, and call the intersection W.

The evaluation matrix of this basis on $\Gamma(V)$ has rank d. Choose d rows with invertible matrix $M = (h_i(\xi_j))_{j,i}$ and set

$$L_V f = \sum_{i=1}^{d} c_i h_i, \qquad (c_i)_i = M^{-1}(f(\xi_j))_j.$$

This is a bounded linear map in the uniform norm, interpolates the selected values, and reproduces every member of $\mathscr{H}_V$. In the vacuous zero-dimensional vector-space case use the zero map. □

Theorem 31.2 (Continuous local representatives from weak curvature data)**.** *Let $\mathscr{P}$ be any compact family in $C(\Omega) \cap \mathcal{C}_b(\Omega)$ for local uniform topology, with $0 \le u \le 1$ for all its members. Fix V, W, L_V as above, shrinking W inside Ω if necessary, and give*

$$\mathscr{A}_W = \{D[u]|_W : u \in \mathscr{P}\}$$

the ordinary weak topology on every original compact test in W. For $A = D[u]|_W$ put

$$s_V(A) = u|_V - L_V(u|_V). \tag{31.1}$$

This is independent of the chosen $u \in \mathscr{P}$, and defines a continuous map $s_V : \mathscr{A}_W \to C(V)$. Its values are actual continuous classical potentials on $\operatorname{Int} V$, with

$$D[s_V(A)] = A|_{\operatorname{Int} V}, \qquad s_V(A)(\xi_j) = 0, \qquad \|s_V(A)\|_V \le 1 + \|L_V\|.$$

Here the curvature equality refers to the unique restriction to the ordinary interior. Each representative is also defined classically on a neighborhood of V, although no common choice outside V is part of the section. In particular, weak convergence $A_j \to A$ within $\mathscr{A}_W$ implies uniform convergence of these specified representatives on V.

Status: `DIRECT_COROLLARY` of the frozen actual potential quotient and compactness, with the local normalization explicitly constructed.

Proof. If $D[u]|_W = D[v]|_W$, the frozen R5 quotient makes $u - v$ a continuous pluriharmonic function on W. Its restriction belongs to $\mathscr{H}_V$, and hence $L_V(u-v) = u - v$ on V. Subtracting proves independence in (31.1).

The curvature map from $\mathscr{P}$ onto $\mathscr{A}_W$ is continuous, as seen on each original compact test. Its domain is compact and its target is Hausdorff, so it is a closed quotient map. The map $u \mapsto u|_V - L_V(u|_V)$ is continuous into $C(V)$ and constant on its fibers. The defining property of a quotient topology therefore gives continuity of s_V. This argument uses no metrizability or choice of convergent lifts.

For any lift u, subtract the fixed finite linear combination of the pluriharmonic basis functions on W. This produces an actual continuous classical potential near V, has unchanged curvature, and restricts to $s_V(A)$. Reproduction and the interpolation identities give the normalizations; the operator norm gives the uniform bound. Different lifts give the same actual function on V, hence the same potential on its ordinary interior. Nothing is asserted about their values outside V. □

Corollary 31.3 (Weighted stability after the specified local normalization)**.** *For $0 \le i \le k$, let $A_{i,j} \to A_i$ be nets in the restricted curvature images of compact normalized families as in the theorem, and put $u_{i,j} = s_V(A_{i,j})$, $u_i = s_V(A_i)$. Let $S_j \to S$ be arbitrary allowed positive separately closed backgrounds on* $\operatorname{Int} V$*. Then*

$$u_{0,j} T_{S_j,k}(u_{1,j}, \dots, u_{k,j}) \longrightarrow u_0 T_{S,k}(u_1, \dots, u_k)$$

on all original compact complementary smooth tests in that interior. The same conclusion holds against a fixed actual degree-one delta-test when it is complementary to the displayed mixed currents.

Status: `DIRECT_COROLLARY`.

Proof. All displayed potentials converge uniformly on V by the theorem. Their restrictions are continuous classical on its interior. Proposition 25.1 proves the original-test assertion. For the degree-one delta assertion, insert its fixed continuous-coefficient Hessian presentation, expand its fixed continuous classical-difference slots, and use the same proposition term by term with the locally uniformly convergent continuous weight. For its PS part the fixed PWS test approximation has the uniform normal coefficient bound, and multiplication by the uniformly bounded weights preserves that bound. A compact cutoff converts each top-degree current convergence into the desired pairing. □

Scope of the normalization. This section does not identify arbitrary actual multipliers with their curvatures. For example 0 and 1 have equal curvature but are distinct multipliers; the specified normalization sends both to zero. Its weighted conclusion applies only after this explicit local choice. The compact families of Theorem 26.3 and Proposition 28.1 are concrete inputs. There is no continuous-section claim for the unrestricted weak space of bounded classical curvatures, and no global compatibility of different choices of V, normalization points, or projections is asserted.

Part III
Sharp relative capacity estimates

32 A sharp Hardy estimate for every normalized relative tester

Fix an arbitrary original-positive symmetric separately closed $(n-1,n-1)$-current S on an allowed ordinary open Ω. For original real compact smooth f write

$$\mathcal{E}_S(f) = \int_\Omega \Gamma_S(f).$$

The first estimate below needs only the frozen bounded BT calculus and its local coefficient-Radon energy measures. The faithful closed form is used subsequently for the capacity and Dirichlet consequences.

Lemma 32.1 (Polynomially weighted local energy limits)**.** *On a fixed local compact regularization stage let $a_{i,j} \downarrow a_i$ be finitely many bounded classical psh rows. Let P, Q, R be real polynomials in these functions and in finitely many fixed smooth local classical differences. Put $P_j = P(a_{1,j}, \dots)$, and similarly for Q_j, R_j. Then*

$$R_j\, \Gamma_S(P_j, Q_j) \longrightarrow R\, \Gamma_S(P, Q)$$

on original compact smooth scalar tests. All functions and measures use their actual representatives.

Status: `DIRECT_COROLLARY` of the weighted one-slot theorem and the definition of Γ_S.

Proof. After shifting the finitely many psh functions by constants on the compact stage, they are nonnegative. Every polynomial in them is a finite real linear combination of powers of their positive sums, by the ordinary polarization formula for monomials. Each such power is classical psh and decreases along the given rows. Fixed smooth local classical differences have the same description after choosing their finite psh decompositions; these functions are constant rows. Thus every polynomial belongs to the local bounded classical-difference algebra. Expand

$$R_j\Gamma_S(P_j, Q_j) = \tfrac{1}{2}\{R_j D(P_jQ_jS) - R_jP_jD(Q_jS) - R_jQ_jD(P_jS)\}.$$

Every term is a finite linear combination of weighted one-slot products of decreasing bounded psh rows. The frozen weighted theorem passes each term. The limiting identity is the Borel multiplier of the coefficient-Radon measure defining $\Gamma_S(P, Q)$. □

Theorem 32.2 (Sharp normalized-tester Hardy inequality)**.** *Let $w \in \mathcal{C}_b(\Omega)$ satisfy $0 \le w \le 1$ everywhere. For every original real $f \in A_c^{0,0}(\Omega)$ and every $\varepsilon > 0$,*

$$\int_\Omega \frac{f^2}{1+\varepsilon-w}\, D(wS) \le \mathcal{E}_S(f). \tag{32.1}$$

Consequently

$$\int_\Omega f^2\, D(wS) \le \mathcal{E}_S(f). \tag{32.2}$$

The coefficient one in (32.2) cannot be decreased uniformly over all allowed domains and backgrounds.

Status: `NEW_PROVED` for the sharp uniform estimate. It strengthens the earlier square-root estimate; that weaker estimate is not used in its proof.

Proof. First suppose w is smooth near the support of f. Put $v = 1 + \varepsilon - w > 0$. The positive covector square gives

$$\Gamma_S(f) - \Gamma_S(v, f^2/v) = \Gamma_S(f) - 2(f/v)\Gamma_S(f, v) + (f^2/v^2)\Gamma_S(v) \geq 0.$$

Compact Green and $D(vS) = -D(wS)$ therefore yield

$$\mathcal{E}_S(f) \geq \int \Gamma_S(v, f^2/v) = \int (f^2/v) D(wS).$$

For bounded possibly discontinuous w, a global smooth row near the entire support is not assumed. Choose finitely many nonnegative original smooth functions χ_i, supported in local compact regularization stages, with $\sum_i \chi_i = 1$ on a neighborhood of $\operatorname{supp} f$. On the stage for χ_i, choose a decreasing original smooth classical row $w_j \downarrow w$. After discarding a prefix, $0 \leq w_j < 1 + \varepsilon/2$ on a fixed neighborhood of the relevant compact support. Indeed the increasing open sublevel sets below $1 + \varepsilon/2$ cover that compact set. Set

$$v_j = 1 + \varepsilon - w_j, \qquad r_j = 1/v_j.$$

The r_j form a bounded decreasing classical psh row on that fixed neighborhood: the scalar reciprocal function of w_j is convex and increasing on its range. The limit is $r = (1 + \varepsilon - w)^{-1}$.

Multiply the smooth positive-square inequality by χ_i before integrating. The localized Green identity is

$$\int \chi_i \Gamma_S(v_j, f^2 r_j) = \int \chi_i f^2 r_j D(w_j S) + \int f^2 r_j \Gamma_S(w_j, \chi_i).$$

The first term passes by weighted one-slot continuity for the two rows r_j, w_j. The second passes by Lemma 32.1, treating r_j as an additional psh row. Fixed original smooth f and χ_i are local classical differences by the frozen smooth compensator construction. Consequently

$$\int \chi_i \Gamma_S(f) \geq \int \chi_i f^2 r D(wS) + \int f^2 r \Gamma_S(w, \chi_i).$$

Sum these finitely many inequalities. Their cross terms cancel exactly: by bilinearity and ordinary locality, $\Gamma_S(w, \sum_i \chi_i) = 0$ near the support of f. Also $\Gamma_S(f)$ is supported there. This proves (32.1) without a partition error or a global regularization assumption. Since $1 + \varepsilon - w \leq 1 + \varepsilon$, it gives $\int f^2 D(wS) \leq (1 + \varepsilon)\mathcal{E}_S(f)$. Let $\varepsilon \downarrow 0$ to obtain (32.2).

For sharpness let $\lambda = -\log|\pi| > 0$, take the original annulus $|\pi| < |T| < |\pi|^{-1}$, put $s = \log|T|/\lambda \in (-1, 1)$, and set $S = [1]$, $w = |s|$. Then

$$D(wS) = (2/\lambda)\delta_{\zeta_{0,1}}, \qquad \mathcal{E}_S(f(s)) = \lambda^{-1} \int_{-1}^{1} |f'(s)|^2 \, ds.$$

Choose smooth compact functions on $(-1, 1)$ converging to $1 - |s|$ in the usual H^1_0 norm and with values at zero tending to one; smoothing slightly truncated tents gives such a sequence. The numerator in (32.2) tends to $2/\lambda$, as does the energy. All functions used as tests are original smooth functions of the unit moment. Thus a universal smaller coefficient is impossible. □

Corollary 32.3 (Weak limits and the energy-capacity bound)**.** *Assume the faithful form exists on $F = \operatorname{supp} S$ as in R4. Every positive weak limit of normalized measures $D(w_j S)$ on all original tests satisfies (32.2). This includes the normalized completed image defined by global normalized classical rows. Moreover, for every subset $A \subset \Omega$,*

$$\operatorname{Cap}^*_S(A; \Omega) \leq \operatorname{cap}_{\mathcal{E},1}(A \cap F). \tag{32.3}$$

For original smooth compact f and $t > 0$ there is the sharper direct level-set bound

$$\mathrm{Cap}_S^*(\{|f| > t\}; \Omega) \le \mathcal{E}_S(f)/t^2. \tag{32.4}$$

These are one-way bounds. Neither the reverse capacity inequality nor the general raw-versus-completed compact equality is asserted.

Status: `DIRECT_COROLLARY` of the sharp estimate and the already verified regular-form compact-core capacity formula.

Proof. For weak limits, f^2 is one fixed original compact smooth scalar test, so its inequality passes directly, also for nets. For a compact $K \subset F$ and original smooth core $f > 1$ on K, every normalized tester satisfies

$$D(wS)(K) \le \int f^2 D(wS) \le \mathcal{E}_S(f) \le \mathcal{E}_1(f, f).$$

The verified compact-core formula gives $D(wS)(K) \le \mathrm{cap}_{\mathcal{E},1}(K)$, with the same bound for all testers. For a relative ordinary open $G \subset F$, inner regularity of each fixed Radon measure therefore gives $D(wS)(G) \le \mathrm{cap}_{\mathcal{E},1}(G)$. Choose an ordinary open $O \subset \Omega$ with $O \cap F = G$. All these measures are supported on F. If G contains $A \cap F$, then $O \cup (\Omega \setminus F)$ contains A, and taking the supremum over testers bounds its raw BT capacity by the energy capacity of G. The two outer infima give (32.3). Only a uniform compact estimate was used before Radon inner regularity; no infimum/supremum exchange has been made.

Finally, Chebyshev and (32.2) give $D(wS)\{|f| > t\} \le \mathcal{E}_S(f)/t^2$ for every normalized tester. The level set is an ordinary open, so its raw capacity equals its outer capacity by definition. Taking the supremum proves (32.4). □

Corollary 32.4 (Linear capacity domination is enough for finite energy)**.** *Let μ be a positive Radon measure supported on a compact subset of $V \Subset \Omega$, with mass $M > 0$, and suppose*

$$\mu(E) \le A \,\mathrm{Cap}_S(E; \Omega)$$

for every Borel subset of that support. Then

$$\|\mu\|_{-1,V} \le 2\sqrt{AM}.$$

It has the unique energy solution of Theorem 53.2. More generally, if $\mu(E) \le A \,\mathrm{Cap}_S(E; \Omega)^\alpha$ with $\alpha > 1/2$, then

$$\|\mu\|_{-1,V} \le \frac{2\alpha}{2\alpha - 1} A^{1/(2\alpha)} M^{1-1/(2\alpha)}.$$

No boundedness or ambient psh property of that solution follows from these assertions alone.

Status: `DIRECT_COROLLARY` by layer cake.

Proof. For a core function of energy e^2, the preceding level-set bound gives

$$\mu\{|f| > t\} \le \min\{M, Ae^{2\alpha} t^{-2\alpha}\}.$$

Split the integral of this tail at $t_0 = (A/M)^{1/(2\alpha)} e$. Since $2\alpha > 1$, its two pieces sum to the stated constant times e. The case $\alpha = 1$ gives $2\sqrt{AM}$. A zero-energy core function has zero integral by the same level-set bound. Taking the core supremum completes the proof. □

Frozen higher-degree tails. For an arbitrary allowed (p,p)-background S_0 and $q = n - p \geq 1$, fix bounded classical potentials $v_2, \dots, v_q$ and put $R = T_{S_0,q-1}(v_2, \dots, v_q)$. The theorem applies verbatim to this fixed degree-one background. Its capacity is $\mathrm{Cap}_R(E;\Omega) = \sup_{0\leq w\leq 1} D(wR)(E)$ and its energy is $\mathcal{E}_R(f) = \int \Gamma_R(f)$. No single quadratic form is thereby asserted to control the nonlinear diagonal family $T_{S_0,q}(w^q)$ when all q slots vary.

33 A sharp strong-capacity inequality and energy traces

Retain the arbitrary fixed degree-one background S and the notation of Section 32. Capacities below use normalized actual classical testers on the entire fixed Ω. All smooth functions and tests are original CLD objects.

Theorem 33.1 (Strong capacitary inequality with optimal constant)**.** *For every real original $f \in A_c^{0,0}(\Omega)$,*

$$\int_0^\infty \mathrm{Cap}_S(\{|f| > t\};\Omega)\, 2t\, dt \leq 4\mathcal{E}_S(f). \tag{33.1}$$

The level sets are ordinary open, so the same formula holds with outer capacity. If a positive Radon measure μ satisfies

$$\mu(E) \leq A\, \mathrm{Cap}_S(E;\Omega) \tag{33.2}$$

for every Borel $E \subset \Omega$, then

$$\int f^2\, d\mu \leq 4A\mathcal{E}_S(f). \tag{33.3}$$

Both universal constants four are optimal, already on one original analytic annulus with $S = [1]$.

Status: `NEW_PROVED` for the sharp capacitary and trace estimates. The proof uses only the preceding Hardy theorem and explicit scalar truncations; no raw-capacity outer regularity is assumed.

Proof. Fix $t > 0$ and put $g_t = \min(f^2/t^2, 1)$. To apply the smooth-core Hardy inequality, choose smooth functions P_δ on the real line such that $P_\delta(0) = 0$, $0 \leq P_\delta' \leq 1$, $P_\delta(s) \to \min(s,1)$ for $s \geq 0$, and $P_\delta'(s) \to 0$ for $s > 1$. They may be obtained by integrating a smooth transition from derivative one below $1-\delta$ to zero above $1+\delta$, chosen symmetrically so the final constant is one. For $\delta < 1/2$ the functions are exactly s near zero and satisfy $0 \leq P_\delta(s) \leq 1$ for $s \geq 0$.

Then $P_\delta(f^2/t^2)$ is an original smooth compact function. For each fixed normalized tester w, Hardy and the ordinary chain rule give

$$\int P_\delta(f^2/t^2)^2 D(wS) \leq \frac{4}{t^4} \int f^2 P_\delta'(f^2/t^2)^2\, \Gamma_S(f).$$

On the fixed compact support both measures are finite. Bounded convergence on the left and dominated upper limits on the right yield

$$D(wS)\{|f| > t\} \leq \int g_t^2 D(wS) \leq \frac{4}{t^4} \int_{\{|f|\leq t\}} f^2\, \Gamma_S(f).$$

The inclusion of the equality level makes this argument independent of any assertion about energy mass on a level set. The right side is independent of w, so the same bound holds for the raw capacity.

Multiply by $2t$ and integrate. Tonelli's theorem for the nonnegative measure $\Gamma_S(f)$ gives

$$\begin{aligned}\int_0^\infty \mathrm{Cap}_S\{|f|>t\}\,2t\,dt &\le 8\int f^2\left(\int_{|f|}^\infty t^{-3}\,dt\right)\Gamma_S(f)\\ &= 4\int_{\{f\neq 0\}}\Gamma_S(f)\le 4\mathcal{E}_S(f).\end{aligned}$$

At $f=0$, the inner-integrand contribution is interpreted before integration as identically zero; no $0\cdot\infty$ expression is used. The capacity of these level sets equals their outer capacity because they are ordinary open. Finally layer cake and (33.2) imply

$$\int f^2\,d\mu=\int_0^\infty \mu\{|f|>t\}\,2t\,dt\le A\int_0^\infty \mathrm{Cap}_S\{|f|>t\}\,2t\,dt,$$

which proves (33.3). □

Example 33.2 (Sharpness on an original annulus)**.** Put $\lambda=-\log|\pi|>0$ and take

$$\Omega=\{1<|T|<|\pi|^{-1}\},\qquad s=\log|T|/\lambda\in(0,1),\qquad S=[1].$$

Let σ_s be the Gauss point of radius $\exp(\lambda s)$. For $0<\varepsilon<1/4$ define the finite compactly supported measure

$$\mu_\varepsilon=\lambda^{-1}\sigma_*(\mathbf{1}_{[\varepsilon,1/2]}s^{-2}ds).$$

It satisfies $\mu_\varepsilon(E)\le\mathrm{Cap}_{[1]}(E;\Omega)$ for every Borel $E\subset\Omega$.

Indeed, if $E_s=\{s\in[\varepsilon,1/2]:\sigma_s\in E\}$ is nonempty, put $a=\inf E_s>0$. Then

$$\mu_\varepsilon(E)\le\frac{1}{\lambda}\int_a^\infty s^{-2}ds=\frac{1}{\lambda a}.$$

For each $r\in E_s$ the globally normalized classical hinge

$$w_r(s)=\max(0,1-s/r)$$

has original curvature $Dw_r=(\lambda r)^{-1}\delta_{\sigma_r}$. Thus the capacity is at least $\sup_{r\in E_s}(\lambda r)^{-1}=(\lambda a)^{-1}$. If E_s is empty the assertion is immediate. No formula for the capacity of a general set is being assumed.

Choose fixed smooth scalar cutoffs α,β with $\alpha=0$ on $(-\infty,1]$, $\alpha=1$ on $[2,\infty)$, $\beta=1$ on $(0,1/2]$, and $\beta=0$ on $[3/4,1)$. Set

$$f_\varepsilon(s)=\alpha(s/\varepsilon)\beta(s)\sqrt{s}.$$

These are original smooth compact tests. Direct integration gives

$$\int f_\varepsilon^2\,d\mu_\varepsilon=\lambda^{-1}\log(1/\varepsilon)+O(1),\qquad \mathcal{E}_S(f_\varepsilon)=(4\lambda)^{-1}\log(1/\varepsilon)+O(1).$$

For the energy, the derivative on $[2\varepsilon,1/2]$ is $1/(2\sqrt{s})$. Its two cutoff collars have uniformly bounded energy: the lower collar has length $O(\varepsilon)$ and derivative $O(\varepsilon^{-1/2})$, and the upper collar is fixed. The mass calculation has the same bounded collar errors. Hence the trace ratio tends to four. Since each μ_ε is dominated by the raw capacity with coefficient one, its layer-cake integral also gives a lower bound for the left side of (33.1). Both constants are therefore optimal. The field and all tests remain the original ones, including Gauss points of non-strict radii.

Corollary 33.3 (Canonical energy traces)**.** *Assume the faithful form exists and μ satisfies (33.2). The original smooth restriction map extends uniquely to a bounded linear map*

$$J_\mu : \mathscr{D} \longrightarrow L^2(\mu), \qquad \|J_\mu u\|_{L^2(\mu)} \le 2\sqrt{A}\,\mathcal{E}(u,u)^{1/2},$$

It is represented by the canonical relative capacitary representative of u, μ-almost everywhere. The same assertion holds on every part space $\mathscr{D}_0(V)$.

Status: `DIRECT_COROLLARY` of the trace bound and the frozen canonical energy representative.

Proof. The estimate makes any energy-Cauchy core sequence Cauchy in $L^2(\mu)$, proving existence, independence and the norm bound. On a fixed compact stage, the same core sequence converges locally in outer relative capacity to the canonical representative by R4. For each Borel gap, its μ-mass is at most A times its raw capacity and hence at most A times its outer capacity. Thus it converges in μ-measure on every compact stage. Its $L^2(\mu)$ limit must be that representative almost everywhere. This argument selects the canonical representative; it does not identify arbitrary modifications on λ-null sets. □

Corollary 33.4 (A controlled negative measure perturbation)**.** *Fix $V \Subset \Omega$, put $H = \mathscr{D}_0(V)$, and let μ be carried by V and satisfy (33.2). For $c \ge 0$ with $4cA < 1$, the bilinear form*

$$\mathcal{B}_c(u,v) = \mathcal{E}(u,v) - c\int (J_\mu u)(J_\mu v)\,d\mu$$

is continuous and coercive on H, with

$$\mathcal{B}_c(u,u) \ge (1-4cA)\mathcal{E}(u,u).$$

For every bounded energy functional $\ell \in H^*$ there is a unique $u \in H$ with $\mathcal{B}_c(u,\phi) = \ell(\phi)$ for all $\phi \in H$, and

$$\|u\|_H \le \frac{\|\ell\|_{H^*}}{1-4cA}.$$

If $\ell(\phi) = -\int \phi\,d\sigma$ for a finite-energy Radon datum, this is the original-current equation

$$D(uS) + c(J_\mu u)\mu = \sigma \quad \text{on } V.$$

The boundary condition is the stated energy part space. No ambient psh, nonlinear Monge–Ampere, or Markov property of the perturbed form is claimed.

Proof. Cauchy–Schwarz in $L^2(\mu)$ and the preceding bound give $|\int J_\mu u J_\mu v\,d\mu| \le 4A\|u\|_H\|v\|_H$. The displayed lower bound follows. Thus $\mathcal{B}_c$ is an equivalent Hilbert inner product, and its Riesz theorem gives the solution and estimate. For an original compact test, the ordinary Green identity identifies its energy term with $-D(uS)$. The trace factor is locally μ-integrable by Cauchy–Schwarz and local finiteness, so it defines the indicated Radon measure. Rearranging the two terms gives the claimed sign in the current equation.

The coefficient threshold is a uniform sufficient threshold. The annular family in Example 33.2, scaled by A, with part domains containing the respective compact test supports, shows that for $4cA > 1$ these forms need not even be nonnegative. At the equality threshold it supplies no uniform positive coercivity constant. This does not decide critical behavior for one fixed measure. □

As throughout this energy discussion, a fixed higher mixed tail can replace S. The resulting capacity and energy belong to that fixed degree-one tail; this does not control all slots of a nonlinear diagonal family by one quadratic form.

34 A square-integrable capacity modulus for finite-energy data

Throughout the first statement, S is an arbitrary allowed original positive separately closed current of bidegree $(n-1,n-1)$ on the fixed ordinary domain Ω. The energy and part spaces are the frozen faithful ones. All capacities use normalized actual classical testers on the whole Ω.

Theorem 34.1 (A modulus criterion from the strong capacity inequality)**.** *Let μ be a finite positive Radon measure, carried by an ordinary $V \subset \Omega$, with total mass M. Suppose that $F : [0,\infty) \to [0,\infty)$ is nondecreasing, finite, locally absolutely continuous, and $F(0) = 0$, and that*

$$\mu(E) \le F(\mathrm{Cap}_S(E;\Omega)) \qquad (E \subset \Omega \text{ Borel}).$$

For infinite capacity, $F(\infty)$ means its monotone extended limit. If $r_0 > 0$ satisfies $F(r_0) \ge M$ and $F' \in L^2(0,r_0)$, then μ has finite energy on V and

$$\|\mu\|_{-1,V} \le 2\left(\int_0^{r_0} |F'(r)|^2\,dr\right)^{1/2}. \tag{34.1}$$

Consequently the energy-class Dirichlet equation has its unique solution in $\mathscr{D}_0(V)$ whenever this coercive part space is defined as in Theorem 53.2.

Status: `DIRECT_COROLLARY` of the new sharp strong-capacity inequality, with the displayed modulus criterion proved explicitly. It is an energy-class solvability statement; no classical or completed-psh membership is asserted.

Proof. Let f be one original real compact smooth core function in V, extended by zero to Ω, and put

$$c(t) = \mathrm{Cap}_S(\{|f| > t\};\Omega), \qquad L(r) = \big|\{t > 0 : c(t) > r\}\big|.$$

The function c is nonincreasing. Its superlevel sets in the variable t are therefore initial intervals, up to endpoints. Tonelli's theorem and Theorem 33.1 give

$$\int_0^\infty L(r)^2\,dr = \int_0^\infty \int_0^{c(t)} 2t\,dr\,dt = \int_0^\infty 2tc(t)\,dt \le 4\mathcal{E}_S(f).$$

All analytic level sets here are ordinary open, so neither a raw/outer identification on general Borel sets nor Choquet regularity is used. Since $F(r_0) \ge M$, the datum bound gives

$$\begin{aligned}\int |f|\,d\mu &= \int_0^\infty \mu\{|f| > t\}\,dt \\ &\le \int_0^\infty F(\min\{c(t), r_0\})\,dt \\ &= \int_0^{r_0} F'(r)L(r)\,dr \\ &\le 2\left(\int_0^{r_0} |F'(r)|^2\,dr\right)^{1/2} \mathcal{E}_S(f)^{1/2}.\end{aligned}$$

Absolute continuity and $F(0) = 0$ justify the second Tonelli identity; its derivative is nonnegative almost everywhere. This proves the core dual bound, and the frozen energy Riesz theorem supplies the stated solution. If $M = 0$, the same conclusion is immediate. □

Corollary 34.2 (The critical logarithmic family)**.** *For*

$$F_\beta(r) = A\,r^{1/2}[\log(e + r^{-1})]^{-\beta}, \qquad A > 0, \quad F_\beta(0) = 0,$$

every finite measure satisfying the preceding capacity-growth bound has finite energy if $\beta > 1/2$. In contrast, for every $0 \le \beta \le 1/2$ there is a finite positive measure on the original unit disc, compactly supported inside one fixed ordinary smaller disc, satisfying that same bound with $A = 1$, whose energy dual norm is infinite.

Status: `NEW_PROVED` for the exact universal logarithmic threshold within this specified family. The sufficient direction is the preceding direct criterion; the counterexample uses the frozen original radial calibration. This is not an if-and-only-if criterion for an individual measure.

Proof. Write $L(r) = \log(e + r^{-1})$. For $r > 0$,

$$F'_\beta(r) = Ar^{-1/2}L(r)^{-\beta}\left(\frac{1}{2} + \frac{\beta}{(er+1)L(r)}\right).$$

For $\beta \ge 0$ this is positive, and its square is integrable near zero exactly when $2\beta > 1$. The function is unbounded at infinity, so a finite r_0 with $F_\beta(r_0) \ge M$ always exists. This proves the sufficient direction.

For the opposite direction fix $\lambda = -\log|\pi|$, let $t_j = 2^j\lambda$, $z_j = \zeta_{0,e^{-t_j}}$, and put

$$W_j = F_\beta(t_j^{-1}) = t_j^{-1/2}[\log(e+t_j)]^{-\beta}, \qquad m_j = W_j - W_{j+1} > 0, \qquad \mu = \sum_{j\ge1} m_j\delta_{z_j}.$$

The total mass is W_1, the support is contained in $\{0\} \cup \{z_j\}$, and $\mu\{0\} = 0$. All nonzero-mass support points are original divisorial Gauss points. As in Proposition 35.1, take the fixed $V = \{|T| < e^{-\lambda/2}\} \Subset \Omega = \{|T| < 1\}$. The normalized hinge with curvature $t_j^{-1}\delta_{z_j}$ proves for every Borel E containing a least-index support atom z_j that

$$\mu(E) \le W_j \le F_\beta(\mathrm{Cap}_{[1]}(E;\Omega)).$$

The inequality is immediate for sets containing no support atom.

Use the finite test functions from that proposition with these new coefficients,

$$f_N(t) = \sum_{j=1}^{N} m_j \min\{(t-\lambda)_+, t_j - \lambda\}.$$

Their energy and partial-measure pairing are again equal:

$$E_N = \sum_{j=1}^{N}(t_j - t_{j-1})(W_j - W_{N+1})^2 = \int f_N\, d\mu_N \le \int f_N\, d\mu, \qquad t_0 = \lambda.$$

The quantities E_N increase to

$$\sum_{j\ge1}(t_j - t_{j-1})W_j^2 = \frac{1}{2}\sum_{j\ge1}[\log(e+t_j)]^{-2\beta} = \infty \quad (0 \le \beta \le 1/2).$$

Here monotone convergence is applied to the nonnegative discrete sum; no rate estimate for m_j/W_j is needed. The same original smooth compact approximations inside the fixed V as in Proposition 35.1 pass both energy and integrals against the finite μ. A finite core dual norm would imply $E_N \le C\sqrt{E_N}$, a contradiction. □

Power bounds and scope. For $F(r) = Ar^\alpha$, $\alpha > 1/2$, taking $r_0 = (M/A)^{1/\alpha}$ gives

$$\|\mu\|_{-1,V} \le \frac{2\alpha}{\sqrt{2\alpha - 1}} A^{1/(2\alpha)} M^{1-1/(2\alpha)}.$$

One may use the smaller of this bound and the earlier direct layer-cake bound. The new improvement is at the half-power endpoint; the linear case still gives $2\sqrt{AM}$. A fixed higher-degree background with frozen mixed slots may be replaced by its resulting degree-one current, with that current's own capacity and energy. No bound for a nonlinear capacity with all slots varying is inferred.

35 Sharpness of the power-growth threshold for finite energy

The following construction reuses the frozen radial Gauss-point calibration and first-atom tail comparison from Section R4–60. Its new conclusion concerns finite faithful energy, rather than bounded classical Monge–Ampere solvability.

Proposition 35.1 (The exponent one half cannot be included)**.** *For every $0 < \alpha \le 1/2$ there is a finite positive Radon measure μ on the original open unit disc Ω, supported on a fixed compact subset of an ordinary $V \Subset \Omega$, such that*

$$\mu(E) \le \mathrm{Cap}_{[1]}(E;\Omega)^\alpha \qquad (E \subset \Omega \text{ Borel}), \tag{35.1}$$

but $\|\mu\|_{-1,V} = \infty$ for the faithful degree-one energy of $S = [1]$. Therefore it has no solution in $\mathscr{D}_0(V)$ of $D(uS) = \mu$ on all original scalar tests. The measure is carried by divisorial Gauss points and gives zero mass to their rigid limit point.

Status: `NEW_PROVED` as the sharpness application for the R6 finite-energy criterion. The radial curvature and original integration normalization are frozen inputs, not new results of this proposition. No additional bounded-classical Dirichlet conclusion is claimed.

Proof. Fix $\lambda = -\log|\pi| > 0$ and put

$$t_j = 2^j\lambda, \qquad z_j = \zeta_{0,e^{-t_j}}, \qquad W_j = t_j^{-\alpha}, \qquad m_j = W_j - W_{j+1} = (1 - 2^{-\alpha})t_j^{-\alpha}.$$

Set $\mu = \sum_{j\ge1} m_j\delta_{z_j}$. Its total mass is $W_1 < \infty$. Since $e^{-t_j} = |\pi|^{2^j}$, all z_j are points of the original K-analytic disc and are divisorial. They converge to its rigid origin, so $K_0 = \{0\} \cup \{z_j : j \ge 1\}$ is compact. The measure gives no mass to zero. Take the fixed ordinary disc

$$V = \{|T| < e^{-\lambda/2}\} \Subset \Omega.$$

It contains K_0.

For each j, the actual normalized classical function on the whole Ω

$$w_j(T) = \max\{0, 1 + t_j^{-1}\log|T|\}$$

is assigned value zero near the origin and has $Dw_j = t_j^{-1}\delta_{z_j}$, by the original radial calibration. If E contains some support atom, choose its least index j. Then

$$\mu(E) \le \sum_{k\ge j} m_k = W_j = t_j^{-\alpha} \le \mathrm{Cap}_{[1]}(E;\Omega)^\alpha.$$

If it contains none, its mass is zero. This proves the bound for every Borel set, including sets containing the limit point. It uses neither compact outer regularity nor an optimizing envelope.

On the punctured unit disc put $t = -\log|T|$. For $N \geq 1$ define the nonnegative continuous radial function

$$f_N(t) = \sum_{j=1}^{N} m_j \min\{(t-\lambda)_+, t_j - \lambda\},$$

with its constant terminal value at the origin. It is zero for $t \leq \lambda$, and constant for $t \geq t_N$. For $t_{j-1} < t < t_j$, where $t_0 = \lambda$, its derivative is

$$M_{j,N} = \sum_{k=j}^{N} m_k.$$

The one-dimensional original energy is consequently

$$E_N = \int_\lambda^\infty |f_N'(t)|^2 dt = \sum_{j=1}^{N} (t_j - t_{j-1}) M_{j,N}^2 \geq \frac{(1-2^{-\alpha})^2}{2} \sum_{j=1}^{N} t_j^{1-2\alpha}. \tag{35.2}$$

This tends to infinity for every $0 < \alpha \leq 1/2$. No factor of λ is missing: t is the unnormalized unit moment $-\log|T|$, so its energy calibration is dt itself.

Let $\mu_N = \sum_{j \leq N} m_j \delta_{z_j}$. A direct finite sum gives

$$\begin{aligned} \int f_N \, d\mu_N &= \sum_{i,j \leq N} m_i m_j \min(t_i - \lambda, t_j - \lambda) \\ &= \int_\lambda^\infty \left(\sum_{j \leq N} m_j \mathbf{1}_{\{t < t_j\}} \right)^2 dt = E_N. \end{aligned}$$

Since $\mu \geq \mu_N$ and $f_N \geq 0$, one has $\int f_N \, d\mu \geq E_N$.

