# CONTINUOUS CLD-POSITIVE POTENTIALS AND MIXED CURRENTS

Abstract. Let $K$ be a non-Archimedean completion of a number field and let $X/K$ be smooth and quasi-projective. A finite continuous function on an ordinary open of $X^{\mathrm{an}}$ has positive Chambert–Loir–Ducros curvature current if and only if it admits smooth positive approximations on a fixed neighbourhood of each point. The proof combines restriction to curves, deterministic complete-intersection approximation, contact rigidity, and continuity of semipositive envelopes. Contact rigidity identifies a continuous curve-positive metric from its contact with a uniform Fubini–Study limit. On a fixed regular model, sums of divisor test ideals and global generation give one-sided $O(m^{-1})$ Fubini–Study approximations for model-obstacle envelopes. A single projective extension then completes local regularization. As consequences, continuous current-positive potentials admit symmetric mixed products with any positive closed CLD current, and these products depend only on their curvature currents. We also obtain a projective curve criterion under a regular-model hypothesis and positivity of the prescribed Yuan–Zhang local mixed measures.

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## 1. Introduction

Positivity of a Chambert–Loir–Ducros current is a condition on smooth test forms. To construct positive mixed products, one must also control $d'd''(uT)$ for each intermediate positive closed current $T$. Multiplication by a continuous potential is available, but positivity after differentiation requires an additional argument. Local approximation by smooth functions with positive curvature provides that argument. The first question addressed here is whether the current inequality itself supplies such an approximation.

1.1. **Local regularization.** Fix a number field $k$, a non-Archimedean place $v$, and $K = k_v$, with its given absolute value. Its valuation ring is $R$. Choose a uniformizer $\pi$ and set $\lambda = -\log|\pi| > 0$. All smooth forms, positive cones, and currents in this paper are those of the first version of Chambert–Loir–Ducros [3]. In particular, positivity of currents is dual to the positive cone of [3, Definition 5.1.1(c)]. An *ordinary open* is open in the Berkovich topology.

Let $X/K$ be smooth of pure dimension $n$, let $O \subset X^{\mathrm{an}}$ be an ordinary open, and let $u : O \to \mathbf{R}$ be finite and continuous. For $n \geq 1$, the condition $d'd''[u] \geq 0$ means

$$\langle d'd''[u], \eta \rangle = \int_O u\, d'd''\eta \geq 0 \quad \text{for every positive } \eta \in \mathcal{A}_c^{n-1,n-1}(O). \tag{1.1}$$

Every compactly supported smooth test is allowed. For components of different dimensions the condition is imposed componentwise, and there is no curvature condition in dimension zero.

We write $u \in \mathcal{P}_{\mathrm{loc}}(O)$ if every $x \in O$ has one ordinary neighbourhood $V \subset O$ on which there are smooth functions $u_j$ with $d'd''u_j \geq 0$ as forms and $u_j \to u$ uniformly on compact subsets of $V$. The neighbourhood is fixed for the entire sequence.

**Theorem 1.1** (Local regularization)**.** *Let $K = k_v$ be a non-Archimedean completion of a number field, with its given absolute value. Let $X/K$ be smooth and quasi-projective, $O \subset X^{\mathrm{an}}$ an ordinary open, and $u : O \to \mathbf{R}$ finite continuous. Then the following are equivalent:*

(i) *The original CLD curvature current $d'd''[u]$ is positive on $O$ in the sense of (1.1).*
(ii) *The function $u$ belongs to $\mathcal{P}_{\mathrm{loc}}(O)$.*

*After shrinking the fixed neighbourhood of each point once, the approximating sequence can be chosen decreasing and uniformly convergent on that neighbourhood.*

The implication from smooth positive approximation to current positivity follows by testing locally uniform convergence against $d'd''\eta$. The reverse implication does not assume that $u$ is a difference of approachable functions. The field and continuity hypotheses remain in force throughout the theorem; in particular, the assertion does not concern arbitrary upper-semicontinuous potentials. The proof is completed in Section 7.2.

1.2. **Contact rigidity.** The projective argument uses a criterion that identifies a metric from its contact with a semipositive lower metric. Recall that a *pure Fubini–Study metric* on an ample line bundle $L$ is given by weights $m^{-1}\max_i \log|s_i|$, where the sections $s_i$ generate $mL$. A *uniform FS limit* is a uniform limit of such metrics. A continuous metric is *curve-positive* if its pullback by every algebraic morphism from a smooth $K$-curve has positive Chern current. These conventions are developed in Section 3.

*Contact rigidity* (Theorem 5.3). Let $Y/K$ be smooth, projective, and integral of dimension $n \geq 1$, and let $L$ be ample. Let $p$ be a uniform FS limit on $L$, and let $r$ be a finite continuous curve-positive metric on $L$. Then

$$r \geq p, \qquad \int_{Y^{\mathrm{an}}} (r-p)c_1(L,p)^n = 0 \quad \Longrightarrow \quad r = p. \tag{1.2}$$

The metric $p$ need not be constructed as an envelope of $r$. The criterion uses the Monge–Ampère measure of the lower metric and positivity of the upper metric only on curves. It therefore applies before a higher-dimensional positive product for $r$ is available. The proof in Section 5.3 passes curve Green identities through complete-intersection approximation and then transfers the contact identity to smooth divisors.

1.3. **Continuous mixed currents.** A current is called *closed* below if both $d'T = 0$ and $d''T = 0$. Positive currents admit multiplication by continuous functions (Lemma 2.2). Thus, for a positive closed current $S$, the recursive rule

$$\mathcal{B}_S() = S, \qquad \mathcal{B}_S(u_1,\dots,u_m) = d'd''(u_1\mathcal{B}_S(u_2,\dots,u_m)) \tag{1.3}$$

is meaningful whenever its intermediate currents are positive. This is the usual Bedford–Taylor construction.

**Theorem 1.2** (Continuous potentials on a positive closed background)**.** *Let $K = k_v$, let $X/K$ be smooth quasi-projective of pure dimension $n$, and let $O \subset X^{\mathrm{an}}$ be an ordinary open. Let $S$ be an original CLD positive current of bidegree $(p,p)$, where $0 \leq p \leq n$, with $d'S = d''S = 0$. Suppose $0 \leq m \leq n-p$ and $u_1,\dots,u_m : O \to \mathbf{R}$ are finite continuous functions with $d'd''[u_i] \geq 0$. Then (1.3) defines a positive closed current of bidegree $(p+m,p+m)$. It has the following properties.*

(i) *It is local, symmetric in the potentials, additive and positively homogeneous in each potential and in the background.*

(ii) *If $S_a \to S$ on every compact smooth test, where all $S_a$ are positive and closed of bidegree $(p,p)$, and the finite continuous functions $u_{i,a} : O \to \mathbf{R}$ converge to $u_i$ locally uniformly with $d'd''[u_{i,a}] \geq 0$, then*

$$\mathcal{B}_{S_a}(u_{1,a},\dots,u_{m,a}) \longrightarrow \mathcal{B}_S(u_1,\dots,u_m)$$

*on every compact smooth test.*

(iii) *Each potential enters only through its curvature current: if $w$ is finite continuous and $d'd''[w] = d'd''[u_i]$, replacing $u_i$ by $w$ leaves the product unchanged.*

*In top degree the product is a positive Radon measure.*

The recursion is described in [3, Section 5.6.8]. For continuous psh potentials, its positivity, continuity, and symmetry are established in [10, Lemma 8.5.5 and Propositions 8.7.2, 8.7.4]. Theorem 1.1 allows us to start from current positivity alone. Potential independence then has a short two-sided proof. If $d'd''[h] = 0$, then both $h$ and $-h$ satisfy Theorem 1.1. For every positive closed $T$ this gives

(1.4)
$$d'd''(hT) \geq 0, \qquad -d'd''(hT) = d'd''((-h)T) \geq 0, \qquad \text{hence } d'd''(hT) = 0.$$

No representation of $h$ as a piecewise-linear or logarithmic function is required. Theorem 1.2 concerns continuous potentials; it does not assert a monotone convergence theorem for bounded discontinuous ones. The theorem is proved in Section 8.1.

1.4. **Adelic mixed measures.** For the arithmetic application, use the rational boundary completion of [9, Sections 2.4–2.5]. Its objects retain the specified line-bundle identifications. A strongly nef object has a Cauchy presentation by nef model objects, and an integrable object is a difference of two strongly nef objects, with rational tensor powers allowed. The order defining the completion compares the full Cartier-and-metric boundary datum, as recalled in Section 9.

**Theorem 1.3** (Positivity of the prescribed adelic mixed measure)**.** *Let $k$ be a number field, $v$ a non-Archimedean place, and $K = k_v$. Let $U/k$ be smooth quasi-projective of pure dimension $n \geq 1$. Let $\overline{L}_1, \dots, \overline{L}_n$ be rational integrable adelic line bundles with their specified identifications in the boundary completion of* [9, Sections 2.4–2.5]*. Suppose their local weights at $v$ are finite continuous and have positive original CLD curvature currents. Then the signed measure prescribed in* [9, Sections 3.6.7–3.6.8] *is positive:*

$$c_1(\overline{L}_1)_v \cdots c_1(\overline{L}_n)_v \geq 0 \quad \text{on } U_K^{\mathrm{an}}.$$

The signed measure is defined by expanding integrable presentations. Its positivity follows by identifying that expansion with the local positive product of the target weights. This conclusion concerns the given measure and does not assert arithmetic nefness of the factors. The comparison is given in Section 9.1.

1.5. **Relation to earlier work.** Chambert–Loir–Ducros distinguish distributional positivity from local psh approachability in [3, Definition 5.6.3 and Remark 5.6.4]; their stable products are treated in [3, Corollary 5.6.5 and Section 5.6.8]. The later continuous psh class is defined by uniform closure [10, Definition 8.1.7]. We do not identify a continuous current-positive function with a member of that class before proving Theorem 1.1.

On curves, Wanner compares CLD and Thuillier subharmonicity [11, Corollaries 4.8 and 5.4]. Gubler–Jell–Rabinoff provide descent, harmonic tropicalizations, and smooth approximation [4, 5]. The finite residue field is

important here: weakly smooth functions on the relevant curves are CLD-smooth. Section 3 retains the passage from the original current inequality over $K$ to those curve results.

Torus-invariant pluripotential theory is developed in [12]. Piecewise-linear semipositivity and positivity tested on curves are studied in [14, 15], and Wu compares local approximation classes in [16, Appendix A]. Our starting hypothesis is instead the original current inequality (1.1) for an arbitrary finite continuous potential on a smooth variety.

The envelope argument follows the test-ideal method of Fang–Gubler–Künnemann [13], using the mixed-characteristic test ideals of [6]. We use test ideals of section divisors directly. The comparison on a higher normal model allows an obstacle on a higher normal model while subadditivity is applied on a fixed regular model. It is not a direct application of [13, Theorems 5.1 and 7.1]. Orthogonality, once continuity is established, is [8, Theorem 6.7]. The final adelic application identifies a measure already constructed by Yuan–Zhang; it does not introduce another mixed measure.

1.6. **Proof strategy.** The proof separates restriction, contact, and envelopes so that none uses regularization in a form still to be established.

