An intrinsic non-Archimedean Bedford–Taylor program: analytic relations, relative capacities and source geometry

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Abstract

The local Bedford–Taylor construction with a positive closed background provides a common setting for curvature products, relative capacity, finite energy and measure equations on Berkovich spaces. We formulate a program for extending this construction to an intrinsic calculus and for determining the correct range of its potential theory. Its principal conjecture asserts that every actual analytic relation between admissible finite PL or piecewise-smooth curvature products acts trivially on every allowed background. The quadratic problem has a degree-one transgression; it is settled for finite presentations of residue rank at most two. On supplied smooth normal-three model germs, the remaining action is a central coefficient expressed by a paired spherical tensor and a realized positive closed tangent current. We use these results to identify the next geometric cancellation problem. Two further questions concern Borel regularity of relative outer capacity and strong continuity of literal energy-dual measures. The nonlinear equation is organized around its closed source image and the exact normal-probe obstruction. Explicit counterexamples determine which capacity, rooftop and solvability principles can belong to the theory. We conclude with precise extension problems for domains, singularities, Green currents and arithmetic intersection theory.

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1 Introduction

The Bedford–Taylor operation begins with a simple recursion. If a potential can be multiplied by a closed current, its curvature can be inserted by applying a second derivative. The force of the construction comes from proving that the resulting product depends on the curvature, is symmetric in its differentiated slots, and survives monotone limits. In the non-Archimedean setting these requirements interact with an additional geometric issue: a differential expression is represented in local moment coordinates, and a relation can become zero only after it is pulled back to the analytic space. Such a relation must remain zero when an arbitrary positive closed background is inserted.

The differential forms and currents of Chambert-Loir–Ducros provide the analytic language for this question [1, 2]. Delta-forms retain polyhedral boundary terms and connect local calculations with tropical and arithmetic intersection theory [3, 4]. The global theory of semipositive metrics and finite-energy potentials provides a second source of constructions and questions [5, 6, 7]. The program developed here starts from the established local calculus with a fixed positive closed background and asks how far its intrinsic differential, capacitary and variational structures extend.

Recent tropical work makes the equation side of this distinction particularly concrete. Amini–Piquerez–Wu develop regularization, Bedford–Taylor measures and energy on compact prekähler tropical varieties, and prove failure of the tropical Calabi–Yau theorem when a codimension-two star satisfies their genericity condition [11, Theorems 5.13 and 11.4]. Their comparison with non-Archimedean solutions uses a strongly faithful tropicalization and specified support hypotheses [11, Theorem 12.11]. These results make source compatibility and the analytic realization of tropical data central questions in the present program.

The choice of background is essential. A current of positive degree can detect a small analytic set and can impose additional restrictions on the sources of a Monge–Ampère operator. Consequently, the relative theory contains more than a change of notation in the ambient theory. Its basic objects are a potential class, an allowed background, a specified capacity, and, for a global equation, a specified boundary completion. We keep these objects explicit throughout.

1.1 The analytic category and the main conjecture

Fix a finite extension $K/\mathbf{Q}_p$ with its given absolute value. An *analytic space* in this paper is a good, Hausdorff, boundaryless, quasi-smooth Berkovich K-space, locally of pure dimension n. Opens, germs and sheaves use its ordinary topology. We write

$$\mathrm{D} = d'd'', \qquad d'd'' = -d''d'.$$

Current identities use the original compactly supported smooth CLD forms of complementary degree. A finite moment presentation is refined by adjoining actual analytic units; every face of the refined image is retained, including faces on which its tangent map loses rank.

Definition 1.1 (Background and classical potential)**.** An *allowed background* on an ordinary open U is a symmetric CLD middle-positive current S of bidegree (p,p) satisfying $d'S = d''S = 0$. Middle positivity is the CLD cone detected by the signed column-square tests $\eta \wedge J\eta$ in complementary degree; its coefficient matrices are positive semidefinite in those presentations. Put $q = n - p$.

The class $\mathcal{C}_b(U)$ consists of real-valued, locally bounded, upper-semicontinuous classical psh functions. Classical psh means that pullback to every analytic curve, after every complete isometric extension of the ground field, is subharmonic. The bounded Bedford–Taylor product

is denoted by

$$T_{S,0} = S, \qquad T_{S,b}(u_1, \dots, u_b) = \mathrm{D}\big(u_1 T_{S,b-1}(u_2, \dots, u_b)\big), \quad b \le q. \tag{1.1}$$

For finite differences of such potentials the notation denotes the signed multilinear extension.

The existence and properties of (1.1) are established inputs, recalled in Section 2. Let $\mathscr{B}^{a,b} = B^{a,b}_{\mathrm{Mih}}$ denote the actual analytic delta-form sheaf, with its full differentials.

Definition 1.2 (Admissible finite scalar presentations)**.** A finite family of continuous real scalars is *admissible* if, locally on one ordinary representative, it is expressed through an actual unit tuple $t_f = (-\log|f_1|, \dots, -\log|f_N|)$ on its complete moment image, with one finite coefficient complex, and has the following properties. Each scalar is PL or piecewise smooth with a smooth extension near each closed coefficient cell. It belongs to the actual degree-zero delta-form sheaf and has its full curvature $\mathrm{D}_\delta h \in \mathscr{B}^{1,1}$. It is also a finite difference of continuous classical psh functions on that representative.

Write $\mathscr{P}_{\mathrm{adm}}$ for the resulting local scalar sheaf. Its PL subsheaf is denoted by $\mathscr{P}$. Every finite continuous PL presentation in actual unit moments belongs to this class by local convex-difference decomposition.

Both the actual delta product and the signed bounded BT product are therefore defined for an admissible family. The finite closed-cell PS extension used in the quadratic results has exactly this scope [20, 18].

For a finite real presentation

$$P = \sum_{\nu=1}^{m} c_\nu [h_{\nu 1}] \cdots [h_{\nu b}] \in \mathrm{Sym}^b_{\mathbf{R}}\, \mathscr{P}_{\mathrm{adm}}(U),$$

define two maps with different targets:

$$\kappa_b(P) = \sum_\nu c_\nu\, \mathrm{D}_\delta h_{\nu 1} \wedge \cdots \wedge \mathrm{D}_\delta h_{\nu b} \in \mathscr{B}^{b,b}(U), \tag{1.2}$$

$$\mathcal{A}_S(P) = \sum_\nu c_\nu T_{S,b}(h_{\nu 1}, \dots, h_{\nu b}). \tag{1.3}$$

The brackets in P are formal symmetric-product symbols over $\mathbf{R}$. They do not denote pointwise multiplication of functions. We call P an *actual analytic relation* when $\kappa_b(P) = 0$ in the analytic delta-form sheaf. Its action (1.3) is an already defined finite signed coefficient-Radon current.

Conjecture 1.3 (Descent of actual analytic relations)**.** *Let $n \ge 1$, $1 \le b \le n$, and let P be a finite signed admissible PL/PS presentation at an ordinary germ (X, x) over the fixed K. Suppose $\kappa_b(P) = 0$ as an actual analytic delta-form germ. There is an ordinary representative U of this germ, chosen for P, such that for every $0 \le p \le n - b$ and every allowed background S of bidegree (p, p) on U,*

$$\mathcal{A}_S(P) = 0 \quad \text{on } U. \tag{1.4}$$

The equality holds on every original compact smooth complementary CLD test and after every actual refinement of the presentation. The field, its absolute value and the carrier of S remain the given ones.

The quantifiers express the intrinsic content of the conjecture. The relation and an ordinary representative are fixed before the background and its tests are chosen. A proof would define an action of the actual curvature-product sheaf $\operatorname{im}\kappa_b$ by choosing any finite local presentation and using (1.3). Ordinary locality would then glue these actions. A common

bounded curvature tail can be absorbed into the background, so the same statement controls its insertion in larger products.