We now realize these comparisons by original smooth core functions, rather than testing a current with a merely PL function. Extend f_N to the real line by zero to the left and its terminal constant to the right, and convolve with smooth nonnegative kernels of radius $\delta < \lambda/4$. The resulting $f_{N,\delta}(t)$ is zero for $t \leq \lambda - \delta$ and constant for $t \geq t_N + \delta$. It therefore extends as an original smooth function across the rigid origin and has compact support inside the fixed V. For each fixed N the approximation is uniform on the entire analytic disc, and its derivative converges in $L^2(dt)$ to f_N'. The original calibration gives

$$\mathcal{E}_S(f_{N,\delta}) \longrightarrow E_N, \qquad \int f_{N,\delta} \, d\mu \longrightarrow \int f_N \, d\mu.$$

The second assertion uses the finite total mass of μ and uniform approximation, not a singular-test convergence theorem.

If $\|\mu\|_{-1,V} = C < \infty$, apply its core bound to $f_{N,\delta}$ and let $\delta \downarrow 0$. It would give

$$E_N \leq \int f_N \, d\mu \leq C\sqrt{E_N}$$

for every N, contradicting (35.2). This proves infinite energy. Any energy solution in the specified part space would make the core datum bounded by its own energy norm through the original Green identity, so such a solution cannot exist. □

Ambient dimension and scope. The same remaining-degree-one example can be embedded in every ambient dimension by taking S to be the integration current of the coordinate disc in a product polydisc. The radial normalized testers pull back from its first coordinate, while the core functions are multiplied by fixed smooth cutoffs equal to one on the other coordinate origins. Their restriction and energy are exactly the displayed ones. This preserves arbitrary original-test meanings. It does not turn this linear energy obstruction into a claim about a nonlinear operator with several moving energy slots.

36 The universal finite-energy criterion for concave moduli

The word "universal" in the following statement quantifies over all measures obeying a fixed growth modulus. It does not characterize the energy of an individual measure and does not characterize classical Monge–Ampere solvability.

Theorem 36.1 (An exact criterion within the concave class)**.** *Let $F : [0,\infty) \to [0,\infty)$ be finite, nondecreasing and concave, with $F(0) = 0$ and F continuous at zero. The following universal finite-energy assertion holds if and only if*

$$\int_0^a |F'(r)|^2\, dr < \infty \quad \text{for some } a > 0 : \tag{36.1}$$

for every allowed original degree-one background S on a fixed ordinary domain Ω carrying the faithful energy form, and every finite positive Radon measure μ supported on a compact $K \Subset V \Subset \Omega$ satisfying

$$\mu(E) \le F(\mathrm{Cap}_S(E;\Omega)) \qquad (E \subset \Omega \text{ Borel}),$$

the core dual norm $\|\mu\|_{-1,V}$ is finite.

If (36.1) fails, the counterexample already occurs for $S = [1]$ on the original open unit disc. Its support is a fixed compact sequence of divisorial Gauss points and their rigid limit, inside one fixed smaller ordinary disc; the limit point has mass zero. Thus failure of the criterion does not require a varying domain, an atomic rigid pole, or a complicated background.

Status: `NEW_PROVED` as the exact universal threshold for this specified concave class, obtained by combining the new strong-capacity criterion with the explicit radial construction. All original Gauss and energy calibrations are frozen inputs.

Proof. A finite concave function is locally absolutely continuous away from zero. Its nonnegative decreasing almost-everywhere derivative integrates to F also from zero, by continuity there. In particular, it is absolutely continuous on every finite closed interval. On intervals bounded away from zero its derivative is bounded. Thus (36.1) for one a implies the same condition on every finite interval.

Suppose first that the condition holds. Relative CLN gives $C_K = \mathrm{Cap}_S(K;\Omega) < \infty$. If $C_K = 0$, the growth bound gives $\mu = 0$. Otherwise $\mu(\Omega) = \mu(K) \le F(C_K)$. Apply Theorem 34.1 with $r_0 = C_K$. It gives, explicitly,

$$\|\mu\|_{-1,V} \le 2\left(\int_0^{C_K} |F'(r)|^2\, dr\right)^{1/2} < \infty.$$

This proves the full arbitrary-fixed-background direction.

Conversely, assume that the integral of $|F'|^2$ diverges near zero. Concavity and $F(0) = 0$ imply, almost everywhere,

$$0 \le F'(r) \le \frac{F(r)}{r}.$$

Indeed the right-hand side is the average of the decreasing derivative on $(0, r)$. Hence $\int_0^a F(r)^2 r^{-2}\, dr = \infty$ for every $a > 0$. Fix $\lambda = -\log|\pi|$, let $t_j = 2^j\lambda$ and $r_j = t_j^{-1}$. Monotonicity of F gives

$$\int_{r_{j+1}}^{r_j} \frac{F(r)^2}{r^2}\, dr \le F(r_j)^2\left(\frac{1}{r_{j+1}} - \frac{1}{r_j}\right) = t_j F(t_j^{-1})^2.$$

It follows that $\sum_j t_j F(t_j^{-1})^2 = \infty$.

Set $W_j = F(t_j^{-1})$, $m_j = W_j - W_{j+1} \geq 0$ and

$$z_j = \zeta_{0,e^{-t_j}}, \qquad \mu = \sum_{j\geq 1} m_j \delta_{z_j}.$$

Continuity at zero implies $W_j \downarrow 0$, so μ is finite with total mass W_1. It is supported in $\{0\} \cup \{z_j\} \Subset V = \{|T| < e^{-\lambda/2}\}$ and gives zero mass to zero. The normalized original hinge of curvature $t_j^{-1}\delta_{z_j}$ proves the growth bound for every Borel set: if j is the least index of an atom of positive mass in that set, its mass is at most W_j, and its capacity is at least t_j^{-1}. If no such index exists, its mass is zero. No capacity regularity is used.

The finite radial test functions

$$f_N(t) = \sum_{j=1}^{N} m_j \min\{(t-\lambda)_+, t_j - \lambda\}, \qquad t = -\log|T|,$$

have, with $t_0 = \lambda$,

$$\mathcal{E}_{[1]}(f_N) = \sum_{j=1}^{N} (t_j - t_{j-1})(W_j - W_{N+1})^2 = \int f_N \, d\mu_N \leq \int f_N \, d\mu.$$

Their energies increase to

$$\sum_{j\geq 1} (t_j - t_{j-1}) W_j^2 = \frac{1}{2} \sum_{j\geq 1} t_j F(t_j^{-1})^2 = \infty.$$

The original smooth compact approximations inside this same fixed V are exactly those justified in Proposition 35.1: mollify each finite real PL profile, retaining its zero initial region and its constant terminal region. Uniform convergence and $L^2(dt)$ derivative convergence pass both the finite-measure pairing and the energy. A finite core dual norm would therefore bound the displayed diverging energy by a constant times its square root, which is impossible. This completes the necessity assertion. □

Qualification. Concavity is used only for the sharp converse. For a general nondecreasing absolutely continuous modulus, the square-integrable derivative condition remains sufficient, but this proof asserts no necessity. Neither direction identifies the energy capacity with relative BT capacity. The conclusion supplies the energy-class linear equation through the Riesz theorem; all psh feasibility and trace constraints remain separate.

37 Capacity domination does not characterize the normalized image

This is a scope consequence of the two sharp estimates just proved. It concerns a fixed normalization on a fixed ordinary domain; it is neither a counterexample to C1 nor an assertion of Dirichlet nonexistence after removing the normalization.

Corollary 37.1 (A strict distinction on an original annulus)**.** *There are an ordinary annulus Ω over the original field K and a finite positive Radon measure μ, compactly supported in Ω, such that*

$$\mu(E) \leq \mathrm{Cap}_{[1]}(E;\Omega) \qquad (E \subset \Omega \text{ Borel}),$$

but μ does not belong to the weak closure, on all original compact smooth scalar tests, of

$$\mathcal{M} = \{Dw : w \in \mathcal{C}_b(\Omega),\ 0 \leq w \leq 1 \text{ on the whole } \Omega\}.$$

In particular it is not the curvature of a normalized completed potential defined by a same-domain row in this normalized classical family.

Status: `DIRECT_COROLLARY` of Theorem 32.2 and Example 33.2. No new compact-regularity assertion is used.

Proof. Put $\lambda = -\log|\pi|$ and $\Omega = \{1 < |T| < e^{\lambda}\}$. Write $s = \log|T|/\lambda \in (0,1)$, and let $\sigma(s)$ be its Gauss section. For $0 < \varepsilon < 1/4$ take

$$\mu_\varepsilon = \lambda^{-1}\sigma_*(\mathbf{1}_{[\varepsilon,1/2]}s^{-2}\,ds).$$

These measures have compact support and finite mass. The normalized hinges $w_r(s) = \max(0, 1 - s/r)$ have $Dw_r = (\lambda r)^{-1}\delta_{\sigma(r)}$ on this same annulus. For a Borel set E meeting the support of μ_ε, put $a = \inf\{s \in [\varepsilon, 1/2] : \sigma(s) \in E\}$. Then

$$\mu_\varepsilon(E) \le \frac{1}{\lambda a} = \sup_{\sigma(r)\in E,\ r\in[\varepsilon,1/2]} \frac{1}{\lambda r} \le \mathrm{Cap}_{[1]}(E;\Omega).$$

The bound is immediate when E misses this support. Attainment of the infimum is unnecessary.

The original smooth compact tests constructed in Example 33.2 satisfy

$$\frac{\int f_\varepsilon^2\,d\mu_\varepsilon}{\mathcal{E}_{[1]}(f_\varepsilon)} \longrightarrow 4.$$

Choose and then fix one sufficiently small ε for which this ratio is greater than one. On the other hand, the sharp Hardy estimate gives

$$\int f_\varepsilon^2\,d\nu \le \mathcal{E}_{[1]}(f_\varepsilon) \qquad (\nu \in \mathcal{M}).$$

The same inequality holds in the weak closure of $\mathcal{M}$ because f_ε^2 is one fixed original compact smooth scalar test. Thus the displayed strict inequality separates μ_ε from that weak closure, for sequences and nets alike. The completed-image conclusion follows from the fixed-bound completed continuity in Corollary 32.3. □

Direction of the implication. C1 asks whether a weak-limit or completed measure is dominated, on compact sets, by the raw classical-test capacity. The present measure already has that domination and fails to be a weak limit. It disproves the converse characterization, not C1. In particular a capacitary upper bound cannot replace the support-functional constraints in Theorem 51.2. No assertion is made here about unnormalized classical potentials or arbitrary boundary values.

38 Relative BT and energy capacities can have different null sets

The sharp one-way energy bound cannot be reversed, even at the level of null sets. The counterexample below uses a smooth original-positive background on a full torus polyannulus. It is not a counterexample to compact no-gain or to ordinary-open outer regularity of BT capacity.

Fix $n \ge 3$, let $Q = (-2,2)^n$, and put

$$\Omega = \mathrm{trop}^{-1}(Q) \subset (\mathbf{G}^n_{m,K})^{\mathrm{an}}, \qquad t_i = \log|T_i|, \qquad \beta = \sum_{i=1}^n d't_i \wedge d''t_i, \qquad S = \frac{\beta^{n-1}}{(n-1)!} \wedge [1].$$

The background S has bidegree $(n-1,n-1)$, is symmetric, original-positive, and separately closed. Write $\sigma : Q \to \Omega$ for the original Gauss section. The original torus calibration gives, for $\varphi \in C_c^\infty(Q)$,

$$D(\varphi \circ \mathrm{trop}) \wedge S = \sigma_*((\Delta\varphi)\,dx), \qquad \mathcal{E}_S(\varphi \circ \mathrm{trop}) = \int_Q |\nabla\varphi|^2\,dx. \tag{38.1}$$

The factorial in S is chosen precisely for these identities. In particular no normalization of the original absolute value is made.

Lemma 38.1 (Uniform Gauss-section growth for all classical testers)**.** *There is a constant C_n such that, for every actual $u \in \mathcal{C}_b(\Omega)$ with $0 \le u \le 1$, and every Euclidean ball $B(x,2r) \subset (-1,1)^n$,*

$$D(uS)(\mathrm{trop}^{-1}\,B(x,r)) \le C_n r^{n-1}. \tag{38.2}$$

Consequently, if $A \subset (-1/2,1/2)^n$ is compact and $\mathcal{H}^{n-1}(A) = 0$, then

$$\mathrm{Cap}_S^*(\sigma(A);\Omega) = 0.$$

Proof. Put $g = u \circ \sigma$. We first justify a restriction property of g rather than assume it from the word "skeleton." Fix all but one of its logarithmic coordinates. Let F be the completed residue field of the Gauss point of the corresponding $(n-1)$-torus over K. Its coordinate elements have the prescribed radii. The analytic map from an annulus over F that varies the remaining coordinate and keeps these elements fixed sends its Gauss point of each radius to the corresponding original K-Gauss point in Ω. This is the iterated Gauss seminorm identity on Laurent polynomials; it concerns actual points, not replacement of full analytic fibers in integration.

By the defining complete-extension curve test for classical psh, the pullback of u is subharmonic. On a closed subannulus, subtract the affine harmonic function of the logarithmic radius that agrees with its two Gauss endpoint values. The subharmonic Shilov maximum principle on that annulus gives a nonpositive difference. If necessary one first extends F to realize the two endpoint radii in its value group; the same actual Gauss identities and classical curve test remain valid. Thus g is convex in the one varying coordinate. Repeating for each coordinate proves separate convexity.

The bound $0 \le g \le 1$ on Q now gives a uniform Lipschitz bound on $(-1,1)^n$: the one-dimensional convex slope bounds hold uniformly in all the fixed other coordinates, and one joins two points by coordinate segments. In particular g has bounded weak first derivatives there, by a constant depending only on n.

Let $\mu_u = D(uS)$, a positive Radon measure by the frozen calculus. For every $\varphi \in C_c^\infty(Q)$, recursion and (38.1) give

$$\langle \mathrm{trop}_*\,\mu_u, \varphi\rangle = \langle uS, D(\varphi \circ \mathrm{trop})\rangle = \int_Q g\Delta\varphi\,dx.$$

Tropicalization is proper on compact subsets of Q, so this is an identity of Radon measures: $\mathrm{trop}_*\,\mu_u = \Delta g$. Choose a nonnegative scalar cutoff equal to one on $B(x,r)$, supported in $B(x,2r)$, with gradient bounded by a fixed multiple of r^{-1}. Integration by parts against the bounded weak gradient of g gives

$$\Delta g(B(x,r)) \le \int \varphi_r\,d\Delta g = -\int \nabla g \cdot \nabla\varphi_r\,dx \le C_n r^{n-1}.$$

The constant is independent of u; this proves (38.2).

Cover A by finitely many sufficiently small balls $B(x_i,r_i)$ with $\sum_i r_i^{n-1} < \epsilon$, keeping the doubled balls in $(-1,1)^n$. Their tropical inverse images form one ordinary open neighborhood

G of $\sigma(A)$ in the same Ω. The uniform estimate and finite subadditivity give $\mathrm{Cap}_S(G;\Omega) \le C_n\epsilon$. Let $\epsilon \downarrow 0$. This proves outer capacity zero directly; no compact outer-regularity theorem has been used. □

Lemma 38.2 (Every original energy dominates the Gauss restriction energy)**.** *For every original real compact smooth f on Ω, the actual function $g = f \circ \sigma$, extended by zero outside Q, belongs to $H^1_0(Q)$ and satisfies*

$$\int_Q |\nabla g|^2 \, dx \le \mathcal{E}_S(f). \tag{38.3}$$

Proof. The function g is continuous and compactly supported in Q. For $\varphi \in C_c^\infty(Q)$, compact Green, the torus calibration, and the original energy Cauchy–Schwarz inequality give

$$\left| \int_Q g \Delta\varphi \, dx \right|^2 = |\mathcal{E}_S(f, \varphi \circ \mathrm{trop})|^2 \le \mathcal{E}_S(f) \int_Q |\nabla\varphi|^2 \, dx. \tag{38.4}$$

This uses every original f, not just a torus-invariant subcore.

Choose a nonnegative even smooth probability mollifier κ_ϵ. For small enough ϵ, both $g_\epsilon = g * \kappa_\epsilon$ and $\varphi_\epsilon = g * \kappa_\epsilon * \kappa_\epsilon$ are smooth and compactly supported in Q. Symmetry, Fubini and ordinary integration by parts give

$$-\int g \Delta\varphi_\epsilon \, dx = \int |\nabla g_\epsilon|^2 \, dx, \qquad \|\nabla\varphi_\epsilon\|_2 \le \|\nabla g_\epsilon\|_2.$$

Substitute into (38.4). After cancellation, or trivially when this norm is zero, one obtains $\|\nabla g_\epsilon\|_2^2 \le \mathcal{E}_S(f)$. As $g_\epsilon \to g$ in L^2, weak compactness in $H^1_0(Q)$ and lower semicontinuity prove (38.3). This argument requires no assertion that a smooth original test has a smooth restriction in one fixed real moment chart. □

Lemma 38.3 (A compact real set with the two needed size properties)**.** *There is a compact $A \subset (-1/2, 1/2)^n$ with $\mathcal{H}^{n-1}(A) = 0$ and a probability measure ϑ on A such that*

$$I(\vartheta) = \iint |x - y|^{2-n} \, d\vartheta(x) d\vartheta(y) < \infty.$$

Moreover there is a constant $c_A > 0$ such that every continuous compactly supported $g \in H^1_0(Q)$ with $g \ge 1$ on A satisfies $\int |\nabla g|^2 \ge c_A$.

Proof. Take a small affine copy C of the middle-third Cantor set, a small $(n-2)$-cube J, and put $A = C \times J \times \{0\}$. At scale 3^{-j} it is covered by at most $C_0 2^j 3^{j(n-2)}$ cubes of that side length. The sum of their $(n-1)$st diameter powers is at most $C_1(2/3)^j$, proving the Hausdorff measure assertion.

Let $d = \log 2/\log 3 > 0$, and let ϑ be the product of the Cantor Bernoulli probability, normalized Lebesgue measure on J, and the point mass at zero. The usual interval count in the Cantor construction gives

$$\vartheta(B(x,r)) \le C_2 r^{n-2+d} \quad (0 < r < 1).$$

Layer cake for $|x-y|^{2-n}$ now bounds its near-diagonal integral by a constant times $\int_0^1 r^{d-1} \, dr < \infty$; the remainder is bounded. This proves the finite energy assertion.

For completeness, this energy supplies the required Sobolev lower bound directly. For a real $g \in C_c^\infty(\mathbf{R}^n)$, the divergence formula for the Newton kernel gives

$$g(x) = c_n \int_{\mathbf{R}^n} \nabla g(y) \cdot \frac{x - y}{|x - y|^n} \, dy$$

with a fixed dimensional constant (its sign is irrelevant below). The convolution of the two nonnegative kernels $|x|^{1-n}$ is $C_n|x|^{2-n}$: translation, rotation and scaling give this form, and the defining integral is finite away from zero since $n \geq 3$. Tonelli and Cauchy–Schwarz therefore imply

$$\left| \int g \, d\vartheta \right|^2 \leq C_n' I(\vartheta) \int |\nabla g|^2 \, dx. \tag{38.5}$$

For continuous compactly supported $g \in H_0^1(Q)$, convolution approximates uniformly as well as in H^1; thus the same inequality holds. If $g \geq 1$ on A, its left side is at least one. Take $c_A = (C_n' I(\vartheta))^{-1} > 0$. □

Theorem 38.4 (Strict separation of the two capacity null ideals)**.** *For the smooth background S above and the compact set $K = \sigma(A) \Subset \Omega$ of Lemma 38.3,*

$$\mathrm{Cap}_S^*(K;\Omega) = 0, \qquad \mathrm{cap}_{\mathcal{E},1}(K) > 0. \tag{38.6}$$

Here the second capacity is that of the frozen faithful regular Dirichlet form on $F = \operatorname{supp} S$. Thus neither a reverse comparison with a finite constant nor equality of the two null ideals holds in general.

Status: `NEW_PROVED` explicit obstruction to the reverse energy-capacity route, in every $n \geq 3$. It makes no negative claim about the general relative C1 or C2 conjectures.

Proof. The first equality follows from Lemma 38.1. Every Gauss point of Q lies in F: a nonnegative original cutoff supported in any neighborhood of it, paired with $\beta \wedge S$, has positive integral on a nonempty real neighborhood of that point. Hence $K \subset F$.

In the verified compact-core formula for $\mathrm{cap}_{\mathcal{E},1}(K)$, let f be any original smooth compact admissible function, so $f > 1$ on K. Its continuous Gauss restriction $g = f \circ \sigma$ is at least one on A. The preceding two lemmas give

$$\mathcal{E}_1(f,f) \geq \mathcal{E}_S(f) \geq \int |\nabla g|^2 \, dx \geq c_A.$$

Taking the infimum proves the second assertion. In particular the failure cannot be repaired by passing to all original smooth tests: the energy lower bound was proved for that full test class. □

Exact scope of the obstruction. This example has a fixed smooth background and remaining degree one. Its BT outer capacity is already zero, so it exhibits no compact weak-closure gain and no raw/outer discrepancy. It shows why the new sharp Hardy bound, although valid for every normalized tester and its weak limits, cannot close C1 merely by identifying the controlling energy capacity with BT capacity. The actual nonlinear psh constraint is visible in the uniform Gauss-section Lipschitz bound.

39 Exact capacity reduction on a Gauss section

Let $n \geq 1$, let $Q \subset \mathbf{R}^n$ be a nonempty bounded convex ordinary open set, and set

$$\Omega = \mathrm{trop}^{-1}(Q) \subset (\mathbf{G}_{m,K}^n)^{\mathrm{an}}, \qquad S = \frac{\beta^{n-1}}{(n-1)!} \wedge [1], \qquad \beta = \sum_i d't_i \wedge d''t_i.$$

Write $\sigma : Q \to \Omega$ for the original Gauss section and define

$$\mathcal{G}(Q) = \{g : Q \to \mathbf{R} : \ g \text{ is convex}, \ 0 \leq g \leq 1\}, \qquad c_Q(E) = \sup_{g \in \mathcal{G}(Q)} \Delta g(E)$$

for Borel $E \subset Q$. The Laplacian is the positive distributional Radon measure on the real domain. All noncompact sets considered below are contained in a fixed compact subset of Q. Let c_Q^* be the outer capacity obtained using ordinary real open neighborhoods in the same Q.

Lemma 39.1 (Joint convexity from complete-extension annuli)**.** *Every actual $u \in \mathcal{C}_b(\Omega)$ with $0 \le u \le 1$ has $u \circ \sigma \in \mathcal{G}(Q)$. Conversely every $g \in \mathcal{G}(Q)$ gives a normalized continuous classical psh function $g \circ \mathrm{trop}$ on the same Ω.*

Proof. The coordinate-annulus argument in Lemma 38.1 applies on every real box compactly contained in Q. It proves separate convexity and local Lipschitz continuity of $g = u \circ \sigma$.

Fix a primitive integral vector $v \in \mathbf{Z}^n$. Complete v to the first column of a matrix $B \in \mathrm{GL}_n(\mathbf{Z})$. The associated monomial change of torus coordinates has logarithmic coordinates $y = B^{-1}t$. Fix $y_2, \dots, y_n$ and form the completed residue field of their Gauss point. The analytic annulus over that field varying y_1 maps into Ω over exactly the interval cut out by $B^{-1}Q$. Its Gauss point at each radius maps to the original Gauss point of Q with those coordinates. This is the monomial Gauss-seminorm identity, not a fiber-integration assertion.

The complete-extension classical curve test and the subharmonic maximum principle on each closed subannulus therefore show that g is convex along every segment parallel to v. Endpoint radii can first be realized after a further complete extension, as in the coordinate case. Convexity follows along every rational direction. For $x, y \in Q$, approximate $y - x$ by rational vectors v_j with $x + v_j \in Q$. Convexity of Q keeps all the corresponding segments inside Q. The already proved continuity of g passes their convex inequalities to the segment $[x, y]$. Thus g is jointly convex.

For the converse, on each relatively compact real box one may approximate the continuous convex g locally uniformly by smooth convex functions. Each such function of the unit-log coordinates is classical psh: after any complete extension and analytic curve map, the coordinates are harmonic and the convex Hessian chain rule is nonnegative. Locally uniform limits preserve the subharmonic curve inequalities. Hence $g \circ \mathrm{trop}$ is classical on Ω, with precisely the original normalization $0 \le g \le 1$. □

Theorem 39.2 (Raw and outer capacities reduce to real convex traces)**.** *For every Borel E contained in a fixed compact core of Q,*

$$\mathrm{Cap}_S(\sigma(E); \Omega) = \mathrm{Cap}_S(\mathrm{trop}^{-1}(E); \Omega) = c_Q(E), \tag{39.1}$$

$$\mathrm{Cap}_S^*(\sigma(E); \Omega) = \mathrm{Cap}_S^*(\mathrm{trop}^{-1}(E); \Omega) = c_Q^*(E). \tag{39.2}$$

Every competitor in these identities is normalized on the entire displayed capacity domain; no smaller-domain tester is substituted.

Status: `DIRECT_COROLLARY` of the actual complete-extension curve test, ordinary torus calibration, and the frozen fixed-background calculus. It identifies a concrete real capacity, rather than proving general relative Borel outer regularity.

Proof. For u as in the preceding lemma, write $\mu_u = D(uS)$ and $g = u \circ \sigma$. The calculation against original toric scalar tests in Lemma 38.1 gives

$$\mathrm{trop}_* \, \mu_u = \Delta g. \tag{39.3}$$

It is an identity of Radon measures because tropicalization is proper over compact subsets of Q. Conversely, for the toric competitor $u = g \circ \mathrm{trop}$ one has the stronger identity

$$D(uS) = \sigma_*(\Delta g). \tag{39.4}$$

For smooth real g this is the original torus calibration on all original scalar tests. Local uniform smooth convex approximation, fixed coefficient-Radon multiplication, and positivity pass it to continuous convex g. The restriction of an arbitrary original test to $\sigma(Q)$ is used only as a continuous compact function.

Equation (39.3) bounds both raw capacities in (39.1) by $c_Q(E)$, while (39.4) supplies the reverse bounds. The sets are Borel: σ is a closed embedding over every compact real core, and tropical inverse images preserve Borel sets.

If $G \subset \Omega$ is any ordinary open set containing $\sigma(E)$, then $O = \sigma^{-1}(G)$ is a real open neighborhood of E. Toric competitors alone give

$$\operatorname{Cap}_S(G;\Omega) \geq \sup_{g\in\mathcal{G}(Q)} \Delta g(O) = c_Q(O) \geq c_Q^*(E).$$

No saturated-neighborhood assumption is made in this lower bound. Conversely, for every real open $O \supset E$, the ordinary saturated open set $\operatorname{trop}^{-1}(O)$ has capacity $c_Q(O)$ by the raw identity. These two bounds prove the outer formula for $\sigma(E)$. Since $\sigma(E) \subset \operatorname{trop}^{-1}(E)$, the same saturated upper bound and monotonicity prove it also for the full inverse image. All infima can be restricted to neighborhoods inside one larger relatively compact core by intersection, so local mass bounds suffice. □

Corollary 39.3 (Compact no-gain on the Gauss section)**.** *The family $\mathcal{G}(Q)$ is compact in the topology of local uniform convergence, and $g \mapsto \Delta g$ is continuous for local weak Radon convergence. Consequently, for compact $A \Subset Q$,*

$$c_Q^*(A) = c_Q(A).$$

If a net of normalized original relative measures $D(u_j S)$ converges weakly to ν on Ω, then

$$\nu(\sigma(A)) \leq \operatorname{Cap}_S(\sigma(A);\Omega).$$

These statements give no Borel identity $c_Q^* = c_Q$ for arbitrary Borel sets.

Status: `DIRECT_COROLLARY`. The C1 consequence for this smooth nonpolar remaining-degree-one background is already within the frozen R3 nonpolar C1 theorem. The new explicit description here is the reduction to $\mathcal{G}(Q)$ and its exact ordinary-open outer counterpart.

Proof. Normalized convex functions have one uniform Lipschitz bound on each compact subset of Q. Arzela–Ascoli on a countable real compact exhaustion gives compactness of $\mathcal{G}(Q)$; local uniform limits retain convexity and the same bounds on all of Q. Distributional convergence of their Laplacians, together with uniform local mass bounds, gives weak Radon convergence. This is compactness of the real normalized family, not a claim that arbitrary relative currents have actual classical representatives.

If $A_j \downarrow A$ are compact sets in one core, choose $g_j \in \mathcal{G}(Q)$ with $\Delta g_j(A_j) \geq c_Q(A_j) - j^{-1}$. Pass to a locally uniformly convergent subsequence with limit g. For every fixed l, compact Portmanteau gives

$$\limsup_j \Delta g_j(A_j) \leq \limsup_j \Delta g_j(A_l) \leq \Delta g(A_l).$$

Let $l \to \infty$ and use finite Radon continuity above. This proves $c_Q(A_j) \downarrow c_Q(A)$. Taking shrinking compact neighborhoods of A now yields $c_Q^*(A) = c_Q(A)$.

For the final assertion, compactness gives a subnet $u_j \circ \sigma \to g \in \mathcal{G}(Q)$ locally uniformly. Equation (39.3), tested first by compact real smooth functions, gives $\operatorname{trop}_* \nu = \Delta g$. Positivity therefore implies

$$\nu(\sigma(A)) \leq \nu(\operatorname{trop}^{-1} A) = \Delta g(A) \leq c_Q(A),$$

as required. The weak limit itself need not be toric. Nothing in this proof exchanges an infimum over open neighborhoods with a supremum over measures on a general Borel set. □

Remaining real question. The equality $c_Q(E) = c_Q^*(E)$ for all Borel $E \Subset Q$ would, by the exact reduction, settle C2 on these Gauss-section sets. Compactness of the convex measure image alone does not prove that equality, as the frozen abstract weak-compact-family counterexample already demonstrates. This restricted real question is distinct from the full arbitrary-background C2 problem.

40 Borel outer regularity on Gauss sets of finite codimension-one size

Retain the full convex torus domain and the smooth remaining-degree-one background of Section 39, and assume $n \geq 2$. Fix a compact real core $Q_0 \Subset Q$. Hausdorff measure below refers to the ordinary Euclidean metric on Q, not to a metric on the Berkovich space.

Proposition 40.1 (An outer bound by codimension-one Hausdorff measure)**.** *There is a constant $C = C(Q_0, Q, n) < \infty$ such that every subset $E \subset Q_0$ satisfies*

$$c_Q^*(E) \leq C\mathcal{H}^{n-1}(E).$$

Equivalently the same bound holds for the ordinary-open outer relative capacities of $\sigma(E)$ and $\mathrm{trop}^{-1}(E)$. For non-Borel sets only these outer capacities are asserted.

Proof. The normalized convex family has a uniform Lipschitz bound on a fixed larger real core. The cutoff calculation in Lemma 38.1 therefore gives

$$c_Q(B(x, r)) \leq C_0 r^{n-1}$$

for all sufficiently small balls centered near Q_0, with their doubled balls in the larger core. Cover E by countably many such balls. Their union is one real open neighborhood; countable subadditivity, valid directly for the supremum of the positive measures Δg, bounds its capacity by $C_0 \sum_i r_i^{n-1}$. Taking the infimum over covers of arbitrarily small diameter proves the bound, up to the fixed normalization of Hausdorff measure.

The exact outer comparison proof of Theorem 39.2 used only open neighborhoods and monotonicity. It therefore applies to arbitrary subsets for the outer capacities, and gives the stated equivalent formulations. Alternatively the saturated inverse images of the covering balls give the Berkovich upper bounds directly. □

Corollary 40.2 (A concrete Borel C2 class)**.** *If $E \subset Q_0$ is Borel and $\mathcal{H}^{n-1}(E) < \infty$, then*

$$c_Q(E) = c_Q^*(E).$$

Consequently raw and ordinary-open outer relative capacities agree for both $\sigma(E)$ and $\mathrm{trop}^{-1}(E)$, with capacity domain the same whole Ω. This includes every Borel subset of a compact smooth real hypersurface in Q of finite area.

If instead E is a countable union of Borel sets of finite $\mathcal{H}^{n-1}$ measure, then

$$c_Q(E) = 0 \quad \Longleftrightarrow \quad c_Q^*(E) = 0.$$

The second assertion is an equality of null classes; it does not claim equality of positive capacity values for every such countable union.

Status: `DIRECT_COROLLARY` of the uniform growth bound and the already proved compact C1 theorem. This is a qualified Borel outer-regularity result, not general C2.

Proof. The finite Borel measure $\mathcal{H}^{n-1}|_E$ on the real core is Radon. For every $\epsilon > 0$ choose compact $K \subset E$ with $\mathcal{H}^{n-1}(E \setminus K) < \epsilon$. Compact C1 and the preceding proposition give

$$c_Q^*(E) \le c_Q^*(K) + c_Q^*(E \setminus K) \le c_Q(K) + C\epsilon \le c_Q(E) + C\epsilon.$$

Let $\epsilon \downarrow 0$ and combine with the always valid reverse inequality. The exact raw and outer reductions then yield the Berkovich statements. No uniform setwise continuity of individual tester measures is asserted or needed.

For the last assertion write $E = \bigcup_i E_i$ with each E_i Borel and of finite Hausdorff measure. If $c_Q(E) = 0$, then $c_Q(E_i) = 0$; the first part gives $c_Q^*(E_i) = 0$. Countable subadditivity gives $c_Q^*(E) = 0$. The converse follows from $c_Q \le c_Q^*$. □

What remains. The proof uses a finite measure on the particular Borel set to approximate it by a compact subset, together with an independent outer estimate on the residual set. Merely knowing compactness of the potential family supplies neither that residual estimate nor ordinary-open outer continuity for all Borel sets. In particular the argument leaves the general real Borel C2 gate, and a fortiori the arbitrary-background Berkovich C2 gate, open.

41 A coordinate-projection criterion for real and Gauss Borel capacities

Retain c_Q, the Gauss section, and the smooth remaining-degree-one background from Section 39, with $n \ge 2$. Fix a compact rectangular core Q_0 and open real boxes

$$Q_0 \Subset W \Subset W^+ \Subset Q.$$

Let $\mathrm{pr}_i : \mathbf{R}^n \to \mathbf{R}^{n-1}$ delete the ith coordinate. Lebesgue outer measure on this target is denoted by $\mathcal{L}^{n-1,*}$.

Lemma 41.1 (Uniform projected-size bound)**.** *There is a finite constant $C = C(W, W^+, Q, n)$ such that every subset $E \subset Q_0$ satisfies*

$$c_Q^*(E) \le C \sum_{i=1}^{n} \mathcal{L}^{n-1,*}(\mathrm{pr}_i E).$$

The same inequality holds for the ordinary-open outer relative capacities of $\sigma(E)$ and $\mathrm{trop}^{-1}(E)$.

Status: `DIRECT_COROLLARY` of the uniform convex slope bound, directional Fubini, and the exact outer Gauss reduction. It is an upper bound, not an identification of capacity with projected size.

Proof. All $g \in \mathcal{G}(Q)$ have one common Lipschitz bound L on W^+. Fix i. On each coordinate line with transverse parameter $y \in \mathrm{pr}_i W$, the restriction g_y is convex, and its one-dimensional second derivative on the interval cut out by W is a positive measure of mass at most $2L$.

Directional distributional Fubini gives

$$(\partial_{ii} g)|_W = \int_{\mathrm{pr}_i W} g_y'' \, dy.$$

For completeness, for a compact smooth test φ this follows by writing $\int_W g \, \partial_{ii}\varphi$ as the iterated integral on these lines and applying the one-dimensional distributional identity. The kernels are weakly Borel: their compact smooth pairings are integrals of the continuous g against second

derivatives, or limits of such pairings after restriction to compact line intervals. The uniform mass bound makes the resulting positive Radon kernel integral well defined and identifies the measures. In particular, for every real open $G \subset W$,

$$\Delta g(G) \le 2L \sum_i \mathcal{L}^{n-1}(\mathrm{pr}_i\, G).$$

For $\varepsilon > 0$ choose open $O_i \supset \mathrm{pr}_i\, E$ with

$$\mathcal{L}^{n-1}(O_i) \le \mathcal{L}^{n-1,*}(\mathrm{pr}_i\, E) + \varepsilon.$$

The real open set $G = W \cap \bigcap_i \mathrm{pr}_i^{-1}(O_i)$ contains E. The preceding inequality is uniform in all normalized convex testers and bounds $c_Q(G)$ by $2L \sum_i \mathcal{L}^{n-1}(O_i)$. Let $\varepsilon \downarrow 0$. The outer comparison proof in Theorem 39.2 applies to arbitrary subsets, so it gives both Berkovich formulations as well. □

Proposition 41.2 (Borel coordinate sheets)**.** *Let $E \subset Q_0$ be Borel. Suppose that for each i the restriction $\mathrm{pr}_i\,|_E$ is one-to-one, its image A_i is Borel, and its inverse $s_i : A_i \to E$ is Borel. Then*

$$c_Q(E) = c_Q^*(E).$$

The corresponding same-domain raw and ordinary-open outer relative capacities agree on both $\sigma(E)$ and $\mathrm{trop}^{-1}(E)$. The same conclusion holds for a finite union of such Borel sheets. No Hausdorff-measure bound on E is assumed.

