Non-Archimedean Geometry Foundations

Analytic, formal and tropical geometry

English reference edition

Working consolidation, version reconstruction RC19 (partial)

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Reconstruction notice

This is a partial reconstruction candidate for the native English Foundations. Its historical sources, numbering, and labels are preserved where recovered. Proofs using confirmed outside inputs have been withheld at the point of use. Open proof obligations accompany the preserved statements. The remaining sources and proof routes still require independent review. Recovered articles outside the native include tree remain separate files.

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Reading guide

This book is organized by mathematical constructions. The original CLD calculus, its 2025 expansion, and the subsequent work of Gubler and his collaborators are read together at the point where they address the same problem. The two CLD versions retain separate names throughout. Their definitions of psh, their conventions for the involution, and the scope of several positivity assertions differ. The source concordance at the end records the fixed versions and the qualifications needed when using them.

The intended reader knows algebraic geometry, basic valuation theory and elementary differential forms. Familiarity with Berkovich spaces is helpful but not assumed for the first chapters. The analytic foundations are developed to the extent needed for the later constructions: completed residue fields, reduction, affinoid domains, tropical multiplicities and skeleta. Deeper existence and comparison theorems are quoted with their hypotheses. This permits a proof to remain readable without hiding its geometric input.

Reading the analytic foundations

The first four chapters now form a continuous introduction to the analytic geometry used throughout the book. They should be read in order on a first encounter. Each construction is followed far enough to identify its role in a later proof. The repeated examples are the closed disc, the annulus, the radius-r field and a formal blow-up of a disc.

Chapter	Central construction and its later use
1	Convergent algebras, Weierstrass division, seminorm points and completed residue fields. The nilpotent example separates points from multiplicity; adjoining a radius prepares the non-strict theory.
2	Domains and gluing, local rings, analytic fibres, boundary, flatness and analytification. The Galois product and the boundary obstruction to naive flatness explain why algebraic-looking formulas need analytic hypotheses.
3	Canonical and graded reduction, Shilov boundary, models, Abhyankar dimension and germ reduction. Formal frames then produce model metrics; residual geometry will later supply intersection degrees.
4	The four types of points, annular homotopy, residual curves, semistable skeleta and the slope formula. This gives the local geometric meaning of tropical directions before general tropical geometry begins.

For a first pass, follow the explicit proofs of Gauss multiplicativity, division, compactness, Laurent gluing, the finite-fibre length formula, the norm/model example, the Gauss residue-dimension formula and the annular retraction. Tate acyclicity, general analytic flatness, GAGA, germ reduction and semistable reduction are stated with precise source inputs. Reading their statements is enough to continue; their full foundational proofs belong to the cited monographs.

Readers returning from version 0.5 should note the numbering change. The former first chapter is replaced by Chapters 1–4; the former Chapters 2–21 are now Chapters 5–24. The Gauss residue-dimension proof and the singleton Shilov criterion have moved from the former Chapter 3 into Chapter 3. Their links and statement identifiers have been retained. Old theorem numbers should therefore be cited with their old version, not silently transferred to this edition.

The order of the book

Chapters 1–6 explain how analytic geometry supplies a tropical chart with integral tangent directions and multiplicities. Chapters 7–10 pass from these charts to forms, currents, integration and divisor identities. Chapters 11–14 study potentials, approximation and continuous products. The dependency is essential: the topology and positive cone of the test space must be fixed before one can assign a meaning to a positive Hessian current.

Chapters 15–18 develop and compare the delta and harmonic calculi. The former retains polyhedral curvature in an intersection algebra; the latter enlarges the functions allowed as smooth coordinates. Chapters 19–21 connect these local objects with model metrics, residual degrees, canonical measures and toric Monge–Ampère equations. Chapters 22–24 treat higher-codimension Green objects, degree normalizations, descent and cohomological limits.

Further source comparisons

For the source repairs, read Sections 11.6 and 19.17; the residual proof obligations are separated in Section 19.18. The delta conventions and graph examples are in Sections 15.9–15.10. Sections 16.7, 16.8 and 19.19 compare polyhedral and non-Archimedean envelopes, including a complete orthogonality counterexample and the relative-volume proof mechanism. Accessible skeleta and weighted measure germs are treated in Sections 6.12 and 21.28. For research involving cycle classes or Green objects, read Sections 24.15, 24.16 and 22.11 together. Section 23.6 supplies the global arithmetic completion framework; Section 24.17 states the remaining comparisons.

Routes for a first reading

A first reading can follow one of the routes below. The references point to the arguments on which the later material depends; they are also useful when returning to the book for a specific calculation.

Question	Route
How does a tropical chart acquire a weight?	Gauss norms and lattices in Chapter 5; initial degenerations in Section 5.3; monomial finite-free degrees in Section 5.7; Shilov points and faithful charts in Sections 6.6 and 3.4.
Which positivity cone is needed?	Chapter 7, followed by coefficient measures and order zero in Chapter 9. Then read the Plücker counterexample in Appendix A and the separate Hessian/corner criterion in Section 10.6.
What does the continuous BT construction prove?	Potential classes in Chapter 11; regularized maxima in Chapter 12; fixed-test estimates and mixed products in Chapter 13; the kernel calculation in Chapter 14.
When does model positivity agree with analytic positivity?	The comparison table in Chapter 18; numerical curve detection and local model approximation in Chapter 19; residual curvature and Chern measures in Chapter 20.
How are concrete measures and potentials computed?	Metric graphs in Chapter 20; real tori, toric roofs and discrete Monge–Ampère inversion in Chapter 21; weighted trace under graph maps in Chapter 23.
What is required for an actual Green form?	The divisor sign in Chapter 10; regular sequences, finite-flat multiplicities and all delta-tests in Chapter 22; the distinction between current descent and representative descent in Chapter 23.
Can tropical Hodge theory be passed to an analytic limit?	The affine inverse limit in Chapter 6; the superform resolution in Chapter 7; filtered cohomology, tropical duality and the explicit flat-torus calculation in Chapter 24.

Recurring distinctions

Several questions recur across the literature because the same formula can describe objects with different information. A tropical set forgets its lattice multiplicities. A current remembers the action on its chosen tests, while a form also has a representative and relations among presentations. A curvature current forgets metric constants. A top-degree measure comparison need not

identify lower-degree forms. The examples throughout the book make these losses of information explicit.

The two kinds of independence in the BT construction deserve particular attention. Approximation independence means that two admitted approximations to the same continuous function define the same product. Potential independence means that two different functions with the same curvature act identically on a background current. The first follows from a limit theorem. The second asks how the Hessian kernel acts on that background. The tropical cross shows that these are separate assertions. On a convex Euclidean domain the kernel is affine and the second assertion has a short proof; on singular supports the kernel can be larger.

There are also two dimensions at a skeletal point. The real dimension of a simplex describes its visible tropical directions. The completed residue field retains the remaining algebraic directions. Section 3.8 computes their sum directly. The same division reappears in the canonical-measure formula: a simplex volume is multiplied by a residual intersection degree. It is the reason that mixed reduction cannot be treated merely as the totally degenerate case with fewer coordinates.

Using the source corrections

Appendix A separates three situations. A false literal statement has a counterexample and a replacement with its own hypotheses. A local formula or index error is corrected by degree, sign or domain checks. A missing proof step remains a stated obligation unless an independent argument supplies it. The conclusions of a paper are not all assigned the status of one problematic lemma. In particular, a counterexample on a reducible abstract tropical support is not automatically a counterexample on a smooth Berkovich space.

For the most consequential issues, the main text includes the repair rather than leaving it in an error list: ordinary-positive coefficient bounds, the convex representative theorem, fixed-neighbourhood approximation, the two parts of a piecewise smooth Hessian, regularized maximum normalization, and the finite-flat comparison at Abhyankar points. The source concordance points back to those proofs. Statements imported from the literature retain their source attribution even when their proof has been reorganized here.

For the current comparison repair, read Section 11.11.1 for harmonic lower contacts, Section 11.11.2 for relative supports through zeros, and Section 11.11.3 for the independent Q-PL identification on unit charts. Theorem 11.11.6 then supplies nef residues, and Corollary 11.11.7 returns to original smooth approximation on one neighbourhood. Section 11.11.5 repairs the interval-support and strict-localization calculations in APW. For vanishing coordinates continue with Section 11.12: its recession argument completes the relative rational PL identification, and Corollary 11.12.6 gives the forward smooth approximation. Section 11.12.6 repairs the Gaussian boundary calculation, and Section 11.12.7 proves fixed-fan coefficient density while exhibiting the obstruction to freezing a general subdivision. The remaining reverse-comparison problems are separated in Section 11.12.8.

Section 11.13 treats a different coefficient problem: an already strict R-PL germ is rationalized on one residue tube by a controlled ample perturbation. Theorem 11.13.6 gives the resulting comparison only within this PL category. Read Section 11.14 alongside the projective transfer: it separates a genuine fixed-model counterexample to GM Lemma 4.12 from the sufficient cofinal ample-model construction. The latter preserves the approximation input of Boucksom–Eriksson Theorem 7.8.

Consulting and citing this version

Continue with Section 11.15 for the passage from formal to algebraic domination and the corrected reduced cofinality theorem. Its norm examples distinguish reducedness, normality, equality of unit balls and equality after scalar extension. The final determinant estimate explains why a bounded discrepancy of norms disappears in relative volume.

Section 11.16 then realizes the fixed-chart obstruction by analytic units and gives a smooth psh example in every positive dimension. The example itself admits a different convex presentation, so it isolates a chart-dependent shortcut and does not disprove the full local comparison. Section 11.17 completes this test case in the original chart: it constructs a decreasing relative rational PL approximation even though ambient convex approximation fails. Read the midpoint defect and the coefficient-rounded formula together. The final monotone-convex composition result gives a higher-dimensional class for which the relative-to-original-smooth passage is available, with all domains fixed before taking limits. Section 11.18 gives a second, polyhedral route:

extend the earlier fixed-fan argument to a finite polytopal complex whose vertices share one value-group scale. Its final examples explain why rational normals and value-group-compatible intercepts must be checked separately.

Section 11.19 separates the arithmetic continuity obstruction from the relative-support condition. Its closure formula and fixed-refinement theorem isolate the remaining arithmetic obstruction; the moving rectangle shows how changing subdivisions can overcome it. The final boundary-extension construction gives cellwise convexity in arbitrary dimension, with the missing cross-cell support condition stated separately. The rectangle example explains why a rational simplicial structure is an additional hypothesis, even when all facet normals are integral.

Section 11.20 completes the moving-subdivision argument for relatively supported real PL functions on coefficient-compatible compact complexes. The proof first controls the normal slopes in convex boundary extension, then preserves supports by a quantitative strict barrier. Read its final scope paragraph before applying the result to real-intercept source approximants or to original smooth functions.

Section 11.21 treats real-intercept PL germs by moving their centres within a stratum. For sequences, read the common-domain hypothesis in Corollary 11.21.6 together with Example 11.21.7. Corollary 11.21.8 states the original-smooth consequence with its analytic hypotheses. Proposition 11.21.10 distinguishes uniform approximation from Lipschitz-small error. Section 11.21.4 separates real-polyhedral, rational-polyhedral and coefficient-compatible PL conventions before using the source regularization.

For the general continuous relative-support input on a fixed compact image, start instead with Section 11.22. Its rational local bumps remove the earlier assumptions on the input subdivision. Theorem 11.22.3 controls coefficients before gluing; Theorem 11.22.8 passes through local maxima of unit logarithms and the fixed projective transfer. Read Example 11.22.11 to distinguish approximation of a limit from preservation of the support definition by that limit. Theorem 11.22.12 and Corollary 11.22.13 extend this construction through zeros, using admissible boundary supports and an exponential strictifier. Proposition 11.22.14 also treats a single original smooth psh input. Its final paragraph explains why the neighbourhood may depend on the input and how arbitrary sequences require a separate argument, now supplied by Section 11.24. The earlier geometric constructions remain useful for stronger quantitative control.

For varying local presentations, read Section 11.24. Exact absolute values on Laurent domains permit one finite extended presentation on a prescribed affinoid. This yields fixed-domain rational replacement and equality of four precisely defined first decreasing stages. Lemma 11.24.5 handles a second decrease when the rows already share one domain. Its final paragraph isolates the remaining sheafification issue.

The English and Chinese editions are separate texts with corresponding numbered statements. The Chinese edition keeps technical terminology in English. There are no exercises. The subject index gives entry points by mathematical term, and the notation table precedes the main text. References to a chapter number of CLD12 or CLD25 are explicitly marked as source references; cross-references within this book are hyperlinks.

This is the full working manuscript, version reconstruction RC19 (partial). A citation to one of its statements should retain the version and theorem or section number; when the statement is an imported theorem, cite the original paper as well. Complete proofs, proof reconstructions using specified external inputs, and remaining comparison questions are distinguished in the text. The manuscript has undergone internal mathematical and editorial checks, but no independent mathematical certification is claimed. The unresolved comparisons are part of the account of the subject, not assertions that all literature gaps have been closed.

Proof completeness and permanent references. The proof architecture distinguishes a complete internal proof from a theorem used as an external input. It also describes the intended dependency graph and permanent identifiers. In particular, a source citation, a reconstruction under named inputs, and a mathematically closed proof are three different states. The separate project charter records the long-term plan; the reference text remains organized by mathematical constructions.

Fixed-affinoid replacement and first stages. Sections 11.23–11.24 separate common unit charts from the stronger method using extended presentations. Read the exact-norm fraction lemma, the saturated-cover globalization, and the fixed-affinoid approximation theorem in that order. Corollary 11.24.4 identifies four explicitly defined first-stage germ classes in the nontrivially valued integral-projective setting. The common-row diagonal handles a further iteration only when all rows already share a domain. Full CLD25/Wu com-

parisons and general sheaf-level stability remain separate. Example 11.23.5 shows, in every positive dimension, why global unit presentations cannot simply be assumed.

Projective approximation and its generator class. Section 11.9 proves a fixed-neighbourhood approximation theorem for decreasing sequences of explicitly specified convex rational PL generators on an integral projective variety over a nontrivially valued field. Its barrier is fixed before the sequence; its scales may vary with the terms. The proof uses the cited model/PL and model-nefness criteria, known uniform Fubini–Study approximation, and the diagonal lemma. No envelope hypothesis is used for this inclusion. Section 11.10 treats ambient analytic generators at the extended tropical boundary, including zeros. Read its boundary factorization and finite-selection proofs, then the fixed-domain corollary in Section 11.10.5. Theorem 11.10.11 gives the broader transfer for locally semipositive rational PL generators, using the maximum theorem and local-to-global model semipositivity. The fixed-affinoid and first-stage comparison is now supplied by Section 11.24; source-category identification, further sheaf-level stability and trivial valuation remain separate. Section A.8 repairs the touching-support formula in Wu's convexity definition.

Common domains and source corrections. Section 11.8 supplies a quantitative diagonal lemma, a uniform tropical localization formula and projective Fubini–Study barriers. Example 11.8.5 refutes only the whole-domain majorization added to APW26 Proposition 5.19; Proposition 11.8.6 preserves its local extension conclusion. Theorem 11.8.9 constructs the geometric data needed in Section 11.7.3. Section 11.9 subsequently supplies a common domain in one direction for the stated rational PL class. The full Wu comparison still requires identification of its precise source conventions and field scope; see Section 11.8.5 and the source table in Section A.7.

Envelope localization. Section 11.7 incorporates Wu's recent local theory and separates its first-stage classes from the generated CLD sheaf. Read the fixed bounded extension in Proposition 11.7.3, then Theorem 11.7.5: bounded decreasing stability is sufficient to identify every continuous ordinal stage with the first stage. Corollary 11.7.6 gives the projective application using ex-

plicitly imported comparison and envelope theorems. Its source dependencies and the general analytic question are kept separate in Remark 11.7.7.

The compact-barrier argument. Sections 11.6.1–11.6.3 separate decreasing iteration on a fixed compact set from the local sheaf construction. The compact upper-barrier theorem gives first-stage approximation whenever a common compact representation is available. The indexed inclusion and the locally integrable hierarchy are proved separately. The general continuous replacement in CLD25 Proposition 8.1.11 remains open here; Proposition 8.1.12 is retained with its valid proof.

The integral PL comparison. Read Sections 19.18.8–19.18.13 after the residual criterion in Theorem 19.18.12. The corrected weighted maximum gives semiample first-stage approximation. The new prescribed graded premodel construction then proves the integral nef implication over arbitrary complete fields, including trivial valuation, on one fixed reduction tube. Corollary 19.18.23 compares original-current positivity, residual nefness and local $\mathcal{C}_{25}$ membership. The general continuous approximation and predecessor-replacement questions remain open in this manuscript.

The version 0.37 foundation increment

This increment adds nineteen English sources containing sixty-nine complete proof blocks. Their permanent labels, exact hypotheses, direct dependencies and separate author and nonauthor checks are recorded in the versioned result register. The two principal modules, curves and models (F5) and tropical geometry (F6), remain partially closed. A registered closed result means closure of its precise statement under the project's declared foundations and AI review procedure; it is not human peer review or formal verification.

The curve increment constructs actual original-field model comparison maps, common models, exact vertex-set correspondences and least marked semistable hulls. It recognizes model vertices by integral Gauss charts, gives a rational completed-degree criterion over a general complete nontrivially valued field, and a numerical criterion over a complete discretely valued field. The realization statements retain the existence of one original-field semistable model and the stated finite étale marking conditions.

The tropical increment proves finite polyhedrality of actual affinoid moment images and their analytic-boundary images, constructs intrinsic module weights and their balanced interior cycles, and proves finite pushforward formulas with generic Artinian lengths. It then constructs finite-cover integrals, ordinary smooth moment germs, compact-support integration, smooth cutoffs and finite partitions near compact sets. Finite projection is proved both for compact domain data and for ordinary compactly supported smooth forms. The ground-field absolute value remains fixed; the analytic statements allow trivial valuation and arbitrary positive presentation radii where their hypotheses say so.

These are internal reconstructions of foundational theory. The classical context for global images and the finite Gauss-fibre boundary mechanism is Ducros, Lemma 3.1 and Theorem 3.2 of *Espaces de Berkovich, polytopes, squelettes et théorie des modèles*, *Confluentes Mathematici* 4 (2012), no. 4, 1250007, doi:10.1142/S1793744212500077. The integration program belongs to the theory of Chambert–Loir and Ducros and its subsequent developments; see [CLD25, Gubler]. Those works supply attribution and comparison targets, not undeclared external proof leaves.

The next general interfaces include pointed tropical-germ stabilization, global Stokes under explicit analytic boundary hypotheses, comparison with specified CLD/GK test conventions and topologies, and wider proper push-forward and higher-dimensional model statements. The exact open tasks are kept separately from the accepted dependency graph. Historical Chinese materials remain frozen; this increment maintains the English mathematical edition only.

Introduction

A local differential theory on a Berkovich space must combine two kinds of data. Analytic functions supply logarithmic coordinates, while integral tangent lattices and multiplicities determine how those coordinates are integrated. The work of Chambert–Loir and Ducros constructs forms and currents from these data. Subsequent work develops delta-form intersections, harmonic coordinates, and comparisons with metrics and algebraic intersection theory. This book arranges these constructions so that their hypotheses and their precise points of contact remain visible.

The first basic issue is integration. Tropical multiplicities are not decorative weights: the change-of-variables formula contains a lattice index, and balancing cancels boundary contributions at internal faces. Chapters 1–6 establish the geometric background, and Chapter 8 explains the passage to an intrinsic analytic sheaf and its integrals. The Gauss-point and lattice calculations are proved directly. Polyhedrality, the general balancing theorem, faithful tropicalization and the analytic calibration theorems remain explicitly cited geometric inputs.

The next issue is positivity. In the real superform algebra, the three positivity cones need not agree in middle degree. The ordinary positive cone controls all symmetric coefficients; the strong cone need not span them. Chapters 7–9 develop this distinction, prove the applicable order-zero estimates, and construct continuous scalar multiplication by measures attached to actual tests. The intrinsic proof is made in an explicit algebraic CLD setting. It does not silently identify every analytic or harmonic quotient with that setting.

For potential theory, the construction $d'd''(uT)$ separates multiplication from differentiation. Positivity of this current requires an admitted class of potentials, and the available limit theorem depends on that class. Chapter 12

proves the normalized regularized maximum and its local gluing properties. Chapter 13 proves mixed-product continuity and symmetry for the explicit smooth-approximation class. Chapter 14 explains why independence of an approximating sequence does not imply independence after changing the potential by an arbitrary current-harmonic function.

The enlarged calculi address different needs. Delta-forms retain corner terms as objects that can participate in intersection products. Weakly smooth forms permit harmonic coordinates which need not be original smooth coordinates. Model and curvewise criteria compare numerical semipositivity with analytic positivity in specified regularity classes. Chapters 15–21 keep these operations separate until a comparison theorem supplies a map between them. The discussion of Green objects and descent then records the additional requirements on representatives, tests and support.

The two CLD manuscripts are treated as different versions of a developing theory. Corrections are attached to exact source statements. A false real-algebra claim, an unsupported analytic proof step, and a typographical formula error have different consequences. The source concordance makes these distinctions explicit. The present text is a working reference manuscript, with full proofs where supplied and identified external inputs elsewhere; it is not a claim that every gap in the literature has been resolved.

The longer-term purpose is to turn this reference manuscript into a proof-complete, searchable foundation for non-Archimedean geometry. An imported result remains an imported result until its proof and the proofs of its non-elementary prerequisites have been incorporated and checked. The proof architecture explains this standard and identifies the first foundational dependencies to be closed.

Conventions and source architecture

0.1 The CLD12 construction and its dependencies

Before CLD, one could attach a top-degree Chambert–Loir measure to a suitable metrized line bundle on a proper variety. A notation such as $c_1(\overline{L})^n$ existed, but a local differential object $c_1(\overline{L})$ was not supplied by that measure construction alone. CLD addresses this mismatch. The analytic space is allowed to be nonproper and need not be algebraic; the calculus must therefore be local and compatible with restriction.

In CLD12, Chapter 1 develops the real algebra of two copies of differential forms. Chapter 2 supplies polyhedral tropical images and the degree information needed to weight them. Chapter 3 uses these images as presentations of forms on the analytic space and proves integration, Stokes, and Green formulas. Chapter 4 takes continuous duals and proves the logarithmic divisor formula. Chapter 5 introduces positivity and constructs products for specified approximation classes. Chapter 6 connects local weights and residual line bundles with the existing model measures.

These references are to the chapters of CLD12, not to the numbering of this book. The positivity statements of CLD12 Chapter 5 use the test spaces constructed in its Chapters 3 and 4. The adjective positive refers to a cone of actual test forms. The equation $\mathrm{d}'\mathrm{d}''[u] \geq 0$ is consequently more than a Hessian calculation on one fixed tropical chart. Similarly, the limit defining a mixed measure is an identity on all compactly supported tests, not merely a sequence of measures on one preferred skeleton.

The geometric and analytic parts meet in the canonical calibration. Algebraic or analytic multiplicities make a tropical integral independent of the

moment map. This is what allows the distributional derivative to be intrinsic. The later delta-calculus retains exactly this requirement while enlarging the types of representatives that can be multiplied.

References. [CLD12, Introduction and Chapters 1–6].

0.2 Conventions, potential classes, and sources

A current is a continuous linear functional on a space of test forms. Thus the expression

$$d'd''[u] \geq 0$$

has a mathematical meaning only after the test space, its topology, the positive cone, and the convention for degrees have been specified. A second question concerns the functions u for which the expression is used. Positivity of a current, existence of smooth positive approximations, and numerical semipositivity of a model are different conditions. A comparison theorem must identify the map between the objects and state the hypotheses under which the conditions agree.

The book develops these comparisons in the following order. The real differential algebra and its cones are constructed in Section 7.1. Original analytic forms and their algebraic presentation are compared in Section 8.1; their continuous duals are defined in Section 9.1. The change from the 2012 to the 2025 definition of psh is treated in Section 11.2. Delta-forms and harmonic tropicalizations are then considered as separate constructions. The numerical and curvewise comparison results are stated with their actual regularity assumptions. The corrections in Section A.1 explain which literal statements from the sources cannot be used in this form.

Two conclusions organize the comparison. For continuous functions, the elementary arguments prove the chain

$$\mathcal{U} \subseteq \mathcal{C}_{25} \subseteq C^0 \cap \mathcal{P}_{25} \subseteq \mathrm{Psh}_{12}^{\mathrm{cont}},$$

with equality between $\mathcal{U}$ and the continuous first decreasing-approximation stage; see Propositions 11.2.3 and 11.2.4. For real PL functions on good Hausdorff boundaryless strictly analytic spaces over a nontrivially valued field, model semipositivity, membership in $\mathcal{C}_{25}$ or $\mathcal{P}_{25}$, and classical curvewise psh are equivalent; see Proposition 19.2.11. This second comparison follows from model approximation and curve tests. Its proof does not assume the stronger general current criterion stated in CLD25.

0.2.1 Scope and conventions

The base field K is complete and non-Archimedean. The original current calculus is described on a good Hausdorff analytic space X of pure dimension n, using test forms whose compact support avoids the analytic boundary. Statements about potentials and products are made on the boundaryless locus. The algebraic results of Gubler and Gubler–Künnemann will impose the additional assumptions that K is algebraically closed and nontrivially valued and that X comes from a separated algebraic variety. The harmonic-tropicalization results instead use good strictly analytic spaces over a nontrivially valued field. Each comparison retains these additional assumptions where they are needed.

An analytic boundary is the boundary of a Berkovich space or morphism. It is not a divisor deleted from an algebraic compactification. An algebraic open subvariety can have an algebraic boundary in a compactification and still have empty analytic boundary over K.

We use additive weights

$$\varphi_e = -\log\|e\|, \qquad dd^c = d'd'', \qquad f_{\mathrm{trop}} = (-\log|f_1|, \dots, -\log|f_r|). \tag{0.2.1}$$

No square of the norm is included. The symbol J denotes the involution of the older Lagerberg convention; the conversion to J_{25} is given in Proposition 7.1.6. Currents are indexed by bidegree unless lower indices are explicitly displayed. A measure on an n-dimensional space has bidegree (n,n) and bidimension $(0,0)$.

0.2.2 Version-specific references

The reference labels below designate fixed texts. A theorem number in a later arXiv revision need not be its number in the published article. The dates in this table refer to the arXiv revisions actually used.

Source	Version used	Principal comparison
CLD12	1204.6277v1; 27 April 2012; 124 pages	Original forms, currents, continuous current-psh functions, and a separate approximation class.

Source	Version used	Principal comparison
CLD25	1204.6277v2; 25 July 2025; 608 pages	New sign convention, generated psh sheaf, G-forms, harmonic charts, and residual comparisons.
Gubler	1303.7364v3; 5 October 2021	Algebraic tropical charts for the original CLD sheaf. The underlying chapter was published in 2016.
GK17	Published ANT article; 2017	Delta-forms, delta-currents, and their intersection calculus.
GK19	1509.09079v2; September 2016	Positivity, functorial psh conditions, and PL metric comparisons.
BGJK	2003.08644v3; February 2021	Real positive cones and counterexamples to strong-cone spanning and weak-positive order-zero claims.
GJR	2111.05741v3; 7 March 2025	Harmonic tropicalizations, weakly smooth forms, and strong currents.
GRPL	2503.05513v1; 7 March 2025	Semipositivity of real PL functions tested on analytic curves.
GRPSH	2506.13548v2; 17 September 2025	Classical psh defined by analytic curve tests after base extension.

The 2025 CLD text is substantially different from its 2012 version. In particular, its psh definition is not obtained merely by renumbering the old definition. Conversely, a corrected statement in a later paper does not justify reading the earlier wording as if the correction were already present. The book therefore records both the mathematical definition and the source in which it occurs.

Remark 0.2.1 (Status of source comparisons). A theorem imported below is cited with a usable statement and its hypotheses. Appendix A uses three different descriptions: a *false literal statement* has a displayed counterexample; a *typographical inconsistency* has a local correction justified by the surrounding definitions; and a *proof issue* identifies a step not justified by the printed argument. A proof issue alone does not prove that the theorem is false. In particular, the deeper PL comparison stated in CLD25 is recorded separately from the statements that we disprove by elementary examples.

0.2.3 Index of notation

Symbol	Meaning	Definition
$A^{p,q}(\Omega)$	Real superforms on an open subset of a real vector space	Definition 7.1.1
J, J_{25}	Older involution and CLD25 graded operator	Proposition 7.1.6
$A^{p,q}_{\mathrm{sm}}$	Original smooth forms from analytic-unit tropicalizations	Definition 8.1.1
$A^{p,q}_{25}$, $A^{p,q}_{G,25}$	Ordinary and G-smooth sheaves in CLD25	Definition 17.3.2
$A^{p,q}_{\mathrm{harm},25}$	CLD25 forms represented by current-harmonic PL charts	Definition 17.3.4
$A^{p,q}_{\mathrm{sm},c}$	Original test forms, with their presentation topology	Definition 9.1.1
$D^{a,b}_{\mathrm{sm}}$	Dual of original tests of degree $(n-a, n-b)$	Definition 9.1.2
$[u]$, $[\alpha]$	Currents obtained by integration against a function or form	Section 9.1
$\mathrm{Psh}^{\mathrm{cont}}_{12}$	Finite continuous functions with positive original curvature current	Definition 11.1.1
$\mathcal{U}$	Local uniform limits of original smooth psh functions	Definition 11.2.1
$\mathcal{P}_{25}$, $\mathcal{C}_{25}$	CLD25 generated psh and continuous generated sheaves	Definition 11.2.2
$\mathcal{P}_{25,\alpha}$	Ordinal stages of the decreasing-limit construction	Definition 11.2.2
$\mathrm{Psh}_{\mathrm{cl}}$	Classical psh defined by analytic extension and curve tests	Definition 19.2.1
$A^{p,q}_{\mathrm{wk}}$	GJR weakly smooth forms from harmonic tropicalizations	Definition 17.2.5
$A^{p,q}_{\mathrm{ps}}$	Piecewise smooth forms with polyhedral derivatives	Theorem 17.2.6
$D^{a,b}_{\mathrm{wk}}$	Strong currents, dual to weakly smooth tests	Definition 17.2.10
d'_P, d''_P	Derivatives taken on individual polyhedral pieces	Section 17.2

Symbol	Meaning	Definition
$B^{p,q}$, $P^{p,q}$	Delta-forms and generalized delta-forms in the algebraic category	Definitions 16.1.1 and 16.1.2
$E^{p,q}$	Delta-currents, dual to compactly supported delta-forms	Definition 16.1.3
$\mathbb{R}$-PL	Piecewise linear functions with real coefficients, defined through local model data	Definition 19.1.4
$c_1(L, \|\cdot\|)$	Curvature current of a metrized line bundle	Definition 11.1.6

0.2.4 Geometric and analytic notation used later

Symbol	Meaning and normalization	Location
Γ	$v(K^\times) \subset \mathbb{R}$, with $v = -\log\|\cdot\|$.	5.2.1
M, N	Character lattice and its integral dual.	5.1.2
$N_{\mathbb{R}}, M_{\mathbb{R}}$	$N \otimes_{\mathbb{Z}} \mathbb{R}$, $M \otimes_{\mathbb{Z}} \mathbb{R}$.	5.1.2
N_σ	$N \cap \operatorname{span}_{\mathbb{R}}(\sigma - \sigma)$; the saturated tangent lattice.	5.6.1
$\operatorname{in}_w(X)$	Initial degeneration, including scheme multiplicities.	5.10
$\sigma(w)$	Gauss section of torus tropicalization.	6.1.1
$\mathcal{H}(x)$	Completed residue field of an analytic point.	1.6.2
$s(x), t(x)$	Residual transcendence degree and relative rational value-group rank.	3.8.1
β	$\sum_i d'x_i \wedge d''x_i = \frac{1}{2} DD\|x\|^2$ in fixed Euclidean coordinates.	9.13
μ_T	$T \wedge \beta^p/p!$ for bidimension (p,p).	9.13
$\nu(T,a)$	Euclidean radial density, not generally an integral tropical weight.	9.14.3
δ_C	Integration current of a weighted tropical cycle.	15.1.1
∂', ∂''	$d'_P - d'$, $d''_P - d''$; boundary operators on delta-forms.	16.5

Symbol	Meaning and normalization	Location
$C \cdot D$	Stable tropical intersection with lattice-index weights.	15.3
$\mathrm{MA}_{\mathbb{R}}(f)$	$E \mapsto \mathrm{vol}(\partial f(E))$; Alexandrov normalization.	21.27
$(DDf)^n$	$n!\,\mathrm{MA}_{\mathbb{R}}(f)$ for a convex real potential.	21.5.1
h_P	$x \mapsto \sup_{p \in P}\langle p, x\rangle$; support function.	21.27
f^*	$p \mapsto \sup_x(\langle p, x\rangle - f(x))$; convex Legendre transform.	21.26.1
ϑ	Concave roof; sign conventions are fixed by the chosen metric weight.	21.8
$E(u)$	Normalized mixed metric energy relative to a fixed reference metric.	20.11.1
$P(u)$	Psh envelope in the stated global metric class.	19.15
Δ_Γ	$-DD$ on a metric graph; positive Dirichlet sign.	20.12.1
$\mathcal{E}(u, v)$	$\sum_e \int_e u'v'\,ds$; graph Dirichlet pairing.	20.15.1
$g_\mu(x, y)$	$\Delta_x g_\mu = \delta_y - \mu$, $\int g_\mu(x, y)\,d\mu(x) = 0$.	20.15.5
Tr_ϕ	Trace weighted by local graph degrees.	23.5.2
F_p, F^p	Multi-tangent coefficient space and its dual.	7.8.2
$H^{p,q}_{BC}, H^{p,q}_{A}$	Bott–Chern and Aeppli groups of the specified double complex.	24.11.4
P_0	Projection to the constant Fourier mode on a real torus.	24.11.1

The symbol P can denote a polytope or an envelope operator, but never both in the same formula. Graph Laplacians have the sign $-DD$; Euclidean Laplacians in the distributional potential proof have the usual sign $\sum_i \partial_i^2$. These conventions serve different integration-by-parts formulas and must be retained when transferring a calculation.

0.3 Chapter concordance for CLD25

The CLD25 source architecture has a useful division of labour. Its Chapters 1–9 build a calculus on abstract tropical spaces. Chapters 10–16 prove that

Berkovich analytic spaces supply the required structures and establish analytic current theorems. Chapters 17–19 apply the calculus to metrized bundles, residual intersection theory, higher-codimension formulas, and abelian varieties.

Chapters	Objects constructed	Why they are needed
1–3	Superforms, paralinéaire spaces, G-tropical spaces and G-forms	A chart-independent language for differential expressions that may only be smooth on pieces.
4–6	Calibrations, integration, refined tropical structures and tropical spaces	Intrinsic integrals, compatible smooth forms, balancing, and Stokes.
7	Currents, positive and skeletal currents, harmonic charts	Duality, measure coefficients, structured singular currents, and extra coordinates.
8	Generated psh classes, CLN estimates, continuous products, dpsh, polar sets	A local pluripotential calculus with explicit approximation and independence conditions.
9	Products of spaces and currents, Gysin construction, tropical intersection	Intersections expressed through differential operations on products.
10–11	Graded reductions, residual intersection numbers, analytic geometry	Dimension, multiplicity, descent, and the residue geometry behind calibrations.
12–14	Analytic skeleta, richness, analytic degrees, canonical calibrations	Verification that the abstract tropical machinery applies intrinsically to analytic spaces.
15–16	Analytic forms and currents, finite maps, cycles, Poincaré–Lelong, support and polar results	The analytic realization of the first part and theorems involving analytic singular sets.
17	Metrized vector bundles, PL and formal metrics, residual bundles	Geometric sources of potentials and numerical positivity.

Chapters	Objects constructed	Why they are needed
18	Curvature products, curve currents, residual formulas, Bochner–Martinelli and Levine	Explicit Chern and Green constructions, including higher codimension.
19	Cubist metrics, uniformization, positivity and canonical measures	A substantial application where harmonic coordinates and residual geometry interact.

The abstract first part allows a differential argument to be proved once on a tropical space and then applied analytically. The price is that the hypotheses defining a tropical space are substantial. One cannot take an arbitrary weighted topological polyhedron, call it tropical, and inherit every theorem. Nor can one replace a skeletal current by an arbitrary current supported on a skeleton: the definition imposes compatibility with all charts.

Three changes especially affect reading. The psh sheaf is now generated from smooth positive functions by a closure procedure, so current positivity is a comparison statement. G-forms and harmonic forms are explicitly distinguished from ordinary smooth forms. Finally, the metric chapters use residual bundles and structured currents to obtain results beyond the original approximation-based measure theory. Each change creates new comparison questions, rather than identifying every old and new object by definition.

References. [CLD25, Table of contents, Introduction, Chapters 1–19].

Proof architecture and dependency closure

The purpose of a reference work is to let a reader recover the hypotheses and proof of a result without reconstructing an informal chain of citations. The present volume has not yet reached that standard for every result. This section specifies how later editions will close the gaps while preserving the precise mathematics already recorded here. It is a description of the intended proof discipline, not a certification of the existing chapters.

When a proof is complete

A statement is marked CLOSED only after its argument, exact scope, and complete dependency list have been reviewed. Every dependency must lead, without a cycle, to an internal proof, the declared elementary base, or an explicitly registered and verified result of the Stacks Project. By the project owner's instruction of 15 November 2026, results already proved in the Stacks Project need not be reproved here. Their permanent tags, applicable parts, hypotheses, proof locations, and consulted versions are recorded as authorized external foundations. They are not counted as proofs written in this project. This authorization does not discharge an unresolved input from another source.

Until that standard is met, a result is described as imported, proved under named inputs, conditional, or unresolved in this manuscript. A counterexample to the literal wording of a source is recorded separately from a missing step in a published proof. The same assertion may also have different statuses in different field or regularity classes. In particular, equality of measures,

currents, and actual forms is not interchangeable; neither is agreement on a restricted test family agreement on all original compactly supported tests.

The information attached to a theorem

The permanent record for a numbered item will contain its stable identifier, semantic LaTeX label, statement type, exact hypotheses, conclusion, original sources, internal proof location, direct dependencies, and review status. Where relevant it will also record the base field and valuation normalization, analytic boundary, support, test space, regularity, and convergence mode. Chapter and theorem numbers remain useful for reading, but they are not permanent identifiers. Moving a result must not change its identifier.

An automated build can verify labels, references, bibliography keys and the consistency of the two editions. It cannot establish a proof's mathematical validity. The proof record will therefore distinguish mechanical checks, internal source-based review, independent mathematical review, and formal verification when available. None of these statuses is inferred merely from the presence of a proof environment.

The analytic foundation and its continuation

The analytic foundation now contains internal proofs of generalized Gerritzen–Grauert, Tate acyclicity for finite modules on finite affinoid-domain covers, and Kiehl's coherent-module equivalence on the full analytic G-site. The route begins with rational localization and bounded Laurent gluing, then passes through simultaneous strictification and descent. The theorems Theorems 2.3.1, 2.11.3 and 2.11.5 point to these complete arguments, including their norm and field hypotheses.

The subsequent foundation develops analytic points and fibres, boundary and properness, quasi-smoothness, proper GAGA, and canonical, graded and germ reduction. Formal models and admissible modifications are treated in their stated nontrivially valued and finiteness categories. The current result register attaches the exact proof and review status to each theorem. The next geometric dependencies are semistable curves and the slope formula in Theorems 4.6.2 and 4.7.1, followed by higher-dimensional tropical multiplic-

ities, balancing, and analytic calibration. Their incorporation will follow the proof dependency graph rather than the present page order.

The construction of intrinsic test forms and currents will come only after the geometric and integral inputs on which it depends. Products involving positive closed currents, higher-codimension Green representatives, and tropical-to-analytic cohomological comparison remain separate later modules. Their open interfaces are recorded in the current research register and are not treated as completed foundational facts.

How to use the present edition

Readers may use a complete proof in this edition together with the prerequisites it explicitly names. Authorized Stacks foundations can be followed directly through their permanent tags. Other imported inputs remain visible proof obligations unless a later, explicitly recorded authorization or complete proof changes their status. Any upgrade still requires scope and dependency checks and agreement of the English and Chinese statements. The project charter and the proof-base register specify the current rules; earlier editions are preserved in the history.

Part I

Analytic and tropical foundations

Chapter 1

Valued fields, Banach algebras and analytic points

A non-Archimedean analytic space is locally described by convergent functions, but its points need not be evaluations at elements of the ground field. The purpose of this chapter is to construct both parts of this description: the algebras of functions and the valued fields in which their values lie. The next chapter supplies domains and sheaves. This order is important for multiplicities: a seminorm detects the value of a function, whereas the algebra also retains nilpotents and finite-module lengths.

1.1 The ground field and convergence

Definition 1.1.1. Throughout the foundations, K is a field complete for a real-valued non-Archimedean absolute value. We permit the trivial absolute value unless a statement explicitly excludes it. Set

$$v(a) = -\log|a|, \qquad \Gamma = v(K^\times),$$
$$K^\circ = \{a \in K : |a| \leq 1\}, \qquad K^{\circ\circ} = \{a \in K : |a| < 1\},$$
$$\widetilde{K} = K^\circ / K^{\circ\circ}.$$

A complete extension L/K always carries an absolute value restricting to the specified one on K. We do not rescale it when changing the ground field.

The inequality $|a+b| \leq \max(|a|, |b|)$ makes K° a ring and $K^{\circ\circ}$ its maximal ideal. An element of K° is a unit there exactly when its absolute value is one.

Thus reduction remembers whether a bounded element is a unit or becomes small. The notation Γ is additive; its multiplicative counterpart is $|K^\times|$. A radius can be a positive real number without belonging to this group.

Lemma 1.1.2. *A series $\sum_{j\geq 0} a_j$ in a complete non-Archimedean normed vector space converges if and only if $a_j \to 0$. More generally, a countable family is summable independently of its enumeration if, for each $\varepsilon > 0$, only finitely many terms have norm at least ε.*

Proof. Convergence forces successive differences of partial sums to tend to zero. Conversely, every finite tail has norm bounded by the largest norm of its terms, so the partial sums are Cauchy. For an unordered family, the same estimate makes the net of finite subsums Cauchy. Any enumeration has the same limit because every prescribed finite set of terms eventually occurs. □

In particular, absolute summability in the Archimedean sense is not the relevant condition. For example $\sum a_j T^j$ converges at every t with $|t| \leq r$ if $|a_j| r^j \to 0$. This condition also controls the function uniformly on the entire closed disc, which is what an algebra of analytic functions requires.

Definition 1.1.3. A Banach K-algebra is a commutative unital K-algebra A with a complete non-Archimedean norm satisfying $\|ca\| = |c|\|a\|$, $\|1\| = 1$, and $\|ab\| \leq \|a\|\|b\|$. A homomorphism is bounded if $\|\phi(a)\| \leq C\|a\|$ for some constant C. Equivalent norms define the same Banach topology. A surjective bounded map is admissible if its target norm is uniformly equivalent to the quotient norm; equivalently, it admits lifts with a uniform norm bound. Equality of topologies alone is insufficient at trivial valuation; see Definition 1.9.1 and Example 1.9.2.

Lemma 1.1.4 (Geometric inversion)**.** *If $\|h\| < 1$ in a Banach algebra, then $1 - h$ is invertible with inverse $\sum_{j\geq 0} h^j$. If $\|h\| \leq c < 1$, the tail after degree m has norm at most c^{m+1}.*

Proof. The powers tend to zero by submultiplicativity. Multiplying the finite sum by $1 - h$ gives $1 - h^{m+1}$, and passage to the limit proves the identity. The ultrametric inequality gives the tail estimate. □

Completeness of the ground field also supplies a unique extension of the absolute value to every finite field extension. Equipped with it, the finite extension is complete. Lemma 1.12.1 proves these assertions, which will be

used for rigid points and finite morphisms. Infinite complete extensions need not be algebraic; adjoining a radius is an important example below.

References. [TemIntro, §§2.1–2.2]; [Ber90, Chapter 1].

1.2 Tate algebras and radii

Definition 1.2.1. For $r = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$, the generalized Tate algebra is

$$K\{r^{-1}T\} = \left\{ \sum_{\nu \in \mathbb{N}^n} a_\nu T^\nu : |a_\nu| r^\nu \longrightarrow 0 \text{ as } |\nu| \to \infty \right\}.$$

$$\|f\|_r = \max_\nu |a_\nu| r^\nu.$$

Here $r^\nu = \prod_i r_i^{\nu_i}$. When all $r_i = 1$, write $K\langle T_1, \dots, T_n \rangle$.

Proposition 1.2.2. *The weighted Gauss norm makes $K\{r^{-1}T\}$ a Banach algebra and is multiplicative. Polynomials are dense. The same finite-support multiplicativity argument applies to weighted Laurent polynomials.*

Proof. Completeness follows coefficientwise: a Cauchy sequence has Cauchy coefficients, and a uniform tail estimate passes to their limits. Truncating a series proves density. The convolution formula for a product and the ultrametric inequality give submultiplicativity.

For nonzero polynomials f, g, choose the lexicographically largest exponent μ attaining $\|f\|_r$, and ν attaining $\|g\|_r$. In the coefficient of $T^{\mu+\nu}$, the term $a_\mu b_\nu$ has weighted size $\|f\|_r \|g\|_r$. Any other term with that size would involve two maximizing exponents. If the first is smaller than μ, the second is larger than ν, a contradiction. Thus this coefficient has a unique term of maximal size, so cancellation cannot lower it. This proves multiplicativity for polynomials.

For general series, take polynomial truncations whose errors are strictly smaller than the nonzero norms of f and g. Their norms equal those of f, g, and the product errors tend to zero. Continuity of the norm gives the result. The argument for finite Laurent sums is unchanged because lexicographic order on $\mathbb{Z}^n$ is translation invariant. □

Proposition 1.2.3 (Substitution)**.** *Let B be a Banach K-algebra and let commuting elements b_i satisfy $\|b_i\| \leq r_i$. There is a unique bounded homomorphism $K\{r^{-1}T\} \to B$ taking T_i to b_i. It is contractive for the displayed norms.*

Proof. The summands $a_\nu b^\nu$ have norms at most $|a_\nu| r^\nu$, which tend to zero. Their sum defines the map and has norm at most $\|f\|_r$. Multiplication follows first for polynomials, then by density. The same density proves uniqueness. □

If $r_i = |c_i|$ with $c_i \in K^\times$, the change of variable $T_i = c_i U_i$ identifies this algebra isometrically with the usual unit-radius Tate algebra. Without such elements, the radii are genuine data over K. For a trivially valued K, the unit-radius algebra is $K[T]$: its coefficients must eventually vanish. With a radius smaller than one, infinite power series can occur even over that same field. Thus neither arbitrary radii nor the valuation hypothesis can be suppressed.

1.3 A division theorem and its meaning

We prove a useful one-variable form of Weierstrass division. The proof explains why convergence, rather than formal manipulation alone, gives control of analytic equations. Let $K\langle T\rangle$ have the unit Gauss norm.

Theorem 1.3.1. *Suppose $f = P + h$, where $P = T^d + \sum_{i<d} p_i T^i$ has $|p_i| \leq 1$ and $\|h\| < 1$, with h involving only powers larger than d. For each $g \in K\langle T\rangle$ there are unique $q \in K\langle T\rangle$ and $s \in K[T]$, $\deg s < d$, such that $g = qf + s$. They satisfy $\|q\|, \|s\| \leq \|g\|$. For $d = 0$ the remainder is zero.*

Proof. Polynomial division by the monic polynomial P gives operators Q, R with $g = PQ(g) + R(g)$. Each elimination step has coefficients of size at most one, so both operators have norm at most one. They extend to the completion, with R valued in the finite-dimensional space of polynomials of degree less than d.

Applying Q to the desired identity gives $(I + E)q = Q(g)$, where $E(q) = Q(hq)$ and $\|E\| \leq \|h\| < 1$. The convergent operator series $\sum_{j\geq 0}(-E)^j$ is the inverse of $I + E$. Define q by this inverse and put $s = R(g - hq)$. Then $Q(g - hq) = q$, so $g - hq = Pq + s$, as required. The inverse has norm at

most one; hence $\|q\| \leq \|g\|$ and $\|s\| \leq \|g\|$. Any other division satisfies the same invertible operator equation and therefore has the same quotient and remainder. □

Corollary 1.3.2. *Under these assumptions, $f = uW$ with u a unit of $K\langle T\rangle$ and W a monic polynomial of degree d whose lower coefficients have size at most one. The quotient $K\langle T\rangle/(f)$ has K-basis $1, T, \ldots, T^{d-1}$.*

Proof. Divide T^d by f: $T^d = qf + s$. Since $Q(T^d) = 1$, the proof above shows $\|q - 1\| < 1$. Thus q is a unit by Lemma 1.1.4; take $u = q^{-1}$ and $W = T^d - s$. The norm estimate bounds the coefficients of s. Division shows that the listed monomials span the quotient, and uniqueness of the remainder proves independence. □

This computation already exhibits analytic multiplicity. The equation $T^m = 0$ has one underlying point but its quotient algebra has dimension m. Weierstrass theory in several variables gives the general finiteness and noetherian results used below. We state those results explicitly rather than treating the one-variable argument as their proof.

1.4 Affinoid algebras and their basic algebraic properties

Definition 1.4.1. A K-affinoid algebra is a Banach K-algebra admitting an admissible surjection $K\{r^{-1}T\} \twoheadrightarrow A$ for some finite tuple of positive radii. It is strictly K-affinoid if such a presentation exists with all radii equal to one. Morphisms are bounded K-algebra homomorphisms.

Theorem 1.4.2 (Affinoid algebra theorems)**.** *A K-affinoid algebra is noetherian, and its ideals are closed. Quotients by ideals, with quotient topology, are again affinoid. Finite modules carry their canonical complete module topology, and homomorphisms between finite modules are continuous and strict. For a strictly affinoid integral algebra A, there is a finite injective map $K\langle T_1, \ldots, T_d\rangle \to A$, where $d = \dim A$. Residue fields at maximal ideals of a strictly affinoid algebra are finite over K.*

Proof. Theorem 1.18.2, applied to submodules of A, proves Noetherianity and closedness of ideals. Corollary 1.18.4 constructs the actual affinoid quotients. Theorem 1.18.3 gives the complete canonical module norms, their independence of the finite free presentation, and norm-strictness of finite-module maps. For a strict integral algebra, Theorem 1.12.2 gives a finite injective map from a Tate algebra; Proposition 3.21.1 identifies its number of variables with $\dim A$. Theorem 1.12.3 proves the assertion on maximal residue fields. □

References. [TemIntro, 3.1.2.1, 3.1.2.3 and 3.1.3.2]; [Ber90, Chapter 2].

Example 1.4.3. The affinoid algebra $K\langle\varepsilon\rangle/(\varepsilon^2)$ has the same seminorm spectrum as K, since every multiplicative seminorm kills ε. Its algebra of functions is nevertheless two-dimensional over K. Passing to the set of analytic points loses this length. The later tropical multiplicity calculations therefore use scheme-theoretic initial degenerations, not their reduced point sets.

1.5 Berkovich spectra

Definition 1.5.1. For a K-affinoid algebra A, its Berkovich spectrum $\mathcal{M}(A)$ is the set of bounded multiplicative seminorms extending $|\cdot|_K$. Its topology is the weakest topology making every evaluation $x \mapsto |a(x)|$ continuous. The algebra A, not just this topological space, is part of the affinoid object.

1.6 Seminorms, residue fields and compactness

The pointwise language of Berkovich geometry is useful in later proofs only if it is connected to the Banach-algebra constructions. We record that connection in an affinoid domain. Throughout this section, A is a nonzero K-affinoid algebra with a submultiplicative Banach norm defining its topology. A bounded multiplicative seminorm x means a multiplicative map $|\cdot|_x : A \to \mathbb{R}_{\geq 0}$ extending $|\cdot|_K$, satisfying the ultrametric inequality and an estimate $|a|_x \leq C\|a\|$ for some constant C.

Lemma 1.6.1. *Every such seminorm satisfies $|a|_x \leq \|a\|$. Its kernel $\mathfrak{p}_x$ is prime, and the induced norm on $A/\mathfrak{p}_x$ extends uniquely to an absolute value on its fraction field.*

Proof. For every positive integer m, multiplicativity and boundedness give $|a|_x^m \leq C\|a^m\| \leq C\|a\|^m$. Taking mth roots and letting m tend to infinity gives the first assertion. If $ab \in \mathfrak{p}_x$, then $|a|_x|b|_x = 0$, so at least one factor belongs to $\mathfrak{p}_x$. The quotient norm is therefore an absolute value on an integral domain. On fractions it must be $|a/b| = |a|_x/|b|_x$; cross multiplication proves that this is well-defined. □

The completion of this valued fraction field is $\mathcal{H}(x)$. The evaluation homomorphism $A \to \mathcal{H}(x)$ explains the notation $a(x)$: it is an element of a complete valued extension of K, even when x is not a classical point. In particular, $a(x) \neq 0$ and $a \in A^\times$ are different assertions. The former is a pointwise condition; the latter requires an inverse on the whole domain.

Proposition 1.6.2. *Let F/K be a complete valued extension, and let $\phi : A \to F$ be a bounded homomorphism. The seminorm $a \mapsto |\phi(a)|$ defines $x \in \mathcal{M}(A)$. There is a unique isometric embedding $\mathcal{H}(x) \to F$ compatible with ϕ. Its image is the closure of the fraction field generated by $\phi(A)$.*

Proof. The kernel of ϕ is $\mathfrak{p}_x$, so ϕ gives an injective isometry from $A/\mathfrak{p}_x$ into F. It extends to its fraction field by division and then to its completion because F is complete. Isometry gives injectivity after completion. Uniqueness follows from density, as does the description of the image. □

Proposition 1.6.3. *The space $\mathcal{M}(A)$ is compact and Hausdorff. A bounded homomorphism $A \to B$ induces a continuous map $\mathcal{M}(B) \to \mathcal{M}(A)$ and, at each point y over x, an isometric embedding $\mathcal{H}(x) \hookrightarrow \mathcal{H}(y)$.*

Proof. The product used below is compact by [StacksAlg, Tag 08ZU]. Embed $\mathcal{M}(A)$ in $\prod_{a\in A}[0, \|a\|]$ by evaluation. The conditions on the coordinates s_a are $s_0 = 0$, $s_1 = 1$, $s_{ab} = s_a s_b$, $s_{a+b} \leq \max(s_a, s_b)$, and $s_c = |c|$ for $c \in K$. They are closed conditions. Conversely, a point satisfying them is a bounded multiplicative seminorm. Thus $\mathcal{M}(A)$ is a closed subspace of a compact Hausdorff product. Its subspace topology is precisely the evaluation topology. This argument proves compactness; nonemptiness for a nonzero affinoid algebra is the separate analytic existence theorem.

Composition with $A \to B$ preserves the seminorm conditions and is bounded. Each evaluation on A becomes an evaluation on B, which proves continuity. Applying Proposition 1.6.2 to $A \to B \to \mathcal{H}(y)$ gives the completed-residue-field embedding. □

This compactness result supplies many compact sets used in cutoff arguments. It does not say that an affinoid domain has empty analytic boundary. Compactness, ordinary interior and analytic boundary serve different purposes: the first controls measures, the second permits local restriction, and the third governs boundary terms in Stokes' formula.

1.7 Nonemptiness, units and spectral size

In this section A is a nonzero K-affinoid algebra.

Theorem 1.7.1 (Spectrum and spectral radius)**.** *For a nonzero K-affinoid algebra, the spectrum is nonempty. For any submultiplicative Banach norm uniformly equivalent to its defining affinoid norm,*

$$\rho_A(a) := \lim_{m\to\infty} \|a^m\|^{1/m} = \max_{x\in\mathcal{M}(A)} |a(x)|.$$

The function ρ_A is a power-multiplicative seminorm. Its kernel is the nilradical. If A is reduced, it is a complete norm equivalent to the given Banach norm.

Proof. Nonemptiness is Theorem 1.20.1, and the displayed limit and maximum formula are Corollary 1.20.3. Their value is unchanged under a uniformly equivalent Banach norm, since the fixed comparison constants have m-th roots tending to one. The maximum formula implies the ultrametric and submultiplicative inequalities and $\rho_A(a^j) = \rho_A(a)^j$. Proposition 3.21.4 identifies the kernel with the nilradical. For reduced A, Theorem 3.21.5 proves completeness and equivalence with the given norm. □

References. [TemIntro, 2.2.2.3 and 3.1.2.1]; [BGR84, 6.2.4/1].

Proposition 1.7.2. *An element $f \in A$ is a unit if and only if $f(x) \neq 0$ for every $x \in \mathcal{M}(A)$. In that case $|f|$ has a positive minimum on the spectrum.*

Proof. A unit cannot vanish at a multiplicative seminorm. If f is not a unit, $A/(f)$ is a nonzero affinoid algebra by closedness of ideals. Its nonempty spectrum maps to a point of $\mathcal{M}(A)$ at which f vanishes. Finally a positive continuous function on a compact space has a positive minimum. □

Example 1.7.3. For $f = \sum a_j T^j \in K\langle T\rangle$, the condition $|a_0| > \max_{j>0} |a_j|$ implies that f is a unit, by geometric inversion. Conversely every unit has this condition. Indeed, divide by a coefficient of maximal size so that $\|f\| = 1$. Multiplicativity of the Gauss norm and $fg = 1$ show that, after the reciprocal scaling of g, both reductions are polynomials whose product is one in $\widetilde{K}[T]$. Thus the reduction of f is a nonzero constant. The strict inequality follows because the coefficients tend to zero and the maximum is attained.

For a general affinoid algebra, ρ_A is intrinsic, while a quotient norm depends on a presentation. Nilpotents illustrate the difference immediately: they can have positive quotient norm but zero spectral seminorm. Even when A is reduced, multiplicativity of ρ_A is a special property. Chapter 3 relates it to a singleton Shilov boundary.

1.8 A field obtained by adjoining a radius

The following construction explains why the word "strict" is a genuine restriction. Suppose $r \notin \sqrt{|K^\times|}$ and complete $K[T, T^{-1}]$ for the multiplicative norm $|\sum a_m T^m|_r = \max_m |a_m| r^m$. Denote the resulting algebra by K_r. Its elements are Laurent sums with $|a_m| r^m \to 0$ as $|m| \to \infty$.

Proposition 1.8.1. *The algebra K_r is a complete valued field and a K-affinoid algebra. Its spectrum consists of one point. It admits the presentation*

$$K_r \simeq K\{r^{-1}T, rS\}/(TS - 1).$$

Proof. The norm-strict quotient presentation is the one-variable case of Theorem 1.17.2. For distinct i, j, nonzero terms cannot have equal sizes: $|a_i| r^i = |a_j| r^j$ would imply $r^{i-j} \in |K^\times|$. A nonzero Laurent sum therefore has a unique largest term $a_m T^m$. It can be written $a_m T^m (1 + h)$ with $\|h\|_r < 1$, and geometric inversion gives its inverse. The norm remains multiplicative on

completion. The presentation follows by reducing monomials using $TS = 1$, with positive powers assigned to T and negative powers to S; the convergence conditions on the resulting Laurent coefficients are exactly the stated ones.

At any bounded point, the bounds on T, S and $TS = 1$ force $|T| = r$. Every Laurent sum then has the same unique largest term as above, so its value is forced. Thus there is only the Gauss point. □

This one-point space is an affinoid domain of the analytic line, not a zero-dimensional analytic space over K. Analytic dimension must be stable under complete scalar extension; Chapter 2 makes this precise. The construction also illustrates the distinction between a finite extension and an extension that supplies a new real value.

References. [TemIntro, 3.1.1.4 and 3.5.1]; the proofs above give the explicit one-radius calculation.

1.9 Norm conventions and bounded localization

Before constructing localization maps, we fix the quantitative meaning of admissibility. This matters when the field is trivially valued. The correction below replaces the topology-only wording in version 0.26; its original wording is retained in the version record. All later uses of "admissible" refer to equivalent quotient norms.

Definition 1.9.1. For a bounded linear map $u : E \to F$, give $\operatorname{im} u$ both its induced norm from F and its quotient seminorm from E. The map is norm-strict, or admissible, when these are uniformly equivalent. Since u is bounded, the additional requirement is a constant $C > 0$ such that every $y \in \operatorname{im} u$ has a lift x with $\|x\| \le C\|y\|$. The constant may be enlarged by any fixed factor greater than one to approximate an unattained infimum.

References. [TemIntro, Definition/Exercise 2.1.1.1(ii)–(iii), p. 4].

Example 1.9.2 (The topology alone is insufficient)**.** Let K be trivially valued. The two norms on $K[T]$ given by

$$\Big\| \sum a_n T^n \Big\|_1 = \max_n |a_n|, \qquad \Big\| \sum a_n T^n \Big\|_w = \max_n (n+1)|a_n|.$$

They make $K[T]$ complete Banach algebras with the same discrete topology, but they are not uniformly equivalent. The identity from the second norm to the first is bounded and has no bounded inverse.

Proof. The coefficientwise ultrametric inequality gives the triangle rules and K-homogeneity. Submultiplicativity of the second norm follows from $i+j+1 \le (i+1)(j+1)$; its unit has norm one. Every nonzero element has norm at least one for both norms, so both topologies are discrete and both spaces complete. But $\|T^n\|_w = n+1$ and $\|T^n\|_1 = 1$, which disproves a uniform reverse bound.

For $r > 1$, the generalized Tate algebra $K\{r^{-1}S\}$ is $K[S]$. The map $S \mapsto T$ to the second normed algebra is bounded, because $(n+1)/r^n$ is bounded, and is a topological quotient isomorphism. Its quotient norm, however, assigns r^n to T^n, and is not equivalent to $n+1$. Thus the topology-only convention even admits presentations outside the intended affinoid category. □

This example also contradicts the unrestricted topology/equivalence sentence in [TemIntro, 2.1.1.1(iv)]. We use the quantitative definitions in its preceding parts (ii)–(iii), not that sentence. This is a correction to the literal scope of the cited wording; it is not an assertion of an author-issued erratum.

Proposition 1.9.3 (Bounded substitution criterion)**.** *Let A, B be Banach K-algebras, let $\phi : A \to B$ be bounded, let $r_1, \dots, r_s > 0$, and let $b_i \in B$. Substitution $T_i \mapsto b_i$ extends uniquely to a bounded A-algebra homomorphism $A\{r^{-1}T\} \to B$ extending ϕ if and only if there are constants C_i such that*

$$\|b_i^n\| \le C_i r_i^n \qquad (n \ge 0, 1 \le i \le s).$$

Proof. Necessity follows by applying a bound for the map to T_i^n. Conversely, if $\|\phi(a)\| \le C_\phi \|a\|$, then

$$\|\phi(a_\nu) b^\nu\| \le C_\phi \Big(\prod_i C_i \Big) \|a_\nu\| r^\nu.$$

The terms tend to zero outside finite subsets. The summability criterion of Lemma 1.1.2 therefore defines the evaluation and bounds its norm by $C_\phi \prod_i C_i$ times the source norm. Polynomial substitution preserves products; polynomial density and the product estimate extend that identity to series. The same density gives uniqueness. The coefficientwise completeness and convolution proof is the finite-multiindex version of Lemma 1.10.1, with exactly the same finite-tail estimate. □

A spectral bound alone is insufficient for an arbitrary Banach target: in Example 1.9.2, the powers of T have norms $n+1$, so their nth roots tend

to one, but they are not bounded. For affinoid targets, the passage from spectral bounds to these scaled power bounds is a separate theorem. It cannot be inferred from the point calculation in Proposition 2.2.1.

Example 1.9.4 (A finite tensor product need not be complete)**.** Suppose $0 < |\pi| < 1$ in K. Put $A = K\langle T\rangle$, $M = A/(T)$, and $N = c_0(K)$ with its supremum norm. Give N the A-action

$$f(T)(x_n)_{n\geq 1} = (f(\pi^n)x_n)_{n\geq 1}.$$

Then N is a Banach A-module, but $M \otimes_A N \neq 0$ and $M\widehat{\otimes}_A N = 0$.

Proof. A supremum-Cauchy sequence in $c_0(K)$ has coordinate limits with a uniform vanishing tail, so N is complete. Since $|f(\pi^n)| \leq \|f\|$, the displayed formula defines a contractive action preserving c_0. The submodule TN contains every finitely supported sequence and is therefore dense. It is proper: the sequence (π^n) has the unique coordinatewise preimage (1), which is not in c_0.

The generators and relations of $M = A/(T)$ give $M \otimes_A N = N/TN$, which is nonzero. Every tensor class is represented by $1 \otimes x$. Replacing x by $x - z$ for $z \in TN$ does not change the class, and its projective seminorm is at most $\|x - z\|$. Density makes that seminorm zero on every class. Its separated completion is therefore zero. □

Thus the arbitrary-Banach-module wording of [TemIntro, Fact 3.1.3.2(ii), p. 13] cannot be used literally. The finite-module base change needed for analytic localizations has an affinoid algebra as target. Proposition 1.11.6 proves that correct statement in the strict category.

Example 1.9.5 (Ratios must be tested at the same point)**.** Let $0 < |\pi| < 1$, $A = K \times K$, $f = (1, 1)$ and $g = (1, \pi)$. For the maximum norm, $\rho(f) = \rho(g) = 1$ but $\rho(f/g) = |\pi|^{-1} > 1$. The domain $|f| \leq |g|$ contains only the first component. Its rational algebra is K, so the identity of A cannot factor through it.

Proof. The two spectral radii are read componentwise. The rational quotient $A\langle U\rangle/(gU - f)$ is the product of $K\langle U\rangle/(U - 1) = K$ and $K\langle U\rangle/(\pi U - 1) = 0$. The first equality follows by monic division, and the second because $1 - \pi U$ is a unit by geometric inversion. A factorization of the identity would kill the nonzero second idempotent of A, which is impossible. □

This corrects the separate-spectral-maxima formulation in the first sentence of [TemIntro, Definition/Exercise 3.2.2.1(i), p. 14]. Use invertibility of g together with a bound on f_i/g, or the simultaneous pointwise inequalities. The pointwise rational-domain definition already used in this book is retained.

1.10 Weierstrass division with Banach coefficients

The proof of Theorem 1.3.1 extends beyond a field of coefficients and the unit radius. This extension supplies the division step needed in an induction on the number of Tate variables. It also identifies the quotient norm explicitly, so no open-mapping theorem or finite-module completeness theorem enters the argument. The construction is the classical Weierstrass division argument with its norm estimates retained. It does not yet prove general Tate acyclicity.

Throughout this section, A is a commutative unital Banach K-algebra as in Definition 1.1.3. A Banach A-module M means an A-module with a complete, definite, non-Archimedean norm, homogeneous over K, such that $\|am\| \le \|a\|\|m\|$. No algebraic finiteness is assumed. Fix $r > 0$ and put

$$E_r(M) = \left\{ \sum_{j\ge 0} m_j T^j : \|m_j\| r^j \longrightarrow 0 \right\},$$

$$\left\| \sum_{j\ge 0} m_j T^j \right\|_r = \sup_j \|m_j\| r^j.$$

For a nonzero series the supremum is attained. Write $P_{<d}(M)$ for the polynomials of degree less than d, with the same norm; $P_{<0}(M) = 0$. The field may be trivially valued, and r need not belong to $|K^\times|$.

Lemma 1.10.1 (Complete coefficient modules)**.** *The normed module $E_r(M)$ is complete, $M[T]$ is dense in it, and $P_{<d}(M)$ is complete. Coefficient convolution defines a continuous action of the Banach algebra $E_r(A)$ on $E_r(M)$, satisfying*

$$\|fg\|_r \le \|f\|_r \|g\|_r.$$

Proof. Let $g_n = \sum_j m_{n,j} T^j$ be Cauchy. Each coefficient sequence is Cauchy because

$$\|m_{n,j} - m_{n',j}\| \le r^{-j} \|g_n - g_{n'}\|_r.$$

Let m_j be its limit. Given $\varepsilon > 0$, choose n so that $\|g_n - g_{n'}\|_r < \varepsilon/2$ for every sufficiently large n'. Choose J so that $\|m_{n,j}\|r^j < \varepsilon/2$ for $j > J$. Passage to the coefficientwise limit gives

$$\|m_j - m_{n,j}\|r^j \le \varepsilon/2, \qquad \|m_j\|r^j \le \varepsilon/2 \quad (j > J).$$

Thus $g = \sum_j m_j T^j$ belongs to $E_r(M)$, and the first estimate gives $\|g - g_n\|_r \le \varepsilon/2$. This proves completeness. Truncation proves density. The same argument on the first d coordinates proves completeness of $P_{<d}(M)$.

For polynomial products, convolution and the ultrametric inequality give the asserted norm bound. For general f, g, choose polynomial truncations f_n, g_n. They are bounded and Cauchy. The identity

$$f_n g_n - f_m g_m = (f_n - f_m)g_n + f_m(g_n - g_m)$$

shows that the products are Cauchy. Their limit defines fg and is independent of the truncations by the same estimate. Each coefficient is the finite convolution sum. The norm bound passes to the limit; associativity and the module law follow from their polynomial versions by continuity. Taking $M = A$ also proves that $E_r(A)$ is a Banach algebra. □

The first division step concerns a monic polynomial. Its lower coefficients need only satisfy a weak inequality. The strict inequality will enter later, in the perturbation of that polynomial.

Lemma 1.10.2 (Bounded monic division)**.** *Let $d \ge 0$ and*

$$P = T^d + \sum_{i<d} a_i T^i, \qquad \|a_i\|r^i \le r^d.$$

Every $g \in E_r(M)$ has a unique decomposition

$$g = PQ_P(g) + R_P(g), \qquad Q_P(g) \in E_r(M), \quad R_P(g) \in P_{<d}(M).$$

The two operators are A-linear and satisfy

$$\|Q_P(g)\|_r \le r^{-d}\|g\|_r, \qquad \|R_P(g)\|_r \le \|g\|_r.$$

Moreover, $\|Pq\|_r = r^d\|q\|_r$ for every $q \in E_r(M)$.

Proof. First let $g \in M[T]$. Remove its highest nonzero term mT^n, with $n \ge d$, by subtracting $mT^{n-d}P$. Every term in this multiple has weighted norm at

most $\|m\|r^n$. Thus the current dividend never has norm greater than the original $\|g\|_r$, and each quotient contribution has norm at most $r^{-d}\|g\|_r$. The degree strictly decreases, so finitely many steps give the required polynomial decomposition and estimates. Its uniqueness follows because the highest coefficient of a nonzero polynomial Pq is the highest coefficient of q. In particular, this argument requires no division by a coefficient of M.

The estimates extend Q_P, R_P to $E_r(M)$ by polynomial approximation and Lemma 1.10.1. Applying the estimates to differences proves independence of the approximation. The decomposition and linearity pass to the limit. On polynomials one has $Q_P(Pq) = q$ and $R_P(Pq) = 0$; continuity gives these identities for all series q. Since also $Q_P(s) = 0$ and $R_P(s) = s$ for $s \in P_{<d}(M)$, uniqueness holds for series as well.

For the norm equality, take $q = \sum_j m_j T^j \neq 0$ and choose the largest j such that $\|m_j\|r^j = \|q\|_r$. This largest index exists because the weighted coefficients tend to zero. In the coefficient of T^{j+d} in Pq, the monic contribution is m_j. Every other contribution is $a_i m_{j+d-i}$ with $i < d$, and its weighted norm is strictly smaller than $r^d\|q\|_r$, since $j + d - i > j$. A term of strictly larger norm than all other terms cannot cancel in a non-Archimedean normed module: apply the triangle inequality also after solving for that term. Hence this coefficient has weighted norm $r^d\|q\|_r$. Submultiplicativity supplies the reverse inequality. □

Theorem 1.10.3 (Division and strictness over Banach coefficients)**.** *With P as in Lemma 1.10.2, let $h \in E_r(A)$ satisfy $\|h\|_r < r^d$, and set*

$$f = P + h, \qquad \theta = r^{-d}\|h\|_r < 1.$$

Every $g \in E_r(M)$ has a unique decomposition $g = fq + s$, where $q \in E_r(M)$ and $s \in P_{<d}(M)$, with

$$\|q\|_r \leq r^{-d}\|g\|_r, \qquad \|s\|_r \leq \|g\|_r.$$

Multiplication by f rescales norms by r^d and has closed image. The remainder map $R_f(g) = s$ gives a split exact sequence of Banach A-modules

$$0 \longrightarrow E_r(M) \xrightarrow{f} E_r(M) \xrightarrow{R_f} P_{<d}(M) \longrightarrow 0.$$

The maps are strict. More precisely, the quotient norm of the class of g is $\|R_f(g)\|_r$, attained at its unique polynomial representative.

Proof. Put $U(q) = Q_P(hq)$. The previous two lemmas give $\|U(q)\|_r \le \theta\|q\|_r$. For each $v \in E_r(M)$, the finite partial sums of $\sum_{n\ge0}(-U)^n v$ are Cauchy, since their terms have norms at most $\theta^n\|v\|_r$. Completeness therefore defines an A-linear map

$$S(v) = \sum_{n\ge0}(-U)^n v, \qquad \|S(v)\|_r \le \|v\|_r.$$

Multiplying a finite partial sum by $I + U$ leaves v and one final term whose norm tends to zero. Passing to the limit proves $(I + U)S = S(I + U) = I$. This constructs the inverse directly; it does not appeal to an operator inverse theorem.

Set $q = S(Q_P(g))$ and $s = R_P(g - hq)$. Then $Q_P(g - hq) = q$, and the polynomial division identity gives $g - hq = Pq + s$. The bounds on S, Q_P, R_P imply the asserted estimates. Any other decomposition must satisfy $(I + U)q = Q_P(g)$, so it has the same quotient and remainder.

For $q \neq 0$, Lemma 1.10.2 and the perturbation bound give

$$\|Pq\|_r = r^d\|q\|_r, \qquad \|hq\|_r \le \theta r^d\|q\|_r < \|Pq\|_r.$$

Thus $\|fq\|_r = r^d\|q\|_r$. If fq_n converges, this equality makes q_n Cauchy; its limit q satisfies $fq = \lim_n fq_n$. The image is therefore closed.

Finally, R_f is A-linear and contractive, vanishes on $fE_r(M)$, and restricts to the identity on $P_{<d}(M)$. For every $q' \in E_r(M)$,

$$\|R_f(g)\|_r \le \|g + fq'\|_r.$$

Taking $q' = -q$, where $g = fq + R_f(g)$, attains equality. This proves the quotient-norm statement, strictness and exactness. The inclusion of $P_{<d}(M)$ is a bounded right inverse for R_f. Equivalently, the division map identifies $E_r(M)$ isometrically with $E_r(M) \oplus P_{<d}(M)$ when the latter carries the norm $\max(r^d\|q\|_r, \|s\|_r)$. □

For division, the perturbation h may have terms in any degree. Preparation uses the normalized case in which the coefficient of T^d is one and the small tail occurs only above degree d.

Corollary 1.10.4 (Preparation and the quotient norm)**.** *In Theorem 1.10.3, take $M = A$ and suppose that h has only terms of degree greater than d. There is a factorization*

$$f = uW, \qquad W = T^d + \sum_{i<d} w_i T^i, \qquad \|w_i\| r^i \le r^d, \qquad \|u - 1\|_r \le \theta < 1.$$

The element u is a unit. The monic degree-d polynomial W is unique among such factorizations by analytic units. The quotient $E_r(A)/(f)$, with its quotient norm, is isometric as an A-module to $P_{<d}(A)$; it is finite free with basis $1, T, \dots, T^{d-1}$.

Proof. Divide T^d by f, obtaining $T^d = qf + s$. Since $Q_P(T^d) = 1$, the construction of the preceding theorem gives $q = S(1)$ and $\|q - 1\|_r \le \theta$. The geometric series $\sum_{n\ge 0}(1-q)^n$ converges and is an inverse of q, as is checked by multiplying finite sums and passing to the limit. Its difference from one has norm at most θ. Set $u = q^{-1}$ and $W = T^d - s$. Then $f = uW$, and $\|s\|_r \le r^d$ gives the lower-coefficient bounds.

If also $f = vW'$ with v a unit and W' monic of degree d, then $W' = v^{-1}uW$. Apply Lemma 1.10.2 to W. Polynomial division of W' by W has quotient one and remainder $W' - W$, whereas the displayed analytic representation has quotient $v^{-1}u$ and remainder zero. Uniqueness gives $W' = W$ and $v = u$. The quotient and its norm were computed in Theorem 1.10.3; its unique polynomial representatives give both spanning and independence of the stated basis. □

If the distinguished coefficient of an initial series F is a unit a_d rather than one, apply these results to $a_d^{-1}F$ only after checking the displayed bounds for its normalized coefficients. Invertibility of a_d alone does not imply those bounds. When $d = 0$, the remainder space is zero and the theorem reduces to geometric inversion.

The division estimate is the analytic ingredient in the finite-module theory. It yields strict affinoid closedness by induction, and Theorems 1.18.2 and 1.18.3 extend the resulting topology and exactness statements to arbitrary radii.

1.11 Controlled generation in strict Tate algebras

Division over a Banach coefficient algebra supplies an induction in the number of variables. We carry out that induction with estimates: finite generation alone would not prove that an algebraic submodule is closed. Put $R_n = K\langle X_1, \dots, X_n\rangle$, with the unit Gauss norm, and $R_0 = K$. Finite powers carry the maximum norm. The field may be trivially valued. Every constant below is independent of the element being represented.

Lemma 1.11.1 (Complete quotients)**.**

Let E be a complete non-Archimedean normed K-vector space and $L \subset E$ a closed linear subspace. The quotient norm $\|e+L\| = \inf_{l\in L} \|e+l\|$ is definite and complete. For $y \neq 0$ and $c > 1$, there is a lift e with $\|e\| < c\|y\|$.

Proof. Norm zero means that e is a limit of elements of L, so $e \in L$. The scalar and ultrametric rules follow by approximating the infima; the lifting assertion follows from the same definition. Given a Cauchy sequence in E/L, choose a subsequence y_j with $\|y_{j+1}-y_j\| < 2^{-j}$. Lift its successive differences to e_j of norm less than 2^{1-j}, using zero for a zero difference, and choose a lift e_0 of y_1. The ultrametric estimate makes $e_0 + \sum_{j\geq 1} e_j$ convergent in E. Its image is the limit of the subsequence. The original Cauchy sequence has that same limit. □

Lemma 1.11.2 (A closedness criterion)**.** *Let R be a complete non-Archimedean normed K-algebra, let E be a complete non-Archimedean normed R-module with $\|av\| \leq \|a\|\|v\|$, and let $N \subset E$ be an algebraic submodule. Suppose that there are $v_1, \dots, v_s \in N$ and $C > 0$ such that every $v \in N$ has an expression*

$$v = \sum_{i=1}^{s} a_i v_i, \qquad \max_i \|a_i\| \leq C\|v\|.$$

Then N is closed, and $R^s \to N$ is a bounded, norm-strict surjection for the induced norm on N.

Proof. If $v_j \in N$ converges in E, choose coefficients a_{ij} representing $v_{j+1}-v_j$ with $\|a_{ij}\| \leq C\|v_{j+1}-v_j\|$. These coefficients tend to zero. Each series $\sum_j a_{ij}$ therefore converges in R. Express v_1 in the same generators and add these coefficient series. Continuity of the finite sum shows that $\lim_j v_j \in N$. The

generator map is bounded by $\max_i \|v_i\|$, while its quotient norm is at most C times the norm of its image. These two estimates give norm-strictness. □

Lemma 1.11.3 (Distinguished coordinates)**.** *For every nonzero $g \in R_n$, $n \geq 1$, there is an isometric triangular K-algebra automorphism ϕ such that $\phi(g)$, multiplied by a nonzero scalar and a unit of R_{n-1}, has the form*

$$f = T^d + \sum_{i<d} b_i T^i + h, \qquad T = X_n, \quad \|b_i\| \leq 1, \quad \|h\| < 1,$$

where h contains only powers of T greater than d.

Proof. Divide g by a coefficient attaining its nonzero Gauss norm, so that $\|g\| = 1$. The set S of exponents with coefficient norm one is finite and nonempty. Let P be the sum of these terms and write $g = P + h_0$. Then $\|h_0\| < 1$: only finitely many coefficients have norm at least $1/2$, and those among them that are less than one, if any, have a maximum less than one. Taking the maximum with $1/2$ bounds all remaining coefficients strictly below one.

Choose an integer $B \geq 2$ larger than every coordinate of every exponent in S, and put $N_i = B^i$ for $1 \leq i < n$. The integers $w(\nu) = \nu_n + \sum_{i<n} N_i \nu_i$ are distinct on S by uniqueness of base-B expansions. Make the substitution

$$X_i \longmapsto X_i + T^{N_i} \quad (i < n), \qquad T \longmapsto T.$$

Both this map and the substitution with minus signs are contractive by Proposition 1.2.3; they are inverse on polynomials and therefore on their completions. Hence ϕ is isometric. Regrouping coefficients identifies R_n isometrically with $R_{n-1}\langle T\rangle$: vanishing of the joint coefficient tails is equivalent to vanishing in each row and convergence to zero of the row norms.

Set $d = \max_{\nu \in S} w(\nu)$. The coefficient of T^d in $\phi(P)$ is the scalar a attached to the unique maximizing exponent. Every other term has smaller T-degree, and all coefficient norms are at most one, since the binomial coefficients have absolute value at most one. The coefficient of T^d in $\phi(g)$ is thus $c = a + e$ with $|a| = 1$ and $\|e\| < 1$. Geometric inversion gives $c^{-1} \in R_{n-1}$ and $\|c\| = \|c^{-1}\| = 1$. Multiplying by c^{-1} gives the required lower-coefficient bounds and strictly small higher tail. □

Theorem 1.11.4 (Controlled generators in every rank)**.** *Every algebraic R_n-submodule $N \subset R_n^m$ has finitely many generators $v_1, \dots, v_s$ and a constant $C_N > 0$ such that*

$$\forall v \in N \quad \exists a_1, \dots, a_s \in R_n : \quad v = \sum_i a_i v_i, \qquad \max_i \|a_i\| \le C_N \|v\|.$$

In particular, every such submodule is closed. No restriction on the valuation of K is needed.

Proof. **Step 1: induction in the number of variables.** Prove the assertion for all m by induction on n. For $n = 0$, choose a row-reduced basis of the subspace $N \subset K^m$. Each basis coefficient is a selected coordinate of v, so it has absolute value at most $\|v\|$. The zero submodule uses the empty generator list.

Step 2: ideals over R_n. Suppose the assertion is known over R_{n-1} in every rank. Let $I \subset R_n$ be a nonzero ideal. Apply Lemma 1.11.3 to a nonzero element of I, transport the ideal through the isometry, and normalize that element. This gives a distinguished $f \in I$ of degree d. Theorem 1.10.3, with coefficient algebra R_{n-1} and radius one, gives

$$g = qf + R_f(g), \qquad \|q\|, \|R_f(g)\| \le \|g\|, \qquad \deg_T R_f(g) < d.$$

If $d = 0$, then f is a unit and $I = R_n$. Otherwise set $J = I \cap R_{n-1}[T]_{<d}$, an R_{n-1}-submodule of R_{n-1}^d. For $g \in I$, its remainder belongs to J because $g - qf \in I$. Choose controlled generators p_j of J by the induction hypothesis. Then

$$g = qf + \sum_j b_j p_j, \qquad \max(\|q\|, \max_j \|b_j\|) \le \max(1, C_J)\|g\|.$$

Here $b_j \in R_{n-1} \subset R_n$. This proves controlled generation of the transported ideal. The inverse isometry transports both generators and coefficients back to the original ideal, with the same bound. The zero ideal is immediate.

Step 3: induction in the rank. Fix n and assume the assertion for rank $m - 1$. Let I be the first-coordinate image of $N \subset R_n^m$. Choose controlled generators f_i of I and lifts $v_i \in N$. The kernel of the projection identifies with a submodule $N_0 \subset R_n^{m-1}$ and has controlled generators w_j. Given $v \in N$, express its first coordinate as $\sum_i a_i f_i$ with $\max_i \|a_i\| \le C_I \|v\|$. Then

$$v' = v - \sum_i a_i v_i \in N_0, \qquad \|v'\| \le \max(1, C_I \max_i \|v_i\|)\|v\|.$$

Expand v' in the w_j. The two coefficient estimates give a single bound for the combined generators. Empty lists contribute zero. This finishes the two inductions, and Lemma 1.11.2 proves closedness. □

The same estimate controls finite modules over a quotient. The following statement proves the strict-category portion of Theorem 1.4.2 concerning Noetherianity and finite-module norms. It does not include normalization or the affinoid Nullstellensatz.

Theorem 1.11.5 (Finite modules over strict affinoids)**.** *Let A be a strict K-affinoid algebra, with admissibility understood as equivalence to the quotient norm. Then A is Noetherian and its ideals are closed. Every finite A-module has a complete norm, canonical up to uniform equivalence, induced by a finite free presentation. Its submodules are finite and closed. Every A-linear map between finite modules is bounded, has closed image, and is norm-strict onto its image.*

Proof. **Step 1: construct the norm.** Write $A = R_n/I$. Theorem 1.11.4 makes I closed, and Lemma 1.11.1 makes the quotient complete. First use this quotient norm on A; replacing it by a uniformly equivalent norm changes only constants. For a finite A-module M, choose a surjection $R_n^m \to M$. Its kernel is a finitely generated closed submodule by the same theorem. The quotient therefore equips M with a complete norm. It agrees with the successive quotient through A^m.

For clarity, the equality of successive quotient norms follows by taking the infimum first over lifts from A^m to R_n^m and then over lifts of an element of M; the resulting set is exactly all its lifts in R_n^m. Approximating lifts of both $a \in A$ and $m \in M$ gives $\|am\| \leq \|a\|\|m\|$. Coefficients in R_n pass to A without increasing their quotient norms.

For two presentations of M, lift each finite list of generators to the other free module. The resulting matrices define bounded maps. Lifting an element of M with norm less than twice its quotient norm, and applying the corresponding matrix, bounds one quotient norm by a constant times the other. Reversing the roles proves uniform equivalence. This also proves independence of the chosen presentation of A.

Step 2: submodules. For $N \subset M$, take its inverse image $E \subset R_n^m$. It has controlled generators e_i. Given $v \in N$, choose a lift $e \in E$ with $\|e\| < 2\|v\|$ when $v \neq 0$, and expand it using the e_i. Their images generate

N, with a uniform coefficient bound in its induced norm. Lemma 1.11.2, applied with coefficient algebra R_n, makes N closed. These generators also generate over A. In particular, all ideals of A are finitely generated, so A is Noetherian.

Step 3: maps. An A-linear map $u : M \to M'$ is bounded: express an element using a controlled lift to a finite free presentation, and bound its image by the maximum norm of the images of the generators. Its image P is a finite closed submodule of M'. Choose controlled generators p_i of P and any lifts $m_i \in M$. If $p = \sum a_i p_i$ with $\max_i \|a_i\| \leq C\|p\|$, then $\sum a_i m_i$ is a lift of norm at most $C \max_i \|m_i\| \|p\|$. This is the required quotient-norm bound. The kernel is a finite closed submodule, and the quotient is complete by Lemma 1.11.1. □

Proposition 1.11.6 (Finite tensor comparison with an affinoid target)**.** *Let $A \to B$ be a bounded homomorphism of strict K-affinoid algebras and M a finite A-module with its canonical norm. The projective tensor seminorm on $M \otimes_A B$ is equivalent to its complete finite-B-module norm. In particular, the canonical map*

$$M \otimes_A B \longrightarrow M \widehat{\otimes}_A B$$

is a bounded isomorphism. This assertion does not imply flatness of $A \to B$.

Proof. Choose a finite presentation $A^s \to A^m \to M \to 0$. The generators-and-relations definition of tensor product gives $M \otimes_A B = \operatorname{coker}(B^s \to B^m)$: the images of the standard generators generate the tensor product, and its relations are exactly the B-linear combinations of the displayed relations of M. The relation image is closed by Theorem 1.11.5, so this quotient norm is complete.

Use the quotient norm from A^m on M, and write e_i for its standard generator images. Since $\|e_i\| \leq 1$, the expression $\sum e_i \otimes b_i$ bounds the projective norm by the quotient norm. Conversely, write $z = \sum_j m_j \otimes b_j$ and lift each m_j to A^m with coefficient norms at most $2\|m_j\|$. If $\|\phi(a)\| \leq C_\phi \|a\|$, this expresses $z = \sum_i e_i \otimes c_i$ with

$$\max_i \|c_i\| \leq 2C_\phi \max_j \big(\|m_j\| \|b_j\|\big).$$

Taking the infimum over tensor representations gives the reverse norm comparison. Hence the projective seminorm is definite and complete, and separated completion makes no further change. □

Remark 1.11.7. For arbitrary polyradii, the coefficient description of a free-radius extension replaces the strict coordinate changes used here. Theorems 1.18.2 and 1.18.3 descend closedness and the canonical finite-module topology through that extension, giving the general affinoid statements.

References. Classical Tate algebra and finite-module theory; compare [TemIntro, 3.1.2.1 and 3.1.3.2(i)]. The proofs in this section reconstruct the strict case from the preceding division estimates; those source facts are attribution, not proof inputs.

1.12 Finite normalization and points of strict affinoids

We need to turn the absence of a common spectral zero into a unit ideal. The required algebraic input is finite normalization, obtained from the division theorem already proved. We first record how the authorized valuation-ring results give real-valued absolute values on finite residue fields. Throughout this section K is complete with its fixed non-Archimedean absolute value; trivial valuation is allowed.

Lemma 1.12.1 (Finite-dimensional norms and finite field extensions)**.** *Every non-Archimedean norm on a finite-dimensional K-vector space, homogeneous for the fixed absolute value, is uniformly equivalent to a coordinate maximum norm and is complete. Every finite field extension L/K admits exactly one non-Archimedean real-valued absolute value extending that of K. It makes L complete and is uniformly equivalent to every such vector-space norm. No separability assumption is required.*

Proof. For the norm assertion, induct on a basis $e_1, \dots, e_n$. The upper bound by a coordinate norm follows from the ultrametric inequality. By induction the subspace $F = \langle e_1, \dots, e_{n-1} \rangle$ is complete, hence closed: any sequence in F converging in the ambient norm is Cauchy, converges in F, and has the same limit. Therefore

$$\delta = \inf_{u \in F} \|e_n - u\| > 0.$$

Writing $x = u + ae_n$ gives $|a| \le \delta^{-1}\|x\|$ and $\|u\| \le \max(1, \|e_n\|/\delta)\|x\|$. The inductive bound on the coordinates of u proves the reverse norm inequality. Coordinatewise completeness of K^n now proves completeness. The cases $n = 0, 1$ follow directly.

Put $O = \{a \in K : |a| \leq 1\}$ and $\mathfrak{m}_O = \{a : |a| < 1\}$. This is a local subring of L. By [StacksAlg, Tag 00IA, Lemma 10.50.2], choose a valuation ring $V \subset L$ with fraction field L dominating O. Then $V \cap K = O$: if $|a| > 1$, its inverse belongs to $\mathfrak{m}_O \subset \mathfrak{m}_V$, so a cannot belong to V. By [StacksAlg, Tag 00IB, Lemma 10.50.4], for every $b \in L^\times$ either b or b^{-1} belongs to V. Thus

$$\Gamma = L^\times/V^\times, \qquad v(b) \geq v(c) \iff b/c \in V$$

is an ordered abelian group, written additively. The order is well-defined under multiplication by units, total by the cited lemma, and antisymmetric because an element and its inverse are both in V exactly when it is a unit. Its positive elements are represented by $\mathfrak{m}_V \setminus \{0\}$. For a nonzero sum, division by a term of smaller value proves $v(b + c) \geq \min(v(b), v(c))$. The subgroup from $K^\times$ is therefore identified, with its order, with $-\log|K^\times| \subset \mathbb{R}$.

For $b \in L^\times$, take a nonzero polynomial relation $\sum_i c_i b^i = 0$ over K, omitting zero terms. The smallest of the values $v(c_i) + iv(b)$ is attained at least twice. Otherwise, division by its unique minimizing term would give 1 plus elements of $\mathfrak{m}_V$ equal to zero. Hence, for some $i \neq j$,

$$(i - j)v(b) = v(c_j) - v(c_i).$$

Every $\gamma \in \Gamma$ consequently has a positive integer multiple $n\gamma$ in the subgroup from K. Define $w(\gamma) = w_K(n\gamma)/n$, where $w_K(v(a)) = -\log|a|$. Common multiples show that this is independent of n and additive; the order on the subgroup shows that it preserves and reflects positivity. Thus $|b|_L = \exp(-w(v(b)))$, with $|0|_L = 0$, is the required absolute value. This construction also covers trivial valuation, when the ordered group has to be zero.

This absolute value is a vector-space norm on finite-dimensional L, so the first paragraph proves its equivalence to every other such norm and its completeness. Two extending absolute values are therefore bounded by constant multiples of one another. Apply both inequalities to b^n and take nth roots; the constants tend to one, proving equality. □

Theorem 1.12.2 (Finite normalization for strict affinoid algebras)**.** *Fix a bounded admissible presentation $q : R_n = K\langle X_1, \ldots, X_n\rangle \twoheadrightarrow A$ of a nonzero strict K-affinoid algebra. There are an integer $d \geq 0$ and a contractive K-algebra map $\iota : R_d \to R_n$ such that $q\iota : R_d \hookrightarrow A$ is injective and finite. In particular, A admits a bounded finite injection from a Tate algebra; it is*

contractive when A carries this presentation's quotient norm. No reducedness or integrality hypothesis on A is required.

Proof. Induct on the number n of variables in a strict presentation $A = R_n/I$, using its quotient norm; an equivalent admissible norm only changes boundedness constants. If $I = 0$, the presentation itself works. If $I \neq 0$, choose $0 \neq f \in I$. Lemma 1.11.3 provides an isometric coordinate change after which a scalar and a unit multiple of f is distinguished in the last variable, of degree s. Degree zero would make f a unit and $A = 0$, excluded here. By Theorem 1.10.3, the quotient $R_n/(f)$ is generated over R_{n-1} by $1, X_n, \dots, X_n^{s-1}$. The same is true of its quotient A.

Write ϕ for the coordinate automorphism and use the transformed presentation $q\phi^{-1}$. Let J be the kernel of its restriction to R_{n-1} and put $B = R_{n-1}/J$. Theorem 1.11.5 makes J closed, so B is a nonzero strict affinoid. The induced injection $B \hookrightarrow A$ is bounded: approximate quotient-norm lifts in R_{n-1} and use the bounded original map. Apply induction to the displayed presentation of B, obtaining a contractive lift $\iota' : R_d \to R_{n-1}$. If $j : R_{n-1} \hookrightarrow R_n$ is the coordinate inclusion, set $\iota = \phi^{-1} j \iota'$. This map is contractive and lifts the composite injection $R_d \hookrightarrow B \hookrightarrow A$. The composite is finite because products of finite generating lists for B over R_d and for A over B generate A. This finite-generation argument is also [StacksAlg, Tag 00GJ, Lemma 10.7.2]. In the case $I = 0$ take $d = n$ and ι the identity. This proves the presentation-compatible assertion throughout the induction. □

Theorem 1.12.3 (Maximal residue fields and the unit-ideal test)**.** *Let A be a strict K-affinoid algebra. For every maximal ideal $\mathfrak{m}$, $A/\mathfrak{m}$ is a finite extension of K, and its extending absolute value composed with the quotient map is a bounded multiplicative seminorm on A, with kernel $\mathfrak{m}$. Consequently $\mathcal{M}(A) = \varnothing$ if and only if $A = 0$. For any finite list $h_1, \dots, h_t \in A$,*

$$(h_1, \dots, h_t) = A \iff \nexists x \in \mathcal{M}(A) \text{ with } |h_1(x)| = \cdots = |h_t(x)| = 0.$$

In particular $h \in A$ is a unit exactly when it vanishes at no point of $\mathcal{M}(A)$. These conclusions include trivial valuation.

Proof. Ideals are closed by Theorem 1.11.5, so $L = A/\mathfrak{m}$ is a nonzero strict affinoid field. Apply Theorem 1.12.2 to obtain a finite injection $R_d \hookrightarrow L$. By [StacksAlg, Tag 00GR, Lemma 10.36.18], R_d is a field. If $d > 0$, evaluation at

$Y_1 = 0$ is a unital homomorphism $R_d \to R_{d-1}$ and kills the nonzero element Y_1, contradicting that R_d is a field. Thus $d = 0$ and L/K is finite.

The quotient norm on L is a homogeneous vector-space norm. By Lemma 1.12.1, its absolute value is bounded by a constant times this quotient norm. Composing with $A \to L$ gives the asserted bounded seminorm and kernel. It is contractive for any chosen submultiplicative norm on A by Lemma 1.6.1. Every proper ideal is contained in a maximal ideal by [StacksAlg, Tag 00E0]. Thus a proper ideal generated by the h_i produces a common-zero point as above. Conversely, a common-zero point prevents a relation $\sum a_i h_i = 1$, since evaluation would send its right side to one and its left side to zero. Taking the zero ideal proves nonemptiness for $A \neq 0$; a multiplicative seminorm cannot exist on the zero ring because it must send both zero to zero and one to one. □

Finite normalization also computes strict affinoid dimension in Proposition 3.21.1. The pointwise unit test extends to generalized affinoids in Theorem 1.20.1; together with the power bounds, it gives the bounded rational-domain universal property in Theorem 2.31.2.

1.13 Pointwise bounds and bounded powers

A pointwise inequality on a rational domain must produce a bounded substitution on its algebra. The needed assertion is that an element of a strict affinoid algebra has bounded powers exactly when its value at every point has absolute value at most one. We prove this assertion through monic equations with controlled coefficients. Throughout this section K is complete, its fixed absolute value may be trivial, and $R_n = K\langle X_1, \dots, X_n\rangle$ carries the Gauss norm.

Theorem 1.13.1 (Factorization and normality of Tate algebras)**.** *For every integer $n \geq 0$, the ring R_n is a unique factorization domain and is integrally closed in its fraction field.*

Proof. The Gauss norm is multiplicative by Proposition 1.2.2, so R_n is a domain. Induct on n, starting with the field $R_0 = K$. Assume R_{n-1} is a unique factorization domain. Its polynomial ring $R_{n-1}[X_n]$ is a unique factorization domain by [StacksAlg, Tag 0BC1, Lemma 10.120.10].

Let $0 \neq f \in R_n$ be a nonunit. Lemma 1.11.3 and Corollary 1.10.4 supply an isometric coordinate automorphism after which f, up to multiplication by a scalar and a unit, is a monic polynomial $P \in R_{n-1}[X_n]$ of positive degree with $\|P\| = 1$. Factor P in that polynomial ring. Since the product of the leading coefficients is one, each leading coefficient is a unit; after adjusting the factors by units we obtain

$$P = P_1 \cdots P_s, \qquad P_j \in R_{n-1}[X_n] \text{ monic and irreducible.}$$

Each P_j has Gauss norm at least one. Multiplicativity and $\|P\| = 1$ therefore give $\|P_j\| = 1$ for every j. Thus each P_j is a monic polynomial with all lower coefficients of norm at most one. The division theorem identifies

$$R_{n-1}[X_n]/(P_j) \xrightarrow{\sim} R_n/(P_j).$$

Indeed, both quotients have the same unique polynomial representatives of degree less than $\deg P_j$, and the map is induced by the algebra inclusion. The left ring is a domain because P_j is prime in the polynomial unique factorization domain. Consequently P_j is prime in R_n. Reversing the coordinate automorphism gives a factorization of f into prime elements. A domain in which every nonzero nonunit factors into primes is a unique factorization domain, by cancellation of prime factors. Finally, [StacksAlg, Tag 0AFV, Lemma 10.120.11] gives integral closedness. □

To control a coefficient in a Tate algebra, it suffices to test it at maximal residue fields. The next proof derives this fact from the unit-ideal criterion already established; it does not use a spectral-radius formula.

Lemma 1.13.2 (A maximal point detects the Gauss norm)**.** *For every $h \in R_n$ there is a maximal ideal $\mathfrak{m} \subset R_n$ such that, for the extending absolute value on the finite field $R_n/\mathfrak{m}$,*

$$|h \bmod \mathfrak{m}| = \|h\|.$$

Proof. If $h = 0$, evaluate all variables at zero. If $h \neq 0$, the coefficient decay implies that a nonzero coefficient $c \in K$ attains its Gauss norm. Dividing by c, we may assume $\|h\| = 1$. Consider the actual quotient

$$D = R_n\langle S\rangle/(1 - hS).$$

The ideal is closed and D is a strict affinoid algebra by Lemma 2.17.1. If $D = 0$, then $1 = (1 - hS)G$ for a series $G = \sum_{r\geq 0} a_r S^r \in R_n\langle S\rangle$. Comparing

coefficients gives $a_0 = 1$ and $a_r = ha_{r-1}$, hence $a_r = h^r$. Multiplicativity of the Gauss norm gives $\|a_r\| = 1$ for every r, contradicting the coefficient decay required for G. Thus $D \neq 0$.

Theorem 1.12.3 supplies a maximal residue field L of D, finite over K, and a bounded evaluation $D \to L$. Its absolute value is contractive for the quotient norm by Lemma 1.6.1. The relation $hS = 1$ and the bounds $|h| \leq 1$, $|S| \leq 1$ force $|h| = 1$. The image of R_n in L is a finite-dimensional K-algebra and a domain, hence a field by [StacksAlg, Tag 00GS(1), Lemma 10.36.19]. The kernel of $R_n \to L$ is therefore maximal. Uniqueness of extending absolute values in Lemma 1.12.1 identifies its residue absolute value with the restriction from L. Rescaling by c proves the assertion. □

Lemma 1.13.3 (Bounded coefficients of a minimal polynomial)**.** *Let B be a nonzero strict K-affinoid algebra which is a domain, and let $R_d \hookrightarrow B$ be a bounded finite injection. For $f \in B$, its monic minimal polynomial over* $\mathrm{Frac}(R_d)$ *belongs to $R_d[T]$, say*

$$p_f(T) = T^r + a_1 T^{r-1} + \cdots + a_r.$$

If $|f \bmod \mathfrak{q}| \leq 1$ for every maximal ideal $\mathfrak{q}$ of B, using the unique extending residue absolute values, then $\|a_i\| \leq 1$ for $1 \leq i \leq r$.

Proof. Finite ring maps are integral by [StacksAlg, Tag 00GK, Lemma 10.36.3]. The preceding factorization theorem makes R_d normal, so [StacksAlg, Tag 00H7, Lemma 10.38.6] places the minimal polynomial in $R_d[T]$. Polynomial division by this monic polynomial gives

$$S := R_d[f] \;\simeq\; R_d[T]/(p_f).$$

Indeed, any polynomial relation has remainder of degree less than r; viewed in the fraction field of the domain B, minimality forces that remainder to be zero.