*Step 1: restrict the current inequality and extend locally.* In a normal disc, let $\beta_N = d'd''R_{2,\varepsilon}(\log|t|, -N)$, where $R_{2,\varepsilon}$ is a smooth convex regularized maximum. For the zero section $i : B \hookrightarrow B \times D_s$, Poincaré–Lelong and continuous multiplication give

$$\beta_N \longrightarrow i_*[B], \qquad u\beta_N \longrightarrow i_*[u|_B]. \tag{1.5}$$

Testing against pullbacks from $B$ proves positivity on the zero section. The étale normal form of a smooth pair and Berkovich's theorem on étale germs then give restriction to smooth subvarieties. Curve regularization and removal of finitely many closed points prove positivity after every algebraic curve map. An algebraic étale presentation also supplies an ambient analytic retraction, extending $u$ to a continuous curve-positive function $v$ near the original point (Section 4).

*Step 2: approximate mixed currents by curves and prove contact.* For a generating tuple $s_0, \dots, s_b$, choose smooth cuts of the form

$$\sigma_N = \sum_{i=0}^{b} a_{i,N} s_i^N, \qquad a_{i,N} \in \pi^i(1 + \pi R).$$

On a tropical presentation of any fixed smooth test, equality of the norms of two terms lies on an affine hyperplane, apart from finitely many exceptional $N$. Distinct coefficient valuations exclude ties that persist for all $N$. Absolute tropical integration therefore gives an eventual $O(N^{-1})$ error on that test. Successive cuts and weak closure yield normalized individual smooth complete-intersection curves, with fixed intersection degrees (Proposition 5.1).

For the contact criterion (1.2), put $g = r - p$ and $\alpha = c_1(L, p)$. Passing curve Green identities through the approximation of $T = \alpha^{n-1}$ gives

$$\nu = d'd''(gT) = \sigma - \alpha^n, \qquad \sigma \geq 0, \qquad 0 \leq -\int g\, d\nu = -\int g\, d\sigma \leq 0.$$

The nonnegative energy of $g + th$, for smooth $h$ and $t \in \mathbf{R}$, then forces $\nu = 0$. For a smooth divisor $D = \operatorname{div}(s)$ in $mL$, set $\rho_N = \max\{mp, \log|s| + N\}$. Fixed-potential continuity gives

$$c_1(mL, \rho_N) \wedge \alpha^{n-1} \longrightarrow i_*(c_1(L|_D, p|_D)^{n-1}). \tag{1.6}$$

Green symmetry transfers contact to $D$, and induction on dimension proves (1.2) (Section 5). No mixed product with the upper metric $r$ is assumed in this argument.

*Step 3: construct continuous envelopes on a fixed regular model.* Represent a model obstacle on a higher normal model $\mathscr{W}$ as $r = \theta + \log|\mathscr{E}|$, with $\mathscr{E}$ an invertible vertical ideal. For each admissible section $s \in H^0(Y, lL)$, let $D_s$ be its divisor on a fixed regular model $\mathscr{X}$. We take the sum

$$\mathfrak{j}_m = \sum_{l \geq 1} \sum_{0 \neq s \in V_l} \tau_{\mathscr{X}}\left(\frac{m}{l} D_s\right), \qquad V_l = \{s : \|s\|_{lr,\sup} \leq 1\}.$$

Subadditivity gives the lower bound below. Birational comparison, followed by clearing a fixed discrepancy denominator, gives the upper bound:

$$\pi^B \mathfrak{j}_m \mathcal{O}_{\mathscr{W}} \subseteq \mathscr{E}^m, \qquad P_L(r) \leq \theta + \frac{1}{m}\log|\mathfrak{j}_m| \leq r + \frac{B\lambda}{m}. \tag{1.7}$$

Global generation and one fixed auxiliary line bundle turn these bounds into pure-FS approximations of $P_L(r)$ on $L$. On $\mathscr{X} = \mathbf{P}^a_R$, the identity $K_{\mathscr{X}} + (a+1)\mathcal{O}(1) = 0$ cancels the auxiliary twist directly and gives

$$0 \leq q_m - P_L(r) \leq \frac{B\lambda}{m}. \tag{1.8}$$

Model-metric density and the one-Lipschitz property of the envelope treat continuous obstacles. Orthogonality [8, Theorem 6.7] then applies (Section 6.2).

*Step 4: extend once to projective space and apply contact rigidity.* After adding a constant and shrinking the ambient neighbourhood, arrange $0 < v + c < \lambda/2$. Choose one compact polynomial norm box $C$ and an FS metric $q$, with a section $s_0$, such that

$$b = -\log\|s_0\|_q, \qquad b = 0 \text{ near } x, \qquad \{b \leq \lambda\} = C.$$

On $\{b < \lambda\}$, use the weight $\max\{v + c, b\}$; on $\{b > \lambda/2\}$, use $q$. The two definitions agree on the overlap and give a global curve-positive metric $r$ (Lemma 7.1). Its envelope $p = P_L(r)$ satisfies the contact identity, so Theorem 5.3 gives $r = p$. Smooth FS approximation and restriction to one fixed neighbourhood recover $u$. Explicit constant translations make the approximants decreasing (Section 7.2). The same envelope–contact

argument gives the projective curve criterion under the retained regular-model hypothesis (Corollary 8.4).

1.7. **Organization.** Section 2 proves the continuous-coefficient estimate and the continuity of a fixed approachable potential acting on positive closed currents. Section 3 supplies smooth Fubini–Study approximation, the measure normalization, and the Green identities on curves. Section 4 establishes restriction and the ambient extension. Section 5 combines complete-intersection approximation with the direct divisor limit to prove contact rigidity. Section 6 proves envelope continuity on a fixed regular model, with projective space as a corollary. Section 7 constructs the projective extension and completes local regularization. The mixed-current and curve-criterion consequences are proved in Section 8; Section 9 identifies the prescribed adelic measure. Appendix A records a quantitative comparison of mixed measures, independently of the contact argument.

## 2. Continuous coefficients and approachable potentials

We use the smooth cutoff and density results [3, Corollary 3.3.4 and Propositions 3.3.5–3.3.6], the Poincaré–Lelong formula [3, Theorem 4.6.5], and uniform stability of products [3, Corollary 5.6.5]. Weak convergence of currents always means convergence on every compactly supported smooth test. Convergence of measures on continuous tests will be stated explicitly.

**Definition 2.1.** On an ordinary open $O \subset X^{\mathrm{an}}$, a smooth function is locally of the form

$$F(\log|f_1|, \dots, \log|f_r|),$$

where the $f_i$ are analytic units and $F$ is smooth near their logarithmic image. The class $\mathcal{P}_{\mathrm{loc}}(O)$ is defined in Section 1. We write $u \in \mathcal{D}_{\mathrm{loc}}(O)$ if $u$ is locally a difference of two members of $\mathcal{P}_{\mathrm{loc}}$. For metrics, these conditions apply to their weights in local frames. The weight in a frame $e$ is $-\log\|e\|$.

A change of frame changes a weight by the logarithm of the norm of an analytic unit. That function is smooth and has zero $d'd''$. Continuous multiplication of positive currents is known from [3, Corollary 5.4.7] and [10, Lemma 8.5.1]. We give the coefficient construction because it also controls weakly convergent nets.

**Lemma 2.2** (Continuous coefficients)**.** *Let $O$ be an ordinary open in a smooth pure $n$-dimensional $K$-variety. Let $T_a$ be positive currents of bidegree $(n-q, n-q)$, $0 \le q \le n$, converging weakly to $T$. Then $T$ is positive. For $S = T$ and for each $S = T_a$, every finite continuous $f : O \to \mathbf{R}$ determines a unique current $fS$ by*

$$\langle fS, \eta \rangle = \lim_{j\to\infty} \langle S, f_j \eta \rangle,$$

*where $\eta$ is any compactly supported smooth test and $f_j$ are smooth functions converging uniformly to $f$ near $\operatorname{supp}\eta$. The limit exists and is independent of the choice of $f_j$.*

*On a fixed compact test chart, the scalar coefficient measures of $T_a$ have eventually uniformly bounded total variations. The same holds after wedging $T_a$ with any fixed smooth form, in the bidegrees where the wedge is defined. The cone of positive currents is pointed. If $f_a \to f$ locally uniformly, with $f_a, f$ finite continuous, then*

$$f_a T_a \longrightarrow fT \tag{2.1}$$

*on every compact smooth test. Nets are allowed.*

*Proof.* Positivity of $T$ follows by taking the limit on each positive test. We establish the coefficient estimate locally and then use a finite smooth partition. Fix a moment chart with $r$ moment coordinates, a compactly supported $(q,q)$-form $\eta$, and a nonnegative smooth cutoff $\chi$ equal to one near its support. Let $J$ denote the involution interchanging the two differential factors, and write

$$\eta_{\mathrm{s}} = \tfrac{1}{2}(\eta + (-1)^q J\eta), \qquad \eta_{\mathrm{a}} = \eta - \eta_{\mathrm{s}}.$$

Wedging $\eta_{\mathrm{a}}$ with $\gamma \wedge J\gamma$, for an $(r-q,0)$-form $\gamma$, gives zero. In the convention [3, Definition 5.1.1(c)], both signs of $\eta_{\mathrm{a}}$ are therefore positive. A positive current annihilates $\eta_{\mathrm{a}}$, also after multiplication by a smooth real scalar.

For symmetric forms the positivity test is a finite-dimensional positive-semidefinite matrix condition. The form

$$\Omega = \frac{1}{q!}\left(\sum_{j=1}^{r} d'x_j \wedge d''x_j\right)^q \qquad (\Omega = 1 \text{ if } q = 0)$$

has a positive-definite coefficient matrix. Boundedness of the coefficient matrix of $\eta_{\mathrm{s}}$ on the fixed compact set gives $C_\eta$ such that, for every smooth real $h$,

$$C_\eta \|h\|_{\operatorname{supp}\eta} \chi\Omega \pm h\eta_{\mathrm{s}} \ge 0.$$

If $h$ uses additional moment coordinates, pull this inequality to a common refinement; positivity is preserved. The forms extend by zero outside the chart. For $S = T_a$ or $T$, we obtain

$$|\langle S, h\eta \rangle| \le C_\eta \|h\|_{\operatorname{supp}\eta} \langle S, \chi\Omega \rangle. \tag{2.2}$$

At this stage $h$ is smooth, so the pairing is already defined.

The right-hand evaluations for $T_a$ converge and are eventually bounded. Apply (2.2) to the finitely many coordinate test forms. Their scalar

functionals extend uniquely to continuous compactly supported functions, and the Riesz theorem represents them by signed Radon measures. The same estimate bounds their total variations. These extensions agree on overlaps by uniform smooth density. On a fixed test stage $C_\eta$ is bounded by a constant times the supremum norm of the coefficient matrix, so the extended functionals define currents.

We can now define $fS$, for $S = T$ or $S = T_a$, by multiplying these coefficient measures by $f$. Smooth density supplies the functions $f_j$ in the statement; estimate (2.2) proves that their pairings converge to this product independently of the approximation. The same local estimate proves continuity on each smooth test stage.