Status: `DIRECT_COROLLARY` of the projection bound and compact C1. All Borel inverse hypotheses are explicit, so no unproved measurable selection is being invoked.

Proof. For each i, push the finite measure $\mathcal{L}^{n-1}|_{A_i}$ forward by s_i, and call the resulting finite Borel measure on $\mathbf{R}^n$ λ_i. Finite Borel measures on Euclidean space are Radon. Put $\lambda_E = \sum_i \lambda_i$; it is carried by E. For every compact $K \subset E$, injectivity gives

$$\mathrm{pr}_i(E \setminus K) = A_i \setminus \mathrm{pr}_i\, K, \qquad \mathcal{L}^{n-1}(\mathrm{pr}_i(E \setminus K)) = \lambda_i(E \setminus K).$$

These projections are Borel, since $\mathrm{pr}_i\, K$ is compact. Inner Radon regularity on the Borel set E supplies $K \subset E$ compact with $\lambda_E(E \setminus K) < \varepsilon$. The projection bound and compact C1 imply

$$c_Q^*(E) \le c_Q^*(K) + c_Q^*(E \setminus K) \le c_Q(K) + C\varepsilon \le c_Q(E) + C\varepsilon.$$

Let $\varepsilon \downarrow 0$ and use $c_Q \le c_Q^*$.

For a finite union of sheets, add their finitely many projection pullback measures. For every residual $E \setminus K$, its coordinate projections lie in the unions of the corresponding sheet projections, so the same upper bound by $\lambda_E(E \setminus K)$ holds. The same compact approximation proves the result. Finally apply the exact raw and outer Gauss reductions. □

A directly specified class in dimension two. Let A, B be Borel subsets of bounded real intervals, and let $F : A \to B$ be a bijection such that both F and F^{-1} are Borel. If its graph lies in Q_0, it satisfies the proposition: the inverse coordinate maps are $x \mapsto (x, F(x))$ and $y \mapsto (F^{-1}(y), y)$. Thus these Borel Gauss graphs and their full tropical inverse images have exact ordinary-open outer regularity. Continuity or a finite area assumption is not used.

Scope of the remaining gate. The finite Radon measure constructed here lives on the particular Borel set and controls the outer capacity of its residual subsets. It is not a common neighborhood dominator for all tester measures. The proposition gives no such residual control for an arbitrary Borel set, and leaves general real Borel C2, and general relative Berkovich C2, open.

42 A finite gluing mechanism for open sets with compact additions

Retain the bounded convex real domain Q, normalized convex class $\mathcal{G}(Q)$, and Laplacian capacity c_Q of Section 39. All sets below lie in one fixed compact core of Q. The normalization is on the whole Q.

Lemma 42.1 (Pasting one local-uniform row on two fixed pieces)**.** *Let $g_j, g \in \mathcal{G}(Q)$ with $g_j \to g$ locally uniformly, and let $K_1, K_2 \Subset Q$ be disjoint compact sets. There are $v_j \in \mathcal{G}(Q)$ and numbers $\epsilon_j \downarrow 0$ such that, for every Borel $A_i \subset K_i$,*

$$\Delta v_j(A_1 \cup A_2) \geq (1+\epsilon_j)^{-1}(\Delta g(A_1) + \Delta g_j(A_2)).$$

The same v_j works for all these Borel subsets.

Status: `DIRECT_COROLLARY` of the explicit maximum-and-compensator method in frozen Theorem R4–79.4. We give the real proof because only local uniform convergence is assumed here, while the normalization must remain global.

Proof. Choose smooth $0 \leq \chi_i \leq 1$ with disjoint compact supports in Q, each identically one near K_i. A sufficiently large positive quadratic ρ on $\mathbf{R}^n$ is nonnegative on Q and makes $\rho + 2\chi_i$ convex on the whole Q. Since Q is bounded,

$$A = \sup_Q \rho + 2 < \infty.$$

Fix a compact neighborhood $H \Subset Q$ of the two regions where the cutoffs equal one. Choose positive numbers $\delta_j \downarrow 0$ such that $|g_j - g| \leq \delta_j$ on H; this is possible by local uniform convergence, enlarging the errors to decreasing tail suprema if necessary. Set

$$v_j = \frac{\max\{g + \delta_j(\rho + 2\chi_1), g_j + \delta_j(\rho + 2\chi_2)\}}{1 + A\delta_j}.$$

Both branches are convex on the entire Q, and their maximum is convex. The global bounds $0 \leq g, g_j \leq 1$ give $0 \leq v_j \leq 1$. Closeness outside H is neither assumed nor needed.

Near K_1 the first branch strictly exceeds the second by at least δ_j, and near K_2 the second strictly exceeds the first. Ordinary locality of distributional Hessians therefore gives

$$\Delta v_j|_{K_1} \geq (1 + A\delta_j)^{-1}\Delta g|_{K_1}, \qquad \Delta v_j|_{K_2} \geq (1 + A\delta_j)^{-1}\Delta g_j|_{K_2}.$$

Here the positive added quadratic curvature is simply discarded. The assertions first hold on neighborhoods and hence on every Borel subset of the compact pieces. Their disjointness proves the lemma with $\epsilon_j = A\delta_j$. □

Theorem 42.2 (Borel outer regularity for an open set plus a compact set)**.** *Let $O \Subset Q$ be real open and let $K \Subset Q$ be compact. Then*

$$c_Q(O \cup K) = c_Q^*(O \cup K).$$

Consequently raw and ordinary-open outer relative capacities agree on both $\sigma(O \cup K)$ and $\mathrm{trop}^{-1}(O \cup K)$ in the same whole torus domain.

Status: `NEW_PROVED` for this exact set class, using the displayed finite gluing argument. Weak compactness of an arbitrary measure family would not prove this assertion.

Proof. The case $K = \varnothing$ is immediate. Otherwise choose decreasing open metric neighborhoods N_j of K, with compact closures in one fixed core, such that $\bigcap_j \overline{N_j} = K$. They are cofinal among open neighborhoods of the compact K. Thus $G_j = O \cup N_j$ are cofinal among open neighborhoods of $E = O \cup K$ that contain O, and

$$c_Q(G_j) \downarrow c_Q^*(E).$$

Choose $g_j \in \mathcal{G}(Q)$ with $\Delta g_j(G_j) \geq c_Q(G_j) - j^{-1}$. The already proved local-uniform compactness of $\mathcal{G}(Q)$ gives a subsequence converging locally uniformly to $g \in \mathcal{G}(Q)$. Keep this subsequence fixed for the rest of the proof. Its Laplacians converge weakly on the fixed core, and its displayed open-set masses still tend to $c_Q^*(E)$.

Now fix an arbitrary open neighborhood N of K with compact closure in that core. Eventually $N_j \subset N$, so

$$\Delta g_j(G_j) \leq \Delta g_j(O \setminus N) + \Delta g_j(\overline{N}).$$

Compact Portmanteau implies

$$c_Q^*(E) \leq \limsup_j \Delta g_j(O \setminus N) + \Delta g(\overline{N}). \tag{42.1}$$

The sets $K_1 = K$ and $K_2 = \overline{O \setminus N}$ are fixed disjoint compacts: K lies inside the open N, while K_2 lies in its closed complement. Apply the preceding lemma to these pieces, with the Borel subsets $A_1 = K$ and $A_2 = O \setminus N$. Since both subsets lie in E, it gives

$$c_Q(E) \geq \Delta g(K) + \limsup_j \Delta g_j(O \setminus N).$$

This does not discard mass on ∂N: the second cutoff is one on a neighborhood of the entire compact closure K_2. Combining with (42.1) yields

$$c_Q^*(E) \leq c_Q(E) + \Delta g(\overline{N}) - \Delta g(K).$$

Only after taking the row limit do we shrink N through nested neighborhoods of K. The fixed finite Radon measure Δg then makes the last difference tend to zero. This order of choices is essential: the cutoff cost may grow as N shrinks, but for each fixed N the local-uniform error δ_j already tends to zero. The opposite inequality is always true. The exact raw and outer Gauss reductions finish the Berkovich statement. □

Corollary 42.3 (Adding a Borel set with controlled compact residuals)**.** *Let $O \Subset Q$ be open and let B be Borel in one compact core. Suppose that for every $\varepsilon > 0$ there is compact $K \subset B$ with $c_Q^*(B \setminus K) < \varepsilon$. Then*

$$c_Q(O \cup B) = c_Q^*(O \cup B).$$

This applies when B has finite $\mathcal{H}^{n-1}$ measure, or when B is a finite union of the Borel coordinate sheets in Proposition 41.2.

Status: `DIRECT_COROLLARY`.

Proof. Outer subadditivity and the theorem give

$$c_Q^*(O \cup B) \leq c_Q^*(O \cup K) + c_Q^*(B \setminus K) < c_Q(O \cup B) + \varepsilon.$$

Let $\varepsilon \downarrow 0$. The two specified residual estimates were proved in the Hausdorff and projection criteria, respectively. □

The unresolved general Borel step. For a general Borel E, its open neighborhoods need not differ from E only near one compact piece, and no compact-residual estimate has been proved. The argument therefore does not establish continuity of outer capacity from below on arbitrary increasing Borel sets. General real Borel C2 and arbitrary-background C2 remain open.

43 A boundary criterion for qualified C1 and C2 statements

Status: `DIRECT_COROLLARY`. Let S be an arbitrary fixed original-positive symmetric separately closed $(n-1,n-1)$-current on an ordinary norm-box domain Ω with the frozen faithful energy form. Put $F = \operatorname{supp} S$. All capacities use normalized actual classical testers on all of Ω. Fix a compact core $K_0 \Subset \Omega$.

For a Borel $E \subset K_0$, set

$$B_E = \partial_F(E \cap F), \qquad e_E = \operatorname{cap}_{\mathcal{E},1}(B_E).$$

Then

$$0 \le \operatorname{Cap}_S^*(E;\Omega) - \operatorname{Cap}_S(E;\Omega) \le e_E. \tag{B1}$$

The quantities on the left are finite by local CLN. In particular $e_E = 0$ gives Borel ordinary-open outer regularity for this E; for compact E it gives sharp compact C1.

Proof. Every normalized BT measure is carried by the closed F. An ordinary open set $O \subset \Omega$ may be chosen with $O \cap F = \operatorname{int}_F(E \cap F)$, and may be intersected with a fixed relatively compact neighborhood of K_0. Its raw capacity equals that of this relative interior and is at most the raw capacity of E. The part of E outside F is contained in an ordinary open set of zero capacity. Finally

$$E \cap F \subset \operatorname{int}_F(E \cap F) \cup B_E.$$

Finite outer subadditivity and the sharp one-way inequality $\operatorname{Cap}_S^*(B_E;\Omega) \le e_E$ from Corollary 32.3 prove (B1). Only relative topology on F is used; the ambient boundary would give an unnecessarily strong condition for polar-carried backgrounds. □

There is also a quantitative continuity-set statement. Suppose a net of normalized measures $\mu_j = D(u_j S)$ converges weakly on all original tests to a positive Radon measure ν. Then

$$\nu(E) - e_E \le \liminf_j \mu_j(E) \le \limsup_j \mu_j(E) \le \nu(E) + e_E. \tag{B2}$$

Hence $e_E = 0$ makes E a simultaneous weak continuity set.

Proof. The fixed-core mass bounds and smooth scalar density turn the stated weak convergence into ordinary local vague convergence. All measures are carried by F, so this may be considered on F; compact continuous functions there extend locally to the ambient space. The sharp Hardy inequality passes to ν on each fixed original square test. Its compact-core and ordinary-open argument therefore also gives $\nu(B_E) \le e_E$. Apply the open and compact inequalities for vague convergence to the relative interior and closure of $E \cap F$. Their differences from $E \cap F$ lie in B_E, proving (B2). One may insert a fixed original cutoff equal to one near K_0 to reduce every step to finite measures on one compact set. □

These statements do not close general C1: the energy capacity of a boundary can be positive even when its BT outer capacity is zero, as Theorem 38.4 demonstrates. They also do not assert a converse boundary criterion. A fixed higher mixed tail can replace S, with its own faithful energy form. For a family with all higher slots moving, one would need a uniform estimate for those varying tails; no such estimate is implicit here.

44 Sharp regularity under continuous ambient-curvature domination

This criterion applies in every remaining degree and to an arbitrary fixed background. The finite-dimensional curvature hypothesis from R5 is replaced by an order bound in an infinite-dimensional family. The conclusion concerns that explicitly dominated family, or a weakly convergent net lying in it; no uniform domination of all normalized psh testers is assumed to exist.

Let Ω be an allowed ordinary open, let S be symmetric, original-positive and separately closed of bidegree (p,p), and put $q = n - p \geq 1$. Fix an ordinary open $W \subset \Omega$ and a compact core $K_0 \Subset W$. All actual competitors below are defined and normalized on the *whole* domain Ω.

Lemma 44.1 (Ambient order gives mixed relative order)**.** *Let $1 \leq k \leq q$. Suppose $u_1, \dots, u_k \in \mathcal{C}_b(\Omega)$ are continuous on W, and $g_1, \dots, g_k \in \mathcal{C}_b(W) \cap C(W)$. If, for constants $M_i \geq 0$,*

$$0 \leq D[u_i]|_W \leq M_i D[g_i],$$

then on every original complementary test in W one has

$$0 \leq T_{S,k}(u_1, \dots, u_k)|_W \leq \Big(\prod_{i=1}^{k} M_i\Big) T_{S|_W,k}(g_1, \dots, g_k). \tag{44.1}$$

Status: `DIRECT_COROLLARY` of R's continuous regularization, expressed in Lemma 11.1, and the frozen mixed BT calculus.

Proof. For any fixed positive separately closed tail R, the continuous function $h_i = M_i g_i - u_i$ has positive original curvature on W. The cited order-transfer lemma gives

$$D(u_i R) \leq M_i D(g_i R).$$

This uses uniform smooth positive approximation of h_i, not a claim that an arbitrary discontinuous difference with positive distributional curvature is classical psh. Symmetry permits each slot to be placed first, while the other bounded psh slots form a positive separately closed tail. Replace the slots one at a time. All intermediate tails retain positivity and separate closedness, which proves (44.1), including zero values of the M_i. □

Theorem 44.2 (An all-degree dominated-family capacity criterion)**.** *Fix continuous classical $g_1, \dots, g_q$ on W and constants $M_1, \dots, M_q \geq 0$. For each i, let $\mathscr{P}_i$ be any subfamily of actual $u \in \mathcal{C}_b(\Omega)$ satisfying*

$$0 \leq u \leq 1 \text{ on } \Omega, \qquad u|_W \in C(W), \qquad D[u]|_W \leq M_i D[g_i].$$

Let $\mathscr{M}$ be the family of positive measures $T_{S,q}(u_1, \dots, u_q)$ with $u_i \in \mathscr{P}_i$, or any subfamily of these measures. Define its restricted raw capacity by

$$c_{\mathscr{M}}(E) = \sup_{\mu \in \mathscr{M}} \mu(E), \qquad \sup \varnothing := 0.$$

Its outer regularization uses ordinary open neighborhoods in the same Ω. Then:

1. *For every Borel $E \subset K_0$, $c_{\mathscr{M}}(E) = c^*_{\mathscr{M}}(E)$.*
2. *If a net from $\mathscr{M}$ converges on all original compact smooth scalar tests in W to a Radon measure μ, then $\int b\, d\mu_j \to \int b\, d\mu$ for every bounded Borel b supported in K_0. In particular $\mu(E) \leq c_{\mathscr{M}}(E)$ for every such Borel E.*

3. *The same restricted capacities have continuity from above on decreasing Borel sequences in K_0. Together with their usual increasing-sequence continuity and Borel inner regularity, this supplies these finite compact-core regularity properties for the restricted family.*

No conclusion of convergence in total variation is made.

Status: `DIRECT_COROLLARY` with an explicit infinite-dimensional sufficient hypothesis. The transfer from *ambient* curvature bounds to a uniform measure bound for an arbitrary S is the substantive input.

Proof. On W put

$$\rho = \Big(\prod_{i=1}^{q} M_i\Big) T_{S|_W,q}(g_1, \dots, g_q).$$

The preceding lemma proves $0 \le \mu \le \rho$ on W for every $\mu \in \mathscr{M}$. Choose an ordinary V with $K_0 \subset V \Subset W$. The positive Radon measure ρ is finite on $\overline{V}$.

For Borel $E \subset K_0$, Radon outer regularity supplies an ordinary open G with $E \subset G \subset V$ and $\rho(G \setminus E) < \epsilon$. For every family member,

$$\mu(G) \le \mu(E) + \rho(G \setminus E) \le c_{\mathscr{M}}(E) + \epsilon.$$

Take the supremum and then $\epsilon \downarrow 0$. Since these are ordinary neighborhoods in Ω, this proves the first assertion. The local nature of the dominator does not change the capacity domain.

For the second assertion, fixed nonnegative smooth tests first imply $0 \le \mu \le \rho$ on W. A bounded Borel b supported in K_0 can be approximated in $L^1(\rho)$ by original smooth compact scalars supported in V: use Radon regularity, compact continuous approximation, and the original smooth scalar density on that compact stage. For such an approximant a,

$$\left|\int (b-a)\, d\mu_j\right| + \left|\int (b-a)\, d\mu\right| \le 2\int |b-a|\, d\rho.$$

The middle smooth pairing converges by hypothesis. Let the $L^1(\rho)$ error tend to zero. This works for nets as well as sequences and proves the asserted fixed-Borel convergence. It is setwise convergence on the core, not uniform convergence over all Borel indicators.

If $E_j \downarrow E$ are Borel subsets of K_0, domination gives

$$0 \le c_{\mathscr{M}}(E_j) - c_{\mathscr{M}}(E) \le \rho(E_j \setminus E) \to 0.$$

For $E_j \uparrow E$, the two suprema over the index and over family members commute, giving increasing continuity. For a Borel E, inner regularity of each fixed member similarly gives $c_{\mathscr{M}}(E) = \sup_{K \subset E \text{ compact}} c_{\mathscr{M}}(K)$. These statements, with the preceding ordinary-open regularity and finite core bound, prove the stated restricted regularity properties. No continuity assertion for arbitrary decreasing Borel nets is used. □

Corollary 44.3 (A maximal dominated subfamily and a sharp C1 test)**.** *For fixed $g \in \mathcal{C}_b(W) \cap C(W)$ and $M \ge 0$, set*

$$\mathscr{P}(g, M; W) = \{u \in \mathcal{C}_b(\Omega) : 0 \le u \le 1 \text{ on } \Omega,\ u|_W \in C(W),\ D[u]|_W \le M D[g]\}.$$

The restricted diagonal *capacity using this entire subfamily is Borel outer regular on K_0, and its weak closure has no compact mass gain. In particular, if normalized u_j lie in this family and $(Du_j)^q S \to \nu$ weakly on W, then for every compact $K \subset K_0$,*

$$\nu(K) = \lim_j (Du_j)^q S(K) \le \mathrm{Cap}_S(K; \Omega).$$

The mixed restricted capacities in the theorem are not identified with diagonal capacities. If the unrestricted normalized family has no such common ambient bound, no full C1 or C2 conclusion follows.

Proof. Use the diagonal subfamily of the preceding theorem, with all $g_i = g$ and $M_i = M$. The last upper bound holds because every member is still a normalized actual tester on all of Ω. □

Example 44.4 (Infinite-dimensional curvature and failure of TV convergence)**.** On the original annulus $e^{-2} < |T| < e^2$ put $t = \log|T|$, take $S = [1]$, and define

$$u_j(t) = \frac{1}{2} + \frac{t^2}{16} - \frac{\sin(jt)}{32j^2}, \qquad u(t) = \frac{1}{2} + \frac{t^2}{16}, \qquad g(t) = \frac{5t^2}{64}.$$

These are smooth classical functions, $0 < u_j < 1$ throughout the annulus, and

$$0 \le Du_j = \left(\frac{1}{8} + \frac{\sin(jt)}{32}\right) d't \wedge d''t \le Dg = \frac{5}{32} d't \wedge d''t.$$

Their curvature currents span an infinite-dimensional real space. They converge weakly, and setwise on every fixed compact core, to Du. On the compact skeleton segment $-1 \le t \le 1$, however, the total variation of the difference is

$$\frac{1}{32}\int_{-1}^{1} |\sin(jt)|\, dt \longrightarrow \frac{1}{8\pi} > 0.$$

All calibrations use $t = \log|T|$ with the original absolute value. Thus the criterion extends beyond exact finite-dimensional curvature families and does not falsely upgrade setwise convergence to TV.

45 Variable curvature bounds and integrable relative density control

The continuous coefficient relation theorem allows upper bounds with variable scalar coefficients. Its measure consequences require a uniform integrability hypothesis, which is stated explicitly. The result concerns the specified family of normalized testers, not all normalized psh functions without that hypothesis.

Proposition 45.1 (Variable continuous curvature bounds)**.** *Fix any allowed (p,p)-background S and $k \le n-p$. Suppose $u_i, w_i \in \mathcal{C}_b(U)$ and $f_i \ge 0$ are finite continuous functions on U, with*

$$D[u_i] \le f_i D[w_i] \quad (1 \le i \le k)$$

as original coefficient-Radon current inequalities. Then

$$0 \le T_{S,k}(u_1, \dots, u_k) \le \left(\prod_{i=1}^{k} f_i\right) T_{S,k}(w_1, \dots, w_k).$$

Neither the w_i nor the u_i are assumed continuous. No closedness of the continuously weighted upper current is required or asserted.

Status: `DIRECT_COROLLARY` of Theorem 21.6.

Proof. For the positive separately closed tail containing all slots other than u_1, apply the module theorem to $f_1 D[w_1] - D[u_1] \ge 0$. This gives the displayed inequality with u_1 replaced by w_1 and scalar factor f_1 outside the product. Use symmetry, apply the same argument to the next remaining u_i inside the still positive separately closed tail, and multiply the result by the already extracted nonnegative scalar factors. Iterating proves the assertion. At no step is a variable-coefficient current used as a closed background. □

Hereafter let $q = n - p$ and fix an ordinary $V \Subset U$ on which

$$\rho = T_{S,q}(w_1, \dots, w_q)$$

has finite mass. Let $\mathscr{F}$ be a nonempty family of actual normalized tuples $(u_1, \dots, u_q)$, each defined on the same whole normalized domain, whose restrictions to V satisfy the preceding bounds with coefficients $f_{i,u}$. For some $1 < r < \infty$ assume

$$\sup_{u \in \mathscr{F}} \left\| \prod_{i=1}^{q} f_{i,u} \right\|_{L^r(V,\rho)} \le C < \infty. \tag{45.1}$$

Put $\mu_u = T_{S,q}(u_1, \dots, u_q)$, and define the restricted raw capacity $c_{\mathscr{F}}(E) = \sup_{u \in \mathscr{F}} \mu_u(E)$. The corresponding star is the infimum over ordinary open neighborhoods in the same normalized domain.

Theorem 45.2 (Qualified C1, C2 and Borel-weighted convergence)**.** *Under* (45.1)*, the following hold.*

1. *For every Borel $E \subset V$,*
$$c_{\mathscr{F}}(E) \le C\rho(E)^{1-1/r}.$$
For every Borel $E \Subset V$, the restricted raw capacity equals its ordinary-open outer capacity.
2. *If a net of measures from this family converges weakly on every original compact scalar test in V to a Radon measure μ, then it converges setwise on every Borel $E \Subset V$. In particular*
$$\mu(E) \le c_{\mathscr{F}}(E)$$
there, giving sharp compact no-gain without asserting that μ has an actual normalized representative.
3. *Write $r' = r/(r-1)$. On compact localizations in V, the Radon–Nikodym densities of the net and its limit converge weakly in $L^r(\rho)$. Consequently Borel weights converging strongly in $L^{r'}(\rho)$ may vary jointly with the measures in their pairings. Locally uniformly bounded weights converging locally in ρ-measure satisfy this condition. This last assertion is valid for nets with convergence in measure; mere pointwise convergence of nets is not substituted for it.*

The analogous conclusions hold with a uniform L^∞ density bound, using L^1 weights and weak-star density convergence. No total- variation convergence follows solely from these hypotheses.

Status: `DIRECT_COROLLARY` of variable curvature order, finite Radon regularity and elementary L^r duality. It enlarges the stated common bounded-density criterion; general unrestricted arbitrary-background C1 and C2 are not asserted.

Proof. The preceding proposition gives $0 \le \mu_u \le F_u\rho$, where $F_u = \prod_i f_{i,u}$. Thus its density h_u satisfies $0 \le h_u \le F_u$ and $\|h_u\|_{L^r(V,\rho)} \le C$. Holder's inequality proves (1)'s mass bound. If $E \Subset V$ is Borel, finite Radon outer regularity gives ordinary open G with $E \subset G \Subset V$ and $\rho(G \setminus E)$ arbitrarily small. For every family member,

$$\mu_u(G) \le \mu_u(E) + C\rho(G \setminus E)^{1-1/r}.$$

Take the supremum and then shrink this error. Ordinary outer monotonicity gives the reverse inequality, proving the asserted C2. The same argument allows opens in the original normalized domain, since the displayed open neighborhoods already belong to it.

For (2) and (3), the closed positive $L^r(V,\rho)$ ball of radius C is weakly compact. Every subnet of densities has a weakly convergent further subnet, with some nonnegative density h of norm at most C. Its integrals against every original compact smooth scalar test equal the assumed current limit. Original smooth scalar tests are uniformly dense on compact stages; the uniform finite-mass bound and cutoffs therefore identify $h\rho = \mu$ on V. Radon uniqueness also makes this density unique up to ρ-null sets. The same identification on every convergent subnet proves weak L^r convergence of the whole density net. This argument does not require a countable exhaustion or a selected convergent lifting of the potentials.

In particular testing with $\mathbf{1}_E \in L^{r'}(V,\rho)$ gives setwise convergence and the sharp capacity bound. If $a_j \to a$ strongly in $L^{r'}(V,\rho)$, then

$$\left|\int a_j h_j \, d\rho - \int a h \, d\rho\right| \le C\|a_j - a\|_{L^{r'}(V,\rho)} + \left|\int a(h_j - h)\, d\rho\right| \longrightarrow 0.$$

For local statements multiply weights by a fixed compact cutoff and apply the same estimate on its support. Locally bounded convergence in measure gives strong $L^{r'}$ convergence on finite-measure compact localizations: split into $|a_j - a| \le \epsilon$ and its complement, use the common bound on the latter, then let $\epsilon \downarrow 0$. This proof works for nets. The L^∞ case uses the weak-star compact ball and the same argument with L^1 tests and weights. □

Interpretation and limitations. The scalar functions $f_{i,u}$ enter through original ambient current inequalities; the dominating Radon measure ρ is the fixed relative BT product on the actual space. Thus the hypothesis supplies a common relative density control, including on carriers that may be ambient pluripolar. It is not an assertion of nonconcentration on all ambient polar sets. The existence of such coefficients with uniform integrable products must be established for the intended tester family; normalizing $0 \le u_i \le 1$ alone does not supply it.

46 Diffuse fiber backgrounds and exact capacity on rectangles

This section gives an all-dimensional class of relative capacities on which sharp ordinary-open outer approximation can be computed. The parameter measure may be diffuse and may be carried by an ambient complete pluripolar set. The set being tested is a rectangle; arbitrary nonrectangular compact sets are not included in the conclusion.

Fix integers $p, q \ge 1$, put $n = p + q$, and let Y and Z be the open unit polydiscs of dimensions q and p over the original K. Write $\Omega = Y \times_K Z$ and let $P \subset Z(K)$ be compact. Here $Z(K)$ means actual K-rational analytic points, not all Berkovich points. Let λ be an arbitrary finite positive Radon measure on P. For $a \in P$, denote the closed fiber embedding by $i_a : Y \hookrightarrow \Omega$, $t \mapsto (t, a)$.

The set $\pi_Z^{-1}(P)$ is naturally homeomorphic to the ordinary topological product $Y \times P$. Indeed every residue field at a point of P is exactly K, so each full analytic fiber is Y. On compact affinoid stages the map $(t, a) \mapsto i_a(t)$ is continuous by uniform polynomial approximation of analytic functions; for a polynomial the assertion follows from the ultrametric estimate when its coefficients vary with a. It is a continuous bijection from a compact space to a Hausdorff space and hence a homeomorphism there. These stages cover $Y \times P$. We use this identification for the Borel rectangles below. It does not replace a general nonrigid analytic fiber by an independent section.

Proposition 46.1 (Original-current disintegration over rigid parameters)**.** *The formula*

$$\langle S, \eta\rangle = \int_P \left(\int_Y i_a^* \eta\right) d\lambda(a), \qquad \eta \in A_c^{q,q}(\Omega), \tag{46.1}$$

defines a symmetric original-positive separately d'- and d''-closed (p,p)-current, carried by $\pi_Z^{-1}(P)$. For any $0 \le k \le q$ and $u_1, \dots, u_k \in \mathcal{C}_b(\Omega)$,

$$\langle T_{S,k}(u_1, \dots, u_k), \eta \rangle = \int_P \Big\langle T_{[1]_Y,k}(u_1 \circ i_a, \dots, u_k \circ i_a), i_a^* \eta \Big\rangle \, d\lambda(a) \tag{46.2}$$

for every original compact smooth complementary test η. At $k = q$ this is an equality of positive Radon measure kernels on the actual product $Y \times P$. In particular it holds on every Borel set, not only on smooth tests or on a selected skeleton.

Status: `DIRECT_COROLLARY` of the frozen fixed-background calculus and CLN estimates, together with the original fiber-integration theorem. The explicit disintegration proof is given to fix the test class, measurability and calibration.

Proof. The projection π_Z is separated, flat, boundaryless, and of pure relative dimension q. Original fiber integration [CLD25, Theorem 16.3.3(d), pp. 447–450] shows that $a \mapsto \int_Y i_a^* \eta$ is continuous, in particular bounded on the compact set P. Its integration in (46.1) is therefore defined. On a fixed compact original moment presentation, dominate a smooth test by finitely many fixed positive controls as in R. Their fiber integrals are continuous and bounded on P. This gives the requisite finite coefficient seminorm bound for the displayed functional and proves continuity in the original test topology. Pullback preserves the original positive cone and symmetry, so the functional is symmetric and positive. For a compact test of one lower degree, pullback commutes with d' and d'', and Stokes on the boundaryless fiber gives zero. Thus S is separately closed. A test supported off $\pi_Z^{-1}(P)$ has zero restriction on every fiber, proving the carrier assertion. The fibers are the original K-space Y; there is no residue-degree factor or absolute-value renormalization in this formula.

Here are the measurability and uniform bounds needed for the higher formula. A compact test is covered by finitely many original local regularization stages for the bounded potentials, with a smooth partition. On each stage take the frozen simultaneous smooth majorizing rows. After restriction to i_a^{-1} of that stage they are legitimate rows on the same original K-space Y. The smooth-row fiber pairings are continuous in a by the same fiber-integration theorem. Their pointwise limits are the fiber BT pairings by the already proved calculus with background $[1]_Y$. They are consequently Borel functions of a.

The relative CLN proof on a finite original presentation is uniform after restriction to these fibers: the real coefficient bounds and cutoff derivatives are the same, while the finitely many background control pairings are fiber integrals of fixed original smooth forms. Those pairings are bounded on P by fiber integration. The local bounds of the u_i on the fixed compact stage give one integrable bound for all pairings under discussion. This argument permits degenerate restricted moment frames; no strictly positive frame on every fiber is assumed.

One can therefore define the right side of (46.2) as a current R_k. It is positive and separately closed by the corresponding fiber properties, and has the finite coefficient controls just established. Thus it has the actual coefficient-Radon property. For fixed tests its coefficient measures are the integrals of the fiber coefficient measures: first test this against original smooth scalar multipliers, then use uniqueness of Radon measures and their bounded Borel multiplication. The same statement is valid with a bounded Borel scalar on a compact stage, including the actual u_i.

Induct on k. The assertion at zero is (46.1). Assuming it for the tail product, the coefficient-measure statement and commutation of pullback with D give

$$\begin{aligned} \langle D(u_1 R_{k-1}), \eta \rangle &= \int_P \Big\langle u_1 \circ i_a \, T_{[1]_Y,k-1}(u_2 \circ i_a, \dots, u_k \circ i_a), D(i_a^* \eta) \Big\rangle \, d\lambda(a) \\ &= \int_P \Big\langle T_{[1]_Y,k}(u_1 \circ i_a, \dots, u_k \circ i_a), i_a^* \eta \Big\rangle \, d\lambda(a). \end{aligned}$$

This is precisely the fixed-background recursion. It proves the formula on all original tests and hence equality with $T_{S,k}$. For $k = q$, the fiber measures are weakly Borel measurable and locally uniformly finite, as proved above. The usual Radon kernel integral on $Y \times P$ is therefore defined. Its compact continuous pairings agree with the current formula; original smooth scalar density identifies the Radon measures, giving the Borel assertion. □

Write $C_Y(A) = \mathrm{Cap}_{[1]_Y}(A; Y)$ and $C_S(E) = \mathrm{Cap}_S(E; \Omega)$, with normalized actual classical testers on the indicated *whole* domains. Stars denote infima over ordinary open neighborhoods in those same domains.

Theorem 46.2 (Sharp rectangular capacity and ordinary-open regularity)**.** *Fix an ordinary core $W \Subset Y$. For every Borel $A \subset W$ and every Borel $B \subset P$, the rectangle $E = i(A \times B)$ satisfies*

$$C_S(E) = \lambda(B) C_Y(A) = C_S^*(E). \tag{46.3}$$

In particular compact rectangles have sharp compact outer regularity, in every dimension and every remaining degree $q \geq 1$.

Status: `DIRECT_COROLLARY` of the preceding disintegration and the frozen *ambient* Borel outer-regularity theorem: R2, Corollary R4–36.9 (Corollary 36.9, printed p. 126 of the frozen R2 report). That theorem applies to Y, an exhaustion polydisc, and to A in its fixed relatively compact core. It is not an assumption of general relative Borel outer regularity.

Proof. For any $0 \leq u \leq 1$ classical on all of Ω, every restriction $u \circ i_a$ is normalized classical on all of Y. Disintegration gives

$$\int_E (Du)^q S = \int_B \left(\int_A (D(u \circ i_a))^q \right) d\lambda(a) \leq \lambda(B) C_Y(A).$$

Conversely a normalized classical v on Y pulls back to the normalized classical $v \circ \pi_Y$ on all of Ω. Its integral over E is $\lambda(B) \int_A (Dv)^q$. Taking suprema proves the raw equality, including the case $\lambda(B) = 0$. All quantities are finite here by CLN on the fixed core and finiteness of λ.