Fix a maximal ideal $\mathfrak{m}$ of R_d and put $k_x = R_d/\mathfrak{m}$, a finite extension of K. By [StacksAlg, Tag 09HU, Lemma 9.16.1], choose a splitting field E of the specialization $\overline{p}_f \in k_x[T]$. It is finite over k_x, since it is generated by finitely many algebraic roots; adjoining them successively gives a finite tower. Equip it with the extending absolute value from Lemma 1.12.1. Let $\alpha \in E$ be any root. Evaluation at α defines a maximal ideal $\mathfrak{n}$ of S, with $S/\mathfrak{n} = k_x[\alpha]$. The algebra B is finite over S: any list generating it over R_d also generates it

over S. Consequently $S \subset B$ is integral. By [StacksAlg, Tag 00GQ, Lemma 10.36.17], there is a prime $\mathfrak{q} \subset B$ over $\mathfrak{n}$. Its quotient is an integral domain integral over the field $S/\mathfrak{n}$, so [StacksAlg, Tag 00GS(2), Lemma 10.36.19] makes $\mathfrak{q}$ maximal.

Theorem 1.12.3 gives a bounded point of B at $\mathfrak{q}$, and $B/\mathfrak{q}$ is finite over K. The inclusion $k_x[\alpha] \hookrightarrow B/\mathfrak{q}$ identifies α with the residue of f. Uniqueness in Lemma 1.12.1 therefore gives $|\alpha| = |f \bmod \mathfrak{q}| \leq 1$. This argument applies to every root, including repeated roots; separability is unnecessary. In E, each coefficient $\overline{a}_i$ is a sum of products of i roots. Its absolute value is consequently at most one.

We have proved $|a_i \bmod \mathfrak{m}| \leq 1$ for every maximal ideal $\mathfrak{m}$ of R_d. Apply Lemma 1.13.2 separately to each a_i to conclude $\|a_i\| \leq 1$. The boundedness of the residue points came from their finite residue fields; no comparison between the norm on B and a finite-R_d-module norm has been assumed. □

Theorem 1.13.4 (Pointwise boundedness and power boundedness)**.** *Let A be a strict norm-admissible K-affinoid algebra, with the zero algebra allowed. Fix a bounded admissible strict presentation $q : R_n \twoheadrightarrow A$, and write $\|\cdot\|_q$ for its quotient norm. For $f \in A$, the following conditions are equivalent:*

1. *$|f(x)| \leq 1$ for every $x \in \mathcal{M}(A)$.*
2. *$|f \bmod \mathfrak{m}| \leq 1$ for every maximal ideal $\mathfrak{m}$ of A, with its extending residue absolute value.*
3. *There are an integer $r \geq 1$ and elements $b_1, \dots, b_r \in A$ satisfying*

$$f^r + b_1 f^{r-1} + \cdots + b_r = 0, \qquad \|b_i\|_q \leq 1 \quad (1 \leq i \leq r).$$

4. *There is a real $C \geq 1$ such that $\|f^m\|_q \leq C$ for every integer $m \geq 0$.*

Condition (4) is unchanged if the norm is replaced by any uniformly equivalent norm. In particular, it holds for the given admissible norm on A. The statement requires neither reducedness nor nontrivial valuation.

Proof. If $A = 0$, conditions (1) and (2) are vacuous, the polynomial T gives (3), and every power has norm zero. Assume $A \neq 0$. Theorem 1.12.3 identifies each maximal residue absolute value with a bounded point, proving (1)⇒(2).

Step 1: an equation on each irreducible quotient. By Theorem 1.11.5, A is Noetherian and every prime ideal is closed. It has finitely

many minimal primes $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ by [StacksAlg, Tag 00FR, Lemma 10.31.6]. Put $B_j = A/\mathfrak{p}_j$ and write f_j for the image of f. Every maximal residue field of B_j is a maximal residue field of A, so condition (2) holds for f_j.

Apply the presentation-compatible form of Theorem 1.12.2 to $R_n \twoheadrightarrow B_j$. It supplies a contractive homomorphism $\iota_j : R_{d_j} \to R_n$ whose composite into B_j is a finite injection. Lemma 1.13.3 gives a monic equation for f_j with coefficients $a_{j,i} \in R_{d_j}$ of Gauss norm at most one. Map these coefficients into A by $q\iota_j$. The resulting monic polynomial $P_j \in A[T]$ has all coefficients of quotient norm at most one and satisfies $P_j(f) \in \mathfrak{p}_j$. This is why the lift to the specified presentation is needed: it controls the coefficients in A, not merely their images in B_j.

Step 2: remove the nilpotent error. Let $P = \prod_{j=1}^{t} P_j$. This is monic, and every coefficient has norm at most one by submultiplicativity and the ultrametric inequality. Moreover $P(f)$ belongs to every minimal prime. By [StacksAlg, Tag 00E0(4),(7), Lemma 10.17.2], the intersection of the minimal primes is the nilradical: every prime contains a minimal prime, and the intersection of all primes is the radical of zero. Thus $P(f)^e = 0$ for some integer $e \geq 1$. The monic polynomial $P(T)^e$ still has all coefficients of norm at most one and annihilates f. This proves (2)⇒(3), including nonreduced A.

Step 3: control all powers. Suppose (3) holds and set

$$C = \max(1, \|f\|_q, \dots, \|f^{r-1}\|_q).$$

Multiplication of the monic equation by f^{m-r} gives, for $m \geq r$,

$$f^m = -\sum_{i=1}^{r} b_i f^{m-i}, \qquad \|f^m\|_q \leq \max_{1 \leq i \leq r} \|f^{m-i}\|_q.$$

Induction proves (4). Finally, a bounded multiplicative seminorm is contractive for $\|\cdot\|_q$ by Lemma 1.6.1; hence (4) gives $|f(x)|^m = |f^m(x)| \leq C$ for every $m \geq 1$. Taking mth roots proves (1). Uniformly equivalent norms have the same bounded subsets, proving the last assertion. □

For a radius $r = |c| \in |K^\times|$, applying the theorem to f/c shows that $|f(x)| \leq r$ at all points is equivalent to a bound $\|f^m\| \leq Cr^m$ for all $m \geq 0$. This consequence is confined to value-group radii; no assertion about arbitrary real radii or generalized affinoid targets is used here.

References. The classical factorization argument is in [BGR84, 5.2.5, Proposition 4, p. 207; 5.2.6, Theorems 1–2, pp. 207–208]. The minimal-polynomial and power-boundedness route is in [Bosch14, 3.1, Lemmas 13–14, pp. 36–38; Theorem 17, p. 39], and [BGR84, 6.2.2, Proposition 4, p. 239; 6.2.3, Proposition 1, p. 240]. The present proof uses the project's division and residue-field results, the cited authorized Stacks algebra, and the explicit quotient argument above for Gauss-norm detection. These classical references give attribution; they are not unproved analytic inputs.

1.14 Finite algebras and bounded epimorphisms

Finite algebra arguments will be used inside the category of strict affinoids. We first check that their algebraic and Banach constructions agree. The distinction matters when a finite algebra already carries an affinoid norm: the finite-module theorem for a fixed coefficient algebra does not, by itself, compare that norm with its finite-module norm.

Lemma 1.14.1 (Bounded lifting for Banach surjections)**.** *Assume that the absolute value on the complete field K is nontrivial. If $u : E \to F$ is a bounded surjective K-linear map between non-Archimedean Banach spaces, there is $C > 0$ such that every $y \in F$ has a lift $x \in E$ with $\|x\| \le C\|y\|$. In particular, u is open and norm-strict.*

Proof. Choose $\pi \in K$ with $0 < |\pi| < 1$, and put $E_1 = \{x : \|x\| \le 1\}$. Surjectivity gives $F = \bigcup_{n\ge 0} u(\pi^{-n}E_1)$. The additive group of F is linearly topologized by its norm balls, complete, and has a countable neighbourhood basis. Hence the Baire theorem in [StacksAlg, Tag 0CQU, Lemma 15.37.3] shows that the closure of one of these images has nonempty interior. The images are additive subgroups, so their closures are subgroups. Translation and scalar multiplication therefore give some $r > 0$ such that

$$\{y : \|y\| \le r\} \subset \overline{u(E_1)}.$$

For nonzero y, choose an integer k with $|\pi| r < \|\pi^k y\| \le r$. Approximate $\pi^k y$ by $u(z)$, $z \in E_1$, so closely that the error is at most $|\pi|\,\|\pi^k y\|$. Then $x = \pi^{-k} z$ satisfies

$$\|x\| \le (|\pi| r)^{-1}\|y\|, \qquad \|y - u(x)\| \le |\pi|\,\|y\|.$$

Repeat this approximation for the successive errors. The resulting corrections x_j have norms at most $(|\pi| r)^{-1}|\pi|^j\|y\|$, so their sum converges in E and maps

to y. The ultrametric inequality gives the asserted bound on the sum. Zero has the zero lift. A small ball in F consequently lifts into any prescribed ball in E, proving openness and the quotient-norm assertion. □

Lemma 1.14.2 (Relative strict presentations)**.** *Every bounded homomorphism $A \to B$ of strict K-affinoid algebras extends to a bounded norm-strict surjection $A\langle T_1, \dots, T_n\rangle \twoheadrightarrow B$. Its kernel is an actual closed ideal. In particular, the norm of B is equivalent to the displayed relative quotient norm. Trivial valuation and zero quotients are allowed.*

Proof. Choose a strict presentation $K\langle T\rangle \twoheadrightarrow B$ and first use its quotient norm on B and a presentation norm on A. The variable classes have norm at most one. The bounded map on coefficients and Proposition 1.9.3 give $A\langle T\rangle \to B$. This map is surjective because it contains the original presentation. That presentation gives lifts of norm less than twice the norm of any nonzero element of B, and its coefficient inclusion into $A\langle T\rangle$ is isometric. These lifts and the bound on the map prove equivalence of its quotient norm and the norm on B. The coefficient algebra is strict by Lemma 2.17.1, so its ideal kernel is closed by Theorem 1.11.5. Equivalent initial norms only change constants. Zero targets are immediate. □

Proposition 1.14.3 (Pushouts of bounded strict-affinoid maps)**.** *For any bounded maps $A \to B$, $A \to C$ of strict K-affinoids, the separated completed projective tensor product $B\widehat{\otimes}_A C$ is a strict affinoid, with zero allowed. It is the pushout for bounded homomorphisms into commutative Banach K-algebras. In relative presentations*

$$B = A\langle T\rangle/I, \qquad C = A\langle S\rangle/J$$

it is canonically $A\langle T, S\rangle/(I, J)$, with bounded inverse comparisons for the given norms. If the displayed quotient norms are used, this identification is isometric. No claim that the ordinary tensor product is complete is made.

Proof. Lemma 1.14.2 supplies these presentations with equivalent quotient norms. Theorem 2.18.2 gives exactly the asserted tensor quotient and its isometry for those norms. Uniform equivalence of the original factor norms passes to projective seminorms and separated completions. For the universal property, two bounded maps $B, C \to E$ agreeing on A give a bounded balanced bilinear map $(b, c) \mapsto bc$ into E. On a finite tensor expression its norm

is bounded by a fixed constant times the maximum of $\|b\|\|c\|$; taking the infimum and completing gives a bounded algebra map. It is unique on the dense algebraic tensor image. This argument also covers a zero tensor quotient: boundedness forces the balanced map to vanish on every seminorm-zero tensor. □

Proposition 1.14.4 (The affinoid norm on a finite algebra)**.** *Let A be a strict K-affinoid algebra over a complete field with its fixed absolute value, possibly trivial. Every commutative finite A-algebra D has a strict K-affinoid structure whose norm is uniformly equivalent to its canonical finite-A-module norm; the structural map $A \to D$ is bounded. This structure is unique up to bounded isomorphism. More precisely, if D already has a strict affinoid norm and $A \to D$ is bounded, that norm is equivalent to the canonical finite-A-module norm. Every A-linear map between such finite algebras is bounded.*

Proof. Use a presentation quotient norm on A, and give D its canonical module norm by Theorem 1.11.5. Multiplication by each $d \in D$ is a bounded A-linear operator L_d on D. If $e_1, \dots, e_s$ generate D as an A-module, controlled lifts in A^s show that $\|L_d\|_{\mathrm{op}} \le C\|d\|$ for a constant independent of d: write $d = \sum a_i e_i$ with $\max \|a_i\| \le 2\|d\|$ and use the finitely many operator bounds for L_{e_i}. Conversely $\|d\| = \|L_d(1)\| \le \|L_d\|_{\mathrm{op}}\|1\|$. Thus

$$\|d\|_* := \|L_d\|_{\mathrm{op}}$$

is an equivalent complete submultiplicative norm, with $\|1\|_* = 1$, and $A \to D$ is contractive. The zero algebra is treated separately with its unique norm.

If the valuation is nontrivial, multiply each e_i by a nonzero scalar so that its $*$-norm is at most one; the rescaled elements still generate D as an A-module. Substitution gives a bounded surjection $A\langle T_1, \dots, T_s\rangle \to D$. It is norm-strict without invoking an open mapping theorem: the polynomials of degree at most one already give controlled lifts, by the chosen finite-module presentation and equivalence of norms. Its kernel is an algebraic ideal, hence closed by Theorem 1.11.5; consequently its quotient norm is equivalent to $\|\cdot\|_*$. Since $A\langle T\rangle$ is strict by the established iterated-presentation construction, this gives the required affinoid structure.

If the valuation is trivial, a strict Tate algebra is a polynomial algebra with the discrete norm (one on every nonzero element). Its presentation quotients and their canonical finite-module norms are also discrete. The

same construction is then the algebraic surjection $A[T_1, \dots, T_s] \to D$, and all the indicated norms are discrete. No choice of a small nonzero scalar is made in this case.

Finally suppose D already has an affinoid norm and the structural map is bounded. The finite generator map $A^s \to D$ is bounded and surjective. Over a nontrivially valued field Lemma 1.14.1 makes it norm-strict, which identifies its quotient norm with the existing norm up to equivalence. Over a trivially valued field both strict presentation norms are equivalent to the discrete norm. The assertion about linear maps now follows from the finite-module theorem with coefficient algebra A. □

Proposition 1.14.5 (Finite base change)**.** *Let $A \to B$ and $A \to C$ be bounded maps of strict K-affinoid algebras, with B finite as an A-module. Then the ordinary algebra $D = B \otimes_A C$ is a strict affinoid, finite over C, and the natural map*

$$B \otimes_A C \longrightarrow B \widehat{\otimes}_A C$$

is a bounded algebra isomorphism when the left side has its canonical finite-C-module norm. Both structural arrows into D are bounded.

Proof. Proposition 1.14.4 identifies the given norm on B with its finite-A-module norm. Proposition 1.11.6, applied to that module and the target C, shows that the ordinary tensor product is already complete with its canonical finite-C-module norm. As a finite C-algebra it is strict by Proposition 1.14.4. The tensor maps $b \mapsto b \otimes 1$ and $c \mapsto 1 \otimes c$ are bounded for the projective tensor norm, hence for the equivalent norm just identified. The multiplication is the ordinary tensor-algebra multiplication and extends without change to its completion. □

Lemma 1.14.6 (Restriction of scalars through a finite field extension)**.** *Let L/K be a finite extension, endowed with the unique extending absolute value. Every strict L-affinoid algebra, with its given admissible norm, is also a strict K-affinoid algebra. Bounded L-algebra homomorphisms remain bounded after restriction of scalars. For every Banach K-space E and every K-basis $\ell_1, \dots, \ell_d$ of L, the map*

$$E^d \longrightarrow E \widehat{\otimes}_K L, \qquad (e_1, \dots, e_d) \longmapsto \sum_i e_i \otimes \ell_i$$

is a bounded linear isomorphism with bounded inverse; the algebraic tensor product is already complete. In particular $L\langle T_1, \dots, T_n\rangle$ is a finite free algebra of rank d over $K\langle T_1, \dots, T_n\rangle$. No separability hypothesis is needed.

Proof. Lemma 1.12.1 gives the absolute value on L and constants bounding its coordinate maps $\lambda_i : L \to K$. On the algebraic tensor product, the displayed basis map is an algebraic isomorphism. Its forward bound follows from the finite expression $\sum e_i \otimes \ell_i$. In the other direction $1_E \otimes \lambda_i$ has bound $\|\lambda_i\|$: apply the ultrametric inequality to an arbitrary finite tensor expression and take its projective infimum. Thus the projective norm is definite and equivalent to the maximum norm on E^d, which is complete. Completing changes nothing and gives the claimed bounded inverse.

Apply the same coordinate bounds to every coefficient of a power series. A coefficient family in L tends to zero exactly when all its d coordinate families in K do, with the same uniform bounds on their supremum norms. Coefficientwise multiplication then identifies $L\langle T\rangle$ as a Banach algebra with $L \otimes_K K\langle T\rangle$. This algebra is finite free over the strict K-affinoid $K\langle T\rangle$, and Proposition 1.14.4 supplies its strict structure with the same norm up to equivalence. A strict L-affinoid is a quotient of $L\langle T\rangle$ by an actual ideal. The ideal is closed, and composing its quotient with a strict K-presentation gives a strict K-presentation, by the successive-quotient argument of Lemma 2.18.1. The quotient norms remain equivalent. The last assertion about homomorphisms follows because the norm inequality is unchanged when scalars are restricted. □

Definition 1.14.7. A bounded homomorphism $A \to B$ of strict K-affinoid algebras is a *strict-affinoid epimorphism* if, for every strict K-affinoid algebra C, two bounded K-algebra maps $B \rightrightarrows C$ agreeing on A are equal. Equivalently, the opposite morphism is a monomorphism in the category of strict affinoid spaces with bounded algebra homomorphisms. This condition is not asserted for arbitrary ring targets.

Lemma 1.14.8 (Finite strict-affinoid epimorphisms)**.** *A finite strict-affinoid epimorphism $A \to B$ is surjective. Conversely, every bounded surjective homomorphism of strict affinoids is a finite strict-affinoid epimorphism. These assertions hold for trivial as well as nontrivial valuation.*

Proof. By Proposition 1.14.5, the ordinary algebra $D = B \otimes_A B$ is a strict affinoid and its two tensor arrows from B are bounded. The epimorphism

condition makes these arrows equal. Therefore every pure tensor satisfies $b \otimes b' = bb' \otimes 1$. The multiplication map $D \to B$ and $b \mapsto b \otimes 1$ are inverse. Thus $A \to B$ is an epimorphism of rings: any two ring maps agreeing on A factor through the tensor product, and the same equality forces them to agree on B. The finite epimorphism theorem [StacksAlg, Tag 04VT, Lemma 10.107.6] now gives surjectivity. The converse follows immediately from surjectivity; as an A-module a quotient algebra is generated by one. □

Lemma 1.14.9 (An affinoid epimorphism over a field)**.** *Let κ be any complete non-Archimedean field with its fixed absolute value, and let $D \neq 0$ be a strict κ-affinoid algebra. If $\kappa \to D$ is a strict-affinoid epimorphism, then it is a bounded isomorphism. No reducedness or separability is assumed.*

Proof. For each maximal ideal $\mathfrak{n}$ of D, the residue field $L = D/\mathfrak{n}$ is finite over κ, and the quotient map $D \to L$ is bounded by Theorem 1.12.3. The finite-dimensional algebra $L \otimes_\kappa L$ is strict by Proposition 1.14.4, and its two maps from L are bounded by finite-dimensional norm equivalence. Compose them with the surjection $D \to L$. The epimorphism condition shows that $a \otimes 1 = 1 \otimes a$ for all $a \in L$. Multiplication $L \otimes_\kappa L \to L$ is consequently an isomorphism, so $[L : \kappa]^2 = [L : \kappa]$ and $L = \kappa$.

The bounded residue maps $D \to \kappa$ for any two maximal ideals must now agree; their kernels agree as well. Hence D has exactly one maximal ideal. Apply finite normalization, Theorem 1.12.2, to obtain a finite injection $\kappa\langle T_1, \dots, T_d\rangle \hookrightarrow D$. If $d > 0$, the two evaluation ideals at $(0, \dots, 0)$ and $(1, 0, \dots, 0)$ are distinct maximal ideals in the Tate algebra. Lying over and the maximal-ideal criterion for integral extensions, [StacksAlg, Tags 00GQ and 00GS], give maximal ideals of D above both of them, a contradiction. Therefore $d = 0$ and D is finite over κ. Lemma 1.14.8 makes $\kappa \to D$ surjective. It is injective because $D \neq 0$, and its inverse is bounded by finite-dimensional norm equivalence. □

1.15 Relative Weierstrass arguments

A relative power series need not have a distinguished coefficient for its original Banach norm, even when every residue-field fibre has one. The following arguments convert this fibrewise information into a finite algebra map. They use maximal residue fields and the already proved normalization theorem; no

structure-sheaf theorem or Gerritzen–Grauert theorem enters. Throughout this section K is complete with its fixed non-Archimedean absolute value, which may be trivial, and all affinoid algebras are strict and norm-admissible.

Theorem 1.15.1 (Strict inequalities at maximal points)**.** *Let $q : R_n = K\langle X_1, \dots, X_n\rangle \twoheadrightarrow A$ be an admissible strict presentation, and let $\|\cdot\|_q$ be its quotient norm. For $f \in A$, the following conditions are equivalent:*

1. *$|f \bmod \mathfrak{m}| < 1$ for every maximal ideal $\mathfrak{m}$ of A, using the unique extending absolute value on its residue field.*
2. *$|f(x)| < 1$ for every $x \in \mathcal{M}(A)$.*
3. *There is a monic equation $f^r + b_1 f^{r-1} + \cdots + b_r = 0$, with $r \geq 1$ and $\|b_i\|_q < 1$ for every i.*
4. *$\|f^m\|_q \to 0$ as $m \to \infty$.*

These conditions imply a uniform bound $|f(x)| \leq \rho < 1$ on $\mathcal{M}(A)$. Condition (4) is unchanged under replacement of the norm by a uniformly equivalent norm. The zero algebra and nonreduced algebras are included.

Proof. The zero algebra is immediate, using the polynomial T. Suppose $A \neq 0$. We first retain the strict inequality in the minimal-polynomial argument of Lemma 1.13.3. If B is a strict affinoid algebra which is a domain, equipped with a bounded finite injection $R_d \hookrightarrow B$, and $u \in B$ has absolute value less than one at every maximal residue field, write its minimal polynomial as $p(T) = T^r + a_1 T^{r-1} + \cdots + a_r \in R_d[T]$. The membership in $R_d[T]$ is already proved in that lemma. Fix a maximal ideal $\mathfrak{m}$ of R_d and a root α of the specialization of p in a finite splitting field of $R_d/\mathfrak{m}$. The proof of the lemma identifies $R_d[u]$ with $R_d[T]/(p)$ and, by lying over for the finite extension $R_d[u] \subset B$, lifts the evaluation at α to a maximal ideal of B. Uniqueness of the extending absolute value then gives $|\alpha| < 1$. Every specialized coefficient is a finite sum of products of roots, so its absolute value is strictly less than one. Lemma 1.13.2, applied to each a_i, now gives $\|a_i\| < 1$. This argument uses precisely the finite splitting-field and integral-extension inputs displayed in Lemma 1.13.3; it does not infer a strict norm bound from pointwise bounds by compactness.

Assume (1). By Theorem 1.11.5 and [StacksAlg, Tag 00FR, Lemma 10.31.6], the algebra A has finitely many closed minimal primes $\mathfrak{p}_1, \dots, \mathfrak{p}_t$. Apply the presentation-compatible normalization theorem, Theorem 1.12.2,

to each quotient $R_n \twoheadrightarrow A/\mathfrak{p}_j$. Its contractive lift $R_{d_j} \to R_n$, together with the strict coefficient bound just proved, yields a monic polynomial $P_j \in A[T]$ whose nonleading coefficients have quotient norm less than one and for which $P_j(f) \in \mathfrak{p}_j$. In a product of monic polynomials with this property, every nonleading coefficient is a finite sum of products containing at least one nonleading coefficient. Thus $P = \prod_j P_j$ has the same strict bounds. Its value $P(f)$ lies in every minimal prime and is nilpotent by [StacksAlg, Tag 00E0(4),(7), Lemma 10.17.2]. Choose $e \geq 1$ with $P(f)^e = 0$. The polynomial $P(T)^e$ is monic, still has all nonleading coefficients of norm less than one, and proves (3).

Assume (3), choose a real δ with $0 < \delta < 1$ and $\|b_i\|_q \leq \delta$ for every i, and put $C = \max_{0 \leq j < r} \|f^j\|_q$. The recurrence $f^m = -\sum_{i=1}^{r} b_i f^{m-i}$ gives, by induction,

$$\|f^m\|_q \leq C\delta^{\lfloor m/r \rfloor} \qquad (m \geq 0).$$

Indeed, for $m \geq r$ one has $\lfloor (m-i)/r \rfloor \geq \lfloor m/r \rfloor - 1$. This proves (4). By Lemma 1.6.1, every bounded multiplicative seminorm satisfies $|f(x)|^m \leq \|f^m\|_q$, so (4) implies (2). Maximal residue absolute values are bounded points by Theorem 1.12.3, giving (2)⇒(1). Finally choose N with $\|f^N\|_q < 1$; then $|f(x)| \leq \|f^N\|_q^{1/N} < 1$ uniformly. Equivalent norms preserve convergence to zero. □

Lemma 1.15.2 (Units in a field Tate algebra)**.** *For a complete non-Archimedean field L, an element $H = \sum_\nu h_\nu U^\nu \in L\langle U_1, \ldots, U_m \rangle$ is a unit if and only if*

$$|h_0| > \sup_{\nu \neq 0} |h_\nu|.$$

For a unit one has $\|H\| = |h_0|$ and $\|H^{-1}\| = |h_0|^{-1}$.

Proof. The strict inequality gives the convergent geometric inverse of $h_0^{-1}H = 1 + (h_0^{-1}H - 1)$ by Lemma 1.1.4. Conversely suppose $HG = 1$. Choose the lexicographically largest exponent μ attaining $\|H\|$ and the corresponding exponent ν for G; these exist because the coefficients tend to zero. In the coefficient of $U^{\mu+\nu}$ in HG, the product of the chosen coefficients is the unique term of norm $\|H\|\|G\|$: another term of that norm would require two maximizing exponents, and decreasing the first would increase the second. It cannot cancel. Hence $\mu + \nu = 0$, since HG is constant. Both exponents are nonnegative, so $\mu = \nu = 0$. No nonconstant exponent can then attain $\|H\|$. Coefficient decay makes the supremum of their absolute

values strictly smaller than $|h_0|$. The inverse-norm assertion follows from multiplicativity of the Gauss norm, Proposition 1.2.2. □

For a maximal ideal $x = \mathfrak{m}_x$ of A, put $k_x = A/\mathfrak{m}_x$ with its complete extending absolute value. Coefficientwise reduction sends $F \in A\langle T_1, \dots, T_n\rangle$ to $F_x \in k_x\langle T_1, \dots, T_n\rangle$. Write $T' = (T_1, \dots, T_{n-1})$ and $F = \sum_{j\geq 0} F_j(T')T_n^j$. We say that F is *T_n-distinguished of order s at x* if $F_{s,x}$ is a unit, $\|F_{j,x}\| \leq \|F_{s,x}\|$ for $j \leq s$, and $\|F_{j,x}\| < \|F_{s,x}\|$ for $j > s$. The order, when it exists, is unique. The preceding lemma describes the unit condition explicitly in coefficients.

Lemma 1.15.3 (Uniform distinguished coordinates)**.** *Let $A \neq 0$, let $n \geq 1$, and suppose the coefficients of $F = \sum_{\nu\in\mathbb{N}^n} a_\nu T^\nu \in A\langle T_1, \dots, T_n\rangle$ generate the unit ideal of A. There is an isometric A-algebra automorphism σ and an integer $s \geq 0$ such that $\sigma(F)$ is T_n-distinguished of some order at most s at every maximal ideal of A. One may take*

$$\sigma(T_i) = T_i + T_n^{t^{n-i}} \ (i < n), \qquad \sigma(T_n) = T_n, \qquad s = t^n - 1$$

for a sufficiently large integer $t \geq 2$.

Proof. Choose finitely many coefficients and an identity $\sum_{j=1}^{r} c_j a_{\nu(j)} = 1$. Contractivity of maximal residue points gives

$$\max_j |a_{\nu(j)}(x)| \geq \delta, \qquad \delta = (\max_j \|c_j\|)^{-1} > 0$$

for every maximal x. Choose $t \geq 2$ so that $\|a_\nu\| < \delta$ whenever some component of ν is at least t. The substitutions defining σ and its inverse, obtained by changing the plus signs to minus signs, are contractive on polynomials and hence on their completions. They are inverse by continuity, so both are isometric.

Fix x and put $M = \|F_x\| \geq \delta$. Every exponent at which a coefficient has absolute value M lies in the box $0 \leq \nu_i < t$. The function

$$w(\nu) = \sum_{i=1}^{n} \nu_i t^{n-i}$$

takes distinct values on that box. Among the maximizing coefficients choose the unique exponent ν_* with largest $w(\nu_*)$, and set $s_x = w(\nu_*) \leq t^n - 1$. Separate F_x into the finite sum of its coefficients of absolute value M and

the remaining series. The remaining series has Gauss norm strictly less than M, since its coefficients tend to zero; its image under σ has the same bound. In the finite sum, the unique term with largest T_n-degree after substitution contributes $a_{\nu_*}(x)T_n^{s_x}$, with constant leading coefficient, and all other contributions have degree less than s_x. Thus the coefficient of $T_n^{s_x}$ in $\sigma(F)_x$ is $a_{\nu_*}(x)$ plus a series of norm less than M; it is a unit by Lemma 1.15.2. Its norm is M, all lower coefficients have norm at most M, and every higher coefficient has norm less than M. This proves the assertion uniformly in x. □

Lemma 1.15.4 (The maximal-order locus)**.** *Let $s \geq 0$ be an integer. Suppose $F \in A\langle T', T_n\rangle$ is T_n-distinguished of some order at most s at every maximal ideal of A. Write $F = \sum_j F_j(T')T_n^j$, and let $a_j \in A$ be the constant coefficient of F_j. Then $a_0, \ldots, a_s$ generate the unit ideal, and*

$$R_s = \{x \in \mathcal{M}(A) : |a_j(x)| \leq |a_s(x)| \ (0 \leq j < s)\}$$

is a strict rational domain. Its maximal residue points are exactly those at which F has order s. On the rational quotient algebra A_s, the image of F has order s at every maximal residue point. For $s = 0$ this domain is all of $\mathcal{M}(A)$.

Proof. At a maximal point x of order $r \leq s$, Lemma 1.15.2 gives

$$\begin{aligned} |a_j(x)| \leq \|F_{j,x}\| \leq |a_r(x)| \quad (j \leq r), \\ |a_j(x)| \leq \|F_{j,x}\| < |a_r(x)| \quad (j > r), \end{aligned} \qquad a_r(x) \neq 0.$$

Thus the $a_0, \ldots, a_s$ have no common maximal zero and generate the unit ideal by Theorem 1.12.3. The strict rational construction therefore applies. The displayed inequalities show that a maximal point belongs to R_s exactly when its order is s. Every maximal residue field of A_s is finite over K; its restriction to A has image a finite-dimensional domain over K, hence a field by [StacksAlg, Tag 00GS(1), Lemma 10.36.19]. It therefore gives a maximal point of A in R_s. The resulting residue-field extension is finite and preserves the absolute value by Lemma 1.12.1. Extension of this field does not change any coefficient norm or the strict inequalities defining the order. This proves the final assertion. If $s = 0$, the same argument makes a_0 a unit and the rational quotient is A. □

Proposition 1.15.5 (Finiteness from a fixed fibrewise order)**.** *Let $n \geq 1$ and $C = A\langle T_1, \ldots, T_{n-1}\rangle$. If $F \in C\langle Z\rangle$ is Z-distinguished of the same order s at every maximal ideal of A, then the natural map*

$$C \longrightarrow D = C\langle Z\rangle/(F)$$

is finite. The quotient norm on D is uniformly equivalent to its canonical finite-C-module norm. If $s = 0$, then $D = 0$.

Proof. The zero coefficient algebra causes no difficulty. Write $F = \sum_{j\geq 0} F_j Z^j$. Any maximal residue field of C is finite over K, and its restriction to A is a maximal residue point, by the finite-dimensional-domain argument in the preceding proof. Evaluation of F_s at such a point factors through the unit $F_{s,x} \in k_x\langle T'\rangle$. It is therefore nonzero. Theorem 1.12.3 makes F_s a unit of C. Replacing F by $F_s^{-1}F$, assume $F_s = 1$. In every maximal residue field of C one then has

$$|F_j| \leq 1 \quad (j < s), \qquad |F_j| < 1 \quad (j > s).$$

Here normalization of each fibre is legitimate because Lemma 1.15.2 gives $\|F_{s,x}^{-1}\| = \|F_{s,x}\|^{-1}$. The normalized coefficients still tend to zero in C. If $s = 0$, each fibre F_x is a unit in $k_x\langle T', Z\rangle$; the same maximal-residue argument makes F a unit globally, so $D = 0$. We may henceforth assume $s \geq 1$ and $D \neq 0$.

Write z for the image of Z in D and put $w = z^s + \sum_{j<s} F_j z^j$. At every maximal residue point y of D, the residue of z has absolute value at most one and the restriction to C is maximal. The relation $F(z) = 0$ gives

$$w(y) = -\sum_{j>s} F_j(y) z(y)^j, \qquad |w(y)| < 1.$$

The strict inequality follows because each summand has absolute value less than one, and their absolute values tend to zero; hence their maximum is less than one. Theorem 1.15.1 shows that w is topologically nilpotent.

By Theorem 1.13.4, the elements $F_0, \ldots, F_{s-1}$ of C are power bounded. Start with an admissible strict presentation of C, and append variables $Y_0, \ldots, Y_{s-1}$ mapping to these elements. Proposition 1.9.3 gives a bounded surjection $p : R \twoheadrightarrow C$, where R is a Tate algebra over K. It is admissible: the old presentation already supplies lifts with a uniform norm bound, while

boundedness of p supplies the reverse norm inequality. Proposition 2.14.1, applied coefficientwise, then gives an admissible surjection

$$\pi : R\langle Z\rangle \twoheadrightarrow D.$$

The polynomial $W = Z^s + \sum_{j<s} Y_j Z^j$ maps to w. Topological nilpotence and admissibility give an integer $r \geq 1$ for which $\|\pi(W^r)\|_\pi < 1$. By the definition of a quotient norm, choose $G \in R\langle Z\rangle$ of norm less than one with $\pi(G) = \pi(W^r)$. Then

$$H = W^r - G \in \ker \pi.$$

The polynomial W^r is monic of degree rs in Z and all its lower coefficients have norm at most one. Theorem 1.10.3, applied with perturbation $-G$, expresses every element of $R\langle Z\rangle/(H)$ as a polynomial in Z of degree less than rs, with the remainder norm no larger than the original norm. Since H lies in the kernel of π, the elements $1, z, \ldots, z^{rs-1}$ generate D over C.

For the norm assertion, lift $d \in D$ through the admissible map π with norm at most a fixed constant times $\|d\|_D$, divide the lift by H, and apply the bounded coefficient map p to the remainder. This gives an expression $d = \sum_{j<rs} c_j z^j$ with $\max_j \|c_j\|_C \leq C_1 \|d\|_D$, where C_1 is independent of d. Conversely the finite generator map $C^{rs} \to D$ is bounded. Its quotient norm is therefore uniformly equivalent to $\|\cdot\|_D$, and this quotient norm is a canonical finite-C-module norm. □

References. The strict smallness argument refines the minimal-polynomial proof of [Bosch14, 3.1, Lemmas 13–14, pp. 36–38; Theorem 17 and Corollary 18, pp. 39–40]. The field-unit criterion is [Bosch14, 2.2, Corollary 4, p. 14], equivalently [BGR84, 5.1.3, Proposition 1, pp. 193–194]. The relative coordinate, maximal-order, and finite-quotient arguments are reconstructed from [Bosch14, 4.2, Lemmas 13–14 and Proposition 15, pp. 77–79]; compare [BGR84, 7.3.5, Lemma 7, Proposition 8, and Proposition 9, pp. 311–312]. The finite-quotient proof above uses an actual small lift and the project's Banach division theorem; it does not import the reduction-theoretic finiteness criterion used in BGR.

1.16 Free polyradii and norm-controlled scalar extension

The strict theory does not by itself settle arbitrary real radii. We begin the passage to that setting with a scalar extension whose elements and bounded

lifts can be described coefficient by coefficient. The arguments below concern arbitrary Banach vector spaces; they do not use affinoid flatness, a structure sheaf, or acyclicity.

Fix a complete real-valued non-Archimedean field K, possibly trivially valued. A finite tuple $r = (r_1, \dots, r_s)$ of positive real numbers is called *free over K* if

$$r^\nu \in |K^\times|, \quad \nu \in \mathbf{Z}^s \qquad \Longrightarrow \qquad \nu = 0.$$

The empty tuple is allowed. For a Banach K-space E, write $c_0(\mathbf{Z}^s, r; E)$ for the families (e_ν) such that, for every $\epsilon > 0$, only finitely many ν satisfy $\|e_\nu\| r^\nu \geq \epsilon$. Its norm is $\sup_\nu \|e_\nu\| r^\nu$. The notation K_r initially means this space for $E = K$, with its Laurent monomials denoted by T^ν. Multiplication and its field property are proved next; they are not part of the notation. All tensor completions below use the non-Archimedean projective seminorm, followed by separation and completion.

Lemma 1.16.1 (The free-polyradius field)**.** *For every finite free tuple r, the space K_r is a complete valued extension field of K. Its multiplication extends Laurent-polynomial multiplication, its absolute value is the displayed weighted supremum norm, and $|T^\nu| = r^\nu$. Every nonzero $a = \sum_\nu a_\nu T^\nu$ has a unique dominant term, and if this term is $a_\mu T^\mu$, then*

$$a = a_\mu T^\mu (1 + h), \qquad \|h\| < 1,$$
$$a^{-1} = a_\mu^{-1} T^{-\mu} \sum_{n \geq 0} (-h)^n, \qquad \|a^{-1}\| = \|a\|^{-1}.$$

No spherical completeness, nontrivial valuation, or restriction on the characteristic is required.

Proof. We first check the underlying normed space. The vanishing condition implies boundedness: there are only finitely many terms of weight at least one, and all other terms have weight less than one. Finite-support families are dense, by retaining only the finitely many terms above a prescribed error threshold. If $(a^{(j)})$ is Cauchy for the weighted norm, each coefficient is Cauchy in K and has a limit a_ν. Given $\epsilon > 0$, choose j so that $\|a^{(j)} - a^{(k)}\| < \epsilon/2$ for every sufficiently large k. Passing to each coefficient limit gives $|a_\nu - a_\nu^{(j)}| r^\nu \leq \epsilon/2$. Outside a finite set the terms of $a^{(j)}$ have weight less than $\epsilon/2$. Thus the limiting family has weight at most $\epsilon/2 < \epsilon$ outside that set. It belongs to K_r, and the same estimate proves convergence in norm. This proves

completeness. The identical argument proves completeness of $c_0(\mathbf{Z}^s, r; E)$ whenever E is complete.

For Laurent polynomials, the ultrametric inequality and $r^{\nu+\mu} = r^\nu r^\mu$ give $\|fg\| \le \|f\|\|g\|$. Approximate each of two elements by Laurent polynomials. This inequality makes the polynomial products a Cauchy sequence and makes its limit independent of the approximations. It defines a bounded bilinear multiplication on K_r. Associativity, commutativity, distributivity and the unit law extend from the dense Laurent-polynomial subalgebra by continuity. In particular, this constructs the product without assuming an infinite convolution or its rearrangement in advance.

Let $a \neq 0$. Its positive supremum is attained: choose a nonzero term of weight d and maximize over the finite nonempty set of terms of weight at least d. Two distinct nonzero terms cannot have the same weight. Such an equality would put $r^{\nu-\mu} = |a_\mu/a_\nu|$ in $|K^\times|$, contrary to freeness. Let $a_\mu T^\mu$ be the unique maximizing term. After deleting it, the remaining terms have supremum strictly less than its weight. Indeed, terms of weight at least half the maximal weight form a finite set, and the others are uniformly smaller. Multiplication by the invertible monomial $a_\mu^{-1} T^{-\mu}$ shifts the coefficients and rescales the norm exactly, so it gives $a = a_\mu T^\mu(1 + h)$ with $\|h\| < 1$.

Submultiplicativity implies $\|h^n\| \le \|h\|^n$, so the geometric series converges. Its finite-sum identity proves that it is the inverse of $1+h$. This inverse equals $1+u$ with $\|u\| < 1$, hence has norm one. The asserted formula for a^{-1} and its norm follow. Thus every nonzero element is invertible. Finally, for $a, b \neq 0$, submultiplicativity applied to $a^{-1}(ab) = b$ gives

$$\|a\|\|b\| \le \|ab\| \le \|a\|\|b\|.$$

The norm is therefore multiplicative. Constant families embed K isometrically, as claimed. □

Theorem 1.16.2 (The coefficient description of completed scalar extension)**.** *Let r be finite and free over K, and let E be any Banach K-space with its given definite homogeneous ultrametric norm. There is a canonical isometric K-linear isomorphism*

$$\Phi_E : E\widehat{\otimes}_K K_r \xrightarrow{\sim} c_0(\mathbf{Z}^s, r; E), \qquad e \otimes \sum_\nu a_\nu T^\nu \longmapsto (a_\nu e)_\nu.$$

It gives $E\widehat{\otimes}_K K_r$ its complete K_r-space norm. The constant-coefficient embedding $\iota_E : E \to E\widehat{\otimes}_K K_r$ is isometric and has a contractive K-linear retraction

p_0 taking the coefficient of T^0. For every bounded K-linear map $u : E \to F$, scalar extension acts coefficientwise and commutes with ι and p_0. The retraction is not asserted to be K_r-linear.

Proof. A finite tensor sum $z = \sum_{j=1}^{m} e_j \otimes f_j$, with $f_j = \sum_\nu a_{j\nu} T^\nu$, has coefficient family $c_\nu = \sum_j a_{j\nu} e_j$. It satisfies the vanishing condition and

$$\sup_\nu \|c_\nu\| r^\nu \le \max_j \|e_j\| \|f_j\|.$$

The bilinear rule is balanced, hence defined on the algebraic tensor product. Taking the infimum over every finite tensor representation gives a contraction for the projective seminorm. It kills the null space, so the completeness proved in Lemma 1.16.1 extends it to a contraction Φ_E on the separated completion.

Conversely, a finite-support family (e_ν) maps to $\sum_\nu e_\nu \otimes T^\nu$ in the separated completed tensor. The projective estimate bounds its norm by $\max_\nu \|e_\nu\| r^\nu$. Density of finite-support families therefore gives a contraction Ψ_E in the opposite direction. On finite-support families, $\Phi_E \Psi_E$ is the identity. On a pure tensor $e \otimes f$, finite Laurent truncations f_J converge to f and

$$\|e \otimes (f - f_J)\|_\pi \le \|e\| \|f - f_J\| \longrightarrow 0.$$

Thus $\Psi_E \Phi_E$ is the identity on pure tensors, hence on their finite sums and, by density, on the completion. Two mutually inverse contractions are isometries. This proves the displayed isomorphism without assuming that an arbitrary Banach tensor map is injective.

Multiplication on the second tensor factor extends to a bounded K_r-action: multiplication by $b \in K_r$ has bound $|b|$ already on every finite tensor representation. For $b \neq 0$, applying the same bound to b^{-1} gives equality of norms, so the action is homogeneous. The constant-family inclusion is isometric and the zeroth coefficient has norm at most the family norm, with $p_0 \iota_E = \mathrm{id}_E$. A bounded map u takes each coefficient to $u(e_\nu)$ and has the same available bound; this identity holds first on pure tensors and then by continuity. It also proves the two asserted commutation relations. □

Theorem 1.16.3 (Preservation and reflection of strict exactness)**.** *Let r be finite and free over K. For any composable bounded maps $E \xrightarrow{i} F \xrightarrow{q} G$ of Banach K-spaces with $qi = 0$, the sequence*

$$0 \longrightarrow E \xrightarrow{i} F \xrightarrow{q} G \longrightarrow 0$$

is norm-strict exact if and only if its separated completed scalar extension to K_r is norm-strict exact. More precisely, suppose $C, D > 0$, $\|e\| \leq D\|i(e)\|$, and every $g \in G$ has a lift $f \in F$ with $\|f\| \leq C\|g\|$. The same constants then work after scalar extension. Conversely, any such constants after scalar extension work before extension. In particular scalar extension detects zero spaces and reflects isomorphisms with bounded inverse. No statement about uncompleted tensor flatness is part of this theorem.

Proof. By the definition of norm-strictness, a strict exact sequence admits positive constants of the stated kind. If a quotient-norm infimum is not attained, enlarge the norm-comparison constant to obtain actual lifts. No open-mapping theorem is needed for this passage.

Use the coefficient isometries of Theorem 1.16.2. Suppose first that the original sequence is norm-strict exact. A coefficient family in the kernel of q_r has all its coefficients in $\ker q = \operatorname{im} i$. Their unique lifts to E have weighted norms bounded by D times the corresponding original weighted norms. They therefore form an element of $c_0(\mathbf{Z}^s, r; E)$. This proves middle exactness and the bound for the inverse of i_r on its image; injectivity follows coefficientwise. For $(g_\nu) \in c_0(\mathbf{Z}^s, r; G)$, choose f_ν with $q(f_\nu) = g_\nu$ and $\|f_\nu\| \leq C\|g_\nu\|$, choosing $f_\nu = 0$ when $g_\nu = 0$. The same estimate shows that (f_ν) belongs to $c_0(\mathbf{Z}^s, r; F)$. This proves surjectivity of q_r with the stated lifting bound. Since its kernel is closed, these facts give norm-strict exactness.

Conversely, suppose the extended sequence is norm-strict exact. Constant-family embeddings show that i is injective. If $f \in \ker q$, exactness after extension gives an $e' \in E\widehat{\otimes}_K K_r$ with $i_r(e') = \iota_F(f)$. Apply the zeroth-coefficient map: naturality gives $i(p_0 e') = f$. Thus $\ker q = \operatorname{im} i$. A lift of $\iota_G(g)$ through q_r, followed by p_0, similarly lifts any g through q. If the extended lift has norm at most $C\|g\|$, its zeroth coefficient has norm at most the same quantity. Applying an extended inverse bound for i_r to a constant family yields $\|e\| \leq D\|i(e)\|$. These are precisely the required bounds. Again $\operatorname{im} i = \ker q$ is closed.

The isometric constant-family embedding detects zero spaces. To see the last assertion directly for a bounded map $u : E \to F$ whose extension is an

isomorphism with bounded inverse, lift every constant $\iota_F(f)$ through u_r and apply p_0; this proves surjectivity and the inverse bound for u. Its injectivity follows from the constant embedding and injectivity of u_r. This argument uses the explicit Banach-space retraction, not an unproved algebraic faithful-flatness statement. □

These results provide the norm-faithful extension step needed for arbitrary radii. Theorem 1.17.6 supplies simultaneous strictification, Proposition 1.18.7 compares ordinary and completed finite-module tensors, and Theorem 2.36.9 identifies affinoid domains after arbitrary complete-extension base change. Together these interfaces let the strict theory descend to the original field and radii.

1.17 Actual presentations and simultaneous strictification

To use the strict theory at arbitrary radii, we need a presentation after scalar extension before invoking any finiteness theorem for the original algebra. We therefore begin with coefficient spaces and quotients by ideals already known to be closed. Only after the extended Tate algebra is strict will we identify a closed extension ideal with its algebraic extension.

Throughout, K is complete for a fixed real-valued non-Archimedean absolute value, possibly trivial. Banach norms are definite, complete, ultrametric and homogeneous over the ground field. Algebra norms are submultiplicative with $\|1\| = 1$; the zero algebra is treated separately with its zero norm. Finite products carry the maximum norm, with maximum zero for an empty product. Norm-admissibility means uniform equivalence to the specified quotient norm, as in Definition 1.9.1. All completed tensor products are separated completions for the non-Archimedean projective seminorm. We use the notation $\mathcal{C}_w(I, E) = c_0(I, E; w)$ of Proposition 2.14.1.

References. Berkovich's scalar-extension method is described in [TemIntro, Remark 3.1.2.4(ii), p. 12]; the radius-field presentation is stated in [TemIntro, Example/Exercise 3.1.1.4, p. 11]. These are a construction and a route to strictification. The coefficient estimates, actual relation images and quotient identifications needed here are proved below.

Lemma 1.17.1 (Rearranging weighted vanishing families)**.** *Let I, J be countable sets, let $u_i, v_j > 0$, and let E be a Banach K-space. There is a natural isometric isomorphism*

$$\mathcal{C}_u\big(I, \mathcal{C}_v(J, E)\big) \xrightarrow{\sim} \mathcal{C}_{u\boxtimes v}(I \times J, E), \qquad (u \boxtimes v)_{ij} = u_i v_j,$$

which sends a family of rows to its joint coefficient family. Consequently any finite number of such coefficient constructions can be regrouped or permuted, preserving norms and the finite-tail condition. These identifications commute with bounded coefficient maps. For coefficient algebras or modules indexed by products of copies of $\mathbf{N}$ and $\mathbf{Z}$, with multiplicative monomial weights, they also preserve the convolution products and module actions. Empty index sets and empty products are allowed.

Proof. Write the joint coefficients as e_{ij} and put $b_i = \sup_j v_j\|e_{ij}\|$. Suppose first that the joint weighted family tends to zero outside finite subsets. Each fixed row belongs to $\mathcal{C}_v(J, E)$, since $u_i > 0$. If $u_i b_i \geq \epsilon > 0$, some entry in that row satisfies $u_i v_j\|e_{ij}\| > \epsilon/2$. There are only finitely many such entries, and hence only finitely many such rows. Thus $(\mathrm{row}_i)_i$ satisfies the outer finite-tail condition.

Conversely, suppose that every row has a vanishing weighted tail and that $(u_i b_i)_i$ tends to zero outside finite subsets. An entry of joint weight at least ϵ lies in a row with $u_i b_i \geq \epsilon$. There are finitely many such rows. In each of them only finitely many entries have joint weight at least ϵ, so the union of these finite sets is finite. This proves the converse. Moreover,

$$\sup_i u_i \sup_j v_j\|e_{ij}\| = \sup_{(i,j)\in I\times J} u_i v_j\|e_{ij}\|.$$

The isomorphism and norm equality follow. Completeness is supplied by Lemma 2.13.1. Repeated application proves the finite regrouping assertion; permutations preserve the joint weights and the same finite subsets. A coefficient map acts on the same e_{ij} in either description and retains its given bound. Finally, finite coefficient convolution is unchanged by these bijections of multiindices. Finite-support families are dense, and Lemma 2.14.3 bounds products and actions. Passing to limits proves the algebra and module assertions. The zero spaces arising from empty index sets and the one-element index set for an empty product satisfy the same formulas. □

Theorem 1.17.2 (The actual multi-Laurent relation quotient)**.** *Let $r = (r_1, \dots, r_d)$ be any finite tuple of positive real numbers, without a freeness assumption, and let E be a Banach K-space. Put*

$$D_r(E) = E\{r^{-1}T, rU\} = \mathcal{C}_{r^{\alpha-\beta}}(\mathbf{N}^d \times \mathbf{N}^d, E),$$
$$L_r(E) = \mathcal{C}_{r^{\nu}}(\mathbf{Z}^d, E), \qquad P_i = T_iU_i - 1.$$

There are K-linear contractions $N : D_r(E) \to L_r(E)$ and $Q_i : D_r(E) \to D_r(E)$, and a K-linear isometric section $S : L_r(E) \to D_r(E)$, such that

$$N(T^{\alpha}U^{\beta}e) = Z^{\alpha-\beta}e, \qquad S\left(\sum_{\nu} e_{\nu}Z^{\nu}\right) = \sum_{\nu} e_{\nu}T^{\nu_+}U^{\nu_-},$$

$$NS = 1, \qquad NP_i = 0, \qquad F - SNF = \sum_{i=1}^{d} P_iQ_iF.$$

Here $(\nu_+)_i = \max(\nu_i, 0)$ and $(\nu_-)_i = \max(-\nu_i, 0)$. In particular,

$$\ker N = \sum_{i=1}^{d} P_iD_r(E), \qquad D_r(E)\Big/\sum_{i=1}^{d} P_iD_r(E) \;\simeq\; L_r(E)$$

with an actual finite sum of relation images and the exact quotient norm. This relation sum is closed. The map $D_r(E)^d \to \ker N$, $(F_i) \mapsto \sum_i P_iF_i$, has a contractive linear right inverse. The quotient norm is attained at SNF. These maps are natural for bounded coefficient maps. For a Banach algebra A and a contractive Banach A-module E, they are A-linear; the quotient identification respects the corresponding algebra map and module action. If r is free over K, they give, in particular, the actual presentations

$$K_r \simeq K\{r^{-1}T, rU\}/(P_1, \dots, P_d),$$
$$E\widehat{\otimes}_K K_r \simeq E\{r^{-1}T, rU\}\Big/\sum_i P_iE\{r^{-1}T, rU\},$$

isometrically, with the field and module structures of Lemma 1.16.1 and Theorem 1.16.2. For $d = 0$ the relations are absent and all assertions reduce to the identity on E.

Proof. Lemma 1.17.1 permits us to regard all pairs except (T_i, U_i) as coefficients. Apply Theorem 2.13.2 to this remaining pair with $a = b = r_i$. Its operators have bounds $\|\mathcal{N}_i\| \le 1$, $\|\mathcal{H}_i\| \le 1$, and $\mathcal{S}_i$ is isometric. On the

original space $D_r(E)$ put $R_i = \mathcal{S}_i\mathcal{N}_i$ and $H_i = \mathcal{H}_i$, leaving the other indices unchanged. Then

$$1 - R_i = P_i H_i, \qquad \|R_i\|, \|H_i\| \le 1.$$

Operators belonging to distinct pairs commute. On finite coefficient polynomials this follows directly from their separate-index formulas; their bounds extend that equality to the complete space. Iterating the normal-form maps and applying the coefficient rearrangement lemma gives a contraction N with the asserted substitution on polynomials. The displayed axis formula defines S: it preserves every coefficient and its weight, so it is isometric and has $NS = 1$. On polynomials, and then by continuity,

$$SN = R_1 \cdots R_d, \qquad NP_i = 0.$$

The finite telescoping identity

$$1 - R_1 \cdots R_d = \sum_{i=1}^{d} R_1 \cdots R_{i-1}(1 - R_i) = \sum_{i=1}^{d} P_i R_1 \cdots R_{i-1} H_i$$

therefore proves the claimed decomposition with $Q_i = R_1 \cdots R_{i-1} H_i$. Each Q_i is contractive. If $NF = 0$, the decomposition represents F as an actual finite sum of relation multiples, with $\max_i \|Q_i F\| \le \|F\|$. The reverse inclusion in the kernel follows from $NP_i = 0$. Thus the relation sum equals the closed kernel of N, and $F \mapsto (Q_i F)_i$ on that kernel is the asserted right inverse. No uniqueness of the individual relation coefficients is claimed.

For every representative of a quotient class, the contraction N bounds the norm of its image by the norm of that representative. The representative SNF lies in the same class and has norm exactly $\|NF\|$. Taking the infimum proves the isometric quotient assertion and attainment. Naturality follows from the one-pair naturality in Proposition 2.13.3 and the rearrangement lemma. Products and actions are bounded coefficient convolutions by Lemma 2.14.3. Substitution preserves them on finite polynomials, hence on their completions. For a free tuple, the resulting coefficient algebra is exactly the field already constructed in Lemma 1.16.1; the module comparison is exactly Theorem 1.16.2. This proves the final assertions without any affinoid finiteness input. □

Proposition 1.17.3 (Closed presentations after adjoining coefficients)**.** *Let D be a Banach K-algebra, let $I \subset D$ be a closed algebraic ideal, and give $A = D/I$ its quotient norm. Let $\Gamma = \mathbf{N}^p \times \mathbf{Z}^q$, with finite $p, q \geq 0$, and let $w_0 = 1$, $w_\gamma > 0$, $w_{\gamma+\delta} \leq w_\gamma w_\delta$. Applying $D \to A$ coefficientwise gives a norm-strict surjection of Banach algebras, with*

$$\ker\big(\mathcal{C}_w(\Gamma, D) \to \mathcal{C}_w(\Gamma, A)\big) = \mathcal{C}_w(\Gamma, I) = \overline{I\,\mathcal{C}_w(\Gamma, D)},$$
$$\mathcal{C}_w(\Gamma, D)\Big/\overline{I\,\mathcal{C}_w(\Gamma, D)} \;\simeq\; \mathcal{C}_w(\Gamma, A) \quad \textit{isometrically.}$$

The ideal inside the closure is the algebraic ideal generated by the constant image of I; no finite generation of I is assumed. If A is equipped instead with a norm uniformly equivalent to its quotient norm, the induced quotient isomorphism is bounded with bounded inverse, with those norm-comparison constants. In particular, if $D = K\{s^{-1}X\}$ and ρ is any finite tuple of positive radii, then $A\{\rho^{-1}Y\}$ is norm-admissible generalized affinoid, with presentation

$$A\{\rho^{-1}Y\} \simeq K\{s^{-1}X, \rho^{-1}Y\}\Big/\overline{IK\{s^{-1}X, \rho^{-1}Y\}}.$$

Any subsequent quotient by a closed ideal is again generalized affinoid with its quotient norm. These statements allow zero quotients, and do not assert that any unclosed extension ideal is already closed.

Proof. The closed ideal I, with its induced norm, is complete. Lemma 1.11.1 makes A complete. The quotient norm is submultiplicative by approximating representatives of two factors; if $A \neq 0$, its unit has norm one, since $0 < \|1_A\| \leq 1$ and $\|1_A\| \leq \|1_A\|^2$. For each fixed $\lambda > 1$, every nonzero quotient coefficient has a lift of norm at most λ times its norm; take the zero lift for zero. Proposition 2.14.1 therefore gives the surjection, its coefficient kernel and the exact quotient norm. Lemma 2.14.3 shows that this map is an algebra homomorphism: it is so on finite-support families, and all maps and products are continuous.

Its kernel is closed and contains $I\,\mathcal{C}_w(\Gamma, D)$, so it contains the closure of that ideal. Conversely, every family with coefficients in I is a norm limit of finite sums $\sum_{\gamma \in F} i_\gamma X^\gamma$. Each summand is the product of the constant $i_\gamma \in I$ and the monomial $X^\gamma \in \mathcal{C}_w(\Gamma, D)$, and hence belongs to the algebraic extension ideal. This proves both inclusions. This argument proves an equality with a closure, not an unproved closed-image assertion.

Uniform comparison of two coefficient norms follows by multiplying the coefficient inequalities by w_γ and taking their supremum. For $D = K\{s^{-1}X\}$ and $\Gamma = \mathbf{N}^n$ with $w_\beta = \rho^\beta$, Lemma 1.17.1 identifies the source isometrically with the joint Tate algebra in X, Y. The displayed quotient is therefore a generalized affinoid presentation. If J is a closed ideal of this quotient, its inverse image under the continuous Tate presentation is closed. The composite quotient is norm-admissible: lift through the two quotient maps with any fixed slack factor greater than one, or take successive infima over representatives. The resulting infimum is the direct quotient norm. This proves the last assertion without a Noetherianity or ordinary flatness hypothesis. □

Proposition 1.17.4 (Presentations after a free-radius extension)**.** *Let A be a norm-admissible generalized K-affinoid with a specified presentation $D = K\{s^{-1}X\} \twoheadrightarrow A$ and closed kernel I. Let r be any finite tuple free over K, put $L = K_r$, and put $A_r = A\widehat{\otimes}_K L$, $D_r = D\widehat{\otimes}_K L$. Then*

$$\begin{aligned} D_r &\simeq L\{s^{-1}X\}, \\ \ker(D_r \to A_r) &= \mathcal{C}_{r^\nu}(\mathbf{Z}^d, I) = \overline{ID_r}, \\ A_r &\simeq L\{s^{-1}X\}/\overline{IL\{s^{-1}X\}}. \end{aligned}$$

The first map is isometric. The quotient map induces precisely the coefficient norm on A_r if A has the displayed quotient norm; for any given uniformly equivalent norm the comparison has the same two constants. Thus A_r is generalized L-affinoid. It is also generalized K-affinoid and has the actual relative presentation

$$A_r \simeq A\{r^{-1}T, rU\}/(T_iU_i - 1)_{1\le i\le d}.$$

This last identification is isometric for the specified norm of A. These constructions are natural for bounded K-algebra maps, so every finite commutative diagram of such maps extends to a commutative diagram of bounded L-algebra maps. No strictification of A_r is asserted for an arbitrary choice of r.

Proof. Theorem 1.16.2 identifies D_r with $\mathcal{C}_{r^\nu}(\mathbf{Z}^d, D)$. Applying Lemma 1.17.1 to the X- and Laurent coefficients gives

$$\mathcal{C}_{r^\nu}\big(\mathbf{Z}^d, \mathcal{C}_{s^\alpha}(\mathbf{N}^n, K)\big) \simeq \mathcal{C}_{s^\alpha}\big(\mathbf{N}^n, \mathcal{C}_{r^\nu}(\mathbf{Z}^d, K)\big) = L\{s^{-1}X\}.$$

On finite coefficient polynomials this is the identity on the monomials $X^\alpha T^\nu$. The convolution bounds show that it is an isometric algebra isomorphism. In

particular it agrees with the algebra structure on the separated completed tensor: the product on finite tensors has the projective bound, and the coefficient description sends it to the same product on a dense subspace. The L-action and its homogeneous norm are those of Theorem 1.16.2.

First give A its quotient norm from D. Proposition 1.17.3, applied with $\Gamma = \mathbf{Z}^d$ and weight r^ν, proves the kernel equality and exact quotient norm. The same coefficient isometry identifies the target with A_r. Restoring the original norm of A preserves the kernel and changes norms by precisely its uniform comparison constants. This gives the displayed generalized L-affinoid presentation without assuming any ideal of D other than the specified I to be closed.

Theorem 1.17.2, with coefficient algebra A, gives the actual relative relation quotient, including its closed kernel and exact norm. By Proposition 1.17.3, $A\{r^{-1}T, rU\}$ is generalized K-affinoid and its quotient by this closed ideal is again generalized. Finally, a bounded algebra map acts on every Laurent coefficient with its original bound, by Theorem 1.16.2. It respects products on finite Laurent sums and therefore on their limits. Identities and compositions are coefficientwise, proving the diagram assertion. □

Lemma 1.17.5 (A free extension for finitely many radii)**.** *Let $s_1, \dots, s_m > 0$ be any finite list and put $\Gamma_K^{\mathrm{div}} = \{t > 0 : t^n \in |K^\times| \text{ for some } n \geq 1\}$. There exist a finite tuple $r = (r_1, \dots, r_d)$ free over K, a common integer $N \geq 1$, elements $c_j \in K^\times$ and vectors $\mu_j \in \mathbf{Z}^d$ such that*

$$s_j^N = |c_j| r^{\mu_j} \qquad (1 \leq j \leq m).$$

The tuple may be chosen from the prescribed radii whose classes form a basis modulo Γ_K^{div}. For $L = K_r$,

$$|L^\times| = |K^\times| r^{\mathbf{Z}^d}, \qquad \Gamma_L^{\mathrm{div}} = \Gamma_K^{\mathrm{div}} r^{\mathbf{Q}^d}, \qquad s_j \in \Gamma_L^{\mathrm{div}}.$$

The empty tuple is allowed. These conclusions preserve the given absolute value on K, allow trivial valuation, and do not generally put s_j in $|L^\times|$ itself.

Proof. Let $\Lambda = \log |K^\times| \subset \mathbf{R}$ and let $\Lambda_{\mathbf{Q}}$ be its rational span. Since Λ is an additive subgroup, clearing finitely many denominators gives $\Lambda_{\mathbf{Q}} = \bigcup_{n \geq 1} n^{-1}\Lambda = \log \Gamma_K^{\mathrm{div}}$. The quotient $\mathbf{R}/\Lambda_{\mathbf{Q}}$ is a vector space over $\mathbf{Q}$. Choose a basis from the finite list of classes $\log s_j$ for their span, and let $r_1, \dots, r_d$

be the corresponding radii. If $r^\nu \in |K^\times|$ for $\nu \in \mathbf{Z}^d$, the basis classes satisfy an integer relation, hence $\nu = 0$. Thus r is free over K.

For each j there are $q_{ji} \in \mathbf{Q}$ with $\log s_j - \sum_i q_{ji} \log r_i \in \Lambda_{\mathbf{Q}}$. Clear all these finitely many denominators, including those needed to put the remainders in Λ, by one integer N. Then $\mu_{ji} = N q_{ji}$ is integral and $N \log s_j - \sum_i \mu_{ji} \log r_i \in \Lambda$. Choosing $c_j \in K^\times$ with this logarithmic absolute value proves the displayed identity.

By Lemma 1.16.1, every nonzero element of L has a unique dominant Laurent monomial aT^ν, and its norm is $|a|r^\nu$. Conversely every such value occurs on that monomial. This proves the equality for $|L^\times|$. Taking divisible closures gives the second equality, or equivalently one can again clear the finitely many denominators in an expression tr^q. The scalar $c_j T^{\mu_j} \in L^\times$ has absolute value s_j^N, proving the last inclusion. The original field embeds isometrically by the same lemma.

If K is trivially valued, then $\Gamma_K^{\mathrm{div}} = \{1\}$ and the same finite-dimensional linear-algebra argument applies in $\mathbf{R}$. If every prescribed radius is one, take the empty tuple; otherwise the chosen tuple contains a radius different from one and L is nontrivially valued. The distinction between the value group and its divisible closure remains necessary: if $|K^\times| = q^{\mathbf{Z}}$ with $0 < q < 1$, the single radius $q^{1/2}$ already belongs to Γ_K^{div}. The empty tuple then suffices, while $q^{1/2} \notin |K^\times|$. □

Theorem 1.17.6 (Simultaneous strictification of closed presentations)**.** *Fix a finite family of norm-admissible generalized K-affinoids A_j, each with a specified presentation*

$$D_j = K\{s_j^{-1}X_j\} \twoheadrightarrow A_j, \qquad I_j = \ker(D_j \to A_j) \text{ closed}, \qquad 1 \le j \le m.$$

Also fix any finite list of additional positive radii $\eta_1, \dots, \eta_t$. There is a finite K-free tuple r, with $L = K_r$, such that every entry of every s_j and every η_k belongs to Γ_L^{div}. For this one extension, each $D_{j,r} = D_j \widehat{\otimes}_K L$ and each nonzero $A_{j,r} = A_j \widehat{\otimes}_K L$ is strict norm-admissible over L, and

$$D_{j,r} \simeq L\{s_j^{-1}X_j\},$$
$$\ker(D_{j,r} \to A_{j,r}) = \mathcal{C}_{r^\nu}(\mathbf{Z}^d, I_j) = \overline{I_j D_{j,r}} = I_j D_{j,r},$$
$$A_{j,r} \simeq D_{j,r}/I_j D_{j,r}.$$

Here $I_j D_{j,r}$ is the actual algebraic extension ideal; it is finitely generated and closed. The last quotient has exactly the coefficient norm if A_j has its

displayed quotient norm, and uniformly equivalent norms otherwise. Zero quotients use the zero-norm convention and are detected by the isometric constant embedding. Any finite diagram of bounded maps between the A_j extends over this same L. The theorem does not assume ordinary flatness of $A_j \to A_{j,r}$ or Noetherianity of the original A_j, and it makes neither conclusion part of the statement.

Proof. Apply Lemma 1.17.5 to the finite union of the presentation radii and the additional radii. It supplies one free tuple r and puts all these radii in Γ_L^{div}. Proposition 1.17.4 identifies $D_{j,r}$ isometrically with the weighted Tate algebra $L\{s_j^{-1}X_j\}$. Since its radii lie in the divisible closure of the value group of L, Lemma 2.26.1, applied over the complete field L with coefficient algebra L, makes this algebra strict norm-admissible. This step uses the radical-radius finite-free construction, not a rescaling that would require the radii themselves to lie in $|L^\times|$.

Only now apply Theorem 1.11.5 to $D_{j,r}$. Its algebraic ideal $I_jD_{j,r}$ is finitely generated and closed, irrespective of whether the original ideal I_j has been shown finite. Proposition 1.17.4 already identified the kernel with $\overline{I_jD_{j,r}}$, so closedness upgrades that equality to the actual ideal equality in the statement. The quotient assertion of Lemma 2.26.1 makes the resulting nonzero quotient strict, and covers zero quotients separately. The exact coefficient-quotient norm, and its uniform comparison with the original norm, are those already proved in Proposition 1.17.4; no new completion or unspecified topology is introduced.

The isometric constant embedding of Theorem 1.16.2 proves $A_{j,r} = 0$ if and only if $A_j = 0$. Naturality in Proposition 1.17.4 extends all specified bounded maps and preserves their identities and compositions. The choice of r depends only on the finite list of radii, so the same field works for the whole diagram. Every step allows trivial valuation on K. The closedness theorem has been used exclusively over the already strict algebras $D_{j,r}$, which is the order needed for subsequent descent of finiteness. □

1.18 Finite modules at arbitrary affinoid radii

A free-radius extension provides two useful operations: extension to a strict affinoid algebra and extraction of the constant coefficient. We use the second operation to descend generators and their norm bounds. This establishes the

finite-module theory before any assertion of algebraic flatness. The tensor comparisons and faithful flatness then follow in that order.

Throughout, K is complete for its fixed real-valued non-Archimedean absolute value, possibly trivial. A generalized K-affinoid algebra means the norm-admissible algebra of Definition 1.4.1, with arbitrary finite positive radii. All admissible norms are uniformly equivalent to the defining quotient norms, as in Definition 1.9.1. Banach algebra norms are homogeneous and submultiplicative, with $\|1\| = 1$; Banach module actions are contractive unless an equivalent norm is explicitly allowed. Finite powers have the maximum norm. The existing zero-quotient convention is retained; statements involving a zero algebra or zero module have their evident zero interpretation.

Lemma 1.18.1 (Coefficient retractions and constant matrices)**.** *Let A be a Banach K-algebra, M a Banach A-module, and r a finite tuple free over K. Write*

$$A_r = A\widehat{\otimes}_K K_r, \qquad M_r = M\widehat{\otimes}_K K_r.$$

The coefficient isometries of Theorem 1.16.2 identify A_r with a Banach K_r-algebra and M_r with a Banach A_r-module, with convolution multiplication and action. The constant embeddings ι_A, ι_M are isometric. Their zeroth-coefficient retractions p_0^A, p_0^M are contractive and A-linear, and satisfy

$$p_0^M \iota_M = \mathrm{id}_M, \qquad p_0^M\big(b\,\iota_M(m)\big) = p_0^A(b)m \quad (b \in A_r,\ m \in M).$$

A bounded A-linear map acts coefficientwise after extension and becomes A_r-linear. In particular, a finite matrix with entries in A acts coefficientwise on A_r^m, and commutes with the constant embeddings and zeroth-coefficient maps. None of the retractions is asserted to be K_r-linear or multiplicative.

Proof. For two finite tensor representations, submultiplicativity in A and multiplicativity in K_r bound the product by the product of their maximum tensor bounds. Taking both representation infima proves submultiplicativity for the projective seminorm. The zero-seminorm space is consequently an ideal, and multiplication extends to the separated completion. The same estimate with the module action constructs the completed action on M_r. These operations agree with Laurent multiplication and action on finite-support coefficient families. Lemma 2.14.3, with index group $\mathbf{Z}^s$ and weight r^ν, and density of finite supports identify them with convolution on all families.

Theorem 1.16.2 supplies the isometries, completeness, the homogeneous K_r-norms and the coefficientwise rule for bounded maps. Multiplication by

a constant $a \in A$ multiplies each coefficient by a. Likewise, the zeroth coefficient of $b\iota_M(m)$ is $b_0 m$. These observations give all the displayed retraction identities. For a bounded A-linear map, A_r-linearity is checked on finite Laurent families and passes to limits by boundedness. Rearranging coefficients in a finite power preserves its maximum norm, so the matrix assertions follow. □

Theorem 1.18.2 (Closed submodules and controlled matrix images)**.** *Let A be a generalized K-affinoid algebra. For every finite $m \geq 0$, every algebraic submodule $N \subset A^m$ is finitely generated and closed. For every chosen finite generating list $n_1, \ldots, n_t$ of N, there is a constant $C > 0$ such that*

$$\forall n \in N \quad \exists a_1, \ldots, a_t \in A : \quad n = \sum_i a_i n_i, \qquad \max_i \|a_i\| \leq C\|n\|.$$

In particular, every finite matrix over A has closed image and a uniform bound for actual lifts from the induced image norm. The constants may depend on the matrix and the chosen norms, but not on the element lifted. Moreover, A is Noetherian.

Proof. We may suppose $A \neq 0$. Theorem 1.17.6 supplies a finite free tuple r such that $R = A_r$, with its coefficient norm, is a strict norm-admissible K_r-affinoid algebra. This step uses the defining presentation of A, not Noetherianity of A. Theorem 1.11.5 therefore applies to R with the indicated norm.

Let N_R be the algebraic R-span of $\iota_A^m(N)$ in R^m. It is finite and closed by the strict finite-module theorem. Choose finitely many generators of N_R and expand each as a finite R-linear combination of elements of N. The union of these finite lists gives $n_1, \ldots, n_t \in N$ that generate N_R over R. If $n \in N$, write $\iota_A^m(n) = \sum_i b_i \iota_A^m(n_i)$ and apply p_0 coordinatewise. Lemma 1.18.1 gives

$$n = \sum_i p_0^A(b_i) n_i.$$

Thus N is finite. The same argument applied to any constant vector lying in N_R proves

$$N = (\iota_A^m)^{-1}(N_R).$$

Since N_R is closed and ι_A^m is continuous, N is closed. Taking $m = 1$ proves Noetherianity.

Now let $D : A^t \to A^m$ be any chosen generating matrix for N. Its scalar extension $D_r : R^t \to R^m$ has image N_R. By Theorem 1.11.5, there is $C > 0$

such that each $\iota_A^m(n)$ has a lift $b \in R^t$ with $\|b\| \le C\|n\|$. Put $a = p_0^t(b)$. The coefficient lemma gives $Da = n$ and $\|a\| \le \|b\|$. This is the asserted actual lifting bound. Finite matrices are bounded by the maximum of the norms of their entries; the reverse bound just proved is precisely norm-strictness onto the induced image. No algebraic flatness or completed-tensor exactness was used in the descent. □

Theorem 1.18.3 (Canonical norms and exactness for finite modules)**.** *Let A be a generalized K-affinoid algebra. Every finite algebraic A-module M has a definite complete norm obtained from a surjection $q : A^m \twoheadrightarrow M$. These quotient norms are uniformly equivalent; a norm uniformly equivalent to one of them is called a canonical finite-module norm. Every algebraic submodule of M is finite and closed, and its induced norm is canonical. Every A-linear map between finite modules with canonical norms is bounded, has closed image, and is norm-strict onto its image. In particular every short exact sequence of finite modules is a norm-strict exact sequence of Banach K-spaces. These assertions include trivially valued K.*

Proof. **Construct and compare the norms.** By Theorem 1.18.2, $\ker q \subset A^m$ is finite and closed. Lemma 1.11.1 makes its quotient norm definite and complete. The action is contractive: a lift v of m gives the lift av of am, and taking infima in $\|av\| \le \|a\|\|v\|$ gives the claim.

For two surjections onto M, lift each finite generating list through the other surjection. The resulting finite matrices are bounded. For a nonzero m, choose a lift with norm less than twice its quotient norm and apply the appropriate matrix. This bounds either quotient norm by a constant times the other. The same argument shows that uniformly equivalent choices of the affinoid norm give equivalent module norms; it uses quantitative equivalence, not equality of topologies.

Submodules. Work first with the quotient norm from q. For $N \subset M$, let $E = q^{-1}(N) \subset A^m$. The preceding theorem supplies finite generators v_i of E and an induced-norm coefficient bound C_E. Their images $n_i = q(v_i)$ generate N. Every nonzero $n \in N$ has a lift $v \in E$ with $\|v\| < 2\|n\|$; expressing v in the v_i shows that n has coefficients of norm at most $2C_E\|n\|$. Lemma 1.11.2 makes N closed in M. The generator map $A^t \to N$ is bounded and has this lifting bound, so its quotient norm and the induced norm on N are uniformly equivalent. The quotient norm is complete by the first paragraph. Thus the induced norm is canonical.

Maps. An algebraic map $u : M \to M'$ is bounded by expressing an element of M through a controlled lift to a finite free presentation and bounding the images of its generators. Its image P is a finite closed submodule of M' by the preceding paragraph. Choose controlled generators p_i of P and arbitrary lifts $m_i \in M$. If $p = \sum_i a_i p_i$ with $\max_i \|a_i\| \leq C_P \|p\|$, then $\sum_i a_i m_i$ lifts p and has norm at most $C_P \max_i \|m_i\| \|p\|$. This proves norm-strictness. Its kernel is finite and closed as well. Replacing any chosen quotient norm by an equivalent canonical norm changes only the constants. Applying these statements to the arrows of a short exact sequence proves the last assertion. □

Corollary 1.18.4 (Actual quotients by arbitrary ideals)**.** *Every algebraic ideal J of a generalized K-affinoid algebra A is finitely generated and closed. The actual quotient A/J, with its quotient norm, is again a generalized K-affinoid algebra or the zero algebra. The quotient map is norm-strict.*

Proof. Closedness and finite generation follow from Theorem 1.18.2; completeness and controlled quotient lifts follow from Lemma 1.11.1. Choose a norm-admissible presentation $D = K\{t^{-1}X\} \twoheadrightarrow A$. The inverse image J' of J is closed, and the algebraic quotient is D/J'. Successive quotient infima are the single infimum over all lifts in D. Consequently the quotient norm on A/J is uniformly equivalent to the quotient norm from D: inequalities comparing the two norms on A remain valid after taking their quotient infima. This is the required admissible affinoid presentation. □

We can now extend finite presentations coefficientwise. The point is that the matrix relation image has already been shown to be closed with actual bounded lifts over A. No finite-module theorem over a new coefficient algebra is required for the next comparison.

Theorem 1.18.5 (Finite presentations in weighted coefficient spaces)**.** *Let A be a generalized K-affinoid algebra and choose a finite presentation*

$$A^s \xrightarrow{D} A^m \xrightarrow{q} M \longrightarrow 0, \qquad R = \ker q.$$

Give M the quotient norm from q and R the induced norm from A^m. For any countable set I and positive weights w_i, let $\mathcal{C}_w(I, E)$ denote weighted c_0 coefficient families in a Banach space E. Then

$$\mathcal{C}_w(I, A)^s \xrightarrow{D_w} \mathcal{C}_w(I, A)^m \xrightarrow{q_w} \mathcal{C}_w(I, M) \longrightarrow 0$$

is exact, with

$$\operatorname{im} D_w = \mathcal{C}_w(I, R) = \ker q_w.$$

This image is closed. If $C_D > 0$ is an actual lifting bound for $D : A^s \to R$, the same bound works for D_w. The map q_w induces an isometry from its normed quotient onto $\mathcal{C}_w(I, M)$; for every fixed $\lambda > 1$, it admits actual lifts with bound λ. The constants are independent of I and w.

Proof. Theorems 1.18.2 and 1.18.3 make R, $\ker D$ and M complete with the stated norms, and give the actual bound C_D. Apply Proposition 2.14.1 to both short exact sequences

$$0 \to \ker D \to A^s \to R \to 0, \qquad 0 \to R \to A^m \to M \to 0.$$

The first gives the surjection onto $\mathcal{C}_w(I, R)$ with bound C_D; the second identifies that space with $\ker q_w$ and proves the asserted quotient isometry and lifts. The coefficient identification $\mathcal{C}_w(I, A^m) = \mathcal{C}_w(I, A)^m$ is isometric: supremum commutes with a finite maximum, and a finite union of finite exceptional sets is finite. Under this identification the maps are exactly the displayed constant matrices. Finally, $\ker q_w$ is closed by continuity of q_w. □

Theorem 1.18.6 (Actual coefficient tensors for finite modules)**.** *Let A be a generalized K-affinoid algebra. Let $\Gamma = \mathbb{N}^p \times \mathbb{Z}^q$ for finite $p, q \geq 0$, and let $w : \Gamma \to \mathbb{R}_{>0}$ satisfy $w_0 = 1$ and $w_{\gamma+\delta} \leq w_\gamma w_\delta$. Put $B = \mathcal{C}_w(\Gamma, A)$ with convolution multiplication. For a finite A-module M with a quotient norm from a finite free surjection, the natural map*

$$\Phi_M : B \otimes_A M \longrightarrow \mathcal{C}_w(\Gamma, M), \qquad b \otimes m \longmapsto (b_\gamma m)_\gamma$$

is a B-linear isometric isomorphism for the projective tensor seminorm. In particular that seminorm is definite and complete, and

$$B \otimes_A M \simeq B \widehat{\otimes}_A M \simeq \mathcal{C}_w(\Gamma, M).$$

With any other canonical norm these are natural bounded isomorphisms with bounded inverses. For a short exact sequence of finite canonical A-modules, both its ordinary and its separated completed tensor sequence with B are exact and norm-strict. In coefficient norms the original inverse bound for the injection and lifting bound for the surjection remain valid. No affinoid or Noetherian hypothesis on B is required.

Proof. Choose a finite presentation $A^s \xrightarrow{D} A^m \xrightarrow{q} M \to 0$, using the surjection that defines the given quotient norm. Its relation module is finite by Theorem 1.18.3. Ordinary tensor right exactness, [StacksAlg, Tag 00DF, Lemma 10.12.10], gives

$$B \otimes_A M \simeq B^m / D_B B^s.$$

The constant matrix D_B is exactly D_w of Theorem 1.18.5. By that theorem, its image is closed and the quotient is isometric to $\mathcal{C}_w(\Gamma, M)$. Lemma 2.14.3 supplies the B-action. The coefficient generator map is B-linear, first on finite supports and then by boundedness and density; the resulting algebraic isomorphism is precisely Φ_M. This identifies actual relation modules before any completion is taken.

For any finite tensor expression, contractivity of the A-action on M gives

$$\left\| \Phi_M \left(\sum_j b_j \otimes m_j \right) \right\|_w \leq \max_j \|b_j\|_w \|m_j\|.$$

Taking infima proves $\|\Phi_M(z)\|_w \leq \|z\|_\pi$. Conversely, let $e_i = q(\mathbf{e}_i)$ be the generator images, of norm at most one. Every lift $(b_i) \in B^m$ of $\Phi_M(z)$ gives the tensor expression $z = \sum_i b_i \otimes e_i$, hence $\|z\|_\pi \leq \max_i \|b_i\|_w$. The infimum of these lift norms is $\|\Phi_M(z)\|_w$ by the preceding quotient isometry. Thus Φ_M is isometric. Weighted coefficient completeness, or the closed quotient already obtained, proves that separated completion changes neither the elements nor the norm.

A bounded A-linear map commutes with Φ_M on pure tensors by its coefficient formula. Theorem 1.18.3 makes every map between finite modules bounded and every short exact sequence norm-strict. Proposition 2.14.1 then gives coefficient exactness with the asserted unchanged constants. Transporting through the natural isomorphisms proves the two tensor exactness assertions. Finally, uniform inequalities between two norms on M induce the same inequalities between their coefficient norms and between their projective norms. This proves the bounded-isomorphism statement for all canonical choices. The tensor assertion is specific to this coefficient algebra, and does not concern an arbitrary Banach target. □

Proposition 1.18.7 (Finite tensors with generalized affinoid targets)**.** *Let L/K be any complete valued extension preserving the given absolute value. Let A be a generalized K-affinoid algebra, B a generalized L-affinoid algebra, and $\phi : A \to B$ a bounded unital K-homomorphism. For every finite canonical A-module M, the projective seminorm on $B \otimes_A M$ is uniformly equivalent to its complete canonical finite-B-module norm. Consequently*

$$B \otimes_A M \longrightarrow B\widehat{\otimes}_A M$$

is a bounded isomorphism with bounded inverse, natural in M. More precisely, if M has the quotient norm from $q : A^m \twoheadrightarrow M$, let Q_B be the quotient norm from the corresponding surjection $B^m \twoheadrightarrow B \otimes_A M$. If $\|\phi(a)\| \le C_\phi \|a\|$, then

$$\|z\|_\pi \le Q_B(z) \le C_\phi \|z\|_\pi.$$

In particular the comparison is isometric when ϕ is contractive and these specified quotient norms are used. This proposition does not assert flatness of $A \to B$.

Proof. The zero target is immediate. Choose a finite presentation $A^s \xrightarrow{D} A^m \to M \to 0$. By [StacksAlg, Tag 00DF, Lemma 10.12.10],

$$B \otimes_A M = B^m / D_B B^s.$$

Theorem 1.18.2, applied over the complete field L, makes the relation image closed. Lemma 1.11.1 makes Q_B complete, and Theorem 1.18.3 identifies it as a canonical finite-B-module norm. No comparison between ordinary and completed tensors is used to obtain this closedness.

Let e_i be the images of the standard generators under q; they have norm at most one. Any vector (b_i) representing z gives $z = \sum_i b_i \otimes e_i$, so taking the infimum of its maximum norm gives $\|z\|_\pi \le Q_B(z)$. For the reverse inequality, write $z = \sum_j b_j \otimes m_j$. Fix $\lambda > 1$ and lift each nonzero m_j to a vector $(a_{ij})_i$ of norm at most $\lambda\|m_j\|$; lift zero by zero. Then z is represented in B^m by $c_i = \sum_j b_j \phi(a_{ij})$, and

$$Q_B(z) \le \max_i \|c_i\| \le \lambda C_\phi \max_j \big(\|b_j\|\|m_j\|\big).$$

Take the infimum over all finite tensor representations and then let $\lambda \downarrow 1$. This proves the stated bound. A seminorm uniformly equivalent to a definite complete norm is itself definite and complete, so its separated completion

changes nothing. Equivalent canonical choices preserve the result by the same norm comparisons. Naturality follows on pure tensors, and the maps are bounded because maps between finite modules are bounded. □

Theorem 1.18.8 (Free-radius extension is algebraically faithfully flat)**.** *Let A be a generalized K-affinoid algebra and let r be any finite tuple free over K, not necessarily one that makes A_r strict. For every finite canonical A-module M there are natural bounded A_r-linear isomorphisms*

$$A_r \otimes_A M \simeq A_r \widehat{\otimes}_A M \simeq M \widehat{\otimes}_K K_r.$$

They are isometric for a specified finite-presentation quotient norm on M. The ring map $A \to A_r$ is faithfully flat for ordinary tensor products on all algebraic A-modules. On finite canonical modules, its ordinary and separated completed tensor sequences are identified with the coefficient sequences and preserve norm-strict short exactness. This statement uses no flatness claim for completed tensors of arbitrary Banach modules.

Proof. By Lemma 1.18.1, A_r is the actual coefficient algebra $\mathcal{C}_r(\mathbf{Z}^s, A)$, with its multiplicative weight.

Apply Theorem 1.18.6 to that algebra and then Theorem 1.16.2 to M. These give the displayed natural isomorphisms, including their norm statements and their compatibility with maps of finite modules. Thus the finite-module tensor comparison has been established before flatness is tested.

For a finitely generated ideal $J \subset A$, the natural map $A_r \otimes_A J \to A_r \otimes_A A = A_r$ is identified with the coefficientwise inclusion of J into A. This map is injective. The finite-module norms needed here are supplied by Theorem 1.18.3; the induced norm on J is one of its canonical norms. The finitely generated ideal criterion, [StacksAlg, Tag 00HD, Lemma 10.39.5(4)], now proves ordinary flatness on all A-modules.

Let N now be any algebraic A-module, with no norm or finite-generation assumption. The map

$$A_r \otimes_A N \longrightarrow N, \qquad b \otimes n \longmapsto p_0^A(b)n$$

is well-defined because p_0^A is A-linear. It retracts $n \mapsto 1 \otimes n$. Thus every nonzero N has nonzero scalar extension. Having already proved flatness, we apply [StacksAlg, Tag 00HP, Lemma 10.39.15(2)] to obtain faithful flatness. This retraction need not be A_r-linear. Finally, the finite-module norm-strict

exactness is the corresponding assertion of Theorem 1.18.6; it was proved coefficientwise, independently of the algebraic flatness criterion. □

There is a further descent issue for a Banach module that is not yet known to be finite. Its completed scalar extension cannot at that stage be replaced by an ordinary tensor product. The following argument instead approximates a finite generating family upstairs and uses the coefficient retraction only after obtaining uniform lifts.

Theorem 1.18.9 (Descending finite Banach generation)**.** *Let A be a generalized K-affinoid algebra, let M be any Banach A-module with contractive action, and let r be finite and free over K. Put $M_r = M\widehat{\otimes}_K K_r$. Suppose that there are $u_1, \dots, u_n \in M_r$ and $C > 0$ such that every $y \in M_r$ has an expression*

$$y = \sum_{i=1}^{n} b_i u_i, \qquad b_i \in A_r, \qquad \max_i \|b_i\| \le C\|y\|.$$

Then there are finitely many $m_1, \dots, m_t \in M$ and $D > 0$ such that every $m \in M$ has an expression $m = \sum_j a_j m_j$ with $\max_j \|a_j\| \le D\|m\|$. In particular, M is algebraically finite and its given norm is uniformly equivalent to its canonical finite-module norm. If the absolute value on K_r is nontrivial, it is enough to assume that M_r is algebraically finite over A_r: the required constant C is then automatic. For trivially valued K_r, the displayed lifting hypothesis is retained.

Proof. If $M_r = 0$, the isometric constant embedding gives $M = 0$. Otherwise let $Q : A_r^n \twoheadrightarrow M_r$ send the standard generators to u_i. Choose $\varepsilon > 0$ with $\theta = C\varepsilon < 1$. By the coefficient description, choose finite Laurent families v_i with $\|u_i - v_i\| \le \varepsilon$, and let Q' be the generator map for the v_i. Its difference from Q has norm at most ε.

The perturbation still has actual lifts with bound C. Indeed, given y, put $y_0 = y$. Choose $b_k \in A_r^n$ with $Qb_k = y_k$ and $\|b_k\| \le C\|y_k\|$, and set $y_{k+1} = y_k - Q'b_k$. Then

$$\|y_{k+1}\| \le \varepsilon\|b_k\| \le \theta\|y_k\|, \qquad \|b_k\| \le C\theta^k\|y\|.$$

The series $b = \sum_{k\ge 0} b_k$ converges in A_r^n, has norm at most $C\|y\|$, and satisfies $Q'b = y$ by the telescoping identity. No linear choice of lifts is needed.

Write $v_i = \sum_{\nu\in F_i} \iota_M(m_{i\nu})T^\nu$ with finite F_i, omitting zero coefficients. The union of these lists is nonempty because Q' surjects onto $M_r \neq 0$. These finitely many coefficients will generate M. Thus $H = \max_{i,\nu\in F_i} r^\nu$ is finite and positive. For $m \in M$, choose a lift $b = (b_i)$ of $\iota_M(m)$ through Q' with norm at most $C\|m\|$. Taking the zeroth coefficient gives, by Lemma 1.18.1,

$$m = \sum_{i,\nu\in F_i} p_0^A(b_iT^\nu)m_{i\nu}, \qquad \|p_0^A(b_iT^\nu)\| \le r^\nu\|b_i\| \le HC\|m\|.$$

This proves finite generation with bound $D = HC$. The generator map from the corresponding finite free A-module is bounded, while this bound controls its quotient norm from above by the given norm. The two norms are therefore uniformly equivalent. Theorem 1.18.3 identifies the quotient norm as canonical and complete.

Finally, if K_r is nontrivially valued and M_r is algebraically finite, choose algebraic generators. The resulting surjection $A_r^n \to M_r$ is bounded by the module inequality. Both sides are Banach K_r-spaces with their homogeneous norms by Lemma 1.18.1 and Theorem 1.16.2. Lemma 1.14.1, applied over K_r, supplies the constant C. This use of the open-mapping theorem is confined to the stated nontrivial valuation case. The proof never replaced M_r by $A_r \otimes_A M$ before proving that M is finite. □

The classical affinoid finite-module statements belong to Tate–Berkovich theory; see the attribution in [Ber90, Chapter 2]. The arguments here supply the explicit free-radius descent and all norm comparisons used in this section. They do not import a general scalar-extension theorem or the false finite-tensor assertion for arbitrary Banach targets. In particular, generalized rational localization maps, covering admissibility, analytic-domain tests and sheaf descent still require their own stated interfaces.

1.19 A continuous section of free scalar extension

The coefficient description of a free scalar extension also lifts spectral points. A new difficulty appears here: radii free over the ground field need not remain free over the valued field determined by a point. The proof below allows several terms of maximal value, rather than assuming that a dominant term is unique.

Theorem 1.19.1 (The Gauss section)**.** *Let K be a complete real-valued non-Archimedean field, possibly trivially valued, and let A be a commutative unital Banach K-algebra with a homogeneous submultiplicative norm. Let $r = (r_1, \ldots, r_s)$ be finite and free over K, allowing the empty tuple. Put $A_r = A\widehat{\otimes}_K K_r$, using separated completed projective tensor products. In the coefficient description of Theorem 1.16.2, the formula*

$$\sigma_r(x)\left(\sum_{\nu\in\mathbf{Z}^s} a_\nu T^\nu\right) = \max_{\nu\in\mathbf{Z}^s} |a_\nu(x)|r^\nu$$

defines a bounded multiplicative seminorm on A_r extending the specified absolute value of K_r, for every $x \in \mathcal{M}_K(A)$; the value is zero when all coefficients have value zero. It gives a continuous section

$$\sigma_r : \mathcal{M}_K(A) \longrightarrow \mathcal{M}_{K_r}(A_r) \quad of \quad \pi_r : \mathcal{M}_{K_r}(A_r) \longrightarrow \mathcal{M}_K(A).$$

Thus π_r is surjective. The sections commute with bounded unital K-algebra homomorphisms. No affinoid, reducedness, spherical-completeness or residue-field freeness assumption is imposed. For the zero algebra both spectra are empty.

Proof. **Step 1: convergence and multiplication.** A bounded multiplicative seminorm x on any such Banach algebra is contractive. Indeed, if its bound is C, then $|a(x)|^n \le C\|a^n\| \le C\|a\|^n$; taking nth roots proves $|a(x)| \le \|a\|$. This also gives $\big||a(x)| - |b(x)|\big| \le \|a - b\|$. Consequently the coefficient weights in the formula tend to zero. A positive supremum is attained, and only finitely many terms attain it. The formula is ultrametric, bounded by the coefficient norm, and agrees with x on constants.

For finite tensor representations $z = \sum_i a_i \otimes u_i$ and $w = \sum_j b_j \otimes v_j$, multiplication gives

$$\|zw\|_\pi \le \max_{i,j} \|a_i\|\|b_j\||u_i||v_j|.$$

Taking both infima proves submultiplicativity of the projective seminorm. Its zero-seminorm subspace is an ideal, so multiplication passes to separation and completion. Finite Laurent families are dense by Theorem 1.16.2. Their products therefore show that multiplication in A_r is convergent coefficient convolution. In particular, if $F = \sum a_\alpha T^\alpha$ and $G = \sum b_\beta T^\beta$, then

$$(FG)_\kappa = \sum_{\alpha+\beta=\kappa} a_\alpha b_\beta.$$

For fixed κ, the terms converge to zero in A: their norms, multiplied by r^κ, are bounded by $(\|a_\alpha\| r^\alpha)(\|b_{\kappa-\alpha}\| r^{\kappa-\alpha})$, the product of a vanishing family and a bounded family. Completeness makes this sum converge independently of its enumeration. Continuous coefficient projection identifies it with the limit of finite products.

Step 2: multiplicativity with tied maximal terms. Write $M = \sigma_r(x)(F)$ and $N = \sigma_r(x)(G)$. Every term of every product coefficient has weighted x-value at most MN. The preceding convergence and continuity of x give $\sigma_r(x)(FG) \leq MN$, including the case $MN = 0$. Suppose $M, N > 0$, and let D, E be the finite sets where the respective maxima are attained. There are $0 \leq \lambda_F, \lambda_G < 1$ bounding all other weights by $\lambda_F M$, respectively $\lambda_G N$: among the finitely many weights at least half the maximum, the non-maximal ones have a strictly smaller maximum, and all remaining weights are at most half the maximum.

Choose the lexicographically greatest indices $\mu \in D$ and $\nu \in E$. The pair (μ, ν) is the only pair in $D \times E$ summing to $\kappa = \mu + \nu$: if $\alpha < \mu$, translation invariance forces $\kappa - \alpha > \nu$. Every other summand of $(FG)_\kappa$ therefore has weighted x-value at most λMN, where $\lambda = \max(\lambda_F, \lambda_G) < 1$. The entire convergent remainder satisfies this same bound. The distinguished term has weighted value MN, so the ultrametric inequality in both directions gives

$$|(FG)_\kappa(x)| r^\kappa = MN.$$

This proves the reverse inequality and hence multiplicativity. Only an order on a finite set of maximal terms was used; no freeness over the field attached to x was needed. For the empty tuple the only index is zero.

Step 3: scalars and topology. On the scalar copy of K_r, the coefficients lie in K, and the formula is its specified absolute value from Lemma 1.16.1. Thus it is a K_r-spectral point. For a finite Laurent polynomial P, the function $x \mapsto \sigma_r(x)(P)$ is a finite maximum of continuous evaluation functions. For every $F \in A_r$ and every such P, uniformly in x,

$$\big|\sigma_r(x)(F) - \sigma_r(x)(P)\big| \leq \|F - P\|_r.$$

Finite truncations approximate F in norm. Its evaluation is therefore a uniform limit of continuous functions, hence continuous. This is the required continuity for the evaluation topology. Constants give $\pi_r \sigma_r = \mathrm{id}$. A bounded homomorphism acts coefficientwise after extension; applying a point to each coefficient proves the asserted naturality. If $A = 0$, the coefficient space is zero and the statement is vacuous. □

The existence of a section does not make the spectral fibres singletons. The following calculation also explains why one must not retain the freeness hypothesis after changing the coefficient field without checking it.

Example 1.19.2 (Two points in a free-extension fibre)**.** Let K be trivially valued, let $r > 1$, and put $A = K_r$, with parameter S. Form $B = A\widehat{\otimes}_K K_r$, using T for the parameter in the second factor. The given absolute value is a point $x \in \mathcal{M}_K(A)$. Then the fibre of $\mathcal{M}_{K_r}(B) \to \mathcal{M}_K(A)$ over x contains two distinct points, and B is not a field.

Proof. The coefficient isometry makes $F = T - S$ nonzero, and Theorem 1.19.1 gives $\sigma_r(x)(F) = r$. Identify the two copies of K_r by sending T to S. The map $a \otimes b \mapsto ab(S)$ on the ordinary tensor product is contractive for every tensor representation, hence for the projective seminorm. It extends to a unital contractive homomorphism $\mu : B \to A$. The point $\delta = |\mu(-)|$ extends the second K_r-absolute value and restricts to x, but $\delta(F) = 0$. The two points are distinct. A nonzero element in the kernel of a unital map to a nonzero field cannot be a unit, so B is not a field. Here r is free over K but not over A, since $|S| = r$; no assertion about zero divisors is being made. □

References. For the classical continuous section, compare Berkovich, [Ber93, Lemma 2.2.5, pp. 41–42]. The proof above reconstructs this part by coefficient convolution. It does not assert that lemma's additional prime-kernel or Zariski-closed conclusions.

1.20 Points and power bounds with arbitrary radii

The finite-module theorem for arbitrary radii lets us pass to actual affinoid quotients. Together with simultaneous strictification, this gives the point and power estimates needed by rational localization. Throughout this section the ground field is complete with its fixed real-valued non-Archimedean absolute value, which may be trivial. Every affinoid norm is uniformly equivalent to a defining quotient norm. Zero algebras are allowed, with zero norm and empty spectrum.

When invoking the preceding finite-module results, one may first use a normalized defining quotient norm. Replacing it by a uniformly equivalent admissible norm preserves bounded spectral points, closed submodules and the existence of all constants in the estimates below. This allows the conclusions to be stated for the original admissible norm.

Theorem 1.20.1 (Nonemptiness and the unit-ideal test)**.** *Let A be a K-affinoid algebra with arbitrary positive presentation radii. Then $\mathcal{M}(A) = \varnothing$ if and only if $A = 0$. For every algebraic ideal $J \subseteq A$,*

$$J \neq A \quad \Longleftrightarrow \quad \text{there is } x \in \mathcal{M}(A) \text{ with } |j(x)| = 0 \text{ for every } j \in J.$$

In particular, $f \in A$ is a unit if and only if $|f(x)| > 0$ at every point. If $A \neq 0$ and f is a unit, $|f|$ has a positive minimum on $\mathcal{M}(A)$. No finiteness of maximal residue fields over K is asserted in this generality.

Proof. For $A \neq 0$, Theorem 1.17.6 supplies a free extension $L = K_r$ for which A_r is strict norm-admissible. The constant embedding is isometric by Theorem 1.16.2, so $A_r \neq 0$. Theorem 1.12.3 gives a bounded L-spectral point of A_r. Its restriction along the bounded map $A \to A_r$ is a point of $\mathcal{M}_K(A)$. Conversely, a seminorm on the zero algebra cannot send its unit to one. Notice that this argument needs only restriction of a point, not surjectivity of the whole spectral map.

Every ideal J is closed by Theorem 1.18.2, and A/J with its quotient norm is affinoid by Corollary 1.18.4. If J is proper, the preceding nonemptiness argument applied to A/J gives the required point of A. Conversely, a point killing J rules out $1 \in J$. Applying this equivalence to $J = fA$ proves the unit criterion; a unit cannot vanish because $|f(x)||f^{-1}(x)| = 1$. Finally Proposition 2.24.1 makes the nonempty spectrum compact, and $|f|$ is continuous and positive on it. Its minimum is therefore attained and positive. □

For a non-strict affinoid, a pointwise bound may have a radius outside the value group. The next argument puts a positive power of that radius into the value group of one auxiliary field, and then returns the resulting estimate to the original norm.

Theorem 1.20.2 (Scaled power bounds for arbitrary positive radii)**.** *Let A be any norm-admissible K-affinoid algebra, let $f \in A$, and let $s > 0$ be any real number. Then*

$$\big(|f(x)| \leq s \text{ for every } x \in \mathcal{M}(A)\big) \quad \Longleftrightarrow \quad \big(\exists C \geq 1 \ \forall m \geq 0 : \ \|f^m\| \leq C s^m\big).$$

The constant may depend on A, its chosen norm, f and s. It is not asserted to be uniform in s. Neither reducedness nor a restriction on the residue characteristic is needed.