For fixed continuous $f$, choose smooth $f_j \to f$ uniformly on an enlarged compact set. For an eventually uniform $M_\eta$,

$$|\langle fT_a - fT, \eta\rangle| \le |\langle T_a - T, f_j\eta\rangle| + M_\eta \|f - f_j\|_\infty.$$

First fix $j$, let $a$ tend to its limit, and then let $j$ increase. This proves $fT_a \to fT$. For varying $f_a$, the additional term satisfies

$$|\langle (f_a - f)T_a, \eta\rangle| \le M_\eta \|f_a - f\|_{\operatorname{supp}\eta} \longrightarrow 0.$$

A finite partition treats any compact test. The coefficient bounds also control wedges with a fixed smooth form, whose coefficients are bounded on compact charts. Finally, if both $S$ and $-S$ are positive, their evaluations on $\chi\Omega$ vanish; applying (2.2) to $S$ gives $S = 0$. □

**Proposition 2.3** (A fixed approachable potential)**.** *Let $T$ be a positive current on $O$ with $d'T = d''T = 0$, and let $u \in \mathcal{P}_{\mathrm{loc}}(O)$. Whenever the bidegrees permit it,*

$$M_u(T) := d'd''(uT)$$

*is positive and closed. For fixed $u$, the operation $M_u$ is continuous on the cone of positive closed currents for weak convergence. The same is true jointly when $u_a \to u$ locally uniformly and all $u_a, u$ belong to $\mathcal{P}_{\mathrm{loc}}$. Successive such operations are symmetric in their potentials.*

*Proof.* On a fixed smaller neighbourhood choose smooth $u_j \to u$ with $d'd''u_j \ge 0$. Closedness gives

$$d'd''(u_jT) = d'd''u_j \wedge T \ge 0.$$

The curvature $d'd''u_j$ is a symmetric positive $(1,1)$-form, hence strongly positive, so its wedge preserves positivity. Lemma 2.2 and continuity of differentiation give the positive limit $d'd''(uT)$. Both differentials annihilate this limit. Equation (2.1) proves the two continuity assertions. Induction gives joint continuity for successive operations. Simultaneous smooth positive approximations of finitely many potentials then give symmetry from the smooth wedge product. □

For a continuously metrized line bundle with approachable weights, write $c_1(L,p)\wedge T$ for this operation in local frames. The local definitions agree because a transition logarithm $\ell$ is smooth, $d'd''\ell = 0$, and $d'd''(\ell T) = 0$.

## 3. Metrics and Green identities on curves

### 3.1. Fubini–Study metrics and normalization.

**Definition 3.1.** Let $L$ be an ample rational line bundle on a projective $Y/K$. A *pure Fubini–Study metric* on $L$ is defined by an integer $d > 0$, with $dL$ an actual line bundle, and a generating tuple $s_0, \dots, s_b \in H^0(Y, dL)$. Its weight in a local frame is

$$q_e = \frac{1}{d} \max_{0\le j\le b} \log |s_j/e^{\otimes d}|.$$

Coefficients multiplying the sections belong to $K$. A *uniform FS limit* is a uniform limit of these metrics. A metric is *globally smoothly approximable* if it is a uniform limit of global smooth metrics with positive curvature forms.

A model metric is *nef* if, after clearing tensor denominators, its model line bundle has nonnegative degree on every integral curve of the special fibre. A continuous metric is *semipositive* if it is a uniform limit of nef model metrics. For an ample rational line, this is equivalent to being a uniform pure-FS limit [7, Corollary 7.6 and Theorem 7.8]. All uniform differences of metrics are scalar differences on the same specified line bundle.

We use the standard regularized maximum from [3, proof of Proposition 6.3.2].

**Proposition 3.2** (Regularized maximum)**.** *Fix an even nonnegative $C^\infty$ function $\rho$, supported in $[-1,1]$, with integral one. For $h \ge 1$ and $\varepsilon > 0$, put*

$$R_{h,\varepsilon}(t) = \int_{\mathbf{R}^h} \max_i (t_i + \varepsilon z_i) \prod_{i=1}^{h} \rho(z_i)\, dz.$$

*This function is smooth, convex, and nondecreasing in each variable. It satisfies*

$$\max_i t_i \le R_{h,\varepsilon}(t) \le \max_i t_i + \varepsilon, \qquad R_{h,\varepsilon}(t + c\mathbf{1}) = R_{h,\varepsilon}(t) + c.$$

*An entry more than $2\varepsilon$ below the maximum is inactive on a neighbourhood of that point. If smooth $a_1, \dots, a_h$ have positive curvature, so does $R_{h,\varepsilon}(a_1, \dots, a_h)$.*

Translation covariance gives the metric transition law, and the inactive-coordinate property handles zeros of the generating sections. These are precisely the features needed in the next lemma.

**Lemma 3.3** (Smooth approximation of FS metrics)**.** *A pure-FS metric, and hence a uniform FS limit, is globally smoothly approximable. If $p$ is globally*

*smoothly approximable on $L$, if $m > 0$ is an integer for which $mL$ is a line bundle, and if $s \in H^0(Y, mL)$, then for each $N \in \mathbf{R}$ the metric*

$$\rho_N = \max\{mp, \log|s| + N\}$$

*on $mL$ is globally smoothly approximable.*

*Proof.* For a pure-FS metric from sections of $dL$, pull back the smooth positive approximations of the canonical metric on $\mathcal{O}_{\mathbf{P}^b}(1)$ from [3, Proposition 6.3.2], and divide their weights by $d$. A diagonal choice treats uniform FS limits.

The second assertion is immediate for $s = 0$. Otherwise, take smooth positive $p_j \to p$ and regularize $\max\{mp_j, \log|s| + N\}$. The logarithmic entry is inactive near the divisor of $s$; elsewhere it is smooth with zero curvature. The resulting global metrics are positive and have error at most $m\|p_j - p\|_\infty + \varepsilon_j$. This is the same regularized-maximum construction, applied to a smooth weight and a section weight. □

**Lemma 3.4** (Degree and comparison of measures)**.** *On a smooth projective $Y/K$ of dimension $n$, globally smoothly approximable metrics on $L_1, \dots, L_n$ have total mixed mass* $\deg_K(L_1 \cdots L_n)$*. For uniform FS limits, the model-theoretic mixed measure agrees with the CLD product.*

The mass assertion is [3, Corollary 6.4.4]. The model comparison is [3, Theorem 6.9.3 and Proposition 6.9.7], with the discretely valued normalization of [3, Section 6.9.6]. For uniform FS limits, use uniform continuity of the model measures [8, Proposition 2.23(d) and Remark 2.24] and of the approachable CLD products. Rational tensor denominators are cleared and divided on both sides.

Degrees in this paper are over the stated ground field. A variety over a finite extension $F/K$, regarded as a $K$-variety, contributes the structural factor $[F : K]$ to intersection and integration. Indeed, a moment map followed by $(\mathbf{G}_m^r)_F \to (\mathbf{G}_m^r)_K$ has that degree. This operation is distinct from scalar extension of a $K$-variety: the normalized scalar-extension integration formula used next has no extra degree factor.

**Definition 3.5.** A continuous function on an ordinary algebraic analytic open is *curve-positive* if its pullback by every algebraic morphism from a smooth $K$-curve has positive CLD curvature current on the inverse-image open. For a continuous metric the same condition is imposed on the pulled-back line bundle. Curves may be disconnected. The definition uses algebraic curve maps over the given $K$; it does not build positivity after arbitrary field extension into the hypothesis.

### 3.2. **Regularization and energy in dimension one.**

**Proposition 3.6** (Curve regularization)**.** *Let $C/K$ be a smooth algebraic curve and $W \subset C^{\mathrm{an}}$ an ordinary open. A finite continuous CLD-positive function on $W$ belongs to $\mathcal{P}_{\mathrm{loc}}(W)$. A finite continuous function with zero Laplacian on a connected proper smooth curve is constant.*

*Proof.* For a finite Galois extension $F/K$, average a nonnegative compact smooth scalar test $\eta$ on $W_F$ over the Galois group. The invariant average descends by [4, Proposition 14.2]. It is smooth because weakly smooth and CLD-smooth forms coincide on these curves [4, Remark 15.10]: their residue fields are algebraic over a finite field. If $\overline{\eta}$ is the descended test, the scalar-extension integration formula [4, Proposition 14.5] gives

$$\int_{W_F} u_F \, d'd''\eta = \int_W u \, d'd''\overline{\eta}.$$

Uniform smooth density permits the continuous coefficient $u$. Consequently current positivity passes to finite Galois extensions.

Let $Y$ be a compact strictly affinoid curve domain inside $W$. After such an extension choose a semistable skeleton $\Gamma$ with retraction $\tau$, edge lengths, and positive weights $w_e$. For an edgewise smooth $\varphi$, constant near vertices and zero near boundary vertices, $\varphi \circ \tau$, extended by zero, is a compact smooth test. Its Laplacian has no vertex atoms, and integration on the annular edges gives

$$\int_\Gamma \varphi \, d\tau_*(d'd''[u]|_Y) = \sum_e w_e \int_e u|_e \, \varphi_e'' \, ds. \tag{3.1}$$

This is also the retraction comparison [5, Proposition 6.3.2]; it does not assume that $u$ factors through $\tau$.

Tests inside an edge show that $u|_e$ is convex. The positive Radon measure $d'd''[u]$ has finite mass on the compact domain. Thus its edge second derivatives have finite mass and finite one-sided derivatives at the endpoints. Integration by parts in (3.1), allowing arbitrary constants near the interior vertices, gives a nonnegative weighted sum of outgoing slopes at each such vertex. The finite-graph maximum principle therefore applies to $u$.

To pass from graph inequalities to subharmonicity, we follow the harmonic-comparison argument of [11, Theorem 4.7]. A harmonic comparison function becomes skeletal after semistable refinement [4, Section 15.6 and Proposition 15.7]. Its boundary comparison with $u$ holds on that skeleton. Any prescribed finite set of type-II points can be included in a refinement after a further finite Galois extension [5, Remark 5.4.2]. The boundary and harmonic compatibilities are [5, Lemma 5.4.1 and Proposition 5.6.8]. Density of these points and continuity prove harmonic comparison on all of $Y$. Hence $u$ is Thuillier-subharmonic.

By [4, Proposition 15.11], $u$ has a decreasing net of weakly smooth psh approximants on a fixed relatively compact open. They are CLD-smooth by the finite-residue-field comparison. On a smaller compact neighbourhood, Dini's finite-cover argument selects a uniformly convergent sequence. On a curve, positivity of the curvature current of a smooth function implies positivity of its curvature form: a negative coefficient on an open one-dimensional tropical cell is detected by a nonnegative scalar test there; continuity handles faces. This proves approachability.

For a zero-Laplacian function on a proper curve, (3.1) makes its restriction to every refined skeleton harmonic. A harmonic function on a connected compact graph without boundary is constant. Refinement through arbitrary type-II points and descent give the final assertion. □

The smooth Green identity is [3, Theorem 3.12.2]. We extend it and the smooth chain rule to the continuous gaps below using curve regularization and bounded Laplacian variation.

**Lemma 3.7** (Green identities on curves)**.** *Let $r, p$ be finite continuous metrics with positive Chern currents on the same rational line $L$ on a smooth proper curve $C/K$. Their gap $g = r - p$ has smooth approximants $a_j \to g$ uniformly with $\sup_j \|d'd''a_j\|_{\mathrm{TV}} < \infty$. The two Chern measures have mass $\deg_K L$, so $\|d'd''g\|_{\mathrm{TV}} \le 2\deg_K L$.*

*On every finite real span $E$ of such gaps and smooth functions,*

$$\int_C a\, d'd''b = \int_C b\, d'd''a, \qquad Q_C(a) := -\int_C a\, d'd''a \ge 0 \quad (a, b \in E). \tag{3.2}$$

*Moreover,*

$$\Gamma_C(a) = d'd''(a^2/2) - a\, d'd''a$$

*is a positive measure of mass $Q_C(a)$, and*

$$d'd''\Phi(a) = \Phi'(a)d'd''a + \Phi''(a)\Gamma_C(a)$$

*for every smooth real $\Phi$ defined near the range of $a$. If $Q_C(a) = 0$, then $d'd''a = 0$ and $a$ is constant on each connected component.*

*Proof.* In finitely many fixed frame neighbourhoods, use Proposition 3.6 to approximate the weights by smooth subharmonic $r_{i,j}, p_{i,j}$. For a compact $F$ in one such neighbourhood, choose a nonnegative smooth cutoff $\xi = 1$ near $F$. Then

$$(d'd''r_{i,j})(F) \le \int \xi\, d'd''r_{i,j} = \int r_{i,j}\, d'd''\xi \le \|r_{i,j}\|_{\operatorname{supp} d'd''\xi} \|d'd''\xi\|_{\mathrm{TV}}.$$

This bounds the local Laplacians of $g_{i,j} = r_{i,j} - p_{i,j}$. For a compactly supported $\rho \ge 0$, the square identity gives

$$\int \rho\, d'g_{i,j} \wedge d''g_{i,j} = \int (g_{i,j}^2/2) d'd''\rho - \int \rho g_{i,j} d'd''g_{i,j},$$

so the local energies are bounded as well.