For $\varepsilon > 0$, the cited ambient theorem supplies an ordinary open O_Y with

$$A \subset O_Y \Subset Y, \qquad C_Y(O_Y) \leq C_Y(A) + \varepsilon.$$

The finite measure λ extended by zero from the compact P to Z is Radon. Choose an ordinary open $O_Z \subset Z$ containing B such that $\lambda(P \cap O_Z) \leq \lambda(B) + \varepsilon$. The ordinary open set

$$G = \pi_Y^{-1}(O_Y) \cap \pi_Z^{-1}(O_Z)$$

contains E. The same raw rectangle formula, and the carrier of S, give

$$C_S^*(E) \leq C_S(G) = \lambda(P \cap O_Z) C_Y(O_Y) \leq (\lambda(B) + \varepsilon)(C_Y(A) + \varepsilon).$$

Let ε decrease to zero. The general inequality $C_S(E) \leq C_S^*(E)$ gives the reverse bound. This proof does not assert that an arbitrary nonsaturated open neighborhood contains one fixed product neighborhood of a noncompact Borel rectangle. □

Example 46.3 (A diffuse complete-pole parameter measure)**.** These backgrounds need not be nonpolar or countable sums of fiber integration currents. For $p = 1$, choose a uniformizer π and put

$$P = \left\{ \sum_{j=1}^{\infty} \epsilon_j \pi^{4^j} : \epsilon_j \in \{0, 1\} \right\} \subset Z(K).$$

The coding by the binary product space is continuous and injective; thus P is compact. Give it the nonatomic Bernoulli probability measure λ. With $\ell = -\log|\pi|$, set

$$\phi_N(z) = \int_P \max(\log|z-a|, -N)\, d\lambda(a), \qquad \phi = \lim_{N\to\infty} \phi_N.$$

Every ϕ_N is continuous classical and nonpositive on Z. Indeed the integrand is continuous jointly on compact stages, so finite positive atomic quadratures converge uniformly on each such stage; each quadrature is continuous classical. Decreasing closure then makes ϕ extended classical.

For $z \in P$, the parameters agreeing with its first j digits have measure 2^{-j} and satisfy $|z-a| \le |\pi|^{4^{j+1}}$. Hence

$$\int_P -\log|z-a|\, d\lambda(a) \ge \ell\, 4^{j+1} 2^{-j} \longrightarrow \infty.$$

Thus $\phi = -\infty$ on P. If an analytic point $z \notin P$, the continuous function $a \mapsto |T-a|(z)$ is everywhere positive on the compact set P and has a positive minimum. Therefore $\phi(z)$ is finite. This proves that P is an exact complete pole set. For larger p embed P as $P \times \{0\}^{p-1}$ and use the witness $\max(\phi(z_1), \log|z_2|, \dots, \log|z_p|)$.

The current in (46.1) is carried by the complete pole set $Y \times P$, yet gives zero coefficient mass to each individual fiber because λ has no atoms. It is consequently not a positive order limit of finite sums of nonzero subcurrents carried by individual fibers. Formula (46.3) nevertheless applies exactly.

The next laminar gate. For a general compact $K \subset Y \times P$, disintegration gives only the fiberwise upper bound by the integral of intrinsic capacities, when that upper envelope is measurable. Choosing an optimizing tester independently on uncountably many fibers does not supply one ambient classical function normalized on Ω. In particular a compact graph of varying fiberwise Gauss points is not a rectangle, and no claim of C1 or C2 for that graph follows from this section. This is a concrete diffuse polar-supported test case for the remaining sharp selection and extension mechanism.

47 Exact capacity on moving Gauss graphs over diffuse pole carriers

Retain the original field, polydiscs, compact rigid parameter set P, finite parameter measure λ, and disintegrated background S of Section 46. The following computation handles a nonrectangular family. Its moving atoms are followed by an actual ambient psh threshold; no countable dense family of fixed fiber testers is substituted for that construction.

For $L > 0$ let $\zeta_{e^{-L}} \in Y$ be the diagonal Gauss point with all q coordinate radii e^{-L}. On the whole unit polydisc Y,

$$\mathrm{Cap}_{[1]}(\{\zeta_{e^{-L}}\}; Y) = \mathrm{Cap}_{[1]}(\{|t_1|, \dots, |t_q| \le e^{-L}\}; Y) = L^{-q}. \tag{47.1}$$

This is the same exact computation as Lemma R4–60.1, now with any real $L > 0$: $U = \max(-1, L^{-1}\max_i \log|t_i|)$ has $(DU)^q = L^{-q}\delta_{\zeta_{e^{-L}}}$ by the original slope-simplex calibration. The normalized tester $1+U$ gives the lower bound. Comparison of $(1+\epsilon)U$ with an arbitrary normalized tester minus one gives the upper bound on the closed polydisc; let $\epsilon \downarrow 0$. No change of the original absolute value, strict-radius assumption, or outer-regularity interchange is involved.

Theorem 47.1 (A sharp formula for moving Gauss graphs)**.** *Let $f \in \mathcal{C}_b(Z) \cap C(Z)$ satisfy $f \le 0$ on all of Z and*

$$0 < m \le -f(a) \le M < \infty \qquad (a \in P).$$

Suppose that P admits finite partitions $P = \bigsqcup_i P_{r,i}$ into compact sets such that each $P_{r,i}$ is an exact complete pole set of a nonpositive extended classical psh function on Z, and

$$\max_i \operatorname{osc}_{P_{r,i}} f \longrightarrow 0.$$

For any Borel $B \subset P$ put

$$E_B = \{i_a(\zeta_{\exp f(a)}) : a \in B\} \subset \Omega.$$

Then, with normalized actual classical testers on the whole Ω,

$$C_S(E_B;\Omega) = C_S^*(E_B;\Omega) = \int_B (-f(a))^{-q}\, d\lambda(a). \tag{47.2}$$

In particular E_P is compact and has sharp compact outer regularity. The theorem holds for every $p, q \ge 1$.

Status: `DIRECT_COROLLARY` of the proved disintegration, exact radial calibration, and frozen finite pole-carrier additivity, with the explicit same-domain testers below. This computes a new nonrectangular class; it does not prove general relative C1 or C2.

Proof. The map $a \mapsto \zeta_{\exp f(a)}$ is continuous: on every polynomial, the Gauss seminorm is a finite maximum of continuous functions of the radius. Together with the product identification in Section 46, this makes E_P a compact graph homeomorphic to P and E_B a Borel subset of it. All its Y-coordinates lie in the one fixed compact polydisc of radius $e^{-m} < 1$.

For a normalized ambient tester u, the fiber function $u \circ i_a$ is normalized on all of Y. The Borel kernel identity and (47.1) therefore give

$$\int_{E_B} (Du)^q S = \int_B (D(u \circ i_a))^q(\{\zeta_{\exp f(a)}\})\, d\lambda(a) \le \int_B (-f(a))^{-q}\, d\lambda(a).$$

This proves the raw upper bound.

Fix one of the finite partitions and abbreviate its parts by P_i. Let S_i be the background obtained by integrating the same full fibers against $\lambda|_{P_i}$. Proposition 46.1 proves directly that each S_i is an allowed positive separately closed current, and $S = \sum_i S_i$. The pullback of the pole witness of P_i has exact pole set $Y \times P_i$ and carries every coefficient of S_i. These closed carriers are pairwise disjoint. Ordinary restriction and locality make every BT measure for S_i vanish on the carrier of S_l for $l \ne i$. Thus the hypotheses of the frozen Theorem R4–79.1 hold for these actual diffuse currents, not merely for individual fiber integration currents. It follows that

$$C_S(E_B;\Omega) = \sum_i C_{S_i}(E_B;\Omega). \tag{47.3}$$

Put $M_i = \max_{P_i}(-f)$ and $m_i = \min_{P_i}(-f)$, omitting empty parts. The globally defined function

$$u_i(t,z) = \max\left\{0,\ 1 + M_i^{-1} \max\left(\log|t_1|, \dots, \log|t_q|, f(z)\right)\right\}$$

is continuous classical psh and satisfies $0 \le u_i \le 1$ on all of Ω. The inner maximum is finite because f is finite; the outer maximum supplies the global lower bound even away from P_i. For $a \in P_i$ the outer floor is redundant, and exact radial calibration on the fiber gives

$$(D(u_i \circ i_a))^q = M_i^{-q} \delta_{\zeta_{\exp f(a)}}.$$

Consequently (47.3) implies

$$C_S(E_B;\Omega) \geq \sum_i M_i^{-q}\lambda(B \cap P_i). \tag{47.4}$$

The carrier theorem takes the independent finite suprema in one same-domain normalized class; no existence of a single exactly optimizing tester is asserted here.

For the outer upper bound set $A_i = \{\zeta_{\exp f(a)} : a \in P_i\} \subset Y$. It is compact, contains the Gauss point of the largest radius e^{-m_i}, and is contained in the closed polydisc of that radius. Equation (47.1) implies $C_Y(A_i) = m_i^{-q}$. The Borel rectangles $i(A_i \times (B \cap P_i))$ cover E_B. The already proved rectangle theorem and finite subadditivity of ordinary-open outer capacity therefore give

$$C_S^*(E_B;\Omega) \leq \sum_i C_S^*(i(A_i \times (B \cap P_i));\Omega) = \sum_i m_i^{-q}\lambda(B \cap P_i). \tag{47.5}$$

The function $x \mapsto x^{-q}$ is uniformly continuous on $[m, M]$. The partition oscillation condition makes the difference of the upper and lower sums tend to zero, bounded by $\lambda(P)$ times that uniform modulus. Both sums converge to the displayed integral in (47.2). The inequalities (47.4), $C_S \leq C_S^*$, and (47.5) prove the theorem. □

Corollary 47.2 (Diffuse complete-polar Cantor parameters)**.** *For the explicit parameter set P in Example 46.3, the preceding theorem applies to every continuous classical $f \leq 0$ satisfying the stated strict negative bounds on P, and to every finite positive Radon measure λ on P, including nonatomic ones.*

Proof. Partition P by its first r binary digits. Each part is compact, and the diameters of the parts tend to zero, so continuity of $f|_P$ gives the required uniform oscillation condition. Each part has an exact complete-pole witness: use the same logarithmic integral as in Example 46.3, with the Bernoulli probability conditioned on that cylinder. The divergent estimate is unchanged up to its fixed positive normalization, and compactness gives finiteness at every analytic point outside that cylinder. For higher p, include the other zero coordinates by the same finite maximum. This construction depends on the parameter set, not on the chosen measure λ. □

Why this does not give arbitrary fiberwise optimization. A weakly dense or countable family of fixed fiber testers can miss all the atoms of a moving graph. The proof above instead follows its radius through the actual joint psh function $f(z)$, and only the normalizing slopes are discretized. For a general compact $K \subset Y \times P$, no such joint psh threshold is supplied. The existence of measurable fiberwise optimizers, even if established, would still not produce normalized ambient classical testers. Neither general compact no-gain nor actual representability of every completed relative measure follows from the graph formula.

48 Exact interpolation on a pole Cantor set and Borel Gauss graphs

Let Z_1 be the ordinary open unit disc over the original field K, and retain the compact set

$$P = \left\{ \sum_{j\geq 1} \epsilon_j \pi^{4^j} : \epsilon_j \in \{0,1\} \right\} \subset Z_1(K)$$

from Example 46.3. A cylinder means the subset obtained by prescribing finitely many initial digits. The following interpolation statement concerns this specified set. It is not an extension theorem for arbitrary pluripolar sets.

Lemma 48.1 (Continuous pole wells on cylinders)**.** *Let $P_1, \dots, P_s$ be the nonempty cylinders of one finite partition of P. There are pairwise disjoint ordinary open discs $N_i \supset P_i$ with the following property. For every $\kappa > 0$ there are continuous classical functions w_i on the whole Z_1 such that*

$$-1 \le w_i \le 0, \qquad w_i|_{P_i} = -1, \qquad -\kappa \le w_i \le 0 \quad \text{on } Z_1 \setminus N_i.$$

Status: `DIRECT_COROLLARY` of the explicit logarithmic pole witnesses, with the uniform bound recorded.

Proof. The finite cylinders have strictly smaller internal diameters than their mutual distances. Choose a disc around each cylinder with radius strictly between these two quantities. The discs are ordinary open, pairwise disjoint, and contain the respective cylinders.

Let τ_i be the Bernoulli probability conditioned on P_i and put

$$\phi_i(z) = \int_{P_i} \log|T - a|(z)\, d\tau_i(a).$$

The construction in Example 46.3 shows that $\phi_i \le 0$ is extended classical and has exact pole set P_i. It is continuous and finite off P_i: on a compact neighborhood of any such point the continuous positive function $|T - a|(z)$ has a uniform positive lower bound for $a \in P_i$, so the logarithmic integral is continuous there. Since ϕ_i is upper semicontinuous,

$$w_{i,N} = \max\{\phi_i/N, -1\}$$

is locally constant equal to -1 near every point of P_i, and is continuous everywhere. It is a bounded classical function by the maximum and decreasing-closure properties.

If N_i has center b_i and radius r_i, then for $z \notin N_i$ and $a \in P_i$ one has $|T - a|(z) \ge r_i$, by the ultrametric inequality and $|a - b_i| < r_i$. Thus $\phi_i(z) \ge \log r_i$ uniformly on the whole complement of N_i. Taking N sufficiently large makes $w_{i,N} \ge -\kappa$ there. This proves the claimed uniform, same-domain wells. □

Theorem 48.2 (Continuous interpolation with arbitrary range slack)**.** *For every $F \in C(P, \mathbf{R})$ and every $\varepsilon > 0$, there is an actual bounded continuous classical function f on the whole Z_1 such that*

$$f|_P = F, \qquad \min_P F - \varepsilon \le f \le \max_P F + \varepsilon \quad \text{on } Z_1.$$

For $P \subset Z_1 \times \{0\}^{p-1} \subset Z$, the same assertion holds on the whole p-dimensional unit polydisc Z, by pullback.

Status: `NEW_PROVED` for this explicit pole Cantor set. The proof uses the displayed continuous wells and a uniformly convergent decreasing construction; it does not infer an ambient representative from relative capacity convergence.

Proof. Write $a = \min_P F$, $b = \max_P F$, and start with the constant classical function $v_0 = b + \varepsilon/4$. Put

$$\eta_j = \frac{\varepsilon}{64}\, 8^{-j}.$$

We construct continuous classical v_j decreasing on Z_1, with continuous residuals $e_j = v_j|_P - F$. Initially $e_0 \ge \varepsilon/4 = 16\eta_0$. At step j assume $e_j \ge 2\eta_j$, and let $H_j = \max_P e_j$. Choose a finite cylinder partition such that the oscillation of e_j on each part P_i is at most η_j. Define

$$a_i = \min_{P_i} e_j - \eta_j > 0.$$

Use the preceding lemma with pairwise disjoint discs N_i, choosing $\kappa > 0$ so small that $sH_j\kappa \le \eta_j/2$. Set

$$v_{j+1} = v_j + \sum_{i=1}^{s} a_i w_i.$$

This is continuous classical and at most v_j.

At a point of P_i, the ith well is exactly -1 and every other well lies in $[-\kappa, 0]$. Therefore

$$\eta_j/2 \le e_{j+1} \le 2\eta_j.$$

In particular $e_{j+1} \ge 4\eta_{j+1}$, so the induction continues. At any point of Z_1 at most one of the disjoint N_i occurs. The corresponding well contributes at most H_j to the absolute decrement; all others contribute together at most $\eta_j/2$. Consequently

$$0 \le v_j - v_{j+1} \le H_j + \eta_j/2 \quad \text{on } Z_1.$$

For $j \ge 1$ one has $H_j \le 2\eta_{j-1}$. The tail decrements are uniformly summable, and v_j converges uniformly on the entire disc to a bounded continuous function f. Decreasing classical closure with bounded limit makes this actual function classical. The residual bounds give $f|_P = F$ exactly.

The upper bound is $f \le v_0 = b + \varepsilon/4$. Also

$$\sum_{j\ge 0} \|v_j - v_{j+1}\|_\infty \le H_0 + \frac{20}{7}\eta_0, \qquad H_0 = b - a + \varepsilon/4.$$

Hence $f \ge a - (20/7)\eta_0 > a - \varepsilon$. This proves the range claim. Pullback by the first coordinate projection gives the final assertion and preserves the original-field classical curve condition. □

Corollary 48.3 (Every bounded Borel radial graph in this lamination)**.** *Let $Y, Z, \Omega = Y \times_K Z$ and $S = \int_P [Y \times \{a\}]\, d\lambda(a)$ be as in Section 46, with this explicit embedded Cantor set P and any finite positive Radon measure λ on P. Let $F : P \to [-M, -m]$ be Borel, where $0 < m \le M < \infty$. For every Borel $B \subset P$, define*

$$E_B = \{i_a(\zeta_{\exp F(a)}) : a \in B\}.$$

Then, in every $p, q \ge 1$, with normalized actual classical testers on the same whole Ω,

$$C_S(E_B; \Omega) = C_S^*(E_B; \Omega) = \int_B (-F(a))^{-q}\, d\lambda(a).$$

Status: `DIRECT_COROLLARY` of the new interpolation theorem, the proved continuous-threshold graph formula, Lusin approximation, and the rectangle formula. The datum F need not be continuous or the trace of a preassigned psh function.

Proof. The real radius interval $[e^{-M}, e^{-m}]$ parametrizes a compact Gauss arc in Y. The graph of the Borel map $a \mapsto \exp F(a)$ is Borel in its product with the compact metrizable P. The original rigid-fiber product identification makes E_B a Borel subset of a fixed compact core of Ω.

For any normalized ambient tester, its fiber restrictions are normalized on the entire Y. The Borel disintegration and exact singleton capacity in (47.1) give the upper bound

$$C_S(E_B) \le I_B, \qquad I_B = \int_B (-F(a))^{-q}\, d\lambda(a).$$

Fix $\delta > 0$. Finite Radon inner regularity and Lusin's theorem give a compact $K \subset B$ with $\lambda(B \setminus K) < \delta$ such that $F|_K$ is continuous. Extend it continuously to $\widetilde{F} : P \to [-M, -m]$ by

the Tietze extension theorem on the compact metrizable space P. Apply Theorem 48.2 with range slack smaller than $m/2$. It gives a continuous classical $f \le -m/2 < 0$ on the whole Z with $f|_P = \widetilde{F}$. Every prefix partition consists of exact complete pole sets and its f-oscillation tends to zero. Thus the already proved Theorem 47.1, applied to the Borel parameter subset K, gives

$$C_S(E_K) = C_S^*(E_K) = I_K.$$

Let A be the compact Gauss arc of radii in $[e^{-M}, e^{-m}]$. Its intrinsic capacity is m^{-q}: it contains the largest-radius Gauss point and is contained in the closed polydisc of that radius. The remaining graph is contained in the Borel rectangle $i(A \times (B \setminus K))$. The rectangle theorem and outer subadditivity therefore imply

$$C_S^*(E_B) \le I_K + m^{-q}\lambda(B \setminus K) \le I_B + m^{-q}\delta.$$

On the other hand $C_S(E_B) \ge C_S(E_K) = I_K \ge I_B - m^{-q}\delta$. Let $\delta \downarrow 0$ and use $C_S \le C_S^*$. Every comparison uses the original whole capacity domain; no fiberwise supremum is interchanged with an integral. □

The remaining selection problem. This result treats one scalar moving radial threshold over the explicit pole Cantor set. It does not extend an arbitrary family of intrinsic psh functions to a jointly psh ambient function, identify the capacity of arbitrary Borel subsets of $Y \times P$, or settle unrestricted C1 or C2 for general backgrounds. The successful mechanism is exact scalar interpolation followed by finite-measure Lusin localization; weakly dense lists of fixed fiber testers would not follow these moving atoms.

49 Filled Borel radial tubes and their intermediate sets

Retain the explicit pole Cantor lamination, the finite measure λ, the background S, and the same whole capacity domain Ω from Corollary 48.3. Let $F : P \to [-M, -m]$ be Borel, $0 < m \le M < \infty$, and let $B \subset P$ be Borel. Write E_B for its diagonal Gauss graph and set

$$H_B = \{i_a(t) : a \in B,\ \max_{1 \le l \le q} |t_l| \le \exp F(a)\}.$$

The latter is the filled moving closed polydisc on each parameter fiber. Both sets are Borel and lie in the fixed compact core consisting of the radius-e^{-m} closed polydisc times P.

Corollary 49.1 (Exact capacity of filled tubes)**.** *Every Borel set E with $E_B \subset E \subset H_B$ satisfies*

$$C_S(E;\Omega) = C_S^*(E;\Omega) = \int_B (-F(a))^{-q}\, d\lambda(a).$$

In particular this holds for the entire filled tube H_B, in every $p, q \ge 1$.

Status: `DIRECT_COROLLARY` of the Borel graph formula and the same-domain rectangle formula. There is no new fiberwise selection or extremal-function assumption.

Proof. For each mesh size $\delta > 0$, partition the compact interval $[m, M]$ into finitely many disjoint Borel intervals I_i of diameter at most δ. Put

$$B_i = \{a \in B : -F(a) \in I_i\}, \qquad s_i = \inf I_i \ge m, \qquad A_i = \{t \in Y : \max_l |t_l| \le e^{-s_i}\}.$$

These are Borel parameter pieces and fixed compact intrinsic polydiscs. Their rectangles cover H_B. The rectangle theorem and the exact radial calibration give

$$C_S^*(H_B;\Omega) \le \sum_i C_S^*(i(A_i \times B_i);\Omega) = \sum_i s_i^{-q}\lambda(B_i).$$

Since $x \mapsto x^{-q}$ is uniformly continuous on $[m, M]$, the right side differs from $\int_B (-F)^{-q}\, d\lambda$ by at most $\lambda(P)$ times its modulus of continuity at δ. Let $\delta \downarrow 0$. The graph inclusion and the proved graph formula give the matching raw lower bound. For any displayed E,

$$C_S(E_B) \le C_S(E) \le C_S^*(E) \le C_S^*(H_B),$$

so every term equals the claimed integral.

The Borel assertion follows from the actual rigid-fiber product identification and the Borel inequality between the continuous function $\max_l |t_l|$ and the parameter function $\exp F(a)$. Every tester used in the rectangle and graph inputs is normalized on the original whole domain Ω. □

This computation includes nonsaturated Borel intermediate sets whenever they contain the Gauss graph. It does not identify the capacity of an arbitrary Borel subset of the lamination or settle general relative C2.

50 Why an ambient upper contact cannot recover relative compact mass

The following example eliminates a specific proposed route to compact outer regularity. It is not a counterexample to compact outer regularity itself. The completed representative and its ambient upper envelope must be kept separate even after the potential-quotient theorem of R5.

Proposition 50.1 (Loss of all relative mass under an ambient upper envelope)**.** *Let $p, q \ge 1$, $n = p + q$, and let Ω be the open unit polydisc over the original K, with coordinates $(t_1, \dots, t_q, z_1, \dots, z_p)$. Put $Y = \{z_1 = \cdots = z_p = 0\}$ and $S = [Y]$. There are continuous classical $a_j \in \mathcal{C}_b(\Omega)$ with $0 \le a_j \le 1$ and $a_j \uparrow a$ such that:*

1. *a belongs to the bounded relative completion for S, and $\mu_{a_j} = (Da_j)^q S = \mu_a$ is a fixed nonzero positive measure;*
2. *the ambient upper semicontinuous regularization is $a^* = 1$, so $(Da^*)^q S = 0$;*
3. *at every point of Y, no normalized upper semicontinuous majorant $b \le 1$ of a on an ordinary neighborhood can contact a. Such a b is identically one on that part of Y.*

In particular, if the majorant is classical, its relative top measure vanishes there. Nevertheless, μ_a itself is realized by a globally normalized continuous classical potential on Ω.

Status: `DIRECT_COROLLARY` of the frozen increasing completion and integration-current compatibility, with an explicit example. The upper-contact strategy it disproves is recorded as `FAILED_ATTEMPT`.

Proof. Choose a uniformizer π, write $\lambda = -\log|\pi| > 0$, and fix $0 < \delta < 1/2$. Set

$$b_i(t_i) = \max\{0, 1 + \lambda^{-1} \log|t_i|\}, \qquad h(t) = \delta + \frac{1 - 2\delta}{q} \sum_{i=1}^{q} b_i(t_i).$$

These are actual continuous classical functions, with $\delta \le h \le 1 - \delta$. Define

$$\psi(z) = \max_{1 \le l \le p} \log|z_l|, \qquad a_j(t, z) = \max\{h(t), 1 + j^{-1}\psi(z)\},$$

with $\psi = -\infty$ on Y. The function ψ is extended classical and nonpositive. Its logarithmic singularity is removed by the finite continuous lower branch h in a_j. Hence a_j is continuous classical, $\delta \le a_j \le 1$, and a_j increases. Its actual limit is

$$a = h \quad \text{on } Y, \qquad a = 1 \quad \text{on } \Omega \setminus Y.$$

The fixed-background increasing-limit theorem puts this actual function in the relative completion. For each classical row, restriction to the smooth analytic subspace gives

$$(Da_j)^q[Y] = i_*((Dh)^q),$$

where $i : Y \hookrightarrow \Omega$. Completed convergence gives exactly the same measure for a. On the original product calibration each Db_i has mass λ^{-1} at the radius-$|\pi|$ Gauss point in its coordinate. Thus, if $y \in Y$ is the Gauss point with all these radii,

$$\mu_a = q! \left(\frac{1-2\delta}{q\lambda}\right)^q \delta_y.$$

Only nonvanishing is needed below; the formula also makes it explicit.

Every point of Y is an ordinary limit of points outside Y. For example, after passage to its completed residue field, take the normal Gauss polydisc of radius $r > 0$ centered at zero and let r tend to zero; its images in Ω approach the original point and lie outside Y. Therefore $a^* = 1$ at every point of Y, and it is already one off Y. If $b \le 1$ is an upper semicontinuous majorant of a on an ordinary neighborhood, upper semicontinuity along the same points forces $b = 1$ on its intersection with Y. Since $h \le 1 - \delta$, there is no contact with a there. When b is classical, integration-current compatibility and ordinary locality imply $(Db)^q[Y] = 0$ on that neighborhood.

Finally the function $h(t)$, pulled back by the coordinate projection, is globally normalized and continuous classical on Ω, and its relative measure is precisely the same μ_a. The example therefore invalidates upper-contact recovery for this representative, without invalidating the raw-capacity inequality for the measure. □

The precise excluded inference. One cannot start from a completed potential a, replace it by a^* or by normalized ambient classical majorants, and use a contact inequality to conclude that the mass of $(Da)^q S$ on a compact set survives. In the example the contact set is empty near its positive atom. Any successful proof of general compact outer regularity must instead choose different testers, or supply an additional mechanism that transfers their compact mass while retaining the normalization on the same whole domain.

Part IV
Relative Dirichlet problems and their constraints

51 A dual solvability criterion for the completed degree-one Dirichlet problem

Throughout this section $K/\mathbf{Q}_p$ is the original finite extension, with its prescribed absolute value. Let Ω be an ordinary algebraic norm-box domain relatively compact in a smooth affine analytification of pure dimension $n \ge 1$. Fix an arbitrary symmetric original-positive,

separately d'- and d''-closed $(n-1,n-1)$-current S. All tests are original compact smooth CLD tests. In particular, no smoothness, nonpolarity or geometric origin of S is assumed.

Use the faithful measure λ and closed coercive form $(\mathcal{E}_S, \mathscr{D}_S)$ of Theorem R4–84.3. Choose

$$L \Subset W \Subset \Omega,$$

where L is compact and lies in a smaller norm box and W is ordinary open. Write $H = \mathscr{D}_{S,0}(W)$, with Hilbert norm $\|x\|_H^2 = \mathcal{E}_S(x,x)$. Fix the smooth classical outer potential h and common bound B from Section R4–80. Let $\mathscr{P}$ be any nonempty convex subfamily of

$$\mathscr{P}_{h,L}^B = \{u \in \mathcal{C}_b(\Omega) : |u| \le B,\ u = h \text{ on } \Omega \setminus L\}.$$

Let $\mathscr{A}$ be its bounded global closure for all-domain local outer S-capacity convergence, modulo local relative-null equality. Thus membership means approximation by this specified classical family, with its one bound and its actual exterior values. For $a \in \mathscr{A}$ put $\mu_a = D(aS)$ and $\mu_h = D(hS)$.

51.1 Identification of the constraint in the faithful energy space

Proposition 51.1 (The completed collar constraint is an energy-closed convex set)**.** *The map*

$$a \longmapsto x_a = a - h \quad \text{in } L^2(\lambda)$$

identifies $\mathscr{A}$ with the nonempty closed bounded convex set

$$C = \overline{\{u - h : u \in \mathscr{P}\}}^{\,H} \subset H.$$

It is injective on the indicated relative classes and satisfies

$$\|x_a - x_b\|_H^2 = I_S(a,b) = \int (a-b)\, d(\mu_b - \mu_a). \tag{51.1}$$

For every original compact smooth scalar ϕ supported in W,

$$\mathcal{E}_S(x_a, \phi) = -\int \phi\, d(\mu_a - \mu_h). \tag{51.2}$$

Each class retains an actual representative bounded by B and equal to h on $\Omega \setminus L$.

Status: `NEW_PROVED` for the identification of this completed constraint with a closed convex subset of the faithful form. Its ingredients are the previously proved energy and completion theorems.

Proof. For $u \in \mathscr{P}$, the actual difference $u - h$ vanishes off L. Choose an original smooth cutoff equal to one near L and supported in W. Proposition R4–86.1, including its support-preserving core approximations, puts $u - h$ in H. The compact Green identity gives

$$\|u - v\|_H^2 = I_S(u,v) \qquad (u, v \in \mathscr{P}).$$

Relative CLN and the common bound give one uniform finite bound on $I_S(u,h)$, so these vectors form a bounded subset of H.

Suppose a defining row $u_j \in \mathscr{P}$ converges in all-domain local outer capacity to a bounded completed potential a. The completed weighted calculus, applied to the four terms defining $I_S(u_j, u_k)$, shows that $I_S(u_j, u_k) \to 0$ as $j, k \to \infty$. In each of these four weighted terms insert the fixed original compact smooth cutoff equal to one near L and supported in W. The full

signed energy is unchanged because $u_j - u_k$ vanishes off L; the individual terms are thereby legitimate compact test pairings. No finite total mass on all of Ω or weak convergence against a Borel indicator is used. For complete detail, failure would give indices $j_l, k_l \to \infty$ with a fixed positive lower bound; all four weighted terms along those two rows have the same completed limits with cancelling signs, a contradiction. Thus $u_j - h$ is Cauchy in H.

Its $L^2(\lambda)$ limit is $a - h$. Indeed the faithful reference potential is normalized and classical on Ω, so on the fixed compact L its measure satisfies $\lambda(E) \le \mathrm{Cap}_S(E;\Omega)$ for Borel E. Capacity convergence, the common bound, and a positive-gap splitting therefore give $L^2(\lambda)$ convergence. Off L all these differences are zero. Passing the energy identity through two defining rows proves (51.1).

Conversely, an H-Cauchy sequence $u_j - h$ with $u_j \in \mathscr{P}$ is capacity Cauchy by the fixed-collar energy-capacity estimate. Reverse framed domain comparison gives the all-domain local condition. The bounded completion theorem supplies a limit in $\mathscr{A}$; the preceding paragraph identifies its H limit. Hence the energy closure is precisely the displayed set C. Convexity follows either from the defining family or from its energy closure. If two classes have the same energy vector, their energy difference is zero; the same energy-capacity estimate makes every positive gap outer-null. This proves injectivity. The actual limsup representative from the bounded completion retains the common bound and exterior equality.

Finally Green's identity holds for each classical defining row by Proposition R4–86.1. Its left side passes by strong form convergence, and its right side by completed weak convergence on the fixed original test. This proves (51.2). □

51.2 An explicit family of linear inequalities on the datum

Let $V \Subset W$ be ordinary open, or let $V = W$. Set $M = \mathscr{D}_{S,0}(V)$, regarded as a closed subspace of H. The choice $V = W$ is allowed, with $M = H$. For an original compact smooth scalar ϕ supported in V, define the support functional of the prescribed classical constraints by

$$\Sigma_{\mathscr{P}}(\phi) = \sup_{u \in \mathscr{P}} \left[-\int_\Omega (u - h)\, D(\phi S) \right]. \tag{51.3}$$

This is finite. It involves only the prescribed classical family and the fixed coefficient measures $D(\phi S)$; it contains no unknown solution and no unknown completed Monge–Ampere measure.

A finite positive Radon datum ν on V is called finite-energy here if $\delta = \nu - \mu_h|_V$ satisfies

$$\|\delta\|_{-1,V} := \sup_{\substack{\phi \in A_c^{0,0}(V;\mathbf{R}) \\ \mathcal{E}_S(\phi,\phi) \le 1}} \left| \int_V \phi \, d\delta \right| < \infty. \tag{51.4}$$

This condition includes annihilation of energy-zero core functions. For example, $\delta = f\lambda|_V$ with $f \in L^2(V,\lambda)$ satisfies it, with norm at most $\sqrt{2}\|f\|_2$, by coercivity. Neither $D(aS) \ll \lambda$ for general bounded potentials nor an equality of energy-null and BT-null ideals is assumed.

Theorem 51.2 (A smooth-test alternative for completed solvability)**.** *For a finite-energy datum ν, the following are equivalent.*

1. *There is $a \in \mathscr{A}$ with $D(aS)|_V = \nu$ on all original compact smooth scalar tests, equivalently as Radon measures on V.*
2. *For every original real compact smooth ϕ supported in V,*

$$-\int_V \phi \, d(\nu - \mu_h) \le \Sigma_{\mathscr{P}}(\phi). \tag{51.5}$$

If these conditions fail, one such original smooth scalar test gives a strict reverse inequality. Thus nonsolvability has a single smooth-test separation certificate.

If the defining family additionally has one fixed actual exterior $u = e$ on $\Omega \setminus V$, then the solution class is unique. If its defining rows are trapped between fixed continuous barriers on $\overline{V}$ having the same boundary value g, the selected solution representative has the ordinary boundary limit g. These exterior and barrier conditions are hypotheses on the nonempty classical family; their availability for arbitrary boundary data is not asserted.

Status: `NEW_PROVED` for this exact finite-energy criterion over arbitrary fixed S. The separating-space argument is elementary Hilbert-space duality. The criterion retains the full prescribed classical constraints through the explicit functional (51.3); it is not a bounded-subsolution criterion and does not assert that every positive datum is feasible.

Proof. The datum functional $\phi \mapsto -\int \phi \, d\delta$ extends continuously to M. Let $y \in M$ be its Riesz vector:

$$\mathcal{E}_S(y, \phi) = -\int \phi \, d\delta \quad (\phi \in A_c^{0,0}(V)).$$

Let $P_M : H \to M$ be orthogonal projection and put $C_V = P_M C$. The set C is closed, bounded and convex in a Hilbert space, hence weakly compact. Its continuous linear image C_V is weakly compact and convex, hence norm closed. In particular it is not necessary to assume that the image of an arbitrary unbounded closed convex set under a linear map is closed.

For every $\phi \in M$ let $\sigma_{C_V}(\phi) = \sup_{z \in C_V} \mathcal{E}_S(z, \phi)$. On the original smooth core this equals $\Sigma_{\mathscr{P}}(\phi)$, by the energy closure defining C and Green's identity. Both support functionals are Lipschitz in the energy norm, with constant $\sup_{x \in C} \|x\|_H$. Core density therefore extends (51.5) to all $\phi \in M$.

If $y \in C_V$, all these inequalities hold. If $y \notin C_V$, let z be its nearest point in this closed convex set. The elementary projection inequality is

$$\mathcal{E}_S(y - z, b - z) \le 0 \qquad (b \in C_V).$$

For $\phi = y - z$ it gives

$$\mathcal{E}_S(y, \phi) - \sigma_{C_V}(\phi) \ge \|y - z\|_H^2 > 0.$$

Approximating ϕ in the energy norm by an original smooth core function preserves a strict positive gap. This proves the stated separation alternative and shows that the inequalities are equivalent to $y \in C_V$.

If $y = P_M x_a$ for some $a \in \mathscr{A}$, equation (51.2) gives the required equality on all original tests in V. Conversely such equality gives $P_M x_a = y$ by core density. This proves the equivalence. Original smooth scalars determine the Radon measures by the same original compact Stone–Weierstrass argument already used for the strong data metric.