Proof. The zero algebra is immediate. Otherwise choose, by Lemma 1.17.5 and Theorem 1.17.6, a single free extension $L = K_r$ for which A_r is strict and $s \in \sqrt{|L^\times|}$. Thus there are $n \geq 1$ and $c \in L^\times$ with $|c| = s^n$. Give A_r its actual coefficient norm, which is admissible for its strict structure. Every point y of $\mathcal{M}_L(A_r)$ restricts to a point of $\mathcal{M}_K(A)$. If the left-hand condition holds, $h = (f \otimes 1)^n / c$ has $|h(y)| \leq 1$ for every such y. Theorem 1.13.4 supplies $C_0 \geq 1$ with $\|h^q\| \leq C_0$ for all $q \geq 0$, also for this equivalent admissible norm. Homogeneity and the isometric constant embedding give

$$\|f^{nq}\| \leq C_0 s^{nq}.$$

Write $m = nq + j$, where $0 \leq j < n$. Submultiplicativity gives $\|f^m\| \leq C_0 s^{nq} \|f^j\|$, so the required bound holds with

$$C = C_0 \max_{0 \leq j < n} s^{-j} \|f^j\|.$$

Conversely, contractivity of bounded multiplicative seminorms, proved by the elementary power argument in Lemma 1.6.1, gives $|f(x)|^m \leq C s^m$. Taking mth roots and letting m increase proves $|f(x)| \leq s$. All absolute values used on auxiliary fields restrict to the specified one on K. □

Corollary 1.20.3 (The spectral-radius formula in the affinoid category)**.** *For every nonzero norm-admissible K-affinoid algebra A and every $f \in A$, the limit exists and satisfies*

$$\lim_{m \to \infty} \|f^m\|^{1/m} = \max_{x \in \mathcal{M}(A)} |f(x)|.$$

This assertion is unchanged by a uniformly equivalent admissible norm. It does not yet assert equivalence of the spectral norm and the Banach norm on every reduced affinoid.

Proof. Theorem 1.20.1 and Proposition 2.24.1 show that the continuous function $|f|$ has a maximum $s_0 \geq 0$ on a nonempty compact space. Contractivity gives $s_0 \leq \|f^m\|^{1/m}$ for every positive m. If $s_0 > 0$, Theorem 1.20.2 gives the reverse bound $\|f^m\|^{1/m} \leq C^{1/m} s_0$, proving convergence to s_0. If $s_0 = 0$, apply the same theorem with any $s > 0$; the upper limit is at most every such s, hence zero. Multiplicative comparison constants between two equivalent norms disappear after taking mth roots. □

These results extend the pointwise and power estimates, not the strict Nullstellensatz. The free-radius field of Lemma 1.16.1 already has a maximal residue field which need not be finite over K. Rational localization will use pointwise inequalities on the whole spectrum, not separate maxima of its numerator and denominator.

Chapter 2

Analytic spaces, morphisms and fibres

Affinoid algebras provide compact analytic pieces. To work locally on them, inequalities must define analytic domains and functions must glue on overlaps. We first construct this local language. We then describe products and fibres, where the completed residue field of a point becomes indispensable. The final sections explain the geometric hypotheses used later in Stokes, finite-flat degree calculations and model comparisons.

2.1 Domains and bounded localization

Definition 2.1.1. Let $X = \mathcal{M}(A)$. An affinoid domain is a subset $V \subset X$ represented by a bounded map $A \to A_V$ such that $\mathcal{M}(A_V) \to X$ identifies its points with V, and every bounded morphism $\mathcal{M}(B) \to X$ whose image lies in V factors uniquely through $\mathcal{M}(A_V)$. The uniqueness is in the affinoid category, not merely among maps of point sets.

The algebra $A[f^{-1}]$ alone is not an analytic localization: it has no prescribed bound on f^{-1}. By adjoining an inverse with a bound, we obtain an analytic domain. The same principle gives three useful presentations:

$$\begin{array}{c|c} |f_i| \le r_i & A\{r^{-1}T\}/(T_i - f_i) \quad \text{(Weierstrass)} \\ |f| \ge r & A\{rS\}/(fS - 1) \quad \text{(Laurent)} \\ |f_i| \le r_i|g| & A\{r^{-1}T\}/(gT_i - f_i) \quad \text{(rational).} \end{array}$$

In the rational case $g, f_1, \dots, f_m$ must generate the unit ideal. Their common zero locus would otherwise introduce points not described by ratios. The universal properties of these presentations are proved in Theorem 2.31.2. Theorem 2.36.6 then gives a finite original-algebra rational cover of every affinoid domain. We begin with the point calculation.

References. [TemIntro, §3.2, especially 3.2.3.4]; [Ber93, §1.3].

2.2 Rational domains and logarithmic coordinates

A rational-domain presentation converts finitely many inequalities into a Banach algebra. Let $g, f_1, \dots, f_m \in A$ generate the unit ideal and let $r_i > 0$. Put

$$D = \{x \in \mathcal{M}(A) : |f_i(x)| \le r_i |g(x)| \ (1 \le i \le m)\},$$

$$B = A\{r_1^{-1}T_1, \dots, r_m^{-1}T_m\}/(gT_i - f_i)_{i=1}^m.$$

The quotient is the affinoid quotient, with its complete quotient topology. Theorem 2.31.2 proves its rational-domain universal property. The following proof identifies its points and the role of the unit-ideal hypothesis.

Proposition 2.2.1. *The map $\mathcal{M}(B) \to \mathcal{M}(A)$ is a homeomorphism onto D. For $x \in D$, the corresponding evaluation sends T_i to $f_i(x)/g(x)$ in $\mathcal{H}(x)$. In particular g is a unit in B.*

Proof. Write $a_0 g + \sum_i a_i f_i = 1$ in A. In B this becomes $g(a_0 + \sum_i a_i T_i) = 1$. Thus g is invertible. A point of B satisfies $|T_i| \le r_i$, and hence its image belongs to D. Conversely, if $x \in D$ and $g(x) = 0$, then every $f_i(x)$ is zero, contradicting the displayed unit-ideal relation. Therefore $g(x) \neq 0$ and the quotients $f_i(x)/g(x)$ have sizes at most r_i. Substitution into a convergent power series defines a bounded homomorphism to $\mathcal{H}(x)$, which kills the defining relations of B. This gives a point over x.

For any point over x, the relations force those same values of the variables in the completed residue field. Values on polynomial expressions are therefore fixed by the seminorm of x on fractions with denominator a power of g. Such polynomial expressions are dense in B, and bounded seminorms are continuous, so the point is unique. Finally, the map is continuous, its source is compact and its target is Hausdorff. A continuous bijection onto D is consequently a homeomorphism. □

Example 2.2.2. The domain $|f| \geq r > 0$ is represented by adjoining f^{-1} with bound r^{-1}. On its interior, $-\log|f|$ is an original smooth function: f is now an analytic unit and defines a one-dimensional moment map. If f has zeros elsewhere, the same logarithm is not a finite smooth function across those zeros. This is the local distinction used when regularized maxima discard inactive logarithmic branches.

Strict inequalities give ordinary open subsets because evaluation is continuous. This elementary fact explains why rational-domain arguments often begin with two sets of bounds: weaker non-strict bounds define a compact affinoid domain, while stronger bounds place the point or support in its ordinary interior. The existence of a suitable presentation around every point is a local geometric input; it should be distinguished from the explicit pointwise construction above.

Example 2.2.3 (Domain versus closed subspace)**.** The closed disc $|T| \leq r$ in the analytic affine line is a domain: it retains the analytic dimension of the ambient curve. The zero locus $T = 0$ is a closed analytic subspace, defined by a quotient of the function algebra. The thickening $T^m = 0$ has the same point but a different structure sheaf. A closed subset in the ordinary topology therefore need not be a closed analytic subspace.

Proposition 2.2.4. *If $f(x) \neq 0$ on an affinoid space, there is an ordinary neighborhood of x on which f is an analytic unit. On any analytic domain where f is a unit, the function $-\log|f|$ is finite and continuous.*

Proof. Choose $0 < r < |f(x)|$. The Laurent domain $|f| \geq r$ contains the ordinary open neighborhood $|f| > r$ of x, and its defining algebra contains f^{-1}. Restrict that inverse to the open neighborhood. Continuity follows from continuity of evaluation and of the logarithm on positive real numbers. □

2.3 G-topology and the structure sheaf

A finite affinoid-domain cover can contain a point only on interfaces; it need not contain that point in the ordinary interior of one member. This is why analytic domains form a site rather than just a basis of ordinary open sets. On an affinoid X, first use compact analytic domains, finite unions of affinoid domains, with finite domain covers. The assignment $V \mapsto A_V$ on affinoid domains extends to the analytic structure sheaf $\mathcal{O}_{X_G}$.

Theorem 2.3.1 (Tate acyclicity)**.** *Let $X = \mathcal{M}(A)$ and let $X = \bigcup_{i=1}^{m} V_i$ be a finite affinoid-domain cover. For a finite A-module M, the augmented Čech complex of the corresponding localized modules is exact. In particular, for $M = A$,*

$$0 \longrightarrow A \longrightarrow \prod_i A_{V_i} \longrightarrow \prod_{i,j} A_{V_i \cap V_j} \longrightarrow \cdots$$

is exact, with the restriction maps and alternating differentials. The maps are admissible for the natural Banach-module topologies.

Proof. Apply Theorem 2.38.1 to A, M and the stated cover. Its all-tuples version gives exactly the displayed differentials. Proposition 1.18.7 identifies the localized ordinary modules with their completed tensors and canonical Banach topologies. Norm-strict exactness is therefore precisely the asserted admissibility and exactness. □

References. [TemIntro, 3.3.2.1]; the original proof routes are recorded in Section 2.11.2.

Proposition 2.3.2 (Gluing across a circle)**.** *Let $0 < s < 1$. Cover the closed unit disc by $V_- = \{|T| \le s\}$ and $V_+ = \{s \le |T| \le 1\}$. With*

$$A_- = K\{s^{-1}T\}, \quad A_+ = K\{T, sT^{-1}\}, \quad C = K\{s^{-1}T, sT^{-1}\}.$$

The sequence $0 \to K\langle T\rangle \to A_- \oplus A_+ \to C \to 0$, with last map $(f_-, f_+) \mapsto f_- - f_+$, is exact.

Proof. A Laurent series in C splits into its nonnegative and negative powers. The nonnegative part converges on V_-; the negative part converges on V_+ because its worst bound is attained at radius s. This proves surjectivity of the difference map. If two restrictions agree on the circle, uniqueness of Laurent coefficients shows that the annular series has no negative powers. Its nonnegative coefficients tend to zero at radius one, since it belongs to A_+. They therefore define an element of $K\langle T\rangle$ restricting to both functions. Injectivity follows from uniqueness of the coefficients. □

For a finite A-module M, the sheaf associated with it has sections $M \otimes_A A_V$ on an affinoid domain V. Finite-module completeness makes this ordinary tensor product sufficient. The converse is Kiehl's coherence theorem: coherent sheaves on X are equivalent to finite A-modules. Thus finite presentations of functions and their relations glue, just as the functions themselves

do. This is used later for ideals of closed subspaces, analytic line bundles and finite-flat modules.

References. [TemIntro, 3.3.4.1].

2.4 From affinoid atlases to analytic spaces

An analytic space is obtained by gluing affinoid spaces along analytic domains with compatible bounded restriction maps. In the general definition, an atlas is a net of compact affinoid pieces: locally, finitely many pieces containing a point cover a neighborhood of it, and the overlap data admit compatible affinoid refinements. Morphisms are locally represented, after such refinements, by bounded maps of affinoid algebras. Section 2.40 constructs the atlas, proves independence under refinement and glues its structure sheaf and morphisms, following [Ber93, §§1.1–1.3].

Definition 2.4.1. An analytic space is good if every point has an affinoid neighborhood: an affinoid domain containing an ordinary open neighborhood of the point. An analytic domain of a general space is a locally represented subset with the corresponding universal factorization property. Its structure is unique when it exists.

For good spaces, ordinary local arguments can be performed inside one affinoid neighborhood. Open subspaces remain good by the rational-neighborhood basis, and the construction in Theorem 2.33.23 makes analytifications of varieties good. In contrast, removing the open bidisc from the closed bidisc of the same radius gives a compact analytic domain which is not good. Theorem 3.23.2 proves this example, recalled from [TemIntro, 4.2.1.4], in every dimension at least two.

The ordinary structure sheaf is obtained from the analytic sheaf on ordinary opens. For a good space X, set $\mathcal{O}_{X,x} = \varinjlim_{x\in U} \mathcal{O}_X(U)$ over ordinary open neighborhoods. A germ is invertible precisely when its value at x is nonzero, by Proposition 2.2.4. Consequently this is a local ring with maximal ideal of germs vanishing at x.

2.5 Local residue fields and completed residue fields

Definition 2.5.1. For a point of a good analytic space, distinguish

$$\kappa_X(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x, \qquad \mathcal{H}(x) = \widehat{\kappa_X(x)}, \qquad \widetilde{\mathcal{H}(x)} = \mathcal{H}(x)^\circ/\mathcal{H}(x)^{\circ\circ}.$$

The first is the residue field of an ordinary local ring, the second its completion for the induced absolute value, and the third the residue field of a valuation ring.

The identification of the middle field with the affinoid construction of Chapter 1 follows from the evaluation maps on neighborhoods. For $X = \mathcal{M}(A)$, the fraction field of $A/\mathfrak{p}_x$ embeds in $\kappa_X(x)$: a denominator nonzero at x is invertible on a neighborhood. Both fields embed in $\mathcal{H}(x)$, and the former is dense by its construction. Thus the completion of $\kappa_X(x)$ is exactly $\mathcal{H}(x)$. This also explains why the completed field is the appropriate invariant for fibres.

If $V \subset X$ is an analytic domain containing x, its completed residue field is canonically the same $\mathcal{H}(x)$, by Corollary 2.33.15. For good V and X, the ordinary local ring and its uncompleted residue field can change when V is not an ordinary neighborhood. If it is a neighborhood, the stalks agree. The example in Section 2.9 uses precisely this change of local ring.

References. [Ber93, §2]; [TemIntro, 4.2.1].

2.6 Completed tensor products and analytic fibres

A product must allow evaluation at points taking values in compatible complete extensions. The algebraic tensor product has the required algebraic universal property, but generally lacks completeness. On $B \otimes_A C$, use the quotient projective seminorm obtained from

$$\|z\|_\pi = \inf_{z=\sum_i b_i \otimes c_i} \max_i \|b_i\|\|c_i\|.$$

After quotienting by its zero seminorm and completing, one obtains $B\widehat{\otimes}_A C$. Its universal property concerns compatible bounded maps. Affinoid completed tensor products are affinoid, and the constructions glue.

Theorem 2.6.1. *Fibre products exist in the analytic category. For affinoid morphisms $A \to B, A \to C$, and for a complete extension L/K,*

$$\mathcal{M}(B) \times_{\mathcal{M}(A)} \mathcal{M}(C) = \mathcal{M}(B \widehat{\otimes}_A C), \qquad \mathcal{M}(A)_L = \mathcal{M}(A \widehat{\otimes}_K L).$$

If $f : Y \to X$ and $x \in X$, its analytic fibre is the $\mathcal{H}(x)$-analytic space $Y_x = Y \times_X \mathcal{M}(\mathcal{H}(x))$. For affinoid X, Y, its algebra is $B \widehat{\otimes}_A \mathcal{H}(x)$. Its underlying topological space identifies with $f^{-1}(x)$.

Proof. Proposition 2.33.8 constructs the affinoid completed tensor product and proves its bounded universal property. Theorem 2.40.11 glues these products on arbitrary analytic atlases. Theorem 2.40.12 supplies complete scalar extension. The actual fibre identification, including its topology and its field of definition, is Theorem 2.33.11 in affinoid charts and Theorem 2.40.13 after gluing. □

References. [Ber93, §1.4]; [TemIntro, 3.1.4.1, 4.1.5–4.1.7].

The fibre and scalar-extension formulas use the analytic category allowing complete extensions of the ground field, as in [TemIntro, 4.1.6]. In particular $\mathcal{M}(\mathcal{H}(x))$ need not be a K-affinoid space. Treating it as an analytic space over its own field is what makes the fibre formula valid for nonclassical points.

Example 2.6.2. For the projection $\mathcal{M}(K\langle S, T\rangle) \to \mathcal{M}(K\langle S\rangle)$, the fibre at x is $\mathcal{M}(\mathcal{H}(x)\langle T\rangle)$. At a nonclassical x, replacing $\mathcal{H}(x)$ by K would discard the value of S and its residual geometry. In the fibre, S is a scalar $S(x) \in \mathcal{H}(x)$, while T remains a variable.

Example 2.6.3. Let L/K be a nontrivial finite Galois extension. Both $\mathcal{M}(L)$ and $\mathcal{M}(K)$ have one point, but

$$\mathcal{M}(L) \times_{\mathcal{M}(K)} \mathcal{M}(L) \simeq \coprod_{\sigma \in \mathrm{Gal}(L/K)} \mathcal{M}(L).$$

Indeed $L \widehat{\otimes}_K L = L \otimes_K L \simeq \prod_\sigma L$, the last map sending $a \otimes b$ to $(a\sigma(b))_\sigma$. Completion adds nothing in finite dimension. Thus the underlying set of an analytic product need not be the Cartesian product of the underlying sets.

Proposition 2.6.4. *Suppose a finite affinoid morphism is given by a finite locally free A-algebra B of constant rank d. For $x \in \mathcal{M}(A)$ put $F = \mathcal{H}(x)$ and $E = B \otimes_A F$. Then E is an Artinian F-algebra of dimension d. Writing it as the product of its local factors E_j with residue fields F_j,*

$$d = \sum_j \operatorname{length}_{E_j}(E_j)[F_j : F].$$

Proof. Finite projectivity commutes with scalar extension, giving $\dim_F E = d$; finite-dimensional algebras are complete by Lemma 1.12.1. The Artinian product decomposition and finite length follow from [StacksAlg, Tag 00JB]. A composition series of E_j as an E_j-module has $\operatorname{length}_{E_j}(E_j)$ factors, each isomorphic to F_j. Taking F-dimensions gives the formula. □

This is the finite-fibre degree formula used in Chapter 22. It retains nilpotent length and residue-field degree separately. It does not determine the normalization of integrals after replacing the ground field; that is the separate comparison in Chapter 23.

2.7 Boundary, finite maps and properness

Definition 2.7.1. For an affinoid morphism $Y = \mathcal{M}(B) \to X = \mathcal{M}(A)$, a point y lies in $\operatorname{Int}(Y/X)$ if there exists an admissible surjection

$$A\{r_1^{-1}T_1, \dots, r_m^{-1}T_m\} \twoheadrightarrow B \quad \text{with} \quad |T_i(y)| < r_i \quad \text{for all } i,$$

Its complement is the relative analytic boundary $\partial(Y/X)$. Absolute interior and boundary are relative to $\mathcal{M}(K)$. For general morphisms these notions are defined locally and are independent of the chosen affinoid descriptions.

For general spaces the precise definition tests all target affinoid domains, as in Definition 2.44.4; Theorem 2.46.3 proves its locality. The strict inequalities above are existential in the relative presentation. The graded criterion in Theorem 3.18.10 gives an intrinsic description.

Theorem 2.7.2. *Relative interior is open and behaves locally on the base under analytic-domain covers. If $V \subset X$ is an analytic domain, $\operatorname{Int}(V/X)$ equals its ordinary topological interior in X. An affinoid morphism $\mathcal{M}(B) \to \mathcal{M}(A)$ is boundaryless if and only if $A \to B$ is finite and admissible. Finite morphisms are stable under base change.*

Proof. Theorem 2.46.2 proves ordinary openness for arbitrary analytic spaces, and Theorem 2.46.3 gives the exact boundary identity on analytic-domain covers of the base. Proposition 2.44.5 identifies the relative interior of every analytic-domain inclusion with its ordinary topological interior. Theorem 2.43.8 proves the affinoid finite criterion, including the canonical finite-module topology and hence admissibility over trivially valued fields. Finally, Proposition 2.33.19 identifies a finite affinoid base change with its finite tensor algebra equipped with that topology. Applying this on affinoid charts of an arbitrary base change and using the analytic-domain gluing theorem proves stability of finite morphisms. □

References. [TemIntro, 3.4.2.3 and 4.2.2].

A closed immersion is locally represented by an admissible quotient of affinoid algebras. A morphism is finite if it is locally represented on the base by finite admissible affinoid-algebra maps. Admissibility matters when the valuation is trivial. A morphism is separated if its diagonal is a closed immersion. A morphism is proper if it is compact and has no relative analytic boundary. Here a compact space is quasi-compact and Hausdorff, and compactness of a morphism means that inverse images of compact analytic domains are compact in this sense, as in Definition 2.53.1. Theorem 2.53.4 proves that a compact boundaryless morphism is separated. For good Hausdorff spaces this agrees with the usual compact-inverse-image formulation.

Example 2.7.3. The closed disc is compact but not proper over K: its algebra is not finite over K, so Theorem 2.7.2 gives nonempty analytic boundary. The analytification of $\mathbb{A}^1_K$ has empty analytic boundary but is not compact. The analytification of $\mathbb{P}^1_K$ is both compact and boundaryless, hence proper. The point at infinity removed from $\mathbb{P}^1$ is an algebraic boundary divisor; it is not an analytic boundary of $\mathbb{A}^{1,\mathrm{an}}$.

These distinctions enter integration concretely. Compact support controls convergence of an integral; support away from analytic boundary removes the boundary contribution in Stokes. A compact affinoid domain meets the first condition automatically for its whole underlying space, but not the second.

References. [TemIntro, 4.2.3–4.2.4 and 5.1.2].

2.8 Dimension and the information carried by a point

Definition 2.8.1. For a strictly affinoid space $\mathcal{M}(A)$, its analytic dimension is the Krull dimension of A. For a general affinoid space, choose a complete extension L/K for which A_L is strictly affinoid and define $\dim \mathcal{M}(A) = \dim A_L$. This is independent of the strictifying extension. Local dimension is the infimum of the dimensions of analytic neighborhoods; a space is pure of dimension n if all its irreducible components have dimension n.

Theorem 3.21.8 proves independence and scalar-extension invariance. The distinction from ordinary Krull dimension is visible in K_r from Proposition 1.8.1: its ring is a field, whereas scalar extension to K_r turns the radius-r circle into a strict one-dimensional affinoid annulus. Thus its analytic dimension over K is one.

The residual transcendence degree and rational value-group rank of $\mathcal{H}(x)$ give a pointwise dimension invariant. Their sum is bounded by local analytic dimension in Theorem 3.22.9. A skeleton can have lower real dimension, as the monomial calculation in Chapter 3 illustrates.

2.9 Flatness and quasi-smoothness

Definition 2.9.1. For $f : Y \to X$ between good spaces, a coherent sheaf $\mathcal{F}$ is naively X-flat at y if $\mathcal{F}_y$ is flat over $\mathcal{O}_{X,f(y)}$. It is analytically X-flat at y if naive flatness holds at every point over y after every good analytic base change, including base changes over arbitrary complete extensions of the ground field. The morphism is flat when this applies to $\mathcal{O}_Y$. The general definition is G-local.

References. [Duc18, 4.1.2–4.1.3 and 4.1.8].

Example 2.9.2 (A boundary obstruction to naive flatness)**.** Take a nontrivially valued K, choose π with $0 < |\pi| < 1$, and put $f(T) = \sum_{n\geq 0} \pi^{\lfloor\sqrt{n}\rfloor} T^n$. This series converges on the closed unit disc but has radius of convergence exactly one. Let Y be that disc and map it to $\mathbb{A}^{2,\mathrm{an}}$ by $\phi(t) = (t, f(t))$. At the image x of its Gauss point η, Theorem 2.60.4 shows that $\mathcal{O}_{\mathbb{A}^{2,\mathrm{an}},x}$ is a field. Consequently ϕ is naively flat at η.

Now base change by the analytic domain $X = \{|T_1| \leq 1\} \subset \mathbb{A}^{2,\mathrm{an}}$. In X, the series $f(T_1)$ is an analytic function and the graph is the closed subspace $T_2 - f(T_1) = 0$. The local ring $\mathcal{O}_{X,x}$ is a domain and $g = T_2 - f(T_1)$ is a nonzero element of its maximal ideal. The quotient by g is not flat: tensoring the injection given by multiplication by g with that quotient produces a zero map on a nonzero module. Thus naive flatness is lost under this domain base change.

The convergence radius follows from $|\pi|^{\lfloor\sqrt{n}\rfloor} \to 0$ and $\lim_n |\pi|^{\lfloor\sqrt{n}\rfloor/n} = 1$. Theorem 2.60.5 proves all the local-ring and base-change assertions used here. This is Ducros's example from [Duc18, 4.4.2–4.4.6].

Relative analytic differentials form a coherent sheaf $\Omega_{Y/X}$ representing analytic derivations. Write $\mathrm{rk}_y \Omega_{Y/X}$ for its fibre dimension over $\mathcal{H}(y)$. A Jacobian presentation at y is a factorization $W \hookrightarrow V \subset \mathbb{A}^{n,\mathrm{an}}_X$, where W is an affinoid domain of Y containing y, V is an affinoid domain, and the closed immersion has an ideal generated by $r = n - \mathrm{rk}_y \Omega_{Y/X}$ elements. A morphism admitting such a presentation is called quasi-smooth at y. These are [Duc18, Definitions 5.2.2 and 5.2.4]; they allow boundary points.

An L-analytic space Z is geometrically regular at z if, for every complete valued extension E/L, every point $z' \in Z_E$ above z, and every affinoid domain $V \subset Z_E$ containing z', the ordinary local ring $\mathcal{O}_{V,z'}$ is regular. Thus the condition is formulated on good charts even when Z itself is not good. For good Z this is equivalent to requiring that $\mathcal{O}_{Z_E,z'}$ be regular for all such E, z'. One implication follows by choosing an ordinary affinoid neighborhood of z'. For the converse, apply the Jacobian-recovery argument in Theorem 2.35.14 to $Z \to \mathcal{M}(L)$: at its rational lift the base local ring is a field, so the flatness needed in that argument is automatic. The resulting Jacobian presentation is preserved under scalar extension and analytic-domain restriction by Theorem 2.41.8; the forward regularity argument of Theorem 2.35.14 then proves regularity on every required chart.

Theorem 2.9.3. *A morphism $f : Y \to X$ is quasi-smooth at y if and only if it is analytically flat there and its $\mathcal{H}(f(y))$-analytic fibre is geometrically regular at y. For morphisms between good spaces, it is smooth at y if and only if it is quasi-smooth and boundaryless there. Analytic-domain inclusions are flat. For a morphism between good spaces, naive flatness of a coherent sheaf at an interior point implies analytic flatness there.*

Proof. The quasi-smoothness criterion, with all complete-extension tests on the analytic fibre and all good analytic base changes for flatness, is Theorem 2.35.14. Its last paragraph passes from good charts to general analytic spaces. The smoothness comparison for good spaces is Theorem 2.50.3, proved with the ordinary-open étale local definition of smoothness. Analytic-domain flatness is Theorem 2.47.1. Finally, for the ordinary-stalk notion of naive flatness in Definition 2.9.1, Theorem 2.62.3 proves the asserted implication at every interior point, over every complete ground field. □

References. The classical statements are [Duc18, Theorem 5.3.4, Corollary 5.4.8, 4.1.11 and Theorem 8.3.4]. The internal proofs organize them through Jacobian charts, ordinary local-ring comparison, smooth envelopes and coherent inner-flatness.

For example, a closed disc is quasi-smooth over K, but its boundary point is not a smooth point of its structure morphism. Its open disc is smooth. Likewise an affinoid neighborhood in a smooth space may acquire boundary when considered as a space over K. This does not prevent its use for local equations or integration on supports contained in its interior.

2.10 Analytification and algebraic comparison

For an affine finite-type K-scheme $Z = \operatorname{Spec} B$, define Z^{an} using multiplicative seminorms on B extending $|\cdot|_K$. There is no single global boundedness condition on this finitely generated algebra. Choose generators b_i; the loci $|b_i| \leq r_i$ are affinoid pieces, obtained from a presentation of B by replacing polynomials with convergent power series of those radii. As the radii grow, these pieces exhaust Z^{an}. The construction is independent of the generators and glues on Zariski opens.

Proposition 2.10.1. *Points of Z^{an} are equivalently pairs consisting of a scheme point $\xi \in Z$ and an absolute value on $\kappa(\xi)$ extending that of K. A scheme morphism induces the corresponding analytic morphism by restriction of seminorms.*

Proof. The kernel of a multiplicative seminorm on B is a prime $\mathfrak{p}$, and its norm on $B/\mathfrak{p}$ extends uniquely to $\kappa(\mathfrak{p})$. Conversely such a valued field gives a seminorm on B. Every chosen finite set of generators has finite sizes, so the point lies in one of the affinoid pieces above. The two constructions are inverse, and composition of algebra homomorphisms gives functoriality. □

Theorem 2.10.2 (Algebraic comparison used in this book)**.** *Analytification of finite-type K-schemes commutes with fibre products and complete scalar extension. It preserves closed immersions, finite morphisms, smooth morphisms and proper morphisms. Analytifications have no analytic boundary over K. For a proper K-scheme Z, analytification gives an equivalence between algebraic and analytic coherent sheaves, compatible with coherent cohomology. In particular algebraic line bundles and their homomorphisms are recovered from the analytic ones.*

Proof. Theorem 2.33.25 proves compatibility with fibre products and every complete scalar extension. Theorem 2.33.26 proves preservation of closed immersions and finite morphisms. Corollary 2.35.3 proves preservation of smooth morphisms, using the ordinary-open étale local definition of analytic smoothness. Theorem 2.33.31 proves preservation of proper morphisms. Applying Proposition 2.33.27 to the structure morphism gives the assertion about the boundary. Finally Theorem 2.34.8 is the asserted equivalence for a proper scheme, including its restriction to line bundles and full faithfulness on their homomorphisms; its cohomology comparison is Theorem 2.34.6. □

References. The classical comparison statements are described in [TemIntro, 5.1.1–5.1.2] and [Ber93, §2.6]. Their internal proofs here use the explicit affinoid construction, projective cohomology, and the Chow support argument.

GAGA concerns coherent algebraic geometry. It does not identify an arbitrary real-valued smooth function with an algebraic function, nor does it provide positivity or a product of currents. Its role here is narrower and concrete: it lets algebraic line bundles, divisors and finite maps be used as analytic geometric inputs. The additional metric and differential constructions occupy the later parts of the book.

2.11 Tate acyclicity and its proof dependencies

The proof of Theorem 2.3.1 proceeds from bounded Laurent calculations to rational covers, then to affinoid-domain covers. The structure sheaf and Kiehl coherence are constructed afterwards. This section describes the complex, records its classical sources, and gives a guide from the original proof tasks to their internal solutions.

2.11.1 The exact complex and its scope

Let K be complete for its fixed real-valued non-Archimedean absolute value, possibly trivial. Let A be a generalized K-affinoid algebra in the norm-admissible convention, $X = \mathcal{M}(A)$, and $V_1, \dots, V_m$ affinoid domains whose union is X. The theorem allows nonreduced A, arbitrary positive radii and boundary points. It assumes neither smoothness nor algebraic closedness. Before forming the complex one needs representing algebras A_I for all finite intersections V_I and compatible bounded restriction maps. Empty intersections contribute the zero module.

For a finite A-module M with its canonical norm, the normalized complex has

$$C^{-1} = M, \qquad C^q = 0 \quad (q \geq m),$$
$$C^q = \prod_{i_0 < \cdots < i_q} M \otimes_A A_{i_0 \cdots i_q} \quad (0 \leq q < m).$$

The differential is the alternating sum of restrictions, and finite products use the maximum norm. Exactness in every degree includes a bounded lifting estimate for each map onto its image; its constant may depend on the cover and norms. The ordered-tuple version, including repeated indices, is compared with this normalized complex by the bounded homotopies in Lemma 2.21.2. Theorem 2.38.1 gives both versions for arbitrary affinoid-domain covers.

Each tensor factor is finite over its own algebra A_I, and usually is not finite over A. Thus Theorem 1.11.5 alone does not prove that the Čech differentials are admissible. Nor does finiteness of M justify tensoring an exact complex: localization flatness or a direct finite-module argument is needed. This is a Čech statement; a derived-sheaf-cohomology assertion requires a separate comparison theorem.

Lemma 2.11.1 (Admissibility of a finite closed cover)**.** *Let $F_1, \dots, F_m$ be closed subsets covering a topological space X. For each $x \in X$, the union of those F_i containing x contains an ordinary open neighborhood of x.*

Proof. The finite union of those F_j not containing x is closed and misses x. Its complement is the required open neighborhood, since the entire family covers X. □

An affinoid domain in an affinoid space is compact and the ambient spectrum is Hausdorff, so the lemma explains the local finite-neighborhood condi-

tion for this G-cover. Compactness and the domain's topological representation are separate analytic inputs. A single covering member need not contain an ordinary neighborhood, and the argument does not apply to arbitrary infinite set-theoretic covers.

2.11.2 Verified source locators

All Temkin introductory-note locators refer to arXiv:1010.2235v2, dated 16 December 2011. Fact 3.3.2.1, printed p. 17, states exactness and admissibility for finite modules. It does not supply a complete proof there; Section 1.2 explicitly describes the use of unproved Facts. Section 3.3 assumes $H \neq 1$. The all-radii category uses $H = \mathbb{R}_{>0}$ and includes trivially valued K; it must not be confused with the strict $H = 1$ site over such a field.

Berkovich's 1993 paper, Section 1.2, printed p. 15, states both Tate and Kiehl in the general analytic setting, with trivial valuation permitted. For the nontrivial strict case it cites BGR 8.2.1/5 and 9.4.3/3; the passage to the general setting points to [Ber90, 2.2.5 and 2.1.11]. Remark 1.2.1(i) on the same page explicitly corrects Ber90 2.2.6(iii): in addition to affinoidness of A_V, one needs bijectivity of $V \to \mathcal{M}(A_V)$. This is an author-issued correction, unlike the three independently checked literal-scope examples in Section 1.9. The original proofs at these two Ber90 locators were not obtained in this round. Their numbers are verified as citations in Ber93, not as independently read Ber90 proof pages.

The original BGR scan was inspected at the following proof passages. The book treats the strict rigid setting over a nontrivially valued complete field.

- Theorem 8.2.1/1, p. 327, gives the structure-presheaf acyclicity statement. Its proof is in Sections 8.2.2–8.2.3, pp. 331–335. The finite-cover reduction uses 8.2.2/2–5, including Gerritzen–Grauert 7.3.5/3 and the small-scalar bound 7.3.4/7. The binary Laurent calculation uses the separate injectivity input 7.3.2/5.
- Corollary 8.2.1/5, p. 330, treats every algebraic A-module. Its free-resolution argument invokes localization flatness 7.3.2/6 and the finite-cover cohomological bound 8.1.3/2. Its text is an algebraic exactness assertion; the required Banach norm estimates must still be tracked for finite modules.

- Theorem 9.4.3/3, p. 378, is proved using Lemmas 9.4.3/5–6, pp. 378–382. These supply quantitative Laurent gluing and global finite generation, using complete module norms, coherent kernels, residue comparisons and Nakayama's lemma. None of these is silently included in the elementary foundation of this project.

References. [Ber93, §1.2, p. 15]; [BGR84, §§8.2, 9.4.3].

2.11.3 Domains, the structure sheaf and coherence

Theorem 2.11.2 (Rational-domain input)**.** *Let A be a generalized K-affinoid algebra and let $g, f_1, \dots, f_s$ generate the unit ideal. For positive radii r_i, the norm-admissible quotient*

$$A\{r^{-1}T\}/(gT_i - f_i)_i$$

represents the domain $|f_i(x)| \leq r_i|g(x)|$. Every bounded map $A \to B$ to a generalized affinoid algebra whose spectrum maps into this domain factors uniquely through that quotient as a bounded map.

Proof. Theorem 2.31.1 constructs the displayed actual quotient as a norm-admissible affinoid algebra and identifies its spectrum homeomorphically with the stated subset. Apply Theorem 2.31.2 to the given generalized affinoid target B. Its geometric criterion is exactly the image condition in the statement, and yields the unique bounded factorization. Both results allow all positive radii and a trivially valued ground field; zero targets have the unique zero factorization. No strictness or rational-cover acyclicity hypothesis is added. □

References. [TemIntro, 3.1.2.5 and 3.2.2.1].

Theorem 2.11.3 (Gerritzen–Grauert input)**.** *Every affinoid domain V of a generalized K-affinoid space X is a finite union of rational domains of X. The field may be trivially valued.*

Proof. Theorem 2.36.6 supplies a finite covering by rational domains defined over the original algebra A. Its proof allows arbitrary positive radii and both trivial and nontrivial absolute values. The domain convention in the statement agrees with that theorem by Theorem 2.36.7. □

References. [Tem05, Theorems 1.1 and 3.1, Corollary 3.2].

Theorem 2.11.4 (Affinoid structure-sheaf input)**.** *On the site of compact analytic domains of $X = \mathcal{M}(A)$ with finite domain coverings, the affinoid assignment $V \mapsto A_V$ extends to a sheaf of Banach algebras with bounded restrictions. For a compact domain covered by affinoid domains, its sections are the compatible tuples of affinoid sections, independently of the chosen finite cover up to canonical bounded isomorphism and uniformly equivalent norms.*

Proof. Restrict the ring sheaf of Theorem 2.39.4 to compact analytic domains. Theorem 2.39.5 realizes its sections as the closed matching kernel for every finite affinoid cover, proves boundedness of restrictions, and identifies the kernels obtained from two covers by mutually bounded maps. Those maps are induced by the unique restriction maps on common affinoid refinements, so the comparison is canonical. Their two boundedness estimates give uniform equivalence of the Banach norms. □

References. [TemIntro, Exercise 3.3.2.2(i)]; [Ber93, §1.2, pp. 15–16].

Theorem 2.11.5 (Kiehl coherence input)**.** *With the preceding affinoid structure sheaf, finite A-modules are equivalent to coherent $\mathcal{O}_{X_G}$-modules. The functor sends M to the sheaf with sections $M \otimes_A A_V$ on affinoid domains; a quasi-inverse is global sections. Here a coherent module is locally finitely generated, and for every analytic domain U, the kernel of every map $\mathcal{O}_{U_G}^n \to \mathcal{F}|_{U_G}$ is locally finitely generated.*

Proof. Theorem 2.38.5 proves the equivalence on the finite affinoid-domain site, including effective gluing of arbitrary finite local modules. Theorem 2.39.6 transports it to the full analytic G-site through the ringed-topos comparison. Its coherence condition is local finite generation together with local finite generation of the stated kernels, so it is exactly the condition here. Evaluating on an affinoid domain gives $M \otimes_A A_V$; evaluating on X recovers M. Restriction to the compact-domain basis preserves both the sheaf and these local properties. □

References. [BGR84, 9.4.3/3 and its Lemmas 5–6]; [Ber93, §1.2, p. 15]; [TemIntro, 3.3.4.1].

2.11.4 The original proof tasks and their solutions

The following task statements retain their original labels. The proof guide following the list identifies their internal solutions, including the complete replacement routes used for the Laurent and Gerritzen–Grauert tasks.

General-radius finite modules
Extend Noetherianity, closed submodules and norm-strict finite-module maps from Theorem 1.11.5 to every generalized affinoid algebra. Any strictification proof must prove exactness and norm reflection under the chosen scalar extension.

Spectra and power bounds
Prove the spectrum/unit criteria and the scaled power bounds needed for bounded rational substitution, including nonreduced algebras and radii not in $|K^\times|$.

Intersection and transitivity
Construct compatible affinoid algebras for intersections and iterated rational domains, with the required quotient norms and universal properties.

Localization flatness
Prove algebraic flatness of rational/affinoid localizations if using the BGR free-resolution route. Finite-module tensor completeness is not flatness.

The Laurent lower bound
For every affinoid A, finite M, $f \in A$ and $r > 0$, establish the lower bound required by Theorem 2.12.1, or supply a different complete strict-exactness proof. Also identify its coefficient quotients with the localized modules.

Čech normalization and refinement
Prove normalization and the finite double-complex comparison in all degrees, with bounded lifts and the required acyclicity on intersections.

Finite rational-cover reduction
Prove the reductions from finite rational covers to the chosen elementary Laurent calculations, including the small-bound and unit-ideal arguments. The BGR small-scalar step must be adapted when the valuation is trivial.

Scalar-extension descent
If radii or a nontrivial valuation are supplied by a complete extension, prove faithful exactness and the descent of norm-controlled primitives, preserving the original absolute value.

Gerritzen–Grauert prerequisites
Reconstruct Gruson tensor injectivity, reduction finiteness, graded reduction and the compactness argument on a separate noncircular branch.

Kiehl descent
Write the quantitative Laurent gluing, global finite generation and relation-kernel argument for coherent local presentations.

The general finite-module and spectral tasks are proved in Theorems 1.18.3, 1.20.1 and 1.20.2. Intersections and localization flatness are established in Corollary 2.36.10 and Proposition 2.32.3, together with Proposition 2.38.2.

For Laurent exactness, we use the strict coefficient estimate and then descend bounded primitives through one simultaneous free-radius extension. The finite comparison and refinement are Lemma 2.21.2 and Proposition 2.22.8; the resulting all-radius theorem is Theorem 2.32.2. Its proof uses the coefficient projection only as a bounded linear map.

Theorem 2.36.6 combines strict Gerritzen–Grauert with the Gauss-section finite truncation of Lemma 2.36.4. The tensor and graded-reduction results are proved on independent branches in Theorems 2.33.3 and 3.18.7. Finally, Theorems 2.28.6, 2.37.3 and 2.39.6 give quantitative gluing, finite-module descent and coherence on the full analytic site.

2.12 A quantitative criterion for an elementary Laurent pair

For an elementary Laurent cover, the coefficient splitting can be separated from the geometric localization theorem. The next result makes this separation precise. Its input is a numerical lower bound for one operator on two-sided Laurent series. It supplies strict exactness once that bound is available; it does not assert the bound for every affinoid module.

Let A be a Banach K-algebra and let M be a Banach A-module with $\|am\| \le \|a\|\|m\|$, as in Section 1.10. Fix $f \in A$ and $r > 0$. Define

$$E_+ = \left\{ \sum_{n\ge 0} m_n T^n : \|m_n\| r^n \longrightarrow 0 \right\},$$
$$E_- = \left\{ \sum_{n\le 0} m_n T^n : \|m_n\| r^n \longrightarrow 0 \text{ as } n \to -\infty \right\},$$
$$E = \left\{ \sum_{n\in\mathbb{Z}} m_n T^n : \|m_n\| r^n \longrightarrow 0 \text{ as } |n| \to \infty \right\}.$$

Each has norm $\max_n \|m_n\| r^n$. The coefficientwise argument of Lemma 1.10.1, with the indicated index sets, proves their completeness. We regard E_+, E_- as closed submodules of E. Multiplication by T is the shift of coefficients and rescales the norm by r. The following operators preserve their indicated modules:

$$Q_+ = T - f : E_+ \longrightarrow E_+, \qquad Q_- = 1 - fT^{-1} : E_- \longrightarrow E_-,$$
$$L = 1 - fT^{-1} : E \longrightarrow E.$$

Theorem 2.12.1 (Quantitative Laurent criterion)**.** *Suppose that there is a real number $c > 0$ such that*

$$\|Lw\| \ge c\|w\| \qquad (w \in E).$$

Endow $P_+ = E_+/Q_+E_+$, $P_- = E_-/Q_-E_-$ and $P_0 = E/LE$ with their quotient norms. They are Banach A-modules, and the sequence

$$0 \longrightarrow M \xrightarrow{\iota} P_+ \oplus P_- \xrightarrow{\delta} P_0 \longrightarrow 0,$$
$$\iota(m) = ([m], [m]), \quad \delta([u], [v]) = [u - v].$$

is strict exact. The direct sum has its maximum norm. More precisely, with

$$C = \max\left(1, \frac{\max(r, \|f\|)}{cr}\right),$$

one has $C^{-1}\|m\| \le \|\iota(m)\| \le \|m\|$. The induced map $(P_+ \oplus P_-)/\iota(M) \to P_0$ is isometric for the quotient norms.

Proof. Step 1: the coefficient identity. Define

$$d_0(u,v) = u - v, \qquad d_1(a,b) = Ta - b, \qquad E_+ \oplus E_- \longrightarrow E.$$

The kernel of d_0 consists of diagonal constants, since $E_+ \cap E_- = M$. Splitting into nonnegative and strictly negative powers gives a right inverse for d_0 of norm at most one. The supports of Ta and b in $d_1(a,b)$ are disjoint: the former has exponents at least one, the latter exponents at most zero. Thus d_1 is bijective and

$$\|d_1(a,b)\| = \max(r\|a\|, \|b\|), \qquad d_0(Q_+a, Q_-b) = Ld_1(a,b).$$

Taking $b = 0$ or $a = 0$ gives $\|Q_+a\| \geq cr\|a\|$ and $\|Q_-b\| \geq c\|b\|$. Each of Q_+, Q_-, L therefore has closed image: convergence of the images makes the original sequence Cauchy by its lower bound, and continuity identifies its limit. Lemma 1.11.1 now proves completeness and definiteness of the three quotient norms. The identity also makes δ well defined.

Step 2: exactness. Surjectivity of δ follows from the coefficient splitting for d_0. Suppose $\delta([u],[v]) = 0$, so that $u - v = Lw$ for some $w \in E$. Write $w = Ta - b$ uniquely. Then

$$u - Q_+a = v - Q_-b.$$

The left side belongs to E_+ and the right side to E_-; hence they equal a constant $m \in M$. This proves exactness in the middle. If $\iota(m) = 0$, write $m = Q_+a = Q_-b$. Then $L(Ta - b) = 0$. The lower bound on L gives $Ta = b$, and their disjoint supports force $a = b = 0$, so $m = 0$.

Step 3: the augmentation estimate. Representatives of the two classes of a constant m have the form $u = m + Q_+a$ and $v = m + Q_-b$. Step 1 gives

$$\max(r\|a\|, \|b\|) \leq c^{-1}\|u - v\| \leq c^{-1}\max(\|u\|, \|v\|).$$

Since $\|Q_+a\| \leq \max(r, \|f\|)\|a\|$, the identity $m = u - Q_+a$ gives $\|m\| \leq C\max(\|u\|, \|v\|)$. Taking infima over both representatives proves the lower bound for ι; using the constant representatives proves its upper bound. Thus ι is strict. Its image is also the kernel of the continuous map δ and is closed.

Step 4: the final quotient norm. The difference map is contractive. Conversely, each representative $z \in E$ of a class in P_0 has a splitting $z = u - v$ with $\max(\|u\|, \|v\|) \leq \|z\|$. The class $([u],[v])$ is a lift with no larger norm. Taking infima over representatives z proves that the quotient norm induced

from $P_+ \oplus P_-$ is at most the norm of the given class in P_0. Contractivity gives the reverse inequality. The induced map is therefore isometric, which proves strictness of δ. □

For finite modules over a strict affinoid algebra and value-group radii, Theorem 2.16.4 supplies the required lower bound. Theorem 2.18.3 identifies these coefficient quotients with their analytic section modules, giving the elementary Laurent step in the finite-cover argument. Simultaneous strictification then yields all positive radii in Theorem 2.32.2; affinoid-domain coverings are treated in Theorem 2.38.1.

2.13 Laurent coefficient normal forms

The Laurent coefficient module in Theorem 2.12.1 was defined directly. We now identify it with a quotient of a two-variable coefficient module, including the quotient norm and the actual image of the relation map. The same construction treats two boundary radii $0 < a \le b$; setting $a = b = r$ gives the equal-radius module used in that theorem.

Let K be complete for its specified real-valued non-Archimedean absolute value, which may be trivial. Let A be a commutative unital Banach K-algebra, and let M be a unital Banach A-module: its norm is definite, non-Archimedean, complete and homogeneous over K, and $\|am\| \le \|a\|\|m\|$. No finiteness hypothesis is imposed on A or M. All coefficient spaces below use these specified norms.

Lemma 2.13.1 (Weighted coefficient completeness)**.** *Let I be a countable set and let $(w_i)_{i\in I}$ be positive real numbers. Put*

$$c_0(I, M; w) = \{(m_i)_{i\in I} : \text{for every } \varepsilon > 0,\ \#\{i : \|m_i\|w_i \ge \varepsilon\} < \infty\},$$
$$\|(m_i)\|_w = \sup_{i\in I} \|m_i\|w_i.$$

We put $\sup \varnothing = 0$ if I is empty. This is a Banach A-module, and its finitely supported families are dense. Each nonzero family attains its norm at a coordinate. The coordinate map at i has bound w_i^{-1}.

Proof. The finite-tail condition makes the supremum finite. The ultrametric inequality, homogeneity, definiteness and the contractive A-action follow coordinatewise. To see that a nonzero family attains its norm, choose a coordinate of positive weighted norm t. Only finitely many coordinates have

weighted norm at least t, and the maximum among them is the supremum of all coordinates.

Let $x^{(j)} = (m_i^{(j)})$ be Cauchy. For each i, the inequality

$$\|m_i^{(j)} - m_i^{(k)}\| \le w_i^{-1}\|x^{(j)} - x^{(k)}\|_w$$

gives a limit $m_i \in M$. Given $\varepsilon > 0$, choose j so that the norm on the right before multiplication by w_i^{-1} is less than $\varepsilon/2$ for all sufficiently large k. Passing to the limit in each coordinate gives

$$\|m_i - m_i^{(j)}\|w_i \le \varepsilon/2 \qquad (i \in I).$$

Outside a finite set, $\|m_i^{(j)}\|w_i < \varepsilon/2$, so $\|m_i\|w_i \le \varepsilon/2$ there. Thus $x = (m_i)$ belongs to the displayed space, and the same estimate proves convergence to x. Finally, retaining all coordinates with weighted norm at least $\varepsilon/2$ gives a finite truncation with error at most $\varepsilon/2$. This proves density and completeness. The coordinate bound is part of the definition of the supremum norm. □

Fix $0 < a \le b$ and write $\lambda = b/a$. We use T, U for two independent variables, and Z for the Laurent variable. Define

$$\begin{aligned}
D_{a,b}(M) &= M\{b^{-1}T, aU\} \\
&= \left\{ \sum_{i,j\ge 0} m_{ij}T^iU^j : \|m_{ij}\|b^ia^{-j} \longrightarrow 0 \right\}, \\
\|g\|_D &= \sup_{i,j\ge 0} \|m_{ij}\|b^ia^{-j}, \\
w_n &= \begin{cases} b^n & n \ge 0, \\ a^n & n < 0, \end{cases} \\
L_{a,b}(M) &= \left\{ \sum_{n\in\mathbb{Z}} c_nZ^n : \|c_n\|w_n \longrightarrow 0 \text{ as } |n| \to \infty \right\}, \\
\|c\|_L &= \sup_{n\in\mathbb{Z}} \|c_n\|w_n.
\end{aligned}$$

In the definition of $D_{a,b}(M)$, convergence to zero means convergence outside finite subsets of $\mathbb{N}^2$. This is a definition by coefficients, not a tensor-product assertion. The preceding lemma proves completeness and polynomial density for both spaces. Multiplication by TU shifts both indices up by one and multiplies the norm by λ. Consequently

$$P = TU - 1 : D_{a,b}(M) \longrightarrow D_{a,b}(M), \qquad \|Pg\|_D \le \lambda\|g\|_D$$

is a bounded A-linear operator.

Theorem 2.13.2 (The Laurent coefficient normal form)**.** *For the data just specified, there are A-linear maps*

$$\begin{aligned}\mathcal{N} &: D_{a,b}(M) \longrightarrow L_{a,b}(M),\\ \mathcal{S} &: L_{a,b}(M) \longrightarrow D_{a,b}(M),\\ \mathcal{H} &: D_{a,b}(M) \longrightarrow D_{a,b}(M).\end{aligned}$$

given by the convergent coefficient formulas

$$\begin{aligned}(\mathcal{N}g)_n &= \sum_{k\geq 0} m_{n_+ +k, n_- +k}, \qquad n_+ = \max(n,0), \quad n_- = \max(-n,0),\\ \mathcal{S}c &= \sum_{n\geq 0} c_n T^n + \sum_{n<0} c_n U^{-n},\\ (\mathcal{H}g)_{ij} &= \sum_{k\geq 1} m_{i+k,j+k}.\end{aligned}$$

They satisfy

$$\|\mathcal{N}g\|_L \leq \|g\|_D, \qquad \|\mathcal{S}c\|_D = \|c\|_L, \qquad \|\mathcal{H}g\|_D \leq \lambda^{-1}\|g\|_D,$$
$$P\mathcal{H} = 1 - \mathcal{S}\mathcal{N}, \qquad \mathcal{H}P = 1, \qquad \mathcal{N}P = 0, \qquad \mathcal{N}\mathcal{S} = 1, \qquad \mathcal{H}\mathcal{S} = 0.$$

In particular, every g has a unique decomposition

$$g = Pq + \mathcal{S}c, \qquad q = \mathcal{H}g, \quad c = \mathcal{N}g,$$
$$\|g\|_D = \max\{\lambda\|q\|_D, \|c\|_L\}.$$

The relation map satisfies $\|Pq\|_D = \lambda\|q\|_D$ and has closed image. Its image is the actual kernel of $\mathcal{N}$, so

$$\ker\mathcal{N} = (TU-1)D_{a,b}(M),$$
$$D_{a,b}(M)/(TU-1)D_{a,b}(M) \xrightarrow{\overline{\mathcal{N}}} L_{a,b}(M)$$

is an isometric isomorphism of Banach A-modules. The quotient norm of $[g]$ is $\|\mathcal{N}g\|_L$, attained at its unique representative supported on the two coordinate axes, with the origin counted once. The associated short exact sequence is split as a sequence of Banach A-modules.

Proof. Step 1: polynomial reduction and its bounds. On a monomial mT^iU^j, put $d = \min(i,j)$ and define

$$\mathcal{N}(mT^iU^j) = mZ^{i-j}, \qquad \mathcal{H}(mT^iU^j) = m\sum_{\ell=0}^{d-1} T^{i-1-\ell}U^{j-1-\ell},$$

where the sum is zero if $d = 0$. Extend these formulas A-linearly to polynomials, and define $\mathcal{S}$ by the displayed axis formula. The weights satisfy

$$b^i a^{-j} = \lambda^d w_{i-j}.$$

Thus $\mathcal{N}$ does not increase the norm of a monomial. Every monomial in its $\mathcal{H}$-image has weight at most $\lambda^{-1} b^i a^{-j}$, because its weight relative to the original one is $\lambda^{-1-\ell}$. Applying the ultrametric inequality to finite sums proves

$$\|\mathcal{N}g\|_L \le \|g\|_D, \qquad \|\mathcal{H}g\|_D \le \lambda^{-1}\|g\|_D \qquad (g \in M[T,U]).$$

The axis formula preserves each nonzero coefficient and its weight, so $\mathcal{S}$ is isometric, on Laurent polynomials and on the full coefficient space.

Step 2: completion and the explicit sums. Choose finite polynomial truncations g_ν converging to g. The preceding bounds make $\mathcal{N}g_\nu$ and $\mathcal{H}g_\nu$ Cauchy in their respective complete coefficient spaces. Their limits define $\mathcal{N}g$ and $\mathcal{H}g$. Applying the same bounds to differences of two approximations proves independence of the truncations, linearity and the asserted bounds. This constructs the extensions directly.

For a fixed n, the terms in the claimed formula for $(\mathcal{N}g)_n$ satisfy

$$\|m_{n_+ + k, n_- + k}\| w_n = \lambda^{-k} \|m_{n_+ + k, n_- + k}\| b^{n_+ + k} a^{-(n_- + k)} \longrightarrow 0.$$

For fixed i, j, the terms in the formula for $(\mathcal{H}g)_{ij}$ satisfy

$$\|m_{i+k, j+k}\| b^i a^{-j} = \lambda^{-k} \|m_{i+k, j+k}\| b^{i+k} a^{-(j+k)} \longrightarrow 0.$$

Both series therefore converge in M by Lemma 1.1.2. The bounded coordinate maps of Lemma 2.13.1 identify their sums with the limits constructed from polynomial truncations. In particular, the formulas define families with the required uniform finite-tail condition; this also follows by approximating their images by the corresponding finite polynomial images.

Step 3: the identities. For a polynomial monomial, telescoping gives

$$(TU - 1) m \sum_{\ell=0}^{d-1} T^{i-1-\ell} U^{j-1-\ell} = m T^i U^j - m T^{i-d} U^{j-d},$$
$$\mathcal{H}(m T^{i+1} U^{j+1}) - \mathcal{H}(m T^i U^j) = m T^i U^j.$$

Summing over monomials proves $P\mathcal{H} = 1 - \mathcal{S}\mathcal{N}$ and $\mathcal{H}P = 1$ on polynomials. Substitution of Z, Z^{-1} gives $\mathcal{N}P = 0$. The axis formula gives $\mathcal{N}\mathcal{S} = 1$

and $\mathcal{HS} = 0$. Every map in these identities is bounded. Taking limits of polynomial truncations proves all five identities on the complete spaces.

Step 4: the relation image and the quotient norm. The identities give the displayed decomposition. Applying $\mathcal{H}$ and $\mathcal{N}$ to any such decomposition proves uniqueness. Their bounds, together with the bound for P, imply

$$\begin{aligned} \|Pq + \mathcal{S}c\|_D &\leq \max\{\lambda\|q\|_D, \|c\|_L\}, \\ \lambda\|q\|_D = \lambda\|\mathcal{H}(Pq + \mathcal{S}c)\|_D &\leq \|Pq + \mathcal{S}c\|_D, \\ \|c\|_L = \|\mathcal{N}(Pq + \mathcal{S}c)\|_L &\leq \|Pq + \mathcal{S}c\|_D. \end{aligned}$$

These are the asserted norm equality. Taking $c = 0$ gives $\|Pq\|_D = \lambda\|q\|_D$, and hence injectivity of P. If Pq_ν converges, this equality makes q_ν Cauchy. Completeness gives a limit q, and boundedness of P identifies the original limit as Pq. Thus $PD_{a,b}(M)$ is closed.

We have $\mathcal{N}P = 0$, while $\mathcal{N}g = 0$ implies $g = P\mathcal{H}g$. This proves the kernel equality with the algebraic image, without first taking its closure. For every $h \in D_{a,b}(M)$,

$$\|\mathcal{N}g\|_L = \|\mathcal{N}(g + Ph)\|_L \leq \|g + Ph\|_D.$$

The choice $h = -\mathcal{H}g$ gives $g + Ph = \mathcal{SN}g$ and attains equality. Hence the quotient norm is exactly $\|\mathcal{N}g\|_L$. The target is complete, so the quotient is a Banach module. The right inverse $\mathcal{S}$, the left inverse $\mathcal{H}$ and the decomposition prove the splitting and norm-strictness. A representative supported on the two axes equals $\mathcal{S}c$ for a unique c; applying $\mathcal{N}$ identifies c with the class, proving the final uniqueness assertion. □

In particular, if $a = b = r$, then $w_n = r^n$, $\lambda = 1$, and the theorem gives

$$\begin{gathered} M\{r^{-1}T, rU\}/(TU - 1)M\{r^{-1}T, rU\} \simeq L_{r,r}(M), \\ L_{r,r}(M) = \left\{ \sum_{n\in\mathbb{Z}} c_n Z^n : \|c_n\| r^n \longrightarrow 0 \text{ as } |n| \to \infty \right\}, \\ \|(TU - 1)q\| = \|q\|. \end{gathered}$$

Here the isomorphism is isometric. Neither the equal-radius assertion nor the two-radius assertion requires a radius in $|K^\times|$.

Proposition 2.13.3 (Multiplication, module actions and naturality)**.**

Coefficient convolution makes $D_{a,b}(A)$ and $L_{a,b}(A)$ Banach K-algebras, and makes $D_{a,b}(M)$ and $L_{a,b}(M)$ contractive modules over them. The map $\mathcal{N}_A$ is an algebra homomorphism, and

$$\mathcal{N}_M(fg) = \mathcal{N}_A(f)\mathcal{N}_M(g) \qquad (f \in D_{a,b}(A),\ g \in D_{a,b}(M)).$$

Consequently, the preceding quotient isomorphism is compatible with the isometric algebra isomorphism

$$D_{a,b}(A)/(TU-1)D_{a,b}(A) \ \simeq\ L_{a,b}(A).$$

All the maps and the quotient identification are natural for bounded coefficient maps. More precisely, let $\phi : A \to A'$ be a bounded homomorphism of Banach K-algebras, let M' be a contractive Banach A'-module, and let $v : M \to M'$ be a K-linear map satisfying

$$v(am) = \phi(a)v(m), \qquad \|v(m)\| \le C\|m\|$$

for some fixed $C > 0$. Applying v to coefficients gives maps $D(v)$ and $L(v)$ bounded by C, commuting with $P, \mathcal{N}, \mathcal{S}, \mathcal{H}$ in their appropriate source and target spaces. The induced map on the relation quotients also has bound C.

Proof. For D, the weight of a product monomial is the product of its weights. For L, write $w_n = \max(a^n, b^n)$; then

$$w_{n+m} \le w_n w_m \qquad (n, m \in \mathbb{Z}).$$

Finite coefficient convolution and the ultrametric inequality therefore give $\|fg\| \le \|f\|\|g\|$ for the polynomial products and actions in both cases. To construct a product of general elements, take polynomial approximations f_ν, g_ν. They are bounded and Cauchy. The identity

$$f_\nu g_\nu - f_\mu g_\mu = (f_\nu - f_\mu)g_\nu + f_\mu(g_\nu - g_\mu)$$

and the product bound make the products Cauchy. Completeness defines their limit; applying the same estimate to two choices of approximations proves independence. The bound persists under limits, and associativity, the identity and the module law follow from the polynomial identities by continuity.

For D, each coefficient is a finite convolution sum. For L, the coefficient of Z^n is the convergent sum

$$\sum_{k\in\mathbb{Z}} f_k g_{n-k}.$$

Indeed,

$$\|f_k g_{n-k}\|w_n \le (\|f_k\|w_k)(\|g_{n-k}\|w_{n-k}) \le (\|f_k\|w_k)\|g\|,$$

so the terms tend to zero as $|k| \to \infty$. Lemma 1.1.2 gives convergence independent of enumeration. Bounded coordinate evaluation identifies this sum with the coefficient of the product constructed by truncation.

On polynomials, $\mathcal{N}_A$ is substitution $T \mapsto Z$, $U \mapsto Z^{-1}$; it preserves multiplication, and the corresponding module identity holds. Its boundedness and the product bounds extend both identities to the complete spaces. Theorem 2.13.2 identifies its kernel and quotient norm, proving the algebra and module assertions. The axis section $\mathcal{S}$ is only asserted to be A-linear; it need not preserve multiplication.

Finally, the coefficient bound for v preserves the finite-tail conditions and gives $\|D(v)\|, \|L(v)\| \le C$. On polynomials, commutation with the four displayed operators follows from their finite coefficient formulas; continuity extends it to all series. The relation image is therefore preserved, so there is an induced map on quotients. For every representative g its image has a representative of norm at most $C\|g\|$. Taking the infimum over representatives gives the asserted quotient bound. Semilinearity of the coefficient maps gives the corresponding compatibility with algebra maps and module actions. □

The proof uses completeness, coefficient estimates and finite telescoping identities. It applies at trivial valuation and in every residue characteristic. It makes no identification with sections on an analytic domain, assumes no finite-module tensor comparison, and proves no lower bound for $1 - fZ^{-1}$ with a general $f \in A$. Those are separate inputs to the geometric application of Theorem 2.12.1.

2.13.1 A coefficient gluing sequence

The normal form gives a complete algebraic justification for the coefficient splitting used in Proposition 2.3.2. We state this consequence for Banach module coefficients and arbitrary positive endpoint radii. The objects in the

statement are coefficient spaces; their identification with sections on analytic domains is a separate assertion.

Proposition 2.13.4 (Gluing coefficient series at an intermediate radius)**.** *Let A and M satisfy the standing Banach hypotheses, and let $0 < a \le b$. Write $E_c(M) = M\{c^{-1}X\}$ for $c = a, b$, and write $L_{a,b}(M)$ for the annular Laurent coefficient space, with weight b^n for $n \ge 0$ and a^n for $n < 0$. The coefficient-preserving maps give a strict exact sequence*

$$0 \longrightarrow E_b(M) \xrightarrow{j} E_a(M) \oplus L_{a,b}(M) \xrightarrow{\delta} L_{a,a}(M) \longrightarrow 0,$$
$$j(h) = (h, h), \qquad \delta(u, v) = u - v.$$

Here the middle term has its maximum norm. The map j is isometric, and δ has an A-linear right inverse of norm at most one. Consequently, the norm on $L_{a,a}(M)$ equals the quotient norm induced by δ. For $M = A$, Proposition 2.13.3 identifies the Laurent coefficient spaces with the corresponding quotient algebras.

Proof. For $n \ge 0$, the inequality $a^n \le b^n$ shows that both restriction maps to the smaller coefficient space are contractive. The map from $E_b(M)$ to the nonnegative part of $L_{a,b}(M)$ is isometric, so j is isometric. The difference map δ is contractive.

If $w = \sum_{n\in\mathbb{Z}} m_n X^n \in L_{a,a}(M)$, set

$$s(w) = \left(\sum_{n\ge 0} m_n X^n, -\sum_{n<0} m_n X^n \right).$$

The first series belongs to $E_a(M)$. The second belongs to $L_{a,b}(M)$ because its negative-degree weights are still a^n. These two series have disjoint coefficient supports and their maximum norm is $\|w\|$. Thus s is an A-linear right inverse of δ with norm at most one.

Suppose $u - v = 0$ in $L_{a,a}(M)$. Equality means equality of every coefficient. Since u has no negative powers, neither does v. The nonnegative coefficients of v tend to zero with weights b^n, so they define a unique $h \in E_b(M)$ satisfying $(u, v) = j(h)$. This proves exactness in the middle; injectivity of j is already part of its isometry. Finally, contractivity of δ makes the target norm no larger than its quotient norm, while s gives the reverse inequality. All asserted norm statements follow. □

Taking $A = M = K$, $b = 1$ and $a = s \in (0,1)$ supplies the coefficient normal forms and exactness used in Proposition 2.3.2. This calculation does not prove the Laurent estimate for an arbitrary element $f \in A$, nor does it identify these quotients with all analytic localization modules. Those are the additional inputs required by Theorem 2.12.1.

2.14 Weighted coefficient extensions of finite modules

Uniformly bounded lifts of its relations allow a finite presentation to extend to convergent coefficient families. The strict affinoid finite-module theorem provides precisely this bound. We first prove that taking weighted coefficient families preserves the bound, and then identify the resulting coefficient module with an ordinary tensor product. The weights may be arbitrary positive real numbers; no change of the ground field is made.

Throughout, K is complete for its fixed real-valued non-Archimedean absolute value, which may be trivial. A Banach K-space has a definite, complete, non-Archimedean, K-homogeneous norm. Banach algebras are commutative and unital, with a submultiplicative norm and $\|1\| = 1$; Banach module actions satisfy $\|am\| \le \|a\|\|m\|$. All finite powers have the maximum norm; the maximum over an empty family is zero. Norm-strictness has the quantitative meaning of Definition 1.9.1.

For a countable set I and positive weights $w = (w_i)_{i\in I}$, write $\mathcal{C}_w(I,E) = c_0(I,E;w)$ as in Lemma 2.13.1; thus

$$\mathcal{C}_w(I,E) = \{(e_i)_{i\in I} : \text{for every } \varepsilon > 0, \#\{i : w_i\|e_i\| \ge \varepsilon\} < \infty\},$$
$$\|(e_i)\|_w = \sup_{i\in I} w_i\|e_i\|.$$

We put $\sup \varnothing = 0$. The cited lemma, with coefficient algebra K if necessary, proves completeness and density of finite-support families. A bounded linear map $u : E \to F$ with bound U induces a coefficient map with the same bound, since $w_i\|u(e_i)\| \le Uw_i\|e_i\|$. There is also a canonical isometry

$$\mathcal{C}_w(I,E^m) \simeq \mathcal{C}_w(I,E)^m \qquad (m < \infty).$$

It merely rearranges coefficients: a supremum commutes with a finite maximum, and a finite union of finite sets is finite, so both the norms and the finite-tail conditions agree.

Proposition 2.14.1 (Strict exactness with unchanged lifting bounds)**.** *Let*

$$0 \longrightarrow E' \xrightarrow{\iota} E \xrightarrow{p} E'' \longrightarrow 0$$

be an exact sequence of Banach K-spaces with bounded maps. Suppose there are $D, C > 0$ such that

$$\|x\| \le D\|\iota(x)\| \quad (x \in E'), \qquad \forall z \in E'' \quad \exists y \in E : p(y) = z, \quad \|y\| \le C\|z\|.$$

For every countable I and positive w, the coefficient sequence is exact and norm-strict:

$$0 \longrightarrow \mathcal{C}_w(I, E') \xrightarrow{\iota_w} \mathcal{C}_w(I, E) \xrightarrow{p_w} \mathcal{C}_w(I, E'') \longrightarrow 0.$$

The inverse bound D, the lifting bound C, and the upper bounds of the original maps remain valid. In particular, these constants do not depend on I or w. If the norm on E'' is the quotient norm induced by p, the induced map

$$\mathcal{C}_w(I, E)/\iota_w\mathcal{C}_w(I, E') \longrightarrow \mathcal{C}_w(I, E'')$$

is isometric for the quotient norm. The same assertions hold for bounded maps of Banach modules over a fixed Banach algebra.

Proof. Injectivity and the inclusion $\operatorname{im}\iota_w \subseteq \ker p_w$ hold coefficientwise. If $y = (y_i) \in \ker p_w$, exactness gives unique $x_i \in E'$ with $\iota(x_i) = y_i$. The bound $w_i\|x_i\| \le Dw_i\|y_i\|$ proves that (x_i) lies in the required coefficient space and gives the inverse bound for ι_w.

For $z = (z_i) \in \mathcal{C}_w(I, E'')$, choose a lift y_i satisfying $\|y_i\| \le C\|z_i\|$, taking $y_i = 0$ when $z_i = 0$. Then (y_i) has a vanishing weighted tail and $\|(y_i)\|_w \le C\|z\|_w$. This proves surjectivity and its lifting bound. Only countably many coordinate choices are involved. No linear section of p is asserted or needed. The coefficientwise estimate preceding the proposition gives the upper map bounds. Thus the maps are norm-strict; the middle kernel is closed because p_w is continuous.

Now assume that E'' has the quotient norm. The map p_w is contractive, so its quotient norm is at least $\|z\|_w$. For each fixed $\lambda > 1$, choose lifts coefficientwise with $\|y_i\| \le \lambda\|z_i\|$. They give a lift of z with norm at most $\lambda\|z\|_w$. Taking the infimum over lifts and then letting $\lambda \downarrow 1$ proves equality of the norms. For module maps, the coefficient maps remain A-linear, and the same argument proves exactness and all the estimates. The selected lifts only establish these bounds; they are not asserted to depend linearly on the data. □

We apply this elementary lifting result to a finite presentation over a strict affinoid algebra. The coefficient weights need not be strict: the closed-image theorem is used only over the original coefficient algebra.

Theorem 2.14.2 (Finite presentations after coefficient extension)**.** *Let A be a strict K-affinoid algebra with a submultiplicative Banach norm uniformly equivalent to its defining quotient norm. Let M be a finite A-module. Choose a surjection $q : A^m \to M$ and give M its quotient norm. Choose a finite presentation*

$$A^s \xrightarrow{D} A^m \xrightarrow{q} M \longrightarrow 0.$$

For every countable set I and every family of positive weights w_i, the coefficient sequence

$$\mathcal{C}_w(I,A)^s \xrightarrow{D_w} \mathcal{C}_w(I,A)^m \xrightarrow{q_w} \mathcal{C}_w(I,M) \longrightarrow 0$$

is exact. The relation image is closed; D_w is norm-strict onto that image, and q_w induces an isometry from its normed quotient onto $\mathcal{C}_w(I,M)$. More precisely, there is $C_D > 0$, depending on D and the norms but not on I, w, such that every $v \in \ker q_w$ has a lift a with

$$D_w a = v, \qquad \|a\|_w \le C_D \|v\|_w.$$

For every fixed $\lambda > 1$, each $z \in \mathcal{C}_w(I,M)$ has a lift b with $q_w b = z$ and $\|b\|_w \le \lambda \|z\|_w$.

Proof. Theorem 1.11.5 makes $R = \ker q$ a finite closed submodule of A^m. In particular, the quotient norm on M is complete and is a canonical finite-module norm. Its algebra action is contractive: if v lifts m, then av lifts am, and taking the infimum of $\|av\| \le \|a\|\|v\|$ proves the assertion. A finite set of generators of R provides the stated matrix D.

Give R its induced norm from A^m. The same theorem, applied to the map $D : A^s \to A^m$, supplies C_D such that every $r \in R$ has a preimage of norm at most $C_D\|r\|$. This is a bound for the actual image norm, not merely for a quotient topology. Coefficientwise lifting with this constant gives

$$\operatorname{im} D_w = \mathcal{C}_w(I,R) = \ker q_w.$$

The second equality and the lifting and quotient-norm assertions for q_w follow from Proposition 2.14.1 applied to $0 \to R \to A^m \to M \to 0$. The finite-power identification was verified above. The kernel is closed by continuity

of q_w. Finally the matrix D_w has the same coefficient bound as D, and the displayed controlled lifts prove its norm-strictness. No theorem about arbitrary submodules over the extended coefficient algebra has been used. □

2.14.1 Power-series and Laurent coefficient algebras

To interpret the preceding presentation as scalar extension, the coefficient space must carry multiplication. Let $\Gamma = \mathbb{N}^p \times \mathbb{Z}^q$, where $p, q \geq 0$ are finite. Choose a weight $w : \Gamma \to \mathbb{R}_{>0}$ satisfying

$$w_0 = 1, \qquad w_{\gamma+\delta} \leq w_\gamma w_\delta \quad (\gamma, \delta \in \Gamma).$$

For a Banach A-module M, abbreviate $\mathcal{C}_w(M) = \mathcal{C}_w(\Gamma, M)$ and write its elements as $\sum_\gamma m_\gamma X^\gamma$. The symbols denote coefficient families; convergence refers to the weighted norm just defined.

Lemma 2.14.3 (Convolution with two-sided exponents)**.** *Convolution makes $B = \mathcal{C}_w(A)$ a Banach A-algebra and $\mathcal{C}_w(M)$ a Banach B-module. The constant embedding $A \to B$ is isometric, and*

$$\|bm\|_w \leq \|b\|_w \|m\|_w, \qquad (bm)_\eta = \sum_{\gamma+\delta=\eta} b_\gamma m_\delta.$$

The coefficient sum converges even when it is infinite, and its value is independent of its enumeration.

Proof. For finitely supported families, convolution is a finite sum. The module inequality and the weight inequality give

$$w_\eta \left\| \sum_{\gamma+\delta=\eta} b_\gamma m_\delta \right\| \leq \max_{\gamma+\delta=\eta} (w_\gamma \|b_\gamma\|)(w_\delta \|m_\delta\|) \leq \|b\|_w \|m\|_w.$$

Choose finite-support approximations $b^{(n)} \to b$ and $m^{(n)} \to m$. These sequences are bounded, and

$$b^{(n)} m^{(n)} - b^{(k)} m^{(k)} = (b^{(n)} - b^{(k)}) m^{(n)} + b^{(k)} (m^{(n)} - m^{(k)}).$$

The finite-product estimate makes their products Cauchy. Lemma 2.13.1 gives a limit independent of the approximations, with the same norm bound. Associativity, commutativity for B, the unit identity and the module law

pass from finite supports to limits by this estimate. The constant embedding is isometric because $w_0 = 1$.

To identify the coefficient formula, fix η. The two nonnegative real families $w_\gamma\|b_\gamma\|$ and $w_\delta\|m_\delta\|$ are bounded and tend to zero outside finite subsets. For every $\varepsilon > 0$, only finitely many pairs have product at least ε: if both supremums are nonzero, each factor must be at least ε divided by the other supremum. For pairs with $\gamma + \delta = \eta$,

$$w_\eta\|b_\gamma m_\delta\| \le (w_\gamma\|b_\gamma\|)(w_\delta\|m_\delta\|).$$

The summability criterion of Lemma 1.1.2 therefore applies. Coordinate evaluation is continuous, so the sum equals the coefficient of the product constructed by finite-support approximation. If either family is zero, this assertion is immediate. □

Two choices cover the coefficient spaces needed below. For arbitrary positive radii $r_1, \dots, r_p$ and $t_1, \dots, t_q$, the weight

$$w_{\alpha,\beta} = r^\alpha t^\beta, \qquad (\alpha, \beta) \in \mathbb{N}^p \times \mathbb{Z}^q,$$

is multiplicative. It gives multivariable power series and two-sided Laurent series at fixed radii. More generally, for $0 < a_j \le b_j$ use

$$w_{\alpha,\beta} = r^\alpha \prod_{j=1}^{q} \max(a_j^{\beta_j}, b_j^{\beta_j}).$$

This weight is submultiplicative: for each Laurent coordinate it is the maximum of the two multiplicative weights at the endpoints. The one-variable case is the coefficient algebra of Proposition 2.13.3. These are coefficient-space definitions; their identification with particular analytic-domain quotients is a separate matter.

Theorem 2.14.4 (Ordinary and completed coefficient tensor products)**.** *Let A, M, q, D be as in Theorem 2.14.2, and let $B = \mathcal{C}_w(A)$ have the convolution structure above. The natural map*

$$\Phi : B \otimes_A M \longrightarrow \mathcal{C}_w(M), \qquad b \otimes m \longmapsto (b_\gamma m)_{\gamma\in\Gamma},$$

is a B-linear isomorphism. Give the algebraic tensor product its projective seminorm

$$\|z\|_\pi = \inf_{z=\sum_{j=1}^{l} b_j\otimes m_j} \max_{1\le j\le l} \big(\|b_j\|_w\|m_j\|\big),$$

where only finite tensor expressions occur. Then

$$\|z\|_\pi = \|\Phi(z)\|_w.$$

In particular this seminorm is definite and complete, and the canonical map to its separated completion is an isometric isomorphism:

$$B \otimes_A M \simeq B\widehat{\otimes}_A M \simeq \mathcal{C}_w(M).$$

These identifications are natural in finite A-modules. No strictness or Noetherianity assumption on B is required.

Proof. Write $e_1, \dots, e_m$ for the images of the standard generators under q. To see the algebraic presentation without assuming exactness of any completed tensor functor, put $Q = B^m/D_B B^s$, where D_B is the same matrix with its entries regarded as constants in B. Sending a vector (b_i) to $\sum_i b_i \otimes e_i$ defines a map

$$Q \longrightarrow B \otimes_A M.$$

Conversely, given $b \in B$ and $m \in M$, choose $a \in A^m$ with $q(a) = m$ and send (b, m) to the class of ba in Q. Two such lifts differ by Dc for some $c \in A^s$, so their classes differ by $D_B(bc)$ and agree. Sums of lifts and scalar multiples of lifts show additivity and A-balancedness. The defining property of the algebraic tensor product therefore gives an inverse map. Thus $B \otimes_A M = Q$ by an explicit generators-and-relations argument.

Under the coefficient identifications, D_B and the generator map $B^m \to \mathcal{C}_w(M)$ are exactly D_w and q_w. This is first checked on finitely supported families; continuity and Lemma 2.14.3 give B-linearity on all families. Theorem 2.14.2 identifies Q with $\mathcal{C}_w(M)$, with its coefficient norm equal to the quotient norm from B^m. The resulting map is precisely Φ. This proves its algebraic bijectivity and shows why its relation image is closed.

For every tensor expression, contractivity of the A-action on M gives

$$\left\| \Phi\left(\sum_j b_j \otimes m_j \right) \right\|_w \leq \max_j \|b_j\|_w \|m_j\|.$$

Taking infima proves $\|\Phi(z)\|_w \leq \|z\|_\pi$. For the reverse inequality, $\|e_i\| \leq 1$ because M has the quotient norm. Every lift $(b_i) \in B^m$ of $\Phi(z)$ gives the tensor expression $z = \sum_i b_i \otimes e_i$, and hence

$$\|z\|_\pi \leq \max_i \|b_i\|_w.$$

The infimum of the right side is $\|\Phi(z)\|_w$ by the quotient-norm assertion already proved. This establishes isometry. Completeness and definiteness follow from Lemma 2.13.1, so separated completion adds no elements and removes no nonzero elements.

For an A-linear map $u : M \to N$, coefficientwise application of u sends $(b_\gamma m)_\gamma$ to $(b_\gamma u(m))_\gamma$. Thus the identifications commute with $1 \otimes u$ on pure tensors, and hence on the whole algebraic tensor product. The maps are bounded by Theorem 1.11.5, so the same assertion holds after completion. □

The isometry uses the specified quotient norm on M. With another uniformly equivalent norm $\|\cdot\|_*$, choose $0 < a \le b$ such that $a\|m\| \le \|m\|_* \le b\|m\|$. The corresponding coefficient and projective norms each satisfy the same comparisons with the original ones. Consequently

$$\frac{a}{b}\|\Phi(z)\|_{w,*} \le \|z\|_{\pi,*} \le \frac{b}{a}\|\Phi(z)\|_{w,*}.$$

Thus the bounded isomorphism and completeness are independent of the chosen canonical norm. Uniform equivalence, rather than equality of topologies, is the relevant hypothesis.

Corollary 2.14.5 (Exact scalar extension on finite modules)**.** *With A, B as above, let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finite A-modules with canonical norms. Then both its ordinary tensor sequence and its separated completed tensor sequence with B are exact. Under their coefficient identifications they are norm-strict. If the original injection has inverse bound D on its image and the original surjection has lifting bound C, these same constants work for the coefficient sequence.*

Proof. The original maps are bounded and norm-strict by Theorem 1.11.5. Proposition 2.14.1 gives exactness and the stated constants for coefficient families. Apply the natural identifications of Theorem 2.14.4. For quotient norms these are isometric. For other canonical norms they are uniformly bounded isomorphisms by the preceding comparison, which preserves norm-strictness; the stated constants continue to refer to the coefficient norms themselves. □

Remark 2.14.6 (The Laurent interface and its remaining input). Let A be strict affinoid and M a finite A-module with its canonical norm. For one variable and $r > 0$, write

$$B_+ = \left\{ \sum_{n \geq 0} a_n T^n : \|a_n\| r^n \to 0 \right\}, \qquad B_- = \left\{ \sum_{n \leq 0} a_n T^n : \|a_n\| r^n \to 0 \right\},$$

$$B_0 = \left\{ \sum_{n \in \mathbb{Z}} a_n T^n : \|a_n\| r^n \to 0 \text{ as } |n| \to \infty \right\}.$$

The three coefficient modules E_+, E_-, E used in Theorem 2.12.1 are now canonically $B_+ \otimes_A M$, $B_- \otimes_A M$, $B_0 \otimes_A M$, respectively, and agree with their separated completed tensors. For B_-, use the variable $S = T^{-1}$ and radius r^{-1}. The identifications preserve the inclusions into bilateral series and multiplication by every element of A. All assertions allow arbitrary $r > 0$ and a trivially valued K, while A remains strict.

The closed relation images proved here are those of matrices whose entries lie in the original algebra A. The operator $1 - fT^{-1}$ has a nonconstant coefficient in B_0, so its injectivity, closed image and a uniform lower bound have not been deduced from this result. Those are the next Laurent inputs. Likewise no arbitrary finite B_0-module theorem or analytic-domain universal property is asserted here. The ordinary/completed tensor comparison is proved for this coefficient extension; it is not the false assertion for a finite module tensored with an arbitrary Banach module.

2.15 Algebraic separation from the Stacks foundations

We use the stated parts of two Stacks Project results as declared foundations. Only the short application needed by the Laurent argument is proved below. For an R-module N and $g \in R$, the localization N_g consists of fractions x/g^a, $a \geq 0$, with

$$\frac{x}{g^a} = \frac{y}{g^b} \quad \Longleftrightarrow \quad \text{there is } e \geq 0 \text{ with } g^e(g^b x - g^a y) = 0.$$

In particular, $x/1 = 0$ means that $g^e x = 0$ for some $e \geq 0$. This is the defining equivalence relation, not an additional exactness theorem.

Corollary 2.15.1 (Detection by maximal-ideal powers)**.** *Let R be a commutative Noetherian ring with identity and let M be a finite unital R-module. Then*

$$\bigcap_{\mathfrak{m}\in \mathrm{Max}(R)} \bigcap_{k\geq 1} \mathfrak{m}^k M = 0.$$

Proof. If $M = 0$ there is nothing to prove; this includes the zero ring. We interpret an empty intersection as M. Suppose that a nonzero x belongs to the displayed intersection. Its annihilator $J = \mathrm{Ann}_R(x)$ is proper. Apply [StacksAlg, Tag 00E0, Lemma 10.17.2(2)] to the nonzero ring R/J and take the inverse image of a maximal ideal. This gives a maximal ideal $\mathfrak{m}$ of R containing J. Put $N = \bigcap_{k\geq 1} \mathfrak{m}^k M$.

Apply [StacksAlg, Tag 00IQ, Lemma 10.51.5(1)] with $I = \mathfrak{p} = \mathfrak{m}$. It gives $g \notin \mathfrak{m}$ such that $N_g = 0$. Since $x \in N$, the defining equivalence relation for localization gives $g^e x = 0$ for some integer $e \geq 0$. Hence $g^e \in J \subseteq \mathfrak{m}$. This is impossible, since no nonnegative power of $g \notin \mathfrak{m}$ belongs to the prime ideal $\mathfrak{m}$. □

References. The localization convention is the fraction construction in [StacksAlg, Tag 00CM, immediately before Definition 10.9.6 (Tag 07JZ)]. The two cited theorem parts are used as the declared Stacks foundation; the argument above is only their short application to the displayed detection statement.

The conclusion concerns the intersection over all maximal ideals. An individual maximal-adic intersection need not vanish over a nonlocal ring: for a field F, take $R = F \times F$, $M = F$ with action through the first projection, and $\mathfrak{m} = F \times 0$. Then $\mathfrak{m}$ is maximal and $\mathfrak{m}^k M = M$ for every $k \geq 1$.

2.16 Injectivity and lower bounds for Laurent operators

The coefficient criterion in Theorem 2.12.1 reduces an elementary Laurent pair to a lower bound for $1 - fT^{-1}$. We now prove injectivity for every positive radius over a strict affinoid algebra. When the radius lies in the value group of the ground field, the coefficient algebra is again strict affinoid, and the finite-module theorem supplies the lower bound. The algebraic separation input is the Stacks-based Corollary 2.15.1; the arguments involving norms are given here.

Keep the fixed complete non-Archimedean field K, with trivial valuation permitted. For a Banach K-space V and $r > 0$, put

$$E_r(V) = \{(v_n)_{n\in\mathbb{Z}} : \|v_n\|r^n \longrightarrow 0 \text{ as } |n| \longrightarrow \infty\}, \qquad \|v\|_r = \sup_{n\in\mathbb{Z}} \|v_n\|r^n.$$

We also write its elements as Laurent series $\sum_n v_n T^n$. Lemma 2.13.1 gives completeness. For a bounded operator f on V, the operator $L_f = 1 - fT^{-1}$ is defined by

$$(L_f v)_n = v_n - f v_{n+1}.$$

Lemma 2.16.1 (Scalar recurrence over a normed residue field)**.** *Let F be a field which is a Banach K-algebra with a definite submultiplicative norm. Let V be a finite-dimensional F-vector space with a Banach K-norm uniformly equivalent to the maximum coordinate norm for an F-basis. For every $\lambda \in F$ and every $r > 0$, the operator L_λ on $E_r(V)$ is bounded and injective. No multiplicativity of the norm on F, and no finiteness of F/K, is assumed.*

Proof. Choose a basis and comparison constants $C, D > 0$ such that, for $v = \sum_j a_j e_j$,

$$\max_j \|a_j\|_F \le C\|v\|, \qquad \|v\| \le D \max_j \|a_j\|_F.$$

These inequalities identify $E_r(V)$, by a bounded isomorphism with bounded inverse, with a finite power of $E_r(F)$. Multiplication by λ has bound $DC\|\lambda\|_F$, so L_λ is bounded. If $\lambda = 0$, it is the identity. Suppose $\lambda \neq 0$ and $L_\lambda v = 0$. Each coordinate sequence then satisfies

$$a_n = \lambda a_{n+1}, \qquad a_n = \lambda^{-n} a_0 \quad (n \in \mathbb{Z}).$$

If $a_0 \neq 0$, then $a_n a_{-n} = a_0^2 \neq 0$, whereas submultiplicativity gives

$$0 < \|a_0^2\|_F \le \big(\|a_n\|_F r^n\big)\big(\|a_{-n}\|_F r^{-n}\big) \longrightarrow 0 \qquad (n \longrightarrow +\infty).$$

This is impossible. Thus every coordinate has $a_0 = 0$, and the recurrence makes every coefficient zero. The argument also covers $V = 0$. □

Theorem 2.16.2 (Laurent injectivity for finite strict affinoid modules)**.** *Let A be a strict K-affinoid algebra with a submultiplicative Banach norm uniformly equivalent to its defining quotient norm. Let M be a finite A-module with a canonical Banach norm and contractive A-action. For every $f \in A$ and every real $r > 0$,*

$$L_f = 1 - fT^{-1} : E_r(M) \longrightarrow E_r(M)$$

is injective. This assertion concerns injectivity alone; it does not assert a lower norm bound for every r.

Proof. Residue-field quotients. By Theorem 1.11.5, A is Noetherian, its ideals are closed, and finite modules and their subquotients have canonical complete norms. Fix a maximal ideal $\mathfrak{m}$. The field $F = A/\mathfrak{m}$, with the quotient norm, is a Banach K-algebra. Indeed, definiteness and completeness follow from Lemma 1.11.1; approximating representatives in a product proves submultiplicativity. Also $\|1_F\| \leq 1$ and $0 < \|1_F\| \leq \|1_F\|^2$, so $\|1_F\| = 1$. The scalar homogeneity follows from the quotient definition.

Every finite A-module V annihilated by $\mathfrak{m}$ is a finite-dimensional F-vector space. An F-basis gives an A-linear isomorphism $F^d \to V$. Applying Theorem 1.11.5 to this isomorphism and its inverse proves uniform equivalence of the canonical norm on V and the maximum coordinate norm. Lemma 2.16.1 therefore proves injectivity on $E_r(V)$ for the action of f mod $\mathfrak{m}$.

Powers of a maximal ideal. We prove injectivity on $E_r(M/\mathfrak{m}^k M)$ by induction on $k \geq 1$. The case $k = 1$ is the preceding case. For $k > 1$, use

$$0 \longrightarrow \mathfrak{m}^{k-1}M/\mathfrak{m}^k M \longrightarrow M/\mathfrak{m}^k M \longrightarrow M/\mathfrak{m}^{k-1}M \longrightarrow 0.$$

All three modules are finite; their maps are bounded and norm-strict by Theorem 1.11.5. Proposition 2.14.1, with weights r^n, makes the corresponding coefficient sequence exact. The left term is annihilated by $\mathfrak{m}$, so the residue-field argument applies to it. If a coefficient family in the middle is killed by L_f, its image on the right is zero by induction. It comes from the left, and injectivity of the coefficient inclusion shows that this preimage is also killed by L_f. Injectivity on the left makes the preimage zero. This proves the induction.

Separation of coefficients. Let $v \in E_r(M)$ satisfy $L_f v = 0$. Its image in every $E_r(M/\mathfrak{m}^k M)$ is zero by the induction. Each coefficient v_n thus belongs to $\mathfrak{m}^k M$ for every maximal $\mathfrak{m}$ and every $k \geq 1$. Corollary 2.15.1 gives $v_n = 0$.

Hence $v = 0$. At no point did we use finite-dimensionality of F over K, a multiplicative quotient norm, or a point of the analytic spectrum. □

Injectivity does not by itself provide the numerical input of the Laurent criterion. For the unit radius, the coefficient module is finite over another strict affinoid algebra. This permits a second application of the quantitative finite-module theorem.

Lemma 2.16.3 (The unit Laurent coefficient algebra is strict)**.** *For A as in Theorem 2.16.2, the convolution algebra $B = E_1(A)$ is strict K-affinoid with its coefficient norm. More precisely, the normal-form map gives an isometric algebra isomorphism*

$$A\langle T, U\rangle/(TU-1) \xrightarrow{\sim} B, \qquad T \longmapsto Z, \quad U \longmapsto Z^{-1}.$$

For every finite A-module M with a canonical norm, $E_1(M)$ is a finite B-module and its coefficient norm is uniformly equivalent to its canonical finite-B-module norm. If M has the quotient norm from a chosen $A^m \to M$, then $E_1(M)$ has exactly the quotient norm from the induced $B^m \to E_1(M)$.

Proof. Theorem 2.13.2 and Proposition 2.13.3, with $a = b = 1$, give the displayed algebra isomorphism, the actual kernel $(TU-1)A\langle T, U\rangle$, and equality of the quotient and coefficient norms. To verify strictness rather than assume it, choose an admissible bounded surjection

$$\pi : R_d = K\langle X_1, \dots, X_d\rangle \longrightarrow A.$$

Its kernel is closed, and admissibility supplies a uniform bound for lifts of elements of A. Applying Proposition 2.14.1 to $0 \to \ker\pi \to R_d \to A \to 0$, with index set $\mathbb{N}^2$ and unit weights, extends π to a bounded surjection with the same lifting bound

$$R_d\langle T, U\rangle \longrightarrow A\langle T, U\rangle.$$

Regrouping coefficients gives an isometric algebra identification

$$R_d\langle T, U\rangle = K\langle X_1, \dots, X_d, T, U\rangle.$$

To check the convergence condition, joint coefficients tending to zero outside finite subsets imply that each (T, U)-row has the same property and that the row norms tend to zero. Conversely, only finitely many rows can have

norm at least a fixed $\varepsilon > 0$, and in each of those rows only finitely many coefficients can have norm at least ε. The norms are equal because the two suprema commute. Multiplication agrees first for polynomials and then by continuity.

The normal-form quotient onto B has an isometric A-linear section. Composing it with the preceding surjection gives a bounded surjection from this unit-radius Tate algebra to B, with uniform lifts. Thus the coefficient norm on B is uniformly equivalent to its defining quotient norm, as required by the definition of strict affinoid. Finally Theorem 2.14.4, for $\mathbb{Z}$ and unit weights, identifies $E_1(M)$ with the finite B-module $B \otimes_A M$. The coefficient presentation in Theorem 2.14.2 gives the asserted equality with the quotient norm from B^m. Replacing the chosen canonical norm on M by an equivalent one preserves uniform equivalence of coefficient norms. □

Theorem 2.16.4 (A lower Laurent bound at value-group radii)**.** *Under the hypotheses on A, M in Theorem 2.16.2, let $f \in A$ and $r \in |K^\times|$. There exists a real constant $c = c(A, M, f, r, \|\cdot\|) > 0$ such that*

$$\|L_f w\|_r \geq c\|w\|_r \qquad (w \in E_r(M)).$$

The constant is independent of w. Consequently L_f has closed image and is norm-strict onto its image. For a trivially valued ground field this theorem includes the radius $r = 1$.

Proof. First let $r = 1$. Lemma 2.16.3 makes $E_1(M)$ a finite module with canonical norm over the strict affinoid algebra $B = E_1(A)$. The operator L_f is multiplication by $1 - fZ^{-1} \in B$, hence is B-linear. By Theorem 1.11.5, it has closed image and is norm-strict. Thus some $C > 0$ has the property that every y in the image has a preimage of norm at most $C\|y\|_1$. Theorem 2.16.2 makes that preimage unique. Substituting $y = L_f w$ gives $\|w\|_1 \leq C\|L_f w\|_1$ and hence the required bound with $c = C^{-1}$.

For general $r \in |K^\times|$, choose $s \in K^\times$ with $|s| = r$. The substitution $T = sZ$ gives an isometry

$$\Phi_s : E_r(M) \longrightarrow E_1(M), \qquad (m_n)_n \longmapsto (s^n m_n)_n.$$

The norms and vanishing tails agree term by term. A coefficient calculation gives

$$\Phi_s L_f = L_{f/s} \Phi_s.$$

Applying the unit-radius bound to f/s proves the assertion. The image and norm-strictness statements follow either by this isometry or directly from the lower bound and completeness: convergence of images makes preimages Cauchy, and continuity identifies their limit. No scalar extension or change in the absolute value is involved. □

Corollary 2.16.5 (Strict exactness for a coefficient Laurent pair)**.** *Let A, M, f, r satisfy Theorem 2.16.4. Define E_+, E_-, E, the relation operators, and P_+, P_-, P_0 exactly as in Theorem 2.12.1, with these M, f, r. All three quotients are Banach A-modules for their quotient norms, and*

$$0 \longrightarrow M \xrightarrow{m\mapsto([m],[m])} P_+ \oplus P_- \xrightarrow{([u],[v])\mapsto[u-v]} P_0 \longrightarrow 0$$

is strict exact. With c supplied by Theorem 2.16.4, the augmentation estimate and the final quotient isometry of Theorem 2.12.1 hold. This applies to every $f \in A$, without a condition that f be a coordinate or a unit.

Proof. Theorem 2.16.4 verifies the sole additional lower-bound hypothesis of Theorem 2.12.1. Apply that theorem to the specified coefficient modules and their actual relation images. Its proof gives closedness of all relation images, completeness of the quotients, both exactness assertions, and the stated quantitative estimates. □

The coefficient argument feeds into the analytic Laurent sequence of Theorem 2.18.3. Finite refinement and simultaneous strictification extend it to the rational Čech complexes of Theorem 2.32.2. The subsequent domain and sheaf arguments culminate in Theorems 2.38.1 and 2.39.6.

2.17 Strict quotient presentations and finite modules

The coefficient construction is useful for localization only after its relation quotients are controlled. In this section the new radii lie in the value group. This allows the finite-module theorem over strict affinoids to control the actual images of the relation maps. No statement about a structure sheaf is used.

Fix a complete real-valued non-Archimedean field K, allowing the trivial absolute value. A nonzero algebra below carries the submultiplicative norm

and norm-admissibility convention of Definition 1.1.3. We also allow a relation ideal to be the whole algebra: its quotient is the zero ring with its zero norm, and every unital module over it is zero. Statements about such a quotient have this explicit degenerate interpretation; the normalization $\|1\| = 1$ is required only for nonzero algebras.

Lemma 2.17.1 (Strict weighted adjunction and relation quotients)**.** *Let A be a nonzero strict K-affinoid algebra with a Banach norm uniformly equivalent to a defining quotient norm. Let $n \geq 0$ and choose $\rho_1, \dots, \rho_n \in |K^\times|$. Put*

$$D = A\{\rho^{-1}X\}, \qquad \left\| \sum_{\alpha \in \mathbb{N}^n} a_\alpha X^\alpha \right\|_\rho = \sup_\alpha \|a_\alpha\| \rho^\alpha.$$

Then D is strict K-affinoid in the norm-admissible sense. Every algebraic ideal $J \subset D$ is finitely generated and closed. With the quotient norm, $B = D/J$ is a strict affinoid quotient, or the zero quotient under the convention above. If $J = (h_1, \dots, h_t)$, the map

$$D^t \longrightarrow D, \qquad (d_j) \longmapsto \sum_j h_j d_j$$

has actual image J and is norm-strict onto that image. No unit-ideal hypothesis on other elements of A is needed for this assertion.

Proof. Choose a bounded admissible surjection $p : R_d = K\langle Z_1, \dots, Z_d\rangle \to A$. There are constants $C_p, L_p > 0$ such that $\|p(z)\| \leq C_p\|z\|$ and every $a \in A$ has a lift of norm at most $L_p\|a\|$, choosing zero for $a = 0$. Applying the coefficient lifting argument of Proposition 2.14.1, with index set $\mathbb{N}^n$ and unit weights, gives a surjection

$$R_d\langle Y_1, \dots, Y_n\rangle \longrightarrow A\langle Y_1, \dots, Y_n\rangle$$

with the same upper bound and lifting bound. This is an algebra homomorphism: the assertion holds on polynomial coefficient families and passes to their limits by Lemma 2.14.3. Regrouping the joint coefficients identifies its source isometrically with R_{d+n}. Indeed, a joint family tends to zero outside finite subsets exactly when each row does so and the row norms tend to zero; finite unions of the finitely many relevant rows prove the reverse implication.

Choose $c_i \in K^\times$ with $|c_i| = \rho_i$. The substitution $X_i = c_i Y_i$ is an isometric algebra isomorphism from D to $A\langle Y_1, \dots, Y_n\rangle$: it sends a_α to $c^\alpha a_\alpha$, preserving

both the weighted norm and the finite-tail condition. Thus the displayed surjection supplies a bounded admissible unit-radius Tate presentation of D. Theorem 1.11.5, now applied to D, proves finite generation and closedness of J, as well as norm-strictness of the displayed relation map.

Lemma 1.11.1 makes B complete with a definite quotient norm. Approximating lifts in D shows that this norm is submultiplicative. If $B \neq 0$, its unit has norm at most one; writing $u = \|1_B\| > 0$, submultiplicativity gives $u \leq u^2$, hence $u = 1$. The quotient map has lifts bounded by any fixed factor $\lambda > 1$. Composing these lifts with the preceding Tate-presentation lifts gives a bounded admissible surjection from R_{d+n} to B. This proves strictness when $B \neq 0$; when $J = D$, all quotient assertions reduce to the zero space. □

The same presentation also controls the relations on a finite module. It is useful to retain exact quotient norms here, because these give an isometry rather than only an unspecified bounded comparison.

Theorem 2.17.2 (Finite modules over a strict relation quotient)**.** *Use the notation of Lemma 2.17.1. Let M be a finite A-module, choose a surjection $q : A^m \to M$, and give M its quotient norm. Write*

$$E = M\{\rho^{-1}X\}, \qquad JE = \left\{ \sum_{j=1}^{t} h_j e_j : e_j \in E \right\}, \qquad Q = E/JE.$$

Here $J = (h_1, \dots, h_t)$, and E has its coefficient norm. Then E is a finite D-module with a canonical norm; JE is its actual algebraic submodule, is closed, and does not depend on the chosen generators of J. The relation map $E^t \to E$ is norm-strict onto JE. The coefficient and tensor maps induce natural isometric B-linear isomorphisms

$$B \otimes_A M \;\simeq\; B\widehat{\otimes}_A M \;\simeq\; Q, \qquad \overline{d} \otimes m \longmapsto [d\, m].$$

The algebraic tensor product carries the projective seminorm, the completed tensor product is its separated completion, and Q has the quotient norm. This is also the finite-B-module quotient norm induced by the generators $q(e_1), \dots, q(e_m)$. For any other uniformly equivalent canonical norm on M, the same maps are bounded isomorphisms with uniformly equivalent norms.

Proof. **Step 1: the actual relation module.** Choose a finite presentation

$$A^s \xrightarrow{U} A^m \xrightarrow{q} M \longrightarrow 0.$$

Theorems 2.14.2 and 2.14.4, for the multiplicative weights ρ^α, identify

$$E = D^m/U_D D^s = D \otimes_A M,$$

with its coefficient norm equal to the quotient norm from D^m. Thus it is a finite D-module with a canonical norm. Lemma 2.17.1 makes D strict. Theorem 1.11.5 applied over D now shows that the map $(e_j) \mapsto \sum_j h_j e_j$ has closed image and is norm-strict. Its image is the algebraic submodule JE, not merely its closure. If one changes generators of J, each new generator is a finite D-linear combination of the old ones, so this submodule is unchanged.