Take a finite smooth partition $\chi_i$ and set $a_j = \sum_i \chi_i g_{i,j}$. Cauchy–Schwarz for integration on the curve bounds the cross terms by

$$\|d'\chi_i \wedge d''g_{i,j} + d'g_{i,j} \wedge d''\chi_i\|_{\mathrm{TV}} \le 2Q_C(\chi_i)^{1/2} \left( \int \rho_i\, d'g_{i,j} \wedge d''g_{i,j} \right)^{1/2},$$

where $\rho_i = 1$ near $\operatorname{supp} \chi_i$. The product rule therefore gives the uniform global variation bound.

The mass formula is [10, Corollary 18.1.7]. Uniform convergence and the variation bound imply weak convergence of $d'd''a_j$ on continuous tests and

$$-\int a_j d'd''a_j \longrightarrow -\int a d'd''a.$$

The cited smooth Green formula and $d'd''(a_j^2/2) - a_j d'd''a_j = d'a_j \wedge d''a_j$ pass to the limit. The latter positive measures have convergent masses. The smooth chain rule passes as well. Choosing approximations simultaneously proves these assertions on a fixed finite span $E$. If $Q_C(a) = 0$, the positive measure $\Gamma_C(a)$ is zero. The nonnegative quadratic $Q_C(a + th)$, for smooth $h$ and arbitrary $t \in \mathbf{R}$, has zero linear coefficient. By Green symmetry, $\int h\, d'd''a = 0$ for every such $h$. Thus $d'd''a = 0$, and Proposition 3.6 gives constancy. □

## 4. Restriction and an ambient extension

Restriction of a current to a subvariety is not a formal operation. For continuous potentials it follows from a positive kernel in the normal direction.

**Lemma 4.1** (A normal kernel)**.** *Let $B$ be an ordinary open in the analytification of a smooth $K$-variety of pure dimension $d \geq 1$, let $D_s = \{|t| < s\}$, and let $i : B \hookrightarrow B \times_K D_s$ be the zero section. For a fixed $\varepsilon > 0$, put*

$$v_N = R_{2,\varepsilon}(\log|t|, -N), \qquad \beta_N = d'd''v_N.$$

*Then, for every finite continuous $u$ on the product,*

$$\beta_N \longrightarrow i_*[B], \qquad u\beta_N \longrightarrow i_*[u|_B]$$

*as currents. In particular, $d'd''[u] \geq 0$ implies $d'd''[u|_B] \geq 0$.*

*Proof.* The form $\beta_N$ is smooth and strongly positive, with support in

$$e^{-N-2\varepsilon} \leq |t| \leq e^{-N+2\varepsilon}.$$

The logarithm is locally integrable. Dominated convergence and Poincaré–Lelong give $d'd''[v_N] \to d'd''[\log|t|] = i_*[B]$, with multiplicity one. Lemma 2.2 permits the fixed multiplier $u$. For a positive $\eta \in \mathcal{A}_c^{d-1,d-1}(B)$, $\mathrm{pr}^*\,\eta \wedge \beta_N$ is a positive compact test for large $N$. Indeed, its support lies above a fixed compact subset of $B$ and inside a fixed closed normal disc. Thus

$$\begin{aligned} 0 &\leq \langle d'd''[u], \mathrm{pr}^*\,\eta \wedge \beta_N\rangle \\ &= \langle u\beta_N, \mathrm{pr}^*(d'd''\eta)\rangle \longrightarrow \langle d'd''[u|_B], \eta\rangle. \end{aligned}$$

Fixed base and normal cutoffs make the middle expression a compact test pairing without altering it. □

**Lemma 4.2** (A local product neighbourhood)**.** *Let $i : Z \hookrightarrow Y$ be a closed immersion of smooth $K$-varieties, let $x \in Z^{\mathrm{an}}$, and let $r$ be the codimension near $x$. There are ordinary neighbourhoods $V \subset Z^{\mathrm{an}}$ and $W \subset Y^{\mathrm{an}}$, a radius $s > 0$, and an analytic isomorphism*

$$\Phi : V \times_K D_s^r \xrightarrow{\sim} W$$

*whose zero section is $i|_V$. Consequently restriction of a finite continuous positive potential along $i$ preserves current positivity.*

*Proof.* Use the étale local form of a smooth pair [2, Lemma 37.17.1, Tag 0FUE]. After Zariski shrinking there is an étale map

$$\Psi = (z, F) : Y \longrightarrow \mathbf{A}_K^{d+r}, \qquad Z = \Psi^{-1}(\mathbf{A}_K^d \times \{0\}),$$

where $d = \dim Z$ and $z|_Z$ is étale. Form $E = Y \times_{\mathbf{A}_K^d} Z$. The two maps

$$E \xrightarrow{p} Y, \qquad E \xrightarrow{q=(\mathrm{pr}_Z, F\circ p)} Z \times \mathbf{A}_K^r$$

are étale by base change. At the diagonal point $e$ above $x$, both induce the identity on the completed residue field $\mathscr{H}(x)$. Berkovich's equivalence for étale germs [1, Theorem 3.4.1] therefore makes both analytic germs isomorphisms. By the definition of these germs [1, Section 3.4], they are represented by isomorphisms of ordinary open neighbourhoods. Their composition identifies a neighbourhood of $x$ in $Y^{\mathrm{an}}$ with an ordinary open $G \subset Z^{\mathrm{an}} \times \mathbf{A}_K^{r,\mathrm{an}}$, matching $Z$ with the zero section.

Choose a compact affinoid neighbourhood $V_0$ of $x$ in $Z^{\mathrm{an}}$ whose zero section lies in $G$. In the product of $V_0$ with a fixed closed normal polydisc, the complement of $G$ is compact and disjoint from the zero section. If nonempty, it has a positive minimum of $\max_i |t_i|$. Choose $s$ smaller than the initial polydisc radius and, when the complement is nonempty, smaller than that minimum. Then $V \times D_s^r \subset G$, where $V = \operatorname{Int} V_0$. Its image is the required ordinary open $W$. Finally apply Lemma 4.1 in each normal coordinate; a zero-dimensional target has no curvature condition. □

**Proposition 4.3** (Curve maps and extension)**.** *Let $u$ be a finite continuous CLD-positive function on an ordinary open $O \subset X^{\mathrm{an}}$, where $X/K$ is smooth. Then:*

(i) *For every algebraic morphism $f : C \to X$ from a smooth $K$-curve, $u \circ f^{\mathrm{an}}$ is CLD-positive on $(f^{\mathrm{an}})^{-1}(O)$.*
(ii) *Near each point of $O$, an affine embedding of $X$ has a finite continuous curve-positive ambient extension $v$ of $u$.*

*Proof.* Argue componentwise. For a nonconstant map the algebraic image is a curve. Away from its finitely many singular points, Lemma 4.2 restricts $u$ to the smooth image. Proposition 3.6 supplies smooth positive approximations, whose pullbacks prove positivity on the source outside finitely many closed points.

Continuity removes each such point $P$. A sufficiently small neighbourhood is a disc over $K(P)$, viewed over $K$. Choose $0 \le w_N \le 1$, equal to one near $P$, with supports shrinking to $P$, by translating one smooth profile in $\log|t|$. Then

$$\int d'd''w_N = 0, \qquad \sup_N \|d'd''w_N\|_{\mathrm{TV}} < \infty.$$

The bound contains only the fixed structural degree $[K(P):K]$. A smooth scalar $\varphi$ is constant near $P$, since each of its finitely many analytic unit norms is constant there. For large $N$,

$$|\langle d'd''[u], \varphi w_N\rangle| \le |\varphi(P)| \sup_{\operatorname{supp} d'd''w_N} |u-u(P)|\, \|d'd''w_N\|_{\mathrm{TV}} \longrightarrow 0.$$

Apply positivity off $P$ to $\varphi(1-w_N)$. Constant curve maps pull back $u$ to a constant.

For the extension, choose affine coordinates $(z,y)$ of an embedding $X \subset \mathbf{A}_K^{d+r}$, with $z : X \to \mathbf{A}_K^d$ étale. The algebraic morphism

$$\Theta : X \times \mathbf{A}_K^r \longrightarrow \mathbf{A}_K^{d+r}, \qquad (a,h) \longmapsto (z(a), y(a)+h)$$

is étale: after translating the second coordinate it is a base change of $z$. At $(x,0)$, the map induces an isomorphism of completed residue fields. The same germ theorem [1, Theorem 3.4.1] gives an analytic inverse branch

$$\iota(w) = (\rho(w), y(w) - y(\rho(w)))$$

on one ordinary ambient open. This branch agrees with the diagonal on $X$. Shrinking so that $\rho$ takes values in $O$, set $v = u \circ \rho$. For any smooth algebraic curve mapping to the ambient affine space, the fibre product with $\Theta$ is a smooth algebraic curve. Its map to $X$ is covered by the first part, and the inverse branch identifies the relevant ordinary curve open with an open of that fibre product. Thus $v$ is curve-positive over $K$. □

## 5. Curve approximation and contact rigidity

We first approximate the mixed currents of approachable metrics by algebraic curves. The coefficients are chosen over the original field $K$, and the normalized intersection degrees remain fixed. These degrees supply the variation bounds needed to pass the curve Green identities to a higher-dimensional mixed current.

### 5.1. **Complete-intersection approximation.**

**Proposition 5.1** (Complete-intersection approximation)**.** *Let $Y/K$ be smooth projective of dimension $n \ge 2$. For $1 \le i < n$, let $p_i$ be a uniform FS limit on an ample rational line bundle $L_i$. Then*

$$T = \bigwedge_{i=1}^{n-1} c_1(L_i, p_i)$$

*is a weak limit, on all compact smooth tests, of individual normalized smooth complete-intersection curve cycles $T_a$ defined over $K$. For every line bundle $H$, each $T_a$ has normalized $H$-degree* $\deg_K(HL_1\cdots L_{n-1})$.

*Proof. Step 1: a smooth cut with prescribed coefficient valuations.* Let $Z \subset Y$ be smooth projective of pure dimension $d \ge 1$. First consider an ample line

bundle $M$ generated by $s_0, \dots, s_b$, with FS weight $q = \max_i \log|s_i|$. For every $N \geq 1$, choose

$$a_{i,N} \in \pi^i(1+\pi R), \qquad \sigma_N = \sum_{i=0}^{b} a_{i,N} s_i^N \tag{5.1}$$

so that $D_N = \mathrm{div}_Z(\sigma_N)$ is smooth. Such a choice exists: the pure powers $s_i^N$ have no common zero, and characteristic-zero Bertini gives a nonempty Zariski-open smooth locus in their coefficient space. The product $\prod_i \pi^i(1+\pi R)$ is Zariski dense, since a nonzero polynomial cannot vanish on a product of infinite subsets of $K$. If $Z$ has several components, impose the finitely many smoothness conditions simultaneously. These choices are made independently of all test forms.