There are two reasons to begin with $b = 2$. First, this is the first degree in which actual relations can couple distinct curvature factors. Second, the quadratic action has a degree-one transgression:

$$\alpha_P = \frac{1}{2}\sum_{\nu} c_\nu(h_\nu \mathrm{D}_\delta k_\nu + k_\nu \mathrm{D}_\delta h_\nu), \qquad \mathcal{A}_S(P) = \mathrm{D}M_S(\alpha_P), \tag{1.5}$$

where M_S is the established degree-one action. This identity turns the descent problem into an exactness or controlled-approximation problem in a module whose action is already available. In finite residue rank at most two that construction has been completed [20]. In the next model case, normal dimension three, it reduces the action to a central angular pairing [21].

1.2 Three interacting directions

The intrinsic relation problem is one branch of the program. A second branch asks which regularity and comparison principles belong to the relative capacity defined by the full class of normalized classical testers. A third studies literal measures on the strong energy completion and the image of its source map. These branches share the bounded calculus, but their remaining arguments have different inputs. The capacity and source questions can be studied with the existing potential calculus while relation descent remains open.

This distinction determines the form of the statements below. Conjecture 1.3 is a positive structural conjecture. Borel inner regularity of relative outer capacity and literal energy-dual continuity are posed as questions: their full original formulations remain undecided. Nonlinear source compatibility is a classification problem, because positive smooth density perturbations can already leave the source image. The examples in Section 5 and Section 6 are therefore part of the mathematical specification of the program.

2 The established potential calculus

We recall the results on which the open problems rest. Detailed proofs of the bounded construction, curvature quotient and energy interfaces are assembled in the companion article [12]; the original constructions and more extensive energy theory are in [13, 14, 15]. The statements in this section are used with their printed domain and boundary hypotheses.

2.1 Bounded products and the curvature quotient

The scalar input identifies locally bounded classical psh functions with limits of decreasing semipositive rational PL sequences on a single fixed ordinary neighborhood [14]. In the finite continuous setting, current positivity is also related to locally uniform smooth positive approximation in the geometric scope of [13]. These regularizations serve different purposes: decreasing approximation constructs bounded products, whereas uniform approximation is suited to continuous coefficients and moving continuous data.

For comparison, Wu constructs top-degree local mixed Monge–Ampère measures for his locally PL-regularizable bounded psh class, with weighted convergence for decreasing PL approximating rows [10, Theorem 4.2, pp. 25–26]. The dependence of a regularizing neighborhood on the member of a general decreasing psh sequence is isolated in his Remark 4.6. His Appendix A compares the one-step smooth- and PL-approximable classes; comparison with the iterated CLD closure uses an envelope property [10, Remark A.2 and Theorem A.3,

pp. 52–53]. The scalar input used here specifies the classical class over the original K and one fixed neighborhood; the relative theorem below then addresses every legal degree and an arbitrary allowed background.

Theorem 2.1 (Bounded relative calculus)**.** *Let S be an allowed background of degree (p,p) on U, and let $u_1,\dots,u_b\in\mathcal{C}_b(U)$, where $b\le n-p$. The recursion (1.1) defines a coefficient-Radon current, independent of the decreasing regularizations used in its construction. It is positive, separately d'- and d''-closed, symmetric in the differentiated slots, and additive and positively homogeneous in each slot. It commutes with ordinary restriction and gluing.*

For decreasing rows $u_{i,j}\downarrow u_i$ in $\mathcal{C}_b(U)$, including a bounded weight slot $u_{0,j}\downarrow u_0$, one has

$$u_{0,j}T_{S,b}(u_{1,j},\dots,u_{b,j})\longrightarrow u_0T_{S,b}(u_1,\dots,u_b) \tag{2.1}$$

on all original compact smooth tests. In top degree this is local weak convergence of measures on compact continuous scalar tests. Each differentiated slot depends only on its curvature current.

The local CLN bounds control the coefficient measures against a finite positive frame. This is the estimate that makes the recursion and its limiting operations analytic-space constructions. In (2.1) the Borel weight is its given pointwise representative. The closure and symmetry assertions concern the unweighted differentiated product.

Potential independence is stronger than regularization independence. For bounded classical potentials u,v it gives

$$\mathrm{D}[u]=\mathrm{D}[v]\quad\Longrightarrow\quad\mathrm{D}((u-v)S)=0. \tag{2.2}$$

The actual bounded classical curvature kernel is continuous pluriharmonic. In the finite continuous class, current harmonicity, classical pluriharmonicity and ordinary local finite real sums of logarithms of analytic units agree [17]. The same quotient argument transports curvature order: if $h=a-b$ is a difference of locally bounded classical potentials, then $\mathrm{D}[h]\ge 0$ precisely when h is classical psh in this class.

Consider the sheaf of actual currents

$$\mathcal{M}^1(U)=\left\{\sum_{i=1}^{m}f_i\mathrm{D}[g_i]:f_i\in C(U,\mathbf{R}),\ g_i\in\mathcal{C}_b(U),\ m<\infty\right\}, \tag{2.3}$$

with signed presentations interpreted by finite expansion. Products of elements of this module with a fixed S are already independent of their continuous-coefficient presentations. This gives a substantial intrinsic calculus. The main conjecture concerns the additional higher-degree relations arising in the actual delta-form quotient.

2.2 Order control and what it measures

For a positive represented curvature current Q, set

$$E_Q=\{A\in\mathcal{M}^1:\ -cQ\le A\le cQ\text{ for some }c\ge 0\},\qquad \|A\|_Q=\inf\{c:\ -cQ\le A\le cQ\}. \tag{2.4}$$

Let $\Phi_{S,b}$ denote the established mixed module product. The positive order estimate is

$$-M_iQ_i\le A_i\le M_iQ_i\quad(1\le i\le b)\quad\Longrightarrow\quad -\Big(\prod_i M_i\Big)R_S\le\Phi_{S,b}(A_1,\dots,A_b)\le\Big(\prod_i M_i\Big)R_S,\qquad R_S=\Phi_{S,b}(Q_1,\dots,Q_b). \tag{2.5}$$

Its polarized difference estimate is obtained by replacing one factor at a time. The completion of E_Q embeds faithfully into actual coefficient-Radon currents. The controls Q and R are

compatible through $Q+R$, and ordinary localization gives the sheaf $\mathcal{M}^1_{\mathrm{ord}}$. The mixed products extend uniquely, with positivity, $C(U)$-multilinearity, locality and support intersection [19].

Order convergence supplies a quantitative comparison with one fixed positive measure after inserting every background. Ordinary weak convergence alone has no such control. Thus (2.4) is useful for relation descent only when the approximants and the fixed Q are actually constructed. Its completion theorem does not supply density of an arbitrary class of currents.

2.3 Truncation, no loss and finite energy

Let V be a strict Laurent box of the following form. For a compact original affinoid B, choose analytic functions and positive thresholds such that

$$\begin{aligned} K_0 &= \{|f_a| \le R_a,\ |g_b| \ge r_b\} \Subset \mathrm{Int}_X(B), \\ V &= \{|f_a| < R_a,\ |g_b| > r_b\} \neq \varnothing, \end{aligned} \tag{2.6}$$

and suppose that S is defined on an ordinary neighborhood of K_0. These assumptions will remain in force for every global energy or source statement in this paper. Write $q = n-p \ge 1$.