Step 2: the algebraic identification and the quotient norm. Apply right exactness of the ordinary tensor product, in [StacksAlg, Tag 00DF, Lemma 10.12.10], to the displayed finite presentation. It gives

$$B \otimes_A M = B^m/U_B B^s.$$

Under the surjection $D^m \to E$, the inverse image of JE is exactly

$$U_D D^s + JD^m.$$

Indeed, lift the finitely many terms of an expression $\sum_j h_j e_j$ to D^m; their difference from any other lift lies in $U_D D^s$. The reverse inclusion is immediate. Under $D^m \to B^m$, the inverse image of $U_B B^s$ is the same submodule: lift a finite vector in B^s to D^s and subtract its image under U_D. Consequently both Q and $B \otimes_A M$ are the quotient of D^m by this identical actual submodule. This proves the displayed algebraic identification, including the formula $\overline{d} \otimes m \mapsto [d\,m]$.

The maximum norm on B^m is exactly the quotient norm from D^m. To check the only nontrivial inequality, lift each of the finitely many coordinates with any fixed factor $\lambda > 1$, then let $\lambda \downarrow 1$. Successive quotient norms therefore give on both modules the infimum of the norms of all their lifts in D^m. They agree. This also shows that the norm on Q is its indicated finite-B-module norm. Since JE is closed, Lemma 1.11.1 makes it definite and complete.

Step 3: the projective norm. Let ν denote this common quotient norm on $P = B \otimes_A M$, and write $m_i = q(e_i)$. Since $\|m_i\| \le 1$, every lift $(b_i) \in B^m$ of z gives

$$z = \sum_i b_i \otimes m_i, \qquad \|z\|_\pi \le \max_i \|b_i\|.$$

Taking the infimum gives $\|z\|_\pi \le \nu(z)$. Conversely, fix a finite expression $z = \sum_j b_j \otimes v_j$ and $\lambda > 1$. Choose lifts $a_j = (a_{ij}) \in A^m$ with $\|a_j\| \le \lambda\|v_j\|$, using zero when $v_j = 0$. The map $A \to B$ is contractive, because the constant inclusion in D is isometric and $D \to B$ is a norm quotient. Thus

$$z = \sum_i \left(\sum_j b_j \overline{a_{ij}} \right) \otimes m_i, \qquad \nu(z) \le \lambda \max_j \big(\|b_j\| \|v_j\| \big).$$

Letting $\lambda \downarrow 1$ and taking the infimum over finite tensor expressions proves $\nu(z) \le \|z\|_\pi$. The projective seminorm therefore equals the complete definite quotient norm. Its separated completion neither adds elements nor kills nonzero elements. If $B = 0$ or $M = 0$, the same conclusions hold on the zero module.

Step 4: functoriality and other norms. For an A-linear map $u : M \to M'$, coefficientwise application is bounded by Theorem 1.11.5. It commutes with each multiplication by h_j, so it descends to the relation quotients. On pure tensors the resulting map is $\overline{d} \otimes m \mapsto \overline{d} \otimes u(m)$; this proves naturality for both ordinary and completed tensors. Finally, if $a\|m\| \le \|m\|_* \le b\|m\|$ for $0 < a \le b$, then coefficient, quotient and projective norms each satisfy the same comparisons with their original norms. Thus the same identifications remain bounded isomorphisms for any other canonical norm. □

In particular, one may take $J = (gX_1 - f_1, \dots, gX_n - f_n)$ for arbitrary $g, f_i \in A$. The relation quotient and its finite-module comparison are already defined in this generality. A unit-ideal condition on $(g, f_1, \dots, f_n)$ enters only when this quotient is to represent a rational analytic domain. No flatness of $A \to B$ follows from the theorem: it identifies a single tensor quotient and does not assert preservation of injections.

2.18 Iterated quotients and elementary Laurent diagrams

The preceding quotient construction permits a direct comparison of presentations. We first show that adjoining a second block of variables commutes with taking a quotient, with equality of the resulting norms. We then identify the coefficient Laurent sequence with a sequence of finite modules over explicit strict affinoid quotient algebras. These are statements about Banach

algebras and modules; identifying them with sections on analytic domains requires the geometric localization theorem.

Throughout this section, K is complete for its fixed real-valued non-Archimedean absolute value, which may be trivial, and A is a strict K-affinoid algebra with the norm conventions of Lemma 2.17.1. All displayed radii belong to $|K^\times|$. A relation quotient is either a strict affinoid algebra or the zero algebra with its zero norm. The latter is allowed in this section as a separate case; it does not satisfy the normalization $\|1\| = 1$ in Definition 1.1.3. Every unital module over it is zero. All quotients below use the actual algebraic ideals or submodules, and their indicated quotient norms.

Lemma 2.18.1 (Adjoining variables after a quotient)**.** *Let $\rho = (\rho_1, \dots, \rho_p)$ and $\sigma = (\sigma_1, \dots, \sigma_q)$ be finite tuples in $|K^\times|$, allowing $p = 0$ or $q = 0$. Put*

$$D = A\{\rho^{-1}X\}, \qquad B = D/I, \qquad F = A\{\rho^{-1}X, \sigma^{-1}Y\},$$

where I is an ideal of D. The coefficient quotient induces an isometric algebra isomorphism

$$F/IF \xrightarrow{\sim} B\{\sigma^{-1}Y\}.$$

In particular, its kernel is the actual extended ideal IF, which is closed. If $J = (q_1, \dots, q_t)$ is an ideal of $B\{\sigma^{-1}Y\}$ and $Q_j \in F$ are any lifts of its displayed generators, then

$$\big(B\{\sigma^{-1}Y\}\big)/J \simeq F/(IF + (Q_1, \dots, Q_t))$$

isometrically. The last ideal and the resulting isomorphism are independent of the chosen lifts.

Proof. Regrouping coefficients identifies F isometrically with $D\{\sigma^{-1}Y\}$. Indeed, for the coefficients $a_{\alpha\beta}$, the joint finite-tail condition with weights $\rho^\alpha\sigma^\beta$ is equivalent to the following two conditions: each β-row belongs to D, and the row norms multiplied by σ^β tend to zero outside finite sets. In the reverse direction, only finitely many rows can have weighted norm at least a fixed $\varepsilon > 0$, and each of these rows contributes only finitely many coefficients of that size. Both norms equal $\sup_{\alpha,\beta} \|a_{\alpha\beta}\|\rho^\alpha\sigma^\beta$. Multiplication agrees on polynomials and hence everywhere by density and continuity.

By Lemma 2.17.1 and Theorem 1.11.5, D is strict, I is finite and closed, and $D \to B$ is a quotient map of Banach spaces. Proposition 2.14.1 therefore shows that its coefficient extension is surjective, has kernel

$$c_0(\mathbb{N}^q, I; \sigma^\beta),$$

and induces precisely the coefficient norm on the quotient. Choose generators $u_1, \dots, u_s$ of I. The generator map $D^s \to I$, where I has its induced norm, is norm-strict by Theorem 1.11.5. Thus some $C > 0$ permits every $i \in I$ to be written as $i = \sum_\ell u_\ell d_\ell$ with $\max_\ell \|d_\ell\| \le C\|i\|$. Applying this bound coefficientwise to any element of the displayed kernel produces s series in $D\{\sigma^{-1}Y\}$, proving that the kernel is IF, without taking a closure. The reverse inclusion follows by applying the quotient map.

Write $\pi : F \to B\{\sigma^{-1}Y\}$ for this surjection. If $h \in \pi^{-1}(J)$, express $\pi(h) = \sum_j q_j b_j$ and lift each of the finitely many b_j to $H_j \in F$. Then $h - \sum_j Q_j H_j \in IF$. Hence $\pi^{-1}(J) = IF + (Q_1, \dots, Q_t)$ as actual ideals. This ideal is closed because F is strict. The two successive quotient norms agree with the direct quotient norm: taking the infimum over lifts through the intermediate quotient and then over representatives modulo J is the same as taking the infimum over all representatives modulo $\pi^{-1}(J)$. Approximation of each infimum by any fixed factor greater than one proves both inequalities even when an infimum is not attained. Changing a lift Q_j adds an element of IF, so does not change the ideal. If $I = D$, both first quotients are zero and the same statements hold directly. □

Theorem 2.18.2 (The completed tensor product of two quotient presentations)**.** *Let*

$$\begin{aligned} B &= A\{\rho^{-1}X\}/I, \\ C &= A\{\sigma^{-1}Y\}/J, \\ H &= A\{\rho^{-1}X, \sigma^{-1}Y\}/(I, J). \end{aligned}$$

where (I, J) denotes the sum of the two extended ideals. Use the stated quotient norms on B, C, H, and the projective seminorm on $B \otimes_A C$. The multiplication of the two canonical maps into H induces an isometric algebra isomorphism

$$B\widehat{\otimes}_A C \xrightarrow{\sim} H.$$

This is a statement about the separated completion; it does not assert that the algebraic tensor product is already complete or has definite projective seminorm. The isomorphism commutes with the two maps from B, C.

Proof. If either B or C is zero, H is zero as well and the assertion follows. Assume otherwise, and put $G = B\widehat{\otimes}_A C$. To make the analytic construction explicit, the seminorm on the algebraic tensor product is

$$\|z\|_\pi = \inf_{z = \sum_{j=1}^n b_j \otimes c_j} \max_j \|b_j\| \|c_j\|.$$

Multiplying two finite tensor expressions and taking infima gives $\|zz'\|_\pi \leq \|z\|_\pi\|z'\|_\pi$. Thus its seminorm-zero subspace is an ideal, multiplication descends to the separated quotient and extends continuously to G. The maps $b \mapsto b \otimes 1$ and $c \mapsto 1 \otimes c$ are contractive. The unit has norm at most one; if $G \neq 0$, submultiplicativity then forces its norm to be one. The construction also gives a commutative unital Banach algebra when nonzero.

The canonical maps $B \to H$ and $C \to H$ are contractive: the maps from their coefficient algebras preserve their weights and then pass to quotient norms. Their product is an A-balanced bilinear map with bound one. On a finite tensor expression its image consequently has norm at most $\max_j \|b_j\|\|c_j\|$. Taking the infimum and then completing gives a contractive algebra homomorphism $\mu : G \to H$.

In the other direction, let $\bar{X}_i \in B$ and $\bar{Y}_j \in C$ denote the variable classes. Their powers satisfy

$$\|\bar{X}_i^n \otimes 1\|_G \leq \rho_i^n, \qquad \|1 \otimes \bar{Y}_j^n\|_G \leq \sigma_j^n \qquad (n \geq 0).$$

The two maps from A agree by the balancing relation. Proposition 1.9.3, with all bounds equal to one, therefore defines a contractive map from $A\{\rho^{-1}X, \sigma^{-1}Y\}$ to G, sending the variables to these tensors. If $G = 0$, the zero map has the same properties directly. Its restriction to the first coefficient algebra factors through B, as can first be checked on polynomials and then on convergent series. Its restriction to the second factors through C. It therefore kills I and J and induces a contractive map $\nu : H \to G$.

The composite $\mu\nu$ fixes every coefficient and variable, hence fixes the dense polynomial image in H and is the identity. Likewise $\nu\mu$ fixes the images of B and C, since each is a quotient of a coefficient algebra with dense polynomial image. It fixes every pure tensor and hence the dense algebraic tensor image in G. It is therefore the identity. Two inverse contractive maps are isometric, proving the assertion. □

Theorem 2.18.3 (The quotient-algebra Laurent sequence)**.** *Let M be a finite A-module, give it the quotient norm from a chosen surjection $A^m \to M$, and fix $f \in A$ and $r \in |K^\times|$. Put*

$$\begin{aligned} A_+ &= A\{r^{-1}T\}/(T - f), \\ A_- &= A\{rU\}/(1 - fU), \\ A_0 &= A\{r^{-1}T, rU\}/(T - f, TU - 1), \\ M_\tau &= A_\tau \otimes_A M \qquad (\tau \in \{+, -, 0\}). \end{aligned}$$

Give each M_τ its quotient norm from A_τ^m. These are finite canonical modules over the corresponding strict quotient algebras, and the ordinary-to-completed tensor maps are isometries. There are canonical isometric identifications

$$M_+ \simeq P_+, \qquad M_- \simeq P_-, \qquad M_0 \simeq P_0,$$

where P_+, P_-, P_0 are the coefficient quotients of Corollary 2.16.5. Under them the quotient-algebra maps give exactly the coefficient inclusions used there, and hence

$$0 \longrightarrow M \xrightarrow{m\mapsto(1\otimes m,1\otimes m)} M_+ \oplus M_- \xrightarrow{(u,v)\mapsto u|_0-v|_0} M_0 \longrightarrow 0$$

is strict exact with the same augmentation estimate and final quotient isometry. Moreover, the two maps to A_0 induce an isometric isomorphism $A_+\widehat{\otimes}_A A_- \simeq A_0$. If the norm on M is replaced by any uniformly equivalent canonical norm, the identifications are uniformly bounded isomorphisms and the sequence remains strict exact.

Proof. Apply Lemma 2.17.1 and Theorem 2.17.2 to each displayed presentation. For the first presentation the coefficient module is $E_+ = M\{r^{-1}T\}$, with relation image $(T-f)E_+$. For the second, the isometry

$$M\{rU\} \longrightarrow E_-, \qquad \sum_{j\geq 0} m_j U^j \longmapsto \sum_{j\geq 0} m_j Z^{-j}$$

takes multiplication by $1-fU$ to multiplication by $1-fZ^{-1}$. This proves the first two identifications, including their norms and actual relation images.

For the third, set $D_M = M\{r^{-1}T, rU\}$. Theorem 2.13.2 with $a = b = r$ gives an isometric quotient map

$$\mathcal{N} : D_M \longrightarrow E_r(M), \qquad \ker\mathcal{N} = (TU-1)D_M.$$

By Proposition 2.13.3, this map respects the algebra actions and sends $(T-f)D_M$ onto $(Z-f)E_r(M)$. Since multiplication by Z is a bounded automorphism of $E_r(M)$ commuting with $L_f = 1-fZ^{-1}$,

$$(Z-f)E_r(M) = L_f E_r(M).$$

Consequently the preimage of $L_f E_r(M)$ is exactly $(T-f)D_M + (TU-1)D_M$. The third identification follows by taking the second quotient. Its norm is unchanged, because successive quotient infima equal the direct quotient

infimum, as in Lemma 2.18.1. Theorem 2.17.2 identifies that same direct quotient with M_0 and its stated canonical norm.

On coefficients, the maps to M_0 send $T^j m$ to $Z^j m$ and $U^j m$ to $Z^{-j} m$. Constant elements m map to the same constant class. These identities hold first for polynomial families and then for all series by continuity. They verify every arrow of the displayed sequence, including the sign of the difference map. Corollary 2.16.5 now proves strict exactness and all its stated norm estimates.

Theorem 2.18.2 identifies $A_+ \widehat{\otimes}_A A_-$ isometrically with

$$A\{r^{-1}T, rU\}/(T - f, 1 - fU).$$

The ideals $(T - f, 1 - fU)$ and $(T - f, TU - 1)$ are equal, because

$$1 - fU = U(T - f) - (TU - 1), \qquad TU - 1 = U(T - f) - (1 - fU).$$

They are quotients of the same normed coefficient algebra by the same actual ideal, so their quotient norms also agree. This gives the last algebra isometry with the asserted maps. Finally, uniform comparison constants for two norms on M pass to their coefficient spaces, projective tensor seminorms and quotients. This proves the assertion about another canonical norm. The proof includes $f = 0$, nonunits, and zero quotient algebras; it uses no localization flatness. □

Proposition 2.18.4 (Finite diagrams of elementary constraints)**.** *Fix a finite labelled list of data (f_i, r_i, ϵ_i), where $f_i \in A$, $r_i \in |K^\times|$ and $\epsilon_i \in \{+, -\}$. For each sublist S, give X_i radius $R_i = r_i$ if $\epsilon_i = +$ and radius $R_i = r_i^{-1}$ if $\epsilon_i = -$, and put*

$$h_i = \begin{cases} X_i - f_i & \epsilon_i = +, \\ 1 - f_i X_i & \epsilon_i = -, \end{cases} \qquad A(S) = A\{(R_i^{-1} X_i)_{i \in S}\}/(h_i : i \in S),$$

$$A(\varnothing) = A, \qquad M(S) = A(S) \otimes_A M,$$

where M and all finite-module norms are chosen as in the preceding theorem. These presentations have the following properties. For $S \subseteq T$, the variable inclusions give contractive maps $A(S) \to A(T)$ and $M(S) \to M(T)$, compatible with composition and with the augmentation from A, M. Successive imposition of the listed constraints gives the same quotient algebra and the same quotient norm as simultaneous imposition. For any S, T,

$$A(S) \widehat{\otimes}_A A(T) \;\simeq\; A(S \cup T)$$

is isometric, with the evident maps, where a constraint appearing in both lists is imposed only once on the right. Permuting constraints or repeating an identical constraint gives canonical isometric identifications. All these maps commute. For $S \subseteq T$, the corresponding finite-module identification is

$$A(T) \otimes_{A(S)} M(S) \;\simeq\; A(T)\widehat{\otimes}_{A(S)} M(S) \;\simeq\; M(T),$$

isometrically. The assertions remain valid when one of the quotient algebras is zero. The algebra $A(S \cup T)$ is a simultaneous constraint algebra here; no assertion that it represents an analytic-domain intersection is included.

Proof. Each variable has powers bounded by its displayed radius to the corresponding power. The coefficient inclusions therefore extend contractively by Proposition 1.9.3, kill the indicated relations, and pass to quotient norms. The same coefficient inclusions for M are contractive and descend through Theorem 2.17.2. Equality on coefficients and variables proves compatibility with composition; polynomial density extends this equality to the algebras and coefficient modules.

Lemma 2.18.1 identifies the algebra obtained by adjoining the variables in $T \setminus S$ over $A(S)$ and imposing their relations with $A(T)$, including equality of the quotient norms. Applying the lemma repeatedly proves the assertion about any order of successive imposition. A permutation of variables preserves the weights term by term and gives the stated isometry.

To remove an identical repeated constraint, first retain separate variables X, X' with the same radius. The map sending both to X is contractive; the map in the other direction keeps the first variable. In the repeated quotient the two variables are equal. For a positive constraint, both equal f. For a negative constraint, $fX = fX' = 1$, so $X = XfX' = X'$. Thus the maps are inverse on every coefficient and variable, and hence everywhere by continuity. They are inverse contractions, so are isometries. Theorem 2.18.2, followed by these repetitions and permutations, now identifies the completed tensor product with $A(S \cup T)$.

For the module statement, ordinary tensor reparenthesization is supplied by [StacksAlg, Tag 00D2, Lemma 10.12.7], with $(A, A(S))$-bimodule $A(S)$; we verify the required norm identification here. Choose a fixed finite presentation $A^s \xrightarrow{D} A^m \to M \to 0$. By Theorem 2.17.2, $M(S)$ has the quotient presentation $A(S)^s \xrightarrow{D} A(S)^m \to M(S) \to 0$, with its displayed quotient norm. Apply that theorem again to the presentation of $A(T)$ over $A(S)$

supplied by Lemma 2.18.1. Both resulting modules have the quotient presentation $A(T)^s \xrightarrow{D} A(T)^m$ and the same norm from $A(T)^m$. This gives the asserted isometries, with $a \otimes (b \otimes m)$ corresponding to $ab \otimes m$. All identities of maps are checked on these generating tensors. If $A(S) = 0$, its further relation quotients and all unital modules are zero, so these assertions hold directly. □

The last proposition concerns a fixed finite list with all f_i in the original algebra A. An arbitrary tree in which new functions are chosen on different intermediate quotients still requires its own mixed-presentation comparison. These results supply the algebra and norm identifications needed to reapply the elementary Laurent sequence after a verified strict quotient construction. They do not assert exactness of an arbitrary localization functor, admissibility of a geometric covering, or higher Čech acyclicity. Such assertions require their own hypotheses and proofs. In particular, no tensoring of the displayed exact Laurent sequence by an unproved flat algebra has been used.

2.19 Bounded factorization and points of strict rational quotients

The quotient construction records relations and bounds on powers. These data first give a universal property for bounded maps. The description of points is a separate consequence, proved below directly from polynomial approximation. We retain the zero-quotient convention of Section 2.17; its spectrum is empty, since a multiplicative seminorm must take zero to zero and the unit to one.

Proposition 2.19.1 (The bounded universal property)**.** *Let A be a nonzero strict norm-admissible K-affinoid algebra, let $\rho = (\rho_1, \dots, \rho_s) \in |K^\times|^s$, and put $D = A\{\rho^{-1}X\}$. For an ideal $J = (h_1, \dots, h_t) \subset D$, give $B = D/J$ its quotient norm and write x_i for the image of X_i. Let C be any Banach K-algebra, with the zero case allowed, and let $\phi : A \to C$ be a bounded unital K-algebra homomorphism. Bounded A-algebra maps $\psi : B \to C$ are in bijection with tuples $b = (b_1, \dots, b_s) \in C^s$ satisfying both conditions:*

1. *For each i there is $C_i \geq 1$ such that $\|b_i^n\| \leq C_i \rho_i^n$ for every integer $n \geq 0$.*
2. *The convergent evaluations $h_j(b)$, with coefficients mapped by ϕ, vanish for $1 \leq j \leq t$.*

The correspondence sends ψ to $(\psi(x_i))_i$. If $\|\phi(a)\| \le C_\phi\|a\|$, the map obtained from such a tuple satisfies

$$\|\psi(v)\| \le C_\phi \prod_{i=1}^{s} C_i \, \|v\| \qquad (v \in B).$$

Proof. Lemma 2.17.1 gives a closed actual ideal J and a complete quotient B. If ψ is bounded by C, then $\|x_i^n\| \le \rho_i^n$ gives $\|\psi(x_i)^n\| \le C\rho_i^n$; enlarge C to at least one. Composing with $D \to B$ shows that every defining relation evaluates to zero.

Conversely, Proposition 1.9.3 extends the tuple to a bounded homomorphism $\Phi : D \to C$ with bound $L = C_\phi \prod_i C_i$. The second condition kills the actual ideal J, not just its chosen generators. Thus Φ induces an algebraic map $\psi : B \to C$. For every lift F of v and every $j \in J$,

$$\|\psi(v)\| = \|\Phi(F + j)\| \le L\|F + j\|.$$

Taking the infimum proves the stated quotient-norm bound; attainment of that infimum is unnecessary. The images of polynomials are dense in B. Two bounded maps with the same values on A and the x_i therefore agree. All constructions include the zero target; a map from the zero source to a nonzero unital target cannot occur, because it would identify zero and one. □

Corollary 2.19.2 (Power-controlled rational factorization)**.** *Keep A, ρ, C, ϕ as above. Let $g, f_1, \dots, f_s$ generate the unit ideal of A, and put*

$$B = A\{\rho^{-1}X\}/(gX_i - f_i)_{i=1}^{s}.$$

There is a bounded factorization $A \to B \to C$ if and only if $\phi(g)$ is a unit and, for each i, the ratios $b_i = \phi(f_i)\phi(g)^{-1}$ satisfy $\|b_i^n\| \le C_i\rho_i^n$ for all $n \ge 0$ and some C_i. Such a factorization is unique.

Proof. Choose $a_0 g + \sum_i a_i f_i = 1$. In B the equality becomes

$$g\left(a_0 + \sum_i a_i x_i\right) = 1.$$

Thus every factorization makes $\phi(g)$ invertible and forces $\psi(x_i) = \phi(f_i)\phi(g)^{-1}$. Proposition 2.19.1 supplies the necessary power bounds. Conversely, the stated ratios satisfy $\phi(g)b_i = \phi(f_i)$, so that proposition

constructs the unique factorization. In the zero ring, the unit equation is interpreted with $1 = 0$; the assertion then concerns the unique map to zero. □

A point supplies multiplicative bounds on every power. This lets us identify the spectrum of this quotient without invoking a spectral-radius theorem or compactness. The following proof uses the already written elementary seminorm calculation in Lemma 1.6.1; its contractivity and valued-fraction-field assertions have been assessed separately.

Theorem 2.19.3 (Points of a strict rational quotient)**.** *With A, g, f_i, ρ, B as in Corollary 2.19.2, the map induced by $A \to B$ is a homeomorphism*

$$\mathcal{M}(B) \xrightarrow{\sim} V = \{x \in \mathcal{M}(A) : |f_i|_x \le \rho_i |g|_x \text{ for } 1 \le i \le s\},$$

where both spectra carry the evaluation topology and V the subspace topology. The subset V is closed. This includes $B = 0$ and an empty spectrum; it does not assert that an algebra with empty spectrum must be zero.

Proof. **Step 1: construct the point over x.** The unit-ideal identity in the preceding proof shows that $|g|_x > 0$ for every $x \in V$. By Lemma 1.6.1, x defines an absolute value on the field $F_x = \mathrm{Frac}(A/\ker x)$, with $|a|_x \le \|a\|$ for $a \in A$. Evaluate polynomials at $X_i = f_i/g$ in F_x. The resulting multiplicative seminorm α_x on $A[X]$ satisfies

$$\alpha_x\left(\sum_\nu a_\nu X^\nu\right) \le \max_\nu |a_\nu|_x \prod_i |f_i/g|_x^{\nu_i} \le \max_\nu \|a_\nu\| \rho^\nu.$$

For polynomials P, Q, the ultrametric inequality also gives

$$|\alpha_x(P) - \alpha_x(Q)| \le \alpha_x(P - Q) \le \|P - Q\|.$$

Given $F \in D = A\{\rho^{-1}X\}$, choose polynomial truncations $P_n \to F$. The displayed estimate makes $\alpha_x(P_n)$ a Cauchy sequence of real numbers and makes its limit independent of the truncations. Define $\alpha_x(F)$ to be this limit. Polynomial density follows from Lemma 2.13.1. Addition, multiplication and the scalar rule pass to the limit: for multiplication, polynomial approximations are norm-bounded and their products converge by submultiplicativity. Thus the extension is a multiplicative seminorm with

$$\alpha_x(F) \le \|F\|, \qquad |\alpha_x(F) - \alpha_x(G)| \le \|F - G\|.$$

It kills every $gX_i - f_i$ and therefore the actual ideal J. If $F - G \in J$, the ultrametric inequality gives $\alpha_x(F) = \alpha_x(G)$. The induced seminorm on B is bounded by its quotient norm, by taking the infimum over lifts. It extends the fixed absolute value on K and defines a point y_x over x. In particular, when $B = 0$, this construction rules out any $x \in V$. It uses no completion of F_x.

Step 2: image and uniqueness. A bounded multiplicative seminorm on the nonzero strict affinoid algebra B is contractive by Lemma 1.6.1. Hence $|x_i|_y \leq \rho_i$, and the relations show that its restriction belongs to V. For a polynomial P of total degree at most N, write

$$Q_P = \sum_{\nu} a_\nu f^\nu g^{N-|\nu|} \in A, \qquad g^N P(x_1, \ldots, x_s) = Q_P \quad \text{in } B.$$

For every point y over x, this gives $|P|_y = |Q_P|_x / |g|_x^N$. Thus its values on polynomial images are fixed. These images are dense in B, and contractivity gives the same continuity estimate used in Step 1. Therefore $y = y_x$. Restriction is continuous because each evaluation on A is an evaluation on B.

Step 3: continuity of the inverse. For each polynomial P, the function $x \mapsto |Q_P|_x / |g|_x^N$ is continuous on V. Fix $v \in B$ and any lift $F \in D$. Polynomial truncations satisfy, uniformly for $x \in V$,

$$\big| |v|_{y_x} - |P_n|_{y_x} \big| \leq \|F - P_n\| \longrightarrow 0.$$

It follows that $x \mapsto |v|_{y_x}$ is continuous. Explicitly, given a point of V and $\varepsilon > 0$, choose a truncation with error less than $\varepsilon/3$ and use continuity of its evaluation to bound the difference of its values by $\varepsilon/3$ on a neighborhood. The two approximation errors then bound the difference for v by ε. Since the evaluation topology is the weakest topology making all these functions continuous, $x \mapsto y_x$ is continuous. Finally, the finitely many inequalities defining V are closed conditions on continuous real-valued functions. □

The bounded substitution criterion becomes a pointwise universal property in Theorem 2.31.2: Theorem 1.20.1 supplies the required unit, and Theorem 1.20.2 supplies the scaled power bounds at the prescribed radius. The same point theorem identifies an empty affinoid spectrum with the zero algebra.

References. For the classical special affinoid quotients, compare [Bosch14, §3.3, Proposition 11, pp. 50–53]; for the general affinoid formulation, see [TemIntro,

3.2.2.1] with the correction recorded in Example 1.9.5. The arguments here establish only the stated bounded and pointwise interfaces and do not invoke those domain theorems as proof inputs.

2.20 Flatness of elementary strict localizations

The preceding tensor comparisons identify the relation quotients, but they do not by themselves show that tensor products preserve injections. For an elementary localization, the missing information is available: its relation operator is injective on the coefficient extension of every finite module. This permits a direct use of the snake lemma, followed by the ideal criterion for flatness. We use both algebraic results from the Stacks Project and prove the required coefficient and norm comparisons here.

Here an algebraic R-module N is flat if the ordinary tensor functor $-\otimes_R N$ preserves exact sequences; a ring map $R \to S$ is flat when S is flat as an R-module. These are the definitions in [StacksAlg, Tag 00HB, Definition 10.39.1(1),(3)]. No completion or norm is part of this definition.

Lemma 2.20.1 (One-sided relation operators)**.** *Let A be a nonzero strict norm-admissible K-affinoid algebra, let M be a finite A-module with a canonical Banach norm, and let $f \in A$ and $r > 0$. On the one-sided coefficient spaces*

$$E_+(M) = M\{r^{-1}T\}, \qquad E_-(M) = M\{rU\}$$

the bounded A-linear operators

$$Q_+ = T - f : E_+(M) \to E_+(M), \qquad Q_- = 1 - fU : E_-(M) \to E_-(M)$$

are injective. No value-group restriction on r is needed for this injectivity statement.

Proof. First give M a quotient norm from a finite free presentation, so that its A-action is contractive. Embed both spaces isometrically into the bilateral coefficient module $E_r(M)$ of Section 2.16, by sending T to Z and U to Z^{-1}. Their images consist of the nonnegative and the nonpositive powers of Z, respectively. Multiplication by Z is a bounded automorphism of $E_r(M)$; its norm scales by r and its inverse scales by r^{-1}. Under these embeddings the two operators become

$$Z - f = Z(1 - fZ^{-1}), \qquad 1 - fZ^{-1}.$$

Theorem 2.16.2 makes $1 - fZ^{-1}$ injective for every $r > 0$, so both restrictions are injective. Their boundedness follows from the shift bounds and the bounded action of f. Finally, replacing the quotient norm on M by a uniformly equivalent canonical norm leaves each coefficient space unchanged as a set and gives a bounded identity map with bounded inverse. It therefore preserves boundedness and injectivity of the same algebraic operators. □

Theorem 2.20.2 (Elementary flatness and strict finite-module exactness)**.** *Let A be as above, fix $f \in A$ and $r \in |K^\times|$, and put*

$$B_+ = A\{r^{-1}T\}/(T - f), \qquad B_- = A\{rU\}/(1 - fU),$$

with their quotient norms and the explicit zero-quotient convention. For either sign $\epsilon \in \{+, -\}$ the ring map $A \to B_\epsilon$ is flat. For every short exact sequence of finite A-modules with canonical norms,

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

its ordinary tensor sequence and its separated completed tensor sequence with B_ϵ are exact and norm-strict. The resulting modules have their canonical finite-B_ϵ-module norms, equivalently the corresponding projective norms. Flatness here concerns ordinary algebraic tensor products on all A-modules; the Banach assertion is restricted to finite modules with these norms.

Proof. **Step 1: exactness before imposing the relation.** Choose quotient norms from finite free presentations of M', M, M''. All maps remain bounded and norm-strict by Theorem 1.11.5. Corollary 2.14.5, with the one-sided multiplicative weights r^n or r^{-n}, gives an exact coefficient sequence for either sign. Multiplication by the chosen relation commutes with its arrows, since the original maps are A-linear. We therefore have the following diagram of underlying A-modules, with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_\epsilon(M') & \longrightarrow & E_\epsilon(M) & \longrightarrow & E_\epsilon(M'') & \longrightarrow & 0 \\ & & \downarrow Q_\epsilon & & \downarrow Q_\epsilon & & \downarrow Q_\epsilon & & \\ 0 & \longrightarrow & E_\epsilon(M') & \longrightarrow & E_\epsilon(M) & \longrightarrow & E_\epsilon(M'') & \longrightarrow & 0. \end{array}$$

Step 2: pass to the actual cokernels. The right vertical map is injective by Lemma 2.20.1. The snake lemma, [StacksAlg, Tag 010H, Lemma

12.5.17(2)], applied in the abelian category of algebraic A-modules, now gives the exact sequence

$$\begin{aligned} 0 \longrightarrow & E_\epsilon(M')/Q_\epsilon E_\epsilon(M') \longrightarrow E_\epsilon(M)/Q_\epsilon E_\epsilon(M) \\ & \longrightarrow E_\epsilon(M'')/Q_\epsilon E_\epsilon(M'') \longrightarrow 0. \end{aligned}$$

The zero at the left comes from $\ker(Q_\epsilon|_{E_\epsilon(M'')}) = 0$ in the snake sequence; the rightmost surjectivity comes from the surjectivity in the bottom row. Theorem 2.17.2 identifies each actual cokernel naturally with $B_\epsilon \otimes_A$ the corresponding finite module, and with its separated completed tensor. That theorem also proves that the displayed relation images are closed and that the quotient norms are canonical. This is where the condition $r \in |K^\times|$ is used.

The induced arrows are B_ϵ-linear by the same naturality. If $B_\epsilon \neq 0$, it is strict by Lemma 2.17.1, so Theorem 1.11.5 over B_ϵ makes these arrows bounded with closed images and norm-strict. If $B_\epsilon = 0$, all terms are zero. Uniformly equivalent choices of the original canonical norms preserve these conclusions. We have used the snake lemma only on algebraic modules, not on a category of Banach spaces.

Step 3: test algebraic flatness. Let $I \subset A$ be a finitely generated ideal. Both I and A/I are finite A-modules, with the canonical norms supplied by Theorem 1.11.5. Apply the exactness just proved to

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0.$$

It gives injectivity of $I \otimes_A B_\epsilon \to A \otimes_A B_\epsilon = B_\epsilon$, with the natural multiplication map. The finitely generated ideal criterion [StacksAlg, Tag 00HD, Lemma 10.39.5(4)] proves that B_ϵ is a flat A-module. This criterion upgrades the verified finite-module statement to algebraic flatness on all modules; it adds no assertion about completed tensors of arbitrary Banach modules. □

Corollary 2.20.3 (Finite compositions of elementary localizations)**.** *Start with a nonzero strict norm-admissible $C_0 = A$. For $1 \leq j \leq n$, choose $f_j \in C_{j-1}$ and $r_j \in |K^\times|$, and let C_j be either*

$$C_{j-1}\{r_j^{-1}T_j\}/(T_j - f_j) \quad \text{or} \quad C_{j-1}\{r_jU_j\}/(1 - f_jU_j),$$

with the quotient norm; once a zero algebra occurs, every subsequent algebra is zero. Every map in the sequence and every composite is algebraically flat.

For finite modules over any nonzero intermediate algebra, extension to a later algebra preserves short exact sequences, and the ordinary and separated completed tensor sequences agree by bounded isomorphisms and are norm-strict for canonical norms. In particular, the maps $A(S) \to A(T)$ for $S \subseteq T$ in Proposition 2.18.4 are flat, as are the maps from A to A_+, A_-, A_0 in Theorem 2.18.3.

Proof. At each nonzero stage Lemma 2.17.1 shows that the next algebra is a strict quotient or zero, and Theorem 2.20.2 proves flatness of the next map. The zero module is flat because its ordinary tensor functor is zero. Flatness of every finite composite now follows directly from [StacksAlg, Tag 00HC, Lemma 10.39.4].

Every resulting map is bounded, being a composite of contractive coefficient inclusions followed by quotient maps. For finite modules, algebraic flatness gives exactness, Proposition 1.11.6 identifies ordinary and completed tensors by bounded isomorphisms, and Theorem 1.11.5 over the final nonzero algebra gives norm-strictness. The zero target is immediate. This proves the assertion for any pair of intermediate stages.

For $A(S) \to A(T)$, impose the constraints in $T \setminus S$ one at a time and use Proposition 2.18.4 to identify the result with the simultaneous presentation. Finally, after forming A_+, impose the negative relation $1 - \bar{f}U$ with U of radius r^{-1}. Lemma 2.18.1 and the ideal equality in the proof of Theorem 2.18.3 identify this two-stage quotient isometrically with

$$A\{r^{-1}T, rU\}/(T - f, 1 - fU) = A\{r^{-1}T, rU\}/(T - f, TU - 1) = A_0.$$

This proves flatness of the stated Laurent intersection map. □

The finite-composition assertion permits a new element f_j of the current algebra at each step. The geometric rational universal property is proved in Theorem 2.31.2; Proposition 2.38.2 gives flatness for arbitrary affinoid-domain restrictions. The full covering comparison and higher Čech exactness are established in Theorems 2.39.4 and 2.38.1.

2.21 Finite Laurent coverings and strict Čech exactness

The elementary Laurent sequence controls one binary splitting. To pass to all degrees for a finite family, one must also compare the mixed intersections. We carry out that comparison on a finite diagram of Banach modules. The homological comparison is taken from the Stacks Project; the norm estimates are proved by finite successive lifting. In particular, no exactness of a completed tensor functor is inferred from algebraic flatness.

Definition 2.21.1 (Finite intersection coefficient diagrams)**.** Let $\mathscr{R}$ be a finite partially ordered set with a greatest element X and a meet $U \cap V$ for every pair of elements. A Banach coefficient diagram N on $\mathscr{R}$ assigns a non-Archimedean Banach K-space $N(U)$ to each U and a bounded restriction $N(U) \to N(V)$ whenever $V \leq U$, with identity restrictions and compatibility with composition. Additional linear actions may be retained throughout. For an ordered finite family $\mathcal{U} = (U_i)_{i \in I}$ and $V \in \mathscr{R}$, put

$$C^p(\mathcal{U}, N; V) = \prod_{i_0 < \cdots < i_p} N(V \cap U_{i_0} \cap \cdots \cap U_{i_p}), \qquad p \geq 0,$$

with the maximum norm; an empty product of modules is zero. The differential is the alternating sum of restrictions, and the augmentation is $N(V) \to C^0(\mathcal{U}, N; V)$. We say that this augmented complex is strictly acyclic if it is exact, the augmentation has a bounded inverse on its image, and each positive-degree cocycle has a primitive with norm bounded by a fixed multiple of its norm. The multiplier may depend on the diagram and the degree. Since the ordered complex is finite, one multiplier may be chosen for all degrees. The same definition applies to a family rooted at U, using $N(U \cap V)$ for its augmentation.

For comparison with the usual Čech convention, the complex using all tuples has degree p term $\prod_{I^{p+1}} N(V \cap U_{i_0} \cap \cdots \cap U_{i_p})$, including repeated indices. Repetitions and permutations refer to the same meet, so their coefficient identifications are the identity.

These definitions do not assert that a member of $\mathscr{R}$ is an analytic domain. They also do not require a sheaf. The formal Čech results for presheaves apply to the intersection terms as follows. Give $\mathscr{R}$ the topology of downward

closed subsets and write $O_U = \{V : V \leq U\}$. Then $O_U \cap O_V = O_{U\cap V}$. The prescription

$$\mathcal{F}(W) = \varprojlim_{O_U \subseteq W} N(U), \qquad \mathcal{F}(\varnothing) = 0.$$

Compatibility in this finite limit defines a presheaf. On O_U its sections are naturally $N(U)$: the U-component determines every other component by restriction. These identifications are bounded with bounded inverses, and are isometric when restrictions are contractive. The family (O_{U_i}) is an open covering of its union, with exactly the required intersection terms. Its augmentation from $N(V)$ is the restriction from O_V and is preserved by the comparison maps below. Thus the formal comparisons require no unproved geometric localization statement.

Lemma 2.21.2 (Quantitative comparison of a finite double complex)**.** *Let $D^{p,q}$ be a double complex of Banach K-spaces in a finite rectangle $0 \leq p \leq a$, $0 \leq q \leq b$, with bounded commuting differentials d_h, d_v. Use $d = d_h + (-1)^p d_v$ on its total complex and maximum norms on finite sums. Let $L^\bullet$ be a complex of Banach K-spaces with bounded differentials, concentrated in degrees $0, \dots, b$, and let $\alpha^q : L^q \to D^{0,q}$ be a bounded compatible horizontal augmentation. Assume that every augmented horizontal row*

$$0 \longrightarrow L^q \xrightarrow{\alpha^q} D^{0,q} \xrightarrow{d_h} D^{1,q} \longrightarrow \cdots \longrightarrow D^{a,q} \longrightarrow 0$$

is strictly exact. Then $\alpha : L^\bullet \to \mathrm{Tot}(D)$ is a quasi-isomorphism of underlying algebraic complexes. Let P be a Banach K-space and let $\epsilon : P \to L^0$ be a bounded augmentation with $d_L\epsilon = 0$, placing P in degree -1. Its augmented complex is strictly acyclic if and only if $P \to \mathrm{Tot}(D)$ is strictly acyclic. All lifting and inverse bounds for either complex can be obtained from finitely many bounds for the rows, the differentials, α, and the other complex. If all these maps are contractive and every horizontal row has a contracting homotopy of norm at most one, a strict acyclicity bound $Q \geq 1$ for $P \to L^\bullet$ also works for $P \to \mathrm{Tot}(D)$. The corresponding assertions with the two axes interchanged hold as well.

Proof. The algebraic assertion is the version with interchanged variables of [StacksAlg, Tag 0133, Lemma 12.25.4], in the abelian category of K-vector spaces, or of the retained algebraic modules. Its finite-diagonal hypothesis holds because the rectangle is finite. We prove the additional norm statement directly. Choose $B, C \geq 1$ so that B bounds the differentials and α, and every

horizontal cycle of positive horizontal degree has a preimage bounded by C times its norm, while $\|u\| \le C\|\alpha u\|$. Such a common C exists because there are finitely many strictly exact rows and terms. Lifts need not be linear or chosen compatibly.

Let $z \in \mathrm{Tot}^k(D)$ be a cocycle. Starting with the largest $p > 0$ for which its $(p, k-p)$ component is nonzero, that component is horizontally closed: the $(p+1, k-p)$ component of $dz = 0$ says so, because the component one column farther to the right is zero. Lift it horizontally to $y \in D^{p-1,k-p}$ and replace z by $z - dy$. This removes that component and introduces a possible new component only one column to the left. After at most a repetitions, only $z_{0,k}$ remains. It equals αu for a unique $u \in L^k$, and $d_L u = 0$ follows from $d_v \alpha u = 0$ and injectivity of α. Thus

$$z = \alpha u + dy, \qquad d_L u = 0.$$

At each elimination the current norm increases by at most BC. Consequently $\|u\|$ and the norm of the sum of the chosen lifts are bounded by $C(BC)^a\|z\|$. This bound is independent of k. If $u = d_L v$, then $y + \alpha v$ is a primitive of z. In degree zero the same construction, followed by the inverse bound for the augmentation $P \to L^0$, controls the unique preimage in P. Injectivity of that augmentation is also preserved. This proves one implication.

For the converse, let $u \in L^k$ be a positive-degree cocycle and suppose $\alpha u = dw$. Apply the same elimination to w. Although w need not be a cocycle, every component of dw of positive horizontal degree is zero, which is precisely the condition used in each elimination. The result is $w - dy = \alpha v$ with $\|v\| \le C(BC)^a\|w\|$. Therefore $\alpha u = d\alpha v = \alpha d_L v$, and injectivity gives $u = d_L v$. A bounded primitive for αu thus yields one for u. In degree zero, a preimage of αu in P maps to u by injectivity of α, with its norm bounded by B times the corresponding total-complex bound. These estimates also prove the converse injectivity statement. For example, multiplying the bound for the other complex by $(BC)^{a+2}$ is sufficient in either direction.

If the row contractions and all maps have norm at most one, use the row contraction to choose each horizontal lift and its degree-zero preimage. No elimination increases the norm. Thus $\|y\|, \|u\| \le \|z\|$, and the asserted unchanged bound follows. Interchanging the two axes changes the total-complex convention by the isometric sign map $x^{p,q} \mapsto (-1)^{pq} x^{p,q}$; hence the same argument proves the symmetric version. Finally, exactness identifies each differential image with the kernel of the next bounded map. The images are therefore closed, and the primitive bounds are exactly the needed bounds

for the inverse maps from image norms to quotient norms. This is norm-strictness, without an appeal to an open mapping theorem. □

Theorem 2.21.3 (Finite binary splitting trees)**.** *Let N be a Banach coefficient diagram as above. Take a finite rooted binary tree whose nodes are elements $U \in \mathscr{R}$, with root X and children $U_+, U_- \leq U$. Assume that for every internal node U and every $V \in \mathscr{R}$ the sequence*

$$0 \longrightarrow N(U \cap V) \longrightarrow N(U_+ \cap V) \oplus N(U_- \cap V) \xrightarrow{(a,b) \mapsto b|-a|} N(U_+ \cap U_- \cap V) \longrightarrow 0$$

is strictly exact. Then the augmented ordered Čech complex of the leaves, restricted to any $V \in \mathscr{R}$, is strictly acyclic in every degree. Its bounds depend only on finitely many restriction bounds and the assumed binary inverse and lifting bounds. The assertion is also valid for the complex using all tuples. It concerns the specified coefficient diagram: verifying the displayed hypothesis on all node intersections is part of applying the theorem.

Proof. We prove the ordered assertion simultaneously for all V, by induction on the number of internal nodes. With no internal node the leaf is X, and the complex is the identity $N(V) \to N(V)$. For the induction step, let $\mathcal{P} = (P_i)$ and $\mathcal{Q} = (Q_j)$ be the leaves of the two subtrees rooted at U_+ and U_-. Order all P_i before all Q_j. The induction hypothesis applies to either subtree after restriction by any finite intersection in $\mathscr{R}$.

Write $W = V \cap U_+ \cap U_-$. The mixed terms are organized by the finite double complex

$$D^{p,q} = \prod_{i_0 < \cdots < i_p} \prod_{j_0 < \cdots < j_q} N(V \cap P_{i_0} \cap \cdots \cap P_{i_p} \cap Q_{j_0} \cap \cdots \cap Q_{j_q}).$$

For a fixed j-tuple, its horizontal augmentation is the Čech complex of the left subtree restricted to $V \cap Q_{j_0} \cap \cdots \cap Q_{j_q}$. By induction this row is strictly exact, with augmentation term $N(U_+ \cap V \cap Q_{j_0} \cap \cdots \cap Q_{j_q})$. These augmentation terms form $L^\bullet = C^\bullet(\mathcal{Q}, N; W)$, which is strictly acyclic after augmentation by $N(W)$, again by induction. Lemma 2.21.2 therefore proves that $N(W) \to \mathrm{Tot}(D)$ is strictly acyclic. The constants are finite because there are finitely many rows and terms.

To see exactly how this total complex occurs, let $C^\bullet$ be the ordered complex of all leaves, without its augmentation, and let $K^\bullet$ be the kernel of restriction

to the two pure subcomplexes. A mixed tuple has $p+1$ left indices followed by $q+1$ right indices and lies in degree $p+q+1$. The identification

$$\mathrm{Tot}(D)[-1] \xrightarrow{\sim} K^\bullet, \qquad a^{p,q} \longmapsto (-1)^p a^{p,q}$$

is isometric and respects the differential. Indeed, deleting a left index gives d_h, while deleting a right index contributes $(-1)^{p+1}d_v$ in the leaf complex. Conjugation by $(-1)^p$ gives $-d_h - (-1)^p d_v$, which is the differential of the shift $[-1]$. This also fixes the sign of the degree-one obstruction.

Let $z \in C^k$ be a cocycle with $k \geq 1$. Its two pure restrictions have bounded primitives by induction. Extend these primitives by zero on mixed tuples and subtract their differential. The remaining cocycle z' belongs to K^k, and its norm is bounded by a fixed multiple of $\|z\|$. If $k \geq 2$, strict acyclicity of $\mathrm{Tot}(D)$ supplies a bounded primitive in K^{k-1}. If $k = 1$, exactness at degree zero of the augmented total complex gives a unique $w \in N(W)$ whose restrictions are the mixed components of z', with a bound for $\|w\|$. The assumed binary surjection at the root gives $a \in N(U_+ \cap V)$ and $b \in N(U_- \cap V)$ with $b| - a| = w$ and a bounded maximum norm. Restrict a, b to the leaves of their respective subtrees. Their differential is zero on pure tuples and equals z' on mixed pairs. This supplies the required primitive also in degree one.

In degree zero, a compatible family on the leaves gives unique elements $a \in N(U_+ \cap V)$ and $b \in N(U_- \cap V)$ by induction. Their difference in $N(W)$ restricts to zero on every mixed pair. The augmentation $N(W) \to \mathrm{Tot}^0(D)$ is injective, so $b| - a| = 0$. Binary exactness gives a unique element of $N(V)$ inducing the family, with its norm bounded by the binary inverse bound and the two subtree bounds. If an element of $N(V)$ restricts to zero on all leaves, the same injectivity argument makes it zero. This proves strict acyclicity of the ordered complex.

Finally, [StacksAlg, Tag 01FM, Lemma 20.23.6] identifies the ordered (equivalently alternating) complex and the complex using all tuples by homotopy equivalences. The finite-poset presheaf above verifies its scope. The sorting and repeated-index homotopies in equations (20.23.6.1)–(20.23.6.2) use only finite sums with coefficients ± 1 and the identity identifications of repeated intersections. Hence they are bounded in each degree, with norm at most one for the maximum norms here. The projection to increasing tuples and alternating extension are also contractive. If f, g are the resulting comparison maps and $gf - 1 = dh + hd$, a positive-degree cocycle z with $fz = dy$ has primitive $gy - hz$. This proves the required norm bounds for the full complex. The maps preserve the augmentation and the degree-zero

cycles, so exactness and the inverse bound at the augmentation are preserved as well. □

Theorem 2.21.4 (All-sign elementary Laurent Čech exactness)**.** *Let K be complete for its fixed real-valued non-Archimedean absolute value, allowing the trivial valuation. Let A be a nonzero strict norm-admissible K-affinoid algebra with its normalized submultiplicative norm, and let M be a finite A-module with the quotient norm from a chosen surjection $A^m \to M$. Fix $n \geq 0$, $f_1, \dots, f_n \in A$, and $r_1, \dots, r_n \in |K^\times|$. For a set S of signed constraints $(i,+), (i,-)$, use the simultaneous presentation*

$$A(S) = \frac{A\{(r_i^{-1}T_i)_{(i,+)\in S}, (r_iU_i)_{(i,-)\in S}\}}{(T_i - f_i : (i,+) \in S,\ 1 - f_iU_i : (i,-) \in S)}, \qquad M(S) = A(S) \otimes_A M,$$

with the exact quotient norms, including the zero-algebra convention. For $\sigma \in \{+,-\}^n$ put $S_\sigma = \{(i,\sigma_i) : 1 \leq i \leq n\}$. Then

$$0 \longrightarrow M \longrightarrow \prod_{\sigma_0} M(S_{\sigma_0}) \longrightarrow \prod_{\sigma_0 < \sigma_1} M(S_{\sigma_0} \cup S_{\sigma_1}) \longrightarrow \cdots \longrightarrow 0$$

is strictly exact in every degree, with the usual alternating restriction differential and maximum norms. The same is true when all tuples, including repetitions, are used. For each fixed datum there is $Q \geq 1$ such that the augmentation has inverse bound Q and every positive-degree cocycle z has a primitive of norm at most $Q\|z\|$. The same conclusions hold after imposing any fixed finite collection of elementary constraints first, and replacing M by its finite module over that quotient. Uniformly equivalent canonical norms preserve strict exactness.

Proof. Take $\mathscr{R}$ to be the signed-constraint subsets, ordered by reverse inclusion, with meet given by union, and put $N(S) = M(S)$. Proposition 2.18.4 supplies its compatible contractive restrictions, all repeated-constraint identifications, and the exact norm identifications for successive constructions. Lemma 2.17.1 and Theorem 2.17.2 show that each nonzero $A(S)$ is strict and that $M(S)$ is a finite canonical Banach module. A zero algebra has the zero module.

Split successively according to $f_1, \dots, f_n$, creating both signs at every node. Consider any such binary splitting after any additional constraint set S has already been imposed. If $A(S) = 0$, its entire binary sequence is zero. Otherwise apply Theorem 2.18.3 over the strict algebra $A(S)$, to the finite

module $M(S)$ and the image of the relevant f_i, with radius r_i. The simultaneous and iterated presentations, and every arrow of the resulting sequence, agree isometrically by Proposition 2.18.4. If that constraint already occurs in S, repetition is removed by the same proposition. Changing the sign of the last arrow gives the difference convention of Theorem 2.21.3. Thus the latter theorem applies to every required intersection. Its leaves are exactly the 2^n sets S_σ, and it proves the displayed strict exactness. This argument reapplies the proved binary theorem over each quotient; it does not tensor an exact sequence through an unverified completed functor.

A useful bound can be read from the proof. Split off f_n first and write $Q_+, Q_-, Q_0 \geq 1$ for bounds for the remaining $n-1$ functions over A_+, A_-, A_0. In the mixed double complex the two index sets are the same old all-sign set. Fixing a nonempty second tuple J, choose $j \in J$. The first cover restricted to that intersection has its jth member equal to the whole intersection. The insertion homotopy of [StacksAlg, Tag 0G6T, Lemma 20.23.7, second proof], with its sign chosen so that $dh+hd = 1$, has norm at most one. In alternating notation it sends a cochain to its component with j prepended; repeated indices give zero. The coefficient meet is unchanged because j already belongs to J. The horizontal augmentation is isometric since one of its coordinates is the identity. Thus Lemma 2.21.2 gives the bound Q_0 for the mixed total complex.

Let $E \geq 1$ be the inverse bound for the binary augmentation $M \to M_+ \oplus M_-$. Its final map is a quotient isometry by Theorem 2.18.3, so every element of M_0 has a lift of norm at most twice its norm (zero is lifted by zero). The induction proof then permits

$$Q_n = \max\{E,\, 2Q_0\} \max\{Q_+, Q_-\}, \qquad Q_{\text{empty list}} = 1.$$

This is a bound for a fixed list and module, not a bound uniform over all data. The insertion homotopies and the ordered-to-all-tuples homotopies do not increase it. Finally, starting with a further fixed constraint quotient is precisely the same argument with that quotient as base. Uniform norm comparisons pass to the finitely presented modules and finite products, proving the assertion for other canonical norms. □

The comparison with two coverings is classical; see [BGR84, §8.1.4, Lemma 1 and Corollaries 3–4, pp. 325–327]. We use that double-complex arrangement, giving the coefficient maps and finite norm estimates explicitly. The

cited BGR results are historical sources here; the proof inputs are the stated diagram hypotheses and the proved comparison lemma.

Proposition 2.21.5 (Strict acyclic refinement and common refinement)**.** *Let N be a finite intersection coefficient diagram, and let $\mathcal{U} = (U_i)_{i\in I}$ and $\mathcal{V} = (V_j)_{j\in J}$ be nonempty finite families. Suppose that for every nonempty tuple of $\mathcal{U}$-indices, the augmented ordered complex of $\mathcal{V}$ restricted to $U_{i_0} \cap \cdots \cap U_{i_p}$ is strictly acyclic. No coefficient term is discarded on the basis of point-set emptiness.*

1. *If $\mathcal{V}$ refines $\mathcal{U}$, meaning that a map $t : J \to I$ satisfies $V_j \leq U_{t(j)}$, then the augmented complexes of $\mathcal{U}$ and $\mathcal{V}$ are strictly acyclic simultaneously. The usual refinement map is a quasi-isomorphism, compatible with augmentation; its cohomology map is independent of t.*
2. *Without a refinement assumption, put $\mathcal{W} = (U_i \cap V_j)_{(i,j)\in I\times J}$. Then the augmented complexes of $\mathcal{U}$ and $\mathcal{W}$ are strictly acyclic simultaneously. The hypotheses on every $\mathcal{U}$-intersection remain necessary in this assertion.*

All conclusions are valid for all-tuples complexes as well, with bounded comparison maps and degreewise bounded homotopies. Only finitely many of the stipulated local inverse and lifting bounds are needed.

Proof. For part (1), form the ordered double complex

$$D^{p,q} = \prod_{i_0<\cdots<i_p} \prod_{j_0<\cdots<j_q} N(U_{i_0} \cap \cdots \cap U_{i_p} \cap V_{j_0} \cap \cdots \cap V_{j_q}).$$

The assumed vertical augmented rows compare $C^\bullet(\mathcal{U}, N)$ with $\mathrm{Tot}(D)$ by Lemma 2.21.2. In the other direction, fix a nonempty j-tuple and choose any one of its indices j_0. Since $V_{j_0} \cap \cdots \cap V_{j_q} \leq U_{t(j_0)}$, the restricted $\mathcal{U}$-family has a member equal to the entire intersection. Its augmented complex has the insertion contraction of [StacksAlg, Tag 0G6T, Lemma 20.23.7]. With the maximum norms, insertion has norm at most one; no local sheaf theorem is used. The same comparison lemma now compares $C^\bullet(\mathcal{V}, N)$ with $\mathrm{Tot}(D)$. The two applications, preserving the common augmentation $N(X)$, prove strict acyclicity in both directions.

We record the actual maps. In alternating notation the refinement map is

$$(T_t c)_{j_0\ldots j_k} = c_{t(j_0)\ldots t(j_k)}\big|_{V_{j_0}\cap\cdots\cap V_{j_k}}.$$

Repeated indices on the right give zero. Let a and b be the augmentations of the two complexes into the total double complex, placing them in bidegrees $(k,0)$ and $(0,k)$ respectively. The bounded homotopy

$$(Hc)^{p,q}_{i_0\dots i_p;j_0\dots j_q} = (-1)^p c_{i_0\dots i_p\, t(j_0)\dots t(j_q)}\big|, \qquad p+q=k-1,$$

satisfies $dH+Hd = bT_t - a$. All restrictions are to the indicated mixed intersection. To check the sign, the terms obtained by deleting an i-index cancel between dH and Hd in positive horizontal degree; the terms obtained by deleting a $t(j)$-index cancel in positive vertical degree. The boundary terms are bT_t at $p=0$ and $-a$ at $q=0$. Thus the quasi-isomorphism supplied by the double complex is induced by the usual refinement map. The prism formula in [StacksAlg, Tag 01FP, Section 20.25, paragraph "Refinements", following (20.25.0.2)] gives a homotopy for two choices t, t'. It consists of finitely many signed restrictions and is bounded in each degree. In the finite-poset realization, one may restrict the first open family to the union of the refining family before applying that formula; pulling it back gives precisely the displayed coefficient maps. This verifies the presheaf scope without identifying $N(X)$ with sections of an unproved sheaf.

For part (2), $\mathcal{W}$ refines $\mathcal{U}$. Fix an intersection $Y = U_{i_0} \cap \cdots \cap U_{i_p}$ and choose i_0 among its indices. On Y, the family $\mathcal{W}|_Y$ refines $\mathcal{V}|_Y$ by $(i,j) \mapsto j$, while $\mathcal{V}|_Y$ refines $\mathcal{W}|_Y$ by $j \mapsto (i_0, j)$ because $Y \cap V_j = Y \cap U_{i_0} \cap V_j$. The composites are homotopic to the identity by the same prism formula. These bounded homotopies preserve augmentation. For clarity, if f, g are such maps with $gf - 1 = dh + hd$ and $fz = dy$, then $gy - hz$ is a primitive of z; its norm is controlled by the norms of g, h and the given primitive. The analogous degree-zero assertion follows from preservation of augmentation and injectivity. Hence strict acyclicity of $\mathcal{V}|_Y$ implies that of $\mathcal{W}|_Y$. Apply part (1) to $\mathcal{W} \to \mathcal{U}$. Finally apply Tag 01FM as in Theorem 2.21.3 to obtain the all-tuples versions. □

The finite rational-cover argument uses these comparisons after identifying every mixed intersection with its actual coefficient algebra. The required refinements lead to Theorem 2.22.9; Theorem 2.32.2 then treats arbitrary radii by simultaneous strictification.

2.22 Rational presentations and Laurent refinements

We work over a complete non-Archimedean field K, allowing trivial valuation except in the small-scalar lemmas where nontriviality is stated. Every nonzero algebra has a strict norm-admissible Banach norm, and every relation quotient has its actual quotient norm. The zero algebra is admitted separately with its zero norm. All radii in this section are one; a radius in $|K^\times|$ can first be reduced to this convention by scaling the corresponding numerator. These conventions concern the displayed presentations, not a spectral norm.

For a tuple $g, f_1, \dots, f_s$ generating the unit ideal of A, write

$$A\langle f/g\rangle = A\{T_1, \dots, T_s\}/(gT_j - f_j)_{1\le j\le s}.$$

For a unit-ideal tuple $f_0, \dots, f_r$, its *standard rational family* consists of the $r+1$ presentations

$$R_i = A\{T_j : j \neq i\}/(f_i T_j - f_j)_{j\neq i}, \qquad 0 \le i \le r.$$

We put $T_i = 1$ when using a formula involving all indices. The joint presentation of a finite family means the quotient with disjoint variable blocks and all their relations. Theorem 2.18.2 identifies it with the separated completed tensor product. Its module attached to a finite A-module M is its ordinary tensor with M, equipped with the quotient norm from a fixed finite free presentation of M. Theorem 2.17.2 gives its comparison with the completed tensor. These choices specify the arrows and norms before a geometric sheaf is available.

Lemma 2.22.1 (A rational quotient as an elementary chain)**.** *Let $B = A\langle f/g\rangle$ and choose*

$$a_0 g + \sum_{j=1}^{s} a_j f_j = 1.$$

Choose $a \in K^\times$ with $|a| \max_{0\le j\le s} \|a_j\| \le 1$. Such a choice is always possible for nontrivial valuation. For trivial valuation use a defining quotient norm on A and take $a = 1$; this replaces any given admissible norm by a uniformly

equivalent one. The following isometry uses the norm for which the stated scalar inequality holds. Put

$$\begin{aligned} C_0 &= A\{U\}/(gU - a), \\ h_j &= f_j\bar{U}/a \in C_0, \\ C_1 &= C_0\{V_1, \ldots, V_s\}/(V_j - h_j)_j. \end{aligned}$$

There is an isometric A-algebra isomorphism $B \simeq C_1$ sending T_j to V_j. Consequently $A \to B$ is algebraically flat. It preserves short exact sequences of finite canonical A-modules under ordinary or separated completed tensor product; the resulting sequences are norm-strict. The assertion about completed tensor products is restricted to these finite modules.

Proof. For nontrivial valuation, powers of a scalar of absolute value less than one give the choice of a. For trivial valuation, the strict Tate algebra is a polynomial ring with norm one on each nonzero polynomial, so every element of its quotient has norm at most one. Thus $a = 1$ works with the stipulated defining norm. In B set $v = a_0 + \sum_j a_j\bar{T}_j$. The defining relations give $gv = 1$, and $\|av\| \leq 1$. In C_0 the equation $g\bar{U} = a$ makes g a unit with inverse $\bar{U}/a$. Hence in C_1 one has $g\bar{V}_j = f_j$, with $\|\bar{V}_j\| \leq 1$. The bounded universal property, Proposition 2.19.1, therefore supplies a contractive map $B \to C_1$.

Conversely, send U to av and V_j to $\bar{T}_j$ in B. These elements have norm at most one. The first relation vanishes because $gav = a$, and the remaining relations vanish because $f_jv = \bar{T}_j$. Thus the same universal property and Lemma 2.18.1 give a contractive map $C_1 \to B$. The composites fix A and all the T_j, V_j. They also fix U, since g is a unit in C_1 and both possible images equal ag^{-1}. Polynomial density proves that the maps are inverse, and two inverse contractions are isometric. This argument also covers zero quotients; every displayed map is a map of actual relation quotients.

Multiplication of $gU - a$ by the nonzero scalar $-a^{-1}$ identifies C_0 with the negative elementary localization for g/a at radius one. Adjoin the V_j one at a time; each next map is a positive elementary localization for the image of h_j. Corollary 2.20.3 gives flatness of the composite and the finite-module assertions, which transfer across the isometry just proved. Uniform equivalence of the original admissible norms preserves these assertions, including in the trivial case. No general relation quotient is asserted to be flat. □

Lemma 2.22.2 (A redundant bounded rational constraint)**.** *Let $B = A\langle f/g\rangle$ and let C be a strict relation quotient over A. Suppose g is a unit in C and the elements $b_j = f_j g^{-1} \in C$ satisfy $\|b_j\| \le 1$. Then the canonical map*

$$C \longrightarrow C\{T_1, \dots, T_s\}/(gT_j - f_j)_j \simeq C\widehat{\otimes}_A B$$

is an isometry. More generally, if each b_j is power bounded, with a fixed bound on $\|b_j^n\|$ for all $n \ge 0$, the same map is a bounded isomorphism with bounded inverse. The isomorphisms are compatible with subsequent bounded maps and with the associated finite-module quotients.

Proof. The image of g is a unit before adjoining the variables, so the actual ideal generated by $gT_j - f_j$ equals the ideal generated by $T_j - b_j$. Substitution at b_j gives an inverse to the coefficient inclusion by Proposition 2.19.1. In the first case both maps are contractive; in the second their bounds are finite. The two composites fix the coefficients and variables, hence are identities by polynomial density. The completed-tensor comparison is Theorem 2.18.2. The same maps on a fixed finite free presentation of a module descend to its relation quotient by Theorem 2.17.2; they retain the stated norm bounds. The formulae show compatibility. The zero case is immediate. □

Lemma 2.22.3 (A Laurent refinement of a unit-generated standard family)**.** *Suppose $f_0, \dots, f_r$ are units of A. Form the all-sign Laurent family for the fixed elements $q_{ij} = f_i/f_j$, $0 \le i < j \le r$, at radius one. Thus each branch D_ϵ imposes, for each pair, either $X_{ij} = q_{ij}$ or $q_{ij}X_{ij} = 1$, with $\|X_{ij}\| \le 1$. For every branch there is an index $k = k(\epsilon)$ for which all f_i/f_k have norm at most one in D_ϵ. In particular, $R_k \to D_\epsilon$ is a contractive map and*

$$D_\epsilon \widehat{\otimes}_A R_k \simeq D_\epsilon$$

isometrically. The same conclusion holds after imposing any additional elementary constraints on that branch.

Proof. The assertion for the zero branch has its stated degenerate meaning. Otherwise, direct an edge $i \to j$ when the chosen relation gives $\|f_i/f_j\| \le 1$, and direct $j \to i$ for the other sign. This defines one directed edge between each pair of vertices. There is a vertex reachable from every vertex by a directed path. To prove this, begin with one vertex k. When adding a new vertex v, keep k if the edge is $v \to k$, and replace k by v if the edge is

$k \to v$. The previously constructed paths followed by this last edge establish the assertion inductively.

For a path $i = i_0 \to i_1 \to \cdots \to i_m = k$, multiply its bounded ratios. Their product is f_i/f_k, so submultiplicativity gives $\|f_i/f_k\| \le 1$. The bounded universal property gives the map from R_k, and Lemma 2.22.2 gives the isometry. Subsequent quotient maps are contractive, preserve the unit equations, and preserve the inequalities just established. Thus the conclusion persists on every further joint presentation. This is a norm calculation; it does not infer a power bound from a pointwise inequality. □

Lemma 2.22.4 (Separating the small members of a standard family)**.** *Assume the valuation of K is nontrivial. Let $f_0, \dots, f_r$ generate the unit ideal and fix $\sum_i a_i f_i = 1$. Choose $\kappa \in K^\times$ with*

$$\theta = |\kappa| \max_i \|a_i\| < 1.$$

Form the all-sign Laurent family for f_i/κ. For a branch C, let P be the positive indices and N the negative indices, and write its relations as

$$f_p = \kappa X_p \quad (p \in P), \qquad f_n X_n = \kappa \quad (n \in N), \qquad \|X_i\| \le 1.$$

If N is empty, C is the actual zero quotient. If $i \in P$, the joint quotient $C\widehat{\otimes}_A R_i$ is zero. For $i \in N$, the element f_i is a unit in C, and there is an isometric identification

$$C\widehat{\otimes}_A R_i \simeq C\{T_j : j \in N \setminus \{i\}\}/(T_j - f_j/f_i)_j.$$

These identifications commute with all intersection maps. They persist after further elementary constraints on C. Consequently the restricted standard rational family is, after removing actual zero terms, the standard family generated by the units $(f_i)_{i \in N}$.

Proof. If every index is positive, the relation ideal contains

$$1 - \kappa \sum_i a_i X_i = \sum_i a_i (f_i - \kappa X_i).$$

The subtracted element has norm at most $\theta < 1$ in the coefficient algebra. Hence this displayed element is a unit by Lemma 1.1.4. The relation ideal is the whole coefficient algebra. This proves the actual zero assertion without using an empty-spectrum implication.

For $i \in P$, work in the coefficient algebra for the joint presentation with R_i, and again put $T_i = 1$. Its relation ideal contains

$$1 - \kappa X_i \sum_j a_j T_j = \sum_j a_j (f_j - f_i T_j) + (f_i - \kappa X_i) \sum_j a_j T_j.$$

The norm of $\kappa X_i \sum_j a_j T_j$ is at most θ. The same geometric inverse proves that this joint ideal is the whole algebra. In particular, every further intersection involving such a term is an actual zero quotient.

If $i \in N$, then $f_i^{-1} = X_i/\kappa$ in C. Multiplying the rational relations by this unit replaces them by $T_j - f_j/f_i$ without changing the actual ideal. For $p \in P$ one has

$$f_p/f_i = X_p X_i, \qquad \|X_p X_i\| \le 1.$$

Thus all variables indexed by P can be eliminated by the two contractive substitution maps of Lemma 2.22.2. What remains is precisely the displayed presentation. The maps fix coefficients and send each surviving variable to the same ratio, so they commute with adding further index blocks. Their norm inequalities and the explicit unit equations survive all later contractive quotient maps. This proves the assertions about intersections and further constraints. □

The preceding two lemmas construct a finite two-level Laurent refinement of a standard rational family. The first level uses functions in A. On a first-level branch the second level uses ratios of units in that branch algebra. We do not identify this construction with a single all-sign family of functions in the original algebra. Every mixed intersection needed below is instead defined by its joint presentation, using Lemma 2.18.1. The actual-zero computations are retained on those intersections.

References. The classical reduction is due to Tate. Compare [Bosch14, §4.3, Lemmas 5–6, pp. 85–87] and [BGR84, §8.2.2, Lemmas 3–4, pp. 333–334]. The proofs here replace the smallness choice by a specified unit-ideal certificate and produce bounded refinement maps by multiplication along directed paths. For the elementary-chain construction compare [Bosch14, §3.3, Proposition 16, p. 56]; its norm proof above uses the same unit-ideal certificate rather than an unproved maximum principle.

2.22.1 The strict-target domain interface

The next interface uses two analytic facts separately: Theorem 1.12.3 detects a unit ideal from points, and Theorem 1.13.4 converts a pointwise bound into

a bound on all powers in a strict affinoid algebra. The earlier presentation arguments did not require the second fact.

Theorem 2.22.5 (Rational factorization for strict targets)**.** *Let $B = A\langle f/g\rangle$, and let*

$$V = \{x \in \mathcal{M}(A) : |f_j(x)| \le |g(x)| \text{ for every } j\}.$$

For every strict norm-admissible affinoid C and bounded unital K-algebra map $\phi : A \to C$, there is a bounded factorization $A \to B \to C$ if and only if the induced map $\mathcal{M}(C) \to \mathcal{M}(A)$ has image in V. The factorization is unique. Zero targets and zero quotients are included. Consequently two strict rational presentations of the same subset of $\mathcal{M}(A)$ have a unique bounded A-algebra isomorphism with bounded inverse; an inclusion of such subsets gives the unique bounded restriction map in the reverse direction. These maps commute with composition. No assertion about generalized non-strict targets is made.

Proof. A bounded homomorphism pulls a bounded multiplicative seminorm back to one, so the stated map on spectra is defined. If the factorization exists, Theorem 2.19.3 gives its image condition. Conversely, assume this condition and first take $C \neq 0$. At a point $y \in \mathcal{M}(C)$, the inequalities and the image of the unit-ideal identity imply $|\phi(g)(y)| > 0$. Theorem 1.12.3 therefore makes $\phi(g)$ a unit of C. Put $b_j = \phi(f_j)\phi(g)^{-1}$. For every y one has $|b_j(y)| \le 1$; Theorem 1.13.4 gives a finite bound for all $\|b_j^n\|$, $n \ge 0$. Corollary 2.19.2 constructs the unique bounded factorization. If $C = 0$, the unique map has the stated properties directly. In particular, a zero B cannot factor into a nonzero C; the preceding construction would contradict its unit equation.

For two presentations with the same subset, apply the first assertion to each presentation as a target of the other. The composites and identity maps agree on A, so uniqueness makes the composites identities. The same uniqueness proves composition compatibility for inclusions. The isomorphisms need not preserve the chosen presentation norms; boundedness in both directions is the conclusion. □

Proposition 2.22.6 (Rational intersections with their actual presentations)**.** *Let $B = A\langle f/g\rangle$ and $C = A\langle e/h\rangle$, with both displayed tuples generating the unit ideal. The joint quotient*

$$H = A\{T_1, \dots, T_s, S_1, \dots, S_t\}/(gT_i - f_i,\ hS_j - e_j)_{i,j}$$

is isometric to the rational presentation with denominator gh and numerator list

$$(hf_i)_i, \qquad (ge_j)_j, \qquad (f_i e_j)_{i,j}.$$

Its point set is the intersection of the point sets for B and C, and $H \simeq B\widehat{\otimes}_A C$. Finite iteration gives the analogous statement for every finite intersection, including repeated members. The intersection and restriction maps are independent of the chosen rational presentations up to the unique bounded isomorphisms of Theorem 2.22.5. For a finite A-module, the corresponding maps on relation-quotient modules are bounded and satisfy the same composition identities.

Proof. Multiplying unit-ideal identities for (g, f_i) and (h, e_j) shows that gh and the listed numerators generate the unit ideal. Let P be their rational presentation, with variables Y_i, Z_j, W_{ij} in the listed order. In H, both g and h are units by their separate unit-ideal identities. Send

$$Y_i \longmapsto \bar{T}_i, \qquad Z_j \longmapsto \bar{S}_j, \qquad W_{ij} \longmapsto \bar{T}_i\bar{S}_j.$$

The images have norm at most one and satisfy all relations, so this gives a contractive map $P \to H$. In P, the product gh is a unit; hence g and h are units separately. Dividing the defining equations gives $gY_i = f_i$, $hZ_j = e_j$, and $W_{ij} = Y_iZ_j$. Thus $T_i \mapsto \bar{Y}_i$, $S_j \mapsto \bar{Z}_j$ gives a contractive map $H \to P$. The two maps are inverse on coefficients and variables, hence on the complete quotients by density. They are isometric.

Theorem 2.19.3 identifies the point set of P. On that set gh is nonzero. The inequalities for hf_i and ge_j can therefore be divided by $|h|$ and $|g|$, giving the two original sets of inequalities. Conversely the original inequalities imply all three product inequalities. The completed-tensor assertion is Theorem 2.18.2. Repeat this construction to treat finite intersections; each step uses actual ideals and the quotient norms, by Lemma 2.18.1. A repeated member is redundant by Lemma 2.22.2.

The strict-target universal property gives the asserted representation independence and makes every composition identity a consequence of uniqueness.

A bounded algebra map induces a bounded map on the finite free modules in a fixed presentation of M, hence on their actual quotient modules. Theorem 2.17.2 identifies these with the ordinary and completed extensions. Their maps coincide on the images of the fixed generators, which proves the same compatibility for modules. □

2.22.2 From all-sign families to finite rational covers

For a finite indexed rational family $\mathcal{U} = (B_i)_{i\in I}$ and a finite module M, let B_J be its joint presentation for a nonempty subset $J \subset I$ and put $M_J = B_J \otimes_A M$. Order I. The ordered augmented complex is

$$0 \longrightarrow M \longrightarrow \prod_{i\in I} M_{\{i\}} \longrightarrow \prod_{i_0<i_1} M_{\{i_0,i_1\}} \longrightarrow \cdots .$$

$$(dc)_{i_0\dots i_{q+1}} = \sum_{\nu=0}^{q+1} (-1)^{\nu} c_{i_0\dots \widehat{i_\nu}\dots i_{q+1}}\big|_{i_0\dots i_{q+1}}.$$

Use the maximum norm on each finite product. The compatibility just proved makes $d^2 = 0$. The all-tuples convention, including repeated indices, is compared with this one by the bounded normalization argument at the end of the proof of Theorem 2.21.3. Strict exactness means that each image equals the next kernel and the quotient norm on that image is uniformly equivalent to the induced norm; all constants may depend on the finite family, M, and the degree.

Theorem 2.22.7 (Strict Čech exactness for a standard rational family)**.** *Assume that K is nontrivially valued. For every unit-ideal tuple $f_0, \dots, f_r \in A$ and every finite canonical A-module M, the augmented ordered Čech complex of its standard rational family is strict exact in every degree. The same is true after replacing A by any strict relation quotient and using the image tuple and any finite module over that quotient.*

Proof. First assume that all f_i are units. Use the all-sign Laurent family $\mathcal{W}$ on the ratios f_i/f_j from Lemma 2.22.3. Theorem 2.21.4 gives strict acyclicity of $\mathcal{W}$. It also gives strict acyclicity of $\mathcal{W}$ restricted to every joint rational presentation R_J: the images of the ratios are elements of that strict algebra, and the restricted family is their all-sign family. The module at the base of this restricted complex is the finite R_J-module $R_J \otimes_A M$. No tensoring of an exact Banach complex by an arbitrary module is involved.

On every nonempty joint presentation of members of $\mathcal{W}$, select one of its members W. Lemma 2.22.3 supplies $k(W)$, and Lemma 2.22.2 identifies adjoining $R_{k(W)}$ with the identity on this joint presentation, isometrically. Thus the restricted rational family contains an identity member. Inserting its index contracts its augmented Čech complex; the maps are bounded, and in the matched quotient norms have bound one. The common-refinement comparison, Lemma 2.21.2, now applies in both directions: the Laurent columns are strict exact, and the rational rows on their intersections are contracted. It identifies the two augmentations through the common double complex and proves strict exactness for $(R_i)_i$. These are the two local acyclicity inputs required for the comparison. Equivalently, one can apply Proposition 2.21.5(1) to this refinement.

For a general unit-ideal tuple, take the first-level all-sign family $\mathcal{V}$ of Lemma 2.22.4. It and its restriction to every R_J are strict acyclic by Theorem 2.21.4, applied afresh over that strict algebra and its finite module. On any nonempty intersection of members of $\mathcal{V}$, choose one member and its sets P, N. The explicit zero identities from Lemma 2.22.4 remain valid there. The terms indexed by P can therefore be discarded, and every term involving one of those indices is zero. The remaining rational family is the standard family on the units $(f_i)_{i\in N}$, by the same lemma and its intersection compatibility; some of its members may also be zero. Its complex is strict exact by the unit case just proved. If the intersection algebra is zero, all its modules and complexes are zero. The same common-refinement comparison therefore proves the theorem for the original tuple.

Every application uses finitely many complexes in each relevant degree. The comparison theorem supplies uniform bounds for their successive lifts, which is the norm-strictness assertion; algebraic exactness alone would not supply those bounds. All auxiliary algebras are verified strict actual quotients by Lemma 2.17.1. Replacing A by such an algebra merely repeats the argument, and equivalent canonical module norms preserve strict exactness. □

Proposition 2.22.8 (A standard refinement of a finite rational cover)**.** *Let $\mathcal{U} = (U_i)_{1\le i\le n}$ be a finite cover of $\mathcal{M}(A)$, where*

$$U_i = \{x : |f_j^{(i)}(x)| \le |f_0^{(i)}(x)|, 1 \le j \le r_i\}, \qquad (f_0^{(i)}, \dots, f_{r_i}^{(i)}) = A.$$

Write B_i for the associated rational presentation. Put

$$\Lambda = \prod_{i=1}^{n} \{0, \dots, r_i\}, \qquad \Lambda_0 = \{\lambda \in \Lambda : \lambda_i = 0 \text{ for some } i\}, \qquad F_\lambda = \prod_{i=1}^{n} f_{\lambda_i}^{(i)}.$$

The tuple $(F_\lambda)_{\lambda\in\Lambda_0}$ generates the unit ideal. Its standard rational family $(S_\lambda)_{\lambda\in\Lambda_0}$ covers $\mathcal{M}(A)$. For any choice $i(\lambda)$ with $\lambda_{i(\lambda)} = 0$, the point set of S_λ is contained in $U_{i(\lambda)}$, and there is a unique bounded map $B_{i(\lambda)} \to S_\lambda$ over A. These maps give a compatible refinement of all finite intersection diagrams.

Proof. Suppose all F_λ, $\lambda \in \Lambda_0$, vanished at x. Choose i with $x \in U_i$. The unit-ideal identity for row i implies $f_0^{(i)}(x) \neq 0$. In every other row, at least one entry is nonzero at x, again by its unit-ideal identity. Choosing such entries and taking $\lambda_i = 0$ gives a nonzero $F_\lambda(x)$, a contradiction. Theorem 1.12.3 now implies that these F_λ generate the unit ideal. The standard family covers because at each point one of the finitely many nonzero values $|F_\lambda|$ is maximal; Theorem 2.19.3 identifies its quotient point sets.

Fix $\lambda \in \Lambda_0$, write $k = i(\lambda)$, and let x be a point of its standard rational set. Here $F_\lambda(x) \neq 0$. Choose j with $x \in U_j$. If $j = k$, the desired inclusion holds. Otherwise, for $0 \le m \le r_k$, replace the jth slot of λ by zero and the kth slot by m, leaving the others unchanged. The resulting index μ belongs to Λ_0, and

$$\left|\frac{F_\mu(x)}{F_\lambda(x)}\right| = \left|\frac{f_0^{(j)}(x)}{f_{\lambda_j}^{(j)}(x)}\right| \left|\frac{f_m^{(k)}(x)}{f_0^{(k)}(x)}\right| \le 1.$$

The first factor is at least one because $x \in U_j$. Thus the second factor is at most one for every m, proving $x \in U_k$. All divisions are legitimate: the factors of $F_\lambda(x)$ are nonzero, and $f_0^{(j)}(x)$ is nonzero on U_j.

Theorem 2.22.5 now gives the bounded map $B_k \to S_\lambda$ and its uniqueness. In particular, this step uses the pointwise-to-power-bound theorem: a numerator with the only zero slot replaced need not occur in Λ_0, so its ratio is not directly one of the norm-one variables of S_λ. We have proved its pointwise bound and then used precisely the analytic interface required to obtain a bounded map. Uniqueness and Proposition 2.22.6 give compatibility on every finite intersection. □

Theorem 2.22.9 (Strict finite rational Čech acyclicity)**.** *Let K be complete, allowing the trivial absolute value. Let A be a strict norm-admissible K-affinoid algebra and let $\mathcal{U}$ be a finite cover of $\mathcal{M}(A)$ by strict rational subsets with radii in $|K^\times|$. For every finite canonical A-module M, the augmented ordered Čech complex of the rational quotient modules is strict exact in all degrees. The all-tuples complex, with repeated indices, has the corresponding*

strict exactness. This statement concerns finite rational covers and finite modules; it asserts neither acyclicity for arbitrary affinoid covers nor exactness of completed base change on arbitrary Banach modules.

Proof. Scale the finitely many numerators to reduce the radii to one, using the isometric change of weighted variables. If $A = 0$ every term is zero. Assume first that $A \neq 0$ and K is nontrivially valued. Proposition 2.22.8 provides a standard rational refinement $\mathcal{V}$. By Theorem 2.22.7, its augmented complex is strict exact. For every finite intersection of members of $\mathcal{U}$, its joint algebra B_J is strict by Proposition 2.22.6. The pullback of $\mathcal{V}$ is the standard rational family generated by the images of the same unit-ideal tuple (F_λ). Theorem 2.22.7, now over B_J and for the finite B_J-module $B_J \otimes_A M$, proves strict exactness on that intersection.

In the other direction, every nonempty intersection of members of $\mathcal{V}$ maps into one chosen member of $\mathcal{U}$, because one of its factors already does. Theorem 2.22.5 and Lemma 2.22.2 identify that restricted member with the identity, by bounded inverse maps. Inserting its index gives a bounded contraction of the restricted coarse complex. Thus both local inputs of Lemma 2.21.2 are verified, or equivalently all hypotheses of Proposition 2.21.5(1). It transfers strict exactness from $\mathcal{V}$ to $\mathcal{U}$ with the stated augmentation. This is the common-refinement argument in all degrees, not merely the degree-zero gluing implication.

It remains to treat trivial valuation. Choose a defining strict quotient norm on A. Then $K\langle X_1, \dots, X_d\rangle = K[X_1, \dots, X_d]$ with norm one on every nonzero polynomial: the finite-tail condition permits only finitely many nonzero coefficients. The quotient norm of every nonzero class in A, and the quotient norm of every nonzero vector in a finite canonical module, is also one. The original admissible norms are uniformly equivalent to these discrete norms. With these choices each rational algebra is the ordinary polynomial quotient

$$B_i = A[T_1, \dots, T_{r_i}]/(g_i T_j - f_j^{(i)})_j \simeq A[1/g_i], \qquad g_i = f_0^{(i)}.$$

Indeed the unit-ideal identity gives an inverse of g_i in the displayed quotient, and the inverse map sends T_j to $f_j^{(i)}/g_i$. These maps are inverse algebraically and bounded for the discrete norms. Joint quotients and their modules are therefore $A[1/\prod_{i\in J} g_i]$ and $M[1/\prod_{i\in J} g_i]$, with the usual localization maps.

For every maximal ideal $\mathfrak{m} \subset A$, Theorem 1.12.3 provides the bounded point from the finite residue field $A/\mathfrak{m}$. Its extending absolute value is trivial

by Lemma 1.12.1. Since this point belongs to some U_i, the unit-ideal identity for that row forces $g_i \notin \mathfrak{m}$. Thus $(g_1, \dots, g_n) = A$. This argument uses these particular maximal-residue-field points; it does not assert that every point of $\mathcal{M}(A)$ is trivially valued.

Apply [StacksAlg, Tag 0G6K, Lemma 15.30.5(4)] to the ordinary extended alternating localization complex, with $f = 1 \in (g_1, \dots, g_n)$. It is exactly our ordered augmented complex, with its degree numbering shifted by one, and is exact in every degree. For discrete norms every linear map is bounded, every image is closed, and the quotient norm of a nonzero image element is one. The complex is consequently norm-strict. Uniform equivalence transfers this conclusion to the original canonical norms. Finally the bounded comparison between ordered and all-tuples complexes at the end of the proof of Theorem 2.21.3 gives the all-tuples assertion in both valuation cases. □

References. For the classical product refinement compare [Bosch14, §4.3, Lemma 4, pp. 84–85] and [BGR84, §8.2.2, Lemma 2, pp. 332–333]. The passage to higher degrees is governed by the two-sided refinement comparison, cf. [BGR84, §8.1.4, Corollaries 3–4, p. 327; §8.2.2, Proposition 5, p. 334]. Bosch's Proposition 4.3/7 proves the sheaf reduction; his discussion of the full acyclicity theorem explicitly refers to BGR rather than supplying its proof. The argument above supplies the finite-module and norm comparisons within the stated strict rational scope.

2.23 The structure sheaf on the fixed rational site

The acyclicity theorem already gives a useful sheaf construction. We specify the site so that no theorem about arbitrary affinoid domains is concealed in its definition. Let A be a strict K-affinoid algebra over a complete field with its fixed non-Archimedean absolute value, allowing trivial valuation, and put $X = \mathcal{M}(A)$.

Definition 2.23.1 (The fixed rational pretopology)**.** Let $\mathcal{R}_X$ be the partially ordered category of subsets of X admitting a rational presentation over the original algebra A with radii in $|K^\times|$. It contains X and the empty set. Arrows are inclusions. A covering family of $V \in \mathcal{R}_X$ is a finite family $(V_i \subseteq V)$ of objects of $\mathcal{R}_X$ whose union is V. The empty family covers the empty set. Choose one rational presentation algebra A_V for each V, with $A_X = A$ and $A_\varnothing = 0$, and use the canonical bounded restriction maps of Theorem 2.22.5. For a finite A-module M, write

$$\mathcal{O}_X^{\mathrm{rat}}(V) = A_V, \qquad \widetilde{M}^{\mathrm{rat}}(V) = A_V \otimes_A M.$$

Corollary 2.23.2 (Rational structure sheaves and finite modules)**.** *The covering families just defined form a pretopology. The assignments* $\mathcal{O}_X^{\mathrm{rat}}$ *and* $\widetilde{M}^{\mathrm{rat}}$ *are respectively a sheaf of rings and a sheaf of modules on the resulting site, independent up to canonical isomorphism of the presentation choices. One has*

$$\Gamma(X, \mathcal{O}_X^{\mathrm{rat}}) = A, \qquad \Gamma(X, \widetilde{M}^{\mathrm{rat}}) = M.$$

The functor $M \mapsto \widetilde{M}^{\mathrm{rat}}$ *from finite* A*-modules to* $\mathcal{O}_X^{\mathrm{rat}}$*-modules is exact and fully faithful. Each finite covering in this site has a norm-strict exact augmented Čech complex for* $\widetilde{M}^{\mathrm{rat}}$*, with canonical finite-module norms.*

Proof. Proposition 2.22.6 represents $V \cap W$ by a rational algebra over A. It is the categorical fiber product in this partially ordered category. Identity coverings, pullbacks of finite set-theoretic coverings, and finite composites of such coverings satisfy the pretopology axioms, since intersection commutes with finite union. This uses only objects rational over the fixed A. It does not require a theorem saying that every rational domain over every intermediate algebra has a rational presentation over A.

Uniqueness in Theorem 2.22.5 proves that restriction maps compose, that identity restrictions are identities, and that two choices of a presentation give compatible bounded inverse isomorphisms. If $V_i \subseteq V$, the joint algebra $A_V \widehat{\otimes}_A A_{V_i}$ is canonically A_{V_i} by the same theorem. Viewed over A_V, its defining tuple is the image of a unit-ideal tuple from A, hence still generates one. Thus any finite covering (V_i) of V becomes a finite rational covering of $\mathcal{M}(A_V)$ in the scope of Theorem 2.22.9. Apply that theorem to the finite A_V-module $A_V \otimes_A M$. The ordinary tensor associativity of [StacksAlg, Tag 00D2, Lemma 10.12.7], together with Theorem 2.17.2, identifies all its terms and arrows with those in the asserted Čech complex.

Degree-zero exactness is precisely the equalizer condition for a sheaf. The value on the empty object is zero, as required by its empty covering. Taking $M = A$ gives the ring sheaf, and multiplication on the local tensor modules gives the module sheaves. Their global sections have the displayed values by the chosen presentation of X.

For a short exact sequence of finite A-modules, rational flatness from Lemma 2.22.1 proves exactness on every object when the valuation is nontrivial. If the valuation is trivial, equip A with a defining quotient norm; all nonzero elements have norm one. In the proof of that lemma the unit certificate then has coefficient norms at most one, so one may take $a = 1$. Its same elemen-

tary chain proves rational flatness in this case as well, using Corollary 2.20.3, which allows trivial valuation. Proposition 1.11.6 gives the corresponding normed finite-module identifications. Objectwise exactness implies exactness of the sequence of sheaves.

An A-linear map $u : M \to N$ produces the sheaf map $1 \otimes u$. Conversely, let $\psi : \widetilde{M}^{\mathrm{rat}} \to \widetilde{N}^{\mathrm{rat}}$ be a morphism of $\mathcal{O}_X^{\mathrm{rat}}$-modules. Its map on X is an A-linear $u : M \to N$. For every V, compatibility with restriction gives $\psi_V(1 \otimes m) = 1 \otimes u(m)$. Since every element of $A_V \otimes_A M$ is a finite sum of elements $b \otimes m$, A_V-linearity forces $\psi_V = 1 \otimes u$. Thus the map on Hom groups is a bijection. The norm-strict Čech assertion was obtained from the same application of the theorem. □

Theorem 2.28.12 supplies essential surjectivity on this rational site by effective descent of finite modules. The corresponding equivalence for generalized affinoid domains is Theorem 2.38.5; its extension to the full analytic G-site, together with Theorem B, is Theorem 2.39.6.

2.24 Compact spectra and finite rational coverings

The algebraic unit-ideal test uses maximal residue fields. A covering theorem, however, concerns every Berkovich point. We record the topological bridge and the exact finite-cover test that will be used below.

Proposition 2.24.1 (Compactness of the bounded spectrum)**.** *Let K be a complete field with its fixed non-Archimedean absolute value, and let A be a commutative unital Banach K-algebra with a submultiplicative homogeneous norm. Its bounded multiplicative spectrum $\mathcal{M}(A)$, with the evaluation topology, is compact and Hausdorff; it may be empty. Every bounded unital homomorphism $A \to B$ induces a continuous map $\mathcal{M}(B) \to \mathcal{M}(A)$. Any subset defined by finitely many inequalities $|f_i(x)| \le r_i|g_i(x)|$, with $r_i > 0$, is closed and therefore compact. No nonemptiness theorem is part of this statement.*

Proof. The power argument of Lemma 1.6.1 applies verbatim to a bounded multiplicative seminorm on any such A: $|a|_x^n \le C\|a\|^n$ and taking nth roots gives $|a|_x \le \|a\|$. Hence evaluation embeds the spectrum in

$$P = \prod_{a \in A} [0, \|a\|].$$

Each closed real interval is compact. For completeness, if an open cover of an interval had no finite subcover, repeatedly bisect and keep a half without a finite subcover. The nested intervals have lengths tending to zero and a common point, by completeness of the real numbers. An open set of the cover containing that point contains a sufficiently small retained interval, a contradiction. By the authorized Tychonov theorem [StacksAlg, Tag 08ZU, Theorem 5.14.4], P is compact. It is Hausdorff because two distinct points differ in a coordinate, where disjoint real neighbourhoods separate them.

The image of the spectrum is cut out by

$$s_0 = 0, \quad s_1 = 1, \quad s_c = |c| \ (c \in K), \quad s_{ab} = s_a s_b, \quad s_{a+b} \leq \max(s_a, s_b).$$

All these conditions are closed, since real addition, multiplication and maximum are continuous. Conversely they give exactly a multiplicative ultrametric seminorm extending the fixed absolute value and bounded by the chosen norm. Thus $\mathcal{M}(A)$ is a closed subspace of P, with exactly its evaluation topology. A closed subspace of a compact space is compact: adjoin its open complement to an open cover and take a finite subcover of the ambient space. Hausdorffness is inherited. Composition with a bounded homomorphism preserves boundedness and turns each evaluation into another evaluation, proving continuity. Finally each stated inequality is the inverse image of a closed subset of $\mathbb{R}^2$ under the pair of evaluations; their finite intersection is closed. □

Theorem 2.24.2 (Testing a finite rational cover at maximal points)**.** *Let A be a strict K-affinoid algebra over any complete non-Archimedean field, possibly trivially valued. A finite family of value-group rational subsets of $\mathcal{M}(A)$ covers the spectrum if and only if it contains the bounded point associated to every maximal ideal of A. The assertion includes the empty family when $A = 0$.*

Proof. Scale the finitely many radii by nonzero elements of K, so that the members have presentations

$$U_i = \{x : |f_{ij}(x)| \leq |g_i(x)| \text{ for all } j\}, \qquad (g_i, f_{i1}, \ldots, f_{im_i}) = A.$$

The forward implication is immediate. For the converse first suppose the valuation is nontrivial, and fix $\pi \in K$ with $0 < |\pi| < 1$. If x lies outside

every U_i, choose for each i an index j_i and put $h_i = f_{ij_i}$, so $|h_i(x)| > |g_i(x)|$. Choose an integer $N_i \geq 1$ and a nonzero scalar c_i with

$$|g_i(x)|^{N_i} \leq |\pi|\,|h_i(x)|^{N_i}, \qquad |c_i| \leq |h_i(x)|^{N_i}.$$

This is possible because the ratio is strictly less than one and $|h_i(x)| > 0$. Let V_i be the rational subset with denominator $h_i^{N_i}$ and numerators $\pi^{-1}g_i^{N_i}, c_i$. Its defining tuple generates the unit ideal since $c_i \neq 0$. It contains x, and every $y \in V_i$ satisfies

$$|h_i(y)| > 0, \qquad |g_i(y)| \leq |\pi|^{1/N_i}|h_i(y)| < |h_i(y)|.$$

Hence $V_i \cap U_i = \varnothing$. The finite intersection $V = \bigcap_i V_i$ has a strict rational presentation by Proposition 2.22.6. Its presentation algebra A_V is nonzero: the point $x \in V$ lifts to its spectrum by Theorem 2.19.3. Theorem 1.12.3 supplies a bounded maximal-residue evaluation $A_V \to L$, with L/K finite. Its restriction to A has maximal kernel: its image is a finite-dimensional K-domain, hence a field by [StacksAlg, Tag 00GS(1)]. Uniqueness of the extending absolute value, Lemma 1.12.1, identifies the resulting point with the maximal point of that kernel. This point belongs to V and to none of the U_i, contrary to the hypothesis.

If the valuation is trivial, equip A with a strict presentation quotient norm, which is discrete. Every element has pointwise value at most one. A unit-ideal certificate for $(g_i, f_{i1}, \ldots, f_{im_i})$ consequently shows that

$$U_i = \{x : |g_i(x)| = 1\}.$$

Indeed, all the numerators have value at most one, and if the denominator and numerators all had value strictly less than one, their unit-ideal relation would contradict $|1| = 1$. At a maximal point $|g_i| = 1$ is equivalent to $g_i \notin \mathfrak{m}$, since finite extensions of a trivially valued field have the trivial extending absolute value. The assumed covering of maximal points therefore gives $(g_1, \ldots, g_t) = A$ by [StacksAlg, Tag 00E0(2)]. Evaluating a unit certificate at any x shows that some $|g_i(x)| = 1$, so that $x \in U_i$. The empty family is dealt with by the nonemptiness criterion of Theorem 1.12.3. □

The trivial-valuation argument is not a density argument. For example, if K is trivially valued and $A = K[T]$ with its discrete norm, the Gauss seminorm of radius $r \in (0, 1)$ is a bounded point. Every maximal residue point gives $|T| = 0$ or 1, whereas an open band $a < |T| < b$ with $0 < a < r <$

$b < 1$ contains that Gauss point and no maximal point. The finite rational-cover test remains valid because its inequalities have value-group radii, not arbitrary real radii.

2.25 Rational domains inside rational domains

The fixed rational site can be compared with the rational site of one of its members only after a transitivity theorem has been proved. The issue is that a function on a rational domain need not come from the original algebra. We first control small changes of its defining functions, and then replace finitely many functions by fractions with one common denominator. The resulting algebra comparison will be proved on the actual relation quotients.

Throughout, K is a complete real-valued non-Archimedean field; trivial valuation is allowed. Nonzero affinoid algebras are strict and norm-admissible, with submultiplicative norm and unit of norm one. Quotients use their actual quotient norms. The zero algebra and its zero modules have the separate zero-norm interpretation of Section 2.17. A rational presentation has a unit-ideal tuple. We use unit radii in the lemmas; radii in $|K^\times|$ are reduced to this case by scaling variables.

Lemma 2.25.1 (Stability under a small change of a rational tuple)**.** *Let E be a nonzero strict norm-admissible affinoid algebra. Fix $a_0, \dots, a_t, b_0, \dots, b_t \in E$ with*

$$\sum_{j=0}^{t} b_j a_j = 1, \qquad H = \max(1, \|b_0\|, \dots, \|b_t\|).$$

Suppose $a'_0, \dots, a'_t \in E$ and put

$$\delta = \max_{0 \le j \le t} \|a'_j - a_j\| < H^{-1}.$$

Then $(a'_0, \dots, a'_t) = E$, with a unit-ideal certificate whose coefficient norms are at most H. The two rational subsets defined by the denominators a_0 and a'_0 are equal. Their actual presentation algebras

$$Q = E\{S_1, \dots, S_t\}/(a_0 S_j - a_j)_j, \qquad Q' = E\{S'_1, \dots, S'_t\}/(a'_0 S'_j - a'_j)_j$$

have a unique bounded E-algebra isomorphism, and this isomorphism is isometric for the displayed quotient norms. For a finite E-module with a fixed finite free presentation, the induced comparison of its two relation-quotient

modules is also isometric. These assertions include zero presentation algebras.

Proof. Write $\epsilon_j = a'_j - a_j$ and $u = \sum_j b_j a'_j = 1 + \sum_j b_j \epsilon_j$. Since $\|u - 1\| \leq H\delta < 1$, Lemma 1.1.4 gives u^{-1} with $\|u^{-1}\| \leq 1$. Thus $b'_j = u^{-1} b_j$ satisfies $\sum_j b'_j a'_j = 1$ and $\|b'_j\| \leq H$. All the ideals and quotients appearing here are actual closed ideals and complete quotients by Lemma 2.17.1.

In Q, put $s_j = \overline{S}_j$. The certificate gives $a_0^{-1} = b_0 + \sum_{j \geq 1} b_j s_j$, hence $\|a_0^{-1}\| \leq H$. Consequently $a'_0 = a_0(1 + \epsilon_0 a_0^{-1})$ is a unit and

$$\frac{a'_j}{a'_0} = \left(s_j + \epsilon_j a_0^{-1}\right)\left(1 + \epsilon_0 a_0^{-1}\right)^{-1}, \qquad \left\| \frac{a'_j}{a'_0} \right\| \leq 1.$$

Proposition 2.19.1 therefore gives a contractive map $Q' \to Q$ sending S'_j to the displayed ratio. In Q', use the certificate (b'_j) and the differences $-\epsilon_j$. The same calculation gives a contractive map $Q \to Q'$. The composites fix E, and each variable is forced to be the corresponding numerator divided by its unit denominator. Thus the composites fix the variables and, by polynomial density, are identities. Two inverse contractions are isometric. The same argument shows uniqueness among bounded E-algebra maps. It also applies when a quotient is zero: unit equations are interpreted with $1 = 0$, and the two constructed maps force both quotients to be zero together.