*Step 2: ties have zero mass for each smooth test.* Fix a smooth $(d-1,d-1)$-test $\eta$ on $Z^{\mathrm{an}}$, and let $\nu_\eta$ be the signed measure defined by $d'd''\eta$. On each component, discard the generators that vanish identically, retaining the original indices $i$. By [3, Lemma 3.2.5], the compact support of $d'd''\eta$ avoids the zero divisors of the remaining generators. Cover it by finitely many compact moment charts, refined by the unit ratios $s_i/s_j$. On each cell of the resulting finite polyhedral complexes, the functions $\ell_{ij} = \log|s_i/s_j|$ are affine.

The absolute-value tropical integration formula [3, Section 3.7 and Proposition 3.7.2] implies that $|\nu_\eta|$ gives zero mass to inverse images of proper affine hyperplanes in the $d$-dimensional cells; lower-dimensional cells carry no top-degree mass. Equivalently, on each compact chart one can dominate both signs of its Lagerberg representative by a positive top form and apply [4, Proposition 11.1(3)]. This is a statement about total variation, not just about the signed pushforward.

Since $|a_{i,N}| = |\pi|^i$, two summands in (5.1) have the same norm exactly when

$$N\ell_{ij} = (i-j)\lambda.$$

If $\ell_{ij}$ is nonconstant on a top-dimensional cell, this is a null hyperplane. If it is the constant $c$, the equality is impossible for $c = 0$ and holds for at most one positive integer $N$ when $c \neq 0$. There are finitely many cells and pairs. Thus, for all sufficiently large $N$, the largest summand is unique at $|\nu_\eta|$-almost every point.

*Step 3: the one-cut estimate.* Put $w_N = Nq - \log|\sigma_N| \geq 0$. Unique dominance in the ultrametric inequality gives

$$\log|\sigma_N| = \max_i\{N\log|s_i| - i\lambda\}, \qquad 0 \leq w_N \leq b\lambda \quad |\nu_\eta|\text{-almost everywhere}$$

for all sufficiently large $N$. Poincaré–Lelong therefore gives

$$\begin{aligned}\left|\left\langle \frac{[D_N]}{N} - c_1(M,q)\wedge[Z], \eta\right\rangle\right| &= \frac{1}{N}\left|\int_{Z^{\mathrm{an}}} w_N\, d'd''\eta\right| \\ &\leq \frac{b\lambda}{N}\|d'd''\eta\|_{\mathrm{TV}}.\end{aligned} \tag{5.2}$$

Here the threshold in $N$ depends on $\eta$, but the chosen sequence $(\sigma_N)$ does not. The formula is interpreted intrinsically on $Z$, or after pushforward to $Y$. It proves one-cut convergence on every original smooth test. The bound is a statement about integration against $d'd''\eta$: $w_N$ is unbounded along $D_N$. The argument does not give the same estimate against an arbitrary positive background current, whose coefficient measures may charge the tie loci.

*Step 4: successive cuts and uniform limits.* For pure-FS metrics $p_i$, let the generating tuple lie in $d_iL_i$, and put $\alpha_i = c_1(L_i, p_i)$. At the $i$-th cut, restrict that tuple to the preceding smooth intersection and apply Steps 1–3. Its divisor has class $N_id_iL_i$, and its current is divided by $N_id_i$. The Bertini conditions can also be imposed on the ambient divisors.

For clarity, let $\mathcal{C}_k$ be the currents of these smooth codimension-$k$ complete intersections, normalized by $\prod_{i=1}^{k} N_id_i$, with $\mathcal{C}_0 = \{[Y]\}$. The one-cut limit places $\alpha_k \wedge S$ in the weak closure of $\mathcal{C}_k$ for every $S \in \mathcal{C}_{k-1}$. Continuity in the positive background (Proposition 2.3) then gives inductively

$$\alpha_1 \wedge \cdots \wedge \alpha_k \in \overline{\mathcal{C}_k}^{\,\mathrm{weak}}.$$

Indexing a weak neighbourhood by finitely many smooth tests and an error tolerance gives the asserted net of individual curves. Each member has normalized $H$-degree

$$\frac{\deg_K\Big(H\prod_{i=1}^{n-1}(N_id_iL_i)\Big)}{\prod_{i=1}^{n-1}N_id_i} = \deg_K(HL_1\cdots L_{n-1}).$$

Finally, approximate the $p_i$ uniformly by pure-FS metrics and take the union of the corresponding families $\mathcal{C}_{n-1}$ over all approximating tuples and tensor degrees. Every member is still an individual normalized smooth complete intersection with the displayed degree. Joint continuity in Proposition 2.3 puts the limiting mixed current in the weak closure of this union. □

5.2. **Concentration on a divisor.** The next limit uses only continuity in the positive background. In particular, the other metrics remain fixed while the truncated section metric tends to its divisor current.

**Lemma 5.2** (Concentration on a smooth divisor)**.** *Let $Y/K$ be smooth projective of dimension $n \geq 1$, and let $s$ define a smooth effective Cartier divisor $i : D \hookrightarrow Y$ in a line bundle $M$. Let $q, p_1, \ldots, p_{n-1}$ be globally smoothly approximable metrics on $M, L_1, \ldots, L_{n-1}$, respectively. Then*

$$\begin{aligned} &c_1(M, \max\{q, \log|s| + N\}) \wedge \bigwedge_{j=1}^{n-1} c_1(L_j, p_j) \\ &\longrightarrow i_*\left(\bigwedge_{j=1}^{n-1} c_1(L_j|_D, p_j|_D)\right) \end{aligned} \tag{5.3}$$

*as $N \to \infty$, on every continuous scalar test on $Y^{\mathrm{an}}$.*

*Proof.* Put $\rho_N = \max\{q, \log|s| + N\}$, which is smoothly approachable by Lemma 3.3. In a local frame, $\rho_N - N = \max\{q - N, \log|s|\} \searrow \log|s|$. Local integrability and Poincaré–Lelong give $c_1(M, \rho_N) \to [D]$. Apply the fixed-potential continuity of Proposition 2.3 in the remaining slots. The projection formula for approachable products [10, Proposition 8.7.5] identifies the limit with the right-hand side of (5.3). All measures are positive and have the common mass $\deg_K(ML_1 \cdots L_{n-1})$, by Lemma 3.4. Uniform smooth density extends convergence to continuous tests. □

5.3. **Contact rigidity.** The following criterion compares two metrics on the same ample line. Only the lower metric is assumed to be a uniform FS limit; the upper metric is tested on algebraic curves. The proof uses the preceding curve approximation and divisor limit, independently of envelope continuity.

**Theorem 5.3** (Contact rigidity)**.** *Let $Y/K$ be smooth, projective, and integral of dimension $n \geq 1$, and let $L$ be ample. Let $p$ be a uniform FS limit on $L$, and let $r$ be a finite continuous metric on $L$ whose pullback by every morphism from a smooth algebraic $K$-curve has positive Chern current. If*

$$g := r - p \geq 0, \qquad \int_{Y^{\mathrm{an}}} g\, c_1(L,p)^n = 0,$$

*then $r = p$.*

*Proof. Step 1: pass from curves to mixed measures.* Put

$$\alpha = c_1(L,p), \qquad T = \alpha^{n-1}, \qquad \mu = \alpha^n, \qquad D = \deg_K(L^n) > 0.$$

For $n > 1$, approximate $T$ by the normalized smooth curve cycles $T_a$ of Proposition 5.1; for $n = 1$, take $T_a = [Y]$. The pullbacks of $r$ and $p$ to each curve have positive Chern measures. Their normalized pushforwards $\sigma_a, \mu_a$ have mass $D$, and

$$\nu_a = d'd''(gT_a) = \sigma_a - \mu_a, \qquad \|\nu_a\|_{\mathrm{TV}} \leq 2D.$$

By Proposition 2.3, $\mu_a \to \mu$. Lemma 2.2 gives $\nu_a \to \nu := d'd''(gT)$ on smooth tests. The variation bounds extend these limits to continuous scalar tests. Thus $\sigma_a \to \sigma := \nu + \mu$, where $\sigma$ is a positive measure of mass $D$.

*Step 2: use zero energy to annihilate the mixed Laplacian.* We pass the curve Green identities of Lemma 3.7 to the limit. For a fixed smooth function $h$, Lemma 2.2 bounds the total variations of $d'd''h \wedge T_a$: a finite collection of local positive test evaluations bounds the fixed coefficient form $d'd''h$. Consequently,

$$\int g\, d'd''(hT) = \int h\, d\nu, \qquad -\int (g+th)d'd''((g+th)T) \geq 0 \quad (t \in \mathbf{R}). \tag{5.4}$$

All integrals are over $Y^{\mathrm{an}}$. The contact assumption implies

$$0 \leq -\int g\, d\nu = \int g\, d\mu - \int g\, d\sigma \leq 0.$$

The quadratic polynomial in $t$ from (5.4) therefore has zero constant term. Green symmetry writes its linear term as $-2t\int h\, d\nu$. Nonnegativity for every

$t \in \mathbf{R}$ forces that coefficient to vanish. Since this holds for every smooth $h$, we have $\nu = 0$.

If $\rho$ is any finite continuous curve-positive metric on $mL$, set $b = \rho - mp$. Its normalized curve Laplacians have variations at most $2mD$. The same passage of Green symmetry, now for the two continuous gaps $g$ and $b$, gives

$$\int g\, d'd''(bT) = \int b\, d\nu = 0. \tag{5.5}$$

Only products with the approachable lower metric $p$ have been used; the upper metric $r$ enters through its curve restrictions.

*Step 3: restrict the contact identity and argue by dimension.* When $n = 1$, Lemma 3.7 and the zero energy already proved make $g$ constant. Its integral against the positive measure $\mu$ of mass $D$ makes that constant zero. Suppose $n > 1$ and proceed by induction. Fix a closed point $x \in Y$. For sufficiently large $m$, choose a smooth $H = \operatorname{div}(s)$ in $mL$ through $x$. To impose this condition, use $H^0(Y, mL \otimes \mathcal{I}_x)$: for $m \gg 0$ it is generated away from $x$ and realizes arbitrary first jets at $x$. A nonzero first jet ensures smoothness there, and characteristic-zero Bertini ensures smoothness elsewhere. The nonempty open conditions admit a $K$-rational choice since $K$ is infinite.

Set $\rho_N = \max\{mp, \log|s| + N\}$. These metrics are smoothly approximable by Lemma 3.3, hence curve-positive. Equation (5.5), together with $\int g\, d\mu = 0$, yields

$$\int_{Y^{\mathrm{an}}} g\, c_1(mL, \rho_N) \wedge \alpha^{n-1} = 0.$$

Lemma 5.2 now gives

$$\int_{H^{\mathrm{an}}} g|_H\, c_1(L|_H, p|_H)^{n-1} = 0.$$

The integrand is nonnegative. It therefore has zero integral on each smooth integral component of $H$. The curve condition passes to that component by composition, $L|_H$ is ample, and $p|_H$ is a uniform FS limit by restriction of the generating tuples. Induction gives $g(x) = 0$. Closed points give a dense set of rigid points of $Y^{\mathrm{an}}$; continuity implies $g = 0$ everywhere. □

## 6. Envelopes on a fixed regular model

For an ample line bundle $L$ on a projective $K$-variety and a continuous metric $r$, set

$$P_L(r) = \sup\{q \le r : q \text{ is a continuous semipositive metric on } L\}. \tag{6.1}$$

We use the standard finiteness, translation, scaling, and monotonicity properties of this envelope [8, Remark 6.2 and Proposition 6.3]. In particular,

$$\|P_L(r) - P_L(r')\|_\infty \le \|r - r'\|_\infty. \tag{6.2}$$

The envelope is not assumed continuous at this stage.