For uniqueness under fixed actual exterior values, choose the actual representatives supplied by the common defining rows. Then $a - b = 0$ at every point of $\Omega \setminus V$, including ∂V. The completed weighted product is therefore exactly

$$(a - b)(\mu_b - \mu_a) = \mathbf{1}_V (a - b)(\mu_b - \mu_a) = 0$$

when the two interior equations agree. Thus $I_S(a, b) = 0$, and (51.1) and the capacity estimate give equality of the relative classes. No equality of their boundary measures is used. The actual limsup construction preserves the exterior values and both barriers, so continuity of the barriers at each ordinary boundary point gives the asserted boundary limit. □

51.3 Best feasible data and a quantitative certificate

Corollary 51.3 (The energy-distance defect is exactly a support gap)**.** *With the preceding notation, set* $d = \operatorname{dist}_H(y, C_V)$*. Then*

$$d = \sup_{\substack{\phi \in A_c^{0,0}(V;\mathbf{R}) \\ \|\phi\|_H \le 1}} \left[-\int_V \phi \, d(\nu - \mu_h) - \Sigma_{\mathscr{P}}(\phi) \right] = \min_{a \in \mathscr{A}} \|\nu - \mu_a|_V\|_{-1,V}. \tag{51.6}$$

The minimum is attained. Its image measure is unique; under the fixed actual exterior hypothesis its completed potential class is unique. The retraction onto this feasible image is firmly nonexpansive in the negative-energy norm. More precisely, if y, y' *are the Riesz vectors of two data and* z, z' *their nearest points in* C_V*, then*

$$\|z - z'\|_H^2 \le \mathcal{E}_S(z - z', y - y').$$

For $V = W$*, the feasible potential vectors themselves are the projections* z, z'*, so their energy distance is at most the data distance. For a proper interior domain the assertion concerns the projected vectors and the interior data; no unproved energy bound for the inverse of* $P_M|_C$ *is asserted.*

Proof. For any $z \in C_V$ and $\|\phi\|_H \le 1$, the bracket in (51.6) is at most $\mathcal{E}_S(y - z, \phi) \le \|y - z\|_H$. For the nearest point z, if $y \ne z$, the direction $\phi = (y - z)/\|y - z\|_H$ attains equality in the Hilbert space. Approximation by the smooth core, with rescaling to the unit ball, gives the same supremum over the displayed original tests. If $y = z$, every bracket is nonpositive and the zero test gives zero. The current pairing identifies the negative-energy norm of a data difference with $\|y - P_M x_a\|_H$, proving the minimum formula. Every point of C_V has a lift in C by its definition, so attainment gives an actual completed potential. Hilbert projection is unique, and original scalar tests distinguish the corresponding measures. Adding the two projection inequalities gives the displayed firm nonexpansiveness. The uniqueness and full-domain claims then follow from the theorem and from $P_M = \mathrm{id}$ when $V = W$. □

When the datum also lies in the admissible class of R4 and $V = W$, this projection agrees with the previous variational retraction. Indeed for $x = a - h$ the degree-one Aubin–Mabuchi energy is

$$\mathcal{E}_{\mathrm{AM}}(a) = \int x \, d\mu_h - \tfrac{1}{2}\|x\|_H^2,$$

so its datum objective equals $-\frac{1}{2}\|x - y\|_H^2 + \frac{1}{2}\|y\|_H^2$. For completed potentials, this identity follows by classical defining rows and their strong energy limits; when actual datum integrals are used, R4's capacity absolute continuity permits the same limit. The present finite-energy construction does not require that extra absolute continuity merely to define the Hilbert projection.

Frozen mixed slots. For an arbitrary original-positive closed (p, p)-background S_0, put $q = n - p \ge 1$ and fix $v_2, \dots, v_q \in \mathcal{C}_b(\Omega)$. Apply this entire section to $R = T_{S_0,q-1}(v_2, \dots, v_q)$. It gives the same exact criterion for the linear mixed equation

$$T_{S_0,q}(a, v_2, \dots, v_q)|_V = \nu.$$

Here a belongs to the completed constraint constructed for this fixed degree-one tail R, and the displayed left side means $D(aR)$. This does not assert that $a \in \mathcal{A}_{S_0,b}$ or that all of its slots may instead be varied in a jointly completed product over S_0. This is a qualified all-dimensional mixed result with one free slot. It does not turn the diagonal equation with $q > 1$ into a linear problem, and it does not establish an unrestricted bounded-subsolution theorem.

52 The feasibility inequalities themselves force finite energy

Status: `DIRECT_COROLLARY` of Theorem 51.2. This removes an a priori data assumption from the solvability alternative. Finite energy is still required for the quantitative Hilbert projection of arbitrary infeasible data.

Retain its fixed degree-one background, prescribed nonempty convex classical family $\mathscr{P}$, completed constraint $\mathscr{A}$, and ordinary interior domain V. Let ν be any positive Radon measure on V, initially only locally finite. Define $\delta = \nu - \mu_h|_V$ on compact subsets and use the same support functional $\Sigma_{\mathscr{P}}$ on original smooth compact scalars. Then the following are equivalent:

1. $D(aS)|_V = \nu$ for some $a \in \mathscr{A}$;
2. for every real original $\phi \in A_c^{0,0}(V)$,

$$-\int_V \phi \, d\delta \le \Sigma_{\mathscr{P}}(\phi).$$

If they hold, ν is in fact finite on V, δ has finite negative-energy norm, and all boundary and uniqueness conclusions of the preceding theorem apply. If they fail, there is an original compact smooth scalar giving a strict reverse inequality.

Proof. Set $R = \sup_{x \in C} \|x\|_H < \infty$. By the support-functional identity and Cauchy–Schwarz,

$$\Sigma_{\mathscr{P}}(\pm\phi) \le R\|\phi\|_H.$$

Applying the assumed feasibility inequality to both signs therefore gives

$$\left| \int_V \phi \, d\delta \right| \le R\|\phi\|_H.$$

Thus the datum defines a bounded functional on the original form core, annihilates its null vectors, and extends to the part space. The Riesz and projection proof of Theorem 51.2 now applies verbatim: its construction uses the datum only on these compact tests until it produces a completed potential. It gives $D(aS)|_V = \nu$ as Radon measures. The left side has finite mass on V, since $\overline{V}$ lies in the fixed compact collar stage and relative CLN applies to the common defining bound. Hence $\nu(V) < \infty$.

Conversely, a completed solution has its energy vector in C and hence satisfies every displayed support inequality. The strict failure alternative is the logical negation of these inequalities; for a finite-energy infeasible datum it is also supplied by the explicit Hilbert separation proof. If the datum has infinite negative-energy norm, the uniform bound R shows directly that at least one signed original smooth test violates the inequality. □

This is not a replacement of the condition by positivity or by a bounded subsolution. The right side retains the full prescribed ambient classical constraints, and the toric certificate shows that those constraints exclude some smooth positive dominated data. The same corollary applies to a fixed mixed tail exactly in the qualified sense stated at the end of Section 51.

53 Measure data and bounded subsolutions in the relative energy class

This section deliberately changes the admissible solution class. The solutions belong to the faithful relative energy space; they need not be ambient classical psh or belong to a bounded classical capacity completion. This distinction is essential even for smooth nonpolar backgrounds.

Fix an arbitrary original-positive symmetric separately closed $(n-1,n-1)$-current S on an ordinary norm-box neighborhood Ω in the allowed space. Let $F = \operatorname{supp} S$, let λ and $(\mathcal{E}, \mathscr{D})$ be the faithful measure and coercive regular form of Theorem R4–84.3, and fix an ordinary open $V \Subset \Omega$. Put

$$H = \mathscr{D}_0(V), \qquad \|u\|_H = \mathcal{E}(u,u)^{1/2}.$$

The part-space closure is the energy boundary condition. No pointwise boundary trace is built into that definition.

For a signed Radon measure σ on V, define its possibly infinite core dual norm by

$$\|\sigma\|_{-1,V} = \sup\left\{ \left| \int_V \phi \, d\sigma \right| : \phi \in A_c^{0,0}(V;\mathbf{R}),\ \|\phi\|_H \le 1 \right\}.$$

Finiteness includes vanishing on zero-energy core functions. Total mass on V need not be finite for this definition.

Lemma 53.1 (Positive finite-energy functionals preserve order)**.** *Suppose a positive Radon measure τ has finite core dual norm. Its core integral extends to a bounded functional ℓ_τ on H, and $\ell_\tau(v) \ge 0$ for every $v \in H$ with $v \ge 0$ λ-almost everywhere. If $\tau_1 \le \tau_2$ and both have finite core dual norm, their extended functionals have the same order.*

For every real $v \in H$,

$$\mathcal{E}(v, v^+) \ge \mathcal{E}(v^+, v^+).$$

Proof. Core density gives the unique bounded extension. To verify positivity without assuming an integral formula for arbitrary energy vectors, take core $v_j \to v$ in H. Smooth nonnegative normal contractions approximating t^+ give nonnegative core functions w_j whose energies are bounded by those of v_j, and with $w_j - v_j^+ \to 0$ in L^2. For example choose the smoothing error small enough on the finite λ-mass of each individual core support. Since $v_j^+ \to v$ in L^2, every weakly convergent subsequence of the bounded w_j in H has limit v. Their nonnegative functional values therefore give $\ell_\tau(v) \ge 0$. Apply this argument to $\tau_2 - \tau_1$ for the order assertion.

The normal-contraction property from the frozen form theorem gives $v^+, v^- \in H$ and $\mathcal{E}(|v|,|v|) \le \mathcal{E}(v,v)$. Expanding the two quadratic forms yields $\mathcal{E}(v^+, v^-) \le 0$. Since $v = v^+ - v^-$, the displayed inequality follows. Thus no extra strong-local chain rule is needed for the order argument. □

Theorem 53.2 (Finite-energy measure Dirichlet problem)**.** *Every signed Radon measure σ with finite $\|\sigma\|_{-1,V}$ has a unique $u_\sigma \in H$ satisfying*

$$\mathcal{E}(u_\sigma, \phi) = -\ell_\sigma(\phi) \qquad (\phi \in H).$$

Its coefficient-defined original current satisfies

$$D(u_\sigma S) = \sigma \quad \text{on } V$$

on every original compact smooth scalar test. The solution map is linear and

$$\|u_\sigma\|_H = \|\sigma\|_{-1,V}, \qquad \|u_\sigma - u_{\sigma'}\|_H = \|\sigma - \sigma'\|_{-1,V}.$$

For positive data the solution is nonpositive almost everywhere. If $0 \le \sigma \le \sigma'$ and both norms are finite, then $u_{\sigma'} \le u_\sigma \le 0$ almost everywhere and for the canonical energy-quasicontinuous representatives outside an energy-polar set.

Status: `DIRECT_COROLLARY` of the R4 faithful coercive form, now applied to finite-energy Radon data instead of only $L^2(\lambda)$ densities. It is not an ambient psh existence theorem.

Proof. Riesz representation in the energy Hilbert space gives the solution, uniqueness, and both equalities. Coefficient domination and coercivity make $u_\sigma S$ a well-defined current: on every compact stage its coefficient variations are bounded by a fixed multiple of λ, and $u_\sigma \in L^2(\lambda) \subset L^1_{\mathrm{loc}}(\lambda)$. Approximation by the original core passes compact Green's identity to

$$\mathcal{E}(u_\sigma, \phi) = -\langle D(u_\sigma S), \phi\rangle \qquad (\phi \in A_c^{0,0}(V; \mathbf{R})).$$

This proves the asserted full original-test equation.

For positive σ, test the weak equation with u_σ^+. The preceding lemma gives

$$0 \le \mathcal{E}(u_\sigma^+, u_\sigma^+) \le \mathcal{E}(u_\sigma, u_\sigma^+) = -\ell_\sigma(u_\sigma^+) \le 0.$$

Coercivity implies $u_\sigma \le 0$. Apply the identical argument to $u_{\sigma'} - u_\sigma$, whose datum is positive, for order comparison. A zero vector has zero canonical quasicontinuous representative outside an energy-polar set; normal contraction gives the stated representative version of these inequalities. This does not authorize arbitrary changes on merely λ-null sets. □

Theorem 53.3 (Bounded subsolution domination in the energy class)**.** *Suppose $w \in H \cap L^\infty(\lambda)$ and its original curvature $\tau = D(wS)$ on V is a positive Radon measure. Then every positive Radon measure $0 \le \mu \le \tau$ has finite energy and admits the unique solution from the preceding theorem, with*

$$w \le u_\mu \le 0, \qquad \|u_\mu\|_H \le \|w\|_H, \qquad \|u_\mu\|_\infty \le \|w\|_\infty.$$

The order and bound first hold almost everywhere, and hold for canonical energy representatives outside an energy-polar set.

If in addition w has a continuous energy representative w_0 on $V \cap F$ extending to a bounded continuous function on $\overline{V}$, with $w_0 \le 0$ on V and $w_0 = 0$ on the ordinary boundary, then u_μ has a bounded actual Borel representative u_0 on V such that

$$w_0 \le u_0 \le 0$$

at every point. Consequently u_0 has ordinary boundary limit zero. It represents the same current and canonical relative energy class. No interior continuity or classical psh property is asserted.

Status: `DIRECT_COROLLARY` with a complete proof of the datum bound. This is a bounded-subsolution criterion in the larger relative energy class; the corresponding unrestricted claim in the bounded classical completion remains false.

Proof. For a real core function ϕ set $P_\epsilon(t) = \sqrt{t^2 + \epsilon^2} - \epsilon$. It is nonnegative, smooth, vanishes at zero, and has derivative of absolute value at most one. Thus $P_\epsilon(\phi)$ is a nonnegative original core function and its energy is at most that of ϕ. The original current equation for w and compact Green give

$$\int P_\epsilon(\phi)\, d\tau = -\mathcal{E}(w, P_\epsilon(\phi)) \le \|w\|_H \|\phi\|_H.$$

Let $\epsilon \downarrow 0$ using monotone convergence on the compact support. Since $\mu \le \tau$,

$$\left|\int \phi\, d\mu\right| \le \int |\phi|\, d\mu \le \int |\phi|\, d\tau \le \|w\|_H \|\phi\|_H.$$

This proves the required dual bound without assuming finite total mass near ∂V. The same argument shows that τ has finite energy. Its energy solution is exactly w, by uniqueness and the original core equation. The order comparison gives $w \le u_\mu \le 0$ and the stated bounds.

For the last assertion take the canonical energy representative on F. Outside one energy-polar exceptional set it is finite and lies between w_0 and zero. Assign value zero on that exceptional set and off F. This gives a Borel representative with the displayed pointwise bounds. The exceptional set is λ-null, hence these changes preserve the coefficient-defined current; by the one-way energy-to-BT capacity theorem they also preserve the canonical relative capacitary class. The squeeze between w_0 and zero proves the ordinary boundary limit. The role of w_0 is explicit; a part-space boundary condition alone was not turned into an ordinary pointwise boundary assertion. □

Proposition 53.4 (Relative Dini-capacity growth supplies finite energy)**.** *Let $H_0 \Subset V$ be compact and let μ be a positive Radon measure supported in H_0, of mass $M > 0$. Let $\omega : (0,\infty) \to [0,\infty)$ be finite and nondecreasing, and assume*

$$\int_0^r \omega(s)\,\frac{ds}{s} < \infty \quad \textit{for some } r > 0.$$

Suppose for every Borel $E \subset H_0$,

$$\mu(E) \le A\ \mathrm{Cap}_S(E;\Omega)\omega(\mathrm{Cap}_S(E;\Omega)),$$

where the right side is zero when the capacity is zero. Then μ has finite energy and hence a unique relative energy Dirichlet solution. If C_{H_0} is the constant in the frozen one-way capacity comparison, then

$$\|\mu\|_{-1,V} \le \sqrt{3}\,C_{H_0} \inf_{r>0}\left\{\frac{M}{r} + A\int_0^r \omega(s)\,\frac{ds}{s}\right\}. \tag{53.1}$$

For $\omega(s) = s^\alpha$, $\alpha > 0$, the right side is at most

$$\sqrt{3}\,C_{H_0}(1+\alpha^{-1})A^{1/(1+\alpha)}M^{\alpha/(1+\alpha)}.$$

These hypotheses alone do not assert boundedness, ambient psh, or membership in the bounded classical completion of the solution.

Status: `DIRECT_COROLLARY` of Theorem R4–85.2, by the following layer-cake calculation. The two capacities are not identified.

Proof. For an original real core function ϕ, put $e = \|\phi\|_H$. The energy capacity definition, applied to $|\phi|/t$, and coercivity give

$$\mathrm{cap}_{\mathcal{E},1}(\{|\phi| > t\} \cap F) \le \frac{\mathcal{E}_1(\phi,\phi)}{t^2} \le \frac{3e^2}{t^2}.$$

Consequently, with $C = \sqrt{3}\,C_{H_0}$,

$$\mathrm{Cap}_S(H_0 \cap \{|\phi| > t\};\Omega) \le \mathrm{Cap}_S^*(H_0 \cap \{|\phi| > t\};\Omega) \le Ce/t.$$

The growth hypothesis and monotonicity of $s\omega(s)$ imply

$$\mu\{|\phi| > t\} \le \min\{M, A(Ce/t)\omega(Ce/t)\}.$$

If $e = 0$, the same capacity estimate makes the integral zero. Otherwise split its layer-cake integral at $t = Ce/r$, and substitute $s = Ce/t$ in the upper part. One obtains

$$\int |\phi|\,d\mu \le Ce\left[\frac{M}{r} + A\int_0^r \omega(s)\,\frac{ds}{s}\right].$$

Take the core supremum and then the infimum over r. The Dini assumption makes these quantities finite (also for every other finite r, by local boundedness of ω). For a power choose $r = (M/A)^{1/(1+\alpha)}$ when $A > 0$; the zero-datum case is trivial. □

Frozen mixed slots. All statements apply with $R = T_{S_0,q-1}(v_2, \dots, v_q)$ in place of S, for any arbitrary allowed original background S_0 and fixed bounded classical slots. The solution then lies in the faithful part energy space for R and satisfies $D(uR) = \mu$. This is a linear equation in its one unfrozen potential, not a diagonal nonlinear equation with q moving energy potentials. In particular no new arbitrary-current wedge or unproved completion inclusion is being introduced.

54 Sharp Hardy control from a bounded energy subsolution

Retain the arbitrary degree-one background, faithful measure λ, and part Hilbert space $H = \mathscr{D}_0(V)$ of Section 53. The potential in this section is an energy vector, not an assumed classical or completed psh function. In particular its values on an arbitrary λ-null set are never used to define a measure multiplier.

Lemma 54.1 (The bounded chain test in the closed core)**.** *Let $h \in H$ satisfy $-B \le h \le 0$ λ-almost everywhere, where $B < \infty$. For an original real compact smooth f in V and $\varepsilon > 0$, the function class*

$$\psi = \frac{f^2}{\varepsilon - h}$$

belongs to H and has support in the compact support of f. Its form pairing satisfies

$$-\mathcal{E}(h, \psi) \le \mathcal{E}(f, f). \tag{54.1}$$

No integral of ψ against $D(hS)$ is required to formulate or prove this assertion.

Status: `DIRECT_COROLLARY` of the closed-core gradient calculus. The following construction records the approximation and admissibility explicitly.

Proof. For original core functions a, b, their energy measures $\Gamma_S(a, b)$ are absolutely continuous with respect to λ. The original positive-square inequality gives their measure-valued Cauchy–Schwarz inequality. One may therefore form the Hilbert module generated by symbols da with bounded Borel coefficients, with

$$\left\| \sum_i c_i \, da_i \right\|^2 = \int \sum_{i,j} c_i c_j \, d\Gamma_S(a_i, a_j),$$

first for finite sums and then by quotient and completion. Positivity holds pointwise after taking densities; for any finite family the matrix is positive semidefinite almost everywhere, as can be tested on rational coefficient vectors. Bounded Borel multiplication is well defined in this module.

If h_j is an original core sequence converging to h in H, then dh_j is Cauchy and defines dh. The energy-measure inequality also shows that $\Gamma_S(h_j, h_j)$ is Cauchy in total variation, with limit absolutely continuous with respect to λ. In particular

$$\|dh\|^2 = \mathcal{E}(h, h), \qquad \|c \, dh\|^2 = \int c^2 \, d\Gamma_S(h).$$

These are closed-form objects, independent of any actual representative of h on a null set.

For a smooth function $\Phi : \mathbf{R} \to \mathbf{R}$ with $\Phi(0) = 0$ and bounded derivative, this construction proves the chain rule directly. Pass to a subsequence with $h_j \to h$ λ-almost everywhere. The smooth core rule and

$$\begin{aligned} \Phi'(h_j) dh_j - \Phi'(h) dh &= \Phi'(h_j)(dh_j - dh) \\ &\quad + (\Phi'(h_j) - \Phi'(h)) dh \end{aligned}$$

show convergence in the gradient module. The second term tends to zero by bounded convergence against the fixed $\Gamma_S(h) \ll \lambda$. Lipschitz continuity gives the $L^2(\lambda)$ limit $\Phi(h)$, so closability yields $\Phi(h) \in H$ and $d\Phi(h) = \Phi'(h)dh$. The approximants are still original core functions in V, since $\Phi(0) = 0$.

Choose such a smooth global Φ agreeing with $(\varepsilon - s)^{-1} - \varepsilon^{-1}$ on $[-B, 0]$. Multiplication by a fixed original compact smooth function is bounded in the form norm, by the frozen coefficient estimate. Hence

$$\psi = \varepsilon^{-1} f^2 + f^2 \Phi(h) \in H,$$

with original core approximations supported inside $\operatorname{supp} f$. The product and chain rules give

$$d\psi = \frac{2f}{\varepsilon - h}\, df + \frac{f^2}{(\varepsilon - h)^2}\, dh.$$

The nonnegative module square is now exactly

$$\left\| df + \frac{f}{\varepsilon - h} dh \right\|^2 = \mathcal{E}(f, f) + \mathcal{E}(h, \psi) \geq 0.$$

This proves (54.1). Writing $v = \varepsilon - h$ would give the usual Picone identity, but no global form membership of the constant ε is needed. □

Theorem 54.2 (The energy-subsolution Hardy constant)**.** *Let $h \in H \cap L^\infty(\lambda)$ and suppose that $\tau = D(hS)$ is a positive Radon measure on V. Put $B = \|h\|_{L^\infty(\lambda)}$. For every positive Radon measure $0 \leq \mu \leq \tau$ and every original real compact smooth f in V,*

$$\int f^2 \, d\mu \leq B\mathcal{E}(f, f). \tag{54.2}$$

The coefficient one multiplying B is optimal.

The core restriction map extends uniquely to

$$J_\mu : H \longrightarrow L^2(\mu), \qquad \|J_\mu u\|_2 \leq \sqrt{B}\, \|u\|_H.$$

It is the trace of the canonical part-form energy-quasicontinuous representative, not of an arbitrary λ-almost-everywhere representative. The bounded energy solution from Theorem 53.3 still satisfies $h \leq u_\mu \leq 0$. If $M = \mu(V) < \infty$, its norm also obeys

$$\|u_\mu\|_H = \|\mu\|_{-1,V} \leq \min\{\|h\|_H, \sqrt{BM}\}. \tag{54.3}$$

No membership of h or u_μ in a psh class is asserted.

Status: `DIRECT_COROLLARY` of the closed-core Picone identity and the already proved positive-functional and bounded-energy-subsolution theorems. The sharp constant and the measure-trace consequence are included in full.

Proof. The original core Green identity identifies $\ell_\tau = -\mathcal{E}(h, \cdot)$, so τ has finite core dual norm. The bounded-energy-subsolution theorem gives finite energy for μ, and the positive-functional lemma gives their extensions with $0 \leq \ell_\mu \leq \ell_\tau$ on the positive cone of H. That same theorem gives $-B \leq h \leq 0$ in the energy order.

For the test ψ of the preceding lemma, the almost-everywhere inequality $\psi \geq f^2/(\varepsilon + B)$ is an inequality between vectors of H. Consequently

$$\frac{1}{\varepsilon + B} \int f^2 \, d\mu \leq \ell_\mu(\psi) \leq \ell_\tau(\psi) = -\mathcal{E}(h, \psi) \leq \mathcal{E}(f, f).$$

Only the first integral is an integral of an original smooth core function. In particular this argument has not presumed an actual integral formula for ψ, nor used the trace inequality it is proving. Let $\varepsilon \downarrow 0$ to obtain (54.2); the case $B = 0$ follows too.

The inequality makes original core approximations Cauchy in $L^2(\mu)$ and gives the unique bounded map J_μ. It also shows that μ charges no part-form energy-polar Borel set: for a compact set, apply the compact-core formula for the part energy capacity to functions at least one on it, and then use Radon inner regularity. A form-convergent core sequence has a subsequence converging outside an energy-polar set to the canonical energy-quasicontinuous representative. Its $L^2(\mu)$ limit is therefore that representative μ-almost everywhere. This identifies the trace without authorizing arbitrary changes on a merely λ-null set.

The already proved bounded-energy-subsolution theorem supplies the order bounds and $\|u_\mu\|_H \le \|h\|_H$. If $M < \infty$, ordinary Cauchy–Schwarz for a core f and (54.2) give

$$\left| \int f \, d\mu \right| \le \sqrt{M} \left(\int f^2 \, d\mu \right)^{1/2} \le \sqrt{BM} \, \|f\|_H .$$

Riesz isometry proves (54.3). □

Example 54.3 (Sharpness with an original annular tent)**.** Let $S = [1]$ on an annulus containing $V = \{|\pi| < |T| < |\pi|^{-1}\}$, and put $s = \log|T|/\lambda$, $\lambda = -\log|\pi|$. For any $B > 0$ the function

$$h = B(|s| - 1) \quad (-1 < s < 1)$$

belongs to $\mathscr{D}_0(V)$, has essential supremum norm B, and satisfies $Dh = (2B/\lambda)\delta_{\sigma(0)}$ on V. Its part-space membership follows by cutting off its zero-boundary linear tails and smoothing; both the real one-dimensional energy and the bounded actual functions converge. Original smooth compact approximations of $f(s) = 1 - |s|$ have

$$\int f^2 \, d(Dh) \longrightarrow 2B/\lambda, \qquad \mathcal{E}(f, f) \longrightarrow 2/\lambda.$$

Thus the constant B in the trace inequality cannot be reduced. The same example attains the height-mass norm bound in the limit.

Corollary 54.4 (Form-small perturbations from a bounded energy barrier)**.** *Under the theorem's assumptions, if $c \ge 0$ and $cB < 1$, then*

$$\mathcal{B}_c(u, v) = \mathcal{E}(u, v) - c \int (J_\mu u)(J_\mu v) \, d\mu$$

is a bounded coercive form on H, with $\mathcal{B}_c(u, u) \ge (1 - cB)\|u\|_H^2$. Every functional $L \in H^*$ therefore has a unique weak solution $\mathcal{B}_c(u, \phi) = L(\phi)$, and $\|u\|_H \le \|L\|/(1 - cB)$. For $L = -\ell_\sigma$ this is, on all original compact tests,

$$D(uS) + c(J_\mu u)\mu = \sigma.$$

The product is locally integrable by the trace bound and local Radon finiteness. This is a linear energy equation; no Markov property of the perturbed form and no classical psh solution are asserted.

Proof. The trace inequality gives boundedness and the displayed lower bound. The coercive Hilbert-space Riesz theorem proves existence, uniqueness and the norm estimate. The original core Green identity gives the stated sign of the distributional equation. □

For the annular example, with $\mu = Dh$, one has $\int h^2 \, d\mu = B\mathcal{E}(h, h)$. At $cB = 1$ the nonzero vector h is even a weak kernel vector, since $h = -B$ at the support of μ and $\mathcal{E}(h, \phi) = -\int \phi \, d\mu$. For $cB > 1$ the form is negative on h. Thus the universal strict coercivity threshold cannot be enlarged under only the stated barrier-height hypothesis.

55 A bounded continuous energy solution with no local classical potential

Retain $n \geq 3$, the full torus box Ω, its smooth background $S = \beta^{n-1}/(n-1)!$, the real compact set A, and the probability ϑ from Section 38. Scale A inside a real ball B with $\overline{B} \Subset Q$ and put

$$V = \operatorname{trop}^{-1}(B), \qquad K = \sigma(A), \qquad \nu = \sigma_* \vartheta.$$

The ordinary domain V is relatively compact in Ω.

Lemma 55.1 (A bounded local potential can be extended near Gauss points)**.** *Let $x \in Q$, let O be an ordinary neighborhood of $\sigma(x)$, and let $v \in \mathcal{C}_b(O)$. There are a real box $Q_x \ni x$, an ordinary neighborhood N_x of $\sigma(x)$ inside $O \cap \operatorname{trop}^{-1}(Q_x)$, and a bounded classical psh function U_x on the* full *polyannulus $\operatorname{trop}^{-1}(Q_x)$ such that*

$$U_x - v = M \log |P| - c \quad \text{on } N_x$$

for a Laurent polynomial P that is an analytic unit there, and finite constants $M \geq 0, c \in \mathbf{R}$. Consequently $D(U_x S) = D(vS)$ on N_x.

If a family of such v has one bound $|v| \leq H$ on a fixed ordinary neighborhood, the box, N_x, P, M, c, and the bound for U_x can all be chosen independently of the member of that family.

Status: `NEW_PROVED` elementary local extension for this Gauss-section geometry. A general ordinary neighborhood is not assumed to contain a full tropical box.

Proof. Shrink the given ordinary neighborhood so that $|v| \leq H$ there. The topology of the affine torus analytification is generated by absolute values of Laurent polynomials. A finite basic neighborhood therefore gives nonzero Laurent polynomials $F_1, \dots, F_m$ and strict lower and upper bounds around their values at $\sigma(x)$. For every analytic point y of the torus,

$$|F_l(y)| \leq |F_l(\sigma(\operatorname{trop} y))|$$

by the ultrametric inequality on its Laurent monomials. The Gauss norms on the right vary continuously with the real coordinates. After shrinking a real box Q_x around x, all upper bounds of the basic neighborhood consequently hold on its entire tropical inverse image. Thus for suitable $a_l < |F_l(\sigma(x))|$ the set

$$D = \{y : \operatorname{trop} y \in Q_x,\ |F_l(y)| > a_l \ (1 \leq l \leq m)\}$$

lies in the fixed ordinary neighborhood where the bound holds. If no lower bound is needed, this is already a full polyannulus; take $U_x = v + H$ and the assertion is immediate.

Otherwise set $c_l = \log |F_l(\sigma(x))|$, $\delta = \min_l(c_l - \log a_l) > 0$, $C = \sum_l c_l$, and $P = \prod_l F_l$. Shrink Q_x further so that

$$\left| \log |F_l(\sigma(s))| - c_l \right| < \delta/(4m) \quad (s \in Q_x).$$

On the Gauss section over Q_x one has $\log |P| \geq C - \delta/4$. At any relative boundary point of D inside the full polyannulus, at least one lower bound is an equality; the remaining Gauss upper bounds imply $\log |P| \leq C - 3\delta/4$ there. Put $L = C - \delta/2$ and choose M with $M\delta/8 > H + 1$. Define

$$U_x = \max\{0, v + M(\log |P| - L)\} \quad \text{on } D, \qquad U_x = 0 \quad \text{off } D.$$

Near each relative boundary point the expression inside the maximum is negative, by the uniform bound on v and continuity of $\log |P|$. Thus the two definitions agree with the constant

zero on an ordinary neighborhood of every gluing point. They define a bounded classical psh function on the full polyannulus. Its bound is at most $\max(0, H + 3M\delta/4)$.

On the ordinary open subset of D where $\log|P| - L > (H+1)/M$, the second branch is strictly positive. This subset contains the whole Gauss section over the small box. There $U_x - v = M\log|P| - ML$, an original smooth pluriharmonic function since P is a unit. Separate closedness of S gives the current equality. All choices used only the fixed neighborhood and the bound H, proving the uniform assertion. □

Proposition 55.2 (Local outer-nullity for every bounded classical trace)**.** *Let O be any ordinary neighborhood of K in the torus, with S restricted there. Every $v \in \mathcal{C}_b(O)$ satisfies*

$$D(vS)(K) = 0.$$

More locally, on each sufficiently small ordinary neighborhood of a point of K, the corresponding portion of K has relative outer BT capacity zero for normalized testers on that neighborhood. The same nonconcentration holds for locally uniformly bounded completed relative potentials.

Proof. At each point of K apply the preceding lemma, and then restrict to a smaller real box with compact closure in Q_x. The full-box Gauss-section growth estimate of Lemma 38.1, after a fixed affine normalization of the bounded U_x, implies that $D(U_xS)$ gives zero mass to every Gauss subset of zero $\mathcal{H}^{n-1}$ measure in that smaller box. On the protected ordinary neighborhood its curvature equals $D(vS)$. Finitely many such neighborhoods cover K, proving the first assertion.

For the capacity assertion start with normalized $0 \le v \le 1$ on one fixed ordinary neighborhood. The extension lemma uses the same data and the same bound for every such tester. Cover a compact portion of K in the protected neighborhood by small real balls with arbitrarily small sum of $(n-1)$st radius powers. Intersect their ordinary tropical inverse images with a fixed relatively compact protected neighborhood. The full-box mass estimate gives arbitrarily small capacity for these ordinary open neighborhoods, uniformly over the original normalized class. This proves the local outer-null assertion on its stated capacity domain.

For completed potentials use their locally uniformly bounded defining rows on smaller ordinary neighborhoods. The same construction and normalization give the same uniform open-neighborhood mass bound. Ordinary weak convergence and the open-set inequality pass that bound to the positive completed curvature measure. Letting the bound tend to zero proves the last assertion. No restriction to K is passed directly through a weak limit. □

Theorem 55.3 (Finite energy and bounded continuous solution without a psh trace)**.** *The probability ν has finite core energy norm on V. There is a bounded continuous actual function a on $\overline{V}$, zero on the ordinary boundary, such that*

$$a \in \mathscr{D}_0(V), \qquad D(aS) = \nu \quad \text{on } V$$

on every original compact smooth scalar test. This is the unique energy-class solution.

Nevertheless, no ordinary neighborhood O of K has a locally bounded actual classical psh potential v with $D(vS) = \nu|_O$. Nor can such an equation be solved by a locally uniformly bounded completed relative potential. In particular there is no bounded classical subsolution whose relative curvature dominates ν near K.

Status: `NEW_PROVED` separation of solution classes and local representability, using the preceding null-set construction. The energy Riesz theorem itself remains a direct input.

Proof. For an original compact smooth f in V, the Newton trace inequality and the all-original-test restriction-energy bound give

$$\left|\int f\,d\nu\right|^2 = \left|\int f\circ\sigma\,d\vartheta\right|^2 \le C_n I(\vartheta)\int |\nabla(f\circ\sigma)|^2 \le C_n I(\vartheta)\mathcal{E}_S(f).$$

Thus ν has finite energy.

We construct its bounded continuous solution explicitly. Let $G(x) = c_n|x|^{2-n}$ be the positive Newton kernel with $-\Delta G = \delta_0$, and put $P = G * \vartheta$. The Frostman bound used in Lemma 38.3 implies uniformly in x

$$\int_{|x-y|<r} |x-y|^{2-n}\,d\vartheta(y) \le Cr^d, \qquad d = \log 2/\log 3 > 0.$$

Splitting the kernel into this small tail and its continuous far part proves that P is bounded and continuous. Its weak gradient is locally square integrable: the vector-kernel convolution bound from the same lemma, applied first to smooth approximations of ϑ, bounds its gradient norm by a fixed multiple of $I(\vartheta)^{1/2}$. Passing weakly and using local uniform convergence identifies the gradient of P.

The measure is supported away from ∂B, so P is smooth near that sphere. Let H be the ordinary real Poisson extension to B of its boundary values, and set $U = H - P$. Equivalently, $-U$ is the positive Dirichlet Green potential of ϑ in the ball. Hence

$$U \in C(\overline{B}) \cap H^1(B), \quad U < 0 \text{ on } B, \quad U|_{\partial B} = 0, \quad \Delta U = \vartheta.$$

The standard ball Poisson kernel gives the displayed harmonic extension; positivity of its Green kernel gives the sign. These are ordinary real integral identities, not an elliptic regularity claim for a Berkovich distribution.