For completeness, equality of the point sets can also be checked without an algebra comparison. At $x \in \mathcal{M}(E)$ put $m(x) = \max_j |a_j(x)|$. The certificate and Lemma 1.6.1 give $m(x) \geq H^{-1} > \delta$. A coordinate attaining this maximum retains its absolute value after perturbation, and every other perturbed coordinate has value at most $m(x)$. If $|a_0(x)| = m(x)$, the same holds for a'_0; if $|a_0(x)| < m(x)$, then $|a'_0(x)| < m(x)$. This is exactly the equivalence of the two rational conditions. Finally choose $E^v \xrightarrow{F} E^w \to N \to 0$; such a finite presentation exists by Theorem 1.11.5. The two induced modules are $Q^w/F_Q Q^v$ and $(Q')^w/F_{Q'}(Q')^v$ by Theorem 2.17.2. The coordinatewise isometry carries one actual relation image to the other, so it induces an isometry of their quotient norms. □

The strict inequality on the error is essential. Over $E = K$, the tuples $(1, 0)$ and $(0, 1)$ are at distance one and both generate the unit ideal, but their rational conditions define respectively all of $\mathcal{M}(K)$ and the empty set. In particular, a pointwise bound is not being silently substituted for a strict error estimate.

Lemma 2.25.2 (Approximation by fractions with one denominator)**.** *Let $g, f_1, \dots, f_s$ generate the unit ideal of A, and set $B = A\langle f/g\rangle$. The element g is a unit in B, and the image of $A[g^{-1}]$ is dense in B. More explicitly, given $v_0, \dots, v_t \in B$ and $\eta > 0$, there exist $N \geq 0$ and $p_0, \dots, p_t \in A$ such that*

$$\|v_j - g^{-N} p_j\|_B < \eta \qquad (0 \leq j \leq t).$$

If K is trivially valued and A carries a defining strict quotient norm, all these approximations may be chosen to be equalities. In that case every nonzero element of A or B has norm one, and B is the ordinary localization $A[g^{-1}]$ with that discrete norm.

Proof. Choose $c_0 g + \sum_i c_i f_i = 1$. In B, with $x_i = \overline{X}_i$, one has $g^{-1} = c_0 + \sum_i c_i x_i$. Polynomial images are dense in B by Lemma 2.13.1 and the quotient construction of Lemma 2.17.1. For a polynomial $P(X) = \sum_\alpha e_\alpha X^\alpha$ and an integer N at least its total degree, put

$$p_P = \sum_\alpha e_\alpha f^\alpha g^{N-|\alpha|} \in A.$$

The defining relations give $g^N P(x) = p_P$ in the actual quotient. Choose polynomial approximations to the finitely many v_j, and then choose one N at least all their degrees. This proves the claimed simultaneous approximation. No injectivity of $A \to B$ or of $A[g^{-1}] \to B$ has been assumed.

For trivial valuation, a strict Tate algebra over K is a polynomial ring: a coefficient family tending to zero has finite support. Its norm is one on nonzero elements. A defining quotient norm on A therefore has the same property. Consequently $A\{X\} = A[X]$, and every class of B already has a polynomial representative. The quotient norm of a nonzero class is one. The unit equation for g and the maps $X_i \mapsto f_i/g$ identify this polynomial quotient with $A[g^{-1}]$. The assertions for a zero quotient have their stated degenerate meaning. □

Lemma 2.25.3 (Flattening a rational presentation by exact fractions)**.** *Let $B = A\langle f/g\rangle$ and fix certificates*

$$c_0 g + \sum_{i=1}^{s} c_i f_i = 1 \quad \text{in } A, \qquad \sum_{j=0}^{t} b_j a_j = 1 \quad \text{in } B.$$

Suppose $a_j = g^{-N} p_j$ in B for one integer $N \geq 0$ and $p_j \in A$. Put

$$L = \max(1, \|c_0\|, \dots, \|c_s\|), \qquad H = \max(1, \|b_0\|, \dots, \|b_t\|),$$

and choose $\gamma \in K^\times$ *satisfying* $|\gamma| L^{N+1} H \le 1$. *Set* $d = g p_0$. *Then*

$$C' = B\{S_1, \dots, S_t\}/(a_0 S_j - a_j)_j,$$
$$D = A\{Y_1, \dots, Y_s, Z_1, \dots, Z_t, V\}/(dY_i - p_0 f_i,\ dZ_j - g p_j,\ dV - \gamma)_{i,j}$$

are canonically isometric as A*-algebras. The tuple defining* D *generates the unit ideal because it contains* γ*; thus* D *is a rational presentation over* A*. The isometry identifies its variables by*

$$Y_i \longmapsto \overline{X}_i, \qquad Z_j \longmapsto \overline{S}_j, \qquad V \longmapsto \gamma g^{-(N+1)} a_0^{-1} \quad \text{in } C'.$$

Zero quotients are included. No unit-ideal condition on $(p_0, \dots, p_t)$ *in* A *is required.*

Proof. All displayed ideals are actual closed ideals by Lemma 2.17.1. In C', the two certificates give

$$g^{-1} = c_0 + \sum_i c_i \overline{X}_i, \qquad a_0^{-1} = b_0 + \sum_{j=1}^{t} b_j \overline{S}_j.$$

Their norms are at most L and H, respectively. Since $p_0 = g^N a_0$, the element $d = g^{N+1} a_0$ is a unit and $\|\gamma d^{-1}\| \le |\gamma| L^{N+1} H \le 1$. The proposed images of Y_i, Z_j, V all have norm at most one. The relations for the first two blocks follow from $g\overline{X}_i = f_i$ and $a_0 \overline{S}_j = a_j = g^{-N} p_j$; the last relation follows from the definition of γd^{-1}. Proposition 2.19.1 therefore gives a contractive map $D \to C'$.

Conversely, in D the equation $d\overline{V} = \gamma$ makes d a unit. Therefore g and p_0 are units separately. Dividing $d\overline{Y}_i = p_0 f_i$ by p_0 gives $g\overline{Y}_i = f_i$. The images of the Y_i have norm at most one, so Proposition 2.19.1 gives a contractive coefficient map $B \to D$. Under this map the equality $a_j = g^{-N} p_j$ persists. Dividing $d\overline{Z}_j = g p_j$ by g and multiplying by g^{-N} gives $a_0 \overline{Z}_j = a_j$. The same proposition now gives a contractive map $C' \to D$.

The composites fix A, X_i, S_j, Y_i and Z_j. They also fix V: the relation $dV = \gamma$ and the invertibility of d uniquely determine this element. Polynomial density then makes the composites identities, proving the isometry. If $B = 0$, the already constructed unital map $B \to D$ forces $D = 0$, and $C' = 0$ directly; the second universal-property step is then replaced by the unique map of zero algebras. All other zero-quotient cases are covered by the same calculations. Thus the proof never identifies an empty spectrum with a zero ring without an algebraic argument. □

The scalar γ serves two purposes. It gives a unit-ideal certificate in the original algebra, and its chosen size makes the additional variable contractive on the iterated quotient. The existence of such a scalar is automatic over a nontrivially valued field. Over a trivially valued field it will follow from the defining discrete norms, with $L = H = 1$.

Theorem 2.25.4 (Transitivity of strict rational presentations)**.** *Let A be a strict norm-admissible K-affinoid algebra. Let B be a rational presentation over A, and let C be a rational presentation over B, with all radii in $|K^\times|$. Then there is a rational presentation D over A and a unique bounded A-algebra isomorphism $D \simeq C$ with bounded inverse. Under the point homeomorphisms of Theorem 2.19.3, the image of $\mathcal{M}(C)$ in $\mathcal{M}(A)$ is precisely the rational subset represented by D.*

If the valuation of K is nontrivial, D can be chosen so that this isomorphism is isometric for the given actual presentation norms. For trivial valuation, the same holds after replacing the norm on A by a defining strict quotient norm and using the induced quotient norms on the presentations; for the original norms the conclusion is a bounded isomorphism in both directions. The construction and conclusion allow zero quotients.

For every finite canonical A-module M, the resulting identifications

$$D \otimes_A M \;\simeq\; C \otimes_A M \;\simeq\; C \otimes_B (B \otimes_A M)$$

are natural bounded isomorphisms with bounded inverses. The corresponding assertions hold with separated completed tensors. With norms from one fixed finite free presentation of M, the module comparisons are isometric whenever the algebra comparison is isometric. This completed-tensor assertion concerns these finite modules.

Proof. By the weighted change of variables in Lemma 2.17.1, reduce every radius to one. This change is isometric. Write

$$B = A\langle f/g\rangle, \qquad C = B\langle h/k\rangle, \qquad h = (h_1, \dots, h_t).$$

If $A = 0$ or $B = 0$, then $C = 0$; the rational presentation $A\{V\}/(0V - 1)$ is zero and gives the conclusion, including the module assertions. We may therefore assume $A, B \neq 0$. Choose certificates

$$c_0 g + \sum_i c_i f_i = 1, \qquad b_0 k + \sum_{j=1}^{t} b_j h_j = 1,$$

and define $L = \max(1, \|c_i\|)$ and $H = \max(1, \|b_j\|)$. Choose $0 < \eta < H^{-1}$. Lemma 2.25.2 supplies a common N and $p_0, \dots, p_t \in A$ such that

$$a'_0 = g^{-N} p_0, \qquad a'_j = g^{-N} p_j, \qquad \|k - a'_0\| < \eta, \quad \|h_j - a'_j\| < \eta.$$

Lemma 2.25.1 gives a unit-ideal certificate for (a'_j) with coefficient norms at most H, and an isometric B-algebra identification

$$C \simeq C' = B\langle a'_1/a'_0, \dots, a'_t/a'_0 \rangle.$$

If K is nontrivially valued, powers of a scalar of absolute value less than one give $\gamma \in K^\times$ with $|\gamma| L^{N+1} H \leq 1$. Lemma 2.25.3 then gives the required D, with denominator gp_0 and numerator list

$$(p_0 f_i)_{1 \leq i \leq s}, \qquad (g p_j)_{1 \leq j \leq t}, \qquad \gamma.$$

Both comparisons are isometric, so $D \simeq C$ is isometric. Notice that adding γ is indispensable to this construction: clearing denominators only proves a unit-ideal statement in B, and does not establish $(p_0, \dots, p_t) = A$.

For trivial valuation, replace the norm on A by a defining strict quotient norm. Uniformly equivalent coefficient norms give uniformly equivalent Tate and quotient norms, with the same underlying complete algebras: the coefficient convergence condition is unchanged, and taking infima over representatives preserves both norm comparisons. By Lemma 2.25.2, every nonzero element of A and B now has norm one. Hence $L = H = 1$, the approximations may be exact, and $\gamma = 1$ meets the required inequality. The same two lemmas give an isometric comparison for these norms. Returning to the original uniformly equivalent norms gives the asserted bounded inverse maps.

The isomorphism is over A. Applying Theorem 2.19.3 first to $A \to B$ and then to $B \to C$ identifies $\mathcal{M}(C)$ homeomorphically with its image in $\mathcal{M}(A)$. The same theorem for $A \to D$, together with the constructed isomorphism, identifies this image with the rational subset of the displayed tuple. This proves the point assertion from the algebra comparison; no spectral-radius or maximum principle is needed in this proof. Any bounded A-algebra endomorphism of C restricts on B to its unique bounded rational factorization and then has its values on the second-stage variables forced by their unit denominator. Thus it is the identity. The same argument after composing with the constructed inverse proves the asserted uniqueness. Different choices of approximations and of γ are consequently compared by unique bounded A-algebra maps.

For the module assertion choose $A^v \xrightarrow{F} A^w \to M \to 0$. Ordinary tensor associativity, [StacksAlg, Tag 00D2, Lemma 10.12.7], gives $C \otimes_B (B \otimes_A M) \simeq C \otimes_A M$, sending $c \otimes (b \otimes m)$ to $cb \otimes m$. Theorem 2.17.2 identifies the successive relation-quotient module with $C^w/F_C C^v$, with its quotient norm. The algebra comparison gives a bounded inverse comparison with $D^w/F_D D^v$, and it is isometric if the coefficient comparison is isometric. All these maps agree on the images of the chosen generators and are therefore natural in M. Finally, Theorem 2.17.2 and Proposition 1.11.6 identify each finite ordinary tensor here with its separated completion. No passage from algebraic flatness to an assertion about arbitrary completed tensors is involved. □

References. The transitivity theorem is classical: see [BGR84, §7.2.4, Theorem 2, statement and proof on pp. 285–286] and [Bosch14, §3.3, Proposition 17, pp. 56–57]. For the stability of special domains under small changes compare [BGR84, §7.2.3, Proposition 3, p. 281]. The proofs above reconstruct the rational case with explicit norm bounds, actual quotient maps, and a separate argument at trivial valuation. The classical texts provide attribution and comparison; their transitivity or maximum-principle statements are not proof inputs.

Proposition 2.25.5 (The rational site of a rational member)**.** *Let $U \subseteq X = \mathcal{M}(A)$ be represented by a strict rational algebra B with value-group radii. Under $\mathcal{M}(B) \simeq U$, the rational subsets defined over B are exactly the subsets of U which are rational over A. Hence the rational site of $\mathcal{M}(B)$ is identified with the slice $\mathcal{R}_X/U$ of Definition 2.23.1, with finite set-theoretic covers on both sides. The structure sheaves agree under this identification, independently up to their unique bounded isomorphisms of all presentation choices. For every finite A-module M, the restriction of $\widetilde{M}^{\mathrm{rat}}$ corresponds to the rational module sheaf associated to $B \otimes_A M$. The coefficient and finite-module comparison maps are bounded in both directions.*

Proof. Every rational subset over B is rational over A by Theorem 2.25.4. Conversely, let $V \subseteq U$ have a rational presentation A_V over A. Map its unit-ideal tuple into B. This gives a rational presentation over B, and Theorem 2.19.3 identifies its point set with the inverse image of V, which is V under $\mathcal{M}(B) \simeq U$. Proposition 2.22.6 identifies the joint algebra with $B \widehat{\otimes}_A A_V$. Its point set is $U \cap V = V$. Theorem 2.22.5 therefore gives its unique bounded A-algebra isomorphism with A_V, with bounded inverse. The coefficient map from B agrees under this isomorphism by uniqueness of the factorization through U.

The two categories have now been identified as partially ordered sets: arrows on both sides are the same inclusions of subsets. Their finite covering families are the same families of subsets with the same union, including the empty covering of the empty object. On every object the coefficient algebras have just been identified. Restriction maps commute with these comparisons, since each is the unique bounded factorization for the corresponding inclusion. Hence the ring presheaves, and therefore the sheaves of Corollary 2.23.2, agree. Theorem 2.25.4 supplies the compatible finite-module comparison

$$C \otimes_B (B \otimes_A M) \simeq C \otimes_A M \simeq A_V \otimes_A M$$

whenever C is a chosen B-rational presentation of V. These maps agree on pure tensors, so their compatibility with restrictions and with morphisms of M follows directly. This proves the assertion for module sheaves and their canonical norms. □

Corollary 2.25.6 (Finite trees with functions on intermediate rational domains)**.** *Take a finite rooted tree of strict rational presentations, with root A and with every child a rational presentation over its parent, using value-group radii. Then every vertex represents a rational subset of $\mathcal{M}(A)$. All finite intersections of these subsets admit rational presentations over A. Their coefficient algebras, restriction maps and associated finite-A-module quotients are independent of the chosen flattenings up to unique compatible bounded isomorphisms. A presentation of an intersection obtained by a further finite rational tower has the same comparison.*

Suppose in addition that the children of each nonterminal vertex cover its spectrum. Then the leaves form a finite rational cover of $\mathcal{M}(A)$. For every finite canonical A-module, its augmented ordered and all-tuples leaf Čech complexes, with these intersection identifications, are norm-strict exact in every degree. Repeated indices and actual zero quotients are retained with the conventions already stated.

Proof. Apply Theorem 2.25.4 inductively along the path from the root to each vertex. This terminates because the tree is finite. The point comparison identifies every parent-child map with the corresponding inclusion in $\mathcal{M}(A)$. Proposition 2.22.6 now gives each finite intersection over A, with its actual relation quotient. Theorem 2.22.5 gives uniqueness and compatibility of its restriction maps. The finite-module identifications follow from Theorem 2.25.4 on the paths and Theorem 2.17.2 on the intersections. An

additional finite rational tower is flattened by the same induction, and its presentation of the same subset is then compared by the same uniqueness theorem.

If the children cover each nonterminal vertex, induction from the leaves towards the root shows that the union of the leaves is the root spectrum. Apply Theorem 2.22.9 to this finite rational cover of $\mathcal{M}(A)$. The preceding compatible algebra and module maps identify its terms and differentials with the stated leaf complex. In every fixed degree only finitely many terms occur, so the product of the comparison maps and its inverse are bounded. Such a bounded isomorphism of complexes preserves exactness, closed images and the quotient-to-induced norm comparison on each image. The theorem therefore gives the claimed norm-strict exactness, including its all-tuples convention. □

Finite rational constructions can therefore be expressed over the original algebra, with compatible section modules. This is the geometric input to the effective descent argument in Theorem 2.28.12. The comparison with affinoid-domain sites is established in Theorems 2.30.6 and 2.38.4.

2.26 Extending dense image across a rational boundary

A dense-image statement on a rational domain can persist when its defining bounds are enlarged slightly. To prove this with quotient norms, the radii must be allowed to vary through a dense subgroup of $\mathbb{R}_{>0}$. We first realize the required radii over the original field. No extension of scalars will be used. Put

$$\Gamma_K^{\mathrm{div}} = \{r > 0 : r^n \in |K^\times| \text{ for some integer } n \geq 1\}.$$

The field K is complete. All preparatory statements allow trivial valuation, although the hypothesis $r_0 < r_1$ in Γ_K^{div} has no instance in that case. The extension theorem explicitly assumes nontrivial valuation. Nonzero strict affinoid algebras have norm-admissible Banach norms, and zero quotients use the zero-norm convention.

Lemma 2.26.1 (Radii in the divisible closure of the value group)**.**

Let A be strict norm-admissible and let $r_1, \dots, r_n \in \Gamma_K^{\mathrm{div}}$. Then the weighted coefficient algebra $D = A\{r^{-1}T\}$ is strict norm-admissible. Every algebraic ideal of D is finite and closed, and its actual quotient is strict, with zero allowed. Adjoining another finite block with radii in Γ_K^{div} commutes isometrically with taking such a quotient: for $B = D/I$ and $F = D\{s^{-1}Y\}$,

$$F/IF \simeq B\{s^{-1}Y\}$$

isometrically, with actual kernel IF. A further finite relation quotient is the direct quotient by the sum of the lifted actual ideals, with the same norm.

Proof. For each i, choose an integer $n_i \geq 1$ and $c_i \in K^\times$ with $|c_i| = r_i^{n_i}$. Write $S_i = c_i^{-1} T_i^{n_i}$ and put $R = A\langle S_1, \dots, S_n \rangle$. Grouping exponents modulo the n_i gives a unique expansion

$$D = \bigoplus_{0 \leq \beta_i < n_i} R\,T^\beta, \qquad \left\| \sum_\beta F_\beta(S) T^\beta \right\|_r = \max_\beta r^\beta \|F_\beta\|.$$

Indeed, the coefficient of $S^\nu T^\beta$ is $a_{n\nu+\beta} c^\nu$, whose weighted norm is $\|a_{n\nu+\beta}\| r^{n\nu+\beta}$. There are only finitely many β, so the finite-tail conditions on both sides are equivalent. Multiplication is determined by $T_i^{n_i} = c_i S_i$ and agrees on polynomials, hence by continuity on the coefficient algebras. Thus D is a finite free R-algebra. Its displayed norm is uniformly equivalent to the maximum norm in this finite basis. Lemma 2.17.1 makes R strict, and Proposition 1.14.4 makes D strict with precisely this norm-admissibility. No assertion about arbitrary positive radii follows from this finite regrouping.

Theorem 1.11.5 now gives finite generation and closedness of every ideal of D. Its quotient is complete by Lemma 1.11.1; composing bounded quotient lifts with a strict Tate presentation of D proves that the quotient norm is admissible. This also covers the zero quotient.

For the iterated assertion, regrouping coefficients first identifies the joint coefficient algebra with $F = D\{s^{-1}Y\}$, with equality of norms. Proposition 2.14.1, applied to $D \to D/I$, gives a surjective coefficient map with quotient norm equal to the coefficient norm on $B\{s^{-1}Y\}$. Its kernel consists of the convergent coefficient families in I. Choose generators $u_1, \dots, u_q$ of I. By Theorem 1.11.5, every $i \in I$ can be written $i = \sum_\ell u_\ell d_\ell$ with $\max_\ell \|d_\ell\| \leq C\|i\|$ for a fixed C. Applying these lifts coefficientwise proves

that the kernel is the actual ideal IF. This repeats the coefficient-lifting argument of Lemma 2.18.1, whose needed strictness is now supplied by the preceding paragraph. Finally the inverse image of a finitely generated relation ideal is IF plus its lifted generators: lift the finitely many coefficients of any relation expression and subtract. Taking successive infima over representatives gives the direct quotient norm. No infimum need be attained. □

Proposition 2.26.2 (Power presentations for rational inequalities)**.** *Let $(g, f_1, \dots, f_s) = A$ and let $r_i \in \Gamma_K^{\mathrm{div}}$. Choose a common integer $n \geq 1$ and $c_i \in K^\times$ with $|c_i| = r_i^n$. Then*

$$Q = A\{r^{-1}T\}/(gT_i - f_i)_i, \qquad P = A\{W_1, \dots, W_s\}/(g^n W_i - c_i^{-1} f_i^n)_i$$

are canonically isomorphic by bounded inverse A-algebra maps. The tuple defining P generates the unit ideal. Thus Q is a strict affinoid, and its spectrum identifies homeomorphically with

$$\{x \in \mathcal{M}(A) : |f_i(x)| \leq r_i |g(x)| \text{ for all } i\}.$$

It has the strict-target domain universal property of Theorem 2.22.5. These conclusions concern the stated actual quotient norms, with bounded comparison; an isometry with the power presentation is not asserted.

Proof. Raise a unit-ideal identity for $(g, f_1, \dots, f_s)$ to the exponent $(s+1)(n-1)+1$. In every term, at least one of these $s+1$ elements occurs to an exponent at least n. Thus $(g^n, f_1^n, \dots, f_s^n) = A$, and multiplying individual entries by c_i^{-1} preserves that ideal. Lemma 2.26.1 makes Q a strict actual quotient; the ordinary strict quotient theorem does the same for P. In Q the element g is a unit by the original certificate. Sending W_i to T_i^n/c_i, which has norm at most one, gives a contractive map $P \to Q$.

In P, its unit-ideal certificate makes g^n, and hence g, a unit. Put $b_i = f_i/g$. The relation gives $b_i^n = c_i W_i$. If $m = qn + j$ with $0 \leq j < n$, then

$$\|b_i^m\| \leq |c_i|^q \|b_i^j\| \leq C_i r_i^m, \qquad C_i = \max_{0 \leq j < n} r_i^{-j} \|b_i^j\|.$$

Proposition 1.9.3 therefore defines a bounded map $A\{r^{-1}T\} \to P$ sending T_i to b_i. It kills the actual relation ideal and descends boundedly to Q by taking infima over lifts. The two composites fix the variables: in P, $f_i^n/(c_i g^n) = W_i$, and in Q, $f_i/g = T_i$. Polynomial density proves that they are inverse. Zero targets and zero quotients satisfy the same conclusion directly.

The point theorem and the strict-target universal property for the unit-radius presentation P now apply. Its inequalities are $|f_i|^n \le |c_i|\,|g|^n$, exactly the displayed inequalities for Q. Transferring across the bounded inverse isomorphism proves both assertions. In particular, every such domain also has a unit-radius rational presentation over the original field. □

Lemma 2.26.3 (Perturbing affinoid generators)**.** *Let $\pi_0 : A\langle X_1, \dots, X_n\rangle \twoheadrightarrow B$ be a surjective homomorphism, with A strict norm-admissible and B carrying its actual quotient norm. Write $h_i^0 = \pi_0(X_i)$. If $h_i \in B$ satisfy*

$$\delta = \max_i \|h_i - h_i^0\| < 1,$$

then substitution $X_i \mapsto h_i$ defines a contractive surjection $\pi : A\langle X\rangle \to B$. The quotient norm induced by π equals the given norm on B. The case $n = 0$ uses $\delta = 0$.

Proof. Each h_i^0 and h_i has norm at most one. The bounded substitution criterion gives π. A telescoping product expansion gives $\|h^\alpha - (h^0)^\alpha\| \le \delta$ for every nonconstant monomial, and the constant monomial has difference zero. The ultrametric inequality, followed by polynomial approximation, therefore gives

$$\|\pi(F) - \pi_0(F)\| \le \delta\|F\| \qquad (F \in A\langle X\rangle).$$

Fix $\lambda > 1$ with $q = \lambda\delta < 1$. For $b \in B$, choose a π_0-lift F_0 with $\|F_0\| \le \lambda\|b\|$, taking zero when $b = 0$. Then $e_1 = b - \pi(F_0)$ has norm at most $q\|b\|$. Repeat for each successive error. This produces F_j with

$$\|F_j\| \le \lambda q^j \|b\|, \qquad \left\| b - \pi\left(\sum_{j=0}^{m} F_j\right)\right\| \le q^{m+1}\|b\|.$$

The sum $F = \sum_j F_j$ converges, has norm at most $\lambda\|b\|$, and satisfies $\pi(F) = b$. Thus π is surjective. Its quotient norm is at least $\|b\|$ because it is contractive, and at most $\lambda\|b\|$ for every sufficiently small $\lambda > 1$. Letting $\lambda \downarrow 1$ proves equality. This proof requires only approximate quotient lifts; it does not assume that a norm-minimizing lift exists. If $B = 0$, the statement is immediate. □

Proposition 2.26.4 (Dense image and a Weierstrass factorization)**.** *For a bounded homomorphism $\phi : A \to B$ of strict K-affinoids, the following are equivalent: $\phi(A)$ is dense in B; and there are finitely many $a_1, \dots, a_n \in A$ and a bounded surjection*

$$A\langle a_1, \dots, a_n\rangle := A\{X_1, \dots, X_n\}/(X_i - a_i)_i \twoheadrightarrow B$$

whose composite with A is ϕ. Thus density supplies the algebraic factorization called a Runge immersion in the classical terminology, without using a stalk criterion.

Proof. Choose a relative presentation $\pi_0 : A\langle X\rangle \twoheadrightarrow B$ by Lemma 1.14.2, and use its quotient norm on B. This replaces its original norm by a uniformly equivalent one. If $\phi(A)$ is dense, choose $a_i \in A$ so that $\|\pi_0(X_i) - \phi(a_i)\| < 1$. Lemma 2.26.3 makes substitution $X_i \mapsto \phi(a_i)$ surjective. It kills each $X_i - a_i$ and hence factors through the displayed actual quotient as a bounded surjection.

Conversely, in that quotient every polynomial in the variables is the image of an element of A, because $X_i = a_i$. Polynomial density therefore makes A dense in the Weierstrass quotient. A bounded surjection from it to B sends this dense image to a dense subset of B: approximate any preimage of a given element. Thus $\phi(A)$ is dense. The argument includes zero algebras and the empty list of variables. □

The next estimate concerns a fixed finite list of elements and a fixed relation ideal. It is enough for the extension theorem; it is not a claim that quotient norms vary continuously for every element of an arbitrary family.

Lemma 2.26.5 (Preserving a strict quotient-norm bound)**.**

Let R be strict norm-admissible, let $r_0 < r_1$ belong to Γ_K^{div}, and let $T = (T_1, \dots, T_s)$. For each $r \in \Gamma_K^{\mathrm{div}} \cap [r_0, r_1]$, put $E_r = R\{r^{-1}T\}$. Fix polynomials $H_1, \dots, H_q \in R[T]$, let $J_r = (H_1, \dots, H_q)E_r$, and give $Q_r = E_r/J_r$ its actual quotient norm. If finitely many $F_1, \dots, F_n \in E_{r_1}$ satisfy $\|[F_i]\|_{Q_{r_0}} < 1$, then there is a real number r_2 with $r_0 < r_2 \le r_1$ such that

$$\|[F_i]\|_{Q_r} < 1 \quad \textit{for all } i \textit{ and } r \in \Gamma_K^{\mathrm{div}} \cap [r_0, r_2].$$

Proof. All the ideals in the statement are actual closed ideals by Lemma 2.26.1. For each i, the quotient-norm inequality gives $j_i \in J_{r_0}$ with

$\|F_i + j_i\|_{r_0} < 1$. Write $j_i = \sum_\nu u_{i\nu} H_\nu$, where $u_{i\nu} \in E_{r_0}$. Approximate the finitely many $u_{i\nu}$ by polynomials $v_{i\nu} \in R[T]$ closely enough that

$$G_i = F_i + \sum_\nu v_{i\nu} H_\nu \in E_{r_1}, \qquad \|G_i\|_{r_0} < 1.$$

This step uses an expression in the actual finitely generated ideal, rather than merely membership in a closure. It does not require a norm-minimizing representative. Choose $\theta < 1$ strictly larger than every $\|G_i\|_{r_0}$. Write $G_i = \sum_\alpha a_{i\alpha} T^\alpha$. Since $G_i \in E_{r_1}$, outside a finite set of indices one has $\|a_{i\alpha}\| r_1^{|\alpha|} < \theta$, simultaneously for all i. Those coefficients satisfy the same bound for every $r \le r_1$. For each of the finitely many remaining indices, the real function $r \mapsto \|a_{i\alpha}\| r^{|\alpha|}$ is continuous and is strictly less than θ at r_0. Hence it remains less than θ on some common interval $[r_0, r_2]$ with $r_2 > r_0$. Shrink r_2 to at most r_1. Then $\|G_i\|_r \le \theta < 1$ throughout this interval. Since $G_i - F_i \in J_r$, the same strict bound holds in Q_r. An empty list of elements requires no estimate. □

Theorem 2.26.6 (Extending dense image to a larger rational domain)**.** *Assume that K is complete and nontrivially valued. Let $\phi : A \to B$ be a bounded homomorphism of strict norm-admissible K-affinoids, and let $(g, f_1, \dots, f_s) = A$. For $r \in \Gamma_K^{\mathrm{div}}$, set*

$$A_r = A\{r^{-1}T\}/(gT_i - f_i)_i, \qquad B_r = B\{r^{-1}T\}/(\phi(g)T_i - \phi(f_i))_i.$$

Use their actual quotient norms and let $\phi_r : A_r \to B_r$ be the induced bounded map. If $\phi_{r_0}(A_{r_0})$ is dense in B_{r_0} for some $r_0 \in \Gamma_K^{\mathrm{div}}$, then there is a real $r_2 > r_0$ such that $\phi_r(A_r)$ is dense in B_r for every $r \in \Gamma_K^{\mathrm{div}} \cap [r_0, r_2]$. In particular, the conclusion holds at some strictly larger radius. Each such dense map has the Weierstrass quotient factorization of Proposition 2.26.4.

Proof. All displayed algebras are strict by Lemma 2.26.1; Proposition 2.26.2 identifies their spectra with the stated rational inequalities. The map on coefficients induced by ϕ is bounded with the same bound on every weighted coefficient algebra. It descends to the actual quotients by taking infima over representatives. Since K is nontrivially valued, Γ_K^{div} is dense in $\mathbb{R}_{>0}$: for any $\pi \in K$ with $0 < |\pi| < 1$, it contains $|\pi|^{\mathbb{Q}}$. Choose $r_1 \in \Gamma_K^{\mathrm{div}}$ with $r_1 > r_0$.

First reduce to Weierstrass equations over a slightly larger domain. Put $\overline{A} = A_{r_1}$ and $\overline{B} = B_{r_1}$. The unit-ideal certificate makes g a unit in $\overline{A}$, so put

$u_i = f_i/g \in \overline{A}$. For every $r \le r_1$ in the indicated group there are canonical isometric identifications

$$A_r \simeq \overline{A}\{r^{-1}S\}/(S_i - u_i)_i, \qquad B_r \simeq \overline{B}\{r^{-1}S\}/(S_i - \phi(u_i))_i.$$

For the first identification, the map from left to right sends T_i to S_i. The map in the other direction first sends the radius-r_1 variable of $\overline{A}$ to the radius-r variable T_i, and then sends S_i to T_i. Both substitutions are contractive because $r \le r_1$. They kill the actual relations; their composites fix coefficients and variables, so polynomial density makes them inverse. The proof for B is identical, and the squares with ϕ_r commute. These arguments also cover zero quotients. If $\overline{A} = 0$, then $\overline{B} = 0$ and all the algebras at smaller radii vanish, so the conclusion already follows.

Replace A, B by $\overline{A}, \overline{B}$ and use these identifications. Thus it suffices to treat the family $A_r = A\{r^{-1}T\}/(T_i - u_i)$ and the analogous family for B, for $r_0 \le r \le r_1$. The image of A is dense in each A_r, since polynomials in the variables reduce to elements of A. Density of $A_{r_0} \to B_{r_0}$ therefore implies density of the image of A in B_{r_0}.

Choose a relative presentation $\pi_0 : E = A\langle X_1, \dots, X_n\rangle \twoheadrightarrow B$ by Lemma 1.14.2. Replace the norm of B by its quotient norm. This is a uniformly equivalent norm; the resulting coefficient and quotient norms at every radius are again uniformly equivalent to the original ones, so density is unchanged. Its kernel I has finitely many generators $H_1, \dots, H_q \in E$ by Theorem 1.11.5. The iterated quotient assertion of Lemma 2.26.1 gives isometric descriptions

$$B_r \simeq E\{r^{-1}T\}/(H_1, \dots, H_q, T_i - u_i)_i, \qquad A_r\langle X\rangle \twoheadrightarrow B_r,$$

where the second map also induces the displayed norm as its actual quotient norm. In particular, the images h_j^0 of the X_j have norm at most one. Choose $a_j \in A$ with $\|h_j^0 - \phi(a_j)\|_{B_{r_0}} < 1$; this is possible by the density just proved. The elements $F_j = X_j - a_j$ belong to $E \subset E\{r_1^{-1}T\}$. Apply Lemma 2.26.5 with coefficient algebra E and the fixed polynomials H_ν and $T_i - u_i$. It gives some $r_2 > r_0$, at most r_1, such that

$$\|h_j^0 - \phi(a_j)\|_{B_r} < 1 \quad (r \in \Gamma_K^{\mathrm{div}} \cap [r_0, r_2]).$$

For each such r, Lemma 2.26.3 says that substitution $X_j \mapsto \phi(a_j)$ makes $A_r\langle X\rangle \to B_r$ surjective. Every polynomial image is already in the image of A_r, so polynomial density proves that $\phi_r(A_r)$ is dense in B_r. Proposition 2.26.4 supplies its stated factorization. Finally, density of Γ_K^{div} gives a

radius strictly between r_0 and r_2. The proof never changes the scalar field, uses no Gerritzen–Grauert theorem, and includes empty variable lists and actual zero quotients. □

Source and scope. The generator perturbation and the dense-image criterion reconstruct Bosch, [Bosch14, §4.2, Lemma 8 and Proposition 7, pp. 72–73]. The extension argument reconstructs [Bosch14, §4.2, Extension Lemma 9, pp. 73–76]. The proof above replaces norm-attaining lifts by approximate quotient lifts and realizes radii in Γ_K^{div} through a finite algebra over the original field. The extension conclusion is restricted to nontrivial valuation, because the required strictly larger radii do not exist in Γ_K^{div} for a trivially valued field. The finite-algebra realization itself, the perturbation lemma, and the dense-image criterion do allow trivial valuation.

2.27 Gerritzen–Grauert in the strict category

We now use the relative Weierstrass results to obtain a finite rational description of affinoid domains. The argument first treats bounded categorical epimorphisms of strict affinoid algebras. It produces rational pieces on which the image is dense. The domain universal property then removes the closed equations from the resulting Weierstrass factorization. Except in the following definition and its immediate observations, K is complete and its fixed absolute value is nontrivial. All algebra homomorphisms are unital and bounded.

Definition 2.27.1. For any complete non-Archimedean field K, including trivial valuation, let A and B be strict K-affinoid algebras. A *strict-test affinoid-domain presentation* of a subset $U \subset \mathcal{M}(A)$ is a bounded map $\varphi : A \to B$ such that the map $\mathcal{M}(B) \to \mathcal{M}(A)$ identifies its points bijectively with U, and every bounded K-algebra map $A \to C$, with C a strict K-affinoid and with spectrum image contained in U, factors uniquely through a bounded map $B \to C$. The zero algebra represents the empty subset.

This is the universal property of Definition 2.1.1 restricted to strict source algebras, strict representing algebras, and strict test algebras. An affinoid domain in that original definition satisfies the present condition whenever its two algebras are strict. The present definition alone makes no assertion about generalized test algebras.

The empty-set assertion follows from Theorem 1.12.3, which says that a strict algebra with empty spectrum is zero. Every such presentation is a

strict-affinoid epimorphism in the sense of Definition 1.14.7: two maps from B agreeing on A are two factorizations of the same map into a strict test algebra. Also, the point identification is a homeomorphism onto U, since the source spectrum is compact and the target spectrum is Hausdorff by Proposition 2.24.1.

Lemma 2.27.2 (A kernel series with unit coefficient ideal)**.** *Let $\varphi : A \to B$ be a strict-affinoid epimorphism, let $A \neq 0$, and let $\pi : A\langle T_1, \dots, T_n\rangle \twoheadrightarrow B$ be a bounded admissible relative presentation with $n \geq 1$. There is an element $F \in \ker \pi$ whose coefficients generate the unit ideal of A.*

Proof. Fix a maximal ideal $\mathfrak{m}$ of A and put $\kappa = A/\mathfrak{m}$. Theorem 1.12.3 makes κ a complete finite extension of K. Coefficientwise quotient exactness, Proposition 2.14.1 and Theorem 2.14.2, gives

$$\kappa\langle T_1, \dots, T_n\rangle \twoheadrightarrow D = B/\mathfrak{m}B, \qquad D = B\widehat{\otimes}_A\kappa.$$

Here the equality with the completed pushout also follows from Proposition 1.14.3; all quotients are by actual closed ideals. In particular, D is a strict κ-affinoid. The map $\kappa \to D$ is a strict-affinoid epimorphism over κ. Indeed, a strict κ-affinoid test algebra is strict over K by Lemma 1.14.6; two bounded maps from D agreeing on κ give maps from B agreeing on A, hence are equal. By Lemma 1.14.9, either $D = 0$ or $D = \kappa$. In either case the displayed map from $\kappa\langle T\rangle$ has nonzero kernel: for $n \geq 1$ this Tate algebra has the two distinct maximal evaluation ideals at $(0, \dots, 0)$ and $(1, 0, \dots, 0)$ and cannot be a field.

The map $\ker \pi \to \ker(\kappa\langle T\rangle \to D)$ is surjective. To see this, lift a relation to $G \in A\langle T\rangle$. Its image under π lies in $\mathfrak{m}B$, which is the image of $\mathfrak{m}A\langle T\rangle$ because π is surjective. Subtract a lift in that ideal. Thus, at every maximal $\mathfrak{m}$, some series in $\ker \pi$ has a coefficient outside $\mathfrak{m}$. The ideal of A generated by all coefficients of all these series is the unit ideal, by [StacksAlg, Tag 00E0(2), Lemma 10.17.2]. A finite unit certificate uses finitely many series $G_1, \dots, G_r \in \ker \pi$.

The algebra A is Noetherian by Theorem 1.11.5. Choose $d \geq 1$ such that, for every i, the coefficients of G_i of total degree less than d generate the same ideal as all its coefficients. Set

$$F = G_1 + T_n^d G_2 + \cdots + T_n^{(r-1)d} G_r \in \ker \pi.$$

At a maximal point x, let i be the first index with $G_{i,x} \neq 0$. Such an index exists by the unit certificate. Some coefficient of $G_{i,x}$ of total degree less

than d is nonzero. After the indicated shift, its degree is less than id. Earlier summands vanish at x, while every later summand has total degree at least id, so that coefficient cannot cancel. Hence $F_x \neq 0$ for every maximal x. Its coefficients have no common maximal zero and therefore generate the unit ideal, as required. □

Theorem 2.27.3 (Rational-local density for strict-affinoid epimorphisms)**.** *Let $\varphi : A \to B$ be a strict-affinoid epimorphism over the complete nontrivially valued field K. There is a finite rational cover $\mathcal{M}(A) = \bigcup_i R_i$ such that, writing A_i for the rational algebra of R_i and $B_i = B\widehat{\otimes}_A A_i$ for its pullback, every map $A_i \to B_i$ has dense image. All the algebras A_i, B_i are strict; zero members are permitted. The cover concerns all bounded multiplicative seminorms.*

Proof. We first record the base-change facts used in the induction. A categorical epimorphism remains an epimorphism under the completed pushouts of Proposition 1.14.3: two maps from the pushout agreeing on its new base agree on B by the old epimorphism property, and therefore agree everywhere by the pushout property. For a rational base change, this pushout is precisely the rational quotient of B with the pulled-back defining functions, by its universal property. An admissible relative presentation $A\langle T\rangle \twoheadrightarrow B$ gives an admissible relative presentation $A'\langle T\rangle \twoheadrightarrow B\widehat{\otimes}_A A'$ by the same actual-ideal presentation formula. Every maximal point of a strict rational algebra over A restricts to a maximal point of A, and the finite residue extension preserves absolute values, as proved in Lemma 1.15.4. Consequently fibrewise distinguished orders are preserved under these base changes.

Induct on the number n of variables in an admissible relative presentation, which exists by Lemma 1.14.2. For $n = 0$ the map is surjective, and the one-member cover suffices. If $A = 0$ or $B = 0$, the same conclusion is immediate. For $n \geq 1$, Lemma 2.27.2 supplies a series in the kernel with unit coefficient ideal. By Lemma 1.15.3, an isometric coordinate change makes it T_n-distinguished of order at most some integer s at every maximal point. The relative presentation remains admissible.

Assume the theorem proved for presentations with fewer than n variables. For the fixed number n, we use a second induction on s for presentations whose kernel contains such a series F. If $s = 0$, Proposition 1.15.5 gives $A\langle T\rangle/(F) = 0$, so $B = 0$. Suppose $s > 0$. Let $R_s \subset \mathcal{M}(A)$ be the rational maximal-order locus of Lemma 1.15.4, with algebra A_s, and put

$B_s = B\widehat{\otimes}_A A_s$. If $A_s = 0$, all maximal orders are at most $s-1$, and the inner induction applies directly. Otherwise, over A_s the image of F has order exactly s. Proposition 1.15.5 makes

$$C_s = A_s\langle T_1, \dots, T_{n-1}\rangle \longrightarrow B_s$$

finite, since B_s is a quotient of $A_s\langle T\rangle/(F)$. It is also a strict-affinoid epimorphism: two maps agreeing on C_s agree on A_s. Lemma 1.14.8 makes it surjective, and Lemma 1.14.1 makes this surjection admissible. The outer induction therefore yields a finite rational cover $R_s = \bigcup_{i=1}^{r} U_i$ on which the corresponding maps have dense image. By rational transitivity, Theorem 2.25.4, each U_i is rational in $\mathcal{M}(A)$, with the same algebra and pullback algebra up to the unique bounded comparison maps.

Choose presentations

$$U_i = \{x : |f_{ij}(x)| \le |g_i(x)| \ (1 \le j \le m_i)\}, \qquad (g_i, f_{i1}, \dots, f_{im_i}) = A.$$

The extension theorem, Theorem 2.26.6, supplies $\epsilon_i \in \sqrt{|K^\times|}$ with $\epsilon_i > 1$ such that the map on

$$U_i^+ = \{x : |f_{ij}(x)| \le \epsilon_i |g_i(x)| \text{ for all } j\}$$

still has dense image. These sets have strict rational presentations by Proposition 2.26.2. We construct a finite rational family covering what remains, and disjoint from R_s. For each i and $1 \le j \le m_i$, put

$$V_{ij} = \{x : \epsilon_i |g_i(x)| \le |f_{ij}(x)|, \quad |f_{i\ell}(x)| \le |f_{ij}(x)| \ (\ell \ne j)\}.$$

To see that this is strict rational, choose $N_i \ge 1$ and $c_i \in K^\times$ with $|c_i| = \epsilon_i^{N_i}$ and raise the inequalities to the N_ith power. The denominator is $f_{ij}^{N_i}$ and the numerators are $c_i g_i^{N_i}$ and the other $f_{i\ell}^{N_i}$. Their ideal is the unit ideal, since any maximal ideal containing them would contain g_i and all the $f_{i\ell}$. The displayed inequalities show $V_{ij} \cap U_i = \varnothing$: otherwise $(\epsilon_i - 1)|g_i(x)| \le 0$, so all entries of a unit tuple vanish at x. If $x \notin U_i^+$, choose j attaining the maximum of the $|f_{i\ell}(x)|$. Then $x \in V_{ij}$.

Take all finite intersections $V = \bigcap_i V_{ij_i}$. They are strict rational by Proposition 2.22.6, they are disjoint from every U_i and hence from R_s, and together with the U_i^+ they cover all of $\mathcal{M}(A)$. If some $m_i = 0$, then $U_i = \mathcal{M}(A)$ and the already dense map on U_i settles the assertion; thus the intersection family is only needed when every $m_i > 0$. On each nonzero algebra A_V, every

maximal point restricts to a maximal point of A outside R_s. The restricted kernel series consequently has distinguished order at most $s-1$. The inner induction gives a finite rational cover of V with dense maps. By transitivity these final members are rational in $\mathcal{M}(A)$. They, together with the U_i^+, give the desired finite cover. Both induction parameters are nonnegative integers and strictly decrease at their indicated steps. □

Lemma 2.27.4 (A closed strict-test domain is Weierstrass)**.** *Suppose a bounded surjection $q : C \twoheadrightarrow D$ of strict K-affinoid algebras is a strict-test affinoid-domain presentation. Then its kernel is generated by an idempotent $e \in C$. For any $c \in K$ with $|c| > 1$, its image is the Weierstrass domain*

$$\{x \in \mathcal{M}(C) : |ce(x)| \leq 1\},$$

and the natural map $C\langle S\rangle/(S - ce) \to D$ is a bounded isomorphism.

Proof. Put $I = \ker q$. The algebra C is Noetherian and I, I^2 are closed by Theorem 1.11.5. A multiplicative seminorm on C annihilates I^2 if and only if it annihilates I, since $a^2 \in I^2$ for every $a \in I$. A bounded seminorm annihilating an ideal descends boundedly to its quotient: take the infimum of its bound over representatives. Thus $\mathcal{M}(C/I^2)$ and $\mathcal{M}(C/I)$ have the same image in $\mathcal{M}(C)$. The quotient C/I is boundedly isomorphic to D by Lemma 1.14.1. Apply the strict-test domain property to the map $C \to C/I^2$. Its factorization through C/I forces $I \subset I^2$, hence $I = I^2$. The authorized algebraic lemma [StacksAlg, Tag 00EH, Lemma 10.21.5] gives an idempotent e with $I = (e)$.

At every multiplicative point, $|e|$ is zero or one. Thus $|ce| \leq 1$ holds exactly where e vanishes, which is the image of $\mathcal{M}(D)$. To identify the algebras, put $E = C\langle S\rangle/(S - ce)$. In E the element e satisfies $e^m = e$ and $S^m = c^m e$. Since the variable S is power bounded in its actual quotient norm, these identities give $\|e\| \leq |c|^{-m}\|S^m\| \to 0$, so $e = 0$. Consequently $C \to E$ factors through $D = C/(e)$. Conversely $C \to D$ sends ce to zero, so bounded substitution gives $E \to D$ with $S \mapsto 0$. The two maps are inverse on C and on the variable S, hence on the completions. Their boundedness follows from quotient norms and the admissible comparison with D. □

Theorem 2.27.5 (Strict Gerritzen–Grauert)**.** *Let K be complete and non-trivially valued. Let $\varphi : A \to B$ be a strict-test affinoid-domain presentation of $U \subset \mathcal{M}(A)$. There is a finite rational cover $\mathcal{M}(A) = \bigcup_i R_i$ such that $U \cap R_i$ is a Weierstrass domain in R_i, represented by $B_i = B\widehat{\otimes}_A A_i$, where A_i is the rational algebra of R_i. In particular, U is a finite union of rational domains of $\mathcal{M}(A)$. These are assertions about all Berkovich points, with bounded algebra identifications and the fixed absolute value on K.*

Proof. As observed after Definition 2.27.1, φ is a strict-affinoid epimorphism. Theorem 2.27.3 supplies a finite rational cover for which every $A_i \to B_i$ has dense image. This map is again a strict-test domain presentation of $U \cap R_i$. Indeed, Theorem 2.19.3 identifies $\mathcal{M}(B_i)$ with the inverse image of R_i in $\mathcal{M}(B)$. If a strict test map from A_i has image in $U \cap R_i$, its restriction from A factors uniquely through B; the two compatible maps from B and A_i then give the required unique map from the completed pushout B_i.

Apply Proposition 2.26.4 to factor the dense map as

$$A_i \longrightarrow C_i = A_i\langle S_1, \dots, S_t\rangle/(S_j - a_j) \twoheadrightarrow B_i, \qquad a_j \in A_i.$$

The first map represents a Weierstrass domain $W_i \subset R_i$, and the second map is surjective. The latter is itself a strict-test domain presentation. To check its factorization property, take a bounded map $C_i \to E$, where E is a strict test algebra, whose spectrum image lies in $\mathcal{M}(B_i) \subset \mathcal{M}(C_i)$. Composing with A_i gives a map with image in $U \cap R_i$, hence a unique map $B_i \to E$. Its composite from C_i equals the original map because both extend the same map from A_i through the rational domain C_i, whose bounded factorization is unique by Theorem 2.22.5. The point identification follows from the already established identifications into R_i.

By Lemma 2.27.4, the map $C_i \twoheadrightarrow B_i$ is a Weierstrass-domain map. It remains to retain the stronger conclusion that its composite with $A_i \to C_i$ is Weierstrass, not merely rational. Let $e \in C_i$ be the kernel idempotent. The image of A_i is dense in C_i, because polynomials in the adjoined variables reduce to elements of A_i. Choose $a \in A_i$ such that $\|a - e\|_{C_i} < |c|^{-1}$ for some $c \in K$ with $|c| > 1$. Since $|e(x)|$ is zero or one, the ultrametric inequality gives, for every $x \in \mathcal{M}(C_i)$,

$$|ca(x)| \le 1 \quad \Longleftrightarrow \quad e(x) = 0.$$

Consequently $U \cap R_i$ is the Weierstrass subset of R_i defined by the original inequalities $|a_j| \le 1$ together with $|ca| \le 1$. Its standard strict rational

algebra and B_i satisfy the same strict-test universal property, so they are uniquely boundedly isomorphic. Finally, each of these Weierstrass subsets is rational in R_i; rational transitivity makes it rational in $\mathcal{M}(A)$. Their finite union is U because the R_i cover the entire spectrum. □

Theorem 2.36.6 extends the finite rational-cover conclusion to generalized affinoid domains over every complete field. Its proof combines free-radius strictification with a finite truncation on the Gauss section, producing rational domains over the original algebra. This gives the general statement recorded in Theorem 2.11.3.

References. The finite induction and its relative Weierstrass preparations are due to Gerritzen–Grauert; the complete classical presentations used here are [BGR84, 7.3.5, Theorem 1 and Corollaries 2–3, p. 309; proof and preparations, pp. 310–315] and [Bosch14, 4.2, Theorem 10, p. 76; proof, pp. 79–81; Corollaries 11–12, pp. 76–77]. The monomorphism formulation is also discussed in [Tem05, Proposition 1.2, pp. 3–4]; we do not use its proof or the Banach tensor-injectivity input occurring there. The present strict proof replaces analytic-stalk arguments by the explicit finite categorical-epimorphism lemmas and replaces the closed-domain step by the nilpotent-thickening test and Stacks Tag 00EH.

2.28 Finite modules and Laurent descent

Tate exactness concerns the localizations of a module that is already defined over the original algebra. For coherence we must also construct a global module from finite modules given separately on a covering. The additional analytic step is a bounded lifting argument for arbitrary finite Laurent gluing data. We then use maximal residue fields to obtain generators and apply the lifting argument once more to the relations. This reconstructs the method of Kiehl's theorem as developed in [BGR84, §9.4.3, Lemmas 5–6, pp. 378–382], with the finite-module norms and the restricted site specified explicitly.

Unless an additional hypothesis is stated, K is a complete real-valued non-Archimedean field, its valuation may be trivial, and A is a strict K-affinoid algebra with a norm-admissible Banach norm. Relation ideals are actual ideals; their closedness and the canonical norms of finite modules are supplied by Theorem 1.11.5. The zero algebra and its zero module are allowed separately. Radii belong to $|K^\times|$ and can be scaled to one. Ordinary tensor products in this section are algebraic tensor products. A completed-

tensor assertion is made only after its finite-module comparison has been established.

Lemma 2.28.1 (Maximal fibers of a rational localization)**.** *Let $g, f_1, \dots, f_s$ generate the unit ideal of A, put*

$$B = A\{T_1, \dots, T_s\}/(gT_j - f_j)_j,$$
$$V = \{x \in \mathcal{M}(A) : |f_j(x)| \le |g(x)| \text{ for all } j\}.$$

For a maximal ideal $\mathfrak{m} \subset A$, let $L = A/\mathfrak{m}$ and let $x_\mathfrak{m}$ be its bounded residue-field seminorm. There is a canonical bounded isomorphism

$$B/\mathfrak{m}B \;\simeq\; B \otimes_A L \;\simeq\; \begin{cases} L, & x_\mathfrak{m} \in V, \\ 0, & x_\mathfrak{m} \notin V. \end{cases}$$

The nonzero isomorphism has bounded inverse for canonical norms and is evaluation at $f_j(x_\mathfrak{m})/g(x_\mathfrak{m})$. Every maximal ideal $\mathfrak{n} \subset B$ is of the form $\mathfrak{m}B$ for a unique maximal ideal $\mathfrak{m} \subset A$ with $x_\mathfrak{m} \in V$, and $A/\mathfrak{m} \to B/\mathfrak{n}$ is an isomorphism. These fiber identifications commute with rational restriction maps and with finite intersections.

Proof. Theorems 1.12.3 and 2.17.2 identify the fiber, with its canonical norm, with

$$L\{T_1, \dots, T_s\}/(\bar{g}T_j - \bar{f}_j)_j.$$

The coefficient map is the quotient $A \to L$, so this identification respects constants and all displayed variables. The field L is finite over K. By Lemma 1.12.1, its extending absolute value is equivalent to its canonical quotient norm.

Suppose first that $x_\mathfrak{m} \in V$. The unit-ideal relation forces $\bar{g} \neq 0$. Set $b_j = \bar{f}_j/\bar{g}$. Since $|b_j|_L \le 1$, the sequences $(b_j^n)_{n\ge 0}$ are bounded for the canonical norm on L. Proposition 2.19.1 therefore permits substitution $T_j \mapsto b_j$. Because $\bar{g}$ is a unit, the relation ideal equals $(T_j - b_j)_j$. Substitution and the inclusion of constants are bounded inverse maps, by polynomial density. This proves the nonzero case without identifying the canonical norm with the field absolute value.

If $x_\mathfrak{m} \notin V$ and $\bar{g} = 0$, at least one $\bar{f}_j$ is nonzero, so the relation ideal contains a unit. Otherwise some $|\bar{f}_j/\bar{g}|_L > 1$. Put $\lambda = \bar{g}/\bar{f}_j$. The norm equivalence gives $\|\lambda^n\| \le C|\lambda|_L^n \to 0$, so $\sum_{n\ge 0} \lambda^n T_j^n$ belongs to $L\{T_1, \dots, T_s\}$ and is the inverse of $1 - \lambda T_j$, as is checked by its finite partial sums. The

relation $\bar{g}T_j - \bar{f}_j = -\bar{f}_j(1 - \lambda T_j)$ is thus a unit. This proves that the actual fiber algebra is zero.

Now take a maximal ideal $\mathfrak{n}$ of B. Its residue field E is finite over K by Theorem 1.12.3. The image of A in E is a finite-dimensional K-domain and hence a field by [StacksAlg, Tag 00GS, Lemma 10.36.19(1)]. Thus $\mathfrak{m} = \mathfrak{n} \cap A$ is maximal. The absolute value on this subfield is the unique extending one by Lemma 1.12.1. The bounded point of B defined by E restricts to $x_{\mathfrak{m}}$, which belongs to V by Theorem 2.19.3. The already computed fiber $B/\mathfrak{m}B = L$ has no proper nonzero ideal, and its quotient $B/\mathfrak{n}$ is nonzero. Consequently $\mathfrak{n} = \mathfrak{m}B$ and $E = L$. Conversely, for $x_{\mathfrak{m}} \in V$, the computed fiber shows that $\mathfrak{m}B$ is maximal. All nonzero fiber maps fix L and substitute the prescribed ratios; maps involving a zero fiber have their unique possible value. This proves compatibility with the restriction and joint-presentation maps of Proposition 2.22.6. □

Corollary 2.28.2 (Algebraic faithful flatness of a finite rational covering)**.** *If $V_1, \dots, V_t$ are rational subsets covering $\mathcal{M}(A)$, with their strict value-group presentation algebras B_i, then $A \to \prod_{i=1}^{t} B_i$ is a faithfully flat ring map. Faithfulness here tests arbitrary algebraic A-modules. It does not identify $B_i \otimes_A B_j$ with the completed intersection algebra.*

Proof. Lemma 2.22.1 gives algebraic flatness of each B_i. To spell out its trivial-valuation branch, choose a defining strict quotient norm on A. Every nonzero element then has norm one, so every coefficient of a unit certificate has norm at most one; the scalar in that lemma can be taken to be $a = 1$. Its elementary chain is therefore available, and Corollary 2.20.3, which allows trivial valuation, proves the same algebraic flatness. A finite product is a finite direct sum as an A-module, so tensoring by it preserves exactness. For every maximal $\mathfrak{m} \subset A$, its residue point belongs to some V_i. Lemma 2.28.1 then gives $B_i/\mathfrak{m}B_i = A/\mathfrak{m} \neq 0$. The product therefore has nonzero fiber at every maximal ideal. Apply [StacksAlg, Tag 00HP, Lemma 10.39.15, equivalence of (1) and (4)]. If $A = 0$, all modules under consideration are zero and the assertion has its usual degenerate meaning. □

Definition 2.28.3 (Finite Laurent gluing data)**.** For the next three results fix $f \in A$ and use the following actual quotient algebras with their quotient norms:

$$A_+ = A\{T\}/(T-f), \qquad A_- = A\{U\}/(1-fU),$$
$$A_0 = A\{T,U\}/(T-f, TU-1).$$

A finite Laurent gluing datum means finite modules N_+ over A_+ and N_- over A_-, together with an A_0-linear isomorphism between $A_0 \otimes_{A_+} N_+$ and $A_0 \otimes_{A_-} N_-$. Denote the common module by N_0, the two maps by ρ_+, ρ_-, and their difference by δ. Equip the finite modules with canonical norms. A morphism of such data is a pair of local module maps commuting with the given overlap isomorphisms.

The gluing isomorphism and its inverse are bounded by Theorem 1.11.5. Thus boundedness follows from the finite canonical module structure.

Lemma 2.28.4 (Bounded lifting for arbitrary finite Laurent data)**.** *For every finite Laurent gluing datum, the map*

$$\delta : N_+ \oplus N_- \longrightarrow N_0, \qquad (v,w) \longmapsto \rho_+(v) - \rho_-(w)$$

is surjective and norm-strict. More precisely, there is a constant $C > 0$, depending on the datum and the chosen canonical norms, such that every $z \in N_0$ has a preimage (v,w) with $\max(\|v\|, \|w\|) \le C\|z\|$. No linear choice of preimages is asserted.

Proof. First, the image of A_- in A_0 is dense. The identity $TU-1 = (T-f)U + (fU-1)$ shows that the ideals $(T-f, TU-1)$ and $(T-f, fU-1)$ coincide. Lemma 2.18.1 therefore identifies A_0 with $A_-\{T\}/(T-f)$. Polynomials in T are dense before taking this quotient by Lemma 2.13.1, and their classes belong to the image of A_- because $T = f$ there. Second, Theorem 2.18.3 for $M = A$ gives the following scalar estimate: for each fixed $\beta > 1$ and every $a \in A_0$, one can write

$$a = b|_0 - d|_0, \qquad b \in A_+,\ d \in A_-, \qquad \max(\|b\|, \|d\|) \le \beta\|a\|.$$

If $N_0 = 0$, the conclusion is immediate. Otherwise choose generators $e_1, \dots, e_m$ of N_+ and $w_1, \dots, w_n$ of N_-. In this proof use their quotient norms, and give N_0 the quotient norm from the generators $\bar{e}_i = \rho_+(e_i)$.

Changing back to any stipulated canonical norms only changes constants. The elements $\bar{w}_j = \rho_-(w_j)$ also generate N_0. Choose $c_{ji} \in A_0$ such that

$$\bar{e}_i = \sum_{j=1}^{n} c_{ji}\bar{w}_j, \qquad H = \max(1, \|\bar{w}_1\|, \dots, \|\bar{w}_n\|).$$

Fix $0 < \eta < 1$. By density choose $c'_{ji} \in A_-$ whose images in A_0 satisfy

$$\max_{j,i} \|c_{ji} - c'_{ji}|_0\| \le \frac{\eta}{\beta^2 H}, \qquad C_0 = \beta^2 \max(1, \max_{j,i} \|c'_{ji}\|).$$

These are finitely many choices, made once for the datum. For $z \in N_0$, choose $a_i \in A_0$ with $z = \sum_i a_i \bar{e}_i$ and $\max_i \|a_i\| \le \beta\|z\|$; for $z = 0$ take all coefficients zero. Split each $a_i = b_i|_0 - d_i|_0$ by the scalar estimate and put

$$v = \sum_i b_i e_i, \qquad w = \sum_{i,j} d_i c'_{ji} w_j.$$

The quotient module norms satisfy $\|av\| \le \|a\|\|v\|$. Hence

$$\max(\|v\|, \|w\|) \le C_0\|z\|, \qquad z - \delta(v, w) = -\sum_{i,j} d_i|_0\,(c_{ji} - c'_{ji}|_0)\bar{w}_j,$$

and the last expression has norm at most $\beta^2\|z\| \cdot \eta/(\beta^2 H) \cdot H = \eta\|z\|$. We have therefore obtained a uniformly bounded approximate lift with a fixed contraction of the error.

Apply this construction successively to $z_0 = z$ and $z_{q+1} = z_q - \delta(v_q, w_q)$. Then $\|z_q\| \le \eta^q\|z\|$ and $\max(\|v_q\|, \|w_q\|) \le C_0\eta^q\|z\|$. Completeness of the finite canonical modules, from Theorem 1.11.5, implies convergence of $v = \sum_{q\ge0} v_q$ and $w = \sum_{q\ge0} w_q$. Boundedness of δ gives $\delta(v, w) = z$, and the ultrametric inequality gives $\max(\|v\|, \|w\|) \le C_0\|z\|$. The kernel of the bounded map δ is closed. The lift bound and boundedness of δ identify its quotient norm with the target norm up to fixed constants, proving norm-strictness. The argument also works for trivial valuation; neither small scalars nor an open mapping theorem have been used. □

Lemma 2.28.5 (Global matching elements generate the local modules)**.** *For a finite Laurent gluing datum set $N = \ker\delta$. The image of N generates N_+ over A_+ and generates N_- over A_-. Consequently a single finite list of elements of N generates both local modules after restriction. Finiteness of N over A is not assumed here.*

Proof. Fix a maximal ideal $\mathfrak{m}$ of A. Put $I_\tau = \mathfrak{m}N_\tau$ and $Q_\tau = N_\tau/I_\tau$ for $\tau = +, -, 0$. These modules are finite by Theorem 1.11.5. The maps $A_+ \to A_0$ and $A_- \to A_0$ are elementary localizations: they adjoin respectively f^{-1} and a radius-one variable equal to f. Their algebraic flatness follows from Theorem 2.20.2 and Lemma 2.18.1. Tensoring $I_\pm \hookrightarrow N_\pm$ therefore identifies its image with I_0, and right exactness identifies the quotient with Q_0. Thus $I_\pm$ and $Q_\pm$ are themselves finite Laurent gluing data.

Lemma 2.28.4 applied to I shows that

$$N \longrightarrow \ker(Q_+ \oplus Q_- \longrightarrow Q_0)$$

is surjective. Indeed, lift a matching pair in $Q_+ \oplus Q_-$ to $N_+ \oplus N_-$. Its discrepancy lies in I_0; subtract a preimage of this discrepancy from $I_+ \oplus I_-$ to obtain a matching lift. This is an algebraic lifting argument using the proved surjectivity, not a tensor product of an unproved exact sequence.

Let $x = x_\mathfrak{m}$ and $L = A/\mathfrak{m}$. The two Laurent subsets are $V_+ = \{|f| \leq 1\}$ and $V_- = \{|f| \geq 1\}$. Lemma 2.28.1 computes the fibers of A_+, A_-, A_0 as L or zero according as x belongs to V_+, V_-, or their intersection. If x belongs to only one member, the displayed quotient equalizer is exactly its corresponding Q_τ. If x belongs to both, the two maps from $Q_\pm$ to Q_0 are isomorphisms and the same conclusion holds for either projection. Consequently

$$N \longrightarrow N_\tau/\mathfrak{m}N_\tau \quad \text{is surjective whenever } x_\mathfrak{m} \in V_\tau.$$

Let $P_\tau \subseteq N_\tau$ be the A_τ-submodule generated by the image of N, and set $C_\tau = N_\tau/P_\tau$. This quotient is finite. Every maximal ideal $\mathfrak{n}$ of A_τ has the form $\mathfrak{m}A_\tau$ with $x_\mathfrak{m} \in V_\tau$, by Lemma 2.28.1. The surjectivity just proved gives $C_\tau/\mathfrak{n}C_\tau = 0$. After localization at $\mathfrak{n}$, Nakayama's lemma, [StacksAlg, Tag 00DV, Lemma 10.20.1(2)], gives $(C_\tau)_\mathfrak{n} = 0$. The maximal-local zero test, [StacksAlg, Tag 00HN, Lemma 10.23.1(2)], then gives $C_\tau = 0$. Thus $P_\tau = N_\tau$. Each of the two finite modules has a finite generating list, and each generator is a finite linear combination of images of elements of N. The union of the finitely many elements of N so obtained is the required common list. Zero local algebras contribute no condition. □

Theorem 2.28.6 (Effective descent of finite modules for a Laurent pair)**.** *The functor from finite A-modules to finite Laurent gluing data, given by ordinary base change to A_+ and A_-, is an equivalence of categories. Its inverse sends the datum to $N = \ker \delta$. This N is a finite A-module, and its norm induced from $N_+ \oplus N_-$ is uniformly equivalent to its canonical finite-module norm. The canonical maps*

$$A_\tau \otimes_A N \longrightarrow N_\tau \qquad (\tau = +, -, 0)$$

are bounded isomorphisms with bounded inverses, and the same assertions hold for separated completed tensor products. In particular, the sequence $0 \to N \to N_+ \oplus N_- \xrightarrow{\delta} N_0 \to 0$ is norm-strict exact. All assertions extend to a radius $r \in |K^\times|$ by scalar rescaling.

Proof. Choose $n_1, \dots, n_r \in N$ as in Lemma 2.28.5. They give surjections $\alpha_\pm : A_\pm^r \to N_\pm$ and a common surjection $\alpha_0 : A_0^r \to N_0$. Put $J_\tau = \ker \alpha_\tau$. These kernels are finite canonical modules; give them the norms induced from the corresponding free modules, which are canonical by Theorem 1.11.5. Algebraic flatness of $A_\pm \to A_0$, as verified in the preceding proof, identifies $A_0 \otimes_{A_\pm} J_\pm$ with J_0. Thus the J_τ form a finite Laurent gluing datum.

We prove that the generator map $\alpha : A^r \to N$ is surjective with a uniform lift bound. For $n = (n_+, n_-) \in N$, norm-strictness of $\alpha_\pm$ gives lifts $u_\pm \in A_\pm^r$ with

$$\max(\|u_+\|, \|u_-\|) \le C_1 \|n\|_{\mathrm{eq}}, \qquad \|n\|_{\mathrm{eq}} = \max(\|n_+\|, \|n_-\|).$$

The difference $z = u_+|_0 - u_-|_0$ lies in J_0 and has its induced norm at most $C_1\|n\|_{\mathrm{eq}}$. Apply Lemma 2.28.4 to J: there are $j_\pm \in J_\pm$ with $j_+|_0 - j_-|_0 = z$ and $\max(\|j_+\|, \|j_-\|) \le C_2\|z\|$. The corrected free vectors $u_\pm - j_\pm$ match. Theorem 2.18.3 for $M = A^r$ now gives a unique $a \in A^r$ with those restrictions, and its augmentation estimate gives

$$\alpha(a) = n, \qquad \|a\| \le C_3 \max(C_1, C_1C_2)\|n\|_{\mathrm{eq}}.$$

Therefore N is finite over A. Conversely, the fixed images of the standard basis of A^r give $\|\alpha(a)\|_{\mathrm{eq}} \le C_4\|a\|$. These two inequalities show that the quotient norm from A^r and $\|\cdot\|_{\mathrm{eq}}$ are uniformly equivalent. This verifies canonicality of the equalizer norm by explicit lifts; it has not been inferred merely from completeness of an arbitrary norm on a finite module.

Let $J = \ker(A^r \xrightarrow{\alpha} N)$. The free Laurent equalizer identifies J with $\ker(J_+ \oplus J_- \to J_0)$. Lemma 2.28.5, now applied to this kernel datum, says that $A_\pm J = J_\pm$ inside $A^r_\pm$. Ordinary right exactness of tensor products, [StacksAlg, Tag 00DF, Lemma 10.12.10], therefore identifies both $A_\pm \otimes_A N$ and $N_\pm$ with $A^r_\pm/J_\pm$. The same conclusion for A_0 follows by tensor associativity, [StacksAlg, Tag 00D2, Lemma 10.12.7]. These canonical algebraic isomorphisms are bounded with bounded inverses by Theorem 1.11.5. Proposition 1.11.6 then gives the completed-tensor statements. No exactness claim for the completed tensor product of arbitrary Banach modules occurs.

A compatible pair of morphisms of gluing data preserves the two equalizers and hence gives an A-linear map between the descended modules. The canonical base-change isomorphisms recover the original morphisms. Conversely, starting from a finite A-module, Theorem 2.18.3 identifies its equalizer with that same module. These statements verify that the two functors are mutually inverse, including on morphisms. The equalizer is closed, its norm is canonical by the preceding estimates, and the last map is norm-strict by Lemma 2.28.4; thus the asserted sequence is norm-strict exact. For $r = |c|$ with $c \in K^\times$, replace f by f/c, T by T/c and U by cU. This is the isometric radius normalization of the displayed presentations and proves the last assertion. □

2.28.1 Keeping the refinement inside the fixed rational site

Write $X = \mathcal{M}(A)$, and use exactly the site $\mathcal{R}_X$ of Definition 2.23.1: its objects are rational subsets presented over the original A, and its coverings are finite set-theoretic coverings by such objects. In particular, the algebra of an object V is denoted by A_V.

Definition 2.28.7 (A finite Laurent tree in the fixed site)**.** A finite Laurent tree in this site means a finite rooted binary tree with root X, such that each node V is an object of $\mathcal{R}_X$ and each nonterminal node is covered by an elementary Laurent pair for some $h \in A_V$ and radius in $|K^\times|$. Its two child algebras and their intersection algebra must be the corresponding actual elementary quotients, up to the canonical bounded isomorphisms. Empty nodes may be terminal. This definition imposes the presentation requirement; it does not assert its existence.

Proposition 2.28.8 (A Laurent refinement tree over the original algebra)**.** *Assume that the valuation of K is nontrivial. Every finite covering of X by objects of $\mathcal{R}_X$ has a subordinate finite Laurent tree in the sense just specified: each leaf is contained in one covering member. All nodes, including those obtained by comparing ratios in intermediate algebras, remain rational over the original A. The child and intersection maps agree with the actual elementary quotient maps.*

Proof. If $A = 0$, use the single empty node. Otherwise Proposition 2.22.8 gives a standard rational refinement generated by a unit-ideal tuple $f_0, \dots, f_s \in A$. Choose a unit certificate $\sum_i a_i f_i = 1$ and $\kappa \in K^\times$ with $|\kappa| \max_i \|a_i\| < 1$. Split successively by the elementary Laurent pairs for $f_0/\kappa, \dots, f_s/\kappa$. Each partial branch is an intersection of rational subsets over A: its positive condition has denominator 1, and its negative condition has tuple $(f_i; \kappa)$, which generates the unit ideal because $\kappa \neq 0$. Proposition 2.22.6 and Lemma 2.18.1 identify its algebra with the simultaneous relation quotient. Theorem 2.19.3 identifies each binary pair with a covering of its parent.

Consider a completed first-stage branch with algebra C. Let P and N be its positive and negative index sets. Lemma 2.22.4 says that $C = 0$ when N is empty. For $i \in N$, the element f_i is a unit in C and $\|\kappa/f_i\| \leq 1$. For a nonzero branch, split further, successively, by the finitely many ratios $q_{ij} = f_i/f_j$ with $i < j$ in N. We verify that this second-stage operation stays in $\mathcal{R}_X$, rather than assuming rational transitivity.

The two rational subsets of the original X

$$W_{ij}^{+} = \{|f_i| \leq |f_j|,\ |\kappa| \leq |f_j|\},$$
$$W_{ij}^{-} = \{|f_j| \leq |f_i|,\ |\kappa| \leq |f_i|\}$$

have unit-ideal tuples $(f_j; f_i, \kappa)$ and $(f_i; f_j, \kappa)$. Let D be C or any partial second-stage branch. By induction its quotient map from C is contractive, so both f_i, f_j remain units and the bounds $\|\kappa/f_i\|, \|\kappa/f_j\| \leq 1$ persist. Over D, intersection with W_{ij}^{+} has the joint presentation

$$D\{Z,Y\}/(f_j Z - f_i, f_j Y - \kappa) \simeq D\{Z,Y\}/(Z - q_{ij}, Y - \kappa/f_j) \simeq D\{Z\}/(Z - q_{ij}).$$

The first equality uses a unit and does not change the actual ideal. In the second, inclusion and substitution $Y = \kappa/f_j$ are inverse contractions by Lemma 2.22.2 and the iterated-quotient lemma. Thus this is the positive elementary quotient with its actual quotient norm. The negative intersection

is similarly $D\{Z\}/(Z - f_j/f_i)$, whose actual ideal equals $(1 - q_{ij}Z)$ because q_{ij} is a unit. Imposing both signs gives the same joint presentation as their elementary intersection. All maps fix D and the indicated ratios, so these isometries commute with restriction. This proves the inductive presentation requirement for every partial branch. The subsets cover their parent because any two nonnegative real numbers are comparable and the scalar lower bounds already hold there.

At a second-stage leaf, Lemma 2.22.3 chooses $k \in N$ with $\|f_i/f_k\| \leq 1$ for every $i \in N$. For $p \in P$, write $f_p = \kappa X_p$ and $\kappa/f_k = X_k$ from the first-stage relations; then $f_p/f_k = X_pX_k$ also has norm at most one. Hence the leaf factors contractively through the kth member of the standard rational family. That member is contained in an original covering member. The construction has finitely many stages and finitely many branches. Empty branches use the zero algebra, as follows either from their explicit relation or from Theorem 1.12.3. This completes the required finite tree. □

Definition 2.28.9 (A finite module model on a rational slice)**.** For an object $V \in \mathcal{R}_X$, denote the slice site by $\mathcal{R}_X/V$ and its restricted ring sheaf by $\mathcal{O}_V$. If P is a finite A_V-module, put

$$\widetilde{P}_V(W) = A_W \otimes_{A_V} P \qquad (W \subseteq V, W \in \mathcal{R}_X).$$

A finite module model for a sheaf $\mathcal{F}$ on V means an isomorphism $\widetilde{P}_V \simeq \mathcal{F}|_{\mathcal{R}_X/V}$ for such a P. It includes all objects of the slice site, not just the value at V.

Lemma 2.28.10 (Finite-module sheaves on a rational slice)**.** *The assignment $P \mapsto \widetilde{P}_V$ is an exact fully faithful functor from finite A_V-modules to $\mathcal{O}_V$-modules, and $\Gamma(V, \widetilde{P}_V) = P$. It commutes with restriction to $W \subseteq V$ by ordinary tensor associativity and canonical finite-module tensor comparison. Every finite covering of an object of $\mathcal{R}_X/V$ has a norm-strict exact augmented Čech complex for $\widetilde{P}_V$.*

Proof. If $W \subseteq V$ are rational over A, Proposition 2.22.6 and Theorem 2.22.5 identify the joint algebra $A_V \widehat{\otimes}_A A_W$ with A_W. Viewed over A_V, the original unit-ideal tuple presenting W is still a unit-ideal tuple. Thus A_W is a rational quotient over A_V; this direction of the comparison does not require every rational subset over A_V to be rational over A. Every covering in the slice is consequently a finite rational covering over the algebra of its target.

Apply Theorem 2.22.9 to that target algebra and its finite module. Tensor associativity, [StacksAlg, Tag 00D2, Lemma 10.12.7], and Theorem 2.17.2 identify all terms and arrows with the displayed presheaf. Degree-zero exactness proves the sheaf condition, including the value zero on the empty object, and the full theorem gives the asserted norm-strict complex.

Rational flatness from Lemma 2.22.1 proves objectwise exactness for short exact sequences of finite A_V-modules, and hence exactness as sheaves. For trivial valuation use a defining discrete quotient norm on A_V and $a = 1$ in that lemma's elementary chain, exactly as in the proof of Corollary 2.23.2; Corollary 2.20.3 applies in this case as well. Evaluation at V gives $A_V \otimes_{A_V} P = P$. A sheaf map is determined by its value at V: its value on W must send $b \otimes p$ to b times the restriction of the image of p. Conversely, every A_V-linear map has these compatible values. This proves full faithfulness. The same formula and ordinary tensor associativity prove compatibility with further restriction; Proposition 1.11.6 supplies the corresponding bounded finite-module identifications. □

Proposition 2.28.11 (Descent along a finite Laurent tree)**.** *Let $\mathcal{F}$ be an $\mathcal{O}_X^{\mathrm{rat}}$-module sheaf, and suppose there is a finite Laurent tree in $\mathcal{R}_X$ on whose leaves $\mathcal{F}$ has finite module models. Then $M = \mathcal{F}(X)$ is a finite A-module and the canonical morphism $\widetilde{M}^{\mathrm{rat}} \to \mathcal{F}$ is an isomorphism. With any canonical norms on the finite leaf models, the norm of a global matching section obtained by taking the maximum of its leaf restrictions is uniformly equivalent to the canonical norm on M.*

Proof. Proceed from leaves towards the root. A zero node has only the zero sheaf and zero module, by the empty covering. Consider a parent V whose two children V_+, V_- already have finite models P_+, P_-. Their common restriction to $V_0 = V_+ \cap V_-$ gives an isomorphism

$$A_{V_0} \otimes_{A_{V_+}} P_+ \;\simeq\; A_{V_0} \otimes_{A_{V_-}} P_-.$$

This is an isomorphism of finite modules: evaluate the two sheaf models on V_0 and use Lemma 2.28.10. The tree's presentation condition therefore makes them a finite Laurent gluing datum over A_V. Theorem 2.28.6 produces a finite A_V-module P whose two base changes recover the children. The sheaf equalizer identifies P with $\mathcal{F}(V)$.

For every $W \subseteq V$ in the slice, multiplication of restricted sections defines $A_W \otimes_{A_V} P \to \mathcal{F}(W)$. This gives a morphism of sheaves. On each child it

is the base-change isomorphism just proved. For arbitrary W, pull back the binary covering to $W \cap V_+, W \cap V_-$. The morphism is an isomorphism on both members and their intersection, so the two sheaf equalizer diagrams show it is an isomorphism on W. This proves the required model on all of $\mathcal{R}_X/V$, rather than only at the parent object.

The norm comparison at each parent is the explicit equalizer comparison in Theorem 2.28.6. Composing these finitely many two-sided inequalities gives the maximum-of-leaves norm comparison at the root. All constants are finite and may depend on the chosen tree, models and canonical norms. No assertion of a uniform constant over all coverings is made. The finite induction ends at X, proving both conclusions. □

Theorem 2.28.12 (Kiehl equivalence on the fixed rational site)**.** *Let K be complete, with trivial valuation allowed, and let A be strict norm-admissible. On the ringed site $(\mathcal{R}_X, \mathcal{O}_X^{\mathrm{rat}})$, the following conditions on an $\mathcal{O}_X^{\mathrm{rat}}$-module sheaf $\mathcal{F}$ are equivalent:*

1. *$\mathcal{F}$ is coherent in the ringed-site sense of [StacksAlg, Tag 03DL, Definition 18.23.1(8)]: it is of finite type, and on each slice the kernel of every map from a finite free module to it is of finite type.*
2. *$\mathcal{F}$ is of finite presentation in the sense of the same definition, part (7).*
3. *Some finite covering of X in $\mathcal{R}_X$ admits finite module models for $\mathcal{F}$ on every member.*
4. *There is a finite A-module M with $\mathcal{F} \simeq \widetilde{M}^{\mathrm{rat}}$.*

Consequently $M \mapsto \widetilde{M}^{\mathrm{rat}}$ is an exact equivalence from finite A-modules to coherent modules on this ringed site, with inverse $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$. Every such sheaf has a global finite presentation by finite free $\mathcal{O}_X^{\mathrm{rat}}$-modules. For any finite covering in this site, its augmented Čech complex is norm-strict exact with the induced canonical finite-module norms. This is the stated fixed-rational-site result; no identification with the full analytic G-site or with all generalized affinoid domains is included.

Proof. We first compare the local conditions without using descent. If $\mathcal{F}$ is coherent, finite type gives a finite covering of X on whose members there are surjections $\mathcal{O}_V^n \to \mathcal{F}|_V$. By coherence each kernel is of finite type. A further finite covering supplies finitely many generators for that kernel, hence presentations $\mathcal{O}_W^m \to \mathcal{O}_W^n \to \mathcal{F}|_W \to 0$. The finite composite covering proves

(2). More generally, covers of any object are obtained by restriction of these covers, so this is the ringed-site definition, not only a condition at X.

For (2)⇒(3), take a finite covering with the displayed finite presentations. A morphism between finite free sheaves on a slice is a matrix with entries in $\mathcal{O}_W(W) = A_W$: its columns are the images of the standard global basis sections. Let P be the cokernel of this matrix over A_W. It is finite. By exactness of Lemma 2.28.10, the sheaf cokernel is precisely $\widetilde{P}_W$. Thus these are the required finite models.

Assume (3), first with nontrivial valuation. Proposition 2.28.8 gives a subordinate finite Laurent tree whose every node remains in $\mathcal{R}_X$. A leaf contained in a model object inherits its finite model by Lemma 2.28.10. Proposition 2.28.11 then gives $\mathcal{F} \simeq \widetilde{M}^{\mathrm{rat}}$ with $M = \mathcal{F}(X)$ finite. This proves (4) in this valuation case.

For trivial valuation choose a defining strict quotient norm. The finite-tail condition makes each unit-radius Tate algebra an ordinary polynomial algebra, and each nonzero quotient element has norm one. For a covering member with rational tuple $(g_i; f_1^{(i)}, \dots, f_{s_i}^{(i)})$, the unit certificate makes g_i invertible in its actual relation quotient. Substitution $T_j = f_j^{(i)}/g_i$ therefore gives

$$B_i = A_{V_i} \simeq A[1/g_i], \qquad A_{V_i \cap V_j} \simeq A[1/(g_i g_j)] \simeq B_i \otimes_A B_j,$$

and likewise for triple intersections, with the actual restriction maps. This is the discrete-norm computation also used in Theorem 2.22.9; here it verifies ordinary, not merely completed, tensor products. Put $S = \prod_i B_i$. Corollary 2.28.2 makes $A \to S$ faithfully flat. The finite local model modules P_i give the finite S-module $P = \prod_i P_i$. Their sheaf identifications on pairwise intersections, evaluated there, give an ordinary descent datum for $A \to S$: the decomposition of $S \otimes_A S$ by its finitely many factor idempotents has components $B_i \otimes_A B_j$, and the cocycle identity is the identity of the three restriction identifications on each triple intersection.

Apply the authorized algebraic descent theorem [StacksAlg, Tag 023N, Proposition 35.3.9]. It produces an A-module M with $S \otimes_A M \simeq P$, whose restrictions and descent datum recover the given ones. By finite-type descent, [StacksAlg, Tag 03C4, Lemma 10.83.2(1)], M is finite. In particular $B_i \otimes_A M \simeq P_i$. These module isomorphisms give compatible sheaf isomorphisms on the covering slices by Lemma 2.28.10. To see the global isomorphism explicitly, their compatibility identifies each $m \in M$ with a matching family

in $\mathcal{F}(V_i)$, hence with a section of $\mathcal{F}(X)$. Restriction and multiplication define $\widetilde{M}^{\mathrm{rat}} \to \mathcal{F}$. For every object W, use the covering $(W \cap V_i)$ and the two sheaf equalizer diagrams; the morphism is an isomorphism on all its members and intersections, hence on W. This proves (4) also for trivial valuation. The use of algebraic faithfully flat descent is confined to the case in which the ordinary tensor intersection was just checked.

Finally suppose (4). The finite A-module M is finitely presented by Theorem 1.11.5; applying the exact functor of Corollary 2.23.2 gives a global finite free presentation of its sheaf. In particular the sheaf is of finite type. On any slice $\mathcal{R}_X/V$, it has the finite model $M_V = A_V \otimes_A M$. Given any finite list of its sections over V, the associated map $\mathcal{O}_V^n \to \widetilde{M_V}_V$ is induced by $A_V^n \to M_V$. Its kernel is the sheaf associated to the algebraic kernel, by Lemma 2.28.10. The algebraic kernel is finite since A_V is Noetherian. Its sheaf is generated by a finite list of global sections on that slice. Thus the kernel is of finite type for every such map and every object V, which is exactly coherence in (1).

Full faithfulness, exactness and the inverse by global sections are now Corollary 2.23.2, together with the essential surjectivity just proved. On each object the section module is finite and therefore has a canonical norm up to uniform equivalence. The norm-strict Čech statement follows from Lemma 2.28.10, or directly from Theorem 2.22.9 after the global model has been obtained. None of these conclusions changes the object category of the site. □

References. Kiehl's original theorem is due to R. Kiehl, *Theorem A und Theorem B in der nichtarchimedischen Funktionentheorie*, Invent. Math. 2 (1967), 256–273. The proof reconstructed here was checked against the later monograph proof in [BGR84, §9.4.3, Theorem 3 and Lemmas 5–6, pp. 378–382]; no theorem or proof number from the original article is asserted without inspection. The present theorem is an internal proof for the specified strict rational ringed site. Passing to a full affinoid G-site, non-strict radii or general analytic spaces remains a separate comparison problem.

2.29 Strict domains over a trivially valued field

The nontrivial-valuation hypothesis in the local Weierstrass form of Gerritzen–Grauert cannot simply be removed. Over a trivially valued field, strict affinoid algebras carry discrete defining norms. This makes an

algebraic route available for finite rational coverings of a domain, but it also makes the stronger ambient local Weierstrass conclusion false. We prove the positive statement and the obstruction separately.

Except in Corollary 2.29.4, K is complete with its fixed *trivial* absolute value. Algebras are commutative and unital, and a strict affinoid algebra is equipped with a norm-admissible Banach norm. The term strict-test affinoid domain has exactly the meaning of Definition 2.27.1; its targets are strict K-affinoid algebras. In the trivial-valuation assertions, all strict rational and Weierstrass radii lie in $|K^\times| = \{1\}$. We allow the zero algebra with its usual conventions.

Lemma 2.29.1 (The discrete algebraic bridge)**.** *Every strict K-affinoid algebra is a finite-type K-algebra, and each of its defining strict quotient norms is the discrete norm: every nonzero element has norm one. Conversely, every finite-type K-algebra, equipped with this norm, is strict affinoid. Every K-algebra homomorphism between such algebras is bounded for their admissible norms and is of finite presentation. If $A \to B$ and $A \to C$ are such maps, the ordinary algebra $B \otimes_A C$ is strict affinoid, and*

$$B \otimes_A C \longrightarrow B \widehat{\otimes}_A C$$

is a bounded isomorphism. For discrete defining norms, its projective tensor norm is exactly the discrete norm and the map is an isometry.

Proof. A family of coefficients in K tending to zero has only finitely many nonzero terms. Thus $K\langle X_1, \dots, X_n\rangle = K[X_1, \dots, X_n]$, and the Gauss norm is discrete. Every ideal is closed for this norm. In a nonzero quotient class, every representative has norm one, so the quotient norm is one. These observations prove both statements about the algebras and their defining norms. A given norm-admissible norm is uniformly equivalent to a defining norm, so every algebra map, contractive for the discrete norms, remains bounded.

A finite set of K-algebra generators of B also generates it over A. Choose a surjection $P = A[X_1, \dots, X_n] \to B$. The algebra P is itself a finite-type K-algebra and hence strict. Theorem 1.11.5 makes it Noetherian, so the kernel is finitely generated. This proves finite presentation of the map.

The tensor algebra is a quotient of $B \otimes_K C$, hence is of finite type over K. For its projective seminorm, every pure-tensor representation has maximum factor norm at most one. If the represented element is nonzero, at least one term in every representation has both factors nonzero; its factor norm

is then one. The infimum over representations is therefore exactly one on every nonzero element. The seminorm is separated, and this discrete norm is complete. Passing to the separated completion changes nothing. Uniformly equivalent norms on B, C give uniformly equivalent projective tensor norms, by applying the two fixed factor bounds to every representation. This proves the assertion for all stipulated norms. □

Proposition 2.29.2 (A strict-test domain gives an étale ring epimorphism)**.** *Let $\varphi : A \to B$ present a strict-test affinoid domain $U \subseteq \mathcal{M}(A)$. As a map of abstract rings, φ is an epimorphism and is formally étale: for every commutative A-algebra C, every ideal $J \subset C$ with $J^2 = 0$, and every A-algebra map $B \to C/J$, there is a unique lift $B \to C$. No topology or finite-type condition is imposed on C in this assertion. Consequently $A \to B$ is an étale ring map of finite presentation, and is algebraically flat.*

Proof. The strict-test domain property makes $A \to B$ an epimorphism in the strict-affinoid category. Indeed, if $u, v : B \to E$ agree on A, their common composite has spectrum image in U, and the uniqueness part of the domain property gives $u = v$. By Lemma 2.29.1, the ordinary algebra $D = B \otimes_A B$ is an allowed strict target, and the two maps $b \mapsto b \otimes 1$ and $b \mapsto 1 \otimes b$ are bounded. They therefore agree. Now let $u, v : B \to E$ be any two homomorphisms of abstract commutative rings agreeing on A. The balanced map

$$D \longrightarrow E, \qquad b \otimes b' \longmapsto u(b)v(b')$$

takes their equality to $u(b) = v(b)$. Thus φ is an epimorphism against arbitrary ring targets, not just strict targets.

Choose the finite presentation $\pi : P = A[X_1, \dots, X_n] \twoheadrightarrow B$ supplied by the preceding lemma, put $I = \ker \pi$, and set $Q = P/I^2$. The algebra Q is finite type over K, so it is an allowed strict test algebra. Every multiplicative seminorm on Q annihilates I/I^2, since each element of this ideal has square zero. It therefore factors through $B = P/I$. Boundedness descends by taking the infimum over quotient representatives, or directly from the discrete defining norms. Conversely, a bounded point of B pulls back to a bounded point of Q. Consequently the maps $A \to Q$ and $A \to B$ have the same spectrum image.

The domain property now supplies an A-algebra map $\sigma : B \to Q$. If $q : Q \to B$ is the quotient map, the endomorphisms $q\sigma$ and id_B agree on

A. The ring-epimorphism property just proved gives $q\sigma = \mathrm{id}_B$. This single section is obtained using only a finite-type strict test.

To pass to an arbitrary square-zero diagram, let $\psi : B \to C/J$ be as in the statement. Choose in C one lift of each of the finitely many elements $\psi(\pi(X_i))$. They define an A-algebra map $\alpha : P \to C$ whose reduction modulo J is $\psi\pi$. Hence $\alpha(I) \subset J$, and $\alpha(I^2) = 0$. There is an induced map $\bar{\alpha} : Q \to C$, and $\bar{\alpha}\sigma$ lifts ψ because $q\sigma = \mathrm{id}_B$. Any two lifts agree on A, so the ring-epimorphism property gives uniqueness. This proves the asserted lifting property without assuming that the arbitrary target C itself is a strict affinoid.

This is formal étaleness in [StacksAlg, Tag 00UQ, Definition 10.150.1]. Finite presentation was proved in Lemma 2.29.1. Apply [StacksAlg, Tag 00UR, Lemma 10.150.3] to obtain étaleness, and [StacksAlg, Tag 00U2, Lemma 10.143.3(6)] to obtain algebraic flatness. These are authorized external algebraic inputs, not additional analytic assumptions. □

Theorem 2.29.3 (A finite rational covering in the trivial-valuation case)**.** *Let $A \to B$ present a strict-test affinoid domain $U \subseteq \mathcal{M}(A)$. There exist finitely many elements $g_1, \dots, g_t \in A$ such that, with W the image of the ordinary prime spectrum* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$*,*

$$W = \bigcup_{i=1}^{t} D_A(g_i), \quad A[1/g_i] \xrightarrow{\sim} B[1/\varphi(g_i)],$$

$$U = \bigcup_{i=1}^{t} R_i, \qquad R_i = \{x \in \mathcal{M}(A) : |g_i(x)| = 1\}.$$

The elements $\varphi(g_i)$ generate the unit ideal of B. Each R_i is rational over the original A, with strict algebra $A\langle Z\rangle/(g_i Z - 1) = A[1/g_i]$. The displayed algebra isomorphisms and

$$B\widehat{\otimes}_A A[1/g_i] \;\simeq\; B[1/\varphi(g_i)] \;\simeq\; A[1/g_i]$$

are bounded with bounded inverses; they are isometric for the discrete defining norms. They commute with restriction to finite intersections, obtained by inverting products of the g_i. Thus the domain itself has a finite original-A rational covering with the stated algebra identifications. No local Weierstrass assertion on a covering of the whole ambient spectrum is made. The empty domain is represented by $B = 0$ and uses the empty family.

Proof. If $B = 0$, all assertions have the indicated empty meaning. Suppose $B \neq 0$. By Proposition 2.29.2, $A \to B$ is a flat ring map of finite presentation. The authorized openness theorem [StacksAlg, Tag 00I1, Proposition 10.41.8] therefore makes W open in $\mathrm{Spec}(A)$. By the definition of the Zariski topology, every point of W lies in some principal open $D_A(g)$ contained in W: write the closed complement as $V(J)$ and choose $g \in J$ outside that prime.

Let G be the set of all $g \in A$ with $D_A(g) \subseteq W$. Their images generate the unit ideal in B. Otherwise that ideal lies in a maximal ideal $\mathfrak{n}$ of B, by [StacksAlg, Tag 00E0, Lemma 10.17.2(2)] applied to its nonzero quotient. The contraction of $\mathfrak{n}$ belongs to W, and a principal neighbourhood as above supplies $g \in G$ whose image is not in $\mathfrak{n}$, a contradiction. Hence there is a finite certificate

$$\sum_{i=1}^{t} b_i \varphi(g_i) = 1, \qquad b_i \in B, \quad g_i \in G.$$

This also proves $W = \bigcup_i D_A(g_i)$. Indeed, choose a prime of B over any point of W; at least one term $\varphi(g_i)$ lies outside that prime. The reverse inclusion holds by the choice of G.

Fix any $g \in G$. The ring map $A_g \to B_g$, where B_g abbreviates $B[1/\varphi(g)]$, is étale by the base-change assertion [StacksAlg, Tag 00U2, Lemma 10.143.3(3)], hence flat by part (6). It remains an epimorphism: two maps from B_g agreeing on A_g agree on B by Proposition 2.29.2, and then agree on the inverse of $\varphi(g)$. Every maximal ideal $\mathfrak{m}$ of A_g has nonzero fiber in B_g. To verify this, contract it to a prime $\mathfrak{p}$ of A. Since $g \notin \mathfrak{p}$ and $D_A(g) \subseteq W$, there is a prime $\mathfrak{q}$ of B over $\mathfrak{p}$. The fraction ideal $\mathfrak{q}B_g$ is prime and contracts to $\mathfrak{m}$; the fraction description verifies both assertions directly. Its residue field consequently receives a unital map from $B_g \otimes_{A_g} \kappa(\mathfrak{m})$, so this fiber is nonzero.

By [StacksAlg, Tag 00HP, Lemma 10.39.15, (1)⇔(4)], the map $A_g \to B_g$ is faithfully flat. The authorized epimorphism criterion [StacksAlg, Tag 04VU, Lemma 10.107.7] makes it an isomorphism. If $A_g = 0$, its unital target B_g is also zero and the same conclusion holds. Thus the open-immersion consequence is obtained by actual principal-localization isomorphisms; no separate analytic local-ring theorem is involved. All these maps and their inverses are bounded by Lemma 2.29.1.

We now compare the two spectra carefully. For $x \in \mathcal{M}(A)$, contractivity for the discrete norm, Lemma 1.6.1, gives $|a(x)| \leq 1$ for every $a \in A$. The subset

$$\mathfrak{r}_x = \{a \in A : |a(x)| < 1\}$$

is a prime ideal: the ultrametric inequality and the bound by one give the ideal axioms, while $|ab(x)| < 1$ implies $|a(x)| < 1$ or $|b(x)| < 1$. It is a proper ideal because $|1| = 1$. Hence

$$\mathfrak{r}_x \in D_A(g) \quad \Longleftrightarrow \quad |g(x)| = 1.$$

This prime is not in general the kernel of x. Thus the comparison uses the value-one condition, not merely nonvanishing of g at x.

The rational algebra for $\{|g| = 1\}$ is $A\langle Z\rangle/(gZ - 1)$. Its tuple $(g; 1)$ generates the unit ideal, and the inequality $1 \leq |g|$ is equivalent to equality because all values are at most one. By the discrete bridge, this actual quotient is $A[1/g]$. Theorem 2.19.3 identifies its spectrum with that rational subset. The already constructed bounded map $B \to B_g \simeq A_g$ shows that this subset lies in U whenever $g \in G$. Conversely, for $y \in \mathcal{M}(B)$, the displayed unit certificate and the bounds $|b_i(y)| \leq 1$ give $|\varphi(g_i)(y)| = 1$ for at least one i. Thus every point of U lies in some R_i. Equivalently, U is precisely the set of x for which $\mathfrak{r}_x \in W$, as follows from the finite principal covering just proved.

Finally, localization gives the ordinary algebra isomorphism $B \otimes_A A_g \simeq B_g$: the mutually inverse maps send $b \otimes a/g^n$ to $ba/\varphi(g)^n$ and send $b/\varphi(g)^n$ to $b \otimes 1/g^n$. Lemma 2.29.1 identifies the ordinary tensor with its separated completion. Repeating this construction for products of the g_i proves the finite-intersection assertions. Every map is fixed on coefficients and on the specified inverses, so the restriction maps commute. □

Corollary 2.29.4 (A rational basis for strict-test domains over any complete field)**.** *Let K now be any complete real-valued non-Archimedean field, with its fixed absolute value, possibly trivial. Let $A \to B$ be a strict-test affinoid-domain presentation of $U \subseteq \mathcal{M}(A)$. Then U has a finite covering by rational subsets $R_1, \ldots, R_t$ defined over the original A, with unit radii or, equivalently after scaling, radii in $|K^\times|$. For each R_i, and for each intersection R indexed by a nonempty finite list of these members, there is a canonical bounded isomorphism*

$$B\widehat{\otimes}_A A_R \;\simeq\; A_R$$

with bounded inverse. It identifies R as a rational domain over B with the same algebra A_R. All identifications commute with the restriction maps between these members and intersections. Empty resulting intersections have algebra zero; the empty domain uses the empty covering. This statement concerns finite rational coverings of the domain, and does not assert ambient local Weierstrass for trivial valuation.

Proof. If U is empty, the point correspondence and Theorem 1.12.3 give $B = 0$, and the empty covering suffices. For nontrivial valuation, Theorem 2.27.5 supplies a finite original-A rational covering of U. For trivial valuation, Theorem 2.29.3 supplies one. Thus in both cases the members have the asserted scope.

Proposition 2.22.6 makes every indicated finite intersection rational over A. It is contained in U. Apply Lemma 2.30.1 to this inclusion: the image of its unit-ideal tuple in B defines the same domain, and the actual completed pushout is A_R. This gives the displayed bounded isomorphisms for both valuation cases without using an intersection theorem for arbitrary strict-test domains. Their compatibility follows from the same lemma, or directly from uniqueness in the strict-test factorization property: after a further restriction, the two maps from B extend the same map from A, and the rational variables are prescribed. No arbitrary Banach completed-tensor exactness is used. □

Proposition 2.29.5 (Failure of ambient local Weierstrass over trivial valuation)**.** *Let $A = K[T]$ and $B = K[T, T^{-1}]$, both with their discrete norms. The rational map $A \to B = A\langle Z\rangle/(TZ - 1)$ presents the nonempty proper strict-test domain*

$$U = \{x \in \mathcal{M}(A) : |T(x)| = 1\}.$$

There is no finite strict rational covering $\mathcal{M}(A) = \bigcup_i V_i$ for which every $U \cap V_i$ is a Weierstrass domain in V_i with radii in $|K^\times|$. The same obstruction applies if such radii are allowed in $\sqrt{|K^\times|}$, which is also $\{1\}$.

Proof. The rational point and domain universal properties, Theorems 2.19.3 and 2.22.5, prove the asserted domain presentation. The seminorm η that assigns one to every nonzero polynomial and zero to zero is multiplicative and bounded, and belongs to U. Evaluation at $T = 0$ gives another bounded point x_0, which does not belong to U.

Every nonempty strict rational subset of $\mathcal{M}(A)$ contains η. Indeed, write it with a unit-ideal tuple $(g; f_1, \ldots, f_s)$ and inequalities $|f_j| \le |g|$. If $g = 0$,

these inequalities force every f_j to vanish at a point, contradicting the unit certificate, so the subset is empty. Otherwise $|g(\eta)| = 1$ and $|f_j(\eta)| \leq 1$ for every j, so η belongs to it.

For any nonzero strict affinoid algebra C over this K, all elements have values at most one at every bounded point, by Lemma 2.29.1 and Lemma 1.6.1. Every unit-radius Weierstrass condition $|h_j| \leq 1$ is therefore automatic. Algebraically, its presentation is the polynomial quotient $C[S_1, \ldots, S_m]/(S_j - h_j)$, canonically equal to C. Thus a nonempty such Weierstrass domain is the entire spectrum of C.

In any rational covering of $\mathcal{M}(A)$, choose a member V containing x_0. It is nonempty, so it also contains η. Hence $U \cap V$ is nonempty and proper in V. The algebra representing V is strict; the preceding paragraph shows that this intersection cannot be a unit-radius Weierstrass domain in V. This proves the contradiction, even if an empty domain is admitted separately by convention. □

References. The positive result is an algebraic reconstruction for the stated trivial-valuation strict-test category. Its external theorem inputs are the explicitly located Stacks results used above; the finite-presentation flat-epimorphism argument is the affine algebraic form of the open-immersion criterion in [StacksAlg, Tag 025G, Theorem 41.14.1], which is only a comparison reference here. It does not invoke a pre-existing analytic Gerritzen–Grauert theorem. Proposition 2.29.5 concerns the stronger ambient local Weierstrass statement; it does not contradict the finite-rational-union conclusion of Theorem 2.29.3. Generalized affinoids and arbitrary real radii are outside this section.