For negative extended classical psh functions $u_1, \dots, u_b$, put $u_{i,j} = \max(u_i, -j)$ and $E_j = \bigcap_i \{u_i > -j\}$. Strict maximum locality makes

$$\mathbf{1}_{E_j} T_{S,b}(u_{1,j}, \dots, u_{b,j}) \tag{2.7}$$

a consistent increasing family of positive coefficient measures. Its locally coefficient-finite limit is the non-polar product. The limit is separately closed; closedness is a property of the resulting current, obtained by truncation estimates.

For a single negative potential u, impose

$$\overline{\{u < -t\}} \Subset V \quad (t > 0), \qquad \nu_{u,j} := \mathbf{1}_{\{u \le -j\}} T_{S,q}(u_j^{[q]}), \qquad \nu_{u,j}(V) \longrightarrow 0. \tag{2.8}$$

These conditions define $\mathcal{F}_{S,\partial}(V)$. If μ_u denotes the non-polar top measure, define

$$\mathcal{E}^{\sigma}_{S,\partial}(V) = \left\{ u \in \mathcal{F}_{S,\partial}(V) : e_\sigma(u) := \int_V (-u)^\sigma \, \mathrm{d}\mu_u < \infty \right\}, \qquad \sigma > 0. \tag{2.9}$$

The relative capacity of Section 5 gives exact tail characterizations:

$$u \in \mathcal{F}_{S,\partial}(V) \iff t^q C_S(\{u < -t\}; V) \longrightarrow 0, \tag{2.10}$$

$$u \in \mathcal{E}^{\sigma}_{S,\partial}(V) \iff \int_0^\infty t^{q+\sigma-1} C_S(\{u < -t\}; V) \, \mathrm{d}t < \infty, \tag{2.11}$$

within the stated boundary class. The theory also provides mixed products in every legal degree, the corresponding recursive current identities, and mixed integrability estimates. Uniform energy bounds, together with the stated capacity convergence, give subcritical stability; the endpoint requires uniform tail integrability [15].

For first energy, the global Green formula makes

$$B_S(u_0, \dots, u_q) = \int_V (-u_0) \, T_{S,q}(u_1, \dots, u_q) \tag{2.12}$$

symmetric in all $q+1$ slots. Its sharp mixed-energy estimate is

$$B_S(u_0, \dots, u_q) \le \prod_{i=0}^{q} e_1(u_i)^{1/(q+1)}. \tag{2.13}$$

The strong completion $\mathscr{C} = \widehat{\mathcal{E}}^{1}_{\mathrm{str}}(V)$ has actual Borel representatives modulo the relative local outer-null ideal, extended mixed currents $\widehat{N}_b$, and top measures $\widehat{\mu}_u = \widehat{N}_q(u^{[q]})$. Put $d = q+1$ and

$$e(u) = B_S(u^{[d]}), \qquad r(u) = e(u)^{1/d}, \qquad I(u,v) = e(u) + e(v) - B_S(u, v^{[q]}) - B_S(v, u^{[q]}). \tag{2.14}$$

Strong convergence $I(u_j, u) \to 0$ is equivalent to convergence in all local truncated outer capacities together with $e(u_j) \to e(u)$. The boundary of $\mathscr{C}$ is encoded by strong approximation from (2.9). Its members need not have ambient classical representatives.

There are consequently three distinct completions: the order completion of curvature currents, the strong completion of potentials, and the Borel completion defined by relative capacity. Each is useful because it preserves a different set of operations. The later interface problems specify when information passes between them.

3 Analytic relations and quantitative certificates

3.1 The quadratic transgression

The degree-one delta action is a canonical map

$$M_S : \mathscr{B}^{1,1}(U) \longrightarrow \mathcal{D}'^{p+1,p+1}(U), \qquad M_S(f\mathrm{D}_\delta g) = f\mathrm{D}(gS), \tag{3.1}$$

for the finite continuous scalar presentations at issue. It is compatible with actual equality and with the full first derivatives [15, 18]:

$$M_S(d'\beta) = d'(\beta \wedge S), \qquad M_S(d''\zeta) = d''(\zeta \wedge S). \tag{3.2}$$

The polyhedral boundary contributions are included in these formulas.

For $P = \sum c_\nu [h_\nu][k_\nu]$, define

$$\begin{aligned} u_P &= \sum_\nu c_\nu h_\nu k_\nu, & A_{0,P} &= \frac{1}{2}\mathrm{D}_\delta u_P, \\ Z_P &= \frac{1}{2}\sum_\nu c_\nu (d'_P h_\nu \wedge d''_P k_\nu + d'_P k_\nu \wedge d''_P h_\nu), & \alpha_P &= A_{0,P} - Z_P. \end{aligned} \tag{3.3}$$

Here d'_P, d''_P mean the piecewise derivatives of the continuous scalar functions. Full delta Leibniz gives

$$\mathrm{D}_\delta \alpha_P = \kappa_2(P), \qquad \mathcal{A}_S(P) = \mathrm{D}M_S(\alpha_P) = -\mathrm{D}M_S(Z_P). \tag{3.4}$$

Thus the signed quadratic relation retains the *paired* tensor Z_P. Treating the two scalar families independently would lose the cross terms that determine its action.

Let

$$\mathscr{J}^{1,1} = d'\mathscr{B}^{0,1} + d''\mathscr{B}^{1,0}, \qquad \mathscr{Q}^{1,1} = \mathscr{B}^{1,1}/\mathscr{J}^{1,1}, \tag{3.5}$$

where the sums are images in the ordinary sheaf category. The operator $\mathrm{D}M_S$ vanishes on $\mathscr{J}^{1,1}$. For an actual zero relation P, its class $[\alpha_P]$ lies in the Aeppli quotient $\ker \mathrm{D}_\delta/\mathscr{J}^{1,1}$. For a general nonzero quadratic curvature, $[\alpha_P]$ instead belongs to the larger quotient (3.5); its derivative is that curvature.

Proposition 3.1 (Two sufficient quadratic certificates)**.** *Let P be an actual quadratic relation on an ordinary germ. Either of the following constructions proves its action vanishes for every allowed background of degree $p \le n-2$:*

1. an actual local identity $\alpha_P = d'\beta + d''\zeta$;

2. *full-exact symmetric approximants $a_j = d'\beta_j + d''\zeta_j$, one fixed positive represented control Q, and $\varepsilon_j \downarrow 0$, all chosen on an ordinary representative before the background and tests, such that*

$$-\varepsilon_j Q \le [\alpha_P] - [a_j] \le \varepsilon_j Q \tag{3.6}$$

in the established order module, with its all-background degree-one action estimate.

The first implication follows by applying (3.2) and D. For the second, the order estimate bounds the difference against each fixed compact test by ε_j times a fixed positive control measure; distributional continuity of D then applies. These implications are the proved transgression and order-continuation results [18, 19]. The construction of the certificates is the remaining mathematical work.

The singular term $A_{0,P}$ has a useful role. It is already full first-order exact and can be retained while only the regular energy Z_P is approximated. The existing counterexample to a compact continuous-coefficient smooth-form double primitive for every full polar [17] rules out an unnecessary regularity demand. It leaves the actual full-exact certificate in (3.6) available.

3.2 A stronger carrier seminorm route

There is a second way to certify all-background descent, based on a fixed absolute cap carrier. We state it because it exposes the difference between an action theorem and a stronger family of quantitative relation certificates.

Let $\mathcal{D}(U)$ be the real space of finite continuous locally bounded classical psh differences and let $\mathcal{P}^b(U) = \mathrm{Sym}^b_{\mathbf{R}}\,\mathcal{D}(U)$. For a closed carrier $F \subset U$, let $\mathcal{Z}^b_F(U)$ be generated by the following specified relations: a factor whose restriction near each point of F agrees with an ordinary local pluriharmonic function; an ordinary pointwise zero product on all of F; and a permitted common ambient real PL cycle identity together with its actual original pullback. These are the established carrier relation generators of [17]. An arbitrary analytic relation is not included as a generator.