6.1. **Test ideals of admissible section divisors.** Let $\mathscr{X}/R$ be regular, integral, projective and flat, with smooth generic fibre $Y$ of dimension $n \geq 1$. Extend $L$ to a line bundle $\mathscr{L}$ on $\mathscr{X}$, and write $\theta$ for its model metric. Such an extension exists because the closure of a generic Cartier divisor is Cartier on the regular scheme $\mathscr{X}$. After clearing a fixed tensor denominator and adjusting the reference identification by a power of $\pi$, a model metric $r$ can be written

$$r = \theta + \log|\mathscr{E}|, \qquad \mathscr{E} = \mathcal{O}_{\mathscr{W}}(-F) \subseteq \mathcal{O}_{\mathscr{W}}, \tag{6.3}$$

where $f : \mathscr{W} \to \mathscr{X}$ is a normal projective model inducing the identity on $Y$, and $F$ is an effective vertical Cartier divisor. Put $p = P_L(r)$.

**Proposition 6.1** (The test-ideal estimate)**.** *There are coherent vertical ideals $\mathfrak{j}_m \subseteq \mathcal{O}_{\mathscr{X}}$, for $m \geq 1$, and an integer $B \geq 0$, independent of $m$, such that*

$$\pi^B \mathfrak{j}_m \mathcal{O}_{\mathscr{W}} \subseteq \mathscr{E}^m, \qquad p \leq \theta + \psi_m \leq r + \frac{B\lambda}{m}, \qquad \psi_m = \frac{1}{m}\log|\mathfrak{j}_m|. \tag{6.4}$$

*For every globally generated ample Cartier divisor $H$ on $\mathscr{X}$,*

$$\mathfrak{j}_m \mathscr{L}^{\otimes m} \otimes \mathcal{O}_{\mathscr{X}}\big(K_{\mathscr{X}} + (n+1)H\big) \quad \textit{is globally generated.} \tag{6.5}$$

*Proof.* For $l \geq 1$, let

$$V_l = H^0\big(\mathscr{W}, f^*\mathscr{L}^{\otimes l} \otimes \mathscr{E}^l\big).$$

Since $f_*\mathcal{O}_{\mathscr{W}} = \mathcal{O}_{\mathscr{X}}$, every member of $V_l$ is a section of $\mathscr{L}^{\otimes l}$ on $\mathscr{X}$. Normality also gives

$$V_l = \{s \in H^0(Y, lL) : \|s\|_{lr,\sup} \leq 1\}. \tag{6.6}$$

Indeed, a generic section has no horizontal poles, while the norm inequalities at vertical divisorial points are exactly the remaining codimension-one extension conditions on $\mathscr{W}$. For $0 \neq s \in V_l$, denote its effective divisor on $\mathscr{X}$ by $D_s$. Then

$$D_s \sim l\mathscr{L}, \qquad f^*D_s \geq lF.$$

Define

$$\mathfrak{j}_m = \sum_{l \geq 1} \sum_{0 \neq s \in V_l} \tau_{\mathscr{X}}\left(\frac{m}{l} D_s\right). \tag{6.7}$$

This sum equals a finite subsum by Noetherianity. For a sufficiently large $l$, a generating tuple of $H^0(Y, lL)$ can be scaled into $V_l$. At every point of $Y$ one of these sections is nonvanishing. Generic-fibre comparison with multiplier ideals [6, Corollary 7.21(e)] therefore shows that $\mathfrak{j}_m|_Y = \mathcal{O}_Y$. Thus $\mathfrak{j}_m$ is vertical.

*Lower bound.* Write $\tau_0 = \tau_{\mathscr{X}}(0)$. This fixed ideal contains a power of $\pi$, since its generic restriction is $\mathcal{O}_Y$. Cartier translation and subadditivity on the regular scheme $\mathscr{X}$ give, for every $k \geq 1$,

$$\mathcal{O}_{\mathscr{X}}(-kmD_s)\tau_0 = \tau_{\mathscr{X}}(kmD_s) \subseteq \tau_{\mathscr{X}}\left(\frac{m}{l} D_s\right)^{kl} \subseteq \mathfrak{j}_m^{kl}. \tag{6.8}$$

Here we use [6, Proposition 7.16(b) and Theorem 7.22(e)]. Taking logarithmic norms, dividing by $kml$, and letting $k \to \infty$ yields

$$\frac{1}{l} \log \|s\|_{l\theta} \le \psi_m.$$

If a pure-FS metric $q$ lies below $r$, its defining sections belong to the corresponding $V_l$ by (6.6); taking their maximum gives $q \le \theta + \psi_m$. Every continuous semipositive competitor below $r$ is a uniform limit of pure-FS metrics shifted down by constants in $\lambda\mathbf{Q}$ tending to zero [7, Corollary 7.6 and Theorem 7.8]. The lower bound in (6.4) follows.

*Upper bound.* Choose compatible canonical divisors and set $\Delta_{\mathscr{W}} = f^*K_{\mathscr{X}} - K_{\mathscr{W}}$. The fractional test ideal $\tau_{\mathscr{W}}(\Delta_{\mathscr{W}})$ is defined because $K_{\mathscr{W}} + \Delta_{\mathscr{W}}$ is Cartier [6, Definition 7.20]. Birational inclusion, monotonicity, and Cartier translation give

$$\begin{aligned} \tau_{\mathscr{X}}\left(\frac{m}{l}D_s\right)\mathcal{O}_{\mathscr{W}} &\subseteq \tau_{\mathscr{W}}\left(\Delta_{\mathscr{W}} + \frac{m}{l}f^*D_s\right) \\ &\subseteq \tau_{\mathscr{W}}(\Delta_{\mathscr{W}} + mF) = \mathscr{E}^m\tau_{\mathscr{W}}(\Delta_{\mathscr{W}}). \end{aligned} \tag{6.9}$$

The cited properties are [6, Proposition 7.16(a),(b) and Theorem 7.22(d)]. The fixed fractional ideal on the right is coherent and restricts to $\mathcal{O}_Y$. A single power $\pi^B$ clears its denominators on a finite affine cover of $\mathscr{W}$. Summing (6.9) gives $\pi^B \mathfrak{j}_m\mathcal{O}_{\mathscr{W}} \subseteq \mathscr{E}^m$, and hence the upper bound.

*Global generation.* Each effective boundary $\Delta = (m/l)D_s$ is $\mathbf{Q}$-linearly equivalent to $m\mathscr{L}$. For $M = m\mathscr{L} + K_{\mathscr{X}} + H$, one has $M - K_{\mathscr{X}} - \Delta \sim_{\mathbf{Q}} H$. Since the special fibre has dimension $n$, [6, Corollary 7.24] makes $\tau_{\mathscr{X}}(\Delta)\mathscr{L}^{\otimes m}(K_{\mathscr{X}} + (n+1)H)$ globally generated. A finite subsum in (6.7) proves (6.5). □

## 6.2. **Uniform approximation and orthogonality.**

**Theorem 6.2** (Envelopes on a regular model)**.** *Let $\mathscr{X}/R$ be regular, integral, projective and flat, with smooth generic fibre $Y$ of dimension $n$, and let $L$ be an ample rational line bundle on $Y$. For every finite continuous metric $r$ on $L$, the envelope $p = P_L(r)$ is continuous, is a uniform limit of pure-FS metrics, and satisfies*

$$\int_{Y^{\mathrm{an}}} (r - p)c_1(L, p)^n = 0. \tag{6.10}$$

*Here the measure is the CLD product. For a model obstacle, after clearing fixed tensor denominators, there are pure-FS metrics $q_\ell$, for all sufficiently large integers $\ell$, such that*

$$0 \le q_\ell - p \le \frac{C}{\ell}, \tag{6.11}$$

*where $C$ depends on the obstacle and the fixed auxiliary data.*

*Proof.* If $n = 0$, then $Y = \operatorname{Spec} F$ for a finite extension $F/K$, and $Y^{\mathrm{an}}$ is a point. In a fixed frame, weights of pure-FS metrics include $\lambda\mathbf{Q}$, so every continuous metric is semipositive and $p = r$. A model metric is pure-FS after clearing its fixed tensor denominator, which also gives the stated approximation. Assume henceforth that $n \geq 1$.

First treat a model obstacle in the form (6.3). Choose a globally generated ample divisor $H$ sufficiently positive that

$$\mathscr{A} = \mathcal{O}_{\mathscr{X}}\big(K_{\mathscr{X}} + (n+1)H\big)$$

is ample and globally generated. Put $A = \mathscr{A}|_Y$, and let $q_A$ be its model FS metric. Choose $k \geq 1$ such that $kL - A$ is very ample and a pure-FS metric $\nu$ defined by a generating tuple on that line bundle. Multiplying this tuple by a sufficiently large negative power of $\pi$, arrange that

$$h := \frac{q_A + \nu}{k} \geq r.$$

By Proposition 6.1, the generators of $\mathfrak{j}_m \mathscr{L}^{\otimes m}\mathscr{A}$ define a pure-FS metric

$$t_m = m\theta + q_A + \log|\mathfrak{j}_m| \quad \text{on } mL + A.$$

Tensoring with the tuple defining $\nu$ gives a pure-FS metric on $L$,

$$h_m = \frac{t_m + \nu}{m+k} = \frac{m}{m+k}(\theta + \psi_m) + \frac{k}{m+k}h.$$

Consequently,

$$p \leq h_m \leq r + \frac{C}{m+k}, \qquad C = B\lambda + k\,\|h - r\|_\infty .$$

The semipositive metric $h_m - C/(m+k)$ lies below $r$, hence below $p$. Thus

$$0 \leq h_m - p \leq \frac{C}{m+k}.$$

These metrics are represented by generating sections in degree $m + k$. Relabeling $\ell = m + k$ proves (6.11). A fixed tensor denominator is divided out using $P_{dL}(dr) = dP_L(r)$.

For a continuous obstacle, use uniform density of model metrics [7, Proposition 5.16(vi) and Theorem 5.18]. Equation (6.2) shows that their envelopes converge uniformly to $P_L(r)$, and a diagonal choice gives pure-FS approximation. Finally, $Y$ is geometrically reduced, so [8, Theorem 6.7] gives (6.10). Lemma 3.4 identifies its measure with the CLD product. □

**Corollary 6.3** (Projective space)**.** *Let $a, b \geq 1$ and $L = \mathcal{O}_{\mathbf{P}^a_K}(b)$. The conclusions of Theorem 6.2 hold for every continuous metric on $L$. For a model obstacle represented as in (6.3) on a model above $\mathbf{P}^a_R$, the generators of $\mathfrak{j}_m\mathcal{O}(bm)$ give pure-FS metrics $q_m$ satisfying*

$$0 \leq q_m - P_L(r) \leq B\lambda/m.$$

*Proof.* Take $\mathscr{X} = \mathbf{P}^a_R$. For the more precise assertion, choose the hyperplane divisor $H$; then $K_{\mathscr{X}} + (a+1)H = 0$. Equation (6.5) makes $q_m = \theta + m^{-1}\log|\mathrm{j}_m|$ pure-FS. By (6.4), $p \le q_m \le r + B\lambda/m$; the envelope property applied to $q_m - B\lambda/m$ finishes the proof. □

## 7. Proof of local regularization

The next lemma places a local curve-positive function inside one global metric on projective space. It uses only algebraic sections and the maximum property on curves.