2.30 Finite affinoid coverings and coherent modules

Throughout this section K is complete, with trivial valuation allowed. The rational calculations now extend to finite coverings by strict affinoid domains. The geometric input is the finite rational covering of Corollary 2.29.4. Its nontrivial-valuation branch comes from Gerritzen–Grauert; its trivial-valuation branch is the separate algebraic argument just proved. The stronger ambient local Weierstrass assertion is not used. We retain the strict-test domain convention of Definition 2.27.1. Thus every ordinary strict affinoid domain is included, but no assertion about non-strict test algebras is built into the arguments below.

Lemma 2.30.1 (Restriction and rational subdomains)**.** *Let K be complete, with trivial valuation allowed, and let $A \to B$ represent a strict-test domain $V \subseteq X = \mathcal{M}(A)$. If $A \to C$ represents another such domain $W \subseteq V$, there is a unique bounded A-homomorphism $B \to C$, and it represents W as a strict-test domain of $\mathcal{M}(B)$. These restriction maps compose. Composing two strict-test domain morphisms again gives a strict-test domain morphism.*

In particular, if W is rational over A with radii in $|K^\times|$ and $W \subseteq V$, then W is rational over B, its rational presentation algebra is canonically boundedly isomorphic to A_W, and

$$B\widehat{\otimes}_A A_W \;\simeq\; A_W.$$

Proof. The domain property for $A \to B$, tested on the strict algebra C, gives the unique map $B \to C$. Its map of spectra is the inclusion $W \subseteq V$ under the given identifications. Suppose that a bounded $B \to E$, with E strict, has spectral image in W. The domain property for $A \to C$ gives a unique $C \to E$ over A. It is also over B: the two resulting A-maps from B to E agree by the uniqueness clause for V. This proves the relative domain property. The same uniqueness proves composition of restrictions. To prove transitivity of domains, factor any eligible $A \to E$ first through B and then through the second domain algebra. Both existence and uniqueness follow in this order, and the spectral identifications compose.

For the rational assertion use the original tuple defining W over A, now in B. Its entries still generate the unit ideal. Its actual rational quotient is $D = B\widehat{\otimes}_A A_W$, by Proposition 1.14.3 and the joint-presentation calculation of Proposition 2.22.6. By Theorem 2.19.3, its spectrum is the inverse image of W in $\mathcal{M}(B)$, hence is W. The restriction map $B \to A_W$ just constructed, together with the identity of A_W, gives $D \to A_W$. This is inverse to the canonical $A_W \to D$. Indeed, the composite fixes A_W; on B it agrees with the given map by domain uniqueness, since $\mathcal{M}(D)$ maps into $W \subseteq V$. The pushout universal property proves equality on D. All maps are bounded, so this is the asserted Banach-algebra isomorphism. Zero algebras give the empty-domain case. □

Proposition 2.30.2 (Intersections of strict-test domains)**.** *Let K be complete, with trivial valuation allowed. Let $A \to B$ and $A \to C$ represent strict-test domains $V, W \subseteq \mathcal{M}(A)$. Then*

$$D = B\widehat{\otimes}_A C$$

is strict, with zero allowed, and represents the strict-test domain $V \cap W$. Its spectrum maps homeomorphically onto that intersection. Iterated intersections are canonically associative, and their bounded algebra maps agree with restriction. In particular, an empty intersection has the zero algebra.

Proof. Proposition 1.14.3 makes D strict and gives its bounded pushout property. Every point of $\mathcal{M}(D)$ restricts to points of $\mathcal{M}(B)$ and $\mathcal{M}(C)$, so its image lies in $V \cap W$. To prove that this map is bijective, rather than assuming a general theorem about Berkovich fibre products, take the finite rational covering $V = \bigcup_\alpha R_\alpha$ over A supplied by Corollary 2.29.4. Lemma 2.30.1 identifies the algebra of R_α as a rational algebra over B with A_{R_α}. Its pullback to D is rational and has algebra

$$D_\alpha = D\widehat{\otimes}_B A_{R_\alpha} \simeq C\widehat{\otimes}_A A_{R_\alpha}.$$

The bounded isomorphism follows either from the two pushout properties or from the common relative presentation; it is not an assertion that an ordinary tensor product is already complete. The right-hand side is a rational quotient of C. Its spectrum identifies with $W \cap R_\alpha$, by the explicit rational point theorem. The D_α cover $\mathcal{M}(D)$, because the R_α cover $\mathcal{M}(B)$. For any $x \in V \cap W$, choose α with $x \in R_\alpha$; the point in $\mathcal{M}(C)$ above x gives a point of $\mathcal{M}(D_\alpha)$ and hence of $\mathcal{M}(D)$. If two points of $\mathcal{M}(D)$ map to x, both lie in this rational piece, and the injectivity of $\mathcal{M}(D_\alpha) \to \mathcal{M}(C)$ makes them equal. This proves the bijection.

The map is continuous, and compactness and Hausdorffness from Proposition 2.24.1 make it a homeomorphism onto its image. If a bounded $A \to E$ with E strict has spectral image in $V \cap W$, the two domain properties supply unique compatible maps $B, C \to E$, and the pushout supplies a unique $D \to E$. This is exactly the required domain property. Iterating it gives canonical restriction and association isomorphisms by uniqueness. Finally, a nonzero strict affinoid has a point by Theorem 1.12.3; hence an empty spectrum forces $D = 0$. □

Theorem 2.30.3 (Tate acyclicity for finite strict affinoid coverings)**.** *Let K be complete, with trivial valuation allowed, let A be a strict K-affinoid algebra, and let M be a finite A-module. Let $(V_i)_{i\in I}$ be a finite covering of $X = \mathcal{M}(A)$ by strict-test affinoid domains. Give each intersection algebra $A_{V_{i_0}\cap\cdots\cap V_{i_q}}$ a norm-admissible Banach norm, and its finite modules their canonical norms. Then the augmented complex*

$$0 \longrightarrow M \longrightarrow \prod_i A_{V_i} \otimes_A M \longrightarrow \prod_{i_0<i_1} A_{V_{i_0}\cap V_{i_1}} \otimes_A M \longrightarrow \cdots$$

is norm-strict exact in every degree. Equivalently the all-tuples version is norm-strict exact. Every ordinary finite-module tensor in these complexes agrees, by a bounded isomorphism, with its completed tensor. No reducedness, separability or residue-characteristic hypothesis is needed.

Proof. Choose for each V_i the finite covering by A-rational domains supplied by Corollary 2.29.4, and let $\mathcal{W}$ be their finite union as a covering family of X. It refines (V_i) with a specified containing index for each member. Its augmented complex is norm-strict exact by Theorem 2.22.9.

Fix a nonempty tuple of coarse indices, and put $Y = V_{i_0}\cap\cdots\cap V_{i_p}$. Proposition 2.30.2 gives its strict algebra A_Y and all compatible intersection maps. Each member of $\mathcal{W}$ restricts to a rational quotient of A_Y: take its original unit-ideal tuple in A, map it to A_Y, and use the rational point theorem. These quotients cover $\mathcal{M}(A_Y)$. The module $A_Y \otimes_A M$ is finite over A_Y. Apply Theorem 2.22.9 over this algebra. Associativity of ordinary tensors, [StacksAlg, Tag 00D2, Lemma 10.12.7], and Proposition 1.11.6 identify its terms, restrictions and augmentation with the mixed intersection diagram. Thus the required local complexes are norm-strict exact on every coarse intersection. Empty intersections have actual zero algebras by the proposition, so their terms are zero for a proved algebraic reason.

All hypotheses of Proposition 2.21.5(1) now hold. It transfers norm-strict exactness from $\mathcal{W}$ to (V_i), in every degree and with the given augmentation. The same proposition supplies the all-tuples comparison. The finite-module completion statement is Proposition 1.11.6. The lifting bounds may depend on the finite cover, the module and the chosen equivalent norms; no bound uniform over all covers is asserted. If X is empty, $A = 0$ and $M = 0$, and the conclusion also covers its empty covering. □

Proposition 2.30.4 (Flat restriction to a strict affinoid domain)**.** *Under the same field hypothesis, every strict-test domain map $A \to B$ is flat as an ordinary ring map. A finite covering by such domains gives a faithfully flat ordinary ring map $A \to \prod_i A_{V_i}$. These assertions concern algebraic tensor products of arbitrary modules; they do not assert exactness of completed tensor products of arbitrary Banach modules.*

Proof. Choose an A-rational covering (R_j) of the domain by Corollary 2.29.4. Lemma 2.30.1 makes it a rational covering over B with the same algebras. Put $S = \prod_j A_{R_j}$. Corollary 2.28.2 makes $B \to S$ faithfully flat, whereas rational flatness makes $A \to S$ flat. If $N' \to N$ is an injective map of A-modules and Q is the kernel of $B \otimes_A N' \to B \otimes_A N$, tensoring over B with the flat algebra S identifies $S \otimes_B Q$ with the kernel of $S \otimes_A N' \to S \otimes_A N$, which is zero. Faithfulness gives $Q = 0$. Together with right exactness of ordinary tensor products, [StacksAlg, Tag 00DF, Lemma 10.12.10], this is flatness of B over A. For an empty domain, $B = 0$ is flat and the assertion is immediate.

For a finite covering, refine each member by the preceding rational covers. The resulting A-rational cover has faithfully flat product $A \to S$, and S is an algebra over $B' = \prod_i A_{V_i}$. The already proved flatness makes B' flat over A. If $B' \otimes_A N = 0$, then $S \otimes_A N = 0$, so $N = 0$. Thus tensoring with B' is faithful as well as exact. This is the asserted faithful flatness. □

Definition 2.30.5 (The finite strict-domain site)**.** For K complete, with trivial valuation allowed, and $X = \mathcal{M}(A)$ strict, let $\mathcal{D}_X$ be the partially ordered category of strict-test domains in X, with inclusions as arrows. Coverings are finite set-theoretic coverings by objects, including the empty covering of the empty set. Choose representing algebras A_V, with $A_X = A$ and $A_\varnothing = 0$. Use the unique bounded restrictions of Lemma 2.30.1, and set

$$\mathcal{O}_X^{\mathrm{str}}(V) = A_V, \qquad \widetilde{M}^{\mathrm{str}}(V) = A_V \otimes_A M \quad (M \text{ finite over } A).$$

Theorem 2.30.6 (The strict-domain structure sheaf and rational comparison)**.** *The coverings of $\mathcal{D}_X$ form a pretopology. The displayed assignments are a ring sheaf and module sheaves, independent up to canonical isomorphism of the representing algebras. The functor $M \mapsto \widetilde{M}^{\mathrm{str}}$ is exact and fully faithful, and its global sections are M. Inclusion $\mathcal{R}_X \hookrightarrow \mathcal{D}_X$ induces an equivalence of ringed topoi, carrying $\mathcal{O}_X^{\mathrm{str}}$ to $\mathcal{O}_X^{\mathrm{rat}}$ and $\widetilde{M}^{\mathrm{str}}$ to $\widetilde{M}^{\mathrm{rat}}$.*

Proof. Intersections exist by Proposition 2.30.2. Pulling back and composing finite set-theoretic covers proves the pretopology axioms. For a covering of an object V, its members and all intersections are strict-test domains over A_V by Lemma 2.30.1. Apply Theorem 2.30.3 over A_V to the finite module $A_V \otimes_A M$. Tensor associativity identifies its terms with the displayed presheaf. Degree-zero exactness is precisely the sheaf equalizer condition. The empty value is zero. Taking $M = A$ gives the ring sheaf. Uniqueness of all restriction maps proves independence of the choices.

Proposition 2.30.4 gives objectwise exactness for finite-module short exact sequences. At X the value is M. A sheaf map between two such module sheaves is determined by its map u at X, because on each V it must send $b \otimes m$ to $b \otimes u(m)$. Conversely this formula defines a sheaf map for every A-linear u. Thus the functor is exact and fully faithful.

For the site comparison apply the authorized comparison lemma [StacksAlg, Tag 03A0, Lemma 7.29.1]. Its hypotheses can be checked directly here. Inclusion is fully faithful, so both local faithfulness and local fullness hold. It preserves covers and finite intersections; restriction of a sheaf is therefore a sheaf, so it is continuous. For a covering of a rational object by strict-test domains, refine each member by its finite A-rational covering from Corollary 2.29.4. Their union is a covering in $\mathcal{R}_X$ refining the given one, which is cocontinuity. The same corollary covers every object of $\mathcal{D}_X$ by objects from $\mathcal{R}_X$. These are exactly the five hypotheses of the comparison lemma, including the empty object. It gives an equivalence of topoi whose restriction functor is the inclusion-induced restriction of sheaves. On rational objects the chosen representing algebras and restrictions agree by domain uniqueness. Thus the ring sheaves and the finite-module sheaves restrict as asserted, making this an equivalence of ringed topoi. □

Theorem 2.30.7 (Kiehl equivalence on the finite strict-domain site)**.** *For a complete K, with trivial valuation allowed, and a strict K-affinoid $X = \mathcal{M}(A)$, the functors*

$$M \longmapsto \widetilde{M}^{\mathrm{str}}, \qquad \mathcal{F} \longmapsto \Gamma(X, \mathcal{F})$$

give inverse exact equivalences between finite A-modules and coherent $\mathcal{O}_X^{\mathrm{str}}$-modules on $\mathcal{D}_X$. Coherence here is Definition 18.23.1(8), Tag 03DL, of the Stacks Project. It is also equivalent to finite presentation on this site. Every coherent module has a global finite free presentation, and every finite strict-

domain covering has a norm-strict exact augmented Čech complex for it, with the canonical norms on its finite section modules.

Proof. Theorem 2.30.6 identifies the ringed topoi of $\mathcal{D}_X$ and $\mathcal{R}_X$. Coherence and finite presentation are intrinsic under this equivalence, by [StacksAlg, Tag 03DM, Lemma 18.23.2]; the special cocontinuous comparison required in that lemma is precisely the one just verified. Theorem 2.28.12 therefore identifies the coherent modules with the sheaves $\widetilde{M}^{\mathrm{rat}}$ for finite A-modules M. Under the equivalence these are exactly $\widetilde{M}^{\mathrm{str}}$. Full faithfulness and the global-sections inverse were already proved in Theorem 2.30.6, as was exactness. A finite presentation of M over the Noetherian ring A gives a global finite free presentation of its sheaf. Finally apply Theorem 2.30.3 to M. Its terms are exactly the section modules of $\mathcal{F} \simeq \widetilde{M}^{\mathrm{str}}$, proving the last assertion. □

The preceding acyclicity statements concern explicit complexes. To pass to derived sheaf cohomology, one needs a comparison theorem; vanishing for a single covering would not suffice. Here vanishing has been proved on every object and for every covering of the stated site.

Corollary 2.30.8 (Theorem B on the specified affinoid sites)**.** *Let K be complete, with trivial valuation allowed, and let A be strict. If $\mathcal{F}$ is a coherent module on the fixed rational ringed site $\mathcal{R}_X$, then*

$$H^q(V, \mathcal{F}) = 0 \qquad (V \in \mathcal{R}_X,\ q > 0).$$

For the same complete field K, including trivial valuation, the same assertion holds for every coherent module on $\mathcal{D}_X$ and every $V \in \mathcal{D}_X$. Here H^q denotes the right derived functors of sections in the category of abelian sheaves of the specified site, applied to the underlying abelian sheaf.

Proof. For $\mathcal{R}_X$, Theorem 2.28.12 writes $\mathcal{F} = \widetilde{M}^{\mathrm{rat}}$ for a finite A-module. Lemma 2.28.10 gives vanishing of higher Čech cohomology for every covering of every object. For $\mathcal{D}_X$, Theorem 2.30.7 gives the same conclusion. Apply [StacksAlg, Tag 03F9, Lemma 21.10.9], with its set $\mathcal{B}$ equal to all objects and its family Cov equal to all the finite coverings in the chosen pretopology. These coverings are a cofinal system by definition of the site. All iterated fibre products required by that lemma are intersections and remain in the site, by Proposition 2.22.6 or Proposition 2.30.2, respectively. Thus all three hypotheses of the cited lemma hold, and it gives the claimed derived vanishing in every positive degree. This application uses the general site lemma, not the different theorem about open covers of a topological space. □

The strict finite-domain theory supplies the local algebra for the general construction. Theorem 2.36.6 gives rational refinement at arbitrary radii, and Theorems 2.38.5 and 2.39.6 give the corresponding sheaf equivalences. Section 2.40 glues these affinoid sites through analytic nets, including non-good spaces.

2.31 Rational domains with arbitrary radii

Fix a complete non-Archimedean field K with its specified real-valued absolute value, allowing trivial valuation. All affinoids in this section may have arbitrary positive presentation radii. Norms are admissible in the quantitative sense of Definition 1.9.1; quotients use actual algebraic ideals. The generalized finite-module theorem proves that these ideals are closed.

For the specified coefficient and quotient norms, nonzero Banach algebras have homogeneous submultiplicative norms normalized by $\|1\| = 1$, as in the finite-module section. The zero algebra has its zero norm. Statements of isometry use precisely these norms. Replacing them by uniformly equivalent admissible norms preserves the bounded isomorphisms and their bounded inverses, but need not preserve literal isometry.

Theorem 2.31.1 (Points of an arbitrary-radius rational presentation)**.** *Let A be a norm-admissible K-affinoid algebra, let $\rho_1, \dots, \rho_s > 0$, and suppose $(g, f_1, \dots, f_s) = A$. With the coefficient norm and its quotient norm, put*

$$D = A\{\rho^{-1}T\}, \qquad B = D/(gT_i - f_i)_{i=1}^{s},$$
$$V = \{x \in \mathcal{M}(A) : |f_i(x)| \le \rho_i |g(x)| \ \forall i\}.$$

Then B is a norm-admissible K-affinoid algebra, possibly zero, and restriction gives a homeomorphism $\mathcal{M}(B) \simeq V$. Moreover $B = 0$ if and only if V is empty. The case $s = 0$ is allowed and gives $B = A$, since g is then a unit.

Proof. Proposition 1.17.3 makes D affinoid, and Corollary 1.18.4 gives the asserted complete actual quotient B. Choose a unit-ideal identity $a_0 g + \sum_i a_i f_i = 1$. In B it becomes

$$g\left(a_0 + \sum_i a_i \bar{T}_i\right) = 1.$$

Thus g is a unit in B, with the usual zero-ring interpretation. At every $x \in V$, the same identity and the inequalities force $|g(x)| > 0$.

For $x \in V$, Lemma 1.6.1 gives the valued fraction field $F_x = \mathrm{Frac}(A/\ker x)$. Polynomial evaluation at $T_i = f_i/g$ in this field has multiplicative seminorm p_x bounded by the weighted coefficient norm:

$$p_x\left(\sum_\alpha a_\alpha T^\alpha\right) \le \max_\alpha |a_\alpha(x)| \prod_i |f_i/g|_x^{\alpha_i} \le \max_\alpha \|a_\alpha\| \rho^\alpha.$$

For polynomials P, Q, $|p_x(P) - p_x(Q)| \le \|P - Q\|$. Hence norm-convergent polynomial truncations extend p_x uniquely to a seminorm on D. Addition, multiplication and its scalar values pass to the limit, because products of bounded convergent sequences converge. It kills all $gT_i - f_i$ and their ideal. Taking the infimum over representatives gives a contractive multiplicative seminorm y_x on B extending x. No completion of F_x is needed here.

Conversely, a point y of B is contractive for its quotient norm, so $|\bar{T}_i(y)| \le \rho_i$. Its restriction belongs to V. To prove uniqueness over a given x, take a polynomial $P = \sum a_\alpha T^\alpha$ of total degree at most N and form

$$Q_P = \sum_\alpha a_\alpha f^\alpha g^{N-|\alpha|} \in A.$$

In B one has $g^N P(\bar{T}) = Q_P$, so every point over x must satisfy $|P(\bar{T})(y)| = |Q_P(x)|/|g(x)|^N$. The polynomial image is dense in B, and contractivity determines the remaining values uniquely. This proves the bijection. For a polynomial, the displayed value is a continuous function of $x \in V$. For any $b \in B$, choose a lift $F \in D$ and polynomial truncations $P_n \to F$. Uniformly in $x \in V$,

$$\big||b(y_x)| - |P_n(\bar{T})(y_x)|\big| \le \|F - P_n\|.$$

Thus all evaluations along $x \mapsto y_x$ are continuous. Restriction is also continuous by its definition, proving the homeomorphism. Theorem 1.20.1 applied to B proves the assertion about emptiness. Zero algebras and an empty V require no exceptional point construction. □

Theorem 2.31.2 (Bounded and geometric rational factorization)**.** *Keep the rational presentation $A \to B$ and subset V above. For every bounded unital K-algebra homomorphism $\phi : A \to C$ to a Banach K-algebra, a bounded K-algebra factorization $A \to B \to C$ exists if and only if $\phi(g)$ is a unit and, for each i, there is $C_i \ge 1$ with*

$$\|\big(\phi(f_i)\phi(g)^{-1}\big)^m\| \le C_i \rho_i^m \qquad (m \ge 0).$$

The factorization is unique. If C is a norm-admissible affinoid over any specified complete extension L/K, this condition is equivalent to the image of $\mathcal{M}_L(C) \to \mathcal{M}_K(A)$ being contained in V. In particular the rational presentation has the domain universal property for all such complete-extension affinoid tests, not only for strict tests over K. Zero targets are allowed.

Proof. In B, the preceding unit identity makes g invertible and forces any factorization to send $\bar{T}_i$ to $b_i = \phi(f_i)\phi(g)^{-1}$. Since $\|\bar{T}_i^m\| \le \rho_i^m$, boundedness of the factorization gives the required estimates. Conversely, Proposition 1.9.3 gives a bounded evaluation $A\{\rho^{-1}T\} \to C$ at the b_i. It kills the actual ideal generated by $gT_i - f_i$. If its bound is L_0, taking the infimum over lifts of $b \in B$ gives $\|\psi(b)\| \le L_0\|b\|$. Thus it descends boundedly. Its values on the dense polynomial image are forced, proving uniqueness. This proof uses the explicit power estimates for an arbitrary Banach target.

Now let C be affinoid over L. If a factorization exists, restricting its spectral points and using Theorem 2.31.1 gives the image condition. Conversely, at every $y \in \mathcal{M}_L(C)$ the inequalities and the image of the unit-ideal identity imply $|\phi(g)(y)| > 0$. Theorem 1.20.1, over L, makes $\phi(g)$ a unit. Its ratios have $|b_i(y)| \le \rho_i$ at every such point. Theorem 1.20.2, again over L and with the actual radii ρ_i, gives the required scaled power estimates. The Banach factorization just proved applies because C is also a Banach K-algebra. The zero target has its unique factorization directly; the equivalence of spectral emptiness and zero prevents an exceptional nonzero target with vacuous hypotheses. □

Taking one rational algebra as the test target for another now gives a unique bounded restriction map whenever the corresponding subsets are nested. Equal subsets give bounded inverse maps, and uniqueness makes restriction compatible with composition. These comparisons preserve the algebra and its Banach structure up to equivalence, not necessarily a particular presentation norm. We next identify finite intersections by their actual presentations.

Proposition 2.31.3 (Joint presentations and completed pushouts)**.** *For any two arbitrary-radius affinoid quotient presentations over A, put*

$$B = A\{\rho^{-1}T\}/I, \qquad C = A\{\tau^{-1}S\}/J,$$
$$H = A\{\rho^{-1}T, \tau^{-1}S\}/(I, J),$$

where (I, J) is the sum of the actual extended ideals. With the displayed quotient norms there is a canonical isometric algebra isomorphism $B\widehat{\otimes}_A C \simeq H$. This is the separated completed tensor product, not an assertion that the ordinary algebraic tensor is already complete. It has the bounded pushout universal property for commutative Banach targets.

If $I = (gT_i - f_i)$ and $J = (hS_j - e_j)$ are rational relation ideals, then H is isometric to the rational presentation with denominator gh and the following numerator-radius pairs:

$$(hf_i, \rho_i), \qquad (ge_j, \tau_j), \qquad (f_i e_j, \rho_i \tau_j).$$

Its spectrum is the intersection of the two original rational subsets. Finite iteration gives all intersections, including repeated members, with compatible restriction maps. Empty intersections have zero algebra.

Proof. All weighted coefficient algebras here are affinoid by Proposition 1.17.3; their actual ideals are closed and their quotients are complete by Corollary 1.18.4. On the ordinary balanced tensor product, the projective seminorm is submultiplicative by multiplying finite representations and taking their two infima. Its null space is an ideal. Separation and completion thus give a Banach algebra $G = B\widehat{\otimes}_A C$, or zero. The structural maps are contractive. The two contractive maps into H give a contractive map $G \to H$ by their balanced product.

Conversely the elements $\bar{T}_i \otimes 1$ and $1 \otimes \bar{S}_j$ have all powers bounded by ρ_i^m and τ_j^m. Proposition 1.9.3 gives a contractive map from the joint coefficient algebra to G. It kills I and J, first on each whole coefficient algebra by bounded polynomial approximation, hence on their extended ideals. It descends to a contractive $H \to G$. The composites fix all coefficients and variables in H and all pure tensors in G. Density makes them identities, proving the isometry. More generally, two bounded maps from B, C to a Banach algebra agreeing on A have a bounded balanced product; its finite-representation bound extends it uniquely to G. This proves the bounded pushout property.

For the rational assertion, multiply the two unit-ideal identities. They show that gh and the listed numerators generate the unit ideal. Call their rational algebra P, with variables Y_i, Z_j, W_{ij} of the listed radii. In H, both g and h are units. The assignments

$$Y_i \longmapsto \bar{T}_i, \qquad Z_j \longmapsto \bar{S}_j, \qquad W_{ij} \longmapsto \bar{T}_i \bar{S}_j$$

respect all bounds and relations and give a contractive $P \to H$. In P, the product gh is a unit, so both factors are units. Dividing its relations gives $gY_i = f_i$, $hZ_j = e_j$, and $W_{ij} = Y_i Z_j$. Sending T_i to Y_i and S_j to Z_j therefore gives a contractive $H \to P$, inverse on all variables. The point theorem identifies the resulting inequalities with the two original sets: divide the hf_i and ge_j inequalities by the nonzero factors; conversely the two original sets imply all product inequalities.

Iteration gives finite intersections. Their comparison and restriction maps agree by the uniqueness in Theorem 2.31.2; a repeated member is consequently redundant by bounded inverse maps. Theorem 1.20.1 makes an empty intersection algebra zero. Every zero-quotient case in the pushout assertion also follows directly: if one factor ideal contains one, so does the joint ideal. □

The preceding statements concern pointwise rational domains and bounded maps. Comparing separate spectral maxima of f_i and g cannot replace their simultaneous pointwise inequalities; Example 1.9.5 already gives a counterexample even when g is a unit. Likewise, the arbitrary-Banach-target criterion retains bounds on all powers, not only a spectral-radius bound.

2.32 Tate acyclicity for arbitrary-radius rational coverings

A finite rational covering can be made strict over one free-polyradius field. To descend its Čech complex, however, one must compare its actual quotient algebras and finite modules, not just their point sets. We first make those identifications, retaining the maps as well as the objects. All fields below are complete for the specified real-valued non-Archimedean absolute value; trivial valuation and zero affinoids are allowed.

The normalized coefficient and quotient norms from the rational-domain section may be chosen for the proof. All conclusions of bounded isomorphism and norm-strict exactness are invariant under uniformly equivalent admissible norms. Isometries below refer only to the specified coefficient and quotient norms.

Proposition 2.32.1 (Actual quotient and finite-module base change)**.**

Let A be a norm-admissible K-affinoid algebra, let ρ be any finite positive tuple, put $D = A\{\rho^{-1}T\}$, and let $J \subseteq D$ be an algebraic ideal. Give $B = D/J$ its quotient norm. For every finite K-free tuple r, put $L = K_r$ and $E_r = E\widehat{\otimes}_K L$ for Banach K-spaces E. Coefficient and quotient maps induce isometric algebra identifications

$$D_r \simeq A_r\{\rho^{-1}T\},$$
$$B_r \simeq D_r/JD_r.$$

The ideal JD_r here is the actual algebraic extended ideal; it is closed and equals $c_0(\mathbf{Z}^\ell, r; J)$ in the coefficient description, where ℓ is the length of r. For every finite canonical A-module M, the natural maps give a bounded isomorphism with bounded inverse

$$(B \otimes_A M)_r \simeq B_r \otimes_{A_r} M_r.$$

Each ordinary tensor displayed is complete for its canonical finite-module norm and agrees boundedly with its separated completed tensor. These identifications commute with bounded maps of such quotient algebras and with maps of finite A-modules. No arbitrary complete extension is substituted for the specified free extension in the isometric coefficient statements.

Proof. Theorem 1.16.2 and Lemma 1.17.1 identify both descriptions of D_r with the same joint coefficient families, with norm $\sup_{\nu,\alpha} \|a_{\nu\alpha}\| r^\nu \rho^\alpha$. They identify multiplication first on finite polynomials and then by continuity. By Theorem 1.18.2, J is finite and closed and any finite generator map $D^t \to J$ has a uniform lifting bound for the induced norm on J. Applying the coefficient exactness of Proposition 2.14.1 to $0 \to J \to D \to B \to 0$ gives a surjection $D_r \to B_r$ with quotient norm equal to the coefficient norm, and with kernel $c_0(\mathbf{Z}^\ell, r; J)$.

Write $J = (h_1, \dots, h_t)$. For each coefficient j_ν of an element of this kernel, choose $d_{k\nu} \in D$ such that $j_\nu = \sum_k h_k d_{k\nu}$ and $\max_k \|d_{k\nu}\| \le C\|j_\nu\|$, choosing zero lifts for zero coefficients. The same bound makes each $\sum_\nu d_{k\nu} T_r^\nu$ a member of D_r. Thus the kernel is contained in JD_r; the reverse inclusion follows from the quotient map. This proves equality with the actual ideal, including its closedness, without replacing a dense image by its closure. It also proves the second isometry.

Let $M_B = B \otimes_A M$ with a canonical finite-B-module norm. Proposition 1.18.7 gives its Banach interpretation. Apply Theorem 1.18.6 with

weights r^ν, first over B and then over A, and use ordinary tensor associativity [StacksAlg, Tag 00D2, Lemma 10.12.7]. Algebraically the required comparison is

$$\begin{aligned}(M_B)_r &\simeq B_r \otimes_B (B \otimes_A M) \simeq B_r \otimes_A M \\ &\simeq B_r \otimes_{A_r} (A_r \otimes_A M) \simeq B_r \otimes_{A_r} M_r.\end{aligned}$$

These are bounded comparisons, not an assumed associativity of arbitrary completed tensors. Indeed, a finite presentation $A^u \to A^v \to M \to 0$ presents both sides by the same matrix on $B_r^u \to B_r^v$. Coefficient presentation, Theorem 1.18.5, gives its canonical quotient norm, up to the stated equivalences. The finite-module theorem then makes the algebraic comparison and its inverse bounded. The coefficient and ordinary tensor formulas agree on finite Laurent sums and module generators. Density and boundedness therefore prove compatibility with every bounded algebra map and finite-module map in the statement. This includes all quotient restriction maps used below. □

Theorem 2.32.2 (Finite rational Tate acyclicity)**.** *Let A be any norm-admissible K-affinoid algebra and let M be any finite A-module with a canonical norm. Let $(V_i)_{i=1}^n$ be any finite covering of $\mathcal{M}(A)$ by rational domains with arbitrary positive radii. For each finite intersection, let $A_{i_0\cdots i_q}$ be its actual joint rational algebra, with any admissible norm. Then the augmented complex*

$$0 \longrightarrow M \longrightarrow \prod_i A_i \otimes_A M \longrightarrow \prod_{i_0<i_1} A_{i_0 i_1} \otimes_A M \longrightarrow \cdots$$

is norm-strict exact in every degree. The all-tuples complex, with repeated indices and the usual alternating differential, has the same conclusion. Each finite-module ordinary tensor agrees boundedly with its separated completed tensor. For every fixed degree, cycles have primitives with a uniform norm bound depending on the finite cover, M, that degree and the chosen norms. No bound uniform over all covers or all degrees is asserted.

Proof. **Step 1: one strictification for the whole finite diagram.** Choose rational presentations for the V_i. Take a free extension $L = K_r$ as in Theorem 1.17.6, using the finite set consisting of the presentation radii of A and all radii in these rational presentations. Thus A_r is strict and all these rational radii belong to $\sqrt{|L^\times|}$. All products of them that arise in finite joint

intersections have the same property. Proposition 2.32.1 identifies every extended intersection algebra with its actual joint presentation over A_r.

Every L-spectral point of A_r restricts to a K-spectral point of A. The original covering therefore implies that the extended rational subsets cover $\mathcal{M}_L(A_r)$. This only uses restriction of points. By Proposition 2.26.2, each extended rational algebra has a unit-radius rational presentation over A_r, with bounded inverse comparison maps. This step uses powers of the inequalities: belonging to the divisible closure of the value group does not permit direct rescaling of the original variable. The comparison maps on all intersections commute with restrictions by the uniqueness of rational factorization.

The module M_r is finite canonical over A_r by Theorem 1.18.6. Theorem 2.22.9 now gives norm-strict exactness of the strict rational covering complex over L, in both versions. This uses no generalized Gerritzen–Grauert theorem.

Step 2: identify the extended complex, including its arrows. Write $C^{-1} = M$ and let C^q be the displayed finite product in degree $q \geq 0$, with maximum norm. Each factor is a Banach space by Proposition 1.18.7. Coefficient families commute isometrically with finite products: a supremum commutes with a finite maximum, and the finite-tail condition is preserved by finite unions. Proposition 2.32.1 consequently identifies $(C^q)_r$, by bounded inverse maps, with the corresponding term of the upper complex. These maps identify restrictions and augmentation, not just their modules. Thus the extended differential is exactly the coefficientwise extension of $d^q : C^q \to C^{q+1}$, including each alternating sign. Each fixed degree has only finitely many factors, so the comparison constants can be absorbed into norm-strictness.

Step 3: descend cycles and their norm bounds. If $z \in C^q$ is a cycle with $q \geq 0$, its constant embedding is an upper cycle. The strict exactness there supplies $w \in (C^{q-1})_r$ satisfying $d_r w = \iota(z)$ and $\|w\| \leq L_q \|z\|$ for a fixed L_q. Apply the zeroth-coefficient map of Theorem 1.16.2. Its naturality for bounded K-linear maps gives

$$d(p_0 w) = z, \qquad \|p_0 w\| \leq L_q \|z\|.$$

Thus exactness and actual bounded lifts descend without presupposing that a lower differential has closed image. Its image is now the kernel of the next bounded differential and hence is closed. At the augmentation, upper injectivity and the isometric constant embedding give injectivity of $M \to C^0$; applying its upper inverse bound to constant families gives the lower

inverse bound. This proves norm-strict exactness in every degree. The same argument works degree by degree for all tuples, since each degree still has a finite product. If the cover is empty, its covering condition forces $\mathcal{M}(A) = \varnothing$, hence $A = M = 0$ by Theorem 1.20.1. All assertions are then immediate. □

Proposition 2.32.3 (Algebraic flatness of rational localization)**.** *Every arbitrary-radius rational map $A \to B$ above is flat as a homomorphism of ordinary rings. It carries a short exact sequence of finite canonical A-modules to a norm-strict exact sequence under ordinary or separated completed tensor product. A finite rational covering gives a faithfully flat ordinary ring map $A \to \prod_i A_i$. These statements do not assert completed-tensor exactness for arbitrary Banach modules.*

Proof. Choose a single free extension strictifying A and the radii of B. Proposition 2.26.2 makes $A_r \to B_r$ boundedly isomorphic to a unit-radius rational map. Its algebraic flatness and finite-module exactness are already proved in Lemma 2.22.1; no covering theorem is required. A short exact sequence of finite A-modules is norm-strict by Theorem 1.18.3. Its coefficient extension is exact by Theorem 1.16.3, and it consists of finite canonical A_r-modules by Theorem 1.18.6. Tensoring this sequence with B_r is exact.

The resulting maps are norm-strict finite-B_r-module maps. Proposition 2.32.1 identifies this sequence with the completed free scalar extension of the original tensor sequence over B. That original sequence consists of Banach spaces by Proposition 1.18.7. Reflection in Theorem 1.16.3 therefore proves the finite-module assertion before extension. For every finitely generated ideal $I \subseteq A$, apply it to $0 \to I \to A \to A/I \to 0$. The map $I \otimes_A B \to B$ is injective. The finite-ideal criterion [StacksAlg, Tag 00HD, Lemma 10.39.5(4)] now proves ordinary flatness for all algebraic modules.

Write B for the base of the given finite rational cover, B_j for its actual row algebras, and $Q = \prod_j B_j$. The preceding single-row result makes each B_j flat over B, so Q is flat by the finite direct-sum identity. If $B = 0$, the assertion is immediate. For any maximal ideal $\mathfrak{m}$ of B, its actual nonzero quotient is affinoid by Corollary 1.18.4. Nonemptiness in Theorem 1.20.1 gives a bounded point of this quotient, hence a point of the domain that kills $\mathfrak{m}$. Some row in the cover over B contains that point. By Theorem 2.31.1 the point lifts to its actual row algebra, so $Q/\mathfrak{m}Q \neq 0$. For a nonzero B-module choose a cyclic submodule B/J with J proper and choose $\mathfrak{m} \supseteq J$.

The quotient Q/JQ surjects onto $Q/\mathfrak{m}Q$ and injects into the tensor of the given module by flatness. This proves that Q is faithfully flat over B. □

Corollary 2.32.4 (Finite-module sheaves on the finite rational site)**.** *Let $X = \mathcal{M}(A)$. Use as objects the rational subsets of X with arbitrary positive radii, and as coverings their finite set-theoretic rational coverings. For a finite A-module M, the assignment*

$$\widetilde{M}(V) = A_V \otimes_A M$$

with the unique bounded restrictions is a sheaf on this site. The assignment $M \mapsto \widetilde{M}$ is exact, $\Gamma(X, \widetilde{M}) = M$, and $\widetilde{A}$ is a sheaf of rings. This statement specifies a finite rational site; it does not identify it with the entire analytic G-site or assert a new coherence equivalence.

Proof. The identity object is rational. Proposition 2.31.3 gives rational intersections, and finite covers are preserved by pullback through them and by finite composition. These facts define the stated pretopology. Theorem 2.31.2 defines restrictions independently of the chosen presentation, with the required composition identities. If $V_i \subseteq V$ are rational over A, their defining tuples remain unit-generated in A_V. Their rational presentations over A_V represent V_i by Theorem 2.31.1. The identity and restriction maps, together with uniqueness, identify these algebras boundedly with A_{V_i}. Equivalently the rational pullback algebra is $A_V \widehat{\otimes}_A A_{V_i} \simeq A_{V_i}$.

For a finite cover of V, apply Theorem 2.32.2 over the affinoid A_V to its finite module $A_V \otimes_A M$. Ordinary tensor associativity identifies the degree-zero and degree-one terms with the claimed restrictions. Exactness at these degrees is precisely the sheaf equalizer condition. The empty object has zero algebra, so its sheaf value is zero. Proposition 2.32.3 proves exactness of the functor on each object, hence exactness as a functor to sheaves. At $V = X$, the identity presentation gives $A_X = A$, so its sections are M. Taking $M = A$, the restrictions are algebra homomorphisms and the equalizer respects multiplication, giving the asserted ring sheaf. □

Generalized Gerritzen–Grauert, Theorem 2.36.6, provides the rational refinement used to pass from this result to the affinoid-cover theorem 2.38.1. Effective module descent then gives Kiehl equivalence on the affinoid-domain site, and Theorem 2.39.6 carries it to the full analytic G-site.

2.33 Completed tensors, scalar extension and fibres

The algebra of an analytic fibre is obtained by changing coefficients to a completed residue field. We first establish the norm estimates needed for this construction. Throughout this section, K is complete for its specified real-valued non-Archimedean absolute value, which may be trivial. Vector-space norms are homogeneous for that absolute value.

2.33.1 Finite-dimensional projections and tensor norms

For normed K-spaces E, F, the projective tensor seminorm is

$$\|z\|_\pi = \inf_{z=\sum_{j=1}^m e_j\otimes f_j} \max_{1\le j\le m} \|e_j\|\|f_j\|.$$

Its separated completion is denoted $E\widehat{\otimes}_K F$. The next argument is the finite-dimensional form of Gruson's tensor theorem; compare [GrP74, §5, Lemma 5.2].

Lemma 2.33.1 (Almost contractive finite-dimensional projections)**.** *Let V be a finite-dimensional normed K-vector space and $W \subset V$ a subspace. For every real $c > 1$ there is a K-linear map $P : V \to W$ with $P|_W = 1_W$ and $\|P(v)\| \le c\|v\|$ for every $v \in V$.*

Proof. First suppose $\dim(V/W) = 1$. By the finite-dimensional assertion of Lemma 1.12.1, W is complete and therefore closed in V. Choose $v_0 \notin W$ and put $\delta = \inf_{w\in W} \|v_0 - w\| > 0$. There is $w_0 \in W$ such that $v = v_0 - w_0$ satisfies $\|v\| < c\delta$. Define $P(w + av) = w$. If $a \neq 0$, homogeneity and the definition of δ give $|a|\delta \le \|w + av\|$. The same estimate is immediate for $a = 0$. Thus

$$\|P(w + av)\| \le \max\{\|w + av\|, |a|\|v\|\} \le c\|w + av\|.$$

For codimension $d > 1$, choose a flag $W = V_0 \subset V_1 \subset \cdots \subset V_d = V$ with successive codimension one. Compose the corresponding projections with bounds $c^{1/d}$. For $W = V$ use the identity; the zero-dimensional cases are immediate. □

Theorem 2.33.2 (Isometric tensor embeddings)**.** *For any normed K-spaces E, F, the projective seminorm on $E \otimes_K F$ is a norm and satisfies $\|e \otimes f\|_\pi = \|e\|\|f\|$. If $E \hookrightarrow E'$ and $F \hookrightarrow F'$ are isometric linear embeddings, then*

$$E \otimes_K F \longrightarrow E' \otimes_K F', \qquad E \widehat{\otimes}_K F \longrightarrow E' \widehat{\otimes}_K F'$$

are isometric embeddings; the image of the second map is closed.

Proof. Algebraic injectivity follows by extending vector-space bases. We first compare seminorms. Fix $z \in E \otimes_K F$ and finite-dimensional subspaces $U \subset E$, $V \subset F$ containing the factors of one expression for z. An arbitrary competing expression

$$z = \sum_{j=1}^{m} e'_j \otimes f'_j \quad \text{in } E' \otimes_K F'.$$

Let $U' \subset E'$ be spanned by U and the e'_j, and let $V' \subset F'$ be spanned by V and the f'_j. The equality holds already in $U' \otimes_K V'$, by algebraic injectivity. For a fixed $c > 1$, Lemma 2.33.1 gives projections $P : U' \to U$, $Q : V' \to V$ of norm at most c. Applying $P \otimes Q$ yields

$$z = \sum_j P(e'_j) \otimes Q(f'_j), \qquad \|z\|_{\pi,E,F} \le c^2 \max_j \|e'_j\|\|f'_j\|.$$

Taking the infimum over competing expressions and then letting $c \downarrow 1$ gives $\|z\|_{\pi,E,F} \le \|z\|_{\pi,E',F'}$. The reverse inequality follows from the inclusion of the sets of expressions. Only projections between the displayed finite-dimensional spaces have been used.

On a finite-dimensional tensor $U \otimes_K V$, choose bases and their coordinate functionals. These functionals are bounded by Lemma 1.12.1. If the (i,j)-coordinate of z is nonzero, their tensor gives

$$0 < |(\lambda_i \otimes \mu_j)(z)| \le \|\lambda_i\|_{\mathrm{op}}\|\mu_j\|_{\mathrm{op}}\|z\|_\pi.$$

Thus the finite-dimensional projective seminorm is definite. Every nonzero tensor lies in such a subspace, whose inclusion is isometric by the preceding argument; this proves definiteness in general. For one-dimensional factors the norm is the product norm, and the same isometric inclusion proves the pure-tensor formula. An isometry extends to an isometry of metric completions. Its image is complete, hence closed. □

Theorem 2.33.3 (Exactness of completed tensors over a field)**.** *Let $0 \to E \to F \to G \to 0$ be an exact sequence of Banach K-spaces, with E carrying the induced norm of a closed subspace of F and G the quotient norm. For every Banach K-space H, the sequence*

$$0 \longrightarrow E\widehat{\otimes}_K H \longrightarrow F\widehat{\otimes}_K H \longrightarrow G\widehat{\otimes}_K H \longrightarrow 0$$

is exact, its injection is isometric, and its surjection induces an isometry on the quotient. In particular completed tensoring preserves norm-strict short exact sequences with any equivalent Banach norms.

Proof. Algebraic tensoring over K is exact, as follows by choosing an algebraic linear splitting of $F \to G$. On the algebraic quotient $(F \otimes_K H)/(E \otimes_K H)$, compare its quotient projective norm with the projective norm on $G \otimes_K H$. Contractivity of $F \to G$ gives one inequality. For the other, take an expression $z = \sum_j g_j \otimes h_j$ and $c > 1$. Lift each nonzero g_j to $f_j \in F$ with $\|f_j\| \le c\|g_j\|$, and lift zero by zero. The tensor $\sum_j f_j \otimes h_j$ represents z and has norm at most $c \max_j \|g_j\|\|h_j\|$. Taking both infima and then $c \downarrow 1$ proves equality of the two quotient norms.

By Theorem 2.33.2, $E\widehat{\otimes}_K H$ embeds isometrically in $F\widehat{\otimes}_K H$. Its image is exactly the closure of $E \otimes_K H$. Quotienting by this closed image gives a Banach space by Lemma 1.11.1. The image of the algebraic quotient is dense and has the quotient norm just calculated: for an algebraic tensor, taking distance to a subspace or to its closure gives the same infimum. Consequently this Banach quotient is the completion of $G \otimes_K H$ for its projective norm, hence is $G\widehat{\otimes}_K H$. This proves all three assertions. Uniform inequalities between equivalent norms pass to projective norms, quotient norms and completions, giving the last statement. □

Lemma 2.33.4 (Scalar extension of weighted coefficient spaces)**.** *Let I be a countable set, let $w_i > 0$, and let E be a Banach K-space. There is a natural isometric linear isomorphism*

$$\mathcal{C}_w(I,K)\widehat{\otimes}_K E \simeq \mathcal{C}_w(I,E), \qquad (a_i)_i \otimes e \longmapsto (a_i e)_i.$$

For a complete valued extension L/K, this gives the isometric algebra isomorphism $K\{r^{-1}T\}\widehat{\otimes}_K L \simeq L\{r^{-1}T\}$. Also $E \to E\widehat{\otimes}_K L$, $e \mapsto e \otimes 1$, is isometric.

Proof. The displayed bilinear map is contractive for the weighted supremum norm. Write ϵ_i for the coordinate vector of norm w_i. For a finitely supported tensor $z = \sum_i \epsilon_i \otimes e_i$, its displayed expression gives $\|z\|_\pi \le \max_i w_i\|e_i\|$. Conversely the ith coordinate functional on $\mathcal{C}_w(I, K)$ has norm w_i^{-1}. Applying its tensor with 1_E to any representation gives $w_i\|e_i\| \le \|z\|_\pi$. Hence equality holds. Finitely supported vectors are dense on the right. They are also dense on the left: truncate each of the finitely many coefficient families in an algebraic tensor, then use density of algebraic tensors in the completion. This proves the isometry and surjectivity. For Tate algebras it preserves multiplication first on polynomials and then by continuity. Finally, the pure-tensor formula of Theorem 2.33.2 gives $\|e \otimes 1\| = \|e\|$. □

2.33.2 Generalized affinoids over complete extensions

Proposition 2.33.5 (Scalar extension and actual relation ideals)**.** *Let L/K be a complete valued extension, and let $A = K\{r^{-1}T\}/I$ be a generalized K-affinoid presentation with its quotient norm. Then*

$$A_L := A\widehat{\otimes}_K L \simeq L\{r^{-1}T\}/I\,L\{r^{-1}T\}$$

is an isometric algebra isomorphism for the displayed quotient norm. The ideal on the right is the actual extended ideal, which is closed. In particular A_L is a generalized L-affinoid algebra, and $A \to A_L$ is isometric. With any equivalent admissible norm the same maps are bounded isomorphisms with bounded inverses.

Proof. Put $D = K\{r^{-1}T\}$. Theorems 1.18.2 and 1.18.3 make I a closed finite D-module. Choose generators $f_1, \dots, f_s$; the surjection $D^s \to I$ is norm-strict for the induced norm on I. Applying Theorem 2.33.3 to its kernel sequence and to $0 \to I \to D \to A \to 0$ gives

$$(D\widehat{\otimes}_K L)^s \twoheadrightarrow I\widehat{\otimes}_K L \hookrightarrow D\widehat{\otimes}_K L \longrightarrow A\widehat{\otimes}_K L \longrightarrow 0.$$

The injection is isometric and closed; the last map is a quotient isometry. Under Lemma 2.33.4, the first map is the row $(f_1, \dots, f_s)$ over $L\{r^{-1}T\}$. Its image is therefore exactly the algebraic ideal generated by the f_i, not just its closure. The last quotient is the asserted presentation, and the algebra operations agree on dense polynomial tensors. The scalar embedding is isometric by the same lemma. Equivalent norms give the final assertion by taking the corresponding projective and quotient infima. □

Theorem 2.33.6 (Finite modules and faithful scalar extension)**.** *For every complete valued extension L/K and generalized K-affinoid A, the ring map $A \to A_L$ is faithfully flat. For every finite canonical A-module M there are natural bounded isomorphisms*

$$A_L \otimes_A M \simeq A_L \widehat{\otimes}_A M \simeq M \widehat{\otimes}_K L.$$

Their inverses are bounded. For the quotient norm from a fixed finite free presentation of M, and the corresponding quotient norm after extension, the comparisons are isometric. They identify short exact sequences of finite modules with norm-strict short exact sequences after either ordinary or completed scalar extension.

Proof. Choose a finite presentation $A^s \xrightarrow{D} A^m \xrightarrow{q} M \to 0$, and give M its quotient norm. Its relation module $R = \ker q$ is closed and finite; both $A^s \to R$ and q are norm-strict by Theorem 1.18.3. Apply Theorem 2.33.3 to these two kernel sequences. Finite coordinate sums commute isometrically with completed scalar extension, either by their coordinate projections or by Lemma 2.33.4 with finite I. We obtain

$$A_L^s \xrightarrow{D_L} A_L^m \longrightarrow M \widehat{\otimes}_K L \longrightarrow 0, \qquad M \widehat{\otimes}_K L \simeq A_L^m / D_L A_L^s$$

with closed relation image and the indicated quotient norm. Ordinary tensor right exactness, [StacksAlg, Tag 00DF], identifies this algebraic quotient with $A_L \otimes_A M$. The map is the natural one on pure tensors. Proposition 1.18.7, with target A_L, identifies its ordinary and completed tensors over A, including the isometry for the chosen norms. Naturality follows on generators and then by continuity. Theorem 2.33.3 and the finite-module theorem give the exact-sequence assertion.

For a finitely generated ideal $J \subset A$, the algebraic map $J \otimes_A A_L \to A_L$ is now identified with the injective map $J \widehat{\otimes}_K L \to A \widehat{\otimes}_K L$. The induced norm on J is canonical, so Theorem 2.33.2 applies. The finitely generated ideal criterion [StacksAlg, Tag 00HD, Lemma 10.39.5(4)] proves flatness on all algebraic A-modules. If J is proper, the isometric map

$$A/J \hookrightarrow (A/J) \widehat{\otimes}_K L \simeq A_L / J A_L$$

shows that the quotient on the right is nonzero. In particular the fibre at every maximal ideal is nonzero. The faithful-flatness criterion [StacksAlg, Tag 00HP, Lemma 10.39.15] completes the proof. For the zero algebra the module category is zero and the same assertions hold directly. □

Lemma 2.33.7 (Generalized relative presentations)**.** *Every bounded homomorphism $A \to B$ of generalized K-affinoid algebras extends to a bounded norm-strict surjection*

$$A\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \twoheadrightarrow B$$

for suitable positive radii r_i. Its kernel is an actual closed ideal, and its quotient norm is equivalent to the given admissible norm on B.

Proof. Choose an admissible presentation $K\{r^{-1}T\} \twoheadrightarrow B$, and temporarily use its quotient norm on B and a presentation norm on A. The variable images b_i satisfy $\|b_i^m\| \le r_i^m$. The given bounded map on coefficients therefore extends by Proposition 1.9.3 to the displayed bounded map. It is surjective because it extends the original presentation. For a fixed $c > 1$, every nonzero $b \in B$ has a lift in $K\{r^{-1}T\}$ of norm at most $c\|b\|$. Its image in $A\{r^{-1}T\}$ has the same norm, since the coefficient inclusion $K \hookrightarrow A$ is isometric. These lifts and boundedness of the map compare its quotient norm with the norm of B in both directions. Proposition 1.17.3 makes the coefficient algebra affinoid, and Theorem 1.18.2 makes its kernel ideal closed. Equivalent initial norms only change the constants. Zero algebras are immediate. □

Proposition 2.33.8 (Pushouts with complete-extension affinoids)**.** *Let $A \to B$ be a bounded map of generalized K-affinoid algebras. Let L/K be a complete valued extension, let C be a generalized L-affinoid algebra, and let $A \to C$ be a bounded K-homomorphism. Then the separated completed projective tensor algebra $B\widehat{\otimes}_A C$ is a generalized L-affinoid algebra. If*

$$B = A\{r^{-1}T\}/I$$

is a norm-admissible relative presentation, there is a natural bounded algebra isomorphism, with bounded inverse,

$$B\widehat{\otimes}_A C \simeq C\{r^{-1}T\}/I\,C\{r^{-1}T\}.$$

The extended ideal is actual and closed. For the displayed quotient norm on B, the coefficient norm on $C\{r^{-1}T\}$, and a contractive map $A \to C$, this isomorphism is isometric. The tensor algebra is the pushout for compatible bounded homomorphisms to commutative Banach K-algebras. Zero algebras are allowed.

Proof. Lemma 2.33.7 supplies the relative presentation. Put $G = B\widehat{\otimes}_A C$ and $H = C\{r^{-1}T\}/I\,C\{r^{-1}T\}$. By Proposition 1.17.3 and Corollary 1.18.4, applied over L, H is a complete actual quotient and is L-affinoid. On the algebraic tensor, multiplying finite tensor expressions proves submultiplicativity of the projective seminorm. Its zero seminorm subspace is an ideal, so separation and completion make G a Banach algebra.

The natural maps $B \to H$, $C \to H$ are bounded and agree on A. Their balanced product extends to a bounded map $G \to H$. In the other direction, the elements $\bar{T}_i \otimes 1$ in G have their mth powers bounded by r_i^m when B has its relative quotient norm. Substitution, with coefficients in C, gives a bounded map $C\{r^{-1}T\} \to G$. The two maps from $A\{r^{-1}T\}$ to G agree on coefficients and variables and hence, by polynomial density, everywhere. Therefore the substitution kills I and its extended ideal. It descends boundedly to $H \to G$ by taking quotient infima. The composites fix every coefficient and variable in H and every pure tensor in G. Density makes them identity maps. When $A \to C$ is contractive, every map just constructed is contractive, giving the isometry. Equivalent admissible norms give the stated bounded comparisons.

Finally, compatible bounded maps $B, C \to D$ to a commutative Banach K-algebra have bounded balanced product $(b, c) \mapsto bc$. On any finite tensor expression its norm is bounded by the product of their operator bounds times the projective bound. Taking the infimum, separating and completing gives a unique bounded map $G \to D$. Uniqueness follows from the dense algebraic tensor image. □

Corollary 2.33.9 (Iterated base change)**.** *For compatible bounded affinoid maps $A \to B$, $A \to C \to D$, allowing complete coefficient-field extensions for C, D, there is a natural bounded isomorphism with bounded inverse*

$$(B\widehat{\otimes}_A C)\widehat{\otimes}_C D \simeq B\widehat{\otimes}_A D.$$

For complete valued extensions $K \subset L \subset L'$, this includes $(A_L)_{L'} \simeq A_{L'}$. All comparison maps commute with the structural homomorphisms.

Proof. Both algebras represent compatible bounded homomorphisms from B and D agreeing on A, by Proposition 2.33.8. Applying that universal property in both directions gives inverse maps; their composites are identities by uniqueness. The same argument applies to the scalar-extension pushouts. □

2.33.3 Residue fields and pointwise fibres

Recall that $\mathcal{H}(x)$ is the completion of the valued fraction field of $A/\ker x$. This completion is a field: a Cauchy sequence with nonzero limit has eventually constant positive absolute value, and

$$|a_n^{-1} - a_m^{-1}| = \frac{|a_n - a_m|}{|a_n|\,|a_m|}$$

shows that its inverse sequence is Cauchy. The field operations and multiplicativity pass to limits. Proposition 1.6.2 therefore applies to every bounded evaluation in a complete valued target.

Proposition 2.33.10 (Rational restriction preserves completed residue fields)**.** *Let $U \subset \mathcal{M}(A)$ be a generalized rational domain, represented by $A \to A_U$. At every $x \in U$, the induced isometric embedding*

$$\mathcal{H}_A(x) \longrightarrow \mathcal{H}_{A_U}(x)$$

is an isomorphism.

Proof. Write $A_U = A\{\rho^{-1}T\}/(gT_i - f_i)$ with $(g, f_1, \dots, f_s) = A$. At $x \in U$ the denominator is nonzero, and $|f_i(x)/g(x)| \le \rho_i$. Evaluation at these ratios gives a bounded map $A_U \to \mathcal{H}_A(x)$ by Proposition 1.9.3 and quotient infima. Its seminorm is the unique rational point over x by Theorem 2.31.1. Proposition 1.6.2 gives an isometric map $\mathcal{H}_{A_U}(x) \to \mathcal{H}_A(x)$. Its composite with the map in the statement fixes the dense field $\mathrm{Frac}(A/\ker x)$ and is therefore the identity. The reverse composite fixes the image of A_U: this follows first on the coefficients and the forced ratios f_i/g, then on their convergent power series. It consequently fixes the dense fraction field of the evaluated A_U, hence is the identity. □

Theorem 2.33.11 (Affinoid fibres at all Berkovich points)**.** *Let $\phi : A \to B$ be a bounded homomorphism of generalized K-affinoid algebras, with spectral map $f : \mathcal{M}(B) \to \mathcal{M}(A)$. For $x \in \mathcal{M}(A)$ put $F = \mathcal{H}(x)$ and*

$$B_x = B\widehat{\otimes}_A F.$$

Then B_x is a generalized F-affinoid algebra. Restriction induces a homeomorphism

$$\mathcal{M}_F(B_x) \xrightarrow{\sim} f^{-1}(x).$$

For the point z corresponding to $y \in f^{-1}(x)$, the structural map gives a canonical isometric isomorphism $\mathcal{H}_B(y) \simeq \mathcal{H}_{B_x}(z)$ compatible with the embedding $F \hookrightarrow \mathcal{H}_B(y)$.

Proof. The map $A \to F$ is bounded, and F is affinoid over itself. Proposition 2.33.8 proves the affinoid assertion; in a relative presentation $B = A\{r^{-1}T\}/I$, it gives $B_x = F\{r^{-1}T\}/I_F$.

For $y \in f^{-1}(x)$, Proposition 1.6.2 supplies an isometric embedding $j : F \hookrightarrow \mathcal{H}_B(y)$ agreeing with $A \to B \to \mathcal{H}_B(y)$. The compatible maps from B and F extend uniquely to a bounded homomorphism $\rho_y : B_x \to \mathcal{H}_B(y)$. Its absolute value defines a point $z_y \in \mathcal{M}_F(B_x)$ restricting to y. Conversely, a point $z \in \mathcal{M}_F(B_x)$ restricts to a point $y \in \mathcal{M}(B)$ with $f(y) = x$, since its values on the factor F are the prescribed absolute value of F.

To prove uniqueness, let $i : \mathcal{H}_B(y) \hookrightarrow \mathcal{H}_{B_x}(z)$ be the residue embedding induced by $B \to B_x$. The structural embedding $F \to \mathcal{H}_{B_x}(z)$ and $i \circ j$ agree on A, hence on $\mathrm{Frac}(A/\ker x)$, and then on F by density. Consequently evaluation at z equals $i \circ \rho_y$ on $B \otimes_A F$ and, by continuity, on B_x. Its seminorm is therefore z_y. Moreover the image $i(\mathcal{H}_B(y))$ is a closed field containing the evaluated image of B_x. The closure of its fraction field is $\mathcal{H}_{B_x}(z)$; hence i is surjective. This proves the residue-field assertion as well as bijectivity.

Restriction is continuous, since evaluation on $b \in B$ is evaluation on $b \otimes 1$. The source is compact and the target is a subspace of the Hausdorff space $\mathcal{M}(B)$ by Proposition 2.24.1. A continuous bijection from a compact space to a Hausdorff space is a homeomorphism: closed subsets of the source are compact, and their images are compact and therefore closed. The argument includes the empty fibre, when $B_x = 0$. □

Proposition 2.33.12 (Rational domains under complete-extension base change)**.** *Let $U \subset \mathcal{M}(A)$ be a generalized rational domain, let C be a generalized affinoid over a complete extension L/K, and let $\phi : A \to C$ be bounded. Then*

$$A_U \widehat{\otimes}_A C \simeq C_{\phi^{-1}(U)}$$

as bounded C-algebras, where the domain on the right is the inverse image under the spectral map. The comparison commutes with restriction to nested

rational domains and finite intersections. Finite rational covers pull back to finite rational covers. In particular, for $\pi : \mathcal{M}(A_L) \to \mathcal{M}(A)$,

$$(A_U)_L \simeq (A_L)_{\pi^{-1}(U)}.$$

Proof. For the presentation $A_U = A\{\rho^{-1}T\}/(gT_i - f_i)$, Proposition 2.33.8 gives the actual algebra

$$C\{\rho^{-1}T\}/\big(\phi(g)T_i - \phi(f_i)\big).$$

The images of $g, f_1, \ldots, f_s$ still generate the unit ideal. By Theorem 2.31.1, its spectrum is precisely the locus $|\phi(f_i)| \le \rho_i|\phi(g)|$, namely the inverse image of U. The uniqueness in Theorem 2.31.2 gives compatibility with restriction. The joint presentations of Proposition 2.31.3 give the same statement for intersections. The pointwise inverse images of a cover cover the target. For $C = A_L$, both $(A_U)_L$ and $A_U\widehat{\otimes}_A A_L$ are the pushout of A_U and L over K; uniqueness of the bounded pushout identifies them and proves the final formula. □

Theorem 2.33.13 (Surjectivity after an arbitrary complete extension)**.** *For every generalized* K*-affinoid* A *and every complete valued extension* L/K*, the natural map*

$$\pi : \mathcal{M}_L(A_L) \longrightarrow \mathcal{M}_K(A)$$

is a continuous closed surjection.

Proof. Continuity follows from the evaluation topology, and closedness follows from compactness of the source and the Hausdorff property of the target, Proposition 2.24.1. Fix $x \in \mathcal{M}(A)$. Consider all the closed rational domains

$$U^+(a,r) = \{v : |a(v)| \le r\} \qquad (r > |a(x)|),$$
$$U^-(a,r) = \{v : |a(v)| \ge r\} \qquad (0 < r < |a(x)|).$$

Every finite intersection U of these domains is rational by Proposition 2.31.3 and contains x. Its algebra A_U is nonzero. Lemma 2.33.4 gives an isometric embedding $A_U \hookrightarrow (A_U)_L$, so the latter is nonzero. It is L-affinoid by Proposition 2.33.5, and hence has a point by Theorem 1.20.1. By Proposition 2.33.12, this supplies a point of $\pi^{-1}(U)$. Thus the inverse images of the displayed domains have the finite intersection property in the compact space $\mathcal{M}_L(A_L)$.

Choose a point y in their intersection. For every $a \in A$, all upper inequalities give $|a(\pi(y))| \leq |a(x)|$. If $|a(x)| > 0$, all lower inequalities give the reverse inequality; if $|a(x)| = 0$, the upper inequalities already force equality. Hence $\pi(y) = x$. This proves surjectivity. If $A = 0$, both spectra are empty and the assertion is immediate. □

Proposition 2.33.14 (Affinoid products and pairs of points)**.** *For bounded maps $A \to B, A \to C$ of generalized K-affinoid algebras, $\mathcal{M}(B\widehat{\otimes}_A C)$ is their fibre product in the affinoid category. The natural map to the ordinary topological fibre product*

$$\mathcal{M}(B\widehat{\otimes}_A C) \longrightarrow \mathcal{M}(B) \times^{\mathrm{top}}_{\mathcal{M}(A)} \mathcal{M}(C)$$

is continuous, closed and surjective.

Proof. The categorical assertion is the opposite of the bounded pushout property in Proposition 2.33.8. Restriction gives the displayed continuous map. Its target is a closed subspace of the compact Hausdorff product $\mathcal{M}(B)\times\mathcal{M}(C)$: it is the inverse image of the diagonal of $\mathcal{M}(A)$. Compactness of the source therefore proves closedness.

Take $y \in \mathcal{M}(B)$ and $z \in \mathcal{M}(C)$ mapping to $x \in \mathcal{M}(A)$, and put $F = \mathcal{H}(x)$, $E = \mathcal{H}(y)$. Theorem 2.33.11 identifies z with a point of the F-affinoid $C_x = C\widehat{\otimes}_A F$. The isometric embedding $F \hookrightarrow E$ makes E/F a complete valued extension. By Theorem 2.33.13, the point z lifts to a point w of $C_x\widehat{\otimes}_F E$. In $\mathcal{H}(w)$ the maps from C and $B \to E$ agree on A. Their bounded product therefore gives a point of $\mathcal{M}(B\widehat{\otimes}_A C)$. Its restrictions are z and y: the first follows from the choice of w, and the second because evaluation on E is its prescribed absolute value. This proves surjectivity. □

2.33.4 Ordinary stalks and domain flatness

Corollary 2.33.15 (Completed residue fields on affinoid domains)**.** *If $A \to B$ represents an affinoid domain $U \subset \mathcal{M}(A)$, then at every $x \in U$ the canonical map $\mathcal{H}_A(x) \to \mathcal{H}_B(x)$ is an isometric isomorphism.*

Proof. Theorem 2.36.6 gives a finite cover of U by domains rational both over A and over B. Choose a member containing x, with algebra C. Both maps $\mathcal{H}_A(x) \to \mathcal{H}_C(x)$ and $\mathcal{H}_B(x) \to \mathcal{H}_C(x)$ are isomorphisms by Proposition 2.33.10. Their triangle commutes on the evaluated elements of A and therefore on its completed fraction field, which proves the assertion. □

Proposition 2.33.16 (Ordinary affinoid stalks)**.** *Let $X = \mathcal{M}(A)$, and restrict $\mathcal{O}_{X_G}$ to ordinary open subsets to obtain $\mathcal{O}_X$. For every $x \in X$,*

$$\mathcal{O}_{X,x} \simeq \varinjlim_{\substack{V \text{ rational in } X \\ x \in \mathrm{int}_{\mathrm{top}}(V)}} A_V,$$

where the system is directed by shrinking. Evaluation defines a local ring structure: its maximal ideal consists of germs vanishing at x. Its residue field $\kappa_X(x)$ embeds isometrically in $\mathcal{H}_A(x)$ and has completion $\mathcal{H}_A(x)$.

Proof. Proposition 2.39.3 constructs, inside every ordinary neighborhood of x, a rational domain containing an ordinary neighborhood of x. Finite intersections have the same property. Theorem 2.39.4 identifies the sections on these domains with A_V. A section on an ordinary open neighborhood restricts to one of them. Conversely, a section on such a V restricts to its ordinary interior near x. These restrictions induce mutually inverse maps on germs, proving the displayed formula.

By Proposition 2.33.10, the evaluations on all A_V take values compatibly in $\mathcal{H}_A(x)$, so they define a ring map $\mathcal{O}_{X,x} \to \mathcal{H}_A(x)$. If a germ is represented by $b \in A_V$ with $b(x) \neq 0$, choose $0 < c < |b(x)|$. The Laurent domain $|b| \geq c$ in V contains a neighborhood of x in X and has an inverse for b in its algebra. Hence the germ is a unit. A germ vanishing at x cannot be a unit, since evaluation is a unital map into a field. The kernel of evaluation is consequently exactly the unique maximal ideal.

It follows that $\kappa_X(x)$ is identified with the evaluated image, a subfield of $\mathcal{H}_A(x)$. It contains $\mathrm{Frac}(A/\ker x)$: each denominator nonzero at x has the local inverse just constructed. This fraction field is dense in $\mathcal{H}_A(x)$ by definition. The intermediate field $\kappa_X(x)$ is therefore dense as well, proving the completion assertion. □

Theorem 2.33.17 (Domain flatness on ordinary local rings)**.** *Let $U \subset X = \mathcal{M}(A)$ be an affinoid domain. For every $x \in U$, the local homomorphism*

$$\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{U,x}$$

is flat. This assertion remains true after every bounded affinoid base change over every complete valued extension of K.

Proof. Write $R = \mathcal{O}_{X,x}$ and $S = \mathcal{O}_{U,x}$. Fix a rational neighborhood V of x in X. Affinoid neighborhoods W of x in U contained in V form a

cofinal system: intersect an ordinary neighborhood with V, then choose a rational neighborhood inside the affinoid space U, as in Proposition 2.39.3. By transitivity and restriction of affinoid domains, each such W is an affinoid domain of V. Proposition 2.38.2 makes $A_V \to A_W$ flat.

We verify flatness of $S = \varinjlim_W A_W$ over this fixed A_V by the finite-ideal criterion [StacksAlg, Tag 00HD]. For a finitely generated ideal $J \subset A_V$, every element of $J \otimes_{A_V} S$ is represented by a finite sum at some stage A_W. If it maps to zero in S, its image vanishes at a smaller neighborhood W'. Flatness of $A_{W'}$ implies that its representative in $J \otimes_{A_V} A_{W'}$ is zero. Thus $J \otimes_{A_V} S \to S$ is injective. This also spells out the exactness of the filtered passage being used.

Now take a finitely generated ideal $I \subset R$ and $z \in I \otimes_R S$ mapping to zero. Choose generators $a_1, \dots, a_m$ of I and write $z = \sum_i a_i \otimes s_i$ with $s_i \in S$. By Proposition 2.33.16, the a_i lift to elements $\widetilde{a}_i$ of one A_V. Put $J = (\widetilde{a}_1, \dots, \widetilde{a}_m)$. Then $z_V = \sum_i \widetilde{a}_i \otimes s_i \in J \otimes_{A_V} S$ maps to z and has zero image in S. We have just proved S flat over A_V, so $z_V = 0$, and hence $z = 0$. Again applying [StacksAlg, Tag 00HD] proves R-flatness. The map is local because both maximal ideals are defined by vanishing at x, and their residue evaluations are compatible by Corollary 2.33.15.

For a bounded map $A \to C$ with C affinoid over a complete extension, Theorem 2.36.9 identifies the inverse image of U with an affinoid domain of $\mathcal{M}(C)$. Apply the same proof over the complete coefficient field of C. □

2.33.5 Finite morphisms and relative interior

Theorem 2.33.18 (Finite algebras with their canonical norm)**.** *Let A be a generalized K-affinoid algebra. Every commutative finite A-algebra D admits a complete submultiplicative norm, with $\|1\| = 1$ when $D \neq 0$, equivalent to its canonical finite-A-module norm. With this norm it is a generalized K-affinoid algebra and $A \to D$ is bounded. The resulting Banach structure is independent, up to bounded isomorphism, of the chosen canonical module norm. Every A-linear map between finite algebras with these structures is bounded. Moreover*

$$\operatorname{Int}\big(\mathcal{M}(D)/\mathcal{M}(A)\big) = \mathcal{M}(D).$$

Proof. Give A a presentation norm, and D a quotient norm from a finite free A-module. For $d \in D$, multiplication is an A-linear map $L_d : D \to D$ and

is bounded by Theorem 1.18.3. Choose module generators $e_1, \dots, e_m$. Controlled lifts of $d = \sum_i a_i e_i$ and contractivity of the A-action give a constant C independent of d with

$$\|L_d\|_{\mathrm{op}} \le C\|d\|, \qquad \|d\| = \|L_d(1)\| \le \|L_d\|_{\mathrm{op}}\|1\|.$$

Thus $\|d\|_* = \|L_d\|_{\mathrm{op}}$ is equivalent to the complete module norm. It is homogeneous and submultiplicative, since $L_{dd'} = L_d L_{d'}$, and $\|1\|_* = 1$. The map $A \to D$ is contractive for this norm. Treat the zero algebra separately with its zero norm.

Choose real radii $r_i > \|e_i\|_*$. Substitution gives a bounded surjection $A\{r^{-1}T\} \to D$, $T_i \mapsto e_i$. It is norm-strict: a controlled module lift $d = \sum_i a_i e_i$ is represented by the linear polynomial $\sum_i a_i T_i$, whose coefficient norm is at most $(\max_i r_i) \max_i \|a_i\|$. Its kernel is closed, and the coefficient algebra is affinoid by Proposition 1.17.3. Corollary 1.18.4 now realizes D as the required affinoid quotient with an equivalent norm.

Every point $y \in \mathcal{M}(D)$ satisfies $|e_i(y)| \le \|e_i\|_* < r_i$, by Lemma 1.6.1. The same relative presentation therefore puts every y in the interior of Definition 2.7.1. Equivalence of canonical module norms and boundedness of all linear maps follow from Theorem 1.18.3. □

Proposition 2.33.19 (General finite base change)**.** *Let $A \to B$ be a bounded map of generalized K-affinoid algebras, where B is finite over A and its given norm is equivalent to its canonical finite-A-module norm. For every bounded map $A \to C$ to an affinoid over a complete extension L/K, the natural map*

$$B \otimes_A C \longrightarrow B\widehat{\otimes}_A C$$

is a bounded algebra isomorphism with bounded inverse when the source has its canonical finite-C-module norm. The result is finite and norm-admissible over C.

Proof. Apply Proposition 1.18.7 to the finite canonical A-module B and the target C. It identifies the projective tensor norm with the complete canonical finite-C-module norm up to uniform equivalence. The separated completion consequently changes no elements. Multiplication agrees on the algebraic tensor and its completion by continuity. Proposition 2.33.8 gives the affinoid structure, and the norm comparison just obtained gives finite admissibility. □

Example 2.33.20 (Algebraic finiteness and the specified norm)**.** Suppose the absolute value of K is trivial and $0 < s < r < 1$. As rings,

$$A_r = K\{r^{-1}T\} = K[[T]] = K\{s^{-1}T\} = A_s.$$

The identity $A_r \to A_s$ is a bounded algebraic isomorphism, hence algebraically finite, but it is not norm-strict. Indeed, every formal power series converges for each of these radii, and for a nonzero series of order n its two norms are r^n and s^n. The identity is contractive, whereas an inverse bound would require $(r/s)^n$ to remain bounded. Thus the finite-module norm in Proposition 2.33.19 is a genuine condition when the valuation is trivial.

Proposition 2.33.21 (Presentation calculus for relative interior)**.** *For affinoid morphisms, the relative interior of Definition 2.7.1 is open and is unchanged by bounded isomorphisms of the given Banach-algebra presentations. For composable affinoid morphisms $Z \xrightarrow{g} Y \xrightarrow{f} X$,*

$$\operatorname{Int}(Z/Y) \cap g^{-1} \operatorname{Int}(Y/X) \subseteq \operatorname{Int}(Z/X).$$

If $Y = \mathcal{M}(B)$, $X = \mathcal{M}(A)$ and $X' = \mathcal{M}_L(C)$ for a bounded $A \to C$ over a complete extension L/K, then, with $Y' = \mathcal{M}_L(B \widehat{\otimes}_A C)$ and projection $p : Y' \to Y$,

$$p^{-1} \operatorname{Int}(Y/X) \subseteq \operatorname{Int}(Y'/X').$$

Proof. For each norm-admissible relative presentation, the locus defined by its finitely many strict inequalities is open in the evaluation topology. Relative interior is the union of these loci over all presentations, so it is open. A bounded algebra isomorphism with bounded inverse transports norm-admissible surjections to such surjections and preserves the pointwise inequalities. This proves the invariance assertion.

For composition, write $X = \mathcal{M}(A)$, $Y = \mathcal{M}(B)$, $Z = \mathcal{M}(D)$. At a point in the left side choose presentations $A\{r^{-1}T\} \twoheadrightarrow B$ and $B\{s^{-1}S\} \twoheadrightarrow D$ whose coordinate values there satisfy the strict inequalities. Coefficientwise controlled lifts, Proposition 2.14.1, give a norm-strict surjection

$$A\{r^{-1}T, s^{-1}S\} \longrightarrow B\{s^{-1}S\} \longrightarrow D.$$

The iterated coefficient algebra equals the displayed joint algebra by Lemma 1.17.1. Composition multiplies the lifting bounds, so the composite is

norm-strict. Both lists of coordinates satisfy the required strict inequalities at the original point, proving the inclusion.

For base change, use a relative presentation witnessing that $p(y') \in \mathrm{Int}(Y/X)$. Proposition 2.33.8 identifies its base change with the norm-admissible presentation $C\{r^{-1}T\} \twoheadrightarrow B\widehat{\otimes}_A C$. The values of every coordinate at y' equal its values at $p(y')$. The same strict inequalities therefore witness that y' is interior over X'. □

Proposition 2.33.22 (Affinoid diagonals and finite proper maps)**.** *Every affinoid morphism is separated. A finite norm-admissible affinoid morphism is proper, and remains finite, separated and proper after every affinoid base change over a complete valued extension.*

Proof. For $A \to B$, the diagonal is represented by multiplication $\mu : B\widehat{\otimes}_A B \to B$. It is bounded, and $b \mapsto b \otimes 1$ is a bounded linear section. Thus μ is a norm-strict surjection: the section gives a controlled lift for every b. Its kernel is a closed ideal, so the diagonal is a closed immersion. This is separatedness.

If B is finite and norm-admissible over A, Theorem 2.33.18 gives empty relative boundary, and its conclusion is preserved by the equivalent given norm. The map of spectra is continuous between compact Hausdorff spaces. The inverse image of any compact analytic domain in the target is closed in the compact source and hence compact. Together with boundarylessness this is properness. Proposition 2.33.19 preserves finite admissibility under each specified base change, so the same argument applies after base change. □

2.33.6 Analytification from affinoid exhaustions

The following construction expands the proof indicated in [TemIntro, Exercise 5.1.1.1]. Every algebra occurring in a bounded piece is an actual quotient of a weighted Tate algebra. Passing from those pieces to an analytic space uses the atlas gluing of Theorem 2.40.10.

Theorem 2.33.23 (Affine analytification and its mapping property)**.** *Let $R = K[T_1, \dots, T_n]/I$ be a finite-type K-algebra. The set X_R of multiplicative seminorms on R extending the absolute value of K, with its evaluation topolo-*

gy, has a canonical good Hausdorff K-analytic structure. For every $r \in \mathbf{R}^n_{>0}$, its locus $|T_i| \le r_i$ is the affinoid space

$$X_{R,r} = \mathcal{M}\big(K\{r^{-1}T\}/IK\{r^{-1}T\}\big).$$

These pieces form an exhaustion by affinoid neighbourhoods. The construction is independent of the presentation and represents the functor

$$\mathrm{Hom}_{K\text{-}an}(Y, X_R) \simeq \mathrm{Hom}_{K\text{-}alg}\big(R, \Gamma(Y_G, \mathcal{O}_Y)\big)$$

for every K-analytic space Y.

Proof. If x is a seminorm with the stated coordinate bounds, then every polynomial satisfies $|p(x)| \le \max_\nu |a_\nu| r^\nu$. Thus evaluation extends continuously to the weighted Tate algebra, uniquely by polynomial density, and kills the actual ideal $IK\{r^{-1}T\}$. Conversely, a point of the displayed affinoid quotient gives precisely such a seminorm on R. This identifies the sets. Its topology agrees with the topology induced by polynomial evaluations: for a convergent series h and a polynomial truncation p, the inequality $||h(x)| - |p(x)|| \le \|h - p\|$ holds uniformly on this piece. All series evaluations are therefore uniform limits of polynomial evaluations.

If $r_i \le s_i$ for all i, adjoining variables U_i of radii r_i with relations $U_i - T_i = 0$ identifies $X_{R,r}$ with a Weierstrass domain of $X_{R,s}$. Both bounded maps between the resulting presentations are given by these variables; they are inverse on the dense polynomials. Arbitrary intersections are the pieces with radii $\min(r_i, s_i)$. Every point has an open neighbourhood $|T_i| < r_i$ contained in one such piece, by increasing all radii past its coordinate values. Thus the pieces and their affinoid subdomains form an affinoid net. Their identifications satisfy the cocycle condition on the coordinates, giving the asserted analytic space by the atlas construction. Its evaluation topology is Hausdorff because distinct seminorms differ on an element of R, and the affinoid neighbourhoods show goodness. Scalar radii $r_i = m$, for positive integers m, already give an exhaustion.

First let $Y = \mathcal{M}(A)$. Any homomorphism $R \to A$ sends its finitely many generators to elements of finite norm. Choosing radii exceeding those norms gives a bounded map from the corresponding affinoid quotient to A, and therefore an analytic map to X_R. Increasing the radii does not change the map. Conversely, the compact image of any analytic map $\mathcal{M}(A) \to X_R$ lies in some $X_{R,m}$: the open sets $|T_i| < m$ form an increasing cover, so compactness

supplies one m. The domain mapping property of Theorem 2.40.8 factors the map through $X_{R,m}$. Its ring map restricts to a homomorphism $R \to A$. Polynomial density proves that these constructions are inverse, and proves uniqueness of an analytic map with the prescribed coordinate functions.

For general Y, restrict a homomorphism $R \to \Gamma(Y_G, \mathcal{O}_Y)$ to every affinoid chart. The preceding construction gives analytic maps on the charts. They agree on each affinoid subdomain of an overlap because their coordinate functions agree, and hence glue by Theorem 2.40.8. Conversely, pullback of the coordinates along an analytic map gives global analytic functions. The affinoid calculation proves both compositions are identities. This mapping property determines X_R up to a unique analytic isomorphism and therefore proves presentation independence. □

Proposition 2.33.24 (Principal opens and global analytification)**.** *For $g \in R$, the natural map $X_{R_g} \to X_R$ identifies its source with the open analytic domain $|g| > 0$. Consequently affine analytification glues uniquely to a functor from finite-type K-schemes to good K-analytic spaces. Open immersions of schemes give open analytic immersions.*

Proof. Multiplicativity gives a unique extension of a seminorm with $|g| > 0$ to R_g, by $|a/g^m| = |a|/|g|^m$. On the bounded piece $X_{R,r}$, a bound $|g^{-1}| \leq t$ is represented by

$$\big(K\{r^{-1}T\}/I\big)\{t^{-1}V\}/(gV - 1).$$

This is exactly its Laurent domain $|g| \geq t^{-1}$ by the rational universal property. These domains, as r and t grow, form neighbourhoods of every point with $|g| > 0$. They are also the affinoid pieces for the presentation $R_g = R[V]/(gV - 1)$. Thus the pointwise bijection is an analytic isomorphism onto that open domain.

Every open subset of an affine scheme is covered by principal opens. The identifications just constructed agree on intersections, since localizing successively at g and h gives R_{gh} and the same seminorm and ring maps. An affine open cover of a finite-type scheme can therefore be analytified and glued along its open overlaps by Theorem 2.40.10. Any two covers have a common affine-open refinement, and the same local identifications give the unique comparison isomorphism. Ring maps on affine neighbourhoods similarly glue to the map attached to a scheme morphism; their composites agree on coordinates. Goodness holds locally on these open affine analytifications. □

Theorem 2.33.25 (Products and extension of the ground field)**.** *For finite-type K-schemes, analytification commutes with fibre products. For every complete valued extension L/K, there is a natural analytic isomorphism*

$$(Z \times_K L)^{\mathrm{an}} \simeq Z^{\mathrm{an}} \widehat{\otimes}_K L.$$

Both identifications respect all scheme morphisms and iterated complete extensions.

Proof. For affine schemes defined by $R \to S$ and $R \to T$, the mapping property of Theorem 2.33.23 gives, for every analytic Y,

$$\mathrm{Hom}(Y, X_{S \otimes_R T}) \simeq \mathrm{Hom}(Y, X_S) \underset{\mathrm{Hom}(Y, X_R)}{\times} \mathrm{Hom}(Y, X_T).$$

The middle algebraic step is the defining universal property of the ordinary tensor product of commutative algebras. Thus $X_{S \otimes_R T}$ is the analytic fibre product. To pass to schemes, cover the target by affine opens and both sources over these opens by affine opens. Their pairwise fibre products cover the scheme fibre product by affine opens. Proposition 2.33.24 identifies the overlap maps, so the affine isomorphisms glue, with the universal property checked on the same cover.

If $R = K[T]/I$, its scalar extension is $L[T]/IL[T]$. Proposition 2.33.5 gives, on every radius piece,

$$\big(K\{r^{-1}T\}/IK\{r^{-1}T\}\big) \widehat{\otimes}_K L \simeq L\{r^{-1}T\}/IL\{r^{-1}T\}.$$

The transition maps agree on polynomials and hence agree everywhere by continuity. Thus these isomorphisms glue over the exhaustion; they further glue over affine Zariski covers. The formulas on polynomial coordinates also prove compatibility with morphisms. Iterated extension agrees by Corollary 2.33.9, applied piecewise. □

Theorem 2.33.26 (Finite maps and closed immersions under analytification)**.** *Analytification preserves finite morphisms and closed immersions. More precisely, for a finite ring map $R \to S$ of finite-type K-algebras and an affinoid domain $V = \mathcal{M}(A) \subset X_R$, the inverse image of V in X_S is the finite norm-admissible affinoid space*

$$\mathcal{M}(A \otimes_R S),$$

where the algebra has its canonical finite-A-module topology. If $S = R/J$, its algebra is A/JA.

Proof. Set $D = A \otimes_R S$. It is finite over A, so Theorem 2.33.18 gives its canonical affinoid algebra structure. Let $\mathcal{M}(C)$ be any affinoid test space. A pair of maps to V and to X_S with the same composite to X_R is a bounded homomorphism $A \to C$ and an algebra homomorphism $S \to C$ agreeing on R. The algebraic tensor property gives a unique A-linear algebra map $D \to C$. It is bounded: choose finitely many generators e_j of D as an A-module and controlled lifts of its elements from A^m; the finite numbers $\|e_j \mapsto C\|$ and the bound on $A \to C$ give a uniform operator bound. Conversely any bounded $D \to C$ supplies the pair.

The bijection is natural and glues on affinoid covers of any analytic test space. Hence $\mathcal{M}(D)$ is $V \times_{X_R} X_S$. The domain mapping property identifies this fibre product with the inverse-image analytic domain. The affinoid domains V cover X_R, proving finiteness of the analytified morphism. When $S = R/J$, the algebraic tensor quotient is A/JA. The ideal JA is finitely generated and closed by the affinoid finite-module theorem, so this is an actual closed immersion. Both properties are local on the affine-open cover of a scheme target, which proves the global assertion. □

Proposition 2.33.27 (Relative neighbourhoods for algebraic morphisms)**.** *The analytification of every morphism between finite-type K-schemes is boundaryless. More explicitly, for every affinoid domain W in the target and every point y of its inverse image, there is an affinoid neighbourhood D of y in that inverse image with $y \in \mathrm{Int}(D/W)$. In particular every finite-type scheme analytification is boundaryless over K.*

Proof. Fix an affinoid target domain W and $y \in f^{\mathrm{an},-1}(W)$, and write $x = f^{\mathrm{an}}(y)$. Choose affine scheme neighbourhoods $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ of the two underlying scheme points, with $S = R[U_1, \dots, U_m]/J$. The intersection $W \cap X_R$ is an ordinary open neighbourhood of x in W. By Lemma 2.41.1, it contains a rational neighbourhood $V = \mathcal{M}(A)$ of x in W such that $x \in \mathrm{Int}(V/W)$. Choose positive radii $s_j > |U_j(y)|$. The affinoid algebra

$$D = A\{s^{-1}U\}/JA\{s^{-1}U\}.$$

It represents the part of the inverse image of V with $|U_j| \le s_j$. Indeed, for every affinoid test algebra C, a map to that part gives $A \to C$ and the coordinate functions from S. The pointwise bounds on the latter imply the required power bounds by Theorem 1.20.2; substitution gives the unique

bounded map $D \to C$. The converse follows directly from evaluation. The resulting domain identifications agree on overlaps, so this represents the actual analytic domain, not only its set of points.

The inverse image of the interior of V in W, intersected with X_S and the strict inequalities $|U_j| < s_j$, is an open neighbourhood of y in $f^{\mathrm{an},-1}(W)$ contained in $\mathcal{M}(D)$. Thus $\mathcal{M}(D)$ is an affinoid neighbourhood in that inverse image. Its displayed norm-admissible quotient presentation witnesses $y \in \mathrm{Int}(\mathcal{M}(D)/V)$. Together with $x \in \mathrm{Int}(V/W)$, the composition inclusion of Proposition 2.33.21 gives $y \in \mathrm{Int}(\mathcal{M}(D)/W)$. This verifies the boundaryless condition for every target affinoid domain. Taking the target to be $\mathrm{Spec}(K)$ proves the last assertion. □

Proposition 2.33.28 (Residue fields, algebraic fibres and surjectivity)**.** *Let $x \in Z^{\mathrm{an}}$ lie above the scheme point ξ. Then $\mathcal{H}(x)$ is the completion of the valued field $\kappa(\xi)$ defined by x. For a morphism $Y \to Z$ of finite-type K-schemes, its analytic fibre over x is naturally the analytification, over $\mathcal{H}(x)$, of $Y \times_Z \mathrm{Spec}\,\mathcal{H}(x)$. A surjective morphism of finite-type K-schemes induces a surjective map of analytifications.*

Proof. Work first on an affine neighbourhood $\mathrm{Spec}(R)$ of ξ and a radius piece $A = K\{r^{-1}T\}/I$ containing an ordinary neighbourhood of x. Polynomial density shows that the image of R is dense in A, also for the seminorm at x. Its fraction field modulo the point kernel is therefore dense in $\mathcal{H}_A(x)$: approximate each numerator and denominator by polynomials, and keep denominators nonzero by sufficiently close approximation. Corollary 2.33.15 makes the identification independent of the affinoid neighbourhood. Since $\kappa(\xi) = \mathrm{Frac}(R/\ker x)$, this proves the first assertion.

Put $F = \mathcal{H}(x)$. The scheme morphism $\mathrm{Spec}(F) \to Z_F$ determined by x is an F-rational point. Analytifying its fibre square over F, Theorems 2.33.25 and 2.33.26 identify the resulting analytic fibre with the fibre of $Y_F^{\mathrm{an}} \to Z_F^{\mathrm{an}}$ over this rational point. On an affinoid chart $A \to B$ about the original map, that fibre has algebra $B\widehat{\otimes}_A F$, by Corollary 2.33.9 and Theorem 2.33.11. The identification agrees on chart intersections, proving the fibre assertion.

Now assume $Y \to Z$ is surjective. Its scheme fibre over ξ is nonempty. It stays nonempty after extension from $\kappa(\xi)$ to F: a nonzero affine algebra in that fibre remains nonzero under a field extension, as follows by extending a vector-space basis. Thus $Y \times_Z \mathrm{Spec}(F)$ has a nonempty affine open of finite type over F. Choose a maximal ideal in its nonzero coordinate ring. Its

residue field E is finite over F by the Hilbert Nullstellensatz, [StacksAlg, Tag 00FV, Theorem 10.34.1(1)]. Lemma 1.12.1 extends the absolute value to E and makes it complete. This valued closed point gives a point of the analytic fibre and hence a point above x. □

Lemma 2.33.29 (Separated algebraic maps and the underlying topology)**.** *If $Y \to Z$ is a separated morphism of finite-type K-schemes, then $Y^{\mathrm{an}} \to Z^{\mathrm{an}}$ is separated and is a Hausdorff map of topological spaces. In particular the inverse image of a Hausdorff subspace of Z^{an} is Hausdorff.*

Proof. Its algebraic diagonal is a closed immersion. Theorems 2.33.25 and 2.33.26 identify its analytification with the analytic diagonal and preserve its closed-immersion property. A closed immersion restricts on any affinoid target domain to the quotient by its finite defining ideal: this follows first on defining affinoid charts by the quotient base-change presentation, and then on a finite affinoid refinement by Kiehl gluing, Theorem 2.39.6. Thus pulling the diagonal back to any affinoid product gives an affinoid closed subspace.

Consider two points y_1, y_2 above the same point z. Goodness permits affinoid neighbourhoods V of z and U_i of y_i with maps $U_i \to V$. The analytic product $P = U_1 \times_V U_2$ is compact Hausdorff. Its closed diagonal pullback D is compact. By Proposition 2.33.14, the map from P onto the ordinary topological fibre product is closed. The image of D is exactly the ordinary diagonal in $U_1 \times_{|V|} U_2$, and is therefore closed there. Restricting to the ordinary interiors of the chosen neighbourhoods proves that the relative ordinary diagonal is closed locally, hence closed. Distinct points in a common fibre can consequently be separated by neighbourhoods in Y^{an}. Points with distinct images in a Hausdorff subspace are separated by inverse images of disjoint neighbourhoods of their images. These two cases give the last assertion. □

Proposition 2.33.30 (Compactness of analytified projective maps)**.** *The analytification of a projective morphism of finite-type K-schemes is compact: the inverse image of every compact Hausdorff subset of the target is compact Hausdorff. In particular $(\mathbf{P}^n_K)^{\mathrm{an}}$ is compact.*

Proof. First consider $\mathbf{P}^n_R \to \mathrm{Spec}(R)$ and an affinoid domain $V = \mathcal{M}(A) \subset X_R$. For each homogeneous coordinate X_i, the domain

$$V_i = \{\, y \text{ above } V : \ |X_j/X_i|(y) \le 1 \text{ for } j \ne i \,\} \simeq \mathcal{M}\big(A\{T_0, \dots, \widehat{T_i}, \dots, T_n\}\big)$$

is compact. These $n+1$ domains cover the inverse image of V. Indeed, at a scheme point choose homogeneous coordinates in its residue field, not all zero, and evaluate them using its analytic absolute value. Choosing a coordinate of maximal absolute value puts the analytic point in the corresponding domain. The identifications of the domains follow either from their displayed coordinate ratios or from Theorem 2.33.25. The union is Hausdorff by Lemma 2.33.29, because projective space has closed diagonal. It is a finite union of compact spaces and hence compact. A closed subscheme of projective space gives a closed analytic subspace, so its inverse image over V is compact as well.

For a compact subset $C \subset X_R$, finitely many affinoid neighbourhoods cover C. Their inverse images form a finite compact union. Since X_R is Hausdorff, C is closed, and its inverse image is a closed subset of that union. This proves compactness over an affine scheme base. For a general scheme base, cover a given compact Hausdorff subset C by finitely many analytified affine opens. A compact Hausdorff space admits a finite closed refinement of any finite open cover: choose, at each point, an open neighbourhood whose closure lies in one chosen cover member, using regularity of compact Hausdorff spaces, and take a finite subcover. Apply the affine result to these finitely many closed subsets of C. Their inverse images are compact and cover the required inverse image. The latter is Hausdorff by the preceding lemma. Hence it is compact Hausdorff. □

Theorem 2.33.31 (Proper algebraic maps remain proper analytically)**.** *If $f : Y \to Z$ is a proper morphism of finite-type K-schemes, then $f^{\mathrm{an}} : Y^{\mathrm{an}} \to Z^{\mathrm{an}}$ is proper. This includes nonreduced schemes and trivially valued complete ground fields.*

Proof. The target is Noetherian by [StacksAlg, Tag 01T6]. Apply Chow's lemma, [StacksAlg, Tag 0200, Lemma 30.18.1], to f. It gives a proper surjective $\pi : Y' \to Y$ and an immersion $j : Y' \to \mathbf{P}^n_Z$. Since $f\pi$ is proper by [StacksAlg, Tag 01W3, Lemma 29.42.4], and $\mathbf{P}^n_Z \to Z$ is separated, [StacksAlg, Tag 01W6, Lemma 29.42.7] shows that j is proper. Its image is closed. An immersion with closed image is a closed immersion: factor it as a closed immersion into an open subspace of the target, and add the open complement of its closed image; the two local defining ideals glue on this open cover. Thus $f\pi$ is projective.

By Proposition 2.33.28, π^{an} is surjective. Let C be any compact Hausdorff subset of Z^{an}. Proposition 2.33.30 makes $(f\pi)^{\mathrm{an},-1}(C)$ compact. Its continuous image under π^{an} is exactly $f^{\mathrm{an},-1}(C)$, so the latter is quasi-compact. The map f is separated; Lemma 2.33.29 therefore makes this inverse image Hausdorff. Hence f^{an} is compact. Proposition 2.33.27 proves that it is boundaryless. These are the two defining conditions of an analytic proper morphism. □

2.33.7 Flatness of analytification

Lemma 2.33.32 (A rational-point completion calculation)**.** *Let $P = K[T_1, \dots, T_n]$, $C = K\langle T_1, \dots, T_n\rangle$, and $a \in K^n$ with $|a_i| \le 1$. For $\mathfrak{p} = (T_1 - a_1, \dots, T_n - a_n) \subset P$ and $\mathfrak{n} = \mathfrak{p}C$, the local map $P_{\mathfrak{p}} \to C_{\mathfrak{n}}$ is faithfully flat. Both maximal-ideal completions are canonically $K[[U_1, \dots, U_n]]$, where $U_i = T_i - a_i$.*

Proof. Translation by a is a polynomial automorphism and a bounded Tate-algebra automorphism with bounded inverse: substitution by $T_i + a_i$ and by $T_i - a_i$ is contractive. We may therefore assume $a = 0$. For every integer $q \ge 1$, truncation in total degree gives

$$P/(T)^q \simeq C/(T)^q \simeq \bigoplus_{|\nu|<q} KT^{\nu}.$$

The kernel in C is the actual ideal $(T)^q$: partition all terms of total degree at least q among the finitely many monomials of degree q which divide them. Dividing each group by its chosen monomial leaves a convergent series. Every element with nonzero constant term is a unit in this truncated ring, by a finite geometric sum. Localization at (T) therefore leaves the displayed quotients unchanged. Taking their inverse limits proves the completion identity.

Both local rings are Noetherian: for P use [StacksAlg, Tag 00FO, Lemma 10.31.3], and for C use Theorem 1.11.5; localization preserves finite generation of ideals by clearing denominators. By [StacksAlg, Tag 00MC, Lemma 10.97.3], each local ring maps faithfully flatly to the common completion $D = K[[T]]$. We spell out the descent step. For a finitely generated ideal $I \subset P_{(T)}$, let N be the kernel of $I \otimes_{P_{(T)}} C_{(T)} \to C_{(T)}$. Tensoring with the flat $C_{(T)}$-algebra D identifies $N \otimes D$ with the kernel of $I \otimes D \to D$, which is zero because D is flat over $P_{(T)}$. Faithfulness gives $N = 0$. The ideal criterion, [StacksAlg, Tag 00HD], proves flatness. The map is local with nonzero residue field K, so [StacksAlg, Tag 00HP] makes it faithfully flat. □

Theorem 2.33.33 (Polynomial-to-Tate flatness in arbitrary radii)**.** *For every complete non-Archimedean field K and every positive polyradius r, the ring map*

$$K[T_1, \dots, T_n] \longrightarrow K\{r^{-1}T\}$$

is flat. Consequently, if $R = K[T]/I$ and $A_r = K\{r^{-1}T\}/IK\{r^{-1}T\}$, then $R \to A_r$ is flat.

Proof. First take $r = (1, \dots, 1)$ and write $P = K[T]$, $C = K\langle T\rangle$. For a maximal ideal $\mathfrak{m} \subset C$, Theorem 1.12.3 gives the finite complete residue field $E = C/\mathfrak{m}$. Let $a_i \in E$ be the images of T_i; contractivity gives $|a_i| \leq 1$. The scalar extension $C_E = E\langle T\rangle$ is $C \otimes_K E$ as an ordinary finite C-module, by Lemma 2.33.4 and a finite K-basis of E. It is free over C. Evaluation at a has maximal kernel $\mathfrak{n} = (T_i - a_i)$ and restricts to the given quotient $C \to E$, so $\mathfrak{n} \cap C = \mathfrak{m}$.

Localization of a flat module is flat, since fractions preserve exact sequences: a fraction maps to zero precisely when some denominator annihilates its numerator. Hence $C_{\mathfrak{m}} \to (C_E)_{\mathfrak{n}}$ is flat, and it is faithfully flat because it is local. Lemma 2.33.32 says that $(C_E)_{\mathfrak{n}}$ is flat over $E[T]_{(T-a)}$, and therefore over P: the polynomial scalar extension is free and the intervening localization is flat. Applying the kernel descent argument of that lemma to $C_{\mathfrak{m}} \to (C_E)_{\mathfrak{n}}$ proves that $C_{\mathfrak{m}}$ is flat over P. For a finitely generated ideal $J \subset P$, the kernel of $J \otimes_P C \to C$ vanishes after localization at every maximal ideal of C. It is therefore zero by [StacksAlg, Tag 00HN, Lemma 10.23.1(2)]. The finite-ideal criterion proves $P \to C$ flat.

For arbitrary r, the multiplicative Gauss norm of Proposition 1.2.2 extends to the fraction field of $K\{r^{-1}S\}$. Let L be its completion. It contains elements s_i with $|s_i| = r_i$. By scalar extension and rescaling $T_i = s_i V_i$,

$$K\{r^{-1}T\}\widehat{\otimes}_K L \simeq L\{r^{-1}T\} \simeq L\langle V\rangle.$$

The first map from $K\{r^{-1}T\}$ is faithfully flat by Theorem 2.33.6. The strict case proves that the right side is flat over $L[T]$, hence over $K[T]$ by the free scalar extension. The same kernel descent argument proves flatness before extension. Finally $A_r = K\{r^{-1}T\} \otimes_{K[T]} R$ as ordinary algebras. Tensoring a short exact sequence of R-modules through this equality proves the last assertion. □

Theorem 2.33.34 (Exact analytification of coherent modules)**.** *For a finite-type K-scheme Z, the natural map of locally ringed spaces $\pi : Z^{\mathrm{an}} \to Z$ is flat on local rings. Its stalk map at x, over $\xi = \pi(x)$, is faithfully flat. The functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}} = \pi^*\mathcal{F}$ sends coherent modules to coherent modules and is exact. It commutes with tensor products and, for coherent modules $\mathcal{F}, \mathcal{G}$, with the sheaf of homomorphisms.*

Proof. On an affine scheme chart $\operatorname{Spec}(R)$, the map π sends a seminorm to its prime kernel. It is continuous because the inverse image of $D(g)$ is $|g| > 0$. A fraction in R_g defines an analytic function there, compatibly with further restriction, giving the structural sheaf map. At x, a fraction whose value is nonzero is invertible in an analytic neighbourhood by a Laurent localization. Thus the map on stalks is local.