For $L \Subset U$, write $C = F \cap L$ and set $\|h\|_C = \sup_C |h|$ and $\operatorname{osc}_C u = \inf_{a\in\mathbf{R}} \sup_C |u-a|$. Empty-carrier norms are zero. Define

$$\begin{aligned} q^{U,b}_{F,L}(P) = \inf\Big\{ \sum_\nu |c_\nu|\, \|h_\nu\|_C \prod_{j=2}^{b} \operatorname{osc}_C u_{\nu j} : \quad P = Z + \sum_\nu c_\nu [h_\nu] \prod_{j=2}^{b} [u_{\nu j}], \\ Z \in \mathcal{Z}^b_F(U), \quad h_\nu \in \mathcal{D}(U), \quad u_{\nu j} \in \mathcal{C}_b(U) \cap C(U) \Big\}. \end{aligned} \tag{3.7}$$

Every decomposition in the infimum is finite and is made on the same ordinary U.

For the cap reduction, the carrier is

$$F = \operatorname{supp} T_{[O],c}((\log|f_1| - a_1)_+, \dots, (\log|f_c| - a_c)_+), \tag{3.8}$$

where O and f_i are the prescribed ordinary localization and analytic units. The relative cap for S is carried by this absolute set. The carrier is selected by the cap construction, independently of cancellation in the signed relation.

Question 3.2 (Vanishing of the carrier-CLN certificate)**.** Let P be an actual signed PL-zero germ of degree $b \ge 2$. On an ordinary representative, is

$$q^{U,b}_{F,L}(P) = 0$$

for every required compact stage L and the original absolute cap carrier (3.8), with the generator space in (3.7) unchanged?

The established cap and CLN argument sends this zero certificate to $\mathcal{A}_S(P) = 0$ for all allowed S [17]. It is therefore a sufficient route to the PL part of Conjecture 1.3. The reverse implication would require information on every polar of the seminorm, whereas background actions constitute a particular family of functionals. No identification of these two families is presently available. Direct exactness in (3.4) may prove the desired action without settling Question 3.2.

There is also a background-approximation route. Let $\mathcal{W}^p_{\mathrm{loc}}$ be the ordinary-local weak closure, on fixed representatives, of positive separately closed actual delta-form backgrounds of bidegree (p, p). The full actual delta-testing calculus has been established in this class, which is stable under insertion of bounded BT curvature tails [15].

Question 3.3 (Local approximation of backgrounds)**.** Does every allowed background of degree (p, p) belong to $\mathcal{W}^p_{\mathrm{loc}}$? More generally, which intrinsic geometric or coefficient conditions characterize this local weak closure on the original analytic space?

A density theorem would transfer the available testing calculus by its specified weak continuity. It is an alternative source of all-background results; the quadratic exactness route treats S directly and can be pursued independently.

3.3 Higher degrees and differential interfaces

For $b > 2$, a relation can have different curvature tails in its different summands. Absorbing one common tail into S treats a useful class of relations, but leaves this coupled case. A possible proof architecture would produce transgressions for coupled presentations and compatibility relations between them, or prove that their analytic relation ideal is generated by already controlled lower-degree relations. Either route must retain the full boundary differentials on common refinements.

Problem 3.4 (From curvature products to differential calculus)**.** Construct, in the original analytic category, the additional interfaces needed for a graded differential action: descent for higher coupled relations, descent with variable finite piecewise-smooth coefficients, compatibility of the resulting differential operators, and the extension from ordinary-current tests to the intended delta-form test classes. For each interface, identify its object sheaf and prove its local presentation independence, differential identities and stated functoriality.

This formulation allows the different interfaces to use different arguments. For example, equality on all ordinary smooth tests gives an ordinary current; extending the pairing to singular delta tests requires a continuity or integrability theorem for those tests. Similarly, an analytic map pulls back the scalar potentials used in a product in its established range, while a pullback of an arbitrary current requires separate hypotheses. The resulting calculus should record the mapping class as part of each naturality statement.

4 Residue rank and the first critical normal model

The finite moment presentation determines which residue geometry enters the quadratic relation. This gives a measure of complexity that is different from the ambient analytic dimension.

4.1 The completed rank-two construction

For a fixed actual unit tuple $f = (f_1, \dots, f_N)$ at x, put

$$\begin{aligned} \mathscr{F}_f(x) &= \widetilde{K}^{\mathrm{gr}}(\widetilde{f}_1(x), \dots, \widetilde{f}_N(x)), \\ d_f(x) &= \operatorname{trdeg}^{\mathrm{gr}}_{\widetilde{K}^{\mathrm{gr}}} \mathscr{F}_f(x), \\ r_f(x) &= \dim_{\mathbf{Q}} \operatorname{Span}_{\mathbf{Q}}\{-\log|f_i(x)| + \mathbf{Q}\log|K^\times|\}_{i=1}^N, \qquad s_f(x) = d_f(x) - r_f(x). \end{aligned} \tag{4.1}$$

The number $s_f(x)$ is the transcendence degree of the finitely generated field of normalized residues of the tuple. It is bounded by $s_K(x) = \operatorname{trdeg}_{\widetilde{K}} \widetilde{\mathcal{H}(x)}$.

Theorem 4.1 (Quadratic descent in finite residue rank at most two)**.** *Fix a finite continuous PL or PS scalar presentation through f, whose coefficient functions extend smoothly near the closed cells of one finite coefficient complex. Let $P = \sum c_\nu[h_\nu][k_\nu]$ be an actual quadratic relation at x. If $s_f(x) \le 2$, an ordinary neighborhood and a finite actual original-unit refinement carry a continuous finite-PS covector whose full strain is Z_P. In particular Z_P, and hence α_P, is full first-order exact there.*

There are also full-exact symmetric approximants satisfying (3.6) with one fixed represented positive control, chosen before every subsequent background, test and refinement. Consequently $\mathcal{A}_S(P) = 0$ for every allowed S of degree $p \le n-2$.

This is the construction of [20], not merely a reduction to an unproved normal-surface lifting statement. It first realizes the finite graded moment field on a target over the same K. A complete normal-surface construction produces the covector on the whole actual moment image. Its coefficients then return to the original source; their strain identities restrict through every later tangent map. The source-side order estimate retains $A_{0,P}$ and regularizes the energy with one fixed positive control. Thus the ambient dimension and the ranks of later tests are unrestricted. The rank hypothesis belongs to the fixed relation tuple.

There is an intrinsic quotient consequence. Let $\mathscr{K}_2 = \operatorname{im}_{\mathrm{sh}} \kappa_2$. At every point with $s_K(x) \le 2$, the formula

$$\tau_{2,x} : \mathscr{K}_{2,x} \longrightarrow \mathscr{Q}^{1,1}_x, \qquad \tau_{2,x}(\kappa_2(P)) = [\alpha_P] = -[Z_P], \qquad \overline{\mathrm{D}}\,\tau_{2,x} = \mathrm{id} \tag{4.2}$$

is well defined. Comparing two presentations uses their finite union tuple; its residue rank is still at most two under this stalk hypothesis. On an ordinary open set where $s_K \le 2$ everywhere, these maps form a sheaf morphism. In particular they do so on analytic spaces of dimension at most two [20].

At a point of larger residue dimension, separate rank bounds on two presentations do not give a rank bound on their union. For example, at the three-dimensional torus Gauss point the pairs (T_1, T_2) and (T_2, T_3) each generate two residue variables, while their union generates three. This explains why the stalk hypothesis in (4.2) has mathematical content.