### 7.1. **A projective extension.**

**Lemma 7.1** (Projective extension)**.** *Let $W \subset \mathbf{A}^{a,\mathrm{an}}_K$ be an ordinary open, $a \ge 1$, and let $v : W \to \mathbf{R}$ be finite, continuous and curve-positive. For every $x \in W$, there are $d \ge 1$, a finite continuous curve-positive metric $r$ on $\mathcal{O}_{\mathbf{P}^a_K}(d)$, a section $s_0$ nonvanishing near $x$, and a constant $c \in \mathbf{R}$, such that*

$$-\log\|s_0\|_r = v + c \quad \text{on an ordinary neighbourhood of } x.$$

*Proof.* Set $c = \lambda/4 - v(x)$. After shrinking $W$, continuity gives

$$0 < v + c < \lambda/2 \quad \text{on } W. \tag{7.1}$$

Choose a compact polynomial norm box

$$C = \{|F_i| \le |c_i|,\ |G_j| \ge |d_j|\} \subset W$$

with all defining inequalities strict at $x$, including upper bounds on the affine coordinates. Here $F_i, G_j \in K[z_1, \dots, z_a]$ and $c_i, d_j \in K^\times$. Such boxes form a neighbourhood basis: choose thresholds in $|K^\times|^{\mathbf{Q}}$ inside an ordinary basic neighbourhood and encode them by powers of the polynomials.

For $N \ge 1$ sufficiently large, every logarithmic term in the following maximum is negative on a neighbourhood of $x$:

$$b_0 = \max\left\{0,\ \log\left|\frac{F_i^N}{\pi c_i^N}\right|, \log\left|\frac{d_j^N}{\pi G_j^N}\right|\right\} \tag{7.2}$$

On the open where every $G_j$ is nonzero, this function satisfies $b_0 = 0$ near $x$ and $\{b_0 \le \lambda\} = C$.

Put $t_0 = \prod_j G_j^N$, with the empty product equal to 1. Clear the denominators in (7.2) and homogenize

$$t_0, \qquad \frac{t_0 F_i^N}{\pi c_i^N}, \qquad \frac{t_0 d_j^N}{\pi G_j^N}$$

to a common degree $d$, chosen strictly larger than $\deg t_0$. Write $s_0$ for the homogenization of $t_0$. Add sufficiently small $K$-multiples of $Z_0^d, \dots, Z_a^d$, so that their ratios to $s_0$ have norm less than 1 on $C$. The resulting tuple generates $\mathcal{O}(d)$; let $q$ be its FS metric. Then

$$b := -\log\|s_0\|_q \in [0, +\infty]$$

equals $b_0$ on $C$, vanishes near $x$, and satisfies

$$\{b \le \lambda\} = C. \tag{7.3}$$

Indeed, $s_0$ vanishes at infinity because $d > \deg t_0$, and the remaining original ratios impose precisely the inequalities defining $C$. The added sections preserve those inequalities on $C$. We interpret $b = +\infty$ on the zero divisor of $s_0$.

The ordinary opens

$$U = \{b < \lambda\}, \qquad V = \{b > \lambda/2\}$$

cover $\mathbf{P}_K^{a,\mathrm{an}}$, and $U \subset W$. On $U$, define a metric by its $s_0$-weight $\max\{v+c, b\}$; on $V$, use $q$. The definitions agree on $U \cap V$ by (7.1). They therefore give a finite continuous metric $r$. Near $x$, one has $b = 0 < v + c$, so its weight is $v + c$.

On every smooth algebraic $K$-curve, the weights just used are continuous subharmonic functions or maxima of two such functions. Their maxima are subharmonic by the curve regularization of Proposition 3.6 and the regularized maximum construction. Thus $r$ is curve-positive. □

### 7.2. **Completion of the proof.**

*Proof of Theorem 1.1.* Local uniform convergence of smooth positive potentials implies current positivity by testing against $d'd''\eta$. Conversely, fix $x \in O$ on a positive-dimensional component of $X$. Proposition 4.3 supplies an ordinary ambient open $W \subset \mathbf{A}_K^{a,\mathrm{an}}$ and a finite continuous curve-positive function $v$ agreeing with $u$ along $X^{\mathrm{an}} \cap W$. Apply Lemma 7.1, and retain its $r, s_0, c$.

By Corollary 6.3, the envelope $p = P_{\mathcal{O}(d)}(r)$ is a uniform FS limit and

$$p \le r, \qquad \int_{\mathbf{P}_K^{a,\mathrm{an}}} (r - p) c_1(\mathcal{O}(d), p)^a = 0.$$

Theorem 5.3 gives $p = r$. By Lemma 3.3, choose global smooth positive metrics $r_j$ with $\|r_j - r\|_\infty \le 4^{-j}$. On one fixed ordinary neighbourhood $V_x$ of $x$ in $X^{\mathrm{an}}$ where the weight identity holds, put

$$u_j = -\log \|s_0\|_{r_j}\big|_{V_x} - c.$$

These functions are smooth, have positive curvature, and satisfy $\|u_j - u\|_{V_x} \le 4^{-j}$. For the decreasing assertion, define

$$w_j = u_j + 2 \cdot 4^{-j}.$$

Then

$$u + 4^{-j} \le w_j \le u + 3 \cdot 4^{-j}, \qquad w_{j+1} \le u + \tfrac{3}{4}\, 4^{-j} < w_j.$$

Thus $w_j \downarrow u$ uniformly on the same $V_x$. Constant translations preserve curvature. On a zero-dimensional component, every finite continuous function is locally constant and already smooth. □

## 8. Consequences for continuous potentials

We first apply local regularization to the operation constructed in Proposition 2.3. The background current remains arbitrary; in particular, it need not itself be a product of curvature currents.

### 8.1. **Proof of the mixed-current theorem.**

*Proof of Theorem 1.2.* Theorem 1.1 places every $u_i$ in $\mathcal{P}_{\mathrm{loc}}(O)$. Iterating Proposition 2.3 in (1.3) proves positivity, closedness, symmetry, and joint continuity, including nets. Locality, additivity, and positive homogeneity follow from the same recursion. A positive current of top degree is a Radon measure.

It remains to prove independence of the potentials. Let $h$ be finite continuous with $d'd''[h] = 0$, and let $T$ be any positive closed current for which the bidegrees permit the operation. Both $h$ and $-h$ have positive curvature currents, so Theorem 1.1 and Proposition 2.3 give

$$d'd''(hT) \geq 0, \qquad -d'd''(hT) = d'd''((-h)T) \geq 0.$$

The positive cone is pointed by Lemma 2.2; hence

$$d'd''[h] = 0 \quad \Longrightarrow \quad d'd''(hT) = 0. \tag{8.1}$$

If $d'd''[w] = d'd''[u_i]$, apply this implication to $h = w - u_i$ and to the positive closed product of the other slots. Symmetry permits us to place the changed slot first, proving the assertion. □

### 8.2. **Chern products and closure operations.**

**Corollary 8.1** (Continuous Chern products)**.** *Let $X/K$ be smooth quasi-projective of pure dimension $n$, let $O \subset X^{\mathrm{an}}$ be an ordinary open, and let $S$ be a positive closed CLD current of bidegree $(p,p)$ on $O$. Let $\overline{L}_1, \dots, \overline{L}_m$ be continuously metrized analytic line bundles on $O$, with finite local weights and positive CLD curvature currents, where $m \leq n - p$. Their local products glue to a positive closed current*

$$c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_m) \wedge S.$$

*It depends only on the curvature currents and $S$, agrees with the smooth wedge product when all weights are smooth, and is a positive Radon measure if $m+p = n$. The same assertions hold for rational line bundles after clearing and dividing their tensor denominators.*

*Proof.* Apply Theorem 1.2 in local frames. On overlaps, weights differ by logarithms of analytic units, whose curvature is zero by Poincaré–Lelong. Equation (8.1) identifies the products. The remaining assertions, including rational normalization, follow from the same theorem. □

**Corollary 8.2** (Products with a smooth subvariety)**.** *Let $i : Z \hookrightarrow X$ be a closed immersion of smooth quasi-projective $K$-varieties, with $X$ of pure dimension $n$ and $Z$ of pure dimension $d$. Let $O \subset X^{\mathrm{an}}$ be ordinary open,*

*put $O_Z = (i^{\mathrm{an}})^{-1}(O)$, and let $u_1, \dots, u_m$ be finite continuous functions on $O$ with $d'd''[u_j] \geq 0$, where $m \leq d$. Write $[Z]_O = i_*[O_Z]$ for the integration current on $O$, and write $\widetilde{u}_j = u_j|_{O_Z}$. Then*

$$\mathcal{B}_{[Z]_O}(u_1, \dots, u_m) = i_* \mathcal{B}_{[O_Z]}(\widetilde{u}_1, \dots, \widetilde{u}_m). \tag{8.2}$$

*Proof.* Positive smooth approximants from Theorem 1.1 restrict to positive smooth approximants on $O_Z$. The assertion is then the projection formula for approachable mixed products [10, Proposition 8.7.5], applied to the closed immersion $i$. □

Formula (8.2) identifies a product with the integration current of $Z$ with the intrinsic product on $Z$. It does not require a pullback operation for arbitrary currents.

**Corollary 8.3** (Closure operations)**.** *Let $u_1, \dots, u_r$ be finite continuous current-positive functions on an ordinary open $O \subset X^{\mathrm{an}}$, where $X/K$ is smooth quasi-projective. If $F : \mathbf{R}^r \to \mathbf{R}$ is convex and nondecreasing in each coordinate, then $F(u_1, \dots, u_r)$ is current-positive. In particular, finite maxima and sums with nonnegative coefficients preserve this class.*

*If $V \subset Y^{\mathrm{an}}$ is an ordinary open in a smooth quasi-projective $K$-variety and $f : V \to O$ is an analytic morphism, then $u_i \circ f$ is current-positive on $V$.*

*Proof.* By Theorem 1.1, the $u_i$ are locally uniformly approachable by smooth positive potentials. The convex composition assertion is therefore [10, Corollary 8.2.4]. For the pullback assertion, pull back those smooth approximants; smooth pullback preserves positivity, and locally uniform convergence is preserved because continuous maps send compact sets to compact sets. □

### 8.3. **The projective curve criterion.**

**Corollary 8.4** (A projective curve criterion)**.** *Let $\mathscr{Y}/R$ be a regular integral projective flat model with smooth generic fibre $Y$ of positive dimension. Let $L$ be an ample line bundle on $Y$, and let $r$ be a finite continuous metric on $L$. The following conditions are equivalent:*

(i) *For every algebraic morphism $f : C \to Y$ from a smooth $K$-curve, the pulled-back metric $f^*r$ has positive Chern current.*
(ii) *The original CLD current $c_1(L, r)$ is positive.*
(iii) *The metric $r$ is semipositive, equivalently a uniform limit of pure Fubini–Study metrics on $L$.*
(iv) *The metric $r$ is a uniform limit of global smooth metrics on $L$ with positive curvature forms.*

*The criterion also holds for an ample rational line bundle by clearing its tensor denominator.*

*Proof.* Suppose the first condition holds. Theorem 6.2 gives a finite continuous semipositive envelope $p = P_L(r)$, which is a uniform Fubini–Study limit and satisfies

$$p \leq r, \qquad \int_{Y^{\mathrm{an}}} (r - p) c_1(L, p)^{\dim Y} = 0.$$

Theorem 5.3 applies to the curve-positive upper metric $r$ and gives $r = p$. This proves the third condition. Lemma 3.3 gives the fourth condition from the third, and positivity passes to the current limit, giving the second. Finally, Proposition 4.3, applied in local frames, gives the first condition from the second. Positive tensor powers scale each curvature by the same positive integer, which proves the rational-line assertion. □

## 9. Comparison with adelic mixed measures

**Definition 9.1** (Boundary completion)**.** Let $U/k$ be quasi-projective. We use the rational boundary completion of [9, Sections 2.4–2.5], retaining its specified line-bundle identifications. A boundary datum consists of a projective compactification, an effective Cartier boundary with its multiplicities, and strictly positive Green data. On common dominating models a Cauchy presentation satisfies, for every $\varepsilon > 0$ and all sufficiently large $j, l$,

$$-\varepsilon\overline{\Delta}_\partial \le \overline{D}_j - \overline{D}_l \le \varepsilon\overline{\Delta}_\partial. \tag{9.1}$$

In a fixed local identification at $v$, this implies

$$|w_j - w_l| \le \varepsilon b_v \quad \text{on } U_K^{\mathrm{an}},$$

where the boundary Green function $b_v$ is finite continuous on that open. The completion uses the full order (9.1), not merely this scalar inequality. A strongly nef adelic line bundle has a Cauchy presentation by nef model objects. An integrable one is a difference of two strongly nef objects, allowing rational tensor powers and the specified isometries [9, Definition 2.5.2].