Choose a radius neighbourhood $X_{R,r}$ of x. Its algebra A_r is flat over R by Theorem 2.33.33. On each smaller rational neighbourhood V of x, the algebra A_V is flat over A_r by Proposition 2.38.2. The analytic stalk is the filtered colimit of these A_V, by Proposition 2.33.16. Filtered colimits of flat modules are flat: a tensor relation uses finitely many elements, and if it becomes zero in the colimit it becomes zero at some stage, where injectivity follows from flatness. Equivalently one applies the finite-ideal criterion to this finite-stage argument. Thus $\mathcal{O}_{Z^{\mathrm{an}},x}$ is flat over R and over $R_{\mathfrak{p}} = \mathcal{O}_{Z,\xi}$. Its maximal ideal lies above $\mathfrak{p}$, and its residue field is nonzero. The faithful-flat criterion for this local map proves the stalk assertion.

By [StacksAlg, Tag 01XZ, Lemma 30.9.1], $\mathcal{F}$ is represented on $\operatorname{Spec}(R)$ by a finite module M. Thus its pullback to $X_{R,r}$ is the coherent module associated with $A_r \otimes_R M$. These descriptions agree on overlaps by the tensor mapping property, so they give the stated coherent analytification. Flatness of $R \to A_r$ preserves exact sequences of finite modules on every such chart, proving exactness on the analytic G-site as well as on the ordinary topology. Tensor compatibility is algebraic associativity. For homomorphisms, take a finite presentation $R^a \to R^b \to M \to 0$ and apply $\operatorname{Hom}_R(-, N)$. Its first three terms identify $\operatorname{Hom}_R(M, N)$ with the kernel of a matrix $N^b \to N^a$. Flat scalar extension to A_r preserves this kernel and gives exactly $\operatorname{Hom}_{A_r}(A_r \otimes M, A_r \otimes N)$. These natural chartwise identifications glue to the asserted sheaf isomorphism. □

2.33.8 Compact restriction and finite-dimensional cohomology

The following functional-analytic argument reconstructs the absolute-field case of the completely continuous method in [Bosch14, §6.4]. The countable-type hypothesis on the Banach spaces is what supplies the bounded splittings used below.

Lemma 2.33.35 (Bounded splittings in countable type)**.** *Let E be a Banach K-space of countable type, meaning that a countable set has dense linear span. Every closed subspace $N \subset E$ is of countable type and admits a bounded linear projection $E \to N$. More generally, every norm-strict surjection of Banach spaces onto a space of countable type admits a bounded linear section. Finite modules with their canonical norms over generalized K-affinoid algebras are of countable type.*

Proof. Let $p : B \to Q$ be the asserted norm-strict surjection. Replace the norm on Q by the equivalent quotient norm. Choose a family (e_i) as in Lemma 3.21.2, with lower constant $0 < c < 1$. For each i, choose a lift $\widehat{e}_i$ with $\|\widehat{e}_i\| \leq 2\|e_i\|$. If $q = \sum_i a_i e_i$, then $|a_i|\|e_i\| \to 0$. Hence $\sum_i a_i \widehat{e}_i$ converges in B, and defines a linear section $s : Q \to B$ satisfying $\|s(q)\| \leq 2c^{-1}\|q\|$. This proves the splitting assertion.

Now take $Q = E/N$. It is Banach by Lemma 1.11.1. The images of a countable dense spanning set of E span densely in Q, so the preceding argument gives a bounded section s. The map $P = 1 - sp : E \to N$ is a bounded projection. Applying P to that same countable set gives a dense spanning set in N. Finally, a generalized affinoid algebra is a quotient of a finite-variable Tate algebra, whose polynomial monomials form a countable dense spanning set. A finite module is a bounded quotient of a finite power of this algebra by Theorem 1.18.3. Images of the corresponding spanning sets prove the last assertion. □

Lemma 2.33.36 (The completely continuous perturbation lemma)**.** *Assume that K is nontrivially valued. A bounded linear map is called completely continuous if it is an operator-norm limit of bounded maps with finite-dimensional image. Let $f : E \to F$ be a surjective bounded map of Banach K-spaces, and let $g : E \to F$ be completely continuous. Then $(f + g)(E)$ is closed and has finite codimension in F. Completely continuous maps remain such after bounded pre- or postcomposition. If their image lies in a closed subspace*

of a countable-type target, they are completely continuous as maps to that subspace.

Proof. The open mapping theorem, Lemma 1.14.1, gives a constant $C > 0$ such that every $y \in F$ has a lift x with $\|x\| \leq C\|y\|$. Choose a finite-rank map h with $C\|g - h\| < 1$, and put $f' = f + (g - h)$. This map is surjective. Indeed, lift y by f; the error under f' has norm at most $C\|g-h\|\|y\|$. Lift that error and repeat. The errors decrease geometrically, so the successive lifts converge and give a preimage under f'. The same estimates give a bounded lifting constant for f', hence it is open.

Write $N = \ker h$. This is closed, and E/N is finite dimensional. The image of $\ker f'$ in E/N is a finite-dimensional subspace, hence closed by Lemma 1.12.1. Consequently $N + \ker f'$ is closed in E. Since f' is an open quotient map, $f'(N)$ is closed in F. The vector space $F/f'(N)$ is finite dimensional, because it is a quotient of E/N. The image of $f' + h$ contains $f'(N)$ and corresponds to a subspace of this finite-dimensional Hausdorff quotient. It is therefore closed and has finite codimension. As $f' + h = f + g$, this proves the first assertion.

Bounded composition preserves convergence in operator norm and finite rank, proving the second assertion. For the last one, use Lemma 2.33.35 to project the target boundedly onto the closed subspace and compose each finite-rank approximant with that projection. □

Proposition 2.33.37 (Completely continuous restriction)**.** *Let $A = K\{r^{-1}T\}/I$ and let $A \to B$ be a bounded homomorphism of generalized affinoid algebras. Suppose that the image b_i of every T_i has spectral radius strictly smaller than r_i. For a finite A-module M, the natural restriction map*

$$M \longrightarrow B \otimes_A M$$

between the canonical Banach modules is completely continuous. This assertion holds also over a trivially valued field.

Proof. Choose numbers $\rho_B(b_i) < \gamma_i < r_i$. By the spectral-radius formula, Corollary 1.20.3, there are constants C_i with $\|b_i^d\| \leq C_i\gamma_i^d$ for every $d \geq 0$. Choose finite generators $m_1, \ldots, m_s$ of M, and write $N = B \otimes_A M$ and $n_j = 1 \otimes m_j$. The composite

$$K\{r^{-1}T\}^{\,s} \longrightarrow M \longrightarrow N$$

sends $\sum_\nu a_{\nu j}T^\nu$ in component j to $\sum_\nu a_{\nu j}b^\nu n_j$. Let $0<\theta=\max_i(\gamma_i/r_i)<1$, and put $C=(\prod_i C_i)\max_j\|n_j\|$, enlarged if necessary for the module-action bound. Terms with $|\nu|\geq d$ have norm at most $C\theta^d|a_{\nu j}|r^\nu$. Truncation to $|\nu|<d$ is a bounded finite-rank map, and its operator-norm error is at most $C\theta^d$. Hence this composite is completely continuous. The first arrow is norm-strict by the quotient presentation and Theorem 1.18.3; its target is of countable type. Lemma 2.33.35 gives a bounded linear section. Composing with it proves the assertion for $M\to N$. If there are no Tate variables, the source module is finite dimensional over K and the assertion is immediate. □

Theorem 2.33.38 (Finiteness of coherent cohomology on projective space)**.** *For every complete non-Archimedean field K, every integer $n\geq 0$, and every coherent module $\mathcal{F}$ on P_G, where $P=(\mathbf{P}^n_K)^{\mathrm{an}}$, the groups $H^q(P_G,\mathcal{F})$ are finite-dimensional K-vector spaces and vanish for $q>n$. On the standard unit-polydisc cover, the coboundary images are closed in the canonical Banach spaces.*

Proof. First assume that K is nontrivially valued. Fix a real $R>1$ and set

$$U_i=\{|X_j|\leq R|X_i| \text{ for every } j\},$$
$$V_i=\{|X_j|\leq |X_i| \text{ for every } j\},\qquad 0\leq i\leq n.$$

These are affinoid domains, and both families cover P: at any point, choose a homogeneous coordinate of maximal nonzero value. Their finite intersections are affinoid. More explicitly, on U_{i_0} write $z_j=X_j/X_{i_0}$ and $z_{i_0}=1$. For each additional $i\in I$, the conditions defining U_i are $|z_j|\leq R|z_i|$ for every j, including $j=i_0$. They form a rational domain because the numerator 1 is present. The rational presentation adjoins all z_j/z_i with radius R. Consequently $A_I=\mathcal{O}(U_I)$ has a quotient presentation by a Tate algebra whose variables are the finitely many ratios X_j/X_i for $i\in I$, all with radius R. On V_I all these ratios have value at most 1. The spectral-radius formula therefore makes their spectral radii at most $1<R$.

Put $C^q_U=\prod_{|I|=q+1}\mathcal{F}(U_I)$ and define C^q_V similarly. These are Banach spaces of countable type. Kiehl's equivalence on each affinoid identifies restrictions of $\mathcal{F}$ with finite-module base change. Proposition 2.33.37 proves that every component of the restriction $r^q: C^q_U\to C^q_V$ is completely continuous, hence so is their finite product. Let Z^q_U, Z^q_V be the kernels of the

corresponding bounded differentials. They are closed, hence are Banach spaces of countable type by Lemma 2.33.35. Projecting C_V^q boundedly onto Z_V^q shows that the induced map $r : Z_U^q \to Z_V^q$ is completely continuous.

By Proposition 2.42.1, both complexes compute $H^q(P_G, \mathcal{F})$, and restriction induces the identity on this derived-section group. Therefore

$$F_q : C_V^{q-1} \oplus Z_U^q \longrightarrow Z_V^q, \qquad (a, z) \longmapsto da + r(z)$$

is surjective, with $C_V^{-1} = 0$. The map $G_q(a, z) = -r(z)$ is completely continuous. Lemma 2.33.36 shows that $F_q + G_q$, whose image is dC_V^{q-1}, has closed image and finite-dimensional cokernel. That cokernel is exactly $H^q(P_G, \mathcal{F})$. This proves finite dimensionality and closedness, without assuming closed coboundary images in advance. Vanishing above n follows from the length of the ordered complex.

Now suppose that K is trivially valued. On the unit cover (V_i), each algebra $\mathcal{O}(V_I)$ is the corresponding polynomial–Laurent algebra with the norm which is 1 on every nonzero element, by Lemma 2.42.2. Each finite module, with a quotient norm from a finite free presentation, has the same discrete 0/1 norm. Thus every vector subspace of each C_V^q is closed, and every exact sequence among its kernels, images and cohomology spaces is norm-strict. Choose a nontrivially valued complete extension L/K: for example, complete the fraction field of $K\{t^{-1}S\}$ in its Gauss norm for any $0 < t < 1$. The scalar-extension construction and finite-module base change identify $C_V^\bullet \mathbin{\widehat{\otimes}}_K L$ with the unit-cover complex of the coherent pullback $\mathcal{F}_L$ on $(\mathbf{P}_L^n)^{\mathrm{an}}$. By Theorem 2.33.3, its cohomology is

$$H^q(P_G, \mathcal{F}) \mathbin{\widehat{\otimes}}_K L \simeq H^q((\mathbf{P}_L^n)^{\mathrm{an}}_G, \mathcal{F}_L).$$

The right side is finite dimensional by the nontrivial case. If the left-hand source had arbitrarily many linearly independent vectors over K, their finite-dimensional spans would, by Theorem 2.33.2, inject after tensoring into the displayed left side and have the same dimension over L. This contradicts finite dimensionality. Hence $H^q(P_G, \mathcal{F})$ is finite dimensional over K, as required. □

2.34 From projective comparison to proper GAGA

The projective calculation also provides the relative comparison needed for a Chow modification. We give the support argument explicitly, so that pass-

ing from projective to proper schemes requires no analytic proper mapping theorem as an additional input.

Proposition 2.34.1 (The flat analytification morphism of ringed sites)**.** *For a finite-type K-scheme X, inverse image of Zariski opens defines a flat morphism of ringed topoi $\alpha_X : X_G^{\mathrm{an}} \to X_{\mathrm{Zar}}$. If $W = \mathrm{Spec}(R) \subset X$ and $V = \mathcal{M}(A)$ is any affinoid domain in W^{an}, then $R \to A$ is flat. On coherent modules α_X^* is the analytification already constructed, and α_X^* is exact on all modules.*

Proof. The rule $W \mapsto W^{\mathrm{an}}$ preserves finite intersections, the final object and coverings, by Proposition 2.33.24. It therefore defines the stated morphism of sites, with structural ring map given by evaluation of regular functions. Choose a finite generating list of R over K. Its values on the compact space V are bounded. Consequently V lies in a sufficiently large radius affinoid $X_{R,r}$. It is an affinoid domain there by the domain restriction property. The factorization

$$R \longrightarrow A_r \longrightarrow A$$

is flat by Theorem 2.33.33 and Proposition 2.38.2.

Here is a sheaf-level verification of flatness, including modules which are not coherent. A finite relation with coefficients in $\alpha_X^{-1}\mathcal{O}_X$ and values in $\mathcal{O}_{X_G^{\mathrm{an}}}$ can, locally for the analytic site, be represented by coefficients $r_1, \ldots, r_t$ in one affine ring R and values $a_1, \ldots, a_t$ in one affinoid ring A, with $\sum r_i a_i = 0$. To obtain this representation, first use the local definition of an inverse-image sheaf for the finitely many coefficients, and then refine to affinoids. Since R is Noetherian, the relation module $N = \ker(R^t \to R)$ is finite. Flatness of $R \to A$ gives $\ker(A^t \to A) = A \otimes_R N$. Thus the vector (a_i) is a finite A-linear combination of relations already over R. This is the finite-relation criterion for flatness, equivalently the finite-ideal criterion [StacksAlg, Tag 00HD], applied locally to the ring sheaves. It proves that $\mathcal{O}_{X_G^{\mathrm{an}}}$ is flat over $\alpha_X^{-1}\mathcal{O}_X$. Inverse image of sheaves is exact; tensoring with this flat ring sheaf is exact as well. On finite presentations the result is $A \otimes_R M$, agreeing with Theorem 2.33.34. □

Lemma 2.34.2 (Every scheme point has an analytic lift)**.** *For every finite-type K-scheme X, the map $|X^{\mathrm{an}}| \to |X|$ is surjective. Consequently coherent analytification detects zero sheaves and isomorphisms.*

Proof. Let $\xi \in X$ and put $E = \kappa(\xi)$, a finitely generated field extension of K. Choose a transcendence basis from a finite generating list. Then E is finite over $F = K(t_1, \dots, t_d)$. The Gauss absolute value on $K[t_1, \dots, t_d]$ is multiplicative by Proposition 1.2.2; it extends by quotients to F. Let H be its completion. The finite nonzero H-algebra $E \otimes_F H$ has a maximal ideal, and its quotient L is a finite field extension of H. The map $E \to L$ is injective because E is a field and 1 has nonzero image. Lemma 1.12.1 gives L its extending absolute value. Its restriction to E defines a multiplicative seminorm on any affine coordinate ring at ξ, with kernel exactly the prime of ξ. The radius construction of Theorem 2.33.23 makes this an analytic point above ξ. At such a point the local ring map is faithfully flat by Theorem 2.33.34. A coherent sheaf whose pullback is zero therefore has every stalk zero. Apply the same assertion to the kernel and cokernel, using exactness, to detect an isomorphism. □

Theorem 2.34.3 (Relative projective cohomology comparison)**.** *Let $f : Y \to X$ be a projective morphism of finite-type K-schemes, and let $\mathcal{E}$ be coherent on Y. There is a natural quasi-isomorphism*

$$\alpha_X^* R f_* \mathcal{E} \;\simeq\; R f_*^{\mathrm{an}}(\mathcal{E}^{\mathrm{an}}).$$

In particular $(R^q f_* \mathcal{E})^{\mathrm{an}} \simeq R^q f_*^{\mathrm{an}} \mathcal{E}^{\mathrm{an}}$ for every q. If $X = \operatorname{Spec}(R)$ and $V = \mathcal{M}(A) \subset X^{\mathrm{an}}$ is an affinoid domain, the corresponding comparison is

$$A \otimes_R H^q(Y, \mathcal{E}) \;\simeq\; H^q(f^{\mathrm{an},-1}(V)_G, \mathcal{E}^{\mathrm{an}}).$$

Proof. **Projective space.** First take $Y = \mathbf{P}^n_R$. For $\mathcal{O}(m)$ the standard unit-affinoid covering above V has coordinate algebras consisting of convergent Laurent series with coefficients in A. The proof of Lemma 2.42.2 works coefficient by coefficient over the complete Banach algebra A: its reduction, isometric section, and convergent ideal multipliers have norm at most one. The exponent sets and the simplex insertion operators in Theorem 2.42.3 are unchanged. They yield a bounded contraction onto the finite free A-modules indexed by the same surviving monomials. The algebraic computation [StacksAlg, Tag 01XT] over R has the same representatives. Hence the

displayed comparison holds for every twist, and respects all restriction maps in V.

For an arbitrary coherent $\mathcal{E}$, choose, by [StacksAlg, Tag 01YS(1)–(2)], an exact sequence $0 \to \mathcal{R} \to \mathcal{L} \to \mathcal{E} \to 0$, where $\mathcal{L}$ is a finite sum of twists and $\mathcal{R}$ is coherent. Both the algebraic standard affine cover and the analytic unit cover have length n, so all cohomology above n vanishes, using [StacksAlg, Tag 01XB] and Proposition 2.42.1. Tensor the algebraic long exact sequence with A; it stays exact by Proposition 2.34.1. Analytification and restriction to $f^{\mathrm{an},-1}(V)$ also preserve the short exact sequence. Descend on q, simultaneously for all coherent sheaves. The exact diagram first lifts an analytic q-class of $\mathcal{E}$ by lifting its connecting $(q+1)$-class of $\mathcal{R}$ and correcting by a class of $\mathcal{L}$; this proves surjectivity for all $\mathcal{E}$. Apply this surjectivity to $\mathcal{R}$ to lift the correction of a class of $\mathcal{L}$ mapping to zero analytically. Injectivity for $\mathcal{L}$ and the $(q+1)$-term then proves injectivity for $\mathcal{E}$. This is the exact diagram chase of Theorem 2.42.4, now with the flat coefficient extension $R \to A$.

Closed projective subspaces and direct images. Factor a general projective f as a closed immersion into $\mathbf{P}^n_X$ followed by projection. Closed direct image is exact and coherent on both sides, by finite quotient modules and Lemma 2.42.7; it commutes with analytification on each chart. The same affinoid and affine Čech complexes compute cohomology before and after this closed direct image. Thus the preceding argument applies to $i_*\mathcal{E}$. The algebraic modules $H^q(Y, \mathcal{E})$ are finite over R by [StacksAlg, Tag 02O5]; their localizations are the sections of $R^q f_*\mathcal{E}$, as also follows by localizing the finite affine Čech complex. On the analytic side, [StacksAlg, Tag 072W] identifies the higher direct image as the sheafification of the cohomology on inverse images of objects. The displayed isomorphisms, compatible with affinoid restriction, therefore give the asserted isomorphisms of cohomology sheaves.

Finally the commutative analytification square of ringed topoi has the canonical derived comparison map of [StacksAlg, Tag 07A7]. Proposition 2.34.1 makes its derived pullbacks ordinary pullbacks. The cohomology maps just calculated are those of this map: on Čech resolutions they send a regular section to its analytic section, which is the adjunction map used in its construction. They are isomorphisms in every degree on an affinoid basis, so the comparison is a quasi-isomorphism. This also proves its compatibility with the adjunction $\mathcal{F} \to Rf_* f^* \mathcal{F}$ and with restriction. □

Lemma 2.34.4 (Finite covers and closed supports on proper analytifications)**.** *Let X be a proper finite-type K-scheme. Its analytification has a finite affinoid covering all of whose finite intersections are affinoid. If $i : T \hookrightarrow X$ is a closed subscheme, closed direct image preserves coherent modules and their cohomology on both the algebraic and the analytic sites; the resulting cohomology identifications commute with analytification.*

Proof. The space X^{an} is good by the radius construction, compact by Theorem 2.33.31, and has a closed analytic diagonal by Lemma 2.33.29. Choose affinoid neighbourhoods and a finite subcover of their interiors. If V_1, V_2 are two of these domains, their fibre product over X^{an} is the inverse image of its closed diagonal in the affinoid space $V_1 \times_K V_2$. It is therefore affinoid, by the quotient presentation of a closed analytic subspace. The universal properties of domains identify this fibre product with $V_1 \cap V_2$. Iteration proves the assertion for every finite intersection.

The inverse images of this covering on T^{an} are closed affinoid subspaces. Thus Lemma 2.42.7 proves the analytic assertions by the identical finite Čech complexes. Algebraically, a finite affine cover of the separated scheme X also has affine finite intersections: apply its closed diagonal to products of affine opens. A finite module over a quotient R/J is the same as a finite R-module killed by J, so closed direct image is exact and coherent. The same affine Čech complex computes cohomology before and after that direct image, by [StacksAlg, Tags 01XB and 03F7]. On every affine–affinoid chart the comparison sends a regular section to its analytic section. The identical-complex identifications therefore commute with comparison. □

Lemma 2.34.5 (An algebraic thickening captures analytic support)**.** *Let X be proper over K, let $Z \hookrightarrow X$ be a closed subscheme with ideal $\mathcal{I}$, and let $\mathcal{F}$ be a coherent module on X_G^{an}. If $\mathcal{F}$ vanishes on $X^{\mathrm{an}} \setminus Z^{\mathrm{an}}$, then $(\mathcal{I}^{\mathrm{an}})^N \mathcal{F} = 0$ for some integer $N \geq 1$. Hence $\mathcal{F}$ is the closed direct image of a coherent module on the analytification of $Z_N = (X, \mathcal{O}_X/\mathcal{I}^N)$.*

Proof. Use a finite affinoid cover from Lemma 2.34.4. On one member $V = \mathcal{M}(A)$ write $M = \mathcal{F}(V)$ and $J = \mathcal{I}^{\mathrm{an}}(V)$; both are finite over A. For $x \in V$, with prime $\mathfrak{p} = \ker |\cdot|_x$, the fibre $M \otimes_A \mathcal{H}(x)$ is zero exactly when $M_{\mathfrak{p}} = 0$. Indeed, the residue-field extension $\kappa(\mathfrak{p}) \subset \mathcal{H}(x)$ is faithful, and Nakayama's lemma applies to the finite module $M_{\mathfrak{p}}$. Moreover $M_{\mathfrak{p}} = 0$ exactly when $\mathrm{Ann}_A(M) \not\subset \mathfrak{p}$: if the localization vanishes, the product of finitely many

denominators killing a generating list of M lies in the annihilator and outside $\mathfrak{p}$; the converse is immediate.

Put $B = A/\operatorname{Ann}_A(M)$ with its affinoid quotient norm. Every point of $\mathcal{M}(B)$ gives a nonzero fibre of M, so the support hypothesis forces every element of J to vanish there. By the spectral-radius formula and Proposition 3.21.4, its image in B is nilpotent. If $a_1, \dots, a_s$ generate J and $a_i^{e_i}$ annihilates M, then $J^{1+\sum_i (e_i-1)} M = 0$: each monomial of this total degree has some exponent at least e_i. Take the maximum of these finitely many exponents over the cover. The resulting global annihilation and Lemma 2.42.7 give the stated factorization through Z_N^{an}. □

Theorem 2.34.6 (Cohomology comparison for proper schemes)**.** *For every proper finite-type K-scheme X and every coherent $\mathcal{F}$ on X, the natural maps are isomorphisms*

$$H^q(X, \mathcal{F}) \longrightarrow H^q(X_G^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}}) \qquad (q \geq 0).$$

The same comparison is a quasi-isomorphism for every bounded complex with coherent cohomology sheaves.

Proof. We first record how to use a comparison already known for the cohomology sheaves of a bounded complex C. The canonical truncation triangles [StacksAlg, Tag 08J5] express C by finitely many successive extensions of $\mathcal{H}^j(C)[-j]$. Flatness of α_X^* preserves these triangles and their cohomology sheaves. Applying derived global sections and the long exact sequences shows, by induction on the number of nonzero cohomology sheaves, that comparison for each $\mathcal{H}^j(C)$ implies comparison for C. The comparison maps here are the maps induced by regular sections mapping to analytic sections, or equivalently the adjunction maps for the flat morphism α_X. They are natural in distinguished triangles.

Fix X and perform Noetherian induction on closed subsets $Z \subset |X|$. The induction assertion is the comparison for every coherent module on X supported in Z. It is clear for the empty set. Let $\mathcal{F}$ have support contained in Z. If its support is a proper closed subset, the induction hypothesis applies. Otherwise let $T \hookrightarrow X$ be defined by $\operatorname{Ann}(\mathcal{F})$; its underlying space is Z. The module $\mathcal{F}$ is the closed direct image of a coherent module $\mathcal{F}_T$ on T, and Lemma 2.34.4 identifies the two comparison problems.

Apply the algebraic Chow lemma [StacksAlg, Tag 0200] to the proper scheme T. It supplies a proper surjection $\pi : Y \to T$, an isomorphism above

a dense open $U \subset T$, and an immersion $Y \to \mathbf{P}^n_K$. Since Y is proper over K, the immersion is closed: its map to $\mathbf{P}^n_K$ is proper by [StacksAlg, Tag 01W6], and a proper immersion is closed by its open–closed factorization. Thus Y is projective over K. The graph of π is closed because T is separated; combining it with $Y \hookrightarrow \mathbf{P}^n_K$ gives a closed immersion $Y \hookrightarrow \mathbf{P}^n_T$. Consequently π is also projective. Put $\mathcal{E} = \pi^*\mathcal{F}_T$ and $D = R\pi_*\mathcal{E}$. Its cohomology sheaves are coherent by [StacksAlg, Tag 02O5] and vanish in sufficiently large degrees, as the finite standard relative-projective cover computes them.

Take the cone C in the natural triangle

$$\mathcal{F}_T \longrightarrow D \longrightarrow C \longrightarrow \mathcal{F}_T[1].$$

The first map is adjunction, followed by the map to the derived direct image. On U it is the identity, because π is an isomorphism there. Hence every $\mathcal{H}^j(C)$ is coherent and supported in the proper closed subset $T \setminus U$ of Z. Closed direct image to X, the induction hypothesis, and the first paragraph give comparison for C.

For D, the derived composition of global sections and direct image [StacksAlg, Tag 0D6E], together with Theorem 2.34.3, identifies its comparison with

$$R\Gamma(Y, \mathcal{E}) \longrightarrow R\Gamma(Y^{\mathrm{an}}_G, \mathcal{E}^{\mathrm{an}}).$$

This is a quasi-isomorphism by Corollary 2.42.9. Compatibility with adjunction in the relative comparison identifies the maps in the two triangles, not only their objects. The long exact cohomology sequences of these triangles therefore give comparison for $\mathcal{F}_T$ and hence for $\mathcal{F}$. Noetherian induction proves the assertion for all coherent sheaves. The first paragraph then proves the assertion for every bounded coherent complex. □

Proposition 2.34.7 (Hom and extension comparison on proper schemes)**.** *Let X be proper over K, and let $\mathcal{E}, \mathcal{F}$ be coherent modules on X. For every $q \geq 0$, exact analytification induces an isomorphism*

$$\mathrm{Ext}^q_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \;\simeq\; \mathrm{Ext}^q_{\mathcal{O}_{X^{\mathrm{an}}_G}}(\mathcal{E}^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}}).$$

For $q = 1$ this is the map which analytifies short exact sequences; for $q = 0$ it gives full faithfulness.

Proof. We first compare sheaf Ext locally. On an affine open $\mathrm{Spec}(R)$ choose a resolution of the finite module representing $\mathcal{E}$ by finite free R-modules,

continuing as far as necessary. Each successive kernel is finite because R is Noetherian. Flat analytification makes this an exact resolution by finite free analytic modules. Applying Hom into $\mathcal{F}$ and its analytification gives complexes which agree after pullback, since each free term is finite. In each fixed degree only finitely many terms are needed to compute kernel modulo image. Exact pullback thus gives

$$(\mathcal{E}xt^j_X(\mathcal{E},\mathcal{F}))^{\mathrm{an}} \simeq \mathcal{E}xt^j_{X^{\mathrm{an}}_G}(\mathcal{E}^{\mathrm{an}},\mathcal{F}^{\mathrm{an}}).$$

The algebraic Ext sheaves are coherent by the same finite-free calculation. These local identifications are independent of the resolution: they are the cohomology maps of the canonical derived-Hom pullback morphism [StacksAlg, Tag 08JF]. They therefore glue to a quasi-isomorphism

$$\alpha_X^* R\mathcal{H}om_X(\mathcal{E},\mathcal{F}) \simeq R\mathcal{H}om_{X^{\mathrm{an}}_G}(\mathcal{E}^{\mathrm{an}},\mathcal{F}^{\mathrm{an}}).$$

Put $C = R\mathcal{H}om_X(\mathcal{E},\mathcal{F})$. It has coherent cohomology in nonnegative degrees, although it need not be bounded above. Fix q. The truncation $\tau_{\le q}C$ is a bounded coherent complex. The complementary truncation belongs to $D^{\ge q+1}$; its derived global sections also belong to $D^{\ge q+1}$, as a bounded below injective resolution may be chosen to start in that degree. Thus $H^q R\Gamma(C)$ is computed by $\tau_{\le q}C$. The same holds after exact pullback. Theorem 2.34.6 and the truncation triangle now give equality in degree q of the two derived global Hom complexes. By [StacksAlg, Tag 08JA], applied after shifting the second argument by q, these groups are the stated Ext groups. Finally [StacksAlg, Tag 06XV] identifies derived Ext in degree one with short exact extensions. Its identification takes an extension to its connecting morphism. Exact pullback preserves that triangle and connecting morphism, proving the claimed compatibility. □

Theorem 2.34.8 (Proper analytic GAGA)**.** *For every proper finite-type scheme X over a complete non-Archimedean field K, analytification is an exact tensor equivalence*

$$\mathrm{Coh}(X) \xrightarrow{\sim} \mathrm{Coh}(X^{\mathrm{an}}_G).$$

It preserves coherent cohomology and restricts to an equivalence on vector bundles, and hence on line bundles.

Proof. Full faithfulness is Proposition 2.34.7. Exactness and compatibility with tensor products are Theorem 2.33.34; cohomology comparison is Theorem 2.34.6. It remains to prove essential surjectivity. We do this by Noetherian induction on closed subsets $Z \subset |X|$, with the assertion that every analytic coherent module supported in Z^{an} is algebraizable on X. For a closed subset, Z^{an} means the analytification of its reduced closed subscheme; this underlying set is unchanged by a thickening. The empty case is immediate.

Let $\mathcal{F}$ be supported in Z^{an}. Lemma 2.34.5 realizes it as the closed direct image of a coherent module $\mathcal{F}_T$ on T^{an}, for a closed subscheme $T \hookrightarrow X$ with underlying space Z. Apply Chow as in the preceding proof: there is a projective surjection $\pi : Y \to T$ with Y projective over K, which is an isomorphism above a dense open $U \subset T$. The coherent module $(\pi^{\mathrm{an}})^*\mathcal{F}_T$ is algebraizable on Y by Corollary 2.42.9; choose an algebraization $\mathcal{E}$ and its identifying isomorphism. Relative projective comparison in degree zero identifies

$$\mathcal{G} := \pi_*^{\mathrm{an}}(\pi^{\mathrm{an}})^*\mathcal{F}_T \simeq (\pi_*\mathcal{E})^{\mathrm{an}}.$$

In particular $\mathcal{G}$ is coherent. The kernel $\mathcal{K}$ and cokernel $\mathcal{C}$ of the adjunction $\mathcal{F}_T \to \mathcal{G}$ are coherent and vanish on U^{an}. Their closed direct images on X^{an} are supported in the proper closed subset $Z' = T \setminus U$ of Z. By the induction hypothesis they are algebraizations of coherent modules K_X, C_X on X. These algebraizations are annihilated by the ideal of T: the respective multiplication maps by this ideal analytify to zero, so full faithfulness makes them zero already on X. Thus they descend to coherent K_T, C_T on T, whose analytifications are $\mathcal{K}, \mathcal{C}$.

The map $\mathcal{G} \to \mathcal{C}$ algebraizes by full faithfulness on the proper scheme T. Let J_T be the kernel of the resulting map $\pi_*\mathcal{E} \to C_T$. Exactness gives an exact analytic sequence

$$0 \longrightarrow K_T^{\mathrm{an}} \longrightarrow \mathcal{F}_T \longrightarrow J_T^{\mathrm{an}} \longrightarrow 0.$$

Proposition 2.34.7 in degree one lifts its class to $\mathrm{Ext}^1_T(J_T, K_T)$. By [StacksAlg, Tag 06XV] this class is represented by an algebraic short exact sequence $0 \to K_T \to F_T \to J_T \to 0$; its middle term is coherent. Indeed, locally take a finite free surjection $P \to J_T$ and lift its finitely many basis sections into F_T after shrinking. Then $K_T \oplus P \to F_T$ is surjective and its kernel, by projection to P, is isomorphic to the coherent kernel of $P \to J_T$. Thus F_T is a cokernel of a map between coherent modules. The analytic extension is isomorphic to the displayed sequence with identity on its ends. Hence $F_T^{\mathrm{an}} \simeq \mathcal{F}_T$. Closed

direct image to X algebraizes $\mathcal{F}$. This completes the Noetherian induction and proves the equivalence.

Finally finite local freeness descends along a faithfully flat local map. Here is the elementary argument in the form needed. If $R \to S$ is such a map, M is finite over the Noetherian local ring R, and $S \otimes_R M$ is free of rank r, then the residue-field extension shows $\dim_{\kappa(R)} M/\mathfrak{m}_R M = r$. Lift a basis; Nakayama gives a surjection $R^r \to M$. After tensoring with S this is a surjection between two rank-r free modules over the local ring S, and hence an isomorphism (reduce its determinant modulo the maximal ideal). Faithful flatness kills its kernel, so M is free.

For completeness, local freeness in the analytic G-topology also yields free ordinary stalks here. On an affinoid $\mathcal{M}(A)$ representing a finite module M, choose a trivializing affinoid domain $\mathcal{M}(B)$ containing the point in question. The domain map $A \to B$ is flat; localizing at the corresponding primes gives a faithfully flat local map. The preceding argument makes M free at that prime of A. A finite presentation then makes it free after inverting one element not in the prime: lift the basis and clear the finitely many relations for the kernel and cokernel. A small rational neighbourhood on which that element is invertible gives the same free ordinary analytic stalk. Now every scheme point has an analytic lift by Lemma 2.34.2, and its local map is faithfully flat by Theorem 2.33.34. Applying the local descent argument shows that an algebraization of an analytic vector bundle is locally free of the same rank. The converse follows by tensoring a local basis. Rank one gives the line-bundle assertion. □

2.35 Finite neighbourhoods and preservation of smoothness

We use the usual local definitions: an analytic morphism is *quasi-finite* if, near each source point, it restricts on ordinary open neighbourhoods to a finite morphism. It is *étale* if these finite local morphisms are flat and their coherent relative differential sheaf vanishes. It is *smooth* if locally on ordinary opens it factors as an étale morphism to a relative analytic affine space followed by its projection. The next argument verifies this local form directly for algebraic smooth morphisms.

Lemma 2.35.1 (Separating the branches of a finite morphism)**.** *Let $f : Y \to X$ be a norm-admissible finite morphism of good analytic spaces. Its fibre over any $x \in X$ is a finite set $\{y_1, \dots, y_m\}$. Given neighbourhoods N_i of these points, there is an ordinary open neighbourhood U of x such that*

$$f^{-1}(U) = \coprod_{i=1}^{m} W_i, \qquad y_i \in W_i \subset N_i,$$

where each W_i is open and closed in $f^{-1}(U)$, and $W_i \to U$ is finite. On ordinary local rings there is a canonical product decomposition

$$(f_*\mathcal{O}_Y)_x \simeq \prod_{i=1}^{m} \mathcal{O}_{Y,y_i}.$$

If $X = \mathcal{M}(A)$ and $Y = \mathcal{M}(B)$, its left side is $B \otimes_A \mathcal{O}_{X,x}$.

Proof. Work over an affinoid neighbourhood V of x. The fibre has algebra $D = B \otimes_A \mathcal{H}(x)$, finite dimensional over the complete field $\mathcal{H}(x)$, by Proposition 2.33.19 and Theorem 2.33.11. Every prime of a finite-dimensional algebra over a field is maximal, since a finite-dimensional domain is a field: multiplication by a nonzero element is injective and hence surjective. There are only finitely many maximal ideals. Indeed, the Chinese remainder map for any finite collection of distinct maximal ideals is surjective onto their residue fields, so their number is at most $\dim_{\mathcal{H}(x)} D$. Each residue field is finite over $\mathcal{H}(x)$ and has exactly one extending absolute value by Lemma 1.12.1; the quotient map is bounded for the canonical finite-dimensional norms. Consequently $\mathcal{M}(D)$ has exactly one point for each of these finitely many maximal ideals.

The space $f^{-1}(V)$ is affinoid, hence compact Hausdorff, and its map to V is closed. Choose pairwise disjoint ordinary open neighbourhoods $O_i \subset N_i$ of its fibre points. The complement of their union is compact and has closed image not containing x. Remove that image and intersect with the ordinary interior of V to obtain U. Then $W_i = O_i \cap f^{-1}(U)$ give the stated open-and-closed partition. For any affinoid domain $V' \subset U$, $f^{-1}(V') = \mathcal{M}(B')$ is affinoid and finite over V'. The function equal to one on $W_i \cap f^{-1}(V')$ and zero on its complement is analytic by sheaf gluing on this ordinary open partition. It is an idempotent $e_i \in B'$, since the global sections on an affinoid are its algebra. Thus $W_i \cap f^{-1}(V') = \mathcal{M}(e_i B')$. The summand $e_i B'$ is a finite module with its canonical norm: projection by e_i is bounded and finite-module maps are norm-strict. This proves finiteness of $W_i \to U$.

Restriction of germs defines the displayed product map on local rings. Given a germ at each y_i, choose representatives on N_i, separate the neighbourhoods, and shrink U as just done. Their restrictions glue on the disjoint W_i to a section over $f^{-1}(U)$. This proves surjectivity. If all germs of such a section vanish, choose neighbourhoods where they vanish and shrink once more; the section then vanishes on an entire inverse-image neighbourhood. This proves injectivity. In the affinoid case, take the filtered colimit over affinoid neighbourhoods V' of x. Finite base change gives $\mathcal{O}(f^{-1}(V')) = B \otimes_A A_{V'}$; ordinary tensor commutes with this colimit, which is $\mathcal{O}_{X,x}$ by Proposition 2.33.16. This gives the final identification. □

Theorem 2.35.2 (Analytification preserves étale morphisms)**.** *An étale morphism of finite-type K-schemes analytifies to an étale analytic morphism. More precisely, near every analytic source point it restricts on ordinary open neighbourhoods to a finite flat morphism whose relative analytic differentials vanish.*

Proof. Fix a source point y over x. By [StacksAlg, Tag 02GT], choose affine Zariski neighbourhoods containing their scheme images on which the morphism is standard étale. In the notation of [StacksAlg, Tag 00UB], its rings have the form

$$R \longrightarrow S_g, \qquad S = R[T]/(P), \quad P \text{ monic}, \quad P' \in S_g^{\times}.$$

The scheme $\operatorname{Spec}(S_g)$ is the open subscheme $D(g)$ of the finite free scheme $\operatorname{Spec}(S)$ over $\operatorname{Spec}(R)$. Analytification preserves this finite morphism and this open immersion, by Theorem 2.33.26 and Proposition 2.33.24. Take an affinoid neighbourhood $V = \mathcal{M}(A)$ of x in $\operatorname{Spec}(R)^{\mathrm{an}}$. The inverse image in $\operatorname{Spec}(S)^{\mathrm{an}}$ is $\mathcal{M}(B)$ with $B = A \otimes_R S = A[T]/(P)$ and its canonical finite module norm. It is a finite free A-module.

Apply Lemma 2.35.1, requiring the neighbourhood of y to be contained in $D(g)^{\mathrm{an}}$. It gives ordinary open neighbourhoods W of y and U of x for which $W \to U$ is finite. For each affinoid $V' \subset U$, its inverse-image algebra is a summand eB', where $B' = A' \otimes_R S$ and $A' = \mathcal{O}(V')$. Since B' is finite free over A', this summand is projective, and in particular flat, over A'. The element g is a unit in eB': it vanishes at no point of its spectrum, so the affinoid unit criterion applies. Therefore P' is also a unit in eB'.

For the finite algebra $B' = A'[T]/(P)$, a bounded A'-derivation into any finite Banach B'-module is determined by dT with the sole relation $P'dT = 0$.

Conversely this relation defines a derivation on all polynomials modulo P; it is bounded because the monic remainder basis is a finite module basis. Thus its analytic differential module is $B'\,dT/(P'dT)$. Restriction to the open-and-closed summand eB' tensors this module with eB', by Theorem 2.41.5. It is zero because P' is a unit there. The local maps $W \to U$ are therefore finite flat with vanishing analytic differentials, which is the asserted étale local form. □

Corollary 2.35.3 (Analytification preserves smooth morphisms)**.** *If $f : Y \to X$ is a smooth morphism of finite-type K-schemes, then $f^{\mathrm{an}} : Y^{\mathrm{an}} \to X^{\mathrm{an}}$ is smooth.*

Proof. At the scheme image of any analytic source point, [StacksAlg, Tag 054L] gives Zariski open neighbourhoods Y_0, X_0 and a factorization

$$Y_0 \xrightarrow{\text{étale}} \mathbf{A}^d_{X_0} \longrightarrow X_0.$$

Proposition 2.33.24 makes their analytifications ordinary open neighbourhoods. By Theorem 2.33.25, the analytification of the middle space is the relative analytic affine space over X_0^{an}; its projection is the usual analytic projection. The first arrow is étale after analytification by Theorem 2.35.2. This is exactly the defining local smooth factorization. □

References. The local finite-branch argument follows the mechanism of [Ber93, §§3.2–3.3]; using the standard étale form makes the finite covering algebra free and avoids a separate analytic flatness theorem. All algebraic existence inputs in these proofs are the cited Stacks lemmas.

2.35.1 Scalar stalks and descent of analytic flatness

Theorem 2.35.4 (Faithful flatness of scalar-extension stalks)**.** *Let X be a good K-analytic space, let L/K be any complete valued extension, and let $u \in X_L$ map to $x \in X$. The local homomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{X_L,u}$ is faithfully flat. For every coherent $\mathcal{F}$ on X, its analytic scalar pullback satisfies*

$$(\mathcal{F}_L)_u \simeq \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X_L,u}.$$

Proof. Put $R = \mathcal{O}_{X,x}$ and $S = \mathcal{O}_{X_L,u}$. Fix an affinoid neighborhood $V = \mathcal{M}(A)$ of x. Its inverse image V_L is an affinoid neighborhood of u: the inverse

image of the ordinary interior of V is an ordinary neighborhood contained in V_L. Rational neighborhoods W of u inside V_L form a cofinal family. Each composite

$$A \longrightarrow A_L \longrightarrow A_W$$

is flat by Theorem 2.33.6 and Proposition 2.38.2. Proposition 2.33.16 identifies S with the filtered colimit of these A_W. The finite-relation proof used in Theorem 2.33.17 therefore proves that S is flat over this fixed A.

To pass from A to R, let $I \subset R$ be finitely generated and let $z \in I \otimes_R S$ have zero image in S. Choose generators a_i of I and write $z = \sum_i a_i \otimes s_i$ with $s_i \in S$. Lift the a_i to the algebra A of one sufficiently small affinoid neighborhood V. If $J \subset A$ is the ideal generated by the lifts, the corresponding tensor in $J \otimes_A S$ maps to z and has zero image in S. Since S is A-flat, this element is zero. The ideal criterion [StacksAlg, Tag 00HD] gives R-flatness. Evaluation is compatible with the isometric embedding $\mathcal{H}(x) \hookrightarrow \mathcal{H}(u)$, so this map is local. It is faithfully flat by [StacksAlg, Tag 00HP].

On V, Theorem 2.39.6 represents $\mathcal{F}|_V$ by a finite A-module M. Pullback sends a finite free presentation to its cokernel presentation with the same matrix over A_L. Thus its scalar pullback is represented by $M \otimes_A A_L$, with the finite scalar comparison of Theorem 2.33.6. Taking stalks gives $M \otimes_A S$, while $\mathcal{F}_x = M \otimes_A R$. The displayed identity is now the associativity of ordinary tensor products. □

Theorem 2.35.5 (Amalgamation of complete valued fields)**.** *Let E and F be complete valued extensions of a complete valued field K. There is a complete valued field Ω with isometric embeddings $E \hookrightarrow \Omega$ and $F \hookrightarrow \Omega$ agreeing on K.*

Proof. For every finite subset $S \subset F^\times$, consider the bounded evaluation map

$$K\{(|s|^{-1}T_s)_{s\in S}\} \longrightarrow F, \qquad T_s \longmapsto s,$$

and let A_S be its quotient by the kernel. The kernel is closed; the quotient is a nonzero generalized affinoid algebra by Corollary 1.18.4. Evaluation gives a point $x_S \in \mathcal{M}(A_S)$ and an injective map from $K[S]$ into A_S. If $S \subset T$, substitution gives a bounded map $A_S \to A_T$: the variables have norms at most their assigned radii, and an element of the first kernel maps to zero because its evaluation in F is zero. These maps preserve the evaluation points.

By Theorem 2.33.13, the fibre

$$P_S = \{w \in \mathcal{M}_E((A_S)_E) : w|_{A_S} = x_S\}$$

is nonempty and compact. The maps $A_S \to A_T$ define an inverse system of these compact spaces. Its inverse limit is nonempty. Indeed, the product $\prod_S P_S$ is compact by [StacksAlg, Tag 08ZU], and each compatibility equation is closed in it. A finite collection of equations involves only finitely many subsets; choose a finite T containing their union and restrict any point of P_T to all the subsets involved. Thus these closed conditions have the finite intersection property.

Choose a compatible family (w_S). For $S \subset T$, the completed-residue-field construction gives an isometric embedding $\mathcal{H}(w_S) \hookrightarrow \mathcal{H}(w_T)$, by agreement of evaluations. These embeddings satisfy the cocycle identity on the dense evaluated fraction fields. Form their directed union as fields and complete it; call the resulting field Ω. Every $\mathcal{H}(w_S)$ contains E isometrically, and the transition maps fix it. The compatible maps $K[S] \to \mathcal{H}(w_S)$ are injective and preserve the given absolute value on F, because w_S lies above x_S. As the finite subsets range over $F^\times$, their union is all of F. They therefore give an isometric embedding $F \hookrightarrow \Omega$, agreeing with the embedding of E on K. Completion of a valued field is again a valued field: inversion of a Cauchy sequence with nonzero limit is Cauchy once its norms are bounded below. This completes the construction. □

Lemma 2.35.6 (Flatness descent through a square of local rings)**.** *Consider a commutative square of rings $R \to R'$, $R \to S$, $R' \to S'$, $S \to S'$. Suppose R' is flat over R and S' is faithfully flat over S. If an S-module M has $M \otimes_S S'$ flat over R', then M is flat over R.*

Proof. The module $M' = M \otimes_S S'$ is flat over R, by composition of the two exact tensor functors. For a finitely generated ideal $I \subset R$, let N be the kernel of $I \otimes_R M \to M$. This is an S-module. Flatness over S identifies $N \otimes_S S'$ with the kernel of $I \otimes_R M' \to M'$, which is zero. Faithfulness gives $N = 0$. The criterion [StacksAlg, Tag 00HD] concludes. □

Theorem 2.35.7 (Descent from one scalar-extension point)**.** *Let $f : Y \to X$ be a morphism of good K-analytic spaces, let $\mathcal{F}$ be coherent on Y, and let L/K be a complete valued extension. For $y \in Y$ and any point $u \in Y_L$ above y, $\mathcal{F}$ is analytically X-flat at y if and only if $\mathcal{F}_L$ is analytically X_L-flat at u.*

Proof. The forward implication follows directly by composing the base changes quantified in Definition 2.9.1. For the converse, let F/K be any complete valued extension, let $X' \to X$ be a good F-analytic base change, and let $y' \in Y' = Y \times_X X'$ map to y. Write x' for its image in X'. We must prove that $\mathcal{F}_{Y',y'}$ is flat over $\mathcal{O}_{X',x'}$.

Apply Theorem 2.35.5 to the complete extensions $\mathcal{H}(u)$ and $\mathcal{H}(y')$ of $\mathcal{H}(y)$. It gives a complete field Ω containing both compatibly. In particular Ω contains L and F over K. The inclusion $\mathcal{H}(y') \hookrightarrow \Omega$ defines an Ω-rational point ω of Y'_Ω above y', by the pointwise fibre description in Theorem 2.33.11, applied locally. Its image in Y_L is u: on any affinoid chart the evaluations of its analytic functions and of the coefficients in L are exactly those obtained through $\mathcal{H}(u) \hookrightarrow \Omega$. Denote its image in X'_Ω by v.

Associativity of the analytic fibre products identifies Y'_Ω with the pullback of Y_L along $X'_\Omega \to X_L$. The assumed analytic flatness at u therefore makes $(\mathcal{F}_{Y'_\Omega})_\omega$ flat over $\mathcal{O}_{X'_\Omega,v}$. In the commutative local square

$$\begin{array}{ccc} \mathcal{O}_{X',x'} & \longrightarrow & \mathcal{O}_{X'_\Omega,v} \\ \downarrow & & \downarrow \\ \mathcal{O}_{Y',y'} & \longrightarrow & \mathcal{O}_{Y'_\Omega,\omega} \end{array}$$

both horizontal maps are faithfully flat by Theorem 2.35.4. The same theorem identifies the pullback coherent stalk with the ordinary tensor product. Lemma 2.35.6 now proves the required naive flatness at y'. Since the complete extension and analytic base change were arbitrary, $\mathcal{F}$ is analytically flat at y. □

References. The descent architecture follows [Duc18, 4.5.3 and 4.5.6]. Scalar stalk flatness and the compatible complete-field construction are proved above; these citations identify the source of the method.

2.35.2 Convergent germs and ordinary local completions

Lemma 2.35.8 (Noetherianity of convergent germs)**.** *Suppose K is nontrivially valued. Let C_n be the subring of $K[[T_1, \dots, T_n]]$ consisting of the series convergent on some positive polydisc about 0. Then C_n is a Noetherian local ring with maximal ideal $(T_1, \dots, T_n)$, residue field K, and maximal-ideal completion $K[[T_1, \dots, T_n]]$.*

Proof. A series with nonzero constant term becomes a unit on a sufficiently small polydisc: its nonconstant part has norm smaller than the constant term, and geometric inversion applies. A series with zero constant term is a sum $\sum_i T_i g_i$ with each g_i convergent, by partitioning its monomials according to a variable dividing them. Thus the asserted maximal ideal and residue field follow. Invertible linear changes of the variables preserve C_n, since either linear substitution maps a sufficiently small polydisc into any prescribed one.

We prove Noetherianity by induction on n, with $C_0 = K$. Let $I \neq 0$ be an ideal and choose $0 \neq f \in I$. If its constant term is nonzero, $I = C_n$. Otherwise, let P_d be its first nonzero homogeneous part. The field K is infinite because its absolute value is nontrivial. A nonzero polynomial over an infinite field cannot vanish on every point: induct on the number of variables and use the root bound in one variable. Hence, after permuting coordinates if needed, choose $a_1, \dots, a_{n-1} \in K$ with $P_d(a_1, \dots, a_{n-1}, 1) \neq 0$. The invertible linear substitution $T_i \mapsto T_i + a_i T_n$ makes

$$f(0, \dots, 0, T_n) = cT_n^d + \text{terms of higher degree}, \qquad c \neq 0.$$

Write $T' = (T_1, \dots, T_{n-1})$ and $f = \sum_j a_j(T') T_n^j$. For arbitrarily small $s > 0$, one can choose a positive polyradius r' such that a_d is a unit in $A = K\{(r')^{-1}T'\}$ and the normalized series $a_d^{-1} f$ is distinguished of degree d at radius s. Here are the norm checks. First choose s small enough that the scalar tail $\sum_{j>d} a_j(0) T_n^j$ has s-norm strictly less than $|c| s^d$. At this fixed s, convergence leaves only finitely many coefficients outside any prescribed smaller bound for the weighted tail. Shrinking r' makes those finitely many $a_j(T') - a_j(0)$ arbitrarily small, makes $a_d - c$ smaller than $|c|$, and makes $\|a_j\| s^j \leq |c| s^d$ for $j < d$, since $a_j(0) = 0$. The omitted tail remains uniformly small under this shrinking. Thus $\|a_d^{-1}\| = |c|^{-1}$ and all hypotheses of Theorem 1.10.3 hold.

Given any $g \in C_n$, take these radii small enough that g also converges. Division yields $g = qf + \sum_{j<d} b_j(T') T_n^j$ with $q \in C_n$ and $b_j \in C_{n-1}$. Consequently $C_n/(f)$ is finite as a C_{n-1}-module. By induction C_{n-1} is Noetherian,

so the submodule $I/(f)$ is finitely generated over C_{n-1}. To recall the elementary module step, submodules of a finite free module are finitely generated by induction on its rank, using the first-coordinate ideal and its kernel; the statement for a finite module follows by inverse image under a finite free surjection. Lift generators of $I/(f)$ and adjoin f. They generate I over C_n. Undoing the coordinate change proves that every ideal is finitely generated.

For every $q \geq 1$, total-degree truncation identifies $C_n/(T)^q$ with $K[T]/(T)^q$. Its kernel is exactly $(T)^q$: partition the terms of total degree at least q among the finitely many degree-q monomials dividing them; the quotients remain convergent. The inverse limit of these finite polynomial truncations is the formal power-series ring. □

Lemma 2.35.9 (Rational points lie inside analytic domains)**.** *If U is an analytic domain of a good K-analytic space X and $x \in U$ satisfies $\mathcal{H}(x) = K$, then U contains an ordinary neighborhood of x in X. In particular the ordinary stalks on U and X agree at x.*

Proof. Choose an affinoid neighborhood X_0 of x in X and an affinoid domain of $U \cap X_0$ containing x. By Theorem 2.36.6, this domain has a finite cover by domains rational in X_0. Choose one containing x, given by $|f_i| \leq r_i|g|$. The unit-ideal condition implies $g(x) \neq 0$. Write $a_i = f_i(x) \in K$ and $b = g(x) \in K^\times$. On the ordinary neighborhood $|g - b| < |b|$, we have $|g| = |b|$. If $a_i \neq 0$, further impose $|f_i - a_i| < |a_i|$; then $|f_i| = |a_i| \leq r_i|b|$. If $a_i = 0$, impose $|f_i| < r_i|b|$. These finitely many open conditions hold at x and force all the defining rational inequalities. Intersect also with the ordinary interior of X_0. The resulting ordinary neighborhood is contained in U. □

Proposition 2.35.10 (Formal presentations at rational analytic points)**.** *Let K be nontrivially valued and $X = \mathcal{M}(K\{r^{-1}T\}/I)$. At a K-rational point $a = (a_1, \ldots, a_n)$, put $U_i = T_i - a_i$ and let I_a denote the ideal of C_n generated by the translated elements of I. Then*

$$\mathcal{O}_{X,a} \simeq C_n/I_a, \qquad \widehat{\mathcal{O}_{X,a}} \simeq K[[U_1, \ldots, U_n]]/I_a K[[U_1, \ldots, U_n]].$$

If X is replaced near a by an analytic domain, these identifications are unchanged. If $Z \hookrightarrow X$ is a closed analytic subspace with coherent ideal $\mathcal{J}$, then $\mathcal{O}_{Z,a} = \mathcal{O}_{X,a}/\mathcal{J}_a$.

Proof. Small polydiscs centered at a form an ordinary neighborhood basis in the ambient polydisc. To verify cofinality, an ordinary neighborhood is specified by finitely many evaluations of analytic functions. On a sufficiently small centered polydisc, each such function differs uniformly from its value at a by an arbitrarily small amount, by its convergent translated series. For a nonzero value this preserves its absolute value; for a zero value it gives any prescribed positive upper bound. These conditions realize every finite evaluation neighborhood. All sufficiently small centered polydiscs lie in the original polydisc, since $|a_i| \leq r_i$.

Intersecting them with X gives a cofinal system of affinoid neighborhoods. Closed-subspace base change and finite-module exactness identify their algebras with the corresponding Tate algebras modulo the extended ideal I. The ideal I is finitely generated by Theorem 1.18.2. Passing to the filtered colimit therefore gives C_n/I_a: tensoring a finite ideal presentation, or lifting a finite equality to a smaller polydisc, verifies both its surjectivity and its kernel. The same argument applies to any coherent closed ideal $\mathcal{J}$ after choosing finite generators on an affinoid neighborhood.

Lemma 2.35.8 makes C_n Noetherian and computes its completion. Exactness of completion for finite modules, [StacksAlg, Tag 00MA], identifies the completion of its quotient with the displayed formal quotient. Finally Lemma 2.35.9 says that any analytic domain containing a is an ordinary neighborhood there, and hence does not change the stalk. □

Theorem 2.35.11 (Noetherian ordinary local rings)**.** *For every good analytic space over any complete non-Archimedean field, every ordinary local ring is Noetherian.*

Proof. Fix $x \in X$ and choose an affinoid neighborhood. Put $E = \mathcal{H}(x)$. If E is nontrivially valued, take $L = E$; otherwise use Lemma 1.16.1 to adjoin one radius different from 1, obtaining a nontrivially valued complete extension L/E. Evaluation $A \to E \to L$, together with the identity on L, defines an L-rational point $u \in X_L$ above x. The completed-residue-field and relative pushout properties verify this point directly.

By Proposition 2.35.10, $S = \mathcal{O}_{X_L,u}$ is Noetherian. The map $R = \mathcal{O}_{X,x} \to S$ is faithfully flat by Theorem 2.35.4. Noetherianity descends here by an elementary ideal argument. For any ideal $J \subset R$, the ideal JS has finitely many generators. Their expressions use finitely many elements $a_1, \ldots, a_m \in J$, so $JS = (a_1, \ldots, a_m)S$. If $J_0 = (a_1, \ldots, a_m) \subset R$, flatness identifies

$(J/J_0) \otimes_R S$ with $JS/J_0S = 0$, and faithfulness gives $J = J_0$. Thus every ideal of R is finitely generated. □

References. This reconstructs the reduction in [Ber93, Theorem 2.1.4]: the convergent-germ argument, the rational-point completion, and faithful scalar-stalk descent are all explicit.

2.35.3 The Jacobian criterion by completed local rings

Lemma 2.35.12 (The fibre over a rational base point)**.** *Let $f : Y \to X$ be a morphism of good spaces over a nontrivially valued field K, and suppose $x = f(y)$ is K-rational. Put $R = \mathcal{O}_{X,x}$, $S = \mathcal{O}_{Y,y}$. Then*

$$\mathcal{O}_{Y_x,y} \simeq S/\mathfrak{m}_R S.$$

If also y is K-rational and a local relative presentation has n variables and relation ideal I, then the cotangent-space dimension of this fibre local ring is n minus the rank at y of the relative Jacobian matrix of any finite generators of I.

Proof. Choose affinoid neighborhoods $X_0 = \mathcal{M}(A)$ and $Y_0 = \mathcal{M}(B)$ with $f(Y_0) \subset X_0$. Evaluation at x is a bounded quotient $A \to K$. Since K is a finite A-module, Proposition 2.33.19 identifies the fibre algebra with the ordinary quotient $B/\mathfrak{m}_x B$. In an affinoid presentation $A = K\{r^{-1}Z\}/J$, translate x to $Z = 0$. Monomial partition, as in Proposition 2.35.10, shows both that $\mathfrak{m}_x$ is generated by the Z_i and that their germs generate $\mathfrak{m}_R$.

The point y need not be rational. Take a cofinal system of ordinary affinoid neighborhoods V of y in Y_0. Since the fibre is a closed analytic subspace with its subspace topology, the intersections $V \cap Y_x$ are cofinal ordinary affinoid neighborhoods of y in Y_x: any ordinary neighborhood in the closed subspace is the intersection with an ambient open neighborhood, inside which goodness supplies such a V. On every V, closed-subspace base change identifies the fibre algebra with $\mathcal{O}(V)/(Z_1, \dots, Z_d)\mathcal{O}(V)$. Passing to the filtered colimit gives $\mathcal{O}_{Y_x,y} = S/(Z_1, \dots, Z_d)S$: every representative lifts on the same V, and a representative vanishing in the quotient is, on some smaller V, a finite combination of these generators. Because their germs generate $\mathfrak{m}_R$, this is the asserted quotient.

At a rational y, its completed fibre ring is $K[[T_1, \dots, T_n]]/\widehat{I}_x$, after translating its relative coordinates to zero. The maximal ideal modulo its square

is consequently K^n modulo the linear terms of the relations. Those terms are the rows of the relative Jacobian evaluated at y. Completion does not change this vector space: its quotient modulo the square of the maximal ideal is the second finite truncation in the definition of completion. □

Proposition 2.35.13 (The completed Jacobian calculation)**.** *Let K be non-trivially valued, let X be good, and let D be an analytic domain of $\mathbb{A}^{n,\mathrm{an}}_X$. Suppose a closed analytic subspace $Z \subset D$ is defined near a K-rational point z by r functions $g_1, \dots, g_r$ whose relative Jacobian has rank r at z. Write x for its image and $R = \mathcal{O}_{X,x}$, $S = \mathcal{O}_{Z,z}$. Then, as complete local $\widehat{R}$-algebras,*

$$\widehat{S} \simeq \widehat{R}[[V_1, \dots, V_{n-r}]].$$

The local homomorphism $R \to S$ is flat. The fibre local ring at z is regular of dimension $n - r$.

Proof. The point x is K-rational, since its completed residue field embeds into $\mathcal{H}(z) = K$ over K. By Lemma 2.35.9, the domain D is an ordinary neighborhood of z in the relative affine space. Choose local affinoid coordinates $X_0 = \mathcal{M}(K\{s^{-1}U\}/J)$ at x and translate x, z to zero. Proposition 2.35.10, applied also to the relative polydisc, gives

$$\widehat{R} = K[[U]]/\widehat{J}, \qquad \widehat{S} = K[[U,T]]/(\widehat{J}, \widehat{g}_1, \dots, \widehat{g}_r).$$

Here each g_i is a germ modulo J; choose any convergent representative in (U,T). The relative derivatives at the origin are unaffected by this choice, because the added ideal consists of functions of U times other germs, and all elements of J vanish at $U = 0$. After reordering the T_i, the first r columns give an invertible minor. The substitution in the formal ring which fixes all U_i, sends T_i to $\widehat{g}_i$ for $i \le r$, and fixes T_i for $i > r$ has invertible linear part. It is a formal automorphism. Explicitly, invert its linear matrix; then correct an inverse modulo degree q by solving the degree-q discrepancy with this same invertible matrix. The corrections have increasing degree, so their sum converges in the formal maximal-ideal topology. The same recursion gives uniqueness, which makes the resulting left and right inverses equal. This argument works in every characteristic.

This automorphism fixes $K[[U]]$ and hence its extended ideal $\widehat{J}$. It carries the ideal generated by $T_1, \dots, T_r$ to that generated by $\widehat{g}_1, \dots, \widehat{g}_r$. Passing to the quotient gives the asserted $\widehat{R}$-algebra isomorphism.

Both R and S are Noetherian by Theorem 2.35.11; their completions are Noetherian by [StacksAlg, Tag 0316]. The ring $\widehat{R}[[V]]$ is the (V)-adic completion of the Noetherian polynomial ring $\widehat{R}[V]$. It is flat over $\widehat{R}$ by [StacksAlg, Tag 00MB] and the freeness of a polynomial ring over its coefficient ring. Also $\widehat{R}$ is flat over R, and $\widehat{S}$ is faithfully flat over S, by [StacksAlg, Tag 00MC]. For each finitely generated ideal $I \subset R$, the kernel of $I \otimes_R S \to S$ becomes zero after tensoring with $\widehat{S}$, because $\widehat{S}$ is R-flat. Faithfulness makes the original kernel zero; [StacksAlg, Tag 00HD] proves the required flatness.

Finally Lemma 2.35.12 and exact completion identify the completed fibre local ring with $K[[V_1, \dots, V_{n-r}]]$. This is a regular local ring of dimension $n-r$, by [StacksAlg, Tag 032C]. Regularity passes to the original local ring by [StacksAlg, Tag 07NY], and its embedding dimension remains $n-r$ because completion preserves the quotient of the maximal ideal by its square. For a regular local ring dimension equals embedding dimension, proving the last assertion. □

Theorem 2.35.14 (Quasi-smoothness, flatness and geometric regularity)**.** *For a morphism $f : Y \to X$ of analytic spaces, quasi-smoothness at y is equivalent to analytic flatness at y together with geometric regularity at y of the $\mathcal{H}(f(y))$-analytic fibre. For good Y, X, whenever these conditions hold one may choose the Jacobian presentation with its source an ordinary affinoid neighborhood of y. More precisely, after fixing any relative affinoid presentation near y, one can restrict it to an ordinary rational neighborhood of y in its ambient polydisc so that the relation ideal is generated by $n - \operatorname{rk}_y \Omega_{Y/X}$ elements with independent relative differentials.*

Proof. **A Jacobian chart is analytically flat.** Start with a presentation $W \hookrightarrow D \subset \mathbb{A}^{n,\mathrm{an}}_X$ whose ideal is generated by r functions with independent relative differentials at y. First treat the good chart itself, without assuming that W is an ordinary neighborhood in any larger source. After any good analytic base change over any complete extension, the same equations and an invertible Jacobian minor give a presentation at each point over y, by Proposition 2.41.4 and the actual domain base-change theorem. At such a point v, extend the ground field to a nontrivially valued complete field Ω containing $\mathcal{H}(v)$. Evaluation gives an Ω-rational lift w. The base point is then Ω-rational as well. Proposition 2.35.13 proves naive flatness at w. Both scalar-stalk maps are faithfully flat by Theorem 2.35.4, so Lemma 2.35.6

descends naive flatness to v. All the tests in Definition 2.9.1 have now been verified.

Its fibres are geometrically regular. Let F be any complete extension of $\mathcal{H}(f(y))$, and take any point v of the extended fibre above y. The pulled-back chart is still a Jacobian presentation over F. Choose a nontrivially valued complete $\Omega \supset \mathcal{H}(v)$ and its rational lift. The completed local ring at that lift is $\Omega[[T_1, \dots, T_{n-r}]]$ by Proposition 2.35.13. Thus its local ring is regular. Its faithfully flat local map from the original fibre stalk has Noetherian source and target, by Theorem 2.35.11. Regularity descends by [StacksAlg, Tag 00OF]. This proves geometric regularity with every complete-extension test.

If the original quasi-smooth presentation uses only an affinoid domain $W \subset Y$ containing y, analytic flatness descends from W to the good source Y by Corollary 2.47.3. On each complete-extension fibre, the inverse image of W is a domain containing every point over y. Its ordinary local map from the ambient fibre is faithfully flat by Theorem 2.47.1. Applying [StacksAlg, Tag 00OF] again descends regularity from this domain to that fibre. Thus the preceding conclusions hold for the original source, not only for the chosen presentation.

Recovering a Jacobian presentation. Now suppose Y, X good, f analytically flat at y, and its analytic fibre geometrically regular at y. Choose ordinary affinoid neighborhoods $Y_0 = \mathcal{M}(B)$, $X_0 = \mathcal{M}(A)$ with $f(Y_0) \subset X_0$. Lemma 2.33.7 gives $B = P/I$, where $P = A\{s^{-1}T_1, \dots, s_n^{-1}T_n\}$. Let r be the rank at y of the relative differentials of I, so that $r = n - \operatorname{rk}_y \Omega_{B/A}$ by Proposition 2.41.6. Choose $g_1, \dots, g_r \in I$ with independent such differentials, put $J = (g_1, \dots, g_r)$, and let $Z = \mathcal{M}(P/J)$. The image point, still denoted y, has the same completed residue field. An invertible Jacobian minor makes $Z \to X_0$ quasi-smooth at y, so the forward part already proved applies to it.

Choose a nontrivially valued complete $\Omega \supset \mathcal{H}(y)$, with rational lifts $u \in Y_{0,\Omega} \subset Z_\Omega$ and $v \in X_{0,\Omega}$. Put

$$R = \mathcal{O}_{X_{0,\Omega},v}, \quad S = \mathcal{O}_{Z_\Omega,u}, \quad T = \mathcal{O}_{Y_{0,\Omega},u}, \quad N = \ker(S \longrightarrow T).$$

These rings are Noetherian, and N is a finite S-module. The R-module T is flat, by the assumed analytic flatness. The two fibre local rings

$$S/\mathfrak{m}_R S, \qquad T/\mathfrak{m}_R T$$

are regular. For the first this follows from Proposition 2.35.13; for the second it is the assumed geometric regularity, using the identification of this fibre

with the scalar extension of $Y_{f(y)}$ by associativity of completed base change. Both have embedding dimension $n - r$, by Lemma 2.35.12 and invariance of the Jacobian rank under the field embedding. Their dimensions therefore both equal $n-r$. The surjection between them has zero kernel by [StacksAlg, Tag 00NR]: an ideal with regular quotient of the same dimension is generated by zero members of a regular parameter system.

Let R be a commutative ring and let $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules with C flat. For every R-module X, tensoring the sequence with X is exact. To check injectivity on the left, choose a free module F mapping onto X, with kernel G. Lift an element of $X \otimes_R A$ that dies in $X \otimes_R B$ to $v \in F \otimes_R A$. Its image in $F \otimes_R B$ comes from some $w \in G \otimes_R B$. The image of w in $G \otimes_R C$ is zero because it is zero in $F \otimes_R C$ and C is flat. Right exactness lifts w to $G \otimes_R A$. Subtract the image of this lift from v. The difference vanishes in $F \otimes_R B$, hence in $F \otimes_R A$ since F is free. Thus v comes from $G \otimes_R A$, and the original element of $X \otimes_R A$ is zero.

If A is also flat, then B is flat. Given an injection $X \hookrightarrow Y$, tensor the short exact sequence with X and with Y, using the first assertion for the left injections. An element of $X \otimes_R B$ that dies in $Y \otimes_R B$ maps to zero in $X \otimes_R C$, by flatness of C. It comes from $X \otimes_R A$. The image of this preimage in $Y \otimes_R A$ is zero by the left injection in the row for Y; flatness of A kills the preimage. Hence the induced map on tensoring with B is injective for every injection, which is flatness of B.

Apply the first assertion to $0 \to N \to S \to T \to 0$ and $X = R/\mathfrak{m}_R$. The quotient T is flat over R, so $N/\mathfrak{m}_R N$ injects into the zero fibre kernel just computed. Hence $N = \mathfrak{m}_R N$. Because $\mathfrak{m}_R S \subset \mathfrak{m}_S$ and N is finite, $N = 0$. Explicitly, writing finite generators in terms of themselves with coefficients in $\mathfrak{m}_R S$ gives a matrix C with entries in $\mathfrak{m}_S$; the adjugate identity for $1 - C$ and the unit $\det(1 - C)$ annihilate every generator.

The coherent ideal I/J on Z thus has zero scalar-pullback stalk at u. Theorem 2.35.4 identifies that stalk with its tensor product over $\mathcal{O}_{Z,y}$ with the faithfully flat $\mathcal{O}_{Z_\Omega,u}$. Consequently $(I/J)_y = 0$. Its finite generators vanish on an ordinary neighborhood N_y in Z. Since Z is closed in the ambient polydisc, choose an ordinary open neighborhood of y in that polydisc whose intersection with Z is contained in N_y, and then a rational affinoid neighborhood D' inside it. On D', the two closed subspaces Y_0 and Z coincide, with ideal generated by $g_1, \ldots, g_r$. Their intersection with D' is an ordinary

affinoid neighborhood of y in Y, and the minor stays nonzero at y. This is the asserted strengthened Jacobian presentation.

For general analytic spaces, choose affinoid domains in source and target and use the G-local definition of analytic flatness. Quasi-smoothness is invariant under these restrictions by Theorem 2.41.8. Completed residue fields on domains agree, so the local analytic fibres are precisely analytic domains in the original fibre. In the forward direction the chart argument above proves regularity on all such domains after all complete extensions. In the reverse direction geometric regularity of the fibre supplies regularity on its affinoid charts, while analytic flatness supplies the good-space tests there. The good case therefore applies on every required chart and proves the general equivalence. □

References. The equivalence and the strengthened presentation are the classical assertions of [Duc18, 5.3.2–5.3.4]. The proof here organizes their analytic content through rational lifts, explicit completed local rings and faithfully flat descent; the commutative-algebra leaves are the checked Stacks statements displayed above.

2.35.4 Finite direct images and overconvergent restriction

Proposition 2.35.15 (Coherent direct image under a finite morphism)**.** *Let $\pi : Y \to T$ be a norm-admissible finite analytic morphism and let $\mathcal{F}$ be coherent on Y_G. Then $\pi_*\mathcal{F}$ is coherent on T_G, and formation of this direct image is exact and commutes with every analytic base change, including base change over an arbitrary complete extension of the ground field.*

Proof. For an affinoid domain $V = \mathcal{M}(A) \subset T$, write $\pi^{-1}(V) = \mathcal{M}(B)$, where B is a finite A-module with its canonical norm. By Theorem 2.39.6, the restriction of $\mathcal{F}$ is represented by a finite B-module M. It is also finite over A. For every affinoid domain $V' \subset V$, finite base change and the finite-module description of coherent restriction identify its inverse-image sections with $M \otimes_A A_{V'}$. Thus the direct image on V_G is the coherent module sheaf associated to M over A. These identifications agree with the natural restriction maps and hence glue on the full analytic site. Exactness follows on each V from affinoid coherent acyclicity, or equivalently from exactness of the finite-module correspondence and restriction of scalars.

Now take an affinoid piece $\mathcal{M}(C)$ of any analytic base change, with C an L-affinoid algebra and L/K complete. Proposition 2.33.19 identifies the new

finite algebra with $D = B \otimes_A C$, with the canonical finite C-module norm; the ordinary tensor product already gives its completed analytic pushout. The pulled-back coherent module on this finite space is $M \otimes_B D = M \otimes_A C$, by the finite-module pullback construction. This is precisely the module representing the pullback of $\pi_*\mathcal{F}$. These canonical identifications commute with restrictions and with the original base-change map, so they prove that map is an isomorphism globally. □

Lemma 2.35.16 (A flat coherent ordinary stalk is locally free)**.** *Let T be good and $\mathcal{G}$ coherent on T_G. If $\mathcal{G}_t$ is flat over $\mathcal{O}_{T,t}$, then $\mathcal{G}$ is free of finite rank on an ordinary neighborhood of t. In particular it is analytically T-flat at t.*

Proof. Put $R = \mathcal{O}_{T,t}$ and $M = \mathcal{G}_t$. The ring R is Noetherian by Theorem 2.35.11, and M is finite. Lift a basis of $M/\mathfrak{m}_R M$ to m elements of M. Nakayama's determinant argument makes the induced map $R^m \to M$ surjective. Its kernel N is finite. Tensoring with the residue field is left exact at N, since flatness of M kills the first Tor group; the chosen residue basis therefore gives $N/\mathfrak{m}_R N = 0$. Another application of the determinant argument gives $N = 0$. Hence M is free.

Represent the finitely many basis germs by local sections, giving a morphism $\mathcal{O}^m \to \mathcal{G}$ on an ordinary neighborhood. Its kernel and cokernel are coherent by the local finite-module theorem. Each has zero ordinary stalk at t, and finite generators on an affinoid neighborhood therefore vanish after shrinking to an ordinary neighborhood of t. The morphism is an isomorphism there. Any analytic pullback of this isomorphism is again a finite-rank free module, whose ordinary stalks are flat. This checks all complete-extension and good-base-change tests. □

Theorem 2.35.17 (Finite pushforward detects flatness)**.** *Let $Y \xrightarrow{\pi} T \to X$ be analytic morphisms with π finite, let $t \in T$, and write $\pi^{-1}(t) = \{y_1, \dots, y_m\}$. For a coherent sheaf $\mathcal{F}$ on Y_G, if the spaces are good there is a canonical decomposition*

$$(\pi_*\mathcal{F})_t \simeq \bigoplus_{i=1}^{m} \mathcal{F}_{y_i}.$$

Consequently $\pi_*\mathcal{F}$ is naively X-flat at t if and only if $\mathcal{F}$ is naively X-flat at every y_i. The analogous equivalence for analytic X-flatness holds for arbitrary analytic spaces.

Proof. For good spaces, use Lemma 2.35.1. Representatives of finitely many section germs at the y_i may be restricted to its disjoint open-and-closed branches above a smaller ordinary neighborhood of t; they glue to a section of $\mathcal{F}$ on that full inverse image. If all germs of such a section vanish, shrinking the same way makes it vanish on a full inverse-image neighborhood. This proves the displayed isomorphism for module sheaves just as for structure sheaves. It respects the action of $\mathcal{O}_{X,f(t)}$. A finite direct sum of modules is flat exactly when every summand is flat: tensor an injection and inspect its component kernels. This gives the naive equivalence.

Take any good analytic base change $X' \to X$ over any complete extension. The map $\pi' : Y' \to T'$ is still finite, and Proposition 2.35.15 identifies $(\pi_*\mathcal{F})_{T'}$ with $\pi'_*(\mathcal{F}_{Y'})$. At any t' above t, the naive equivalence already proved applies to its finite fibre. Every point of that fibre lies over one of the y_i, and every point above a y_i is in the fibre of its own image t'. Thus the complete collection of naive-flatness tests on the two sides is the same. This proves the analytic equivalence in the good case. For general spaces, restrict to affinoid domains in X, T and their finite inverse-image domains in Y. The general G-local definition and the canonical base-change isomorphism give the same equivalence. □

Corollary 2.35.18 (Naive flatness for a finite morphism)**.** *For a finite morphism $Y \to X$ of good spaces and a coherent sheaf $\mathcal{F}$ on Y_G, naive X-flatness at a point y is equivalent to analytic X-flatness there.*

Proof. The analytic assertion always implies the naive one. For the converse, separate the finite branches over $x = f(y)$ by Lemma 2.35.1, and retain the branch containing y. It is an ordinary neighborhood of y, finite over an ordinary neighborhood U of x, and its fibre over x consists only of y. Denote its finite map by π. Theorem 2.35.17 makes $(\pi_*\mathcal{F})_x$ flat over $\mathcal{O}_{U,x}$. Lemma 2.35.16 makes the direct image locally free near x, hence analytically U-flat. The analytic equivalence in Theorem 2.35.17 gives analytic flatness at y. Ordinary open restrictions do not change any of the pointwise tests, so this is the assertion for the original morphism. □

Proposition 2.35.19 (The overconvergent restriction interface)**.** *Let $Y \to X$ and $Y' \to X$ be morphisms of good spaces. Suppose an ordinary analytic neighborhood W of y in Y is identified over X with a good analytic domain in Y', and suppose coherent sheaves $\mathcal{F}$ on Y and $\mathcal{G}$ on Y' have identified restrictions to W. Then $\mathcal{F}$ is naively X-flat at y if and only if $\mathcal{G}$ is; the same equivalence holds for analytic X-flatness. Both statements are unchanged by restricting X to an ordinary neighborhood of the image point and the sources to its inverse images.*

Proof. The ordinary restriction from Y to W preserves the stalk at y. After any good analytic base change, its inverse image is again an ordinary neighborhood of every point over y, so it preserves all analytic-flatness tests too. On the other side, Corollary 2.47.3 gives the equivalence of both types of flatness for $\mathcal{G}$ and its restriction to the source domain W. Combine these two facts with the given isomorphism. A target ordinary open restriction changes neither the target stalk nor the source stalk; every base-change test at the point factors locally through that target open set. Hence it also preserves both assertions. □

References. These arguments reconstruct the finite-pushforward step and the restriction mechanism used in [Duc18, 4.1.14–4.1.15 and 8.3.1–8.3.4]. The boundaryless inner-flatness theorem is a separate input to the final overconvergent application; the proposition above is the exact transfer needed once that theorem is established.

2.35.5 Finite algebraic tests realized on analytic germs

Theorem 2.35.20 (Realizing a finite ordinary-stalk algebra)**.** *Let X be good, let $x \in X$, and put $R = \mathcal{O}_{X,x}$. For every finite R-algebra C there are an ordinary affinoid neighborhood $V = \mathcal{M}(A)$ of x and a finite A-algebra B, with its canonical affinoid norm, such that*

$$C \simeq B \otimes_A R.$$

Writing $Z = \mathcal{M}(B)$ and $\pi^{-1}(x) = \{z_1, \dots, z_m\}$ for the resulting finite map, one has $C \simeq \prod_i \mathcal{O}_{Z,z_i}$. If C is a nonzero domain, one can choose B to be a domain; then $m = 1$ and $\mathcal{O}_{Z,z_1} \simeq C$.

Proof. The ring R is Noetherian by Theorem 2.35.11. Choose finitely many algebra generators c_j of C; every c_j is integral over R, by the determinant

argument on a finite generating set of the R-module C containing 1. Thus the kernel of $R[T_1, \dots, T_s] \to C$ contains a monic polynomial in each T_j. It is finitely generated, because the polynomial ring is Noetherian by [StacksAlg, Tag 00FN]. Adjoin these monic polynomials to a finite generating set of the kernel.

All coefficients of this finite list of relations are germs represented on a common ordinary affinoid neighborhood V of x, by Proposition 2.33.16. Write $A = \mathcal{O}(V)$ and take the quotient B of $A[T_1, \dots, T_s]$ by these representatives. The monic relations show that bounded powers of the T_j span B over A, so B is finite. Tensoring its presentation with R recovers exactly the chosen presentation of C. Theorem 2.33.18 gives B its actual finite affinoid structure. Lemma 2.35.1 then identifies $B \otimes_A R$ with the displayed product of ordinary local rings.

Suppose C is a nonzero domain. Let I be the kernel of $B \to B \otimes_A R = C$. Since C is a domain, B/I is a domain. It is still finite over A, and its tensor product with R is C: right exactness gives the quotient of C by the image of $I \otimes_A R$, and that image is zero by the definition of I. Replace B by B/I. The resulting product of nonzero local rings is the domain C, so exactly one factor occurs. This gives the last assertion, without normalization or an excellence assumption. □

Corollary 2.35.21 (Henselian ordinary local rings)**.** *Every ordinary local ring R of a good analytic space has the following properties: every finite R-algebra is a finite product of local rings, and every simple residue-field root of a monic polynomial over R lifts uniquely to a root in R. In particular R is henselian.*

Proof. The first assertion is Theorem 2.35.20. To prove the second directly, let $P \in R[T]$ be monic and let $a \in R/\mathfrak{m}_R$ be a simple root of $\overline{P}$. Apply the first assertion to the finite free algebra $B = R[T]/(P)$. Every local factor is finite over R and hence has its maximal ideal over $\mathfrak{m}_R$: a field integral over the image of R forces that image to be a field. The factor C corresponding to the maximal ideal $(\mathfrak{m}_R, T - a)$ therefore satisfies $C/\mathfrak{m}_R C \simeq R/\mathfrak{m}_R$. Here simplicity of the root ensures that the associated local factor of $(R/\mathfrak{m}_R)[T]/(\overline{P})$ has dimension one and no nilpotents; factor $\overline{P} = (T - a)Q$ with $Q(a) \neq 0$ and use the Chinese remainder identity.

Let $\epsilon : R \to C$ be the structure map and put $k = R/\mathfrak{m}_R$. Its cokernel Q_C is finite. The isomorphism $k \to C/\mathfrak{m}_R C$ gives $Q_C/\mathfrak{m}_R Q_C = 0$ by right

exactness. Apply [StacksAlg, Tag 00DV(2)] with $M = Q_C$ and $I = \mathfrak{m}_R$ to see that ϵ is surjective.

Let $p_C : B \to C$ be the projection. Lift the images under p_C of the standard free R-basis of B along ϵ. These lifts define an R-linear map $B \to R$. Its restriction to the direct summand C splits ϵ as a map of R-modules. Thus $N = \ker \epsilon$ is a direct summand of R, hence finite, and reduction of this split sequence identifies $N/\mathfrak{m}_R N$ with the kernel of $k \to C/\mathfrak{m}_R C$. That kernel is zero. Apply [StacksAlg, Tag 00DV(2)] with $M = N$ and $I = \mathfrak{m}_R$ to get $N = 0$. The image of T in $C = R$ is the desired lifted root. If $b, c \in R$ were two lifts, then $P(b) - P(c) = (b - c)Q_1(b, c)$, where $Q_1(b, c)$ reduces to $\overline{P}'(a) \neq 0$; hence $Q_1(b, c)$ is a unit and $b = c$. □

Proposition 2.35.22 (Realization of prime and finite-field fibre tests)**.** *Let $f : Y \to X$ be a morphism of good spaces, let $x = f(y)$, and put $R = \mathcal{O}_{X,x}$, $S = \mathcal{O}_{Y,y}$. For a prime $\mathfrak{p} \subset R$ and a finite extension $E/\operatorname{Frac}(R/\mathfrak{p})$, there are, after ordinary neighborhood restrictions, an integral finite analytic space $Z \to X$ with a unique point z above x, and finitely many points $t_1, \dots, t_m$ above y in $T = Y \times_X Z$, such that*

$$C := \mathcal{O}_{Z,z} \text{ is a domain}, \qquad \operatorname{Frac}(C) = E, \qquad \ker(R \to C) = \mathfrak{p},$$

$$S \otimes_R E \simeq \left(\prod_{i=1}^{m} \mathcal{O}_{T,t_i} \right) \otimes_C E.$$

Every local ring of $\operatorname{Spec}(S \otimes_R E)$ is therefore a localization of some $\mathcal{O}_{T,t_i}$ at a prime whose contraction to the affinoid domain algebra of Z is zero.

Proof. Write $R_0 = R/\mathfrak{p}$ and $F = \operatorname{Frac}(R_0)$. Choose finitely many generators $\alpha_1, \dots, \alpha_s$ of the finite field extension E/F. Each satisfies a monic equation over F. For each j, choose a nonzero $d_j \in R_0$ clearing the finitely many denominators in that equation. Then $\beta_j = d_j \alpha_j$ is integral over R_0: in the transformed monic equation, its ith coefficient is the old ith coefficient multiplied by d_j^i, which lies in R_0. Thus $C = R_0[\beta_1, \dots, \beta_s]$ is a finite integral domain over R_0, and its fraction field is exactly E. This construction uses no integral-closure finiteness theorem.

Apply Theorem 2.35.20 to C, obtaining an ordinary affinoid neighborhood $X_0 = \mathcal{M}(A)$ and a finite domain B over A, with $Z = \mathcal{M}(B)$, a unique point z above x, and $B \otimes_A R = C = \mathcal{O}_{Z,z}$. The construction there may and does arrange $B \hookrightarrow C$, by quotienting the kernel of this map. Choose an affinoid

neighborhood Y_0 of y mapping to X_0; this changes neither R nor S. The projection $T = Y_0 \times_{X_0} Z \to Y_0$ is finite. Lemma 2.35.1 and finite algebra base change give

$$\prod_i \mathcal{O}_{T,t_i} \simeq B \otimes_A S \simeq (B \otimes_A R) \otimes_R S = C \otimes_R S.$$

Localize this identity by $C \setminus \{0\}$. Since $\mathrm{Frac}(C) = E$, it becomes the asserted isomorphism. Every prime of a finite product belongs to exactly one factor: the complementary factor idempotents have product zero and sum one. A prime in the localization avoids every nonzero element of C, and hence every nonzero element of its subring B. Localizing once more at that prime gives precisely the local ring of the original factor at its corresponding prime. This proves the last assertion and explicitly realizes every finite-field test in an algebraic fibre of the ordinary-stalk morphism. □

References. Compare [Duc18, 2.1.6 and 2.3.13–2.3.14]. Clearing finitely many monic-equation denominators replaces the normalization input in the finite-field test. The realization theorem is a reduction of tests, not a proof that those local rings are regular.

2.35.6 Regularity of domains in affine space

Theorem 2.35.23 (Regular affine-space domains)**.** *Let K be any complete non-Archimedean field, and let $V = \mathcal{M}(A)$ be an affinoid domain in $\mathbb{A}_K^{n,\mathrm{an}}$. Every ordinary local ring $\mathcal{O}_{V,v}$ is regular, and A is a regular Noetherian ring. The same assertions hold after every complete valued extension of K.*

Proof. Fix $v \in V$. Choose a nontrivially valued complete extension L of $\mathcal{H}(v)$; if necessary adjoin a free Gauss radius. The evaluated coordinates give an L-rational point w of V_L above v. Indeed, the bounded map $A \to \mathcal{H}(v) \to L$ and the identity of L induce a bounded map $A \widehat{\otimes}_K L \to L$. By analytic domain base change, V_L is an affinoid domain of $\mathbb{A}_L^{n,\mathrm{an}}$. Lemma 2.35.9 makes V_L an ordinary neighborhood of w in that affine space. Consequently

$$\mathcal{O}_{V_L,w} = \mathcal{O}_{\mathbb{A}_L^{n,\mathrm{an}},w}, \qquad \widehat{\mathcal{O}_{V_L,w}} \simeq L[[T_1, \dots, T_n]].$$

The completion formula is Proposition 2.35.10, with the coordinates translated by their values at w. The convergent-germ theorem makes the local ring Noetherian. Its completion is regular by [StacksAlg, Tag 032C]; the regularity/completion criterion [StacksAlg, Tag 07NY] therefore makes $\mathcal{O}_{V_L,w}$ regular.

Theorem 2.35.4 makes $\mathcal{O}_{V,v} \to \mathcal{O}_{V_L,w}$ a faithfully flat local homomorphism. Both rings are Noetherian by Theorem 2.35.11, so regularity descends by [StacksAlg, Tag 00OF].

To pass from ordinary local rings to A, take a maximal ideal $\mathfrak{m}$ of A. It is closed by the finite-module theorem; the nonzero affinoid quotient $A/\mathfrak{m}$ has a point by Theorem 1.7.1. Its point kernel is zero, since this quotient is a field. It therefore gives $v \in V$ with point kernel exactly $\mathfrak{m}$. The map $A \to \mathcal{O}_{V,v}$ is flat: the latter is the filtered colimit of the rational-neighborhood algebras of Proposition 2.33.16, each of which is flat over A by Proposition 2.38.2; the finite-ideal criterion commutes with this filtered colimit. Evaluation now gives a local flat map $A_{\mathfrak{m}} \to \mathcal{O}_{V,v}$. Hence $A_{\mathfrak{m}}$ is regular by the same descent lemma. Every $A_{\mathfrak{p}}$ is a localization of some such $A_{\mathfrak{m}}$ and is regular by [StacksAlg, Tag 0AFS]. Thus A is regular.

Finally, affinoid domain base change identifies V_F with an affinoid domain of $\mathbb{A}^{n,\mathrm{an}}_F$ for every complete F/K. Apply the argument over F. □

References. The proof reconstructs the rational-lift argument of A. Ducros, *Les espaces de Berkovich sont excellents*, Lemma 2.1, using the ordinary-stalk and completion results proved above.

Proposition 2.35.24 (A uniform separable regular locus)**.** *Let D be the affinoid algebra of an affinoid domain in $\mathbb{A}^{n,\mathrm{an}}_K$, assume that D is a domain, and let $D \hookrightarrow A$ be finite with A a domain. Suppose* $\operatorname{Frac}(A)/\operatorname{Frac}(D)$ *is separable. There is $0 \neq d \in D$ such that, for every complete L/K, writing $A_L = A\widehat{\otimes}_K L$ and $X_L = \mathcal{M}(A_L)$, the following rings are regular:*

$$A_L[1/d], \qquad \mathcal{O}_{X_L,y}[1/d], \qquad \widehat{\mathcal{O}_{X_L,y}}[1/d] \quad (y \in X_L).$$

Here a hat denotes completion at the maximal ideal of the ordinary local ring. In particular this conclusion holds for every strict integral K-affinoid algebra A when $\operatorname{char} K = 0$*, by choosing a finite Tate normalization.*

Proof. Put $F = \operatorname{Frac}(D)$ and $E = \operatorname{Frac}(A)$. The finite-dimensional domain $A \otimes_D F$ is a field, and is therefore E. The primitive-element theorem [StacksAlg, Tag 030N] gives $E = F(\theta)$. After multiplying θ by a nonzero element of D, we may take $\theta \in A$. Let $P \in F[T]$ be its monic minimal polynomial. Choose finitely many D-module generators of A, and express each as an F-linear combination of $1, \theta, \dots, \theta^{m-1}$, where $m = \deg P$. Separability

gives $P'(\theta) \neq 0$; express its inverse in the same basis. Choose $0 \neq d \in D$ clearing the coefficients of P and all these finitely many expressions. Then

$$A[1/d] = D[1/d][\theta] \simeq D[1/d][T]/(P), \qquad P'(\theta) \in A[1/d]^{\times}.$$

The displayed presentation is exact: division by the monic polynomial reduces every relation to degree less than m, and $1, \theta, \dots, \theta^{m-1}$ are F-linearly independent. Thus $D[1/d] \to A[1/d]$ is finite standard étale. This uses the standard-étale verification after Definition 59.26.3 in [StacksAlg, Tag 03PA]. Its base changes and product factors are étale by [StacksAlg, Tag 00U2, parts (3) and (11)].

Fix L/K and put $D_L = D\widehat{\otimes}_K L$. Finite norm-admissible base change, Proposition 2.33.19, identifies A_L with $A \otimes_D D_L$. Theorem 2.35.23 makes D_L regular. Hence $A_L[1/d]$ is regular: at every prime it is étale over a localization of the regular ring $D_L[1/d]$, and [StacksAlg, Tag 025N] preserves regularity under such a local étale map.

Let $t \in \mathcal{M}(D_L)$, put $R = \mathcal{O}_{\mathcal{M}(D_L),t}$, and list the finitely many points y_i above t. Finite branch separation gives an isomorphism of finite R-algebras

$$A_L \otimes_{D_L} R \simeq \prod_i S_i, \qquad S_i = \mathcal{O}_{X_L,y_i}.$$

The ring R is regular by Theorem 2.35.23. Localizing the displayed product at d makes each factor finite étale over $R[1/d]$. This proves regularity of all $S_i[1/d]$ by the same local argument.

For completeness, the finite-module completion identity [StacksAlg, Tag 00MA, part (3)] identifies the $\mathfrak{m}_R$-adic completion of the product with $A_L \otimes_{D_L} \widehat{R}$. Each S_i is a finite local R-algebra, so $S_i/\mathfrak{m}_R S_i$ is an Artinian local algebra over the residue field of R. Its maximal ideal is nilpotent: the descending sequence of its powers stabilizes in a finite-dimensional vector space, and Nakayama makes the stable power zero. Thus $\mathfrak{m}_{S_i}^{N_i} \subset \mathfrak{m}_R S_i \subset \mathfrak{m}_{S_i}$ for some N_i. The two adic filtrations are cofinal, giving

$$A_L \otimes_{D_L} \widehat{R} \simeq \prod_i \widehat{S}_i.$$

The ring $\widehat{R}$ is regular by [StacksAlg, Tag 07NY]. After inverting d, each factor on the right is finite étale over $\widehat{R}[1/d]$ and hence regular. This proves the completion assertion. Finally, for strict integral A in characteristic zero, Theorem 1.12.2 supplies $D = K\{T_1, \dots, T_n\}$; the finite extension of fraction fields

is separable because a nonconstant irreducible polynomial in characteristic zero has nonzero derivative and is coprime to that derivative. □

References. This is the separable finite-normalization part of Ducros, *Les espaces de Berkovich sont excellents*, Proposition 2.2, with a single denominator chosen before arbitrary scalar extension.

Lemma 2.35.25 (Strict ordinary neighborhoods)**.** *Suppose the valuation of K is nontrivial and $X = \mathcal{M}(A)$ is strict. Every point of X has a basis of ordinary neighborhoods consisting of strict affinoid domains. Consequently, in Theorem 2.35.20, when X is strict, the neighborhood algebra and the finite algebra may both be chosen strict.*

Proof. The topology of the spectrum is generated by evaluation inequalities. A prescribed open neighborhood of x therefore contains a finite intersection of conditions $|f_i| < a_i$ and $|g_j| > b_j$ holding at x. Since the valuation is nontrivial, $\sqrt{|K^\times|}$ is dense in $\mathbb{R}_{>0}$: the rational powers of any element of absolute value different from one already have logarithms dense in $\mathbb{R}$. Choose r_i, s_j in this group with

$$|f_i(x)| < r_i < a_i, \qquad b_j < s_j < |g_j(x)|.$$

The domain defined by $|f_i| \le r_i$ and $|g_j| \ge s_j$ is an ordinary neighborhood of x contained in the prescribed open set. Write $r_i^{m_i} = |c_i|$ and $s_j^{n_j} = |d_j|$, with nonzero scalars $c_i, d_j \in K$. The inequalities are equivalent to

$$|c_i^{-1} f_i^{m_i}| \le 1, \qquad |d_j^{-1} g_j^{n_j}| \ge 1.$$

They give a finite succession of unit-radius Weierstrass and Laurent domains, hence a strict affinoid domain by the established presentation and intersection theorems. The finite list of germs in the proof of Theorem 2.35.20 can thus be represented on such a strict neighborhood. Its finite algebra is strict by Proposition 1.14.4; taking the quotient by the kernel used there preserves strictness. □

Theorem 2.35.26 (Ordinary regular comparison in the strict characteristic-zero case)**.** *Let K be nontrivially valued of characteristic zero, let A be strict K-affinoid, let $x \in X = \mathcal{M}(A)$, and put $\mathfrak{p} = \ker(x)$ and $R = \mathcal{O}_{X,x}$. Both of the following homomorphisms are regular:*

$$A_{\mathfrak{p}} \longrightarrow R, \qquad R \longrightarrow \widehat{R}.$$

Here regular means flat with geometrically regular algebraic fibres; the fibre fields and tensor products in this assertion are those of ordinary commutative algebra.

Proof. All rings are Noetherian, by the affinoid finite-module theorem, Theorem 2.35.11, and [StacksAlg, Tag 0316]. The map $A_{\mathfrak{p}} \to R$ is flat: R is the filtered colimit of ordinary rational-neighborhood algebras, all flat over A, and elements of $A \setminus \mathfrak{p}$ are units in R. The completion map is faithfully flat by [StacksAlg, Tag 00MC]. It remains to check their algebraic fibres.

Let $\mathfrak{q} \subset \mathfrak{p}$ be any prime of A, set $B = A/\mathfrak{q}$, and put $F = \operatorname{Frac}(B)$. The point x lies on $Z = \mathcal{M}(B)$, and finite branch separation for this closed immersion gives

$$R/\mathfrak{q}R = \mathcal{O}_{Z,x}, \qquad R \otimes_A F = \mathcal{O}_{Z,x} \otimes_B F.$$

The ring B is strict and integral. Proposition 2.35.24 supplies a nonzero $d \in B$ for which $\mathcal{O}_{Z,x}[1/d]$ is regular. The displayed fibre is its further localization by the nonzero elements of B, so it is a regular Noetherian ring. Its ground field F has characteristic zero. By [StacksAlg, Tag 0381], a Noetherian algebra over a characteristic-zero field is geometrically regular once it is regular: the only finite purely inseparable field extension to be tested is the identity extension. This proves geometric regularity of every fibre of $A_{\mathfrak{p}} \to R$, at all primes $\mathfrak{q} \subset \mathfrak{p}$.

Now let $\mathfrak{P}$ be any prime of R and put $C = R/\mathfrak{P}$. By Lemma 2.35.25 and Theorem 2.35.20, there is a strict integral affinoid algebra B, a point $z \in Z = \mathcal{M}(B)$, and an identification

$$B \hookrightarrow \mathcal{O}_{Z,z} = C.$$

The injection is part of the kernel-quotient construction in that theorem. Finite-module completion, [StacksAlg, Tag 00MA, part (3)], gives $\widehat{R} \otimes_R C = \widehat{C}$, since the maximal ideal of the quotient C is the image of the maximal ideal of R. Consequently the algebraic fibre of $R \to \widehat{R}$ at $\mathfrak{P}$ is

$$\widehat{R} \otimes_R \kappa(\mathfrak{P}) = \widehat{C} \otimes_C \operatorname{Frac}(C).$$

Apply Proposition 2.35.24 to B. It gives $0 \neq d \in B$ such that $\widehat{\mathcal{O}_{Z,z}}[1/d] = \widehat{C}[1/d]$ is regular. Because $B \hookrightarrow C$, the image of d is nonzero in C. The last displayed fibre is thus a further localization of this regular ring, and is regular. It is Noetherian and is an algebra over the characteristic-zero field

$\mathrm{Frac}(C)$, so the same Tag 0381 makes it geometrically regular. This treats every prime of R, and proves the second assertion. □

References. The generic-fibre organization follows Ducros, *Les espaces de Berkovich sont excellents*, Proposition 2.9. Here separable normalization supplies the regular locus, and finite ordinary-germ realization handles every prime of the ordinary local ring.

Proposition 2.35.27 (A uniform characteristic-zero regular open set)**.** *Let K be any complete field of characteristic zero and let A be an integral generalized K-affinoid algebra. There exists $0 \neq a \in A$ such that, for every complete L/K, the ring $A_L[1/a]$ is regular and every ordinary local ring of $\mathcal{M}(A_L)$ at a point where a is nonzero is regular.*

Proof. Choose a finite free polyradius r such that $F = K_r$ is nontrivially valued and A_r is strict, by Theorem 1.17.6. The algebra A_r is integral. Indeed, Lemma 2.52.1 supplies $x \in \mathcal{M}(A)$ with kernel zero. Its Gauss lift has value

$$\left|\sum_{\nu} b_\nu T^\nu\right|_{\sigma_r(x)} = \max_{\nu} |b_\nu(x)| r^\nu.$$

This vanishes only when every coefficient is zero. The lift is a multiplicative seminorm by Theorem 1.19.1; its kernel-zero property makes A_r a domain. Apply Proposition 2.35.24 over F, and obtain $0 \neq d \in A_r$ with its conclusions over every complete extension of F. Write $d = \sum_\nu a_\nu T^\nu$, and choose one nonzero coefficient $a = a_{\nu_0} \in A$.