4.2 Ordinary punctured descent in a supplied normal-three germ

The next completed reduction concerns a supplied smooth integral normal-three model germ (Y, y) over K°. Its distinguished analytic point has

$$|\mathcal{H}(y)^\times| = |K^\times|, \qquad F = \widetilde{\mathcal{H}(y)}, \qquad \operatorname{trdeg}_{\widetilde{K}} F = 3. \tag{4.3}$$

Here "supplied model germ" includes the original-field lifting and complete-source comparison of [21]: the finite residue tuple is prepared on a smooth projective simple-normal-crossings model of F; its labelled monomial cones, their calibrations and all lower strata are realized by

actual Gauss families on the ordinary germ. The comparison covers the complete source of the fixed tuple, and the families converge ordinarily to y uniformly in their angular parameters. These are the model hypotheses of the result below.

Theorem 4.2 (Punctured quadratic action and a central residual)**.** *Let (Y, y) be a supplied normal-three model germ as above, and let P be a fixed finite continuous closed-cell PL/PS quadratic presentation with $\kappa_2(P) = 0$. There is an ordinary representative U such that, for every allowed background S of degree $p = 0$ or 1,*

$$\mathcal{A}_S(P)|_{U\setminus\{y\}} = 0. \tag{4.4}$$

For $p = 0$ the action vanishes on U. For $p = 1$ there is a scalar $c_S(P)$ such that

$$\mathcal{A}_S(P) = c_S(P)\delta_y. \tag{4.5}$$

All equalities use the original compact smooth tests and actual refinements. The representative and its ordinary cover are fixed for P before S and its tests are chosen.

The proof in [21] covers a fixed finite old source inventory. Top-cone interiors have local ambient strain extensions. Positive two-faces are treated by proper marked residue curves, simultaneous Picard completion and a final faithful separator. The remaining finite ray families are treated by ordinary products with completed surface core fibers. Off the old source the rank-two construction applies. Each local primitive covers an ordinary neighborhood of the actual point, and ordinary current locality yields (4.4).

The curve calibration explains why the geometric preparation must keep its labels. At a marked residue point a, let $b_a = m_a v_a$ be the actual order vector with v_a primitive in the saturated normal lattice, and let $\delta_a = [\kappa(a) : \widetilde{K}]$. Its weight and the slope of a newly adjoined rational function g satisfy

$$w_a = \delta_a m_a, \qquad w_a h_a(g) = \delta_a \operatorname{ord}_a(g). \tag{4.6}$$

New zero-old-column marks and satellite pieces can enter after the Picard functions are adjoined. Applying the last faithful separator to the completed tuple is what makes the subsequent common-strain Stokes argument account for those pieces.

Once the action is supported at y, coefficient-Radon structure and separate closedness eliminate point support below top degree. In top degree, a Dirac mass remains possible. The current problem therefore has an exact scalar form.

4.3 Critical tangents and the full spherical pairing

Choose a protected finite actual moment map $t : U \to \Omega \subset \mathbf{R}^N$ containing the scalar presentation, with $t(y) = 0$, such that $t^{-1}(L)$ is compact in U for every compact $L \Subset \Omega$. For an allowed degree-one background define the real supercurrent

$$\langle T_S, \eta\rangle = \langle S, t^*\eta\rangle, \qquad \eta \in \mathcal{A}_c^{2,2}(\Omega). \tag{4.7}$$

This uses the entire ordinary inverse image. It is middle-positive, symmetric and separately closed, of real bidimension $(2, 2)$. With

$$\beta = \sum_i d' z_i \wedge d'' z_i, \qquad \mu_S = \frac{1}{2} T_S \wedge \beta^2,$$

the function $r \mapsto \mu_S(B_r)/r^2$ is nondecreasing. Thus

$$\mu_S(B_r) = O(r^2), \qquad \nu = \lim_{r\downarrow 0} \frac{\mu_S(B_r)}{r^2} < \infty. \tag{4.8}$$

Subsequences of the rescaled currents $T_{S,\delta} = \delta^2 (a_\delta)_* T_S$, $a_\delta(z) = z/\delta$, have conic positive separately closed limits T_0 with trace mass $\mu_{T_0}(B_R) = \nu R^2$ [21]. The real growth calculation is the Lelong monotonicity argument for positive closed supercurrents [9, Proposition 3.4 and Corollary 3.5]. Its application here uses the protected transpose (4.7): full compact pullback support retains both closedness equations. The later PS pairing and the comparison with the analytic relation are the additional interfaces proved in [21].

The closed-cell scalar extensions have homogeneous PL first jets $h_\nu^{[1]}, k_\nu^{[1]}$. Their full paired tensor

$$Z_0 = \frac{1}{2} \sum_\nu c_\nu (d'_P h_\nu^{[1]} \wedge d''_P k_\nu^{[1]} + d'_P k_\nu^{[1]} \wedge d''_P h_\nu^{[1]}) \tag{4.9}$$

satisfies $-Cr\beta \le Z_P - Z_0 \le Cr\beta$ on the complete protected inverse image of B_r. The inequality is preserved by rank-losing tangent maps. A cutoff $\rho(z/\delta)$ has second derivatives of size $O(\delta^{-2})$; combining this with (4.8) shows that replacement of Z_P by Z_0 changes its residual pairing by $O(\delta)$.

Theorem 4.3 (Spherical formula for the realized residual)**.** *In the setting of Theorem 4.2, let T_0 be any critical tangent obtained from the same background and protected presentation. There is a finite positive semidefinite matrix-valued measure M on $\mathbb{S}^{N-1}$ such that*

$$\mathsf{M}\theta = 0, \qquad \operatorname{div}_{\mathbb{S}} \mathsf{M} = 0, \qquad \sigma = \operatorname{tr} \mathsf{M}, \qquad \sigma(\mathbb{S}^{N-1}) = 2\nu. \tag{4.10}$$

For the actual first jets in (4.9),

$$\begin{aligned} c_S(P) = \sum_\nu c_\nu \Big(& \int_{\mathbb{S}^{N-1}} h_\nu^{[1]} k_\nu^{[1]} \, \mathrm{d}\sigma \\ & - \int_{\mathbb{S}^{N-1}} d_{\mathbb{S}} h_\nu^{[1]} \otimes d_{\mathbb{S}} k_\nu^{[1]} : \mathrm{d}\mathsf{M} \Big). \end{aligned} \tag{4.11}$$

The coefficient-wall values are their canonical tangential PS values. Equivalently, for every compact smooth cutoff ρ equal to one near zero,

$$c_S(P) = -\langle T_0, Z_0 \wedge \mathrm{D}\rho \rangle, \qquad \mathrm{D}(Z_0 \wedge T_0) = -c_S(P)\delta_0. \tag{4.12}$$

The value is independent of the chosen critical tangent and cutoff.

The matrix measure is the angular coefficient measure of the conic current: its radial coefficient disintegration is $r\,\mathrm{d}r\,\mathrm{d}\mathsf{M}$. Positivity controls all coefficients by the trace, and separate closedness gives the spherical divergence equation. In fact this construction classifies the corresponding real positive closed cones. The analytic cancellation problem concerns the pairs (Z_0, M) realized jointly by an original analytic zero relation and its allowed background.

Conjecture 4.4 (Cancellation of the realized critical pairing)**.** *For every supplied normal-three model germ, every actual quadratic relation in the finite closed-cell PL/PS class, every allowed degree-one background, and every common protected presentation, the right-hand side of (4.11) is zero.*

This is the precise remaining special case of quadratic descent in that model category. It is proved when $\nu = 0$, when Z_0 has an actual full-exact identity on an ordinary neighborhood of the protected central fiber, or when each scalar first jet is one ambient linear function. The last case follows from the angular balance identity

$$\int_{\mathbb{S}^{N-1}} \mathrm{d}\mathsf{M} = \int_{\mathbb{S}^{N-1}} \theta\theta^t \, \mathrm{d}\sigma. \tag{4.13}$$

Bounded-potential and the established weakly approximable background classes provide further known cases [15, 21].