To verify the zero-boundary energy class directly, put $U_\epsilon = \min(U+\epsilon, 0)$. It is continuous with compact support in B. The Sobolev contraction rule and $U < 0$ show $U_\epsilon \to U$ in $H^1(B)$: the gradient difference is supported where $-\epsilon \le U < 0$ and tends to zero by dominated convergence. Mollification then gives real smooth compact functions converging to U in energy and uniformly on B. Their toric pullbacks are original compact smooth functions in V, and (38.1) makes their energy differences exactly the real ones. The frozen coercivity and uniform convergence identify their part-space limit with the actual bounded continuous function $a = U \circ \mathrm{trop}$. Thus $a \in \mathscr{D}_0(V)$. Continuity on $\overline{V}$ and the ordinary zero boundary value follow from the real boundary values and proper tropicalization.

It remains to check the equation on all original tests. Extend U by zero beyond B and convolve on the real box. The resulting smooth U_j converge locally uniformly to U. On each fixed compact subset of B, for sufficiently small convolution radius, $\Delta U_j = \vartheta * \kappa_j$; any boundary contribution is outside that compact set. For an arbitrary original compact smooth scalar test η in V, the original torus calibration gives

$$\langle D((U_j\circ\mathrm{trop})S), \eta\rangle = \int_B (\eta\circ\sigma)\,\Delta U_j\,dx \longrightarrow \int_A \eta\circ\sigma\,d\vartheta.$$

Here $\eta\circ\sigma$ is a continuous compact test, not assumed smooth in one real chart. On the other hand, uniform coefficient multiplication of the fixed S gives convergence of the left-hand currents to $D(aS)$. This proves the equation on the full original test class. Energy uniqueness is the frozen Riesz uniqueness theorem.

Finally, Proposition 55.2 makes every locally bounded classical or locally uniformly bounded completed relative curvature give zero mass to K. Since $\nu(K) = 1$, it cannot equal or be dominated by such a curvature. This proves all the asserted nonrepresentability statements. □

Scope. The background is smooth and fixed, the solution is actually bounded and continuous, and the failure is local near its compact datum. It is therefore stronger than a failure of one prescribed boundary value problem. The successful solution belongs to the larger faithful energy class. It supplies neither a new ambient psh solution nor a counterexample to general relative compact no-gain or Borel outer regularity.

56 Consequences of the toric capacity and energy separation

Status: `DIRECT_COROLLARY`. Retain the notation of Theorems 38.4 and 55.3; in particular $n \geq 3$, $K = \sigma(A)$, and $\nu(K) = 1$.

No local difference of bounded classical psh potentials. There is no locally bounded actual function b near K which is locally a difference of two members of $\mathcal{C}_b$ and satisfies $D(bS) = \nu$. Indeed, on each ordinary neighborhood with $b = u - v$, Proposition 55.2 makes both positive measures $D(uS)$ and $D(vS)$ vanish on the corresponding portion of K. Their signed difference therefore vanishes there. A finite ordinary covering of the compact K gives $D(bS)(K) = 0$, contradicting $\nu(K) = 1$. The same argument applies to local differences of locally uniformly bounded completed relative potentials. In particular the explicit bounded continuous energy solution in Theorem 55.3 lies outside the local bounded DSH class near its datum. This statement does not assert a failure of the frozen potential-quotient theorem, whose hypotheses require an actual classical difference from the start.

Relative BT nullity does not mean ambient pluripolarity. Fix $x \in A$. Choose $M > 0$ larger than every positive coordinate increment $t_i - x_i$ for $t \in Q$, and put

$$w_x(t) = M^{-1} \max(0, t_1 - x_1, \dots, t_n - x_n).$$

It is a normalized continuous convex function on Q, so its toric pullback is an actual ambient classical tester on the whole Ω. The original torus slope-simplex calibration gives

$$\big(D(w_x \circ \mathrm{trop})\big)^n = M^{-n} \delta_{\sigma(x)}.$$

Indeed the only full-dimensional slope polytope is $M^{-1} \operatorname{conv}(0, e_1, \dots, e_n)$ at x; its volume is $M^{-n}/n!$, canceled by the original top-degree factor $n!$. The identity is valid on all original scalar tests, by the same convex smoothing calibration used for the frozen radial examples. Consequently

$$\mathrm{Cap}_{[1]}(K; \Omega) \geq M^{-n} > 0, \qquad \mathrm{Cap}^*_S(K; \Omega) = 0.$$

The tester can be restricted to any ordinary neighborhood of $\sigma(x)$ while preserving its normalization and its atom. Thus the frozen local ambient polar outer-zero theorem (Corollary R4–26.3) excludes a local pluripolar witness at every point $\sigma(x)$ of K. The discrepancy already occurs for the smooth fixed S used here; it does not depend on a background carried by an ambient pluripolar set.

57 An explicit energy-dual certificate for the nonpolar obstruction

Status: `DIRECT_COROLLARY` of the new separation criterion, with the following explicit real-variable calculation. The underlying failure of a bounded-subsolution theorem is the frozen R4 counterexample. The added conclusion is a strictly positive, quantified distance from the

datum to the feasible completed image in the negative-energy norm, witnessed by an original compact smooth test.

Let $n \geq 2$, $t_i = \log |T_i|$ on the original n-torus, and

$$B_1 = \{x \in \mathbf{R}^n : |x| < 1\}, \qquad V = t^{-1}(B_1), \qquad \beta = \sum_i d't_i \wedge d''t_i, \qquad S = [\beta^{n-1}].$$

Choose a larger ordinary norm-box domain Ω in the torus, containing $\overline{V}$, and ordinary W and compact L with $\overline{V} \subset L \Subset W \Subset \Omega$. Put $h(t) = (|t|^2 - 1)/2$ and choose a common bound larger than its bound on Ω. Use the nonempty convex family

$$\mathscr{P} = \{u \in \mathcal{C}_b(\Omega) : u = h \text{ on } \Omega \setminus V, \quad h \leq u \leq 0 \text{ on } V\}.$$

All its members have one common global bound and collar. Its selected completed limits retain ordinary boundary value zero.

Take a nonzero radial $f \in C_c^\infty(B_1)$ with $0 \leq f \leq 1$, supported in $\{|x| < r_0\}$ for some $r_0 < 1$. Set

$$\nu = f(t)\beta^n.$$

The smooth subsolution h has $D(hS) = \beta^n \geq \nu$ on V. The signed datum $\delta = \nu - \mu_h|_V$ has finite energy: on the compact $\overline{V}$, the smooth coefficient measure β^n has a bounded density against the faithful λ, so its density and that of δ belong to $L^2(\lambda)$.

Proposition 57.1 (A positive distance despite a bounded subsolution)**.** *There are $\eta \in C_c^\infty(B_1)$, $\eta \geq 0$, and $c > 0$ such that*

$$\inf_{a \in \mathscr{A}} \|\nu - D(aS)|_V\|_{-1,V} \;\geq \sqrt{(n-1)!}\, \frac{c \int_{B_1} \eta \, dx}{\|\partial_1 \eta\|_{L^2(B_1)}} \;> 0. \tag{57.1}$$

In particular the smooth-test feasibility inequality fails for some original compact smooth scalar on V. Here $\mathscr{A}$ is precisely the bounded global capacity closure of the displayed family, not an unspecified local sheaf.

Proof. Write $f(x) = f_0(|x|)$ and consider the explicit radial function

$$U_f(r) = -\int_r^1 s^{1-n} \left(n \int_0^s f_0(t) t^{n-1} \, dt \right) ds.$$

It is the smooth real solution of $\Delta U_f = nf$ with zero boundary value; the integral is regular at zero because its inner integral is $O(s^n)$. Moreover $h(x) \leq U_f(x) \leq 0$ because $0 \leq f \leq 1$. For $r > r_0$, with

$$M = n \int_0^{r_0} f_0(t) t^{n-1} \, dt > 0,$$

one has $U_f'(r) = Mr^{1-n}$ and $U_f''(r) = (1-n)Mr^{-n} < 0$. Choose $x_* = (r_*, 0, \ldots, 0)$ with $r_0 < r_* < 1$. On a sufficiently small real neighborhood of x_*, $\partial_1^2 U_f \leq -c$ for some $c > 0$. Choose a nonzero nonnegative smooth η compactly supported there.

Let $\varphi \in H_0^1(B_1)$ be the real weak solution of

$$\Delta \varphi = \partial_1^2 \eta.$$

It is obtained directly by Riesz representation for the gradient norm: $\int \nabla\varphi \cdot \nabla v = \int \partial_1\eta \, \partial_1 v$. Consequently

$$\|\nabla \varphi\|_2 \leq \|\partial_1 \eta\|_2. \tag{57.2}$$

Choose $\varphi_m \in C_c^\infty(B_1)$ converging to φ in that norm and set $\Phi_m = \varphi_m(t)$. These are original compact smooth scalars on V. The original torus calibration gives

$$\mathcal{E}_S(\Phi_m - \Phi_k) = (n-1)! \int_{B_1} |\nabla(\varphi_m - \varphi_k)|^2 \, dx.$$

Thus they converge to a vector $\Phi \in \mathscr{D}_{S,0}(V)$, with $\|\Phi\|_H \le \sqrt{(n-1)!}\|\partial_1\eta\|_2$. Only this form-domain vector and its original smooth approximations are needed; no arbitrary energy representative is called a classical psh function.

For $u \in \mathscr{P}$ let $U = u \circ \sigma$ be its actual Gauss trace. The R4 full-annulus argument makes U separately convex and continuous on B_1, and $h \le U \le 0$. These bounds also put $U - h$ in $H_0^1(B_1)$. To see this without an unproved trace assertion, for $x \in B_1$ set $d = 1 - |x|$. On each coordinate segment of radius $d/2$ about x, one has $|U| \le 3d/2$. Convexity bounds each slope at x in absolute value by 3. On interior real boxes, Fubini therefore gives uniformly bounded weak coordinate derivatives. The zero boundary limit, squeezed between h and zero, makes the zero extension a Sobolev function with zero trace. Hence $U - h \in H_0^1(B_1)$.

For each original Φ_m, calibration and Green give

$$\mathcal{E}_S(u - h, \Phi_m) = -(n-1)! \int_{B_1} (U - h)\Delta\varphi_m \, dx.$$

Pass in the real gradient pairing and in the faithful form pairing to obtain

$$\mathcal{E}_S(u - h, \Phi) = -(n-1)! \int_{B_1} (U - h)\partial_1^2\eta \, dx \le (n-1)! \int_{B_1} \eta \, dx.$$

The inequality uses the positive distribution $\partial_1^2 U$ and $\partial_1^2 h = 1$. It is uniform over the entire classical family. Thus its support functional satisfies $\sigma_{C_V}(\Phi) \le (n-1)! \int \eta$.

Let y be the Riesz vector of the datum. The same calibration, together with $\Delta(U_f - h) = n(f-1)$ and zero real boundary values, gives

$$\begin{aligned} \mathcal{E}_S(y, \Phi) &= -n! \int_{B_1} (f-1)\varphi \, dx \\ &= -(n-1)! \int_{B_1} (U_f - h)\partial_1^2\eta \, dx \\ &\ge (n-1)! \left(\int \eta + c \int \eta \right). \end{aligned}$$

Subtract the support bound. The gap is at least $(n-1)!c \int \eta > 0$. Normalize by $\|\Phi\|_H$ and use (57.2) and Corollary 51.3 to obtain (57.1). The support functional is energy-continuous on the bounded constraint, so the strict gap persists for some original Φ_m. This is the promised original-test certificate, not only an abstract nonsmooth separating direction. □

The pointwise lower and upper barriers are satisfied by U_f itself. The obstruction is its negative coordinate Hessian in the outer annulus, which a Gauss trace of a globally defined classical psh row cannot have. This calculation does not identify raw BT capacity with energy capacity, and does not assert that all coefficient measures of S live on the Gauss skeleton. Only the specified toric top pairings are used.

58 A checkable radial criterion for a smooth toric background

This qualified family supplies actual classical solutions as well as an exact obstruction. It is a remaining-degree-one statement for the specified smooth background, not a replacement for the arbitrary-S criteria above.

Fix $n \geq 2$, put $B = \{x \in \mathbf{R}^n : |x| < 1\}$, and let

$$V = \operatorname{trop}^{-1}(B) \subset (\mathbf{G}^n_{m,K})^{\mathrm{an}}, \quad t_i = \log|T_i|, \quad \beta = \sum_i d't_i \wedge d''t_i, \quad S = \frac{\beta^{n-1}}{(n-1)!} \wedge [1].$$

Write $\sigma : B \to V$ for the original Gauss section. The ordinary boundary below is taken in the ambient analytic torus. Its Gauss points include $\sigma(\partial B)$; it is not necessary to identify the entire boundary with every fiber above ∂B.

Proposition 58.1 (Radial classical solvability)**.** *Let $\rho \in C([0,1])$ be nonnegative and set*

$$\mu = \sigma_*(\rho(|x|)\,dx), \qquad M(r) = \int_0^r s^{n-1}\rho(s)\,ds.$$

There exists a bounded actual classical psh u on V such that $D(uS) = \mu$ on every original compact scalar test and

$$\lim_{\substack{y \in V \\ y \to z}} u(y) = 0 \qquad (z \in \partial V)$$

if and only if

$$r^n\rho(r) \geq (n-1)M(r) \qquad (0 < r < 1). \tag{58.1}$$

When this condition holds, one such solution is the continuous torus-invariant function

$$u = U(|t|), \qquad U(r) = -\int_r^1 s^{1-n}M(s)\,ds. \tag{58.2}$$

Every bounded classical solution with the stated boundary values has this same actual Gauss-section trace $U(|x|)$. This assertion does not claim uniqueness of all actual values away from that trace.

Status: `DIRECT_COROLLARY` of the frozen whole-domain Gauss-trace argument, original toric calibration, and the elementary radial computation below. No general relative subsolution theorem is assumed.

Proof. The formula for M gives

$$0 \leq M(r) \leq \|\rho\|_\infty r^n/n, \qquad -\|\rho\|_\infty/(2n) \leq U(r) \leq 0.$$

For $r > 0$,

$$U'(r) = r^{1-n}M(r), \qquad U''(r) = \rho(r) - (n-1)r^{-n}M(r), \qquad U''(r) + (n-1)U'(r)/r = \rho(r).$$

Both $U''(r)$ and $U'(r)/r$ tend to $\rho(0)/n$ at zero. Thus $x \mapsto U(|x|)$ is C^2 on B, continuous on its closure, solves $\Delta U = \rho(|x|)$, and has boundary value zero.

Suppose an actual bounded classical solution u exists, and put $g = u \circ \sigma$. On every coordinate segment contained in B, the complete-extension annulus construction of Lemma 38.1 applies inside this full tropical inverse-image domain. Its two-endpoint Shilov comparison shows that g is separately convex. Its boundedness therefore makes it locally Lipschitz on B.

For each $\varphi \in C^\infty_c(B)$, the pullback $\varphi \circ \operatorname{trop}$ is an original compact smooth scalar on V. Recursion and the original toric calibration give

$$\int_B g\,\Delta\varphi\,dx = \langle uS, D(\varphi \circ \operatorname{trop})\rangle = \int_B \varphi\rho(|x|)\,dx.$$

Only this top trace is calibrated; no assertion about support of every lower coefficient measure of S is used.

Every Gauss point $\sigma(x_0)$ with $|x_0| = 1$ belongs to the ordinary boundary of V, by approaching x_0 through the real ball and using continuity of σ. The specified ordinary boundary limit hence extends g continuously by zero to ∂B. Therefore $g - U(|x|)$ is a continuous distributionally harmonic function on B with zero boundary values, and is zero. One can verify this last uniqueness directly: mollify on successively smaller balls, apply the ordinary smooth harmonic maximum principle there, and let their boundaries approach ∂B, using uniform continuity on the closed ball.

Consequently every such solution has trace $U(|x|)$. Separate convexity on the first coordinate axis, at $(r, 0, \dots, 0)$, implies $U''(r) \geq 0$. The displayed derivative formula is exactly (58.1), proving necessity.

Conversely, suppose that condition holds. Then $U'' \geq 0$ and $U' \geq 0$. The radial Hessian has eigenvalues U'' in the radial direction and U'/r in the tangential directions, with their stated common limit at zero. It is positive semidefinite, so $U(|x|)$ is convex. A continuous convex function of original logarithmic unit moments is classical psh: on compact smaller moment domains one may use convex smooth mollifications, or the defining curve test with harmonic logarithmic unit coordinates. Thus the function in (58.2) is actual continuous classical psh on V.

The calibration $D(\varphi(t)S) = \sigma_*((\Delta\varphi)dx)$ first holds for smooth real φ. Apply ordinary real mollification on compact smaller balls to $U(|x|)$, with C^2 convergence, or pass the fixed continuous multiplier through the original distributional recursion. It gives $D(uS) = \mu$ on every original scalar test. Finally every ordinary boundary point of V has $|t| = 1$, by continuity of t; the continuous formula $U(1) = 0$ gives the required boundary limits. □

Two immediate checks. If ρ is nondecreasing, then $M(r) \leq r^n\rho(r)/n$, so the criterion holds. In particular constant density $\rho = c$ gives $u = c(|t|^2 - 1)/(2n)$. In contrast, if ρ is nonzero and vanishes on some terminal interval $(r_0, 1)$, then $M > 0$ there while $\rho = 0$, and the criterion fails. The latter recovers the frozen compact radial-source obstruction while the former gives a direct solvable family in every ambient dimension. No nonlinear Gauss-pushforward formula is asserted for several moving slots.

59 Admissible variations in every remaining degree

Fix the original field and test sheaf, an arbitrary fixed original-positive separately closed (p, p)-background S, and $q = n - p \geq 1$. Retain the norm-box domain Ω, compact outer collar set L, smooth outer potential h, and common bound B of the R4 completed variational construction. Let V be ordinary open with $\overline{V} \subset L$. As in Section R4–81, prescribe an actual exterior function e on $\Omega \setminus V$ and continuous barriers $\ell \leq r$ on $\overline{V}$ with the same ordinary boundary value g. Assume that the classical family

$$\mathscr{P}_{\mathrm{bd}} = \{u \in \mathscr{P}^B_{h,L} : u = e \text{ on } \Omega \setminus V,\ \ell \leq u \leq r \text{ on } V\}$$

is nonempty. Its bounded global capacity closure is denoted $\mathscr{K}$. It is convex and capacity closed, and its selected actual representatives retain the exterior, both barriers, and boundary value.

Let $\widetilde{\nu} \in \mathfrak{N}_{S,h,L}$ be admissible data in the precise R4 sense, and write $\nu = \widetilde{\nu}|_V$. The completed variational objective is

$$F(a) = \mathcal{E}_{\mathrm{AM}}(a) - \int_L (a - h)\, d\widetilde{\nu}, \qquad m = \max_{a \in \mathscr{K}} F(a).$$

Its unique maximizing class is denoted a_*. Different admissible extensions of ν outside V change F by a constant on $\mathscr{K}$, because every selected representative has the same actual exterior e. Thus they do not change a_*.

59.1 What a verified two-sided variation gives quantitatively

Proposition 59.1 (A stationarity estimate with an explicit admissible radius)**.** *Let $\phi \in A_c^{0,0}(V;\mathbf{R})$ be original smooth. There is a finite C_ϕ, depending only on the fixed collar data, S, the common bound and this test, with the following property. If $a \in \mathscr{K}$ and $a \pm t_0\phi \in \mathscr{K}$ for some $t_0 > 0$, then for $0 < t \le t_0$,*

$$\left|\int_V \phi\, d(\mu_a - \nu)\right| \le \frac{m - F(a)}{t} + \frac{C_\phi t}{2}. \tag{59.1}$$

In particular, at the maximizer the displayed integral is zero whenever some two-sided admissible radius exists for this test. Moreover, for all $a \in \mathscr{K}$,

$$I_S(a, a_*) \le \frac{m - F(a)}{2c_q}, \qquad c_q = \frac{q}{4^{q+1}}. \tag{59.2}$$

The previously proved energy-dual and energy-capacity estimates turn this into the corresponding quantitative errors for the fixed-background measure and potential class.

Status: `DIRECT_COROLLARY` of the proved compact-collar energy polynomial, relative CLN and the R4 midpoint inequality. The displayed radius is an actual admissibility hypothesis; an obstacle gap alone does not provide it.

Proof. Convexity makes $a + s\phi$ admissible for $|s| \le t_0$. The finite-polynomial energy identity and compact Green formula give

$$\frac{d}{ds}F(a + s\phi) = \int \phi\, d(\mu_{a+s\phi} - \widetilde{\nu}),$$

and

$$\frac{d^2}{ds^2}F(a + s\phi) = -q\int \Gamma_{T_{S,q-1}((a+s\phi)^{q-1})}(\phi).$$

These identities for completed potentials follow from their defining classical rows and the completed weighted calculus, exactly as in Lemma R4–82.2. Equivalently the second identity follows by polarization of the finite polynomial; the smooth compact function ϕ is the only differentiated direction. The last integrand is positive. Relative CLN applied to the fixed positive test $d'\phi \wedge d''\phi$ bounds its total mass by C_ϕ/q, uniformly over the admissible segment. Every member has the same bound B. Hence

$$F(a \pm t\phi) - F(a) \ge \pm t\int \phi\, d(\mu_a - \widetilde{\nu}) - \tfrac{1}{2}C_\phi t^2.$$

The left side is at most $m - F(a)$; use both signs and the support of ϕ to obtain (59.1). At a maximizer let $t \downarrow 0$.

For the last assertion, the midpoint belongs to $\mathscr{K}$ and the linear datum term cancels from its midpoint difference. Thus

$$c_q I_S(a, a_*) \le F((a + a_*)/2) - \tfrac{1}{2}F(a) - \tfrac{1}{2}m \le \tfrac{1}{2}(m - F(a)),$$

which is (59.2). □

59.2 A classical curvature reserve that supplies the missing directions

For each original compact smooth ϕ in V, the explicit R4 compensator lemma supplies a bounded nonnegative original smooth classical function ρ_ϕ on Ω such that

$$\rho_\phi + 2\phi, \quad \rho_\phi - 2\phi \quad \text{are classical psh.}$$

Its proof uses finite unit-moment presentations and a positive sum of smooth convex functions of logarithms of affine regular functions. In particular this assertion does not invoke a general equivalence between original-current positivity and classical psh.

Theorem 59.2 (A qualified interior Dirichlet criterion in all degrees)**.** *Suppose there is a classical maximizing sequence $u_j \in \mathscr{P}_{\mathrm{bd}}$, with $F(u_j) \to m$, having the following property for every original $\phi \in A_c^{0,0}(V;\mathbf{R})$. There are an ordinary neighborhood O_ϕ of* $\operatorname{supp}\phi$*, with $O_\phi \subset V$, constants $\epsilon_\phi > 0$, $b_\phi > 0$, and an index j_ϕ, such that for all $j \geq j_\phi$:*

1. *the actual function $u_j - \epsilon_\phi\rho_\phi$ belongs to $\mathcal{C}_b(O_\phi)$;*
2. *on* $\operatorname{supp}\phi$ *one has the simultaneous margins*

$$|u_j| \leq B - b_\phi, \qquad \ell + b_\phi \leq u_j \leq r - b_\phi.$$

Then the maximizing completed class solves

$$T_{S,q}(a_*^q)|_V = \nu$$

on all original compact tests. Its selected representative retains the actual exterior e and ordinary boundary value g. It is the unique relative class in $\mathscr{K}$ solving this interior equation.

Status: `CONDITIONAL` as a Dirichlet existence criterion: the stated uniform reserve and margins are the additional input. Their availability is not claimed for arbitrary data or from a bounded subsolution. The implication itself is proved below for every remaining degree and every allowed fixed background.

Proof. Fix ϕ and choose a positive t_ϕ with

$$t_\phi \leq \epsilon_\phi, \qquad t_\phi\|\phi\|_\infty \leq b_\phi.$$

On O_ϕ the decomposition

$$u_j \pm t_\phi\phi = (u_j - \epsilon_\phi\rho_\phi) + \epsilon_\phi(\rho_\phi \pm (t_\phi/\epsilon_\phi)\phi)$$

is a sum of classical psh functions. The second factor is classical by convexity of the classical cone and the two compensator endpoints. Off the support of ϕ it equals u_j. These local assertions glue on the ordinary topology. The value margins preserve all bounds and barrier inequalities, and the compact support inside V preserves the entire actual exterior. Therefore

$$u_j \pm t_\phi\phi \in \mathscr{P}_{\mathrm{bd}} \qquad (j \geq j_\phi).$$

Every maximizing sequence converges to a_* in all-domain local outer capacity by the R4 variational theorem. Adding the same smooth function does not change its positive gap sets. The two sequences just constructed therefore define $a_* \pm t_\phi\phi$ in the same completed constraint $\mathscr{K}$. The preceding proposition, or the exact R4 Euler inequality applied to these two competitors, gives

$$\int_V \phi\, d(\mu_{a_*} - \nu) = 0.$$

This holds for every original compact smooth scalar test in V, so it is the required Radon-measure equality. No maximum locus has been treated as an ordinary open set, and no contact-support formula is used.

For two solutions $a, b \in \mathscr{K}$, their selected actual difference vanishes on $\Omega \setminus V$, including the ordinary boundary. Consequently

$$I_S(a,b) = \int (a-b)\, d(\mu_b - \mu_a) = \int_V (a-b)\, d(\mu_b - \mu_a) = 0.$$

The fixed-collar energy-capacity estimate proves equality of their relative classes. The preserved continuous barriers give the ordinary boundary limit, as in the R4 interior theorem. □

How the reserve can be checked, and what it does not assert. Condition (1) is an actual classical decomposition on a prescribed ordinary neighborhood. For potentials represented on one finite unit chart, a uniform Hessian margin that dominates the compensator Hessian on an open neighborhood of the compact moment image verifies it by convex composition after every complete-extension curve pullback. For a finite cover one verifies the condition on every member of that cover. A merely positive difference of two original currents is not silently substituted for this actual-function hypothesis.

The radius must be independent of the index for each fixed test. Vanishing margins or reserves along a maximizing row do not justify the limit argument. The smooth nonpolar counterexample of R4 shows why some additional condition of this kind is necessary for this route: ordinary boundary barriers and a bounded subsolution alone do not supply the two-sided psh variations. No strict psh exhaustion of an arbitrary analytic domain is assumed here.

60 Approximate admissible directions and vanishing reserves

Retain the arbitrary fixed original-positive separately closed S, all $q = n - p \geq 1$, the common defining bound and collar, the convex completed constraint $\mathscr{K}$, and the actual boundary data of Section 59. Write $\mu_a = T_{S,q}(a^q)$, let $\widetilde{\nu}$ be the same admissible datum, and put

$$F(a) = \mathcal{E}(a) - \int_L (a-h)\, d\widetilde{\nu}, \qquad m = \max_{\mathscr{K}} F, \qquad \nu = \widetilde{\nu}|_V.$$

All pairings below use actual selected representatives, or equivalently their relative classes: each measure being used annihilates the relevant relative-null sets. Measures are restricted to the fixed compact L when an L^1 norm is written. No whole-domain finite mass is required.

Lemma 60.1 (The completed first-order energy remainder)**.** *For $a, b \in \mathscr{K}$ define*

$$J_a(b) = \mathcal{E}(a) - \mathcal{E}(b) + \int_L (b-a)\, d\mu_a.$$

Then

$$\frac{I_S(a,b)}{q+1} \leq J_a(b) \leq \frac{qI_S(a,b)}{q+1}, \qquad F(b) - F(a) = \int_L (b-a)\, d(\mu_a - \widetilde{\nu}) - J_a(b). \tag{60.1}$$

The map $b \mapsto J_a(b)$ is convex, $J_a(a) = 0$, and for each fixed b one has $J_a(a + t(b-a)) = O(t^2)$ as $t \downarrow 0$.

Status: `DIRECT_COROLLARY` of Corollary R4–38.4 and the completed weighted calculus; it is not a new energy inequality.

Proof. For classical common-collar potentials the frozen polynomial calculation gives

$$J_a(b) = \sum_{r=0}^{q-1} \frac{q-r}{q+1} \int \Gamma_{T_{S,q-1}(b^r,a^{q-1-r})}(b-a).$$

Pass to bounded defining classical rows. Every term is evaluated with one fixed compact original smooth cutoff equal to one near L, so the completed weighted calculus passes each finite term. Positivity gives the two stated bounds. The objective identity is its defining algebra. Convexity follows from concavity of $\mathcal{E}$. Along a fixed segment the energy is a polynomial, and the derivative at zero is precisely $\int (b-a)\, d\mu_a$. Thus its constant and linear remainders cancel, which proves the $O(t^2)$ assertion. □

Theorem 60.2 (A criterion using approximate feasible variations)**.** *Let $a_j \in \mathscr{K}$ be a maximizing sequence and set $\epsilon_j = m - F(a_j) \to 0$; monotonicity of these numbers is unnecessary. Suppose that for every original real $\phi \in A_c^{0,0}(V)$ there are numbers $t_j > 0$ and feasible $b_j^+, b_j^- \in \mathscr{K}$ such that*

$$\epsilon_j/t_j \longrightarrow 0, \qquad I_S(a_j, b_j^+)/t_j \longrightarrow 0, \qquad I_S(a_j, b_j^-)/t_j \longrightarrow 0, \tag{60.2}$$

$$\left\| (b_j^\pm - a_j)/t_j \mp \phi \right\|_{L^1((\mu_{a_j}+\widetilde{\nu})|_L)} \longrightarrow 0. \tag{60.3}$$

Then the maximizing completed class a_* solves

$$\mu_{a_*}|_V = \nu$$

on all original compact smooth tests. It retains the prescribed actual exterior and ordinary boundary trace, and is the unique solution class in $\mathscr{K}$.

More precisely, for each sign and each index,

$$\pm \int_V \phi\, d(\mu_{a_j} - \nu) \le \frac{\epsilon_j}{t_j} + \frac{q I_S(a_j, b_j^\pm)}{(q+1)t_j} + \left\| (b_j^\pm - a_j)/t_j \mp \phi \right\|_{L^1((\mu_{a_j}+\widetilde{\nu})|_L)}. \tag{60.4}$$

Status: `CONDITIONAL` as a solvability criterion, with the explicitly stated feasible-direction conditions. Its implication and error estimate are `DIRECT_COROLLARY` of the frozen energy identities. No availability of these competitors for arbitrary data is asserted.

Proof. Since $F(b_j^\pm) \le m$, the exact identity (60.1) gives

$$\int_L \frac{b_j^\pm - a_j}{t_j}\, d(\mu_{a_j} - \widetilde{\nu}) \le \frac{\epsilon_j}{t_j} + \frac{J_{a_j}(b_j^\pm)}{t_j}.$$

The signed residual is dominated in total variation by $\mu_{a_j} + \widetilde{\nu}$. Subtract the indicated approximate direction and use the upper bound for J to obtain (60.4).

Every maximizing sequence converges to a_* in all-domain local outer relative capacity. The completed calculus therefore gives $\mu_{a_j} \to \mu_{a_*}$ on the fixed original test ϕ. Let $j \to \infty$ in both signs of the estimate. The residual pairing vanishes for every such ϕ, proving equality of the Radon measures on V. The boundary and uniqueness assertions are those of Theorem 59.2: actual differences of two feasible representatives vanish outside V, including its boundary, so the interior equation gives $I_S(a, b) = 0$. No assertion that boundary Monge–Ampere masses themselves agree is needed. □

Corollary 60.3 (An admissible radius may tend to zero)**.** *Suppose for each original $\phi \in A_c^{0,0}(V;\mathbf{R})$ there are positive radii $r_{j,\phi}$ such that*

$$a_j \pm r_{j,\phi}\phi \in \mathscr{K}, \qquad \frac{m - F(a_j)}{r_{j,\phi}} \longrightarrow 0.$$

Then the conclusion of the preceding theorem holds. The radii need not have a positive lower bound.

For classical a_j, these hypotheses are verified by the actual curvature reserve and value margins of Theorem 59.2, with index-dependent positive reserves $\varepsilon_{j,\phi}$ and margins $b_{j,\phi}$, provided one can choose

$$0 < r_{j,\phi} \le \varepsilon_{j,\phi}, \qquad r_{j,\phi}\|\phi\|_\infty \le b_{j,\phi}, \qquad (m - F(a_j))/r_{j,\phi} \longrightarrow 0.$$

Status: `DIRECT_COROLLARY` as an implication; the existence of such a rate-controlled maximizing row is an additional `CONDITIONAL` hypothesis, not a consequence of a bounded subsolution.

Proof. Write $\epsilon_j = m - F(a_j)$. If $\epsilon_j > 0$, choose $t_j = \min(r_{j,\phi}, \sqrt{\epsilon_j})$. Then $t_j \to 0$ and $\epsilon_j/t_j \to 0$. If $\epsilon_j = 0$, choose $t_j = \min(r_{j,\phi}, 1/j)$. Convexity preserves feasibility at the smaller radius. Proposition 59.1 gives

$$\left| \int_V \phi \, d(\mu_{a_j} - \nu) \right| \le \epsilon_j/t_j + C_\phi t_j/2 \longrightarrow 0.$$

The maximizing-sequence limit proves the equation. For the final assertion, the compensator decomposition and ordinary-open gluing in the earlier reserve proof apply separately for each index, with the displayed radius. No fixed-radius perturbation is passed to the limit. □

60.1 The exact tangent cone that these tests must reach

Proposition 60.4 (Feasible tangent directions and the residual)**.** *Fix $a \in \mathscr{K}$ and set $\kappa_a = (\mu_a + \widetilde{\nu})|_L$. In $L^1(\kappa_a)$ let*

$$\mathfrak{T}_a = \overline{\{c(b-a) : c \ge 0,\ b \in \mathscr{K}\}}^{L^1}.$$

This is a closed convex cone. A function ξ belongs to it if and only if there exist $t_j \downarrow 0$ and $b_j \in \mathscr{K}$ with

$$(b_j - a)/t_j \longrightarrow \xi \quad \text{in } L^1(\kappa_a), \qquad J_a(b_j)/t_j \longrightarrow 0.$$

At the variational maximizer the residual has the exact polar condition

$$\int_L \xi \, d(\mu_a - \widetilde{\nu}) \le 0 \qquad (\xi \in \mathfrak{T}_a).$$

Conversely this condition characterizes the maximizer. The residual annihilates the lineality space $\mathfrak{T}_a \cap (-\mathfrak{T}_a)$. Consequently, if that space contains every original compact smooth scalar test in V, the maximizer solves the interior equation.

Status: `DIRECT_COROLLARY` of convexity and the proved energy polynomial. This identifies a precise remaining admissibility condition; it does not assert that this lineality space contains all smooth tests for a general background.

Proof. The set before closure is a convex cone: the sum of two elements is their total nonnegative coefficient times a convex combination of the two competitors minus a. Any sequence in the displayed equivalent description lies in this cone, proving one inclusion.

For the converse choose $\xi_j = c_j(v_j - a) \to \xi$ in L^1. Along the fixed segment to v_j, the last assertion of Lemma 60.1 implies

$$\frac{J_a(a + tc_j(v_j - a))}{t} \longrightarrow 0 \quad (t \downarrow 0)$$

for each fixed j. Choose $t_j > 0$ decreasing to zero, with $t_j c_j \le 1$, and with this quotient at most $1/j$. Then $b_j = a + t_j c_j(v_j - a)$ is feasible, its normalized difference is exactly ξ_j, and it has the required negligible first-order cost.

The functional $\xi \mapsto \int \xi \, d(\mu_a - \widetilde{\nu})$ is continuous on $L^1(\kappa_a)$ with norm at most one. The frozen Euler inequality holds on $\mathscr{K} - a$, hence on its cone and closure. Conversely the polar inequality includes every $b - a$ and therefore is the frozen characterization of maximality. Using both ξ and $-\xi$ gives zero on the lineality space. For original tests this is precisely the stated measure equality. □

The first-order energy requirement does not enlarge this closed feasible tangent cone: the proposition proves equality with the usual L^1 tangent cone. In particular energy estimates cannot silently remove incompatible trace, curvature, or actual-boundary constraints. Vanishing reserves alone give no conclusion; the objective-defect to radius ratio in Corollary 60.3 is the quantitative condition that permits them.