### 9.1. Identification and positivity of the prescribed measure.

*Proof of Theorem 1.3.* Choose integrable presentations with their prescribed isometries,

$$\overline{L}_i^{\otimes r_i} \simeq \overline{P}_i \otimes \overline{Q}_i^{-1}, \qquad r_i \in \mathbf{Z}_{>0},$$

where $\overline{P}_i, \overline{Q}_i$ are strongly nef. Their metrics at $v$ are locally psh-approachable, and their mixed measures are the stable CLD products [9, Lemma 3.6.7 and Sections 3.6.7–3.6.8]. Let $u_i, p_i, q_i$ be weights in compatible local frames. The specified isometries give

$$r_i u_i = p_i - q_i + h_i, \qquad h_i = -\log|a_i|, \qquad d'd''h_i = 0,$$

for analytic units $a_i$. By the definition in [9, Section 3.6.8], the prescribed signed measure is

$$\mu_v^{\mathrm{YZ}} = \frac{1}{\prod_i r_i} \sum_{\epsilon\in\{0,1\}^n} (-1)^{|\epsilon|} \prod_{i=1}^{n} c_1(\overline{R}_{i,\epsilon_i})_v, \qquad \overline{R}_{i,0} = \overline{P}_i, \quad \overline{R}_{i,1} = \overline{Q}_i. \tag{9.2}$$

Its independence of presentation is part of that construction.

Theorem 1.1 makes each target weight $u_i$ approachable. Apply additivity of approachable products to $r_i u_i + q_i = p_i + h_i$ in every slot; the zero-curvature

terms $h_i$ disappear by (8.1). The alternating expansion is exactly

$$\mu_v^{\mathrm{YZ}} = \bigwedge_{i=1}^{n} d'd''u_i. \tag{9.3}$$

The right-hand side is positive by Theorem 1.2. Positivity of Radon measures is local, so this proves the assertion on all of $U_K^{\mathrm{an}}$. □

## Appendix A. A quantitative comparison of mixed measures

For completeness, we record a comparison estimate in which the constant is independent of the metrics being compared. It is separate from the proof of contact rigidity: the divisor limit there uses only the continuity of a fixed potential acting on positive closed currents.

**Lemma A.1** (Global domination of finitely many forms)**.** *Let $Y/K$ be smooth projective, and let $\eta_1, \dots, \eta_s$ be smooth symmetric real $(1,1)$-forms on $Y^{\mathrm{an}}$. There are an ample line bundle $H$, a global smooth metric $\phi$ on $H$, and its positive curvature $\omega = c_1(H, \phi)$, such that*

$$-\omega \le \eta_i \le \omega \qquad (1 \le i \le s).$$

*The inequalities are in the original CLD form cone, and all the constructions take place over $K$.*

*Proof.* Choose a very ample line bundle $A$ on $Y$. We first construct one global positive curvature that controls the square of a local moment coordinate. On a compact polynomial norm box $C$ contained in an affine chart of the embedding defined by $A$, let $f$ be an analytic unit on a neighbourhood of $C$. The box is defined by finitely many upper bounds and positive lower bounds on polynomial functions, with bounds in $|K^\times|^{\mathbb{Q}}$. After clearing powers, its affinoid algebra is the completion of the affine coordinate algebra with the lower-bounded polynomials inverted. Thus $f$ can be uniformly approximated on $C$ by a rational function $h$ from this algebra. Choose the error smaller than $\min_C |f|$. The ultrametric inequality gives

$$|h| = |f| \quad \text{throughout } C.$$

In particular $h$ is a rational unit there. This equality of moment maps gives the same differential forms on $\operatorname{Int} C$, by [3, Lemma 3.1.10].

Homogenizing a numerator and denominator for $h$ expresses it as $h = s_1/s_0$, where $s_0, s_1$ are sections of some positive tensor power $A^d$, both nonzero on $C$. Set $M = \|\log|h|\|_C$, and choose $\varepsilon > 2M$. In the construction of Proposition 3.2, choose $\rho$ strictly positive on $(-1,1)$, and write

$$H_\varepsilon(t) = R_{2,\varepsilon}(t, 0).$$

Then

$$H_\varepsilon''(t) = \frac{1}{\varepsilon}(\rho * \check{\rho})(t/\varepsilon) > 0 \qquad (|t| < 2\varepsilon), \qquad \check{\rho}(t) = \rho(-t).$$

Hence $H_\varepsilon'' \ge c > 0$ on $[-M, M]$.

Take a global generating tuple of $A^d$, multiply its members by sufficiently small elements of $K^\times$, and append them to $s_0, s_1$. If $a$ is any appended section, choose the coefficient so that

$$\log|a/s_0| < -2\varepsilon - 1 \quad \text{on } C$$

where the value is $-\infty$ at a zero. Thus its logarithmic entry lies more than $2\varepsilon$ below the maximum of 0 and $\log|h|$ throughout $C$. Every appended entry is exactly inactive in the regularized maximum on a neighbourhood of $C$. The full tuple generates $A^d$ everywhere, and regularizing its maximum gives a global smooth positive metric by Lemma 3.3. In the frame $s_0$, its weight on $\operatorname{Int} C$ is precisely $H_\varepsilon(\log|h|)$. Its curvature there is

$$H_\varepsilon''(\log|h|)\, d'\log|h| \wedge d''\log|h| \geq c\, d'\log|h| \wedge d''\log|h|.$$

Now cover $Y^{\mathrm{an}}$ by the interiors of finitely many such boxes, using common moment presentations for the finitely many forms $\eta_i$ on each box. Replace the analytic moment units by rational units as above. The symmetric coefficient matrices of all the $\eta_i$ are bounded on these compact boxes. The construction for each moment unit supplies a global positive curvature controlling its coordinate square. Suitable positive integer multiples of these curvatures therefore dominate both signs of every coefficient matrix on the corresponding box. Their sum is a global smooth positive curvature, dominates every $\eta_i$ on the finite cover, and is the curvature of a positive tensor power of $A$. If the list of required coordinate squares is empty, add any smooth positive approximation of a pure-FS metric on $A$. This gives the asserted ample line $H$ and metric $\phi$. □

**Proposition A.2** (Comparison on a fixed smooth test)**.** *Let $Y/K$ be smooth projective of dimension $n \geq 1$. For $1 \leq i \leq n$, let $p_i, q_i$ be globally smoothly approximable metrics on the same specified line bundle $L_i$. Fix a real smooth function $\varphi$ on $Y^{\mathrm{an}}$. Choose an ample $H_\varphi$ and a global smooth positive curvature $\omega_\varphi$ such that*

$$-\omega_\varphi \leq d'd''\varphi \leq \omega_\varphi,$$

*as in Lemma A.1. Then*

$$\begin{aligned} &\left| \int_{Y^{\mathrm{an}}} \varphi \left( \bigwedge_{i=1}^n c_1(L_i, p_i) - \bigwedge_{i=1}^n c_1(L_i, q_i) \right) \right| \\ &\qquad \leq \sum_{i=1}^n \|p_i - q_i\|_\infty \deg_K \left( H_\varphi \prod_{h \neq i} L_h \right). \end{aligned} \tag{A.1}$$

*The norm is the uniform norm of the scalar difference of two weights on the same line. The line $H_\varphi$ is fixed once $\varphi$ and its calibration have been chosen; the degree constants are independent of all the metrics $p_i, q_i$. The assertion also holds for rational line bundles after clearing and dividing tensor denominators.*

*Proof.* Put $\alpha_i = c_1(L_i, p_i)$, $\beta_i = c_1(L_i, q_i)$, and $T_i = \bigwedge_{h<i} \alpha_h \wedge \bigwedge_{h>i} \beta_h$. Telescoping and current duality give

$$\int \varphi \left( \bigwedge_i \alpha_i - \bigwedge_i \beta_i \right) = \sum_i \int (p_i - q_i)\, d'd''\varphi \wedge T_i.$$

Each $T_i$ is positive and closed, so $|d'd''\varphi \wedge T_i| \le \omega_\varphi \wedge T_i$. By Lemma 3.4, the latter measure has mass $\deg_K(H_\varphi \prod_{h\neq i} L_h)$. Estimating each summand by its uniform coefficient norm proves (A.1). Multilinearity gives the rational-line version. □

## Notation

| Symbol | Meaning | Reference |
|---|---|---|
| $K, R, \pi, \lambda$ | Fixed completion, valuation ring, uniformizer, and $-\log\lvert\pi\rvert$ | Section 1 |
| $d'd'', \mathcal{A}_c^{p,q}$ | Original CLD differentials and compact smooth test forms | Section 1 |
| $\mathcal{P}_{\mathrm{loc}}, \mathcal{D}_{\mathrm{loc}}$ | Local smooth positive approximation and its difference class | Definition 2.1 |
| $M_u(T)$ | The operation $d'd''(uT)$ for approachable $u$ | Proposition 2.3 |
| $\mathcal{B}_S$ | Mixed product with a fixed positive closed background | (1.3) |
| $Q_C, \Gamma_C$ | Curve Dirichlet energy and its energy measure | Lemma 3.7 |
| $T_a$ | Normalized smooth complete-intersection curve currents | Proposition 5.1 |
| $p, r, g$ | Lower approachable metric, upper curve-positive metric, and $g = r - p$ | Theorem 5.3 |
| $P_L(r)$ | Semipositive envelope of an obstacle metric | Section 6 |
| $V_l, \mathfrak{j}_m$ | Admissible sections and sums of divisor test ideals | Proposition 6.1 |
| $\mathscr{E}, \Delta_{/\!\!/}$ | Vertical obstacle ideal and canonical discrepancy boundary | Proposition 6.1 |
| $\omega_\varphi, H_\varphi$ | Fixed positive calibration and its ample line | Proposition A.2 |
| $b_v, \overline{\Delta}_\partial$ | Boundary Green function and full boundary datum | Definition 9.1 |
| $\mu_v^{\mathrm{YZ}}$ | Prescribed signed adelic mixed measure | (9.2) |
| $R_{h,\varepsilon}$ | Convex regularized maximum | Proposition 3.2 |

## Declaration on the use of generative AI

Generative AI was used in developing and drafting mathematical arguments and in revising the exposition of this manuscript.

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