The next geometric question is a wall-flux cancellation. Cellwise strain primitives already exist. Their differences on overlaps are affine Killing fields, and smooth interpolation across a wall produces an ε^{-1} derivative. The available normal strip estimate is $O(\varepsilon)$, so the resulting error can have order one. Separate closedness controls the compatible PS pairings, but this critical contribution must be eliminated using the actual zero relation and its joint source geometry. A successful argument should express the wall flux in an actual relation module and then annihilate it under refinement; an estimate that only treats each open cell cannot decide Conjecture 4.4.

Two existing examples delimit this approach. A real five-plane fan has locally normalized angular primitives and a nonzero global Killing obstruction, with a quantitative L^1 gap; it is D-closed but not separately d'- and d''-closed. It rules out automatic global patching from those local data. A separate real coefficient example has a positive closed plane background and a bounded PS tensor whose second derivative is a Dirac mass; it reaches the critical r^2 growth. Neither example is an original analytic zero relation. Their role is to identify the missing structural input, rather than to decide the analytic conjecture [21].

5 Relative capacity and the correct Borel question

5.1 The capacity attached to the full tester class

For S and V as in (2.6), define on Borel sets $E \subset V$

$$C_S(E;V) = \sup\left\{\int_E T_{S,q}(u^{[q]}) : u \in \mathcal{C}_b(V),\ 0 \le u \le 1\right\}, \tag{5.1}$$

$$C_S^*(E;V) = \inf\{C_S(G;V) : E \subset G \subset V,\ G \text{ ordinary open}\}. \tag{5.2}$$

The same outer formula applies to arbitrary subsets. The testers are defined on the whole specified V. They are actual classical potentials, so the definition keeps the original analytic constraint.

Relative quasicontinuity, bounded decreasing convergence in capacity, strict maximum locality, compact-gap comparison and the appropriate relative quasi-everywhere domination are established. The ambient background [1] also has its local polar-set theory and its qualified exhaustion-domain Choquet theory [15]. For a general S, an ambient polar set may carry background mass. Thus relative nullity must be defined by (5.2); it cannot be replaced by ambient pluripolarity.

Question 5.1 (Borel inner regularity of relative outer capacity)**.** For every strict Laurent box (2.6), every allowed extending background S, and every Borel $E \subset V$, is

$$C_S^*(E;V) \stackrel{?}{=} \sup\{C_S^*(H;V) : H \subset E,\ H \text{ compact}\} \tag{5.3}$$

The question includes the local relatively compact case and its passage to an exhaustion of V.

This is an outer-capacity question. The stronger equality $C_S = C_S^*$ is false even for compact sets in the original category. The crossed-box construction gives a full-tester compact gap and also excludes the corresponding continuity assertion for the raw capacity on decreasing compact sets [17]. These examples leave (5.3) undecided.

5.2 Exact examples and a restricted positive result

There is a second obstruction to transplanting the usual extremal-mass proof. Let $X = (\mathbf{G}^n_{m,K})^{\mathrm{an}}$ with $n \geq 2$, $\lambda = -\log|\pi|$, $t_i = \log|T_i|$, and

$$V = \{|t_i| < \lambda\}, \qquad \beta = \sum_i d't_i \wedge d''t_i, \qquad S_0 = \beta^{n-1} \wedge [X].$$

Put $\tau = t/\lambda$, $K_r = \tau^{-1}([-r,r]^n)$, and $A_n = n!2^n\lambda^{n-2}$. The complete original tester class satisfies

$$C_{S_0}(K_r;V) = C^*_{S_0}(K_r;V) = A_n\frac{r^{n-1}}{1-r}, \qquad 0 < r < 1. \tag{5.4}$$

For $h_r = \max\{(\|\tau\|_\infty - r)/(1-r), 0\} - 1$, every bounded actual classical u satisfying $h_r \leq u \leq 0$ and $u \leq -1$ on $\operatorname{Int} K_r$ has

$$\int_V \mathrm{D}u \wedge S_0 = \frac{A_n}{1-r}. \tag{5.5}$$

The ratio of (5.5) to (5.4) is r^{1-n}, which is unbounded as $r \downarrow 0$ [22]. A total relative extremal mass therefore cannot calibrate this capacity by a universal constant. The same original smooth-background setting also supplies failure of strong subadditivity for outer capacity; ordinary subadditivity survives. Strong subadditivity is a possible proof mechanism, rather than a necessary condition for the desired Borel inner regularity.

There is a useful positive range. If B is a Borel subset of a compactly contained moment box and $\mathcal{H}^{n-1}(B) < \infty$, then for the full saturated set $E = \tau^{-1}(B)$,

$$C^*_{S_0}(E;V) = \sup_{H \subset E \text{ compact}} C^*_{S_0}(H;V), \tag{5.6}$$

and saturated compact subsets suffice. The proof obtains a uniform Hausdorff bound for the full tester family; finite-measure inner regularity then makes the discarded remainder small uniformly in the testers. It extends to backgrounds S' for which both S' and $CS_0 - S'$ are allowed positive separately closed currents [22].

This suggests a concrete route for larger classes: obtain local uniform remainder estimates for the original tester measures, and then pass to Borel sets by an exhaustion that preserves those estimates. A countable family of dominating Radon measures, or a replacement by completed sources, would require a separate theorem. The finite-Hausdorff argument works because it proves precisely the uniform remainder bound needed for its specified sets.

Problem 5.2 (Detection and tester recovery)**.** Find intrinsic conditions on (V, S) and a class of sets under which normalized actual classical testers recover the intended compact masses and relative quasi-everywhere statements determine the desired pointwise representatives. The conditions should distinguish the relative null ideal from ambient pluripolarity and remain valid for the full tester class in (5.1).

Such conditions could connect capacity to geometric support or a verified local approximation of S. They would also supply the representative information needed in the singularity and boundary problems below.

6 Literal measures and the geometry of the source image

Fix (2.6) and its strong completion $\mathscr{C}$. The equation of interest is

$$\widehat{\mu}_u = \mu, \qquad u \in \mathscr{C}. \tag{6.1}$$

It has two logically distinct requirements: the prescribed measure must pair continuously with the strong potential topology, and its functional must lie in the source image.

6.1 The energy-difference space and literal continuity

The completed Borel difference space and its norm are

$$\mathbb{D}_S = \mathscr{C} - \mathscr{C}, \qquad \|h\|_{\mathrm{ed}} = \inf_{h=a-b}\{r(a) + r(b) : a, b \in \mathscr{C}\}. \tag{6.2}$$

This is a Banach space, with $\|u\|_{\mathrm{ed}} = r(u)$ for $u \in \mathscr{C}$. A positive Radon measure μ is a *literal energy-dual datum* if it annihilates the relative null ideal, every member of $\mathbb{D}_S$ is literally μ-integrable, and

$$\ell_\mu(h) = -\int_V h \,\mathrm{d}\mu, \qquad |\ell_\mu(h)| \le C_\mu \|h\|_{\mathrm{ed}} \quad (h \in \mathbb{D}_S). \tag{6.3}$$

This definition concerns the actual measure and all completed Borel differences. The measure may have infinite total mass on V.

Question 6.1 (Literal strong continuity)**.** For every fixed allowed S and strict Laurent box, does (6.3) imply

$$I(u_j, u) \longrightarrow 0 \quad \overset{?}{\Longrightarrow} \quad \int_V |u_j - u| \,\mathrm{d}\mu \longrightarrow 0 \qquad (u_j, u \in \mathscr{C}) \tag{6.4}$$

Source-dominated data and specific capacity-dominated classes already satisfy this continuity. The general question asks for the missing uniform integrability mechanism. Strong convergence controls the nonlinear energy discrepancy in (2.14); it need not be convergence in the difference norm (6.2). Boundedness of a functional for the latter norm therefore does not settle (6.4).