61 An exact curvature test for actual admissible variations

The new bounded-difference positivity theorem removes one previously separate feasibility premise in the variational argument. It applies to actual classical potentials and their smooth perturbations. It does not identify an arbitrary relative completed representative with an ambient classical function.

Lemma 61.1 (Smooth directions belong locally to the difference class)**.** *Every real original smooth function ϕ is locally a difference of locally bounded classical psh functions. Consequently, for actual $a \in \mathcal{C}_b(\Omega)$ and $t \in \mathbf{R}$,*

$$a + t\phi \in \mathcal{C}_b(\Omega) \quad \Longleftrightarrow \quad D[a] + tD\phi \ge 0$$

on all original compact complementary positive tests.

Status: `DIRECT_COROLLARY` of Theorem 21.4.

Proof. Near a given point write $\phi = F(t_1, \dots, t_r)$ in original unit logarithms. Shrink to a relatively compact convex real ball on which the Hessian of F is bounded. For a sufficiently large M, both $M \sum t_i^2$ and $M \sum t_i^2 + F(t)$ are convex there. Their pullbacks are bounded classical on a smaller ordinary neighborhood: after every complete extension and curve pullback the unit logarithms are harmonic, and convex composition is subharmonic. Their difference is ϕ. Thus $a + t\phi$ is locally a bounded classical difference for either sign of t. The new positivity criterion, applied locally and glued on ordinary opens, proves the implication from the curvature condition. The converse implication is the established positivity of classical curvature. □

Retain the actual constrained family $\mathscr{P}_{\mathrm{bd}}$ and its data from Section 59. In particular every member is bounded by B, equals the prescribed actual exterior e on $\Omega \setminus V$, and lies between the actual barriers ℓ, r on V. Fix $a \in \mathscr{P}_{\mathrm{bd}}$ and a real original $\phi \in A_c^{0,0}(V)$. On $\operatorname{supp} \phi$ put

$$m_a(x) = \min\{B - a(x), B + a(x), a(x) - \ell(x), r(x) - a(x)\} \ge 0.$$

Define two extended nonnegative numbers by

$$r_{\mathrm{val}}(a,\phi) = \inf_{\phi(x)\neq 0} \frac{m_a(x)}{|\phi(x)|}, \tag{61.1}$$

$$r_{\mathrm{curv}}(a,\phi) = \inf_{\substack{\eta\in A_c^{n-1,n-1}(\Omega),\ \eta\geq 0\\ \langle D\phi,\eta\rangle\neq 0}} \frac{\langle D[a],\eta\rangle}{|\langle D\phi,\eta\rangle|}. \tag{61.2}$$

The infimum of an empty family is $+\infty$. Tests in the second line range over the full original positive cone, with arbitrary compact support and moment refinements.

Proposition 61.2 (Exact two-sided radius in the actual family)**.** *For $t \geq 0$,*

$$a + t\phi,\ a - t\phi \in \mathscr{P}_{\mathrm{bd}} \quad\Longleftrightarrow\quad t \leq \min\{r_{\mathrm{val}}(a,\phi), r_{\mathrm{curv}}(a,\phi)\}.$$

In particular these two explicitly defined quantities give the maximal admissible two-sided radius, including its finite endpoint. The curvature radius is equivalently

$$\sup\{t \geq 0 : -D[a] \leq tD\phi \leq D[a]\}.$$

It is unchanged by adding an actual pluriharmonic function to a; the value radius need not be unchanged.

Status: `DIRECT_COROLLARY`. This is an actual feasibility criterion, not a Dirichlet existence assertion.

Proof. The two original-current inequalities $D[a] \pm tD\phi \geq 0$ hold exactly when

$$t|\langle D\phi,\eta\rangle| \leq \langle D[a],\eta\rangle$$

for every displayed positive test. Tests with zero denominator give an automatic inequality. Thus the curvature inequalities are equivalent to $t \leq r_{\mathrm{curv}}$, including the infimum endpoint. The preceding lemma identifies them with actual classicality of both perturbed functions.

The simultaneous bounds $|a \pm t\phi| \leq B$ and $\ell \leq a \pm t\phi \leq r$ are exactly the inequalities $t|\phi(x)| \leq m_a(x)$ on the support of ϕ. They hold precisely when $t \leq r_{\mathrm{val}}$. Elsewhere nothing changes. Compact support in V preserves the entire prescribed actual exterior, hence also the fixed outer collar. These are all the defining conditions of $\mathscr{P}_{\mathrm{bd}}$, proving the equivalence. The curvature-radius assertion about pluriharmonic functions follows from $D[a+H] = D[a]$; the value constraints refer to the actual chosen function and do not have that invariance. □

Corollary 61.3 (The variational reserve can be checked in current order)**.** *In Theorem 59.2, replace its first hypothesis by the original-current inequality*

$$D[u_j] \geq \epsilon_\phi D[\rho_\phi] \quad \text{on } O_\phi \qquad (j \geq j_\phi).$$

Retain every other hypothesis, in particular the actual classical maximizing sequence, the uniform positive reserve for each fixed test, the actual bound and barrier margins, and the specified datum class. Then the same completed interior Dirichlet conclusion and relative-class uniqueness hold for every remaining degree and every allowed fixed positive closed S.

The vanishing-radius version in Corollary 60.3 can likewise use these current inequalities in place of its actual classical reserve, while retaining the stated quantitative relation between objective deficit and radius.

Status: `DIRECT_COROLLARY` for this replacement. As a Dirichlet criterion it remains `CONDITIONAL` on the displayed reserves, margins and maximizing sequence; their existence is not proved for arbitrary data.

Proof. On O_ϕ the actual function $u_j - \epsilon_\phi \rho_\phi$ is a locally bounded classical difference. Its original curvature is nonnegative by the new hypothesis, so Theorem 21.4 makes that actual function classical. This is precisely the previous first hypothesis. The remaining proof, including the capacity passage of the two actual perturbed rows and preservation of the ordinary boundary data, is unchanged. The same reasoning applies separately at each positive radius in the vanishing-radius argument; it does not supply the required rate automatically. □

What remains essential. Curvature positivity now decides whether an actual bounded classical potential admits the specified smooth perturbation. It does not provide slack in the pointwise obstacle or norm constraints, nor a uniform radius along a maximizing row. For an arbitrary completed potential, an actual ambient difference presentation has not been supplied, so the positivity theorem cannot simply be applied to it. The completed variational inequalities still require either verified feasible directions or a separate argument producing them.

62 An atomic solution with no nonzero two-sided smooth direction

Status: `DIRECT_COROLLARY` of original one-dimensional torus calibration, atomlessness of original smooth top-form measures, and the curve maximum principle. This example limits a sufficient variational criterion; it is not a Dirichlet nonexistence example.

Let

$$\Omega = \{-2 < t < 2\} \subset \mathbf{G}^{\mathrm{an}}_{m,K}, \qquad t = -\log|T|, \qquad S = [1], \qquad a = \frac{1}{2} + \frac{1}{8}|t|.$$

The function a is continuous classical, satisfies $1/2 \le a < 3/4$, and has ordinary boundary limit $3/4$ at both ends of the annulus. With ζ_1 the original unit Gauss point, the exact original curvature is

$$D[a] = \frac{1}{4}\delta_{\zeta_1}.$$

Thus a is an actual bounded classical solution of this atomic Dirichlet datum. In the normalized constraints $0 \le u \le 1$, it has pointwise obstacle margin at least $1/4$ everywhere.

Nevertheless, if $\phi \in A^{0,0}_c(\Omega; \mathbf{R})$ is original smooth and $s > 0$, then

$$a + s\phi,\ a - s\phi \in \mathcal{C}_b(\Omega) \quad \Longrightarrow \quad \phi = 0.$$

In particular the curvature radius in (61.2) is zero for every nonzero compact original smooth direction, despite the uniform value slack.

Proof. The slope jump of $|t|/8$ is $1/4$. Original torus integration therefore gives the displayed Dirac measure with no rescaling of the absolute value. The boundary values and the stated range follow from the formula for a.

Every original smooth $(1,1)$-form on a curve has an atomless coefficient measure. Indeed, on a compact stage a finite original moment presentation and its positive coefficient controls dominate its variation by finitely many positive smooth top-form measures. The pushforward of each such control by that presentation is calibrated integration of a smooth density on finitely many real one-dimensional polyhedra. It gives zero mass to a single real point, hence the positive measure gives zero mass to the inverse image of that point. In particular it gives zero mass to any individual analytic point. If the presentation has rank zero, its top form is zero. Finite localization proves the assertion without claiming that arbitrary positive currents

are atomless. Applied to the compact smooth top form $D\phi$, this shows that its signed Radon measure has no atom at ζ_1.

Positivity of both proposed classical curvatures would imply

$$-\frac{1}{4}\delta_{\zeta_1} \le sD\phi \le \frac{1}{4}\delta_{\zeta_1}.$$

Thus $D\phi$ would be carried by the singleton $\{\zeta_1\}$. Its atomlessness forces $D\phi = 0$. The smooth function ϕ is locally a bounded classical difference by Lemma 61.1; the frozen actual kernel theorem therefore makes it harmonic on the curve. Equivalently, apply the new bounded-difference positivity criterion to both ϕ and $-\phi$.

Choose a closed subannulus compactly contained in Ω whose ordinary interior contains $\operatorname{supp}\phi$. The two Gauss boundary values of ϕ vanish. The maximum principle for both harmonic signs gives $\phi = 0$ on that subannulus, and it is already zero outside its support. This proves the implication. Hence no positive two-sided radius is available for a nonzero direction. □

Meaning for the earlier criterion. An objective can attain the correct atomic Dirichlet solution at a point where its actual psh constraint has no nonzero two-sided original smooth direction. The reserve theorem is therefore a sufficient route to its Euler equation, not a necessary property of all solvable data. The pointwise obstacle margins and the curvature reserve are independent requirements. This example does not alter the exact radius formula or the conditional variational conclusions.

63 Relative Hessian concavity and hyperbolicity can fail

Status of the proposed general assertions: `FALSE`. The following is an explicit construction over the original finite extension $K/\mathbf{Q}_p$, using smooth bounded classical potentials and a smooth strongly positive separately closed background. It excludes two possible extensions of the relative Dirichlet theory: unconditional root-concavity of the relative top density, and an elliptic admissibility cone obtained merely by taking the positive component of that density. It does not challenge the established BT products, gradient Cauchy–Schwarz inequality, or compact-collar energy concavity.

Fix $q \ge 2$, let $n = q+2$, and work on an ordinary bounded full polyannulus Ω in the original n-torus. Write

$$t_i = \log|T_i|, \qquad \beta_i = d't_i \wedge d''t_i, \qquad \gamma = \sum_{i=5}^{n} \beta_i, \qquad dV = \beta_1 \wedge \cdots \wedge \beta_n.$$

The empty sum is zero when $q = 2$. Set

$$S = [\beta_1 \wedge \beta_2 + \beta_3 \wedge \beta_4].$$

Each summand is strongly positive. Pullback to every enlarged original moment presentation preserves this cone, and its wedge with a weakly positive complementary test is a positive top form. Thus S is in particular positive in the original middle CLD cone. It is symmetric and separately closed because all the coordinate logarithmic differentials are closed. Its bidegree is $(2,2)$, so its remaining degree is exactly q. The brackets mean the ordinary current of the displayed smooth form, not a current pushed forward from a skeleton.

Define

$$u = \tfrac{1}{2}(t_1^2 + t_2^2 + \sum_{i=5}^{n} t_i^2), \qquad v = \tfrac{1}{2}(t_3^2 + t_4^2 + \sum_{i=5}^{n} t_i^2).$$

These functions are original smooth and bounded on Ω. They are classical psh after every complete extension: on any analytic curve each unit logarithm is harmonic, and the displayed convex quadratic compositions are subharmonic. Their curvatures are

$$Du = \beta_1 + \beta_2 + \gamma, \qquad Dv = \beta_3 + \beta_4 + \gamma.$$

Proposition 63.1 (A smooth strongly positive counterexample in every remaining degree)**.** *For every $a, b \geq 0$, the following equality holds on all original compact smooth scalar tests:*

$$T_{S,q}((au + bv)^q) = q!(a^2 + b^2)(a + b)^{q-2}\, dV. \tag{63.1}$$

Consequently the qth root of the relative density is not concave on the cone of semipositive Hessians. For $q = 2$ one also has

$$T_{S,2}(u, u) = T_{S,2}(v, v) = 2dV, \qquad T_{S,2}(u, v) = 0.$$

Thus a reverse Cauchy–Schwarz or Alexandrov–Fenchel inequality for these relative mixed top densities is false for arbitrary backgrounds.

Proof. Every β_i^2 is zero, and the $(1,1)$-factors commute. The diagonal curvature of $au + bv$ has coefficients a, a, b, b in its first four directions and $a + b$ in every remaining direction. Wedge with the first summand of S leaves the product with coefficients $b^2(a + b)^{q-2}$, and the second leaves $a^2(a + b)^{q-2}$. Each surviving top monomial occurs $q!$ times in the qth power. This proves the equality as an identity of original smooth forms and hence on every original test and refinement.

The measure dV is nonzero and positive. More explicitly, on this standard torus its original calibration is $dV = \sigma_*(dx)$, whereas $(\sum_i \beta_i)^n = n!dV$. A nonnegative compact smooth scalar depending on t with positive real integral therefore has positive integral against dV. No equality of the lower-degree coefficient measures of S with skeleton-supported measures is needed.

At u and v the density in (63.1) equals $q!$, but at their midpoint it equals $q!/2$. Hence its qth root at the midpoint is strictly smaller than the arithmetic mean of its endpoint roots. When $q = 2$, comparison of the coefficient of ab gives the vanishing mixed term; the two diagonal terms are positive. □

The failure is not caused by a zero eigenvalue in the endpoint toric Hessians. Add to both u and v the same small positive multiple of $\frac{1}{2}\sum_i t_i^2$. Their Hessians in the displayed n coordinate directions become positive definite. The strict failure of the root inequality persists by continuity of the finite polynomial.

For $q > 2$ the two-free-slot reverse Cauchy–Schwarz failure can also be retained by freezing the remaining slots. Put $w = \frac{1}{2}\sum_{i=5}^n t_i^2$ and $R = S \wedge (Dw)^{q-2}$. With $u_0 = (t_1^2 + t_2^2)/2$ and $v_0 = (t_3^2 + t_4^2)/2$ one obtains

$$R \wedge (Du_0)^2 = R \wedge (Dv_0)^2 = 2(q-2)!dV, \qquad R \wedge Du_0 \wedge Dv_0 = 0.$$

This is a computation of legitimate mixed products over the original fixed S, not an assertion about wedging arbitrary currents.

Proposition 63.2 (The positive density component is not an elliptic branch)**.** *For this same background, let $P(H)$ be the coefficient of $S \wedge (\sum_{i,j} H_{ij} d't_i \wedge d''t_j)^q$ relative to dV, where H is a real symmetric matrix. The homogeneous polynomial P is not hyperbolic in the direction I. Moreover the connected component of $\{P > 0\}$ containing I contains matrices at which adding a positive semidefinite direction decreases P.*

Proof. On diagonal matrices the exact polynomial is

$$P(\operatorname{diag}(\lambda_1,\dots,\lambda_n)) = q!(\lambda_1\lambda_2+\lambda_3\lambda_4)\prod_{i=5}^{n}\lambda_i.$$

For $A=\operatorname{diag}(1,1,-1,-1,0,\dots,0)$,

$$P(A+tI) = 2q!(t^2+1)t^{q-2}.$$

It has the nonreal roots i and $-i$, whereas $P(I)=2q!>0$. This violates the defining real-root condition for hyperbolicity.

The path $H_s=\operatorname{diag}(1,1,s,s,1,\dots,1)$, $-1\le s\le 1$, has $P(H_s)=q!(1+s^2)>0$ and joins I to $A_*=\operatorname{diag}(1,1,-1,-1,1,\dots,1)$. But

$$DP_{A_*}(E_{33}) = -q! < 0,$$

although E_{33} is positive semidefinite. This proves the asserted failure of ellipticity on the positive component. □

The indefinite matrix in the last proof is used only to test a proposed enlargement of the admissible class. It is not called a psh Hessian. The first proposition already disproves root-concavity using genuinely classical psh potentials.

Separation from the valid energy inequalities. The proved Cauchy–Schwarz inequality is for first-derivative energy measures $\Gamma_R(f,g)$ over a fixed positive closed degree-one tail R. The failed inequality here concerns second-derivative mixed top densities; it is a different statement. Likewise the valid second variation of compact-collar Aubin–Mabuchi energy is

$$\frac{d^2}{ds^2}\mathcal{E}_{\mathrm{AM}}(a+s\xi) = -q\int \Gamma_{T_{S,q-1}((a+s\xi)^{q-1})}(\xi) \le 0$$

along admissible segments. Its sign needs positivity of the indicated tail and no pointwise root-concavity of P. Consequently all earlier relative energy arguments remain compatible with this counterexample.

64 The actual relative Dirichlet image need not be convex

R3 already disproved the unrestricted classical relative subsolution theorem. The new assertion here is different: two data which each have explicit continuous classical solutions on the same domain can have an unsolvable midpoint. This occurs in every remaining degree $q\ge 2$. It obstructs a convex-image extension of the degree-one Hilbert separation criterion.

Fix $q\ge 2$, set $n=2q$, and write the coordinates as two blocks $X=(X_1,\dots,X_q)$ and $Y=(Y_1,\dots,Y_q)$. On the ordinary unit polydisc

$$\Omega = \{|X_i|<1,\ |Y_i|<1\} \subset (\mathbf{A}_K^{2q})^{\mathrm{an}}$$

take the closed analytic coordinate polydiscs

$$Z_1=\{Y=0\}\cap\Omega, \qquad Z_2=\{X=0\}\cap\Omega, \qquad S=[Z_1]+[Z_2].$$

Their only intersection is the rigid origin o. The fixed S is symmetric, original-positive, separately closed, and has bidegree (q,q). Thus its remaining degree is q. Put $\lambda=-\log|\pi|>0$ and, on a block polydisc, define

$$g_t(Z) = \max\{\log|Z_1|,\dots,\log|Z_q|,-t\}.$$

The maximum is assigned its finite constant value at $Z = 0$. It is continuous classical psh, lies in $[-t, 0]$, and has ordinary boundary limit zero. Let a_1 be the radius-$|\pi|$ Gauss point of Z_1, and let b_1, b_2 be the radius-$|\pi|$, respectively radius-$|\pi|^2$, Gauss points of Z_2. These are original analytic points, intrinsically divisorial on their carriers, and are distinct from o and from the other carrier.

Theorem 64.1 (Nonconvexity with fixed continuous matching boundaries)**.** *Let $\mathscr{I}_S$ consist of the measures $T_{S,q}(w, \dots, w)$ for all bounded actual $w \in \mathcal{C}_b(\Omega)$ having ordinary boundary limit zero. The measures*

$$\mu = \delta_{a_1} + 2^{-q}\delta_{b_2}, \qquad \nu = \delta_{a_1} + \delta_{b_1}$$

belong to $\mathscr{I}_S$ and have continuous classical solutions. Neither $\mu + \nu$ nor $(\mu + \nu)/2$ belongs to $\mathscr{I}_S$. In particular this actual relative zero-boundary image is not convex. Nevertheless each of these two unsolvable data has a bounded continuous classical subsolution with the same zero boundary values.

Status: `NEW_PROVED` for the nonconvexity assertion and its explicit all-q construction. The general failure of the classical relative subsolution theorem is already an R3 result; it is not counted again as a new conclusion.

Proof. We first specify the frozen intrinsic uniqueness input. Each Z_i is a connected ordinary q-polydisc relatively compact in its smooth affine q-space. The function g_λ is a bounded continuous negative classical exhaustion with ordinary boundary value zero. It therefore satisfies condition R4–H. Theorem R4–48.4 gives uniqueness among *all bounded actual classical potentials*, including nontoric and discontinuous ones, for each finite divisorial datum below. We use this exact intrinsic theorem, not uniqueness merely in an energy or completed class.

The original radial calibration gives

$$(Dg_\lambda)^q = \delta_{\zeta_{|\pi|}}, \qquad (D(\tfrac{1}{2}g_{2\lambda}))^q = 2^{-q}\delta_{\zeta_{|\pi|^2}}.$$

At the indicated Gauss point the slope simplex has volume $1/q!$, which the original top-product factor $q!$ cancels; positive scaling gives the second identity. These are all-original-test identities, as in Lemma R4–60.1 and its frozen smoothing calibration.

Define on the whole ambient Ω

$$u = \max\{g_\lambda(X), \tfrac{1}{2}g_{2\lambda}(Y)\}, \qquad v = \max\{g_\lambda(X), g_\lambda(Y)\}.$$

Both are continuous classical, take values in $[-\lambda, 0]$, and have ordinary boundary limit zero. Indeed any ordinary boundary point is a limit of points of Ω with at least one block norm tending to one, where its corresponding branch tends to zero. Their restrictions are

	Z_1	Z_2
u	g_λ	$\frac{1}{2}g_{2\lambda}$
v	g_λ	g_λ.

For example the unused branch of u is exactly $-\lambda$ on each opposite carrier, so it does not change the indicated restriction. Integration-current compatibility and background linearity give, for every bounded actual classical w,

$$T_{S,q}(w^q) = i_{1*}(D(w|_{Z_1}))^q + i_{2*}(D(w|_{Z_2}))^q. \tag{64.1}$$

This is the same all-original-test identity as R4–64.4; it uses the restrictions of classical potentials, not a pullback of an arbitrary current. Thus $T_{S,q}(u^q) = \mu$ and $T_{S,q}(v^q) = \nu$.

Suppose w were a bounded actual classical solution for $\mu+\nu$. Its analytic restrictions remain bounded actual classical. The two positive summands in (64.1) are carried by their respective Z_i. Restriction off o separates them, and the total measure has zero mass at o. Positivity hence forces the intrinsic equations

$$(D(w|_{Z_1}))^q = 2\delta_{a_1}, \qquad (D(w|_{Z_2}))^q = \delta_{b_1} + 2^{-q}\delta_{b_2}.$$

There is no possible cancellation on the intersection. Moreover every intrinsic ordinary boundary point of Z_i is an ambient ordinary boundary point of Ω: it is a limit of points of $Z_i \subset \Omega$ and lies outside Ω because a free block coordinate has unit norm. Thus both restrictions have boundary limit zero.

The first intrinsic solution is $2^{1/q}g_\lambda$, whose actual origin value is $-\lambda 2^{1/q}$. For the second, set

$$A = (1+2^{-q})^{1/q}, \qquad B = \tfrac{1}{2}, \qquad h = (A-B)g_\lambda + Bg_{2\lambda}.$$

This is continuous classical and has zero boundary limit. The exact two-shell radial calculation in R4–60.3–R4–60.4 gives outer mass $A^q - B^q = 1$ and inner mass $B^q = 2^{-q}$. Explicitly, the slope shell at the outer point has volume $(A^q - B^q)/q!$, and the inner slope simplex has volume $B^q/q!$; the original factor $q!$ gives precisely those masses. Its actual origin value is

$$h(o) = -\lambda(A+B).$$

The pinned intrinsic uniqueness theorem therefore forces these two different values for the single actual number $w(o)$. They are indeed different: strict convexity of the ℓ^q norm, for $q > 1$, gives

$$2^{1/q} = \|(1,1)\|_q < \|(1,\tfrac{1}{2})\|_q + \|(0,\tfrac{1}{2})\|_q = (1+2^{-q})^{1/q} + \tfrac{1}{2}.$$

The vectors are not proportional, so the inequality is strict. This contradiction excludes $\mu+\nu$.

If a bounded actual classical zero-boundary potential solved the midpoint datum, multiplication by $2^{1/q}$ would solve $\mu+\nu$, by positive homogeneity. The midpoint is therefore also excluded. Finally mixed positivity gives

$$T_{S,q}((u+v)^q) \geq T_{S,q}(u^q) + T_{S,q}(v^q) = \mu+\nu.$$

Thus $u+v$ is the asserted continuous matching subsolution for the sum, and $2^{-1/q}(u+v)$ is one for the midpoint. This uses only the positive mixed expansion, not an invalid root-concavity inequality for arbitrary relative backgrounds. □

Uniform capacity bounds do not repair the example. The normalized functions $1+u/\lambda$ and $1+v/\lambda$ are actual testers on the whole Ω. Every point in $\{a_1, b_1, b_2\}$ therefore has relative singleton capacity at least $c_0 = 2^{-q}\lambda^{-q}$. Each of μ, ν and their midpoint has mass at most two. Hence for every fixed $\alpha > 0$ they all satisfy, on every Borel E,

$$\eta(E) \leq 2c_0^{-(1+\alpha)}\,\mathrm{Cap}_S(E;\Omega)^{1+\alpha}, \qquad \eta \in \{\mu, \nu, (\mu+\nu)/2\}.$$

If E misses the three support points the inequality is immediate; otherwise the stated singleton lower bound proves it. The failure therefore concerns actual carrier compatibility, even within a fixed finite compact support and fixed common growth bounds.

Linear versus nonlinear image geometry. For $q = 1$ the curvature map is linear, and addition of two classical zero-boundary solutions solves the sum datum. The strict inequality above becomes equality. The construction thus identifies a nonlinear obstruction to extending the degree-one convex-image/Farkas method. It does not disprove a conditional feasible-tangent theorem, and it makes no nonexistence assertion for the larger completed or energy solution classes.

65 An explicit trace-compatible relative Dirichlet criterion

The following qualified theorem gives a numerical solvability test, not a condition formulated as membership in an unknown operator image. It concerns a specified family of analytic integration backgrounds and finite radial data. It does not replace the arbitrary-background energy alternative or assert sufficiency of a general bounded psh subsolution.

Fix integers $q \geq 1$ and $m \geq 2$, and put $n = mq$. On the ordinary unit polydisc $\Omega \subset (\mathbf{A}_K^n)^{\mathrm{an}}$ write the coordinates as m blocks $Z^{(i)}$ of length q. Let Y_i be the coordinate q-polydisc on which all blocks other than i vanish. Thus $Y_i \cap Y_l = \{o\}$ for $i \neq l$, where o is the rigid origin. Choose constants $c_i > 0$ and set

$$S = \sum_{i=1}^{m} c_i [Y_i].$$

This is an allowed original-positive separately closed background of bidegree $(n-q, n-q)$, with remaining degree q.

Put $\lambda = -\log|\pi|$. For each i fix an increasing finite list $0 < t_{i1} < \cdots < t_{iN_i}$ of positive integer multiples of λ, and nonnegative real numbers m_{ij}. Write z_{ij} for the radius-$e^{-t_{ij}}$ Gauss point of Y_i, viewed as a point of the original ambient analytic space. These points are intrinsically divisorial, lie off every other carrier, and are distinct from o. Zero masses are allowed. Define

$$\nu = \sum_{i=1}^{m} \sum_{j=1}^{N_i} m_{ij} \delta_{z_{ij}}, \tag{65.1}$$

$$A_{ij} = \left(c_i^{-1} \sum_{k=j}^{N_i} m_{ik} \right)^{1/q}, \qquad A_{i,N_i+1} = 0, \qquad t_{i0} = 0, \tag{65.2}$$

$$d_i = \sum_{j=1}^{N_i} (t_{ij} - t_{i,j-1}) A_{ij}. \tag{65.3}$$

Theorem 65.1 (Finite radial data on coordinate carriers)**.** *There is a bounded actual $u \in \mathcal{C}_b(\Omega)$ with ordinary boundary limit zero and*

$$T_{S,q}(u^q) = \nu$$

if and only if

$$d_1 = \cdots = d_m.$$

When these depths have common value d, a continuous classical solution is given explicitly by

$$g_t(Z) = \max\{\log|Z_1|, \ldots, \log|Z_q|, -t\}, \tag{65.4}$$

$$h_i(Z) = \sum_{j=1}^{N_i} (A_{ij} - A_{i,j+1}) g_{t_{ij}}(Z), \tag{65.5}$$

$$u(Z^{(1)}, \ldots, Z^{(m)}) = \max_{1 \leq i \leq m} h_i(Z^{(i)}). \tag{65.6}$$

It has $\|u\|_{L^\infty(\Omega)} = d$. Every bounded actual classical zero-boundary solution has restriction h_i on Y_i, and its supremum norm is at least d. Uniqueness off the union of the carriers is not asserted.

Status: `DIRECT_COROLLARY` of the frozen all-original-test radial calibration, intrinsic bounded classical uniqueness, and integration-current compatibility. The explicit equal-depth criterion and extension formula record their consequence for this family; no new general relative Dirichlet existence theorem is being claimed.

Proof. Every $A_{ij} - A_{i,j+1}$ is nonnegative. The functions g_t and h_i are therefore continuous classical on the whole intrinsic polydisc and have ordinary boundary limit zero. Their finite values at the rigid origin are assigned by the displayed maxima. The frozen radial-shell calculation R4–60.3–R4–60.4 gives

$$(Dh_i)^q = \sum_{j=1}^{N_i} (A_{ij}^q - A_{i,j+1}^q)\delta_{z_{ij}} = c_i^{-1} \sum_{j=1}^{N_i} m_{ij}\delta_{z_{ij}}.$$

These are identities against all original compact tests. At a shell, the slope polytope has volume $(A_{ij}^q - A_{i,j+1}^q)/q!$; the original top-product calibration multiplies it by $q!$. Thus no additional factorial or residue-degree normalization occurs. The same calculation is valid when two successive slopes agree, giving zero mass at that shell.

Summation by parts gives

$$h_i(o) = -\sum_j t_{ij}(A_{ij} - A_{i,j+1}) = -d_i.$$

Moreover $-t_{ij} \le g_{t_{ij}} \le 0$, so $-d_i \le h_i \le 0$ everywhere and the lower bound is attained at o. Each intrinsic polydisc satisfies R4–H, for example with the negative exhaustion g_λ. The pinned intrinsic uniqueness theorem R4–48.4 therefore applies to these finite divisorial measures among all bounded actual classical solutions, including nontoric and discontinuous ones.

Suppose an ambient bounded actual classical w solves the stated problem. Its analytic restrictions to Y_i are actual bounded classical, and the frozen integration-current identity gives

$$T_{S,q}(w^q) = \sum_i c_i \, i_{i*}(D(w|_{Y_i}))^q.$$

All summands are positive. Off o their carriers are disjoint, whereas the prescribed total measure has zero mass at o. Consequently each intrinsic measure is exactly the one displayed for h_i. An intrinsic ordinary boundary point is a limit of points of $Y_i \subset \Omega$ and is outside Ω, because a free block coordinate has unit norm. It is therefore an ambient ordinary boundary point, and the restriction of w has boundary limit zero. Intrinsic uniqueness gives $w|_{Y_i} = h_i$ at every actual point. In particular the single number $w(o)$ equals every $-d_i$. This proves necessity, the asserted trace uniqueness, and the lower bound for the norm of any solution.

Conversely suppose all depths equal d. Pulling the continuous classical h_i back by the block projections and taking their finite maximum gives the continuous classical function in (65.6). On Y_i, every other branch has the constant value $h_l(o) = -d$, while $h_i \ge -d$. Hence $u|_{Y_i} = h_i$. The same integration-current identity proves $T_{S,q}(u^q) = \nu$.

The bounds $-d \le u \le 0$ and $u(o) = -d$ give its exact norm. At any ordinary boundary point of Ω, at least one block norm tends to one. Along that block every $g_{t_{ij}}$ tends to zero, so its h_i tends to zero. Since u is at least this branch and at most zero, it has ordinary boundary limit zero. This also handles the case $d = 0$, when all data vanish and $u = 0$. □

A numerical distinction between remaining degrees. For $q = 1$, the depth is the linear expression

$$d_i = c_i^{-1} \sum_j t_{ij} m_{ij}.$$

For higher q it is a sum of qth roots of tail masses. Thus the actual compatibility condition is still a finite explicit condition, but generally not a linear constraint on the data. The nonconvex image in Theorem 64.1 is one instance. Equal depths prove sufficiency here because the specified radial traces have a common attained global minimum and possess the explicit ambient block extensions. No such extension assertion is made for general intersecting analytic carriers or arbitrary intrinsic trace data.

Completed solvability has no equal-depth obstruction. For every datum (65.1), whether or not the d_i agree, the frozen completed class already contains a bounded zero-boundary solution. This is a `DIRECT_COROLLARY` of Proposition R4–66.1, not a new completed existence theorem. To verify its bound explicitly, choose $T \geq \max_{i,j} t_{ij}$ and $C \geq \max_i A_{i1}$, and put

$$G = \max\left(-T, \max_{1\leq i\leq m,\, 1\leq l\leq q} \log|Z_l^{(i)}|\right).$$

On each carrier,

$$CG|_{Y_i} = Cg_T \leq A_{i1} g_T \leq h_i.$$

The last inequality follows by adding the nonnegative terms in (65.5) and using $g_{t_{ij}} \geq g_T$. Apply the frozen coordinate separator construction with branch functions $h_i \circ \mathrm{pr}_i$, backgrounds $c_i[Y_i]$, and the common continuous floor CG. It gives uniformly bounded continuous ambient classical functions increasing to an actual completed potential a with

$$CG \leq a \leq 0, \qquad \mathcal{T}_{S,q}(a^q) = \nu.$$

The same floor gives ordinary boundary limit zero. Multiplying a carrier background by the positive constant c_i preserves every outer-null hypothesis of that construction. Thus the equal-depth condition distinguishes actual classical compatibility at the shared point from the already available completed solution class.

66 Compact radial measures on transverse coordinate carriers

The equal-depth criterion extends from finite shell data to arbitrary finite measures on a compact radial interval. This is an explicit nonatomic family in every remaining degree; the geometry of the carriers and their ambient extensions remains specified.

Retain the coordinate carriers Y_i, weights $c_i > 0$, and background $S = \sum_{i=1}^m c_i[Y_i]$ of Theorem 65.1. For each i, let τ_i be a finite positive Radon measure supported in a compact interval $[a_i, b_i] \subset (0,\infty)$. Let $\sigma_i(t)$ be the radius-e^{-t} diagonal Gauss point of Y_i, viewed in the original ambient space, and put

$$\nu = \sum_i (\sigma_i)_* \tau_i, \qquad A_i(t) = \left(c_i^{-1}\tau_i([t,\infty))\right)^{1/q} \quad (t>0), \qquad d_i = \int_0^\infty A_i(t)\,dt.$$

All depths are finite. The convention at a jump of the tail function does not affect these integrals.

Proposition 66.1 (A radial-measure compatibility criterion)**.** *There is a bounded actual classical u on the whole mq-dimensional unit polydisc, with ordinary boundary limit zero and $T_{S,q}(u^q) = \nu$, if and only if $d_1 = \cdots = d_m$. When the depths agree, write their common value as d and define*

$$H_i(r) = -\int_0^r A_i(t)\,dt \quad (0 \leq r \leq \infty), \qquad h_i(Z) = H_i\left(-\max_{1\leq l\leq q} \log|Z_l|\right),$$

where $h_i(o) = H_i(\infty) = -d_i$. Then

$$u = \max_i h_i(Z^{(i)})$$

is a continuous classical solution with norm d. Every bounded actual classical solution has exactly these restrictions to the carriers. No uniqueness away from their union is asserted.

Status: `DIRECT_COROLLARY` of the finite-shell calibration, locally uniform continuous-potential stability, and frozen pointwise ambient uniqueness. The measure τ_i may be nonatomic, singular continuous, or mixed.

Proof. We first prove the intrinsic equation on one q-polydisc, suppressing the index i and retaining its positive weight c. Choose positive finite atomic measures τ_N with the same total mass as τ, supported in one fixed compact interval $[a/2, 2b]$, and converging weakly to τ. Such measures are obtained by finite interval partitions and one representative per part; the representatives may be chosen at rational multiples of $-\log|\pi|$. Their tails converge to the tail of τ at every continuity point, hence at Lebesgue-almost every t. They are bounded by the same total mass and vanish for $t > 2b$. Therefore

$$A_N(t) = \left(c^{-1}\tau_N([t,\infty))\right)^{1/q} \longrightarrow A(t) \quad \text{in } L^1(0,\infty).$$

For atomic τ_N, summation by parts identifies the displayed integral profile with the finite positive combination of g_t in (65.5). The original radial-shell calibration thus gives

$$(Dh_N)^q = c^{-1}\sigma_*\tau_N.$$

The same calibration is valid at any real radius: it is the slope-polytope computation used in (47.1), with shell masses given by the difference of the qth powers of adjacent slopes. In particular no strict-radius or new-field normalization is required here.

The estimate

$$\|h_N - h\|_{L^\infty(Y)} \le \|A_N - A\|_{L^1(0,\infty)}$$

holds even at the rigid origin, since every profile is constant past $2b$. Thus h is a bounded continuous classical function on the whole intrinsic polydisc. Locally uniform continuous-potential stability, Proposition 25.1 with fixed background [1], gives

$$(Dh)^q = c^{-1}\sigma_*\tau$$

against every original compact smooth test. On the right, continuity of the Gauss section on the fixed compact interval proves weak convergence. The profile is between $-d$ and zero, attains $-d$ at the origin, and tends to zero at every ordinary boundary point.

Intrinsic uniqueness for these possibly diffuse measures is the frozen Corollary R4–22.5, rather than an assumption that a finite-atomic existence theorem includes diffuse data. The unit polydisc is an ordinary open set with compact closure in smooth affine space, so it satisfies R4–(G). Two bounded actual classical functions with ordinary boundary limit zero satisfy the strict compact-gap condition in both orders: a finite open cover of the compact ordinary boundary excludes every fixed positive-margin gap. Equal intrinsic Monge–Ampere measures therefore force actual pointwise equality.

Now let w be any ambient actual bounded classical solution. Its restrictions are bounded classical on each carrier, and original integration-current compatibility decomposes its relative measure as the sum of their positive intrinsic measures weighted by c_i. The carrier intersections consist only of the origin, while ν has no mass there: its radial parameters are bounded above. Positivity therefore identifies every individual intrinsic measure with $c_i^{-1}(\sigma_i)_*\tau_i$. Intrinsic ordinary boundary points are also ambient ordinary boundary points. The preceding uniqueness argument gives $w|_{Y_i} = h_i$, so its single actual origin value equals every $-d_i$. This proves necessity and trace uniqueness.

If the depths agree, each h_i has common attained minimum $-d$. The finite maximum of their block pullbacks is continuous classical; on Y_i all other branches are the constant $-d$, so its restriction is h_i. Integration-current compatibility gives the required relative measure. The bounds and origin value give norm d. At any ordinary boundary point some block norm tends to one; the corresponding h_i tends to zero, and the maximum is squeezed between this branch and zero. This proves the boundary condition. □

Completed potentials and scope. The frozen completed carrier-gluing theorem also applies to these continuous intrinsic profiles for arbitrary depths. Indeed, with $T \geq \max_i b_i$ and $C \geq \max_i (c_i^{-1}\tau_i((0,\infty)))^{1/q}$, one has $Cg_T \leq h_i$ on every carrier, since $A_i \leq C$ and is supported in $[0,T]$. The same global floor CG as in the finite-shell theorem gives a bounded zero-boundary completed solution. Equal depths are precisely the additional actual point-value compatibility for the specified ambient classical construction. This statement neither extends arbitrary intrinsic psh data to ambient classical functions nor asserts a general relative subsolution theorem.

67 A finite-depth criterion for noncompact radial data

Corollary 67.1 (Exact radial tail integrability)**.** *Retain the coordinate carriers and weights of Proposition 66.1, but let each τ_i be any finite positive Radon measure on $(0,\infty)$. Set*

$$\nu = \sum_i (\sigma_i)_* \tau_i, \qquad d_i = \int_0^\infty (c_i^{-1}\tau_i([t,\infty)))^{1/q}\, dt \in [0,\infty].$$

The datum has a bounded actual classical solution on the whole ambient unit polydisc with ordinary boundary limit zero if and only if all d_i are finite and equal. In that event the same integral profiles and finite maximum from the cited proposition give a continuous classical solution. The measures may accumulate at the rigid origin or towards the ordinary boundary; no mass is assigned to the origin itself.

Status: `DIRECT_COROLLARY` of the compact radial-measure construction, frozen actual measure order, and uniform continuous-potential stability. This is a criterion for the specified radial data, not for arbitrary measures.

Proof. On one intrinsic carrier put $\tau_N = \mathbf{1}_{[1/N,N]}\tau$. The corresponding tail roots A_N increase pointwise to $A(t) = (c^{-1}\tau([t,\infty)))^{1/q}$, up to the irrelevant choice of values at endpoints, and $d_N = \int A_N \uparrow d$. The intrinsic construction in the compact radial-measure proposition gives continuous classical zero-boundary h_N with

$$(Dh_N)^q = c^{-1}\sigma_*\tau_N, \qquad h_N(o) = -d_N.$$

If an actual bounded classical zero-boundary h has intrinsic measure $c^{-1}\sigma_*\tau$, then $(Dh)^q \geq (Dh_N)^q$. The frozen measure-order lemma R4–48.5 gives $h \leq h_N$ at every actual point. Thus $h(o) \leq -d_N$ for all N, excluding $d = \infty$. This use of order does not assume the existence of a solution for the limiting datum.

If $d < \infty$, the integral profile $H(r) = -\int_0^r A(t)\, dt$, with $H(\infty) = -d$, satisfies

$$\|h_N - h\|_\infty \leq d - d_N \longrightarrow 0.$$

Hence its radial pullback h is bounded continuous classical on the entire intrinsic polydisc, including the rigid origin. Uniform BT stability and monotone convergence of the finite measures give $(Dh)^q = c^{-1}\sigma_*\tau$ on all original compact tests. Since $A(t) \leq (c^{-1}\tau((0,\infty)))^{1/q}$, the estimate $|H(r)| \leq r(c^{-1}\tau((0,\infty)))^{1/q}$ proves the ordinary boundary limit zero. Intrinsic actual uniqueness follows from Corollary R4–22.5 and its two compact-gap conditions, exactly as in the compact radial-measure proposition.

For an ambient solution, positivity of the carrier decomposition identifies the individual intrinsic measures. The datum gives no mass to the shared origin, and the other carrier parts are disjoint. The preceding argument forces each depth to be finite, and actual intrinsic uniqueness forces its value at the common origin to be every $-d_i$. Conversely finite equal

depths give the same max-of-blocks solution as before. Boundary convergence follows from the displayed linear bound near radial parameter zero. No compact support of the datum is needed for these steps. □

A bounded completed solution when the depths differ. If every d_i is finite, bounded completed solvability remains a direct consequence of the frozen carrier-gluing theorem, even when the depths differ. Choose

$$C > 0, \qquad C \geq \max_i \bigl(c_i^{-1}\tau_i((0,\infty))\bigr)^{1/q}, \qquad T > 0, \qquad CT \geq \max_i d_i.$$

For the intrinsic radial parameter r,

$$h_i(r) \geq -Cr \quad (r \leq T), \qquad h_i(r) \geq -d_i \geq -CT \quad (r \geq T).$$

Thus $Cg_T \leq h_i$ on every carrier, and the same global continuous floor CG supplies bounded completed gluing and the ordinary zero-boundary squeeze. This paragraph asserts sufficiency only in the completed class; it makes no boundary-representative claim for an arbitrary completed solution.

Part V
Failed mechanisms and the next proof gates

68 Small analytic translations do not regularize a transverse Cantor current

Status: `FAILED_ATTEMPT` for a proposed regularization mechanism; the construction and its calculation are `DIRECT_COROLLARY` of torus calibration and the finite polyhedral description of actual delta-forms. This is not a counterexample to weak delta approximation or to the conjectural equality of all allowed backgrounds with the weak approximation class.

Fix the original finite extension $K/\mathbf{Q}_p$, choose a uniformizer π, and put $\lambda = -\log|\pi| > 0$. On the ordinary bidannulus

$$V = \{e^{-2\lambda} < |T_i| < e^{2\lambda} : i = 1, 2\}, \qquad t_i = \log|T_i|,$$

let ν be a nonzero finite atomless singular measure supported on a middle-thirds Cantor subset of $(-\lambda/2, \lambda/2)$. Define

$$g(s) = \int (s - r)_+ \, d\nu(r), \qquad S = D[g(t_1)].$$

The function g is convex and C^1: its derivative is the continuous distribution function of the atomless measure ν. Its distributional second derivative is exactly ν. The potential $g(t_1)$ is continuous, locally bounded and classical psh. Indeed convex piecewise affine approximants are finite sums of maxima of affine functions of the pluriharmonic function t_1, and converge locally uniformly to it. Thus S is an allowed symmetric positive separately closed original $(1,1)$-current on the smooth two-dimensional K-space V.

Proposition 68.1. *The following assertions hold.*

1. *For every $a_1, a_2 \in K$ with $|a_i| < e^{-2\lambda}$, the analytic automorphism*

$$\tau_a(T_1, T_2) = (T_1 + a_1, T_2 + a_2)$$

of V satisfies $\tau_a^* S = S$. Consequently averaging such translates, with any probability law for which the scalar averages are defined, leaves S unchanged.
2. *On any smaller bidannulus whose t_1-interval contains the support of ν, the current S is not the ordinary current of an actual delta-form.*
3. *Nevertheless S belongs to the fixed-domain weak closure of positive separately closed actual delta-forms.*

Proof. At every Berkovich point of V, the ultrametric inequality and $|a_i| < |T_i|$ give $|T_i + a_i| = |T_i|$. The inverse translation has the same property, so τ_a is an automorphism and $g(t_1) \circ \tau_a = g(t_1)$. Naturality of D under an analytic isomorphism proves the first assertion. The scalar average of a fixed current is that same current.

For the second assertion, choose nonnegative smooth real cutoffs χ_1, χ_2 compactly supported in the smaller radius intervals, with $\chi_1 = 1$ on $\operatorname{supp} \nu$ and $\int \chi_2(s)\, ds = 1$. The smooth original form

$$\eta = \chi_1(t_1)\chi_2(t_2) d' t_2 \wedge d'' t_2$$

has compact analytic support. Original torus calibration, first for smooth convex g and then by locally uniform convergence of its convex approximants, shows that the pushforward of the positive top measure $S \wedge \eta$ under t_1 is exactly ν. All constants in this assertion use the original absolute value; the two coordinate moments have calibration one.

Suppose instead that $S = [\sigma]$ for an actual delta-form σ near this compact support. Then $S \wedge \eta = [\sigma \wedge \eta]$. Take a finite common moment presentation of this compactly supported top delta-form, adjoining T_1 to the moments. The integration definition expresses its measure as a finite sum of pushforwards of integrable piecewise smooth densities on calibrated polyhedral cells. On each cell, the coordinate t_1 is affine. If its linear rank is one, pushing the density to the real line is absolutely continuous by Fubini. If its rank is zero, the pushforward is supported at one point. Hence the total pushforward is the sum of an absolutely continuous signed measure and finitely many atoms. This remains true after every original refinement and after adding the finitely many local presentations. It cannot equal the nonzero singular continuous measure ν. This proves the contradiction without restricting a delta-form along an arbitrary analytic embedding.

Finally, take convex polygonal interpolants g_N on nested finite partitions containing the endpoints of an interval which contains $\operatorname{supp} \nu$. Extend them by the unchanged affine tails. They are finite convex piecewise affine functions, and $g_N \to g$ uniformly on compact real intervals. Each $D_\delta(g_N(t_1))$ is an actual delta-form whose ordinary current is positive and separately closed. Distributional differentiation of locally uniform functions gives

$$[D_\delta(g_N(t_1))] \longrightarrow S$$

on every original compact test of V. Thus S lies in the same-domain weak approximation class, despite being fixed by all the indicated analytic translations. □

The failed inference is therefore specific: analytic automorphisms which are sufficiently close to the identity in the non-Archimedean additive coordinates do not move tropical radii at all. A complex-style average over these automorphisms supplies no smoothing in the real transverse variable. Convex polygonal approximation succeeds here by changing that transverse distribution directly; the example supplies no such operation for a general positive closed background.

69 Why one positive stratum decomposition does not settle higher delta relations

Status: `FAILED_ATTEMPT` for a proposed reduction. The explicit nonclosed restriction below is a proved scope counterexample, not a counterexample to a full delta action.

The positive bounded-difference theorem makes every continuous scalar relation among represented $(1,1)$-curvatures act independently of its presentation over any allowed background. It does not alone identify the kernel of the map from polynomials in those curvatures to actual higher-codimension delta-forms. In particular, freezing scalar coefficients in a relation between products still leaves a relation between products. It does not turn that relation into a degree-one curvature relation.

A tempting next step is to stratify the actual moment image, restrict the positive background to each Borel stratum, and apply the closed background cycle calculus to those restrictions. Level tangency does hold after Borel restriction. Separate closedness, however, need not survive, even in a very simple positive actual delta example.

Example 69.1 (A wall restriction of a positive closed current is not closed)**.** Let K be the original field, and work on the ordinary tropical box

$$V = \{|\log|T_1|| < 1/4,\ |\log|T_2|| < 1/4\} \subset (\mathbf{G}^2_{m,K})^{\mathrm{an}}.$$

Write $x = \log|T_1|$ and $y = \log|T_2|$, and set

$$u = x^2 + y^2 + (1+y)x_+, \qquad S = D[u], \qquad H = \{x = 0\}, \qquad R = \mathbf{1}_H S.$$

Then u is continuous classical psh, S is a symmetric positive separately closed original current, and R is coefficient-Radon and positive, but neither $d'R$ nor $d''R$ is zero.

Proof. On this box $1 + y > 0$, so

$$u = \max\{x^2 + y^2, x^2 + y^2 + x + xy\}.$$

The two smooth functions have real Hessian matrices $\mathrm{diag}(2,2)$ and $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. They are original smooth psh, and their maximum is continuous classical psh. Its piecewise smooth expression also defines an actual delta curvature. Therefore S has all the stated positivity and closedness properties. Borel multiplication defines R.

Choose $\phi, \psi \in C_c^\infty((-1/4, 1/4))$, with $\phi(0) = 1$, $\psi \geq 0$, and $\int \psi(y)\,dy > 0$. Put

$$\eta = \phi(x)\psi(y)d''y.$$

This is an original compact smooth $(0,1)$-test. On the toric test subspace, the smooth Hessian pieces of S have two-dimensional Lebesgue coefficient measures, so restricting their pairing with $d'\eta$ to H gives zero. The singular wall contribution is

$$(1+y)\delta_{x=0}\, d'x \wedge d''x.$$

The torus moments have calibration one. Since the term of $d'\eta$ containing $d'x$ is annihilated by this wall factor,

$$\langle R, d'\eta\rangle = \phi(0)\int (1+y)\psi'(y)\,dy = -\int \psi(y)\,dy.$$

With the original total-degree current convention,

$$\langle d'R, \eta\rangle = \int \psi(y)\,dy > 0.$$

Likewise, using $\eta' = \phi(x)\psi(y)d'y$ gives $\langle d''R, \eta' \rangle = -\int \psi(y)\, dy \neq 0$. Thus neither separate derivative vanishes. Every test in this calculation is an original compact smooth test on the actual Berkovich space.

The wall expression was used only to evaluate this toric test subspace. It is not asserted to be a full-current formula for R after all moment refinements. For example, a refined moment can detect smooth coefficient mass in branches on which $|T_1| = 1$. Such mass contributes zero to the displayed toric tests, so it cannot change the nonzero derivative calculation. This distinction avoids an independent false assertion about all refined coefficient measures. □

Exact remaining gate. Let $\mathscr{P}$ denote a finite local presentation algebra generated by continuous piecewise linear curvature operators and compatible piecewise smooth coefficient forms. The established constructions annihilate ambient cycle relations, pointwise piecewise-polynomial zero relations, and actual degree-one curvature relations. For a general analytic moment presentation t, the remaining assertion is that the additional kernel of

$$\mathscr{P} \longrightarrow B_X^{\bullet,\bullet}, \qquad P \longmapsto t^* P,$$

annihilates every allowed positive separately closed background. The statement must be proved with the full original refinement equivalence, not merely on the tangent spaces of one selected image. The example explains why the following argument does not prove it: "restrict the background to each face and reapply closed-background cycle insertion." Those restricted backgrounds need not be closed.

This leaves two mathematically distinct requirements: either an actual local relation theorem reducing the higher kernel to relations already annihilated by the operator calculus, or a separate approximation or continuity theorem controlling the corresponding current actions. No such general relation theorem is asserted here. The already proved matroidal saturated case supplies one sufficient relation theorem; the weak delta approximation class supplies one sufficient continuity mechanism. The example itself is in that approximation class and therefore cannot refute either construction.

The wall $H = \{\log |T_1| = 0\}$ is a real moment-level set, not the complete analytic pole set used in the frozen polar-restriction theorem. The latter theorem's separate closedness is unaffected.

70 Status change from R5 and the remaining minimal gates

70.1 What is already settled before this record

The seven bounded BT properties over an arbitrary fixed original positive separately closed background remain `PROVED` in the frozen calculus. The arbitrary-background potential quotient is also `PROVED` by R5. The present record neither reopens that quotient nor claims those previous proofs as new results. Its main further advance is the ordered potential quotient: positivity of the original curvature of an actual bounded classical difference forces that actual difference to be classical psh. This is stronger than preservation of the zero kernel. Further results concern the actual delta-test category, quantitative relative capacities, compact represented curvature families, and verifiable Dirichlet constraints.

Precise assertion	Before R6	R6 conclusion
Ordinary fixed-S bounded calculus and differentiated potential quotient	`PROVED`	Unchanged input. The positive-cone question explicitly left aside in R5's signed-current section is closed below only for its actual bounded classical difference class.
Positive curvature in the actual bounded classical difference class	R5 settled the zero kernel only	`NEW_PROVED`: $D[a-b] \geq 0$ if and only if the actual $a-b$ is classical psh; all complete-field curve tests and ambient upper semi-continuity are proved.
Represented curvature order over arbitrary S	No general order transfer for discontinuous differences	`DIRECT_COROLLARY`: $D[a] \geq D[b]$ implies $D(aS) \geq D(bS)$ in all differentiated slots; explicit mixed order-error and top-degree variation bounds follow.
Continuous scalar relations among bounded-potential curvatures	Q handled constant differentiated-potential relations	`NEW_PROVED`: positive order freezing transfers all finite continuous-coefficient relations to every S, even with discontinuous classical slots. Positive products of this module are a direct consequence; arbitrary Borel coefficient descent is false.
Actual total-degree $(1,1)$ delta-tests over arbitrary S	No general actual relation theorem	`NEW_PROVED`: canonical ordinary local action, all refinements and ordinary supports; in remaining degree one this gives a full closed complementary-test extension and the specified positivity-class uniqueness.
Ambient real cycle relations	Generator formulas did not remove all relations	`NEW_PROVED`: a piecewise polynomial operator descends through the full ambient cycle ring over arbitrary S, including real directions.
Generation of every real delta-form	Constant-cycle generation alone did not cover all smooth coefficient arrays	`NEW_PROVED`: exact finite anchored-face generation by PS coefficients and constant real cycles, in every tridegree.
Full actual delta-tests in arbitrary remaining degree	Ordinary equality alone was insufficient	`NEW_PROVED` for $\mathcal{W}_{\mathrm{loc}}$, including all its bounded BT tails; weighted joint decreasing convergence and both derivatives hold. All-background higher-degree relation annihilation remains `OPEN`.
Geometric and diffuse examples in $\mathcal{W}$	No R6 closure description	`DIRECT_COROLLARY`: smooth regular flags, singular effective Cartier divisors, curvatures of coefficient-integrable extended classical potentials, same-domain barycenters, and diffuse polar-supported fiber currents.
Saturated torus/matroidal charts	No all-background relation mechanism at this scope	`NEW_PROVED` under the exact fixed-chart and calibration assumptions; locality is saturated unless another ordinary-local theorem applies.

Precise assertion	Before R6	R6 conclusion
Represented curvature compactness	No common-domain compactness mechanism for these families	`NEW_PROVED`: normalized continuous order intervals are locally uniformly compact; the positive DSH criterion removes the initial continuity premise. Compact upper- potential families also work without any common Radon dominator.
Weak curvature and varying-background stability	Unrestricted weak stability is false	`DIRECT_COROLLARY` in the proved compact represented families. Explicit finite Shilov gauges give weighted stability for those chosen local representatives.
Relative energy versus normalized degree-one BT measures	A square-root capacity upper bound was available	`NEW_PROVED`: Hardy constant one and strong capacitary constant four, both optimal. The energy-capacity bound is linear.
Capacity-growth data with finite degree-one energy	A weaker sufficient growth condition	`PROVED`: powers $\alpha > 1/2$ suffice and the endpoint fails in general. The concave-modulus theorem gives its precise universal extension.
Reverse equality of energy and BT capacity null sets	Not established	`FALSE`: a compact Gauss-section Cantor-product set has zero BT outer capacity and positive faithful energy capacity in every $n \geq 3$.
General relative C1 and C2	`OPEN` outside stated subclasses	Still `OPEN`; compact upper-potential families give C1, but Borel C2 and outer continuity from below for their restricted capacities are `FALSE` for every $p, q \geq 1$: explicit smooth-torus examples have raw value $1/2$ and outer value 1. Common Radon domination, uniformly L^r relative density bounds obtained from variable continuous curvature coefficients, and, in remaining degree one, energy-null relative boundaries give qualified C2. On full torus boxes, exact convex reduction gives C2 for finite Hausdorff $(n-1)$-measure Borel sets, Borel coordinate sheets, and relatively compact open sets with compact additions, and null equivalence under sigma-finiteness. Exact diffuse rectangle/graph formulas also remain. A new exact continuous psh interpolation construction on the fast pole Cantor set extends the graph formula to arbitrary bounded Borel radial thresholds and to Borel sets between those graphs and their filled radial tubes.
Capacity domination characterizes the normalized image	Not proved	`FALSE`: one original smooth square separates a capacity-dominated annular measure from the weak normalized image. This is a stronger inverse/image requirement, not a counterexample to C1.

Precise assertion	Before R6	R6 conclusion
Completed degree-one Dirichlet feasibility	A general usable criterion remained open	`NEW_PROVED`: exact energy-dual support inequalities, a separating smooth test, and a distance formula for finite-energy data. Evaluating those inequalities in new geometries can still be difficult.
Relative energy-class Dirichlet solutions	Faithful form available	`DIRECT_COROLLARY`: finite-energy measures and dominated bounded energy subsolutions give solutions in that energy class. Classical psh feasibility is a distinct constraint.
Actual nonlinear Dirichlet image	Earlier subsolution obstructions existed; image convexity was not settled	`FALSE` in general: for every $q \geq 2$, two solvable zero-boundary data have unsolvable sum and midpoint, despite a continuous classical subsolution and all finite power-capacity bounds.
Higher-degree stationary solutions	Feasible variations not automatic	`CONDITIONAL` on the stated reserves and margins. R6 computes the exact two-sided actual variation radius from value margins and original current order; the new positive-DSH theorem removes the formerly separate classicality premise for that reserve.
Unrestricted relative Hessian root concavity and hyperbolicity	No justified general hypothesis	`FALSE`: explicit smooth strongly positive backgrounds give the displayed polynomial counterexamples.

70.2 The eleven classical directions, with current qualifications

Direction	Current mathematical scope
1. Locally bounded psh	B's actual class and regularization are inputs. R5's zero kernel is continuous PH; R6 now identifies the positive cone within the actual bounded classical difference class. Every actual member under a continuous curvature upper bound is continuous. Arbitrary bounded USC/Borel representatives are excluded.
2. Positive closed currents and mixed products	Ordinary arbitrary fixed-S calculus and Q are proved. R6 adds the order-preserving quotient, continuous coefficient products, all-background degree-one actual delta action and full higher-degree testing on the stated weak approximation class.
3. CLN	The earlier finite-control relative CLN estimates remain. They provide uniform masses in the new fixed-test and capacity arguments; no arbitrary moving-background theorem is inferred.

Direction	Current mathematical scope
4. Symmetry and weighted monotonicity	The ordinary fixed-S theorem remains. R6 adds explicit order-error bounds, qualified joint weak curvature/background convergence, and weighted convergence after a fixed local normalization. Full delta-test convergence retains the stated background/degree scopes.
5. Locality and maximum principles	Frozen ordinary and Borel maximum locality remain. The new degree-one action has ordinary locality; saturated chart theorems retain their more limited scope. Strict gaps of two discontinuous psh functions need not be ordinary open.
6. Capacities	General C1/C2 remain open. Sharp energy bounds, compact upper-potential families, exact full-torus convex reduction, finite-Hausdorff Borel sets and explicit diffuse polar carriers add distinct qualified mechanisms.
7. Quasicontinuity and capacity convergence	Frozen bounded and energy-representative results retain their exact hypotheses. New energy traces use these representatives, rather than arbitrary coefficient-null modifications.
8. Pluripolar and negligible sets	Arbitrary backgrounds can charge ambient complete-polar sets. Diffuse examples in this record are not countable sums of fiber subcurrents. The BT and energy capacity null ideals can themselves differ.
9. Comparison and domination	The qualified compact-gap and relative-q.e. principles remain. Energy order comparison and Dirichlet distance statements do not imply unrestricted actual pointwise classical uniqueness.
10. Extremal functions	The frozen constrained and qualified ambient results remain. The upper-contact counterexample rules out a simple ambient-envelope repair of relative compact mass.
11. Monge–Ampere and Dirichlet	Linear energy solutions, completed-class dual feasibility and actual classical nonlinear feasibility are distinguished. The latter image is explicitly nonconvex in every $q \geq 2$; scalar energy solutions and capacity growth do not resolve the carrier compatibility constraint. General existence retains the false/open scopes recorded here.

70.3 Primary comparisons that do not close the remaining gates

Mihatsch's actual sheaf is faithful in ordinary currents [Mih24, Proposition 4.11]. This resolves equality for two actual delta-forms and is not an arbitrary-current extension theorem. The real balanced definition and its product are [Mih23, Theorem 3.3 and Theorem 4.1]. Esterov's ring theorem [Est12, Theorem 3.1] removes ambient real fan relations; Gross–Shokrieh [GS21, Lemma 4.4 and Theorem 4.5] supplies the distinct matroidal relative-duality input. None of these inputs was silently extended to arbitrary singular moment images.

The focused capacity-source check also preserves a genuine distinction. Dhouib–Elkhadhra's discussion of arbitrary-background capacity [DE15, Section 3] does not provide the missing general compact continuity theorem. In Elaini–Elkhadhra, [EE21, Proposition 3] gives basic relative-capacity properties, whereas the extremal-capacity results used in Section 4.2 have the explicit $T = 1$ specialization. Their subsolution theorem has an additional current class and approximation/tail hypotheses. It cannot be cited as unconditional arbitrary-background classical psh Dirichlet existence in this project. Exact printed locations and hypothesis pins

are retained in the source register.

70.4 Minimal proof gates after the proved advances

1. **Higher-degree actual relation annihilation.** Given an actual zero delta-presentation of total complementary degree at least two, show its prescribed finite PL/PS BT expression vanishes over every allowed fixed S. Exact generation is now proved, so surjectivity of the test presentation is no longer the gap. The missing assertion is its analytic kernel. Weak positive closed delta approximation is sufficient, not yet proved universal. The degree-one order argument does not automatically lift to a higher-degree relation theorem. The explicit real-wall example 69.1 shows that Borel stratum restrictions need not be separately closed. They therefore cannot simply be reused as admissible backgrounds, despite level tangency. The Cantor example also rules out small analytic translations as a universal smoothing mechanism; it does not refute weak delta density.
2. **C1 without a uniform domination mechanism.** On the same whole normalized domain, recover the compact mass of a weak-limit or completed measure from actual normalized testers. The new compact upper-potential argument solves this for its entire represented family, even without one common measure dominator. The general problem still needs lower compact-mass approximation, not merely weak density or a capacitary upper bound on the datum.
3. **C2 beyond the established set and family classes.** Control the simultaneous compact extraction and normalized-tester pasting losses needed for arbitrary Borel ordinary-open regularity. Already on a full torus box this reduces exactly to the upper envelope of Laplacian masses of normalized convex functions. Finite Hausdorff measure, Borel coordinate sheets, open sets with compact additions, and sigma-finite nullity are solved there; positive-capacity Borel sets outside those hypotheses remain a concrete gate. Even compact normalized curvature families can have restricted raw/outer values 1/2 and 1, so their compactness alone cannot supply the missing simultaneous extraction. On the explicit pole Cantor lamination, exact scalar interpolation and Lusin localization now handle arbitrary bounded Borel radial graphs. Arbitrary Borel subsets with general fiber geometry still require a joint extension or selection mechanism; independent measurable fiber optimizers do not supply that mechanism.
4. **Classical feasibility of nonlinear Dirichlet data.** Supply verifiable admissible directions or a separate construction inside the actual classical constraint. The exact actual variation radius separates current-order slack from pointwise obstacle slack. For intersecting coordinate carriers, the depths of intrinsic unique solutions must agree at their common actual point; the nonconvexity example violates this compatibility. The radial coordinate-carrier theorems give a necessary and sufficient finite-equal-depth condition for arbitrary finite radial data, with depth given by the integral of the qth root of the weighted tail mass and an explicit max extension when it holds. When every individual depth is finite, a bounded completed solution exists even if the depths differ; no necessity for arbitrary completed solutions is asserted. The relative energy solution operator solves a larger linear problem. Hessian root concavity and hyperbolicity cannot be assumed for every positive background.

70.5 Excluded shortcuts

The following implications have deliberately not been used: ordinary current equality implies equality for arbitrary delta-test functionals; ambient cycle relations remove every analytic

pullback relation; generator formulas alone cover smooth flat coefficient remainders; energy capacity and BT capacity have identical null sets; capacity domination characterizes the normalized measure image; a pointwise obstacle gap supplies a classical psh variation; or an arbitrary positive background has the usual relative Hessian hyperbolicity. Each failure or missing hypothesis is recorded with its own example or proof gate above. The project still defines no wedge of two completely arbitrary positive closed currents and claims no blanket replacement of GK's nonclosed, nonpositive, off-diagonal calculus.

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A The capacity gap for convex compact potential families

This is a supplemental consequence of the accepted restricted-capacity example. Convexity of the entire normalized potential family does not repair its Borel outer regularity. The unrestricted relative BT capacity is not determined by this construction.

Proposition A.1 (Convex compact potential families with a Borel gap)**.** *In the setting of Proposition 30.1, for every $p, q \geq 1$ one may take the upper-potential family and the entire associated normalized competitor family to be convex and compact, while their restricted capacity still satisfies*

$$C(E) = \tfrac{1}{2}, \qquad C^*(E) = 1$$

on the same F_σ set in a fixed compact core. Compactness of the competitor family is in local uniform topology; its actual degree-q measure image is weakly compact.

Status: `DIRECT_COROLLARY` of the accepted product example, its mixed order theorem, and compact upper-potential compactness. The single varying Hessian direction is essential to the linear barycenter formula proved below.

Proof. Use exactly the original field, domain, background and coordinates of Proposition 30.1:

$$\Omega = \{-1 < x_i, y_a < 2\} \subset (\mathbf{G}^{p+q}_{m,K})^{\mathrm{an}}, \qquad A = \{0\} \cup \{1/j : j \geq 1\},$$
$$g_s = \frac{(x_1 - s)_+ + \sum_{i=2}^{q} (x_i)_+}{3q}, \qquad S = \frac{(3q)^q}{q!} \left[\bigwedge_{a=1}^{p} d'y_a \wedge d''y_a \right].$$

For a probability measure λ on the compact real set A, put

$$g_\lambda(z) = \int_A g_s(z)\, d\lambda(s), \qquad \overline{\mathscr{G}} = \{g_\lambda : \lambda \in \mathrm{Prob}(A)\}.$$

These integrals are ordinary real-valued integrals. They do not require an integration operation on psh classes over another field.

We first verify actual classicality and compactness. Collapse the tail $\{1/j : j > N\}$ of a probability λ onto 0, retaining the first N masses, and call the resulting finite probability λ_N. The uniform Lipschitz estimate of the product example gives, on the entire Ω,

$$\|g_{\lambda_N} - g_\lambda\|_\infty \leq \frac{1}{3q(N+1)}.$$

Each g_{λ_N} is a finite convex combination of continuous actual classical functions. The uniform classical-closure property therefore makes g_λ continuous classical. The same bound $0 \leq g_\lambda < 2/3$ holds globally.

The map $\lambda \mapsto g_\lambda$ is continuous from weak probability convergence to the global uniform norm. Indeed, the first N isolated singleton masses are continuous weak-probability coordinates, because their indicator functions are continuous on A. Thus the finite collapsed barycenters converge uniformly for fixed N, and the displayed uniform tail estimate is independent of the probability. Compactness of $\mathrm{Prob}(A)$ now proves compactness of $\overline{\mathscr{G}}$ in the global uniform norm. Its convexity follows from $g_{t\lambda+(1-t)\kappa} = t g_\lambda + (1-t) g_\kappa$.

Let σ be the original full Gauss section. The exact top measure is

$$T_{S,q}(g_\lambda, \dots, g_\lambda) = \nu_\lambda := \sigma_*(\lambda \otimes \delta_0^{\otimes(q-1)} \otimes dy_1 \cdots dy_p) = \int_A \nu_s\, d\lambda(s). \tag{A.1}$$

Here every passive coordinate has range $(-1, 2)$. To justify this identity without a false general nonlinear barycenter rule, write

$$a_\lambda(x_1) = \int_A (x_1 - s)_+\, d\lambda(s).$$

Only this first-coordinate term varies with λ; the other $q-1$ hinge terms remain fixed. Smooth convex one-variable approximations show that two curvature factors in the same coordinate wedge to zero. This holds for bounded continuous BT products by their local uniform continuity. Consequently, in the qth power of the sum only the terms using each of the q coordinates once remain. There are $q!$ such terms, and the coefficient is exactly

$$\frac{(3q)^q}{q!}\frac{q!}{(3q)^q} = 1.$$

For finite λ_N, linearity of the surviving first-coordinate factor and the accepted product calibration give (A.1). Passing $N \to \infty$ uses the global uniform convergence just proved and fixed-background continuous BT convergence. On the measure side it uses weak probability convergence and continuity of the original Gauss section; pairings with every original compact smooth scalar test converge. Thus the identity is on all original tests, not only on tests which factor through the displayed radius coordinates. The product measures are finite Radon, and the same identity gives the displayed Borel barycenter formula.

Now take the entire normalized family

$$\mathscr{P}(\overline{\mathscr{G}}) = \{u \in \mathcal{C}_b(\Omega) : 0 \le u \le 1, \quad D[u] \le D[g_\lambda] \text{ for some } \lambda \in \mathrm{Prob}(A)\}.$$

It is convex: if u, v have upper probabilities λ, κ, then $tu + (1-t)v$ has upper probability $t\lambda + (1-t)\kappa$. Every member is continuous by Corollary 21.7. Proposition 28.1 therefore proves local-uniform compactness of this whole family. Its degree-q measure image is weakly compact by Corollary 28.2.

For each such u, mixed order replacement in all slots gives

$$0 \le T_{S,q}(u, \dots, u) \le \nu_\lambda.$$

For every Borel $B \subset \Omega$, it follows that

$$\begin{aligned} C_{\overline{\mathscr{G}}}(B) &\le \sup_{\lambda \in \mathrm{Prob}(A)} \nu_\lambda(B) \\ &= \sup_{s \in A} \nu_s(B) = C_{\mathscr{G}}(B). \end{aligned}$$

The scalar function $s \mapsto \nu_s(B)$ is Borel by the product kernel formula, so the middle equality follows from averaging and Dirac probabilities. Conversely every g_s is a member of the enlarged family and attains ν_s. Hence equality holds throughout for all Borel B, in particular for every ordinary open set. The raw and outer capacities are therefore identical to those in the accepted all-degree example, which gives the claimed $1/2$ and 1. □

Exact limitation. Convexity here refers to the actual normalized potential family and its upper family. For $q = 1$ its measure image is also convex, by linearity; no such nonlinear image-convexity assertion is needed or made for $q > 1$. Closure under arbitrary maxima is not asserted. The example still concerns a restricted competitor class and does not establish failure of unrestricted relative BT C1 or C2.