Let $P_S(a, b)$ be the greatest member of $\mathscr{C}$ lying below a and b in the relative quasi-everywhere order. Finite rooftops exist. One proved sufficient route to Question 6.1 is the following fixed-background ordered-rooftop continuity assertion:

$$\left.\begin{array}{l} w \preceq_{\mathrm{rel}} u, \quad u_j \longrightarrow u \text{ strongly in } \mathscr{C} \end{array}\right\} \quad \Longrightarrow \quad e(P_S(w, u_j)) \longrightarrow e(w). \tag{6.5}$$

Another sufficient route is sublinear growth of the energy roots of finite common minorants along subsequences. For example, a direct estimate

$$e(P_S(u_j : j \in F)) \le C \sum_{j \in F} e(u_j) \tag{6.6}$$

with C independent of the finite cardinality $|F|$, on the rows needed by the packing criterion, yields a sublinear root budget $O(|F|^{1/(q+1)})$. The established reductions and the exact weaker subsequence criterion are recorded in [15, 23].

The coefficient-one binary energy bound is false. In remaining degree $q = 1$, the universal bound

$$e(P_S(u, v)) \le 2(e(u) + e(v)) \tag{6.7}$$

has optimal constant 2 across allowed backgrounds, already for smooth strongly positive backgrounds and bounded actual obstacles [23]. Repeated use of this binary bound gives a quadratic worst-case energy budget in the number of obstacles and does not supply (6.6). The counterexamples also exclude an energy-only modulus uniform over changing backgrounds; the fixed-S assertion (6.5) remains open. The constant-one mixed-energy inequality (2.13) concerns a different expression and remains available.

6.2 Variational projection and an exact scalar obstruction

Define the source functional

$$J(u)(h) = -\int_V h \, \mathrm{d}\widehat{\mu}_u, \qquad J : \mathscr{C} \longrightarrow \mathbb{D}^*_S. \tag{6.8}$$

The image $J(\mathscr{C})$ is norm closed. The distance $\|J(u) - J(v)\|_*$ is complete and induces the strong topology. Let $\mathbb{D}^*_{S,\mathrm{s}}$ be the norm-dual functionals continuous on every strongly convergent cone row.

Theorem 6.2 (Projection and the normal-probe criterion)**.** *For every $\ell \in \mathbb{D}^*_{S,\mathrm{s}}$ there is a unique $u = \Pi(\ell) \in \mathscr{C}$ maximizing*

$$\ell(v) - \frac{e(v)}{q+1} \quad (v \in \mathscr{C}).$$

Its normal residual $n = \ell - J(u)$ satisfies

$$n(v) \le 0 \quad (v \in \mathscr{C}), \qquad n(u) = 0. \tag{6.9}$$

There is one bounded continuous strictly negative actual classical potential ψ_0, depending only on the box, for which every literal positive Radon datum μ with $\ell_\mu \in \mathbb{D}^*_{S,\mathrm{s}}$ has

$$\Xi_{S,\psi_0}(\mu) := -n(\psi_0) \ge 0, \qquad \Xi_{S,\psi_0}(\mu) = 0 \Longleftrightarrow \widehat{\mu}_{\Pi(\ell_\mu)} = \mu \Longleftrightarrow \mu \in \widehat{\mu}(\mathscr{C}). \tag{6.10}$$

When the equation is soluble, its member of $\mathscr{C}$ is unique. For every original compact smooth scalar test φ there is a finite constant C_φ, independent of S and μ, such that

$$\left| \int_V \varphi \, \mathrm{d}(\widehat{\mu}_{\Pi(\ell_\mu)} - \mu) \right| \le C_\varphi \Xi_{S,\psi_0}(\mu). \tag{6.11}$$

The proof of [16] constructs ψ_0 so that both signs of each original compact smooth direction are admissible small perturbations of it. Testing the normal inequality with these two perturbations proves (6.11). Literal Radon uniqueness then identifies the source measures when the defect is zero. This makes the remaining equation problem explicit: find conditions on the given data that eliminate the variational normal.

Problem 6.3 (A checkable description of compatible sources)**.** For a specified geometry (V, S), characterize substantial classes of literal positive Radon data μ for which $\ell_\mu \in \mathbb{D}^*_{S,\mathrm{s}}$ and $\Xi_{S,\psi_0}(\mu) = 0$. The hypotheses should be verifiable from μ, the geometry and explicitly supplied reference objects. Determine the corresponding existence, uniqueness and stability theorems in $\mathscr{C}$, together with precise obstructions outside these classes.

The zero-defect criterion is a complete test after the variational maximizer is known. Problem 6.3 asks for input conditions that make that equality follow. Assuming the desired source identity or a sharp norm equality involving the unknown optimizer would restate the conclusion.

6.3 Why smooth positive data do not give a universal conjecture

Tropical non-surjectivity is established in [11]; the relative source problem asks for its own geometric constraints on the given analytic domain and background. The fixed-source obstruction already appears in remaining degree one on a two-dimensional analytic torus [24].

Use $V = t^{-1}((-\lambda, \lambda)^2)$ and $S = (d't_1 \wedge d''t_1 + d't_2 \wedge d''t_2) \wedge [X]$. Every completed potential has a canonical convex moment trace g_u with

$$g_u \in H_0^1((-\lambda, \lambda)^2), \qquad t_* \widehat{\mu}_u = \Delta g_u. \tag{6.12}$$

The trace is defined canonically; arbitrary Borel values at individual Gauss points are not used.

There are fixed strict reference sources μ_w and smooth positive densities $f_j \to 1$ uniformly such that

$$f_j \mu_w \notin \widehat{\mu}(\mathscr{C}). \tag{6.13}$$

The data retain literal strong continuity and uniform source domination. Their zero-boundary real Poisson solutions fail convexity. Thus the linear trace identity (6.12) obstructs the existence of a source potential in the full prescribed strong completion. This is a geometric constraint on the positive potential cone, and it survives arbitrarily small uniform density perturbations.

A prospective local existence theorem should therefore choose a compatible data manifold or cone around a known reference w. On a class where the mixed expansion is justified, the linearization is

$$L_w h = q \, \mathrm{D}h \wedge (\mathrm{D}w)^{q-1} \wedge S. \tag{6.14}$$

Three ingredients would make an inverse construction meaningful: an independently described range of L_w with its boundary and moment conditions; a right inverse with estimates in spaces embedded in the required potential completion; and stability of that range under the mixed nonlinear remainder. The last condition is substantive. The available finite-angular-mode constructions impose weighted harmonic moment conditions on the linear inverse, and a second nonlinear residual can fail them [25]. A right inverse on the initial finite modes alone does not control the nonlinear iteration.

6.4 The completed linear equation in its natural space

Remaining degree one means $q = n - p = 1$, in arbitrary ambient dimension. For the specified affinoid part domains and relative faithful energy form, the signed measure equation has a developed renormalized theory [15, 12]. Here H is the energy space of that part form, $T_k v = \max(-k, \min(v, k))$, and Γ_S is its energy measure. A finite signed energy-diffuse Radon datum, meaning that its total variation vanishes on energy-polar sets, has a unique canonical zero-boundary renormalized solution v. Its clippings $T_k v \in H$ belong to the energy space, and its finite energy-quasicontinuous representative $\widetilde{v}$ satisfies

$$\mathrm{D}(\Phi(v)S) = \Phi'(\widetilde{v})\mu + \Phi''(\widetilde{v})\Gamma_S(v) \tag{6.15}$$

for $\Phi \in C^2$ with compactly supported derivative, together with the radiation condition $k^{-1}\mathcal{E}(T_{2k}v - T_k v) \to 0$. There are coarea, level-carrier and total-variation stability results.

This theorem identifies a natural solution space for the linear measure problem. Its solution need not be an ambient classical psh function or a member of the nonlinear source cone $\mathscr{C}$. The compatibility obstruction in (6.13) and the renormalized existence theorem are therefore complementary parts of the relative theory.

7 Domains, singularities and geometric applications

The Laurent-box hypotheses isolate a domain on which barriers, truncation estimates and global integration by parts are established. Extending the theory requires replacing the work performed by each of those hypotheses.

7.1 Boundary and representative problems

Problem 7.1 (A domain theory with explicit boundary control)**.** Find geometric conditions on an ordinary domain V and an allowed background S that imply the barrier construction, the global Green formula and uniform boundary tightness needed for the strong completion. Determine when exhaustion by relatively compact subdomains preserves the no-loss class, the source measures and the specified boundary condition.

A useful theorem would state whether the boundary is an ordinary pointwise value, a capacity boundary value, an energy trace or a boundary defined by approximation. It would then give the comparison between those notions under its geometric assumptions. A negative exhaustion potential by itself does not record all the estimates used in the Laurent-box construction; the missing ingredients include uniform control of cutoff errors and of weighted mass near the boundary.

Problem 7.2 (Classical representatives of completed potentials)**.** Characterize pairs (V, S) and subclasses of $\mathscr{C}$ for which every completed Borel class has a canonical ambient classical psh representative. Establish the continuity, comparison and boundary properties of that representative, and identify a relative detection condition that promotes quasi-everywhere equality to the intended pointwise equality.

The curvature quotient and relative capacity provide two possible inputs. Curvature determines a bounded potential modulo a pluriharmonic term, while capacity controls the exceptional sets of a completed representative. A representative theorem must connect these two facts using actual regularization or detection; it cannot identify their quotient spaces by definition.

7.2 Singularity types and relative mass

For negative actual potentials, a local singularity comparison $u \leq v + O(1)$ defines a preorder, and mutual comparison defines a singularity type. Non-polar products give a way to ask which masses and mixed energies are determined by such a type. In a relative theory the background is part of this question: it may charge an ambient polar locus, and a relative outer-null set need not be an ambient polar set.

Problem 7.3 (Relative singularity invariants)**.** For explicit classes of negative actual potentials and allowed backgrounds, determine which non-polar masses, no-loss conditions and energy integrability properties are invariant under bounded changes of singularity. Construct the associated envelopes and comparison theorems, specifying their ordinary domains, boundary conditions and relative null ideals. Describe how these invariants change under proper geometric modifications in mapping classes for which the required current operations are defined.

The capacity-tail formulas (2.10)–(2.11) provide a concrete starting point: they turn integrability into sublevel-set estimates. A useful first extension would combine these estimates with a controlled model singularity and prove stability of the truncation mass under a specified modification. This would produce information about singularity types without first requiring the entire graded differential interface.

7.3 Green currents and intersection theory

For a cycle Z of codimension r, a Green current has bidegree $(r-1, r-1)$ and satisfies an identity of the form

$$\mathrm{D}g_Z + \delta_Z = \omega_Z, \tag{7.1}$$

with a prescribed representative ω_Z of its class. Existing delta-form theories construct Green objects and intersection formulas in substantial geometric ranges [3, 4]. The relative BT program suggests an extension in which the metric potentials or coefficients have finite energy.

Problem 7.4 (Finite-energy Green products)**.** Construct classes of Green currents satisfying (7.1) for which mixed products and local intersection pairings are defined by integrable truncation limits. Prove independence of the admissible presentations and truncations, Green symmetry, and agreement with the existing smooth, formal and delta-form products on their common domains. State proper-intersection or replacement hypotheses that control products along the cycles.

For curvature-generated Green expressions, relation descent would make the local differential presentation intrinsic. The energy theory would provide bounds for the limiting integrals. Products with a cycle current and their functoriality would still require the stated intersection and integrability arguments. This separates the algebraic role of Conjecture 1.3 from the analytic role of energy estimates in a proposed Green product.

7.4 An arithmetic application with local and global conditions

The continuous theory already connects local current positivity with metric geometry. In the geometric scope of [13], continuous Chern products glue, enjoy the stated smooth-subvariety projection formula, and identify the prescribed local mixed measures of the relevant integrable adelic line bundles. With the exact boundary-completion identifications assumed there, finite continuous local CLD-positive weights give positivity of those prescribed Yuan–Zhang local measures. The separate geometric adelic-nefness input retains its own hypotheses [13, 8].

Problem 7.5 (Adelic finite-energy intersection)**.** For a specified class of adelic metrized line bundles on a quasi-projective variety, define local finite-energy intersection terms that extend the established continuous ones. Find explicit summability and boundary conditions under which their sum defines a global intersection pairing, independent of approximation and compatible with the product formula. Prove the resulting variation, projection and positivity statements under stated geometric positivity hypotheses.

The local and global requirements can be tested separately. At a non-Archimedean place, the local task is to establish the chosen mixed-energy integral and its change under replacement of a metric. Across places, one needs a dominating summable estimate and a compatible treatment of the quasi-projective boundary. Local positivity supplies the sign of a measure; an arithmetic positivity theorem also needs the global geometric assumptions. This division would make a finite-energy extension of adelic intersection theory verifiable one interface at a time.

8 A proof architecture for the program

The program has three immediate mathematical targets. The relation branch seeks cancellation of the critical pairing (4.11), followed by a return from supplied normal-three models to general ordinary germs and by treatment of higher coupled relations. The capacity branch seeks the uniform tester remainder estimates needed for (5.3). The equation branch seeks fixed-background control sufficient for (6.4) and independent conditions describing the closed source image.

These targets suggest distinct forms of progress. For the first, one needs an actual geometric wall-flux identity, an actual full-exact construction, or a counterexample retaining the complete original analytic relation. The real angular classification makes the residual

explicit, but analytic joint realization supplies further constraints that a successful proof must use. For the second, a positive result should exhibit a remainder estimate uniform in the original normalized testers; a negative result should use those same testers and ordinary Borel sets. For the third, an existence theorem should derive compatibility from data and known reference geometry, while an obstruction should identify the exact constraint that the datum violates.

Direction	Established input	Missing mathematical step
Intrinsic relations	Bounded calculus, degree-one action, quadratic transgression, rank-two construction	Critical joint cancellation; general-germ return; higher and coefficient relations
Relative capacity	CLN, quasicontinuity, comparison, qualified Hausdorff control	Uniform Borel remainder control for the original testers
Literal data	Strong completion, energy difference norm, source and capacity dominated classes	Fixed-background uniform integrability or the corresponding actual counterexample
Nonlinear sources	Closed source image, variational projection, normal-probe criterion	Independent compatible data classes and stability of their nonlinear constraints
Geometric applications	Green symmetry, local Chern products, existing delta and adelic theories	Mapping, singularity, intersection and global summability interfaces

The foundations support these directions without a circular dependence. The bounded recursion is available before the analytic relation conjecture. The capacities and strong energy spaces are built from that recursion. Literal continuity would enlarge the data to which the variational projection applies; it would not remove its normal residual. Source compatibility is a further condition. Conversely, explicit source classes can be constructed and analyzed without first settling literal continuity for every energy-dual datum.

A complete theory in this sense would give a coherent account of objects, operations, limits and compatible equations. It would include the nonexistence and strict-inequality phenomena already described here, together with the hypotheses that replace the false universal assertions. The present conjectures and problems specify the mathematical steps from the established potential calculus to that theory.

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