Fix a complete extension L/K and a point $y \in \mathcal{M}(A_L)$ with $|a(y)| > 0$. Let E be the completion of $\mathcal{H}(y)(T_1, \ldots, T_s)$ for the weighted Gauss absolute value of polyradius r. Lemma 3.21.7 gives compatible isometric embeddings $L \subset \mathcal{H}(y) \subset E$ and $F \subset E$. Evaluation $A_L \to \mathcal{H}(y) \subset E$ defines an E-rational lift w in $\mathcal{M}(A_E)$ above y. Associativity identifies

$$A_E = A_r \widehat{\otimes}_F E.$$

The image of d at this lift has absolute value $\max_\nu |a_\nu(y)| r^\nu > 0$. To justify the formula for the infinite expansion, its finite Laurent truncations converge in the Gauss field, since the weighted coefficient values tend to zero; the maximum formula passes to the limit. Thus d is a unit in the ordinary local ring at w. The separable regular-locus proposition makes that local ring regular. The local map

$$\mathcal{O}_{\mathcal{M}(A_L),y} \longrightarrow \mathcal{O}_{\mathcal{M}(A_E),w}$$

is faithfully flat by Theorem 2.35.4. Regularity therefore descends by [StacksAlg, Tag 00OF]. This proves the assertion about all ordinary local rings over $D(a)$, uniformly in L.

For any prime $\mathfrak{q}$ of A_L not containing a, Lemma 2.52.1 gives a point y with kernel $\mathfrak{q}$. It satisfies $|a(y)| > 0$. The map $(A_L)_{\mathfrak{q}} \to \mathcal{O}_{\mathcal{M}(A_L),y}$ is a faithfully flat local map, by the filtered rational-neighborhood description of the stalk and domain flatness. The target is regular by the preceding paragraph, so the source is regular by Tag 00OF. These are all local rings of $A_L[1/a]$, proving its regularity. □

Proposition 2.35.28 (Generic regularity of ordinary germs and their completions)**.** *Under the hypotheses of Proposition 2.35.27, one may choose $0 \neq a \in A$ such that, for every $x \in X = \mathcal{M}(A)$, both of the following rings are regular:*

$$\mathcal{O}_{X,x}[1/a], \qquad \widehat{\mathcal{O}_{X,x}}[1/a].$$

Proof. Take a from Proposition 2.35.27 and a nontrivial free strictification $F = K_r$. Write $B = A_r$, take any $z \in \mathcal{M}(B)$ above x, and put $R = \mathcal{O}_{X,x}$ and $S = \mathcal{O}_{\mathcal{M}(B),z}$. Such a z exists by the Gauss section. The ring $B[1/a]$ is regular. Theorem 2.35.26 says that $B_{\ker(z)} \to S$ and $S \to \widehat{S}$ are regular homomorphisms.

Let $\mathfrak{Q}$ be a prime of S avoiding a, and let $\mathfrak{q}$ be its contraction to B. The local ring $B_{\mathfrak{q}}$ is regular. The map $B_{\mathfrak{q}} \to S_{\mathfrak{Q}}$ is flat and has regular closed fibre, since that fibre is a localization of a geometrically regular fibre of $B_{\ker(z)} \to S$. The ascent lemma [StacksAlg, Tag 031E] makes $S_{\mathfrak{Q}}$ regular. Therefore $S[1/a]$ is regular. Applying exactly the same local argument to the regular map $S \to \widehat{S}$ gives regularity of $\widehat{S}[1/a]$.

The map $R \to S$ is faithfully flat by Theorem 2.35.4. All local rings are Noetherian, so [StacksAlg, Tag 0C4G] makes $\widehat{R} \to \widehat{S}$ flat; it is local, hence faithfully flat by [StacksAlg, Tag 00HP]. Inverting a preserves faithful flatness for both maps. Every prime of either source then lifts to the corresponding target by [StacksAlg, Tag 00HQ]. Localization yields a flat local map with regular target, and Tag 00OF descends regularity. This proves both assertions. □

Theorem 2.35.29 (Characteristic-zero ordinary regular comparison)**.** *Let K be any complete field of characteristic zero, with possibly trivial valuation. Let A be any generalized K-affinoid algebra, let $x \in X = \mathcal{M}(A)$, and put $\mathfrak{p} = \ker(x)$ and $R = \mathcal{O}_{X,x}$. The following maps are regular homomorphisms:*

$$A_{\mathfrak{p}} \longrightarrow R, \qquad R \longrightarrow \widehat{R}.$$

Proof. Flatness and Noetherianity are proved exactly as at the start of Theorem 2.35.26; those arguments use no strictness or nontriviality of the valuation. Let $\mathfrak{q} \subset \mathfrak{p}$, put $B = A/\mathfrak{q}$, and let $F = \operatorname{Frac}(B)$. The finite closed immersion $\mathcal{M}(B) \to X$ gives $R/\mathfrak{q}R = \mathcal{O}_{\mathcal{M}(B),x}$. The integral generalized algebra B satisfies Proposition 2.35.28, so this ordinary local ring becomes regular after inverting one nonzero element of B. Its further localization by all of $B \setminus \{0\}$ is the fibre $R \otimes_A F$, hence is regular. By [StacksAlg, Tag 0381] it is geometrically regular over F. This treats every fibre of the first map.

For a prime $\mathfrak{P} \subset R$, put $C = R/\mathfrak{P}$. Theorem 2.35.20 realizes C as $\mathcal{O}_{Z,z}$ for an integral generalized affinoid $Z = \mathcal{M}(B)$, with $B \hookrightarrow C$. Proposition 2.35.28 gives $0 \neq a \in B$ for which $\widehat{C}[1/a]$ is regular. Since a stays nonzero in C, the fibre

$$\widehat{R} \otimes_R \kappa(\mathfrak{P}) = \widehat{C} \otimes_C \operatorname{Frac}(C)$$

is a localization of a regular ring. The equality follows from finite-module completion, Tag 00MA. The fibre is geometrically regular over its characteristic-zero ground field by Tag 0381. This treats all primes of R and completes the proof. □

References. The coefficientwise regular-open descent and the completion square reconstruct the characteristic-zero part of Ducros, *Les espaces de Berkovich sont excellents*, Propositions 2.2 and 2.9. The field extensions used in the Gauss argument retain the original absolute value.

Theorem 2.35.30 (Ordinary regular maps at quasi-smooth points in characteristic zero)**.** *Let K be any complete field of characteristic zero, let $f : Y \to X$ be a morphism of good K-analytic spaces, and let f be quasi-smooth at y, with $x = f(y)$. Then the ordinary local homomorphism*

$$\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{Y,y}$$

is regular.

Proof. Put $R = \mathcal{O}_{X,x}$ and $S = \mathcal{O}_{Y,y}$. These rings are Noetherian, and the homomorphism is flat by Theorem 2.35.14. We prove regularity of its fibre at every prime $\mathfrak{P} \subset R$. By Theorem 2.35.20, after an ordinary affinoid restriction of X, the domain $C = R/\mathfrak{P}$ has a realization

$$B \hookrightarrow C = \mathcal{O}_{Z,z}, \qquad Z = \mathcal{M}(B) \longrightarrow X$$

with B integral and $Z \to X$ finite. There is a unique point z above x. The base change $T = Y \times_X Z \to Y$ is finite. List its finitely many points t_i above y, and write $S_i = \mathcal{O}_{T,t_i}$. Finite branch separation gives

$$S \otimes_R \operatorname{Frac}(C) \simeq \left(\prod_i S_i \right) \otimes_C \operatorname{Frac}(C).$$

The map $T \to Z$ is quasi-smooth at each t_i, by the Jacobian base-change theorem. Choose an ordinary affinoid neighborhood $T_i = \mathcal{M}(D_i)$ of each t_i on which this morphism is quasi-smooth everywhere. Such a neighborhood exists by the strengthened ordinary-neighborhood presentation in Theorem 2.35.14: the equations generate on that neighborhood, and a nonzero Jacobian minor stays invertible on a smaller rational neighborhood.

Apply Proposition 2.35.27 to B, obtaining $0 \neq a \in B$ such that, after every complete scalar extension, the ordinary local rings of Z are regular at all points where a is nonzero. We claim first that every ordinary local ring of T_i at a point v with $|a(v)| > 0$ is regular. Choose a nontrivially valued complete field $\Omega \supset \mathcal{H}(v)$ and the rational lift w of v in $(T_i)_\Omega$. Its image u in Z_Ω is rational and has $|a(u)| > 0$. Hence $R' = \mathcal{O}_{Z_\Omega,u}$ is regular. The lifted morphism is quasi-smooth at w, and Proposition 2.35.13 gives

$$\widehat{\mathcal{O}_{(T_i)_\Omega,w}} \simeq \widehat{R'}[[V_1, \ldots, V_d]].$$

The right side is regular. Indeed, $\widehat{R'}$ is regular by Tag 07NY; its power-series extension is flat by the completion argument in that proposition, with regular closed fibre $\kappa(u)[[V_1, \ldots, V_d]]$ by Tag 032C. Tag 031E gives regularity of the power-series ring. Tag 07NY then makes the local ring at w regular, and the faithfully flat scalar-stalk map descends regularity to $\mathcal{O}_{T_i,v}$ by Tag 00OF. This proves the claim.

Every prime of D_i avoiding a is the kernel of an analytic point, by Lemma 2.52.1. At that point the ordinary local ring is regular by the claim. Its faithfully flat local map from the corresponding localization of D_i descends

regularity. Thus $D_i[1/a]$ is regular. Theorem 2.35.29 makes $(D_i)_{\ker(t_i)} \to S_i$ regular. For any prime of S_i avoiding a, its contracted localization in D_i is regular, and the local map has regular closed fibre. Tag 031E therefore gives regularity of every such localization of S_i. Hence $S_i[1/a]$ is regular.

Finally a stays nonzero in C, since $B \hookrightarrow C$. The displayed algebraic fibre is therefore a localization of the finite product of regular rings $\prod_i S_i[1/a]$, and is regular. Its ground field $\mathrm{Frac}(C)$ has characteristic zero, so Tag 0381 makes it geometrically regular. Since $\mathfrak{P}$ was arbitrary, all algebraic fibres of the flat homomorphism $R \to S$ are geometrically regular. □

References. This reconstructs, in characteristic zero, the ordinary-local-ring comparison appearing in [Duc18, 5.5.3]. The argument uses all prime fibres of the ordinary local rings, through finite-germ realization and generic regularity.

Theorem 2.35.31 (Quasi-excellence in characteristic zero)**.** *Every generalized affinoid algebra over a complete field of characteristic zero is a quasi-excellent ring and is Nagata. Its algebraic normalization is finite.*

Proof. Let A be such an algebra. It is Noetherian. We first prove that A is a G-ring, meaning that for every prime $\mathfrak{p} \subset A$ the completion map $A_{\mathfrak{p}} \to \widehat{A_{\mathfrak{p}}}$ is regular. Choose $x \in \mathcal{M}(A)$ with kernel $\mathfrak{p}$, using Lemma 2.52.1, and put $D = A_{\mathfrak{p}}$ and $R = \mathcal{O}_{\mathcal{M}(A),x}$. Theorem 2.35.29 makes $D \to R$ and $R \to \widehat{R}$ regular.

We verify the composite on its fibres. For $\mathfrak{q} \in \mathrm{Spec}(D)$, set $F = \kappa(\mathfrak{q})$, $R_F = R \otimes_D F$ and $H_F = \widehat{R} \otimes_D F$. The ring R_F is regular. The map $R_F \to H_F$ is flat. A prime of R_F corresponds to a prime $\mathfrak{Q}$ of R over $\mathfrak{q}$, with the same residue field, because R_F is obtained from R by quotient and localization. Its fibre in H_F is therefore $\widehat{R} \otimes_R \kappa(\mathfrak{Q})$, which is regular. All these rings are Noetherian. At each prime of H_F, the local source is regular and the local map has regular closed fibre; Tag 031E makes the target local ring regular. Thus H_F is regular, and Tag 0381 makes it geometrically regular over F. This proves directly that $D \to \widehat{R}$ is regular.

By Tag 0C4G, $\widehat{D} \to \widehat{R}$ is flat, since $D \to R$ is a flat local map; it is local and hence faithfully flat. After tensoring with each $F = \kappa(\mathfrak{q})$ over D, the resulting map

$$\widehat{D} \otimes_D F \longrightarrow \widehat{R} \otimes_D F$$

remains faithfully flat. Its target is regular by the preceding paragraph. Tags 00HQ and 00OF descend regularity to its source, prime by prime. Tag 0381

then gives geometric regularity. Together with flatness of completion, this proves regularity of $D \to \widehat{D}$ at every algebraic fibre. Hence A is a G-ring.

To prove the J-2 condition, use [StacksAlg, Tag 07PC, equivalence (4)⇒(1)]. For any prime $\mathfrak{p} \subset A$, the field $\kappa(\mathfrak{p})$ has characteristic zero, so its finite purely inseparable extension is itself. Take the finite A-algebra $A/\mathfrak{p}$. It is a domain, has fraction field $\kappa(\mathfrak{p})$, and has a nonempty regular principal open subset by Proposition 2.35.27. These are exactly the hypotheses of that criterion. Thus A is J-2. By definition, a Noetherian G-ring which is J-2 is quasi-excellent. The Nagata conclusion follows from [StacksAlg, Tag 07QV], and finiteness of normalization from [StacksAlg, Tag 035S], applied to $\operatorname{Spec}(A)$. □

Corollary 2.35.32 (Normal loci and quasi-smooth ascent in characteristic zero)**.** *Let K be complete of characteristic zero. For an affinoid $X = \mathcal{M}(A)$ and $x \in X$, put $\mathfrak{p} = \ker(x)$. Then $A_{\mathfrak{p}}$ is normal if and only if $\mathcal{O}_{X,x}$ is normal. The set of such points is a Zariski open subset of X. The normal-point set of a good K-analytic space is therefore ordinarily open. If $f : Y \to X$ is a morphism of good spaces, is quasi-smooth at y, and X is normal at $f(y)$, then Y is normal at y.*

Proof. The map $A_{\mathfrak{p}} \to \mathcal{O}_{X,x}$ is regular by Theorem 2.35.29. Regular rings are normal by [StacksAlg, Tag 0567], so all its fibres are normal. Normality ascends by [StacksAlg, Tag 0C22]. Conversely suppose the target $R = \mathcal{O}_{X,x}$ is normal. This faithfully flat local map is injective, so $A_{\mathfrak{p}}$ is a domain, since R is a domain. If $u/v \in \operatorname{Frac}(A_{\mathfrak{p}})$ is integral over $A_{\mathfrak{p}}$, it is integral over R and belongs to R. Thus $u \in vR$. Faithful flatness contracts the ideal vR to $vA_{\mathfrak{p}}$: the class of u in $A_{\mathfrak{p}}/vA_{\mathfrak{p}}$ becomes zero after tensoring with R, hence was already zero. Consequently $u/v \in A_{\mathfrak{p}}$, proving normality.

For openness, let N be the algebraic normalization of A, namely the product of the integral closures of its finitely many minimal-prime quotients in their fraction fields. It is finite by Theorem 2.35.31. A localization $A_{\mathfrak{p}}$ is normal exactly when the map $A_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is an isomorphism. Indeed normalization commutes with localization: clearing the finitely many denominators of a monic equation proves that integral closure does so, and the minimal-prime factors localize to the surviving components. If $A_{\mathfrak{p}}$ is normal it is reduced with one integral component and already integrally closed; conversely an isomorphism to the localization of N makes it normal.

The kernel and cokernel of $A \to N$ are finite A-modules. Their supports are closed: for a finite module M, its localization at $\mathfrak{p}$ vanishes exactly when some element outside $\mathfrak{p}$ annihilates all members of a fixed finite generating set, equivalently when $\mathfrak{p}$ avoids $\operatorname{Ann}(M)$. Hence the normal locus of $\operatorname{Spec}(A)$ is the complement of these two closed supports. Pulling this open set back by $x \mapsto \ker(x)$ gives a Zariski open subset of $\mathcal{M}(A)$; a principal open pulls back to $|a(x)| > 0$. The normality equivalence already proved identifies it with the ordinary normal-point set. Affinoid ordinary neighborhoods then prove ordinary openness on every good space.

Finally Theorem 2.35.30 makes $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ regular. Its fibres are normal by Tag 0567. Applying Tag 0C22 gives the last assertion. □

2.35.7 Topological p-bases and finite differential criteria

Theorem 2.35.33 (Countable exponent-one extensions)**.** *Let F be a complete non-Archimedean field of characteristic $p > 0$. Let E be a complete intermediate field $F \subset E \subset F^{1/p}$ which has a countable dense F-linear span. There is a finite or countable family $(a_i)_{i\in I}$ in E such that the monomials*

$$a^{\nu} = \prod_{i\in I} a_i^{\nu_i}, \qquad 0 \le \nu_i < p, \qquad \#\{i : \nu_i \neq 0\} < \infty,$$

form a topological F-basis of E. More precisely, the summation map from the weighted c_0-space with weights $|a^{\nu}|$ onto E is a bounded linear isomorphism with bounded inverse.

Proof. We first treat a single extension $L = H(x)$ of degree p, with $x^p \in H$ and H complete. The H-derivation $D = d/dx$ is well defined on L, and is bounded by finite-dimensional norm equivalence, Lemma 1.12.1. Suppose first that $|H^{\times}|$ is dense in $\mathbb{R}_{>0}$. Given $0 < \theta < 1$, choose $0 < \eta < 1$ with $\eta^{2(p-1)} > \theta$. Put $C = \sup_{y\neq 0} |Dy|/|y|$, which is finite and positive. Choose y with $|Dy| > \eta C|y|$, and multiply it by an element of H so that $a = \lambda y$ satisfies $\eta < |a| \le 1$. Then $|Da| > \eta^2 C$. Thus the derivation $\partial = D/(Da)$ satisfies $\partial a = 1$ and $\|\partial\| < \eta^{-2}$.

For $z = \sum_{i=0}^{p-1} c_i a^i$, Taylor's finite identity gives

$$c_i = \sum_{j=i}^{p-1} \frac{(-a)^{j-i}}{i!(j-i)!}\, \partial^j z.$$

All factorials here are nonzero elements of the prime field and have absolute value one. Hence

$$\theta \max_i |c_i| \le \eta^{2(p-1)} \max_i |c_i| \le |z| \le \max_i |c_i|.$$

In particular, a generates L/H and its power basis has a coefficient bound as close to one as desired.

If H is discretely valued, the value group of $L \subset H^{1/p}$ is discrete as well. The bounded set $\{|Dz| : |z| \le 1\}$ therefore has a positive maximum. Choose a in the unit ball attaining it. The derivation $\partial = D/(Da)$ preserves that unit ball. The same finite Taylor identity shows that the coefficients of every z in the unit ball, expressed in $1, a, \dots, a^{p-1}$, lie in H°. Consequently

$$L^\circ = \bigoplus_{i=0}^{p-1} H^\circ a^i.$$

For a trivial absolute value all these assertions follow directly from the algebraic power basis.

Choose a sequence generating a dense subfield $E' \subset E$ over F, and write $E' = \bigcup_n F_n$, where each nontrivial step F_n/F_{n-1} has degree p. Each F_n is complete and contained in $F^{1/p}$. If F is nondiscretely and nontrivially valued, choose numbers $\theta_n \in (0,1)$ with $\theta_* = \prod_n \theta_n > 0$, and use the first construction at the nth step. Iteration of its coefficient estimate gives, for every finite linear combination of the resulting p-monomials,

$$\theta_* \max_\nu |c_\nu| \le \left| \sum_\nu c_\nu a^\nu \right| \le \max_\nu |c_\nu|.$$

If F is discretely valued, use the integral power bases instead. They give $(E')^\circ = \bigoplus_\nu F^\circ a^\nu$. For a uniformizer $\pi \in F$, scaling a nonzero sum into the interval $(|\pi|, 1]$ gives the same estimate with $\theta_* = |\pi|$. For trivial F, take $\theta_* = 1$. In all cases $\theta_* \le |a^\nu| \le 1$, so the displayed norm is equivalent to the weighted maximum norm. Completing this finite-support direct sum gives exactly the indicated weighted c_0-space. Its image is closed by the lower estimate and dense because it contains E'. It is therefore all of E, and both continuity assertions follow. □

References. This is a reconstruction of Kiehl, *Ausgezeichnete Ringe in der nichtarchimedischen analytischen Geometrie*, J. reine angew. Math. 234 (1969), pp. 90–91, Satz 1.2 and Satz 1.4. The derivation estimate above makes the uniform coefficient constant explicit.

Lemma 2.35.34 (Cofinite subfields of a topological p-basis)**.** *Let $(a_i)_{i\in I}$ be a topological p-basis of a complete exponent-one extension E/F. For $J \subset I$, let E_J be the closure of $F(a_j : j \in J)$ in E. It consists precisely of the convergent p-monomial expansions supported on J. If $I \setminus J$ is finite, then*

$$E = \bigoplus_{\substack{0\le\nu_i<p\\ i\in I\setminus J}} E_J a^{\nu}, \qquad [E : E_J] = p^{|I\setminus J|}, \qquad \Omega_{E/E_J} = \bigoplus_{i\in I\setminus J} E\, da_i.$$

Moreover $\bigcap_{I\setminus J \text{ finite}} E_J = F$.

Proof. The subspace of expansions supported on J is closed, because each coordinate functional in a topological basis is continuous. Finite such expansions form the field $F(a_j : j \in J)$: the pth power of every element lies in F, so the inverse of a nonzero element b is b^{p-1}/b^p. Its closure is again a field, since inversion is continuous away from zero. Thus the claimed description of E_J follows. Regrouping the expansion by the finitely many exponents outside J gives the displayed direct sum; bounded coordinate maps justify regrouping and convergence. Independence follows from uniqueness of the full expansion. The presentation

$$E_J[U_i : i \in I \setminus J]/(U_i^p - a_i^p)_i \xrightarrow{\sim} E$$

then computes the differentials: the defining equations have zero differential and the indicated monomials form a basis on both sides. Finally, membership in every E_J forces all coefficients of nonconstant monomials to vanish, since each such monomial involves an index excluded by some cofinite J. Only the constant coefficient remains. □

Lemma 2.35.35 (Finite rank stabilization over descending fields)**.** *Let (F_λ) be a downward directed family of subfields of a field L, and put $F_\infty = \bigcap_\lambda F_\lambda$. For every finite family $v_1, \dots, v_d \in L$, its rank over F_∞ equals its rank over some F_{λ_0}. In particular, a family independent over F_∞ is independent over one member of the family.*

Proof. Choose λ_0 at which the integer rank is maximal, and choose from the v_i a basis $w_1, \dots, w_r$ of their span over F_{λ_0}. For every λ with $F_\lambda \subset F_{\lambda_0}$, these elements stay independent, and maximality makes them a basis of the span over F_λ. The coefficients expressing any v_i in this basis are unique among coefficient tuples in F_{λ_0}, hence independent of λ, and lie in every such F_λ. This subfamily is cofinal by downward directedness, so those coefficients lie in F_∞. Thus the same w_j span over F_∞ and remain independent over it. □

Lemma 2.35.36 (A triangular presentation on a finite generic open)**.** *Let $D \subset A$ be a finite extension of Noetherian integral domains. There is a nonzero $d \in D$ such that*

$$A[1/d] \simeq D[1/d][U_1, \dots, U_m]/(P_1, \dots, P_m),$$

where P_j is monic in U_j, has coefficients in $D[1/d][U_1, \dots, U_{j-1}]$, and the reduced monomials form a free $D[1/d]$-basis.

Proof. Put $F = \mathrm{Frac}(D)$ and $E = \mathrm{Frac}(A)$. Since A is finite and integral over D, $A \otimes_D F = E$. Choose a tower $F \subset F(u_1) \subset \cdots \subset F(u_1, \dots, u_m) = E$ of simple finite extensions, with the u_j chosen among elements of A generating E. The minimal polynomial of u_j over the preceding field is monic. Express its coefficients in the reduced power basis of that preceding field. Clearing the finitely many denominators in D gives triangular monic polynomials over $D[1/d]$. Their quotient is free with the reduced monomials as basis, by successive division by monic polynomials. Its evaluation map into $A[1/d]$ becomes an isomorphism over F, so its kernel is zero because this free module is torsion-free. The cokernel is a finite $D[1/d]$-module whose localization at F vanishes. A further nonzero multiplier in D kills that cokernel. Replacing d by its product with this multiplier proves the assertion. □

Lemma 2.35.37 (Equal differential ranks and regularity)**.** *Let $S \to B$ be a ring map with B Noetherian and regular, and suppose that $\Omega_{B/S}$ is free of rank N. Let*

$$C = B[U_1, \dots, U_m]/(P_1, \dots, P_m)$$

is finite over B, with the displayed triangular monic presentation. If $\Omega_{C/S}$ is free of rank N, then C is regular. The same conclusion holds for each direct factor of C.

Proof. The differential presentation is

$$C^m \xrightarrow{(dP_j)} C^{N+m} \longrightarrow \Omega_{C/S} \longrightarrow 0.$$

The last module is free of rank N, so its kernel is projective of constant rank m. The first map surjects onto this kernel. After localization at any prime of C, a surjection between free modules of rank m is an isomorphism: reduce modulo the maximal ideal, then its determinant is a unit. Hence the dP_j are independent modulo every such prime. Let $\mathfrak{q}$ be its inverse image in $B[U]$.

This polynomial ring is regular by [StacksAlg, Tag 07NF], so its localization $R = B[U]_{\mathfrak{q}}$ is regular local. A relation $\sum c_j P_j \in \mathfrak{q}^2 R$ would, on differentiating and reducing modulo $\mathfrak{q}$, imply $\sum \overline{c}_j \, dP_j = 0$. Therefore the classes of the P_j are independent in $\mathfrak{q}R/(\mathfrak{q}R)^2$. Extend them to a minimal set of generators; [StacksAlg, Tag 00NQ] shows that $R/(P_1, \ldots, P_m)$ is regular local. These are all the local rings of C, proving regularity. Local rings of a direct factor are among them. $\square$

Definition 2.35.38 (Analytically separable scalar extensions)**.** In characteristic $p > 0$, a complete extension L/K is called analytically separable if multiplication

$$L^p \widehat{\otimes}_{K^p} K \longrightarrow L$$

is injective with closed image and its inverse onto that image is bounded. In characteristic zero every complete extension is called analytically separable.

Lemma 2.35.39 (Free Gauss extensions and preservation of p-bases)**.** *The identity extension and every free Gauss extension K_r/K are analytically separable. If L/K is analytically separable and K_0 is a complete intermediate field $K^p \subset K_0 \subset K$ with a topological p-basis a over K^p, then a remains topologically p-independent over L^p. For every complete subfield $F \subset L$ containing $L^p K_0$ and of countable type over L^p, and every cofinite subset $b \subset a$, there is a complete subfield $F^\sharp \subset F$ containing the closure K_b of $K^p(b)$ such that*

$$K_0 \otimes_{K_b} F^\sharp \xrightarrow{\sim} F.$$

Proof. The assertion for the identity is immediate. If $L = K_r$, the coefficient description gives $L^p = K_{rp}^p$. The map $L^p \widehat{\otimes}_{K^p} K \to L$ identifies with the subspace of Laurent expansions whose exponents are divisible by p. The coefficient formula of Theorem 1.16.2 makes this an isometric embedding with closed image. For general analytically separable L/K, Theorem 2.33.2 embeds $L^p \widehat{\otimes}_{K^p} K_0$ into $L^p \widehat{\otimes}_{K^p} K$. A bounded topological basis and its inverse remain bounded after completed tensor product. Thus the p-monomials in a form a topological L^p-basis of the closure F_0 of $L^p K_0$ in L.

The extension F/F_0 has exponent one and countable type. By Theorem 2.35.33 it has a topological p-basis c. Combining the two bounded basis isomorphisms shows that $a \cup c$ is a topological p-basis of F/L^p. Let $F^\sharp$ be the closed subfield generated by L^p and $b \cup c$. Lemma 2.35.34 shows that $a \setminus b$ is

a finite algebraic p-basis of $F/F^\sharp$. It is also a p-basis of K_0/K_b. The multiplication map in the statement therefore carries the same monomial basis to the same monomial basis, and is an isomorphism. □

Lemma 2.35.40 (Countable descent of completed ordinary germs)**.** *Let L be complete of characteristic $p > 0$, let $F_0 \subset L$ be complete with $L^p \subset F_0$, and let X_0 be an affinoid F_0-space. Put $X = (X_0)_L$, choose $x \in X$, and denote its image in $(X_0)_F$ by x_F. If F ranges over the complete intermediate fields $F_0 \subset F \subset L$ of countable type over F_0, then*

$$\operatorname*{colim}_F \widehat{\mathcal{O}_{(X_0)_F, x_F}} \xrightarrow{\sim} \widehat{\mathcal{O}_{X,x}}.$$

All transition maps, and all maps into the ring on the right, are faithfully flat local maps.

Proof. The ordinary local maps are faithfully flat by Theorem 2.35.4; their completions are flat by Tag 0C4G, and local, hence faithfully flat. In particular they are injective. Every individual germ descends to some F in the stated family. Indeed represent it on a rational affinoid ordinary neighborhood, using finitely many inequalities. Lift the finitely many functions in those inequalities, and then the represented analytic function on the rational algebra, to convergent weighted series. Only countably many scalar coefficients of L occur. Adjoining those coefficients to F_0 and taking the closure gives such an F: each coefficient has its pth power in $L^p \subset F_0$, and the finitely supported p-monomials in countably many generators have countable dense linear span. The inequalities defining the ordinary neighborhood hold in a neighborhood of x_F, since their strict versions hold at x_F exactly when they hold at x.

Write $R = \mathcal{O}_{X,x}$, choose finitely many generators $f_1, \dots, f_d$ of its maximal ideal, and descend them to some F_1. For $g \in \widehat{R}$, choose $g_n \in R$ with $g - g_n \in (f_1, \dots, f_d)^n \widehat{R}$. Descend all g_n and the f_i to one field F; a countable union of the coefficient sets above is still countable. Put $R_F = \mathcal{O}_{(X_0)_F, x_F}$, and denote the descended elements by f_i' and g_n'. The faithfully flat map

$$R_F/(f_i') \longrightarrow R/(f_i)$$

has a field as target, so its source is a field. Thus the f_i' generate the maximal ideal of R_F. Faithful flatness contracts every power of this ideal, and therefore $g_{n+1}' - g_n' \in (f_i')^n$. The compatible residues of the g_n' define an element of

$\widehat{R_F}$ whose image is g. This proves surjectivity of the colimit map; injectivity was proved at the start. □

Theorem 2.35.41 (A uniform regular open in positive characteristic)**.** *Let K be complete of characteristic $p > 0$, let r be a finite K-free polyradius, and let $D = K_r\{T_1, \dots, T_l\} \subset A$ be a finite extension of integral domains. Equip A with its finite affinoid norm. There exists $0 \neq d \in D$ such that, for every analytically separable complete extension L/K and every $x \in X_L = \mathcal{M}(A_L)$,*

$$\mathcal{O}_{X_L,x}[1/d], \qquad \widehat{\mathcal{O}_{X_L,x}}[1/d], \qquad A_L[1/d]$$

are regular rings.

Proof. **Step 1: descend the finite algebra.** Choose finitely many integral generators for A/D, their monic equations, and a finite set of generators for the ideal of relations in a polynomial presentation. Every coefficient in D is a Laurent–Tate series with countably many coefficients in K. Let K_0 be the closed subfield generated by K^p and all these coefficients. It is of countable type over K^p. The descended equations define a finite algebra A_0 over $D_0 = (K_0)_r\{T\}$ and an isomorphism $A_0 \otimes_{D_0} D \simeq A$. Indeed the chosen monic equations make A_0 finite, and the full finite relation list gives that isomorphism. The map $D_0 \to D$ is faithfully flat by Theorem 2.33.6. Thus A_0 injects into A and is an integral domain; also $D_0 \subset A_0$. Put $F_0 = \mathrm{Frac}(D_0)$, $E_0 = \mathrm{Frac}(A_0)$ and $F = \mathrm{Frac}(D)$. A basis $\alpha_1, \dots, \alpha_e$ of E_0/F_0 remains a basis after extension to F, since $E_0 \otimes_{F_0} F = \mathrm{Frac}(A)$.

Step 2: retain finitely many p-directions. Choose a topological p-basis a of K_0/K^p by Theorem 2.35.33. For a cofinite $b \subset a$, let K_b be the closure of $K^p(b)$ and put

$$D_b = (K_b)_{r^p}\{T_1^p, \dots, T_l^p\}, \qquad F_b = \mathrm{Frac}(D_b).$$

Here the Laurent variables in D_b are the pth powers of those in D_0. Grouping their exponents and the T-exponents modulo p, and using Lemma 2.35.34, makes D_0 finite free and radicial over D_b, of rank p^N, where $N = |a \setminus b| + |r| + l$. Its differentials over D_b are free of rank N, with basis given by the retained coefficient directions, the Laurent variables, and the T_i.

We need the equality $\bigcap_b F_b = F^p$. Set $D_a = (K_0)_{r^p}\{T^p\}$. An element of the intersection is g/h with $g, h \in D_a$, $h \neq 0$. Replacing numerator and denominator by gh^{p-1} and h^p puts the denominator in $(K^p)_{r^p}\{T^p\}$. For every

cofinite b, the new numerator belongs both to D_a and to F_b. Since $D_b \to D_a$ is faithfully flat, $D_a \cap F_b = D_b$: a relation $vg = u$ with $u, v \in D_b$, $v \neq 0$, contracts the principal ideal vD_a to vD_b. Hence the numerator belongs to every D_b. Uniqueness of Laurent–Tate coefficients and $\bigcap_b K_b = K^p$ identify their intersection with $(K^p)_{r^p}\{T^p\}$. Taking fractions proves the required equality.

The elements α_i^p are independent over F^p, by taking pth roots of a putative relation and using the F-independence of the α_i. Lemma 2.35.35 therefore supplies a cofinite b for which they are independent over F_b. Since they form a basis of E_0^p/F_0^p and $F_0^p \subset F_b$, this implies

$$[E_0^p F_b : F_b] = e, \qquad [E_0 : E_0^p F_b] = [F_0 : F_b] = p^N.$$

The universal derivation over F_b annihilates $E_0^p F_b$. An algebraic p-basis of the finite exponent-one extension $E_0/E_0^p F_b$ computes its differentials by the equations $U_i^p - c_i$; thus $\dim_{E_0} \Omega_{E_0/F_b} = N$.

Step 3: choose one denominator. The finite A_0-module Ω_{A_0/D_b} consequently becomes free of rank N after inverting a nonzero element of A_0: choose lifts of a generic basis, then kill the finite torsion kernel and cokernel. Any nonzero element of A_0 becomes a unit after inverting a nonzero element of D_0, since $A_0 \otimes_{D_0} F_0$ is a field. Raising that latter element to its pth power puts it in D_b. Combine this denominator with the one in Lemma 2.35.36, again taking a pth power. We obtain $0 \neq d \in D_b$ for which $A_0[1/d]$ has a triangular monic presentation over $D_0[1/d]$ and $\Omega_{A_0[1/d]/D_b[1/d]}$ is free of rank N.

Step 4: completed germs over countable scalar fields. Fix an analytically separable complete extension L/K. Let F_* be the closure of $L^p K_0$ in L, and let H range over the complete fields between F_* and L of countable type over F_*. Lemma 2.35.39 gives $H^\sharp$ with $H = K_0 \otimes_{K_b} H^\sharp$. Write

$$D_H = D_0 \widehat{\otimes}_{K_0} H, \quad B_H = D_b \widehat{\otimes}_{K_b} H^\sharp, \quad A_H = A_0 \otimes_{D_0} D_H.$$

The finite monomial bases prove the actual algebra identity $D_H = D_0 \otimes_{D_b} B_H$. Both D_H and B_H are algebras of affinoid domains in affine space: a product of a polydisc with polyannuli of exact radii r and r^p, respectively. Here the Laurent construction remains a polyannulus even when a radius ceases to be free over H.

Fix the image x_H of x in $\mathcal{M}(A_H)$, and let y and z be its images in $\mathcal{M}(D_H)$ and $\mathcal{M}(B_H)$. The finite radicial map $\mathcal{M}(D_H) \to \mathcal{M}(B_H)$ has a unique point

over z: in a radicial finite fibre there is one residue field, purely inseparable over $\mathcal{H}(z)$, and its absolute value is forced by $|u|^{p^m} = |u^{p^m}|$. The finite branch and completion formulas of Lemma 2.35.1 therefore give, with $S = \widehat{\mathcal{O}_{\mathcal{M}(B_H),z}}$ and $B = \widehat{\mathcal{O}_{\mathcal{M}(D_H),y}}$,

$$B = D_0 \otimes_{D_b} S, \qquad A_0 \otimes_{D_0} B = \prod_{t \mapsto y} \widehat{\mathcal{O}_{\mathcal{M}(A_H),t}}.$$

The ring B is regular by Theorem 2.35.23 and Tag 07NY. After inverting d, its differentials over $S[1/d]$ are free of rank N, and those of $(A_0 \otimes_{D_0} B)[1/d]$ have the same rank, by finite algebraic base change of the differential presentations in Steps 2 and 3. The latter algebra also retains its triangular monic presentation over $B[1/d]$. Lemma 2.35.37 thus proves that it is regular. Each factor, in particular $\widehat{\mathcal{O}_{\mathcal{M}(A_H),x_H}}[1/d]$, is regular.

Step 5: pass to L. Lemma 2.35.40 identifies $\widehat{\mathcal{O}_{X_L,x}}[1/d]$ with the filtered colimit of the regular rings in Step 4. Localize at any prime and its contractions. The resulting system consists of regular local rings with local transition maps and has a Noetherian colimit. By [StacksAlg, Tag 07DX], that colimit is regular local. This proves regularity of the completed localized germ. Completion is faithfully flat, also after inverting d, so Tags 00HQ and 00OF descend regularity to $\mathcal{O}_{X_L,x}[1/d]$. Finally, every prime of A_L avoiding d is the kernel of an analytic point by Lemma 2.52.1. Its faithfully flat local map into the ordinary germ at that point descends regularity once more, proving that $A_L[1/d]$ is regular. □

References. The proof reconstructs the positive-characteristic argument of Ducros, *Les espaces de Berkovich sont excellents*, Proposition 2.2(b) and Lemmas 2.6–2.8, using the explicit p-basis construction and the two finite algebra lemmas above.

Theorem 2.35.42 (Ordinary regular comparison for strict affinoids)**.** *Let K be nontrivially valued, let A be strict K-affinoid, let $x \in X = \mathcal{M}(A)$, and put $\mathfrak{p} = \ker(x)$ and $R = \mathcal{O}_{X,x}$. The homomorphisms*

$$A_{\mathfrak{p}} \longrightarrow R, \qquad R \longrightarrow \widehat{R}$$

are regular: they are flat, and every algebraic fibre is geometrically regular.

Proof. The characteristic-zero case is Theorem 2.35.26. Assume that K has characteristic $p > 0$. Noetherianity and flatness are exactly as in the first paragraph of that proof. By Tag 0381, it is enough to test regularity after

every finite purely inseparable extension of the residue field of every prime. We perform those tests explicitly.

First choose $\mathfrak{q} \subset \mathfrak{p}$ in A, put $A_0 = A/\mathfrak{q}$, and let $E/\operatorname{Frac}(A_0)$ be finite and purely inseparable. Choose generators α_j of this field extension and powers p^{e_j} with $\alpha_j^{p^{e_j}} \in \operatorname{Frac}(A_0)$. Clearing denominators gives nonzero $c_j \in A_0$ such that $\beta_j = c_j\alpha_j$ has $\beta_j^{p^{e_j}} \in A_0$. Thus $B = A_0[\beta_j]$ is a finite integral domain, has fraction field E, and is radicial over A_0. Equip it with its finite affinoid norm; it is strict by Proposition 1.14.4. There is a unique analytic point z of $Z = \mathcal{M}(B)$ above x: the finite fibre has one residue field, purely inseparable over $\mathcal{H}(x)$, whose absolute value is determined by its p-power equations. Finite branch separation gives

$$R \otimes_A B = \mathcal{O}_{Z,z}, \qquad R \otimes_A E = \mathcal{O}_{Z,z} \otimes_B E.$$

Theorem 1.12.2 gives a finite inclusion $K\{T_1, \dots, T_l\} \subset B$. Apply Theorem 2.35.41 with empty free polyradius and the identity scalar extension. It supplies $0 \neq d \in B$ such that $\mathcal{O}_{Z,z}[1/d]$ is regular. The displayed E-fibre is a further localization of this regular ring, since every nonzero element of B is invertible in E. Hence it is regular. This treats every finite purely inseparable test of every fibre of $A_{\mathfrak{p}} \to R$.

For the completion map, choose any prime $\mathfrak{P} \subset R$ and any finite purely inseparable extension $E/\operatorname{Frac}(R/\mathfrak{P})$. Clearing denominators as in Proposition 2.35.22 produces a finite domain C over $R/\mathfrak{P}$ with fraction field E. Theorem 2.35.20 and Lemma 2.35.25 realize it as

$$B \hookrightarrow C = \mathcal{O}_{Z,z}, \qquad Z = \mathcal{M}(B),$$

with B a strict integral affinoid algebra. In particular C is local and finite over R. Its maximal-ideal topology equals its $\mathfrak{m}_R C$-adic topology: the finite algebra $C/\mathfrak{m}_R C$ is local Artinian, so a power of its maximal ideal is zero. The finite-module completion formula, Tag 00MA, therefore gives

$$\widehat{R} \otimes_R C = \widehat{C}, \qquad \widehat{R} \otimes_R E = \widehat{C} \otimes_C E.$$

Apply the same identity-extension case of Theorem 2.35.41 to a strict finite normalization of B. It gives $0 \neq d \in B$ with $\widehat{C}[1/d]$ regular. The injection $B \hookrightarrow C$ ensures that d remains nonzero in C, and hence becomes invertible in E. Thus the last displayed ring is a localization of a regular ring and is regular. Tag 0381 now proves that every algebraic fibre of $R \to \widehat{R}$ is geometrically regular, completing both assertions. □

Proposition 2.35.43 (A regular open preserved by analytically separable extensions)**.** *Let K be any complete field and let A be an integral generalized K-affinoid algebra. There is $0 \neq a \in A$ such that, for every analytically separable complete L/K, the ring $A_L[1/a]$ is regular and all ordinary local rings of $\mathcal{M}(A_L)$ at points where a is nonzero are regular.*

Proof. In characteristic zero this is Proposition 2.35.27. In positive characteristic, choose a finite K-free polyradius r such that K_r is nontrivially valued and A_r is strict. The kernel-zero point and its Gauss lift used in the proof of that proposition also show, in every characteristic, that A_r is integral: the Gauss value of a nonzero Laurent coefficient family is positive. By Theorem 1.12.2 there is a finite inclusion $K_r\{T_1, \dots, T_l\} \subset A_r$. Theorem 2.35.41, applied over the original field K with this polyradius r, gives a nonzero $d \in A_r$ having its stated regularity properties after every analytically separable extension of K. Write

$$d = \sum_{\nu \in \mathbb{Z}^{|r|}} a_\nu U^\nu, \qquad a = a_{\nu_0} \neq 0$$

for one nonzero coefficient. Fix such an extension L/K and $y \in \mathcal{M}(A_L)$ with $|a(y)| > 0$. Associativity of completed tensor products and the coefficient description identify

$$(A_r)_L = A_L\{r^{-1}U, rU^{-1}\}.$$

The right side is the algebra of the relative polyannulus $P \to \mathcal{M}(A_L)$ on which $|U_i| = r_i$. Evaluate its coefficients in $\mathcal{H}(y)$ and then in the completion of $\mathcal{H}(y)(U)$ for the weighted Gauss absolute value. This defines a point z of P above y and gives

$$|d(z)| = \max_\nu |a_\nu(y)| r^\nu > 0.$$

The multiplicativity and extension of this weighted Gauss norm were proved in Lemma 3.21.7; no freedom assumption over L is used. The ring $\mathcal{O}_{P,z}$ is regular by the choice of d. Moreover P is the Laurent domain $|U_i| \geq r_i$ in the relative polydisc $|U_i| \leq r_i$. Corollary 2.48.3 therefore makes $\mathcal{O}_{\mathcal{M}(A_L),y} \to \mathcal{O}_{P,z}$ a flat local map. Tag 00OF descends regularity to the source.

Finally, every prime of A_L avoiding a is the kernel of an analytic point y, by Lemma 2.52.1. Its local ring maps faithfully flatly to the ordinary local ring at y, which is regular by the preceding paragraph. Another application of Tag 00OF proves regularity at that prime. This proves regularity of $A_L[1/a]$ and all the assertions with a single coefficient a independent of L. □

Proposition 2.35.44 (Generic regularity of arbitrary analytic germs)**.** *Let A be an integral generalized affinoid algebra over any complete field K. There is $0 \neq a \in A$ such that, for every $x \in X = \mathcal{M}(A)$, the rings $\mathcal{O}_{X,x}[1/a]$ and $\widehat{\mathcal{O}_{X,x}}[1/a]$ are regular.*

Proof. Choose a by Proposition 2.35.43, and choose a nontrivial free strictification K_r of A. Lemma 2.35.39 shows that this scalar extension is analytically separable, so $B = A_r$ satisfies $B[1/a]$ regular. For $z \in \mathcal{M}(B)$ above x, put $R = \mathcal{O}_{X,x}$ and $S = \mathcal{O}_{\mathcal{M}(B),z}$. Theorem 2.35.42 makes $B_{\ker(z)} \to S$ and $S \to \widehat{S}$ regular. At a prime of S avoiding a, its contracted localization in B is regular and its closed fibre is regular; Tag 031E therefore proves regularity of that localization of S. The same argument for $S \to \widehat{S}$ proves that both $S[1/a]$ and $\widehat{S}[1/a]$ are regular.

The map $R \to S$ is faithfully flat by Theorem 2.35.4, and its completed map is faithfully flat by Tag 0C4G. These properties persist after inverting a. Every prime of either source has a prime above it in the target by Tag 00HQ; the corresponding localized flat local map has regular target. Tag 00OF descends regularity in each case. Thus both $R[1/a]$ and $\widehat{R}[1/a]$ are regular, including at algebraic primes of these rings that arise when $a(x) = 0$. □

Theorem 2.35.45 (Ordinary regular comparison in every characteristic)**.** *Let K be any complete non-Archimedean field, let A be a generalized K-affinoid algebra, and let $x \in X = \mathcal{M}(A)$. Put $\mathfrak{p} = \ker(x)$ and $R = \mathcal{O}_{X,x}$. Then*

$$A_{\mathfrak{p}} \longrightarrow R, \qquad R \longrightarrow \widehat{R}$$

are regular homomorphisms of Noetherian rings.

Proof. The characteristic-zero case was established in Theorem 2.35.29. In positive characteristic, the flatness and Noetherianity argument of Theorem 2.35.42 is unchanged: the ordinary-neighborhood colimit uses arbitrary rational domains, and completion is faithfully flat. We retain its finite purely inseparable fibre tests, but replace strict finite normalization by Proposition 2.35.44. Here are the two substitutions in full.

For $\mathfrak{q} \subset \mathfrak{p}$ and a finite purely inseparable extension $E/\operatorname{Frac}(A/\mathfrak{q})$, construct the finite radicial domain $B \subset E$ over $A/\mathfrak{q}$ by the same denominator-clearing power equations. The finite affinoid norm exists by Theorem 2.33.18. With $Z = \mathcal{M}(B)$, there is a unique z above x, and

$$R \otimes_A E = \mathcal{O}_{Z,z} \otimes_B E.$$

The new generic-germ proposition gives $0 \neq b \in B$ with $\mathcal{O}_{Z,z}[1/b]$ regular. Since $\operatorname{Frac}(B) = E$, the displayed fibre is its further localization and is regular. Tag 0381 proves geometric regularity of every fibre of the first homomorphism.

For $\mathfrak{P} \subset R$ and a finite purely inseparable extension $E/\operatorname{Frac}(R/\mathfrak{P})$, construct a finite domain C over $R/\mathfrak{P}$ with fraction field E, and realize it using Theorem 2.35.20 as $B \hookrightarrow C = \mathcal{O}_{Z,z}$, where B is now an arbitrary integral generalized affinoid algebra. The finite-local completion calculation in the strict proof applies without change and gives

$$\widehat{R} \otimes_R E = \widehat{C} \otimes_C E.$$

Proposition 2.35.44 supplies $0 \neq b \in B$ with $\widehat{C}[1/b]$ regular. Its image in C is nonzero by the injection $B \hookrightarrow C$, hence invertible in E. Again the fibre is a localization of a regular ring. Tag 0381 treats all primes $\mathfrak{P}$ and proves geometric regularity of every completion fibre. This completes the proof over arbitrary radii and with either trivial or nontrivial valuation. □

References. The last four results reconstruct the regular-open and ordinary-local comparison in Ducros, *Les espaces de Berkovich sont excellents*, Propositions 2.2 and 2.9. The reduction to finite purely inseparable field tests is retained explicitly, while completed-germ regularity is supplied by the preceding topological p-basis argument.

Theorem 2.35.46 (Excellence of affinoid algebras and ordinary analytic local rings)**.** *Every generalized affinoid algebra over a complete non-Archimedean field is excellent. Every ordinary local ring of a good analytic space over such a field is excellent. In particular, these rings are Nagata and have finite algebraic normalization.*

Proof. Let A be affinoid and let $R = \mathcal{O}_{\mathcal{M}(A),x}$. Both are Noetherian. Theorem 2.35.45 proves that $R \to \widehat{R}$ is regular. The maximal-ideal criterion [StacksAlg, Tag 07PT] therefore makes R a G-ring. For any prime $\mathfrak{p} \subset A$, choose an analytic point x with kernel $\mathfrak{p}$ and put $D = A_{\mathfrak{p}}$. The same theorem makes $D \to R \to \widehat{R}$ a composite of regular maps. Since the target is Noetherian, the composite is regular by [StacksAlg, Tag 07QI]. The completed map $\widehat{D} \to \widehat{R}$ is a flat local map by Tag 0C4G, hence faithfully flat. Descent of regularity of ring maps, [StacksAlg, Tag 07NT], applied to $D \to \widehat{D} \to \widehat{R}$, makes $D \to \widehat{D}$ regular. This holds for every $\mathfrak{p}$, so A is a G-ring as well.

We verify the J-2 property with Tag 07PC(4). For a prime $\mathfrak{p} \subset A$ and a finite purely inseparable extension $E/\kappa(\mathfrak{p})$, clearing monic-equation denominators gives a finite domain B over $A/\mathfrak{p}$ with fraction field E. Its canonical finite affinoid structure is supplied by Theorem 2.33.18. The identity-extension case of Proposition 2.35.43 gives a nonempty regular principal open in $\operatorname{Spec}(B)$. Thus B is J-0 and the criterion applies to A. For a prime $\mathfrak{P} \subset R$ and a finite purely inseparable extension $E/\kappa(\mathfrak{P})$, take a finite domain C over $R/\mathfrak{P}$ with fraction field E. Realize it by Theorem 2.35.20 as $B \hookrightarrow C = \mathcal{O}_{Z,z}$, with B integral affinoid. Proposition 2.35.44 gives $0 \neq b \in B$ for which $C[1/b]$ is regular. Since $B \hookrightarrow C$, this is a nonempty principal open. Thus C is J-0, and the same criterion proves that R is J-2. Both A and R are now quasi-excellent.

It remains to prove universal catenarity. Present A as a quotient of a generalized Tate algebra $D = K\{r^{-1}T\}$. The ring D is regular by Theorem 2.35.23. Its local rings are Cohen–Macaulay by Tag 00NQ; hence D is universally catenary by [StacksAlg, Tag 00NM]. Quotients preserve universal catenarity by [StacksAlg, Tag 0ECE], so A is universally catenary. For R, the closed immersion $\mathcal{M}(A) \hookrightarrow \mathcal{M}(D)$ identifies R with a quotient of the ordinary local ring $\mathcal{O}_{\mathcal{M}(D),x}$ by the extended defining ideal; this is the finite closed-immersion case of Lemma 2.35.1. That local ring is regular by Theorem 2.35.23. The same two catenarity results therefore apply to its quotient R. A Noetherian, universally catenary, quasi-excellent ring is excellent by definition. Nagata's property and finite normalization now follow from Tags 07QV and 035S. Ordinary affinoid neighborhoods identify every ordinary local ring of a good space with an R of this form. □

Corollary 2.35.47 (Normal loci over arbitrary complete fields)**.** *Let $X = \mathcal{M}(A)$ over an arbitrary complete field, and put $\mathfrak{p} = \ker(x)$ for $x \in X$. Then $A_{\mathfrak{p}}$ is normal if and only if $\mathcal{O}_{X,x}$ is normal. These points form a Zariski open subset of X. Consequently the ordinary normal-point set of every good analytic space is ordinarily open.*

Proof. Put $D = A_{\mathfrak{p}}$ and $R = \mathcal{O}_{X,x}$. By Theorem 2.35.45, $D \to R$ is a faithfully flat regular local map. Its fibres are normal by [StacksAlg, Tag 0567], so normality of D implies normality of R by [StacksAlg, Tag 0C22]. Conversely, if R is normal, faithful flatness embeds D into the domain R. For $u/v \in \operatorname{Frac}(D)$ integral over D, normality of R gives $u/v \in R$, hence

$u \in vR$. Faithful flatness contracts vR to vD, so $u/v \in D$. This proves the equivalence.

Let N be the product of the integral closures of the minimal-prime quotients of A in their fraction fields. It is finite by Theorem 2.35.46. Integral closure commutes with localization, by clearing the finitely many denominators in a monic equation. Thus $A_{\mathfrak{p}}$ is normal exactly when $A_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is an isomorphism: in the normal case the localization is reduced with one integral component and already integrally closed, and in the converse case it is a localization of the normal ring N. The kernel and cokernel of $A \to N$ are finite, so their supports are closed. Their complement is precisely the normal locus in $\operatorname{Spec}(A)$. Its pullback by $x \mapsto \ker(x)$ is Zariski open in X, since a principal open pulls back to $|a(x)| > 0$. The equivalence already proved identifies it with the normal-point set; ordinary affinoid neighborhoods give the assertion for every good space. □

References. These arguments reconstruct Ducros, *Les espaces de Berkovich sont excellents*, Theorem 2.13 and the ensuing normality comparison. The finite-prime field tests supply the J-2 condition, and universal catenarity is reduced to the checked Stacks results for regular rings and their quotients.

Proposition 2.35.48 (Regular algebraic scalar extension)**.** *Let A be a generalized K-affinoid algebra and let L/K be an analytically separable complete extension. The ordinary ring map $A \to A_L$ is regular. Consequently, if B is a finite-type A-algebra, then $B \to B \otimes_A A_L$ is regular, and regular localizations of B have regular inverse images under this base change.*

Proof. The map is flat by Theorem 2.33.6. For any $\mathfrak{p} \subset A$ and any finite purely inseparable extension $E/\kappa(\mathfrak{p})$, clear denominators to find a finite domain C over $A/\mathfrak{p}$ with $\operatorname{Frac}(C) = E$. Give C its canonical finite affinoid structure. Finite-module scalar extension identifies C_L with $C \otimes_A A_L$. Proposition 2.35.43 gives $0 \neq c \in C$ such that $C_L[1/c]$ is regular. Therefore

$$A_L \otimes_A E = C_L \otimes_C E$$

is a localization of a regular ring and is regular. Tag 0381 proves geometric regularity of every algebraic fibre, giving the first assertion. Finite-type base change of regular ring maps is [StacksAlg, Tag 07C1]; its Noetherian fibre hypothesis holds since A_L and its finite-type algebra $B \otimes_A A_L$ are Noetherian. At a prime above a regular localization of B, apply Tag 031E to the flat local map and its regular closed fibre. This proves the last assertion. □

Lemma 2.35.49 (Analytic domains around rigid points)**.** *Let X be good and let $x \in X$ have $[\mathcal{H}(x) : K] < \infty$. Every analytic domain $V \subset X$ containing x contains an ordinary neighborhood of x.*

Proof. Choose an ordinary affinoid neighborhood U of x in X, and then an affinoid domain $W \subset V \cap U$ containing x, using the affinoid domain basis. Choose a finite extension L/K in which the finitely many minimal polynomials of generators of $\mathcal{H}(x)/K$ split. The unique extension of the complete absolute value makes L a complete valued field. Every residue field of $\mathcal{H}(x) \otimes_K L$ is then L, even when the tensor product has nilpotents. Thus all points of the finite fibre of $\pi : U_L \to U$ over x are L-rational. They all lie in W_L, by the domain base-change theorem. Lemma 2.35.9 gives each of them an ordinary neighborhood in U_L contained in W_L. Let O be the union of these neighborhoods.

The space U_L is compact and U is Hausdorff. Hence the continuous image $\pi(U_L \setminus O)$ is closed in U and does not contain x. Its complement is an ordinary neighborhood of x. Scalar-extension spectral surjectivity shows that this complement is contained in W: a point outside W has every lift outside W_L, and hence outside O. The complement therefore lies in V, proving the assertion. □

Proposition 2.35.50 (Polynomial-to-Tate flatness over an affinoid base)**.** *For a generalized K-affinoid algebra A and any positive polyradius r, the ordinary ring homomorphism*

$$A[T_1, \dots, T_n] \longrightarrow A\{r^{-1}T\}$$

is flat. At a K-rational point of the target, the induced map of local rings identifies their maximal-ideal completions.

Proof. We first prove the completion assertion, without strictness assumptions. Write $A = K\{s^{-1}S\}/I$, and let the rational point have coordinates $S = a$, $T = b$. Translation by these coordinates is a bounded automorphism of the relevant Tate algebras, since $|a_i| \le s_i$ and $|b_j| \le r_j$. After translation the point is the origin. Grouping every monomial of total degree at least N by one of the finitely many degree-N monomials proves that reduction modulo $(S, T)^N$ is precisely Taylor truncation. The grouped series remain convergent: division by a fixed monomial multiplies their weighted coefficient norms by a

fixed positive constant. Consequently the completion of the localized target is

$$K[[S,T]]/IK[[S,T]].$$

For the polynomial source, its maximal ideal is generated by the maximal ideal (S) of A together with the T_j. The same finite Taylor truncations, first in T and then in S, give exactly the same quotients by every power of that maximal ideal. Finite-module exactness of completion, Tag 00MA, passes to the quotient by I. This proves the claimed canonical isomorphism. Both local rings are Noetherian; completion is faithfully flat by Tag 00MB. The kernel argument in Lemma 2.33.32 therefore proves flatness of the local map at every rational point.

Suppose next that K is nontrivially valued, A is strict, and $r = (1, \dots, 1)$. Put $P = A[T]$ and $B = A\{T\}$. For a maximal ideal $\mathfrak{m} \subset B$, Theorem 1.12.3 gives the finite complete residue field $E = B/\mathfrak{m}$. Evaluation defines an E-rational maximal ideal $\mathfrak{n}$ of B_E, lying over $\mathfrak{m}$. The ordinary extension $B \to B_E = B \otimes_K E$ is finite free, so $B_{\mathfrak{m}} \to (B_E)_{\mathfrak{n}}$ is faithfully flat. The first paragraph shows that $(B_E)_{\mathfrak{n}}$ is flat over the corresponding localization of $A_E[T]$. Since $A_E[T] = P \otimes_K E$ is flat over P, the same kernel descent proves that $B_{\mathfrak{m}}$ is flat over P. Testing the kernel of $J \otimes_P B \to B$ at every maximal ideal of B, for a finitely generated ideal $J \subset P$, proves that it vanishes by Tag 00HN. The finite-ideal criterion gives flatness of $P \to B$.

Finally choose a complete extension L/K containing elements whose absolute values are all the radii in a fixed presentation of A and all the r_j, and one element of absolute value different from one. Such a field is obtained by completing a rational function field for the weighted Gauss absolute value, as in Lemma 3.21.7. Rescaling coordinates makes A_L strict and $B_L = A_L\{r^{-1}T\}$ a unit-radius Tate algebra over it. The preceding case makes B_L flat over $A_L[T]$, hence over $A[T]$, because $A \to A_L$ is flat. The map $B \to B_L$ is faithfully flat by Theorem 2.33.6. Descent of the same finite-ideal kernels proves that B is flat over $A[T]$. □

Proposition 2.35.51 (Analytification over an affinoid algebra)**.** *Let A be a generalized K-affinoid algebra and let R be a finite-type A-algebra. There is a good Hausdorff analytic space Z^{an} over $\mathcal{M}(A)$, where $Z = \mathrm{Spec}(R)$, constructed by the radius pieces*

$$R = A[T]/I, \qquad Z_r = \mathcal{M}(A\{r^{-1}T\}/I), \qquad Z^{\mathrm{an}} = \bigcup_r Z_r.$$

The construction is independent of the presentation, is compatible with finite morphisms, closed immersions, principal open subsets and complete scalar extensions, and glues on finite-type A-schemes. If $V = \mathcal{M}(C)$ is an affinoid domain in Z^{an}, the ring map $R \to C$ is flat. The map from an algebraic local ring to the corresponding ordinary analytic local ring is faithfully flat.

Proof. The radius-piece construction and its gluing are those of Theorem 2.33.23, with coefficients in A. More explicitly, the intersection of two pieces is the piece obtained by taking the coordinatewise minimum of their radii, and is a Weierstrass domain in both. Every piece is contained in the ordinary interior of a larger piece, by strict coordinate inequalities. Hence the union is good. Two distinct points are separated either by their image in $\mathcal{M}(A)$ or by the absolute value of a polynomial coordinate expression, so the union is Hausdorff. A morphism from an affinoid space over $\mathcal{M}(A)$, together with an A-algebra map R to its analytic functions, has bounded images of the finitely many T_i. It therefore factors through a radius piece. The affinoid universal property gives the unique factorization. Localizing this assertion and gluing morphisms gives the same universal property for arbitrary analytic sources. It also proves presentation independence.

For a finite R-algebra S, its inverse image over a radius piece with algebra B is $\mathcal{M}(B \otimes_R S)$. Indeed this is a finite B-algebra with its canonical complete finite-module norm, and the bounded universal property identifies its maps with the maps to S above that piece. Each of the finitely many integral generators satisfies a monic equation, so its value is bounded there; thus every algebraic valuation point above the piece is included. This proves the finite assertion, and quotient algebras give closed immersions. The principal open $D(g)$ is the union of the Laurent domains $|g| \geq \epsilon > 0$, with the function g^{-1} supplied by their localization algebras. This proves the open assertion and gives the cocycle identifications required for scheme gluing. Completed tensor associativity identifies every radius piece after L/K with the radius piece for $R \otimes_A A_L$, proving scalar compatibility.

For flatness, the finitely many coordinate functions are bounded on the compact affinoid domain V. Choosing larger radii places all of V in a single radius piece Z_r, as an affinoid domain. Proposition 2.35.50, followed by the quotient by I, makes $R \to B = A\{r^{-1}T\}/I$ flat. Proposition 2.38.2 makes $B \to C$ flat. Their composite is flat. Ordinary stalks are filtered colimits over affinoid neighborhoods; the finite-relation argument in Theorem 2.33.34

makes their maps from R flat. Localizing at the image prime gives a flat local map, hence a faithfully flat map. □

Lemma 2.35.52 (Rigid completions in relative analytification)**.** *In the preceding construction, let $x \in Z^{\mathrm{an}}$ be rigid over K, let $\xi \in Z$ be its image, and let $V = \mathcal{M}(C)$ be any affinoid domain containing x, with $\mathfrak{m} = \ker_C(x)$. The natural maps identify all four complete local rings*

$$\widehat{\mathcal{O}_{Z,\xi}}, \qquad \widehat{\mathcal{O}_{Z^{\mathrm{an}},x}}, \qquad \widehat{\mathcal{O}_{V,x}}, \qquad \widehat{C_{\mathfrak{m}}}.$$

Proof. Assume first that x is K-rational. For a presentation $A = K\{s^{-1}S\}/J$ and $R = A[T]/I$, translate the coordinates to their values at x. The finite Taylor calculation of Proposition 2.35.50, followed by the finite-ideal completion formula, identifies the first ring and the local completion of any radius piece containing x with

$$K[[S,T]]/(J,I).$$

Proposition 2.35.10 gives the same ring for the ordinary analytic germ. Lemma 2.35.49 identifies the two ordinary germs, since V contains an ordinary neighborhood of x. Applying the rational Taylor calculation to a presentation of C identifies its algebraic local completion with its ordinary germ completion. This proves all four comparisons in the rational case.

In general $E = \mathcal{H}(x)$ is finite over K. The residue fields of both R and C at the indicated ideals are E: their valued fraction fields are dense in E, while every K-subspace of the finite-dimensional complete field E is closed. Choose a finite extension L/K splitting E/K. All residue fields of $E \otimes_K L$ are L, so all points over x are L-rational. Tensoring any of the four complete local rings with L decomposes it into the product of the complete local rings at these points. For the ordinary analytic rings this is Lemma 2.35.1. For the algebraic rings, the same assertion follows by decomposing the finite algebra modulo the maximal ideal into its local Artinian factors and lifting the resulting idempotents through the complete adic tower; the geometric-series correction solves an approximate idempotent modulo each next power, and Tag 00MA identifies the completed finite algebra. In each factor the maximal-ideal and original-ideal topologies agree because the residue algebra is local Artinian.

The maps in every factor are exactly the rational comparison maps already proved, since finite scalar extension commutes with the relative analytification and domain constructions. They are therefore isomorphisms. Tensoring

with the finite field extension L/K is faithfully flat; the kernels and cokernels of each original comparison map vanish after this tensor product and hence vanish. This descends all four canonical isomorphisms. □

Lemma 2.35.53 (Regular localizations on relative analytic domains)**.** *Let A be generalized affinoid, let R be a finite-type A-algebra, and let $V = \mathcal{M}(C)$ be an affinoid domain in $(\operatorname{Spec} R)^{\mathrm{an}}$. For a prime $\mathfrak{q} \subset C$, write $\mathfrak{p} = \mathfrak{q} \cap R$. If $R_{\mathfrak{p}}$ is regular, then $C_{\mathfrak{q}}$ is regular.*

Proof. Choose an analytic point $v \in V$ with kernel $\mathfrak{q}$, by Lemma 2.52.1. Theorem 2.35.46 makes A excellent, so its J-2 property says that the regular locus of $\operatorname{Spec}(R)$ is open. Choose $g \notin \mathfrak{p}$ such that $R[1/g]$ is regular. Choose a nontrivially valued free extension $L = K_r$ making both A_L and C_L strict, and lift v to a point w of V_L. Lemma 2.35.39 and Proposition 2.35.48 show that $R_L[1/g]$, where $R_L = R \otimes_A A_L$, is regular.

By Lemma 2.35.25, take a strict affinoid ordinary neighborhood $W = \mathcal{M}(D)$ of w in the open set $V_L \cap D(g)^{\mathrm{an}}$. For every maximal ideal $\mathfrak{n} \subset D$, the strict point theorem gives an L-rigid point $t \in W$ with that kernel. Lemma 2.35.52 identifies $\widehat{D_{\mathfrak{n}}}$ with the completion of the local ring of $\operatorname{Spec}(R_L[1/g])$ at the image of t. This algebraic local ring is regular. The regularity/completion criterion, Tag 07NY, shows first that its completion is regular and then that $D_{\mathfrak{n}}$ is regular. Since every prime lies under a maximal ideal and localization preserves regularity, D is regular.

The composite $C \to C_L \to D$ is flat by scalar flatness and domain flatness. At the kernel $\mathfrak{n}_w$ of w on D, its contraction to C is $\mathfrak{q}$. Thus $C_{\mathfrak{q}} \to D_{\mathfrak{n}_w}$ is a flat local map with regular target. Tag 00OF descends regularity and proves the assertion. □

Theorem 2.35.54 (Regular comparison over an affinoid base)**.** *Let R be a finite-type algebra over a generalized affinoid algebra A, let $Z = \operatorname{Spec}(R)$, and let $V = \mathcal{M}(C)$ be an affinoid domain in Z^{an}. For $x \in V$, put $\xi = \ker_R(x)$ and $\mathfrak{c} = \ker_C(x)$. Every arrow in the following ring diagram is regular:*

$$\begin{array}{ccc} R_{\xi} & \longrightarrow & C_{\mathfrak{c}} \\ \downarrow & & \downarrow \\ \mathcal{O}_{Z^{\mathrm{an}},x} & \longrightarrow & \mathcal{O}_{V,x}, \end{array} \qquad R_{\xi} \longrightarrow \mathcal{O}_{V,x}.$$

Proof. All four rings are Noetherian. The relative analytification construction proves flatness of the upper and left arrows, and the right arrow is regular by Theorem 2.35.45. The bottom arrow is a flat local map by analytic-domain stalk flatness. It remains to verify geometric regularity; throughout we test the residue field at every algebraic prime, not only primes supplied by analytic fibres.

First consider the diagonal map $R_\xi \to S = \mathcal{O}_{V,x}$. Choose $\mathfrak{p} \subset \xi$ in R and a finite purely inseparable extension $E/\operatorname{Frac}(R/\mathfrak{p})$. Clearing denominators in the power equations gives a finite radicial domain B over $R/\mathfrak{p}$ with fraction field E. The finite analytification of $\operatorname{Spec}(B) \to Z$ has a unique point y above x; the inverse image W of V is affinoid, and finite branch separation gives

$$S \otimes_R E = \mathcal{O}_{W,y} \otimes_B E.$$

Write $W = \mathcal{M}(D)$ and let $\mathfrak{Q}$ be any prime of $\mathcal{O}_{W,y}$ surviving this localization. Its contraction to B is zero. Its contraction $\mathfrak{q}$ to D therefore lies above the regular generic point of the integral scheme $\operatorname{Spec}(B)$. Lemma 2.35.53 proves that $D_\mathfrak{q}$ is regular. The regular map from D to the ordinary local ring at y, localized at $\mathfrak{Q}$, has regular source and regular closed fibre. Tag 031E shows that $(\mathcal{O}_{W,y})_\mathfrak{Q}$ is regular. Thus the displayed E-fibre is regular at every prime. Tag 0381 proves geometric regularity of all fibres of the diagonal map.

Next consider the bottom arrow. Replace Z^{an} by an ordinary affinoid neighborhood $X_0 = \mathcal{M}(A_0)$ of x, and replace V by an ordinary affinoid neighborhood of x inside $V \cap X_0$. These replacements do not change either ordinary stalk. Put $R_0 = \mathcal{O}_{X_0,x}$. For a prime $\mathfrak{P} \subset R_0$ and a finite purely inseparable extension $E/\kappa(\mathfrak{P})$, Proposition 2.35.22, applied to the domain inclusion, realizes the test as

$$S \otimes_{R_0} E = \left(\prod_i \mathcal{O}_{T,t_i} \right) \otimes_{C_0} E, \qquad B_0 \hookrightarrow C_0 = \mathcal{O}_{Y,z}, \qquad Y = \mathcal{M}(B_0),$$

where B_0 and C_0 are domains, C_0 is finite over R_0, $\operatorname{Frac}(C_0) = E$, and T is the inverse-image domain of V in Y. A prime in any factor surviving the displayed localization contracts to zero in B_0. The diagonal comparison already proved, now with the affinoid base B_0 and its finite-type algebra B_0 itself, makes $B_0 \to \mathcal{O}_{T,t_i}$ regular at the relevant localizations. Their localizations at primes above zero are regular, because the source localization is the field $\operatorname{Frac}(B_0)$ and Tag 031E applies. Hence every local ring of $S \otimes_{R_0} E$ is regular. Tag 0381 proves that the bottom arrow is regular.

The upper arrow is regular by descent, Tag 07NT: its composite with the faithfully flat right arrow is the regular diagonal. Finally, the diagonal assertion applied to an ordinary affinoid neighborhood of x in Z^{an} proves regularity of the left arrow. This establishes every map of the diagram. □

Corollary 2.35.55 (Regular ordinary local maps of analytic domains)**.** *If V is a good analytic domain in a good analytic space X, then $\mathcal{O}_{X,x} \to \mathcal{O}_{V,x}$ is regular for every $x \in V$. The same holds after every complete extension and every analytic base change for which the spaces under consideration are good.*

Proof. Restrict X to an ordinary affinoid neighborhood of x, and V to an ordinary affinoid neighborhood of x inside its inverse image. For the corresponding affinoid algebra A, apply the bottom-arrow assertion of Theorem 2.35.54 to $R = A$. Domain base change remains an analytic domain, so the same argument applies after each of the stated base changes. □

References. The preceding three comparisons reconstruct Ducros, *Les espaces de Berkovich sont excellents*, Lemma 3.2 and Theorem 3.3. The rigid-completion input and the finite-type-over-affinoid analytification used there have been supplied explicitly above.

Proposition 2.35.56 (Ordinary regularity of relative affine space)**.** *Let X be good and let Y be a good analytic domain in $\mathbb{A}^{n,\mathrm{an}}_X$. At every $y \in Y$, with image $x \in X$, the ordinary local homomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is regular.*

Proof. Flatness follows from relative polydisc flatness and analytic-domain flatness, since one can choose a polydisc ordinary neighborhood of y with radii larger than its coordinate values. Put $R = \mathcal{O}_{X,x}$ and $S = \mathcal{O}_{Y,y}$. For every prime $\mathfrak{p} \subset R$ and finite purely inseparable extension $E/\kappa(\mathfrak{p})$, apply Proposition 2.35.22. After ordinary restrictions it supplies an integral affinoid space $Z = \mathcal{M}(B)$, an inclusion $B \hookrightarrow C = \mathcal{O}_{Z,z}$ with $\mathrm{Frac}(C) = E$, and an inverse-image domain T of Y in $\mathbb{A}^{n,\mathrm{an}}_Z$ such that

$$S \otimes_R E = \left(\prod_i \mathcal{O}_{T,t_i} \right) \otimes_C E.$$

Fix any prime $\mathfrak{Q}$ in a factor surviving this localization. It contracts to zero in B. On an ordinary affinoid neighborhood of t_i in T, the polynomial coordinates define a ring map $B[T_1, \dots, T_n] \to \mathcal{O}_{T,t_i}$. Let $\mathfrak{q}$ be the contraction of

$\mathfrak{Q}$ to this polynomial ring. Since $\mathfrak{q} \cap B = 0$, its localization $B[T]_{\mathfrak{q}}$ is a localization of a polynomial algebra over $\mathrm{Frac}(B)$, and is regular. Theorem 2.35.54 makes its map to $(\mathcal{O}_{T,t_i})_{\mathfrak{Q}}$ regular. Tag 031E proves that this target is regular. Every prime of the displayed E-fibre has been treated. Tag 0381 therefore gives geometric regularity of every algebraic fibre of $R \to S$. □

Lemma 2.35.57 (Ordinary local rings of étale morphisms)**.** *For an étale morphism of good analytic spaces, every induced ordinary local homomorphism is regular.*

Proof. Use the ordinary finite neighborhoods in the definition of étaleness. For the corresponding finite étale affinoid algebra $A \to B$ and $R = \mathcal{O}_{X,x}$, Lemma 2.35.1 expresses each source stalk as a local factor of $B \otimes_A R$. Finite flatness and vanishing of algebraic relative differentials survive this base change and passage to a factor, as in Proposition 2.50.2. Hence the local map is flat. At every algebraic prime of R, its fibre is a factor of a finite étale algebra over the residue field. After any field extension it remains finite étale: finite flatness and the standard étale presentations are preserved by tensor product. A finite étale algebra over a field is a product of finite separable fields. Indeed its standard presentations have an invertible polynomial derivative, so after an algebraic closure all their local factors are copies of that field, with no nilpotents; descent gives the asserted finite separable factors. Thus all the algebraic fibres and all their finite field extensions are regular. This proves geometric regularity. □

Theorem 2.35.58 (Ordinary regularity of quasi-smooth morphisms)**.** *Let $f : Y \to X$ be a morphism of good analytic spaces over any complete non-Archimedean field, quasi-smooth at y, and put $x = f(y)$. Then $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is a regular homomorphism of Noetherian local rings. If $X = \mathcal{M}(A)$ and $Y = \mathcal{M}(B)$ are affinoid, then $A_{\ker(x)} \to B_{\ker(y)}$ is regular as well.*

Proof. First suppose that f is smooth. Locally on ordinary neighborhoods it factors as an étale map to an ordinary open subset of $\mathbb{A}^{d,\mathrm{an}}_X$ followed by projection. Proposition 2.35.56 and Lemma 2.35.57 make the two induced ordinary local homomorphisms regular. Their composite is regular by Tag 07QI; all target rings and fibres are Noetherian.

In the quasi-smooth case, Theorem 2.59.4 identifies an ordinary neighborhood V of y in Y with an analytic domain in a smooth X-space T, after an ordinary restriction around x. The map on ordinary local rings factors as

$$\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{T,y} \longrightarrow \mathcal{O}_{V,y} = \mathcal{O}_{Y,y}.$$

The first arrow is regular by the smooth case; the second is regular by Corollary 2.35.55. Tag 07QI again proves regularity of the composite. This reasoning applies in every characteristic and tests all algebraic fibres through the preceding propositions.

For the affinoid assertion, put $\mathfrak{p} = \ker(x)$ and $\mathfrak{q} = \ker(y)$. Theorem 2.35.45 and the first assertion make the composite $A_{\mathfrak{p}} \to \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ regular. It factors through $B_{\mathfrak{q}}$, and the latter map $B_{\mathfrak{q}} \to \mathcal{O}_{Y,y}$ is faithfully flat. Descent of regular ring maps, Tag 07NT, proves that $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ is regular. □

Corollary 2.35.59 (Normality and regularity along quasi-smooth maps)**.** *Under the hypotheses of the preceding theorem, $\mathcal{O}_{Y,y}$ is normal if and only if $\mathcal{O}_{X,x}$ is normal, and it is regular if and only if $\mathcal{O}_{X,x}$ is regular. In particular a quasi-smooth good space over a normal good analytic space is normal. These assertions hold after every analytic base change and complete extension for which the spaces are good.*

Proof. The local map is regular and hence faithfully flat. Regularity ascends by Tag 031E and descends by Tag 00OF. Its geometrically regular fibres are geometrically normal. Normality therefore ascends by Tags 0C22 and 0567. For descent, if the target normal local ring S is a domain, faithful flatness embeds the source R into S, so R is a domain. If $a/b \in \operatorname{Frac}(R)$ is integral over R, its image is integral over S and lies in S. Thus $a \in bS \cap R = bR$, where the equality follows by faithful-flat contraction of the ideal bR. Hence $a/b \in R$, proving normality. Quasi-smoothness and good-domain restrictions are stable under the stated base changes, so the theorem applies again after each of them. □

References. The ordinary-ring argument reconstructs the regular-fibre comparison of [Duc18, 5.5.1–5.5.5]. It uses the smooth-envelope factorization together with finite algebraic-prime tests, so its geometric regularity concerns the ordinary local rings, not merely the analytic fibre spaces.

Normality ascends along smooth maps. Let $f : Y \to X$ be a morphism of good analytic spaces over a complete non-Archimedean field, smooth at y, and let $x = f(y)$. If $\mathcal{O}_{X,x}$ is normal, then $\mathcal{O}_{Y,y}$ is normal.

Use the ordinary open étale factorization of a smooth morphism through a relative analytic affine space. On the actual local rings, Proposition 2.35.56 makes the projection regular and Lemma 2.35.57 makes the étale arrow regular. The composite is regular by [StacksAlg, Tag 07QI]; all of its fibre rings are Noetherian since the target ordinary local ring is Noetherian. In particular the composite is flat and its ordinary algebraic fibres are regular, hence normal by [StacksAlg, Tag 0567]. Apply [StacksAlg, Tag 0C22] to the actual Noetherian local rings to obtain normality.

2.36 Rational refinement over the original field

The strict Gerritzen–Grauert theorem and the free-extension calculations now give rational refinements for general affinoid domains. The essential step is to describe the inverse image of a rational set under the Gauss section. Its defining functions have Laurent expansions, and a unit-ideal identity permits those expansions to be truncated uniformly. Throughout this section K is complete with its fixed real-valued non-Archimedean absolute value, which may be trivial. Affinoid algebras carry norm-admissible Banach norms, and the domain property of Definition 2.1.1 is tested against all generalized affinoid algebras over K.

Lemma 2.36.1 (Affinoid tests over a free extension)**.** *Let r be a finite tuple free over K, and put $L = K_r$. Every norm-admissible L-affinoid algebra C is a norm-admissible generalized K-affinoid algebra. Its bounded multiplicative seminorms over K are exactly those over L:*

$$\mathcal{M}_K(C) = \mathcal{M}_L(C).$$

Proof. Theorem 1.17.2 presents L as a quotient of a generalized Tate algebra over K. Apply Proposition 1.17.3 to a presentation $C = L\{s^{-1}Z\}/J$. It presents $L\{s^{-1}Z\}$ over K, and its further quotient by the closed ideal J is again norm-admissible over K. Uniformly equivalent admissible norms give the same conclusion.

A bounded multiplicative seminorm z on C restricts along the bounded map $L \to C$ to a bounded multiplicative seminorm on the valued field L. For $a \in L^\times$, a bound D gives $|a|_z^n \le D|a|^n$ for all n, hence $|a|_z \le |a|$. Applying the same argument to a^{-1} gives the reverse inequality. Thus z extends the specified absolute value of L. Conversely an L-point is a K-point. Both

spectra have the topology generated by evaluation on the same algebra C, so the identification is topological as well. If $C = 0$, both spaces are empty. □

Lemma 2.36.2 (Spectral injectivity of strict epimorphisms)**.** *Let L be complete and nontrivially valued. A bounded categorical epimorphism $D \to E$ of strict L-affinoid algebras induces an injective map $\mathcal{M}(E) \to \mathcal{M}(D)$.*

Proof. Theorem 2.27.3 supplies a finite rational covering (W_j) of $\mathcal{M}(D)$ such that $D_j \to E_j = E\widehat{\otimes}_D D_j$ has dense image. Suppose $z_1, z_2 \in \mathcal{M}(E)$ have the same image x, and choose j with $x \in W_j$. By Proposition 1.14.3, E_j is the actual strict rational quotient over E for the pulled tuple. The strict rational point comparison, Theorem 2.19.3, lifts each z_i uniquely to a point of E_j. Their restrictions to D_j are both the unique rational lift of x. Thus these two seminorms agree on the dense image of D_j. Bounded multiplicative seminorms are continuous, by Lemma 1.6.1, so they agree on E_j, and hence $z_1 = z_2$. The claim is immediate if the source spectrum is empty. □

Proposition 2.36.3 (Strictification of a domain presentation)**.** *Let $A \to B$ represent an affinoid domain $U \subseteq \mathcal{M}_K(A)$. There is a finite free tuple r such that $L = K_r$ is nontrivially valued, $A_r = A\widehat{\otimes}_K L$ and $B_r = B\widehat{\otimes}_K L$ are strict L-affinoid, and $A_r \to B_r$ is a strict-test affinoid-domain presentation of its spectral image U_r. Writing $\varphi_r : \mathcal{M}_L(B_r) \to \mathcal{M}_L(A_r)$ and $\varphi : \mathcal{M}_K(B) \to \mathcal{M}_K(A)$, one has*

$$\varphi_r \circ \sigma_{r,B} = \sigma_{r,A} \circ \varphi, \qquad \sigma_{r,A}^{-1}(U_r) = U.$$

Proof. Choose r by Theorem 1.17.6 for presentations of A and B. If K is trivially valued and the chosen tuple is empty, append one radius greater than one. It is free over K, and the original presentation radii remain in $\sqrt{|L^\times|}$, so the same strictification applies. All these algebras are also generalized K-affinoid by Proposition 1.17.4.

The map $A_r \to B_r$ is an epimorphism among strict L-affinoid algebras. Indeed, two bounded L-maps $B_r \to C$ agreeing on A_r restrict to maps $B \to C$ agreeing on A. Lemma 2.36.1 makes C a permitted generalized K-affinoid test, and both restrictions factor the same map with spectral image in U. Domain uniqueness makes them equal. The original B and the scalar field L generate a dense subalgebra of B_r, so the two maps agree on B_r. Lemma 2.36.2 now gives injectivity on spectra. Compactness and Hausdorffness, Proposition 2.24.1, identify $\mathcal{M}_L(B_r)$ homeomorphically with U_r.

Suppose a bounded L-map $A_r \to C$, with C strict over L, has spectral image in U_r. Its restriction $A \to C$ has image in U, since restricting any point of B_r to B gives a point over U. By the same lemma, the original domain property provides a unique bounded map $B \to C$. Together with the given scalar map $L \to C$, its bounded balanced product extends to $B\widehat{\otimes}_K L \to C$. This is the required L-factorization; its uniqueness follows by restriction to B and L and density. Thus $A_r \to B_r$ has the strict-test domain property.

The first displayed identity is the naturality in Theorem 1.19.1. It implies $\sigma_{r,A}(U) \subseteq U_r$. Conversely the projection of U_r to $\mathcal{M}_K(A)$ is contained in U, and $\pi_r \sigma_{r,A}$ is the identity. These two inclusions prove the second displayed equality. □

Lemma 2.36.4 (Rational inverse images under a Gauss section)**.** *Let A be a generalized K-affinoid algebra, let r be a finite free tuple, and put $A_r = A\widehat{\otimes}_K K_r$. For every rational domain R' of $\mathcal{M}_{K_r}(A_r)$, the subset $\sigma_{r,A}^{-1}(R')$ is a finite union of rational domains defined over the original algebra A, with positive real radii.*

Proof. The zero algebra gives the empty case. Otherwise choose a rational presentation

$$R' = \{z : |f_i(z)| \le \rho_i |g(z)| \ (1 \le i \le m)\}, \qquad h_0 g + \sum_{i=1}^{m} h_i f_i = 1$$

in A_r, with $\rho_i > 0$. Use the coefficient norm of Theorem 1.16.2 and set

$$C = \max\bigl(\|h_0\|, \rho_1\|h_1\|, \dots, \rho_m\|h_m\|\bigr) > 0, \qquad \delta = C^{-1}.$$

For $x \in \sigma_{r,A}^{-1}(R')$, evaluating the unit identity gives

$$1 \le C\, \sigma_{r,A}(x)(g), \qquad \sigma_{r,A}(x)(g) \ge \delta.$$

Write $g = \sum_\nu g_\nu T^\nu$ and $f_i = \sum_\nu f_{i\nu} T^\nu$. Choose finite index sets $G, F_i \subset \mathbf{Z}^s$ such that the omitted coefficient weights satisfy

$$\|g_\nu\| r^\nu < \delta/2 \quad (\nu \notin G), \qquad \|f_{i\nu}\| r^\nu < \rho_i \delta/2 \quad (\nu \notin F_i).$$

Such sets exist by coefficient convergence. For each $\nu \in G$, define $R_\nu \subseteq \mathcal{M}_K(A)$ by

$$\begin{aligned} \delta &\le r^\nu |g_\nu(x)|, \\ r^\mu |g_\mu(x)| &\le r^\nu |g_\nu(x)| \quad (\mu \in G), \\ r^\mu |f_{i\mu}(x)| &\le \rho_i r^\nu |g_\nu(x)| \quad (1 \le i \le m,\ \mu \in F_i). \end{aligned}$$

This is an original-A rational presentation with denominator g_ν: its numerators are 1, the g_μ, and the $f_{i\mu}$, with respective radii r^ν/δ, $r^{\nu-\mu}$, and $\rho_i r^{\nu-\mu}$. The numerator 1 guarantees the unit-ideal condition. Empty members, including those with $g_\nu = 0$, are allowed.

At a point of $\sigma_{r,A}^{-1}(R')$, the denominator value is at least δ. The omitted g-coefficients have smaller value, so some retained coefficient g_ν attains its Gauss maximum. The rational inequalities and the same Gauss formula then put the point in R_ν. Conversely, on R_ν, the first two conditions and the omitted-coefficient bound give $\sigma_{r,A}(x)(g) = r^\nu|g_\nu(x)| \geq \delta$. The retained f_i-coefficients satisfy the required bound by the third condition, while the omitted coefficients are smaller than $\rho_i\delta$. Therefore $\sigma_{r,A}(x)(f_i) \leq \rho_i\sigma_{r,A}(x)(g)$ for every i. This proves the exact set equality

$$\sigma_{r,A}^{-1}(R') = \bigcup_{\nu\in G} R_\nu.$$

The proof also allows an empty tuple r, zero retained coefficients and an empty inverse image. □

Lemma 2.36.5 (Restriction to a rational subdomain)**.** *Let $A \to B$ represent an affinoid domain U. If $A \to C$ represents an affinoid domain $V \subseteq U$, there is a unique bounded A-map $B \to C$, representing V as a domain of $\mathcal{M}(B)$. These restriction maps compose, and compositions of domain embeddings are domain embeddings. If $V = R$ is rational over A, with algebra A_R, its defining tuple in B presents a rational domain with the same algebra, and*

$$B\widehat{\otimes}_A A_R \simeq A_R$$

by canonical bounded inverse maps. These identifications commute with restriction to finite intersections of original-A rational subsets contained in U.

Proof. Test the domain property of $A \to B$ on C. It supplies the unique map $B \to C$, whose spectral map is $V \subseteq U$. For a generalized affinoid B-test E with image in V, the A-domain property of C supplies $C \to E$. This is a B-map because its two maps from B agree over A and the uniqueness for U applies to E. The same uniqueness proves composition of restrictions. Factoring a test map successively proves transitivity of domains.

For R, write its actual rational algebra as $A_R = A\{\rho^{-1}T\}/(gT_i - f_i)$. The pulled-back tuple defines the actual quotient $D = B\{\rho^{-1}T\}/(\varphi(g)T_i -$

$\varphi(f_i)$). It is affinoid, and its spectrum is the inverse image of R in $\mathcal{M}(B)$, by Theorem 2.31.1. The two bounded rational universal properties identify D with the completed pushout $B\widehat{\otimes}_A A_R$: the variables have the prescribed powers and images, and their finite polynomials are dense.

The restriction $B \to A_R$ and the identity on A_R induce $D \to A_R$, inverse to the canonical map $A_R \to D$. One composite fixes A_R. For the other, its two maps from B agree over A and the spectral image of D is contained in $R \subseteq U$; hence domain uniqueness makes them equal. The pushout property gives equality on D. The same argument on an intersection, which is original-A rational by Proposition 2.31.3, proves compatibility. Every comparison is bounded in both directions. □

Theorem 2.36.6 (Generalized Gerritzen–Grauert)**.** *Every affinoid domain $U \subseteq \mathcal{M}_K(A)$ has a finite covering by rational domains defined over the original A, with positive real radii. If B represents U, each member R_i and each finite intersection R of these members is rational over B with algebra canonically boundedly isomorphic to A_R. In particular*

$$B\widehat{\otimes}_A A_R \simeq A_R,$$

compatibly with all restriction maps. The empty domain admits the empty covering and has algebra zero.

Proof. If U is empty, the point identification and Theorem 1.20.1 give $B = 0$. Otherwise choose the simultaneous free strictification of Proposition 2.36.3. Corollary 2.29.4, over $L = K_r$, gives a finite covering $U_r = \bigcup_j R'_j$, where the R'_j are rational in $\mathcal{M}_L(A_r)$. By that proposition and Lemma 2.36.4,

$$U = \sigma_{r,A}^{-1}(U_r) = \bigcup_j \sigma_{r,A}^{-1}(R'_j),$$

and each inverse image is a finite union of original-A rational domains. Their combined family is the required finite covering. Lemma 2.36.5 supplies all algebra identifications, their bounded inverses and their compatibility on intersections. □

Theorem 2.36.7 (Complete-extension tests for an affinoid domain)**.** *Let $A \to B$ represent an affinoid domain U using generalized tests over K. For every complete isometric extension L/K and every norm-admissible L-affinoid algebra C, a bounded K-map $\alpha : A \to C$ whose spectral image lies in U has a unique bounded K-factorization through B. Thus the same-K generalized domain property and the domain property using all complete-extension affinoid tests are equivalent, with the same representing algebra.*

Proof. Choose the finite original-A rational covering $U = \bigcup_i R_i$ from Theorem 2.36.6, and write $B_i = A_{R_i}$ and $B_{ij} = A_{R_i \cap R_j}$. They form a rational covering of $\mathcal{M}(B)$ by Lemma 2.36.5. Pull the original rational tuples into C. Their actual quotient algebras C_i and C_{ij} form a finite rational covering of $\mathcal{M}_L(C)$ and its intersections, by Theorem 2.31.1 and Proposition 2.31.3. The covering assertion follows from the spectral-image hypothesis.

Theorem 2.31.2, with its complete-extension test clause, gives bounded maps $B_i \to C_i$. After restricting to C_{ij}, the same theorem extends each map through the rational overlap algebra B_{ij}. The two resulting maps $B_{ij} = A_{R_i \cap R_j} \to C_{ij}$ agree: they extend the same map from A and send every rational variable to its prescribed ratio. Composing with $B \to B_i$ consequently gives a bounded map from B to the matching kernel of $\prod_i C_i \rightrightarrows \prod_{i,j} C_{ij}$. Degree-zero norm-strict exactness in Theorem 2.32.2, applied over L to the finite C-module C, identifies this kernel with C, with bounded inverse. The resulting map $B \to C$ is bounded, preserves multiplication and the unit, and extends α, because these assertions hold after the injective restriction into $\prod_i C_i$.

For uniqueness, let $\beta : B \to C$ be any bounded factorization. After restricting to C_i, its spectral image in $\mathcal{M}_K(B)$ lies over R_i, hence in the rational B-domain represented by B_i. The same rational universal property factors $B \to C_i$ through B_i. This map $B_i = A_{R_i} \to C_i$ is the unique map determined by α and the original rational ratios. It is therefore the map used in the construction. Injectivity of $C \to \prod_i C_i$ proves uniqueness. If the spectral image is empty, Theorem 1.20.1 over L gives $C = 0$, for which the factorization is unique. The converse implication follows by taking $L = K$. □

Corollary 2.36.8 (Comparing tests for strict presentations)**.** *Suppose A, B are strict norm-admissible K-affinoid algebras, with trivial valuation on K allowed. A bounded map $A \to B$ is a strict-test affinoid-domain presentation if and only if it is a generalized same-K affinoid-domain presentation, and if and only if it has the affinoid-domain property for all complete-extension affinoid tests. The represented subset and algebra are unchanged.*

Proof. Only the implication from strict tests to all complete-extension tests requires proof. Let U be the represented subset. Corollary 2.29.4 gives a finite covering $U = \bigcup_i R_i$ by original-A rational domains with unit-radius presentations. Lemma 2.30.1 identifies each A_{R_i} and each intersection algebra as an actual rational quotient of B. Thus we have exactly the rational-cover data used in the proof of Theorem 2.36.7, without first enlarging the test class. For a bounded $A \to C$ into any complete-extension affinoid whose spectral image lies in U, pull back these tuples. The rational universal property gives $A_{R_i} \to C_i$ and the maps on their overlaps. Their composites from B match. The norm-strict rational Tate equalizer glues them to a bounded $B \to C$. Any other factorization has the same restrictions, by uniqueness for the actual rational quotients of B; injectivity of the equalizer therefore gives equality. This proves the implication. The other implications restrict the test class. □

Theorem 2.36.9 (Affinoid domains under arbitrary affinoid base change)**.** *Let $A \to B$ represent a generalized K-affinoid domain U. Let L/K be any complete isometric extension, let C be a generalized norm-admissible L-affinoid algebra, and let $\alpha : A \to C$ be bounded over K. Write $p : \mathcal{M}_L(C) \to \mathcal{M}_K(A)$. Then*

$$D = B\widehat{\otimes}_A C$$

is norm-admissible L-affinoid and $C \to D$ represents the affinoid domain $p^{-1}(U)$. For every original-A rational domain $R \subseteq U$, its pullback to $\mathcal{M}_L(D)$ is rational and has algebra

$$D\widehat{\otimes}_B A_R \simeq C\widehat{\otimes}_A A_R,$$

the actual rational quotient of C defined by the pulled-back tuple. These are canonical bounded isomorphisms and commute with finite intersections, restriction and further complete-affinoid base change. Finite affinoid-domain coverings pull back to finite affinoid-domain coverings.

Proof. Proposition 2.33.8 gives the actual generalized L-affinoid algebra D and its bounded pushout property. Choose $U = \bigcup_i R_i$ by Theorem 2.36.6. Each R_i is rational over B with algebra A_{R_i} by Lemma 2.36.5. Pull its tuple into D. By the rational point theorem these rational subsets cover $\mathcal{M}_L(D)$, since the R_i cover $\mathcal{M}_K(B)$. Their actual algebras have the displayed bounded identifications: compatible bounded maps out of either side are precisely compatible maps from C and A_{R_i} over A, because the intervening map from B is the restriction $B \to A_{R_i}$. The rational universal properties, or the actual joint quotient presentations, therefore give bounded inverse maps.

The right-hand algebra is the rational quotient of C for $p^{-1}(R_i)$. Every point of D maps into $p^{-1}(U)$ by restricting to B. Conversely, for $x \in p^{-1}(U)$ choose i with $p(x) \in R_i$. Theorem 2.31.1 gives a unique point above x in this rational quotient of C, hence a point of D. If two points of D lie over the same x, both belong to that ith rational piece; its identification with the rational quotient of C makes them equal. Thus $\mathcal{M}_L(D) \to \mathcal{M}_L(C)$ is bijective onto $p^{-1}(U)$; compactness and Hausdorffness in Proposition 2.24.1 make this a homeomorphism.

For the domain property, let $C \to E$ be a bounded L-map to a generalized L-affinoid algebra whose spectral image lies in $p^{-1}(U)$. The composite $A \to E$ has image in U. Theorem 2.36.7 supplies a unique bounded K-map $B \to E$ over A. Together with $C \to E$, the pushout gives the required bounded L-map $D \to E$. Conversely any such factorization restricts to these two maps, so it is unique. This proves the generalized L-domain property.

The algebra identification for any rational $R \subseteq U$ follows by the same pushout argument. Uniqueness of the bounded factorization makes these identifications compatible with restriction, intersections and a further base change. For successive base changes, the two parenthesizations of the completed pushout represent the same bounded-map problem, so their canonical maps are bounded inverses. A finite domain covering remains a covering because inverse images preserve finite unions. Empty pullbacks have zero algebra by Theorem 1.20.1 over L. □

Corollary 2.36.10 (Intersections and restrictions of affinoid domains)**.** *For affinoid domains $U, V \subseteq \mathcal{M}_K(A)$ represented by B, C, their intersection is affinoid and is represented by $B\widehat{\otimes}_A C$. All finite intersections have the corresponding iterated completed pushout algebras, independently of parenthe-*

sization. If $U \subseteq V$, the canonical restriction $C \to B$ represents an affinoid domain and

$$B\widehat{\otimes}_A C \simeq B.$$

All these identifications are canonical and bounded in both directions.

Proof. Apply Theorem 2.36.9 with $L = K$ to the domain U and the map $A \to C$ representing V. Its inverse image is $U \cap V$, viewed as a domain of V. Domain transitivity in Lemma 2.36.5 makes it a domain of $\mathcal{M}_K(A)$. Iteration and uniqueness give the finite-intersection and parenthesization assertions. If $U \subseteq V$, the restriction part of that lemma and the same uniqueness identify the pushout with B. □

References. The generalized finite rational-refinement theorem is due to Gerritzen–Grauert in the strict setting and to its generalizations; compare [Tem05, Theorem 3.1 and the following paragraph, p. 8]. The proof here reuses the internal strict theorem through the free Gauss section of [Ber93, Lemma 2.2.5]. The complete-extension comparison follows the rational-cover gluing method of [Tem05, Corollary 3.2, pp. 8–9].

2.37 Finite-module descent at arbitrary radii

The finite-module descent theorem is proved first for a finite rational covering. A single free-radius extension makes that entire diagram strict. Its matching module is then finite by the strict Kiehl theorem. Extracting coefficients descends finite generation and identifies the restrictions over the original field. This uses the strict descent argument already reconstructed from [BGR84, §9.4.3]; only the change of radii is added here.

Throughout this section, K is complete for its specified real-valued non-Archimedean absolute value, which may be trivial. Affinoid algebras have arbitrary positive presentation radii and norm-admissible norms. Finite modules carry canonical norms. Rational algebras and their intersections mean the actual quotients of Proposition 2.31.3.

Lemma 2.37.1 (Coefficient extension of matching kernels)**.** *Let $d : E \to F$ be a bounded map of Banach K-spaces, let $N = \ker d$ with its induced norm, and let r be a finite K-free tuple. Under the coefficient identifications for scalar extension to K_r, the natural map identifies*

$$N\widehat{\otimes}_K K_r \xrightarrow{\sim} \ker\bigl(E\widehat{\otimes}_K K_r \longrightarrow F\widehat{\otimes}_K K_r\bigr)$$

isometrically. Coefficient extension also commutes isometrically with finite products equipped with their maximum norms. These identifications respect the actions and maps of Banach modules when the original maps are module-linear.

Proof. The kernel N is closed, hence complete. By Theorem 1.16.2, the extended map applies d to every coefficient. A family (e_ν) belongs to its kernel exactly when every e_ν belongs to N. The induced norm on N makes its weighted vanishing condition and supremum norm exactly those inherited from E. This proves the kernel assertion without a closed-image assumption on d. For finite products, interchange the finite maximum with the coefficient supremum; the vanishing condition is preserved by finite unions of exceptional index sets. Module compatibility follows first on finite Laurent families and then on their limits by Lemma 1.18.1. □

Lemma 2.37.2 (Module sheaves on a rational slice)**.** *Let $X = \mathcal{M}(A)$ and let $\mathcal{R}_X$ be its arbitrary-radius finite rational site. For $V \in \mathcal{R}_X$ and a finite A_V-module P, put*

$$\widetilde{P}_V(W) = A_W \otimes_{A_V} P \qquad (W \in \mathcal{R}_X, W \subseteq V).$$

This is a sheaf on the slice $\mathcal{R}_X/V$. The functor $P \mapsto \widetilde{P}_V$ is exact and fully faithful, its sections over V are P, and it commutes with restriction to smaller rational objects. The augmented Čech complex for every finite covering in this slice is norm-strict exact in all degrees.

Proof. A rational object $W \subseteq V$ is rational over A_V with algebra A_W: pull back its original defining tuple and apply uniqueness in Theorem 2.31.2. Finite intersections have the same identifications. Applying Theorem 2.32.2 over A_V, and then over each A_W, proves the sheaf and Čech assertions. Proposition 2.32.3 proves exactness. At V the value is P. A sheaf homomorphism is determined by its map on P, since its value on $b \otimes p$ must be b times the restriction of that map on p. Conversely every A_V-linear map defines these maps on all objects. Ordinary tensor associativity proves the restriction assertion, with the bounded finite-module comparisons of Proposition 1.18.7. □

Theorem 2.37.3 (Effective finite-module descent for a rational covering)**.** *Let $V_1, \dots, V_n$ be a finite rational covering of $X = \mathcal{M}(A)$, with algebras A_i and joint algebras A_{ij}, A_{ijk}. Suppose finite A_i-modules N_i are given, together with isomorphisms*

$$\alpha_{ij} : A_{ij} \otimes_{A_i} N_i \xrightarrow{\sim} A_{ij} \otimes_{A_j} N_j$$

satisfying the identity and cocycle conditions on all triple intersections. Define the matching A-module

$$N = \left\{ (n_i) \in \prod_i N_i : \alpha_{ij}(1 \otimes n_i) = 1 \otimes n_j \text{in } A_{ij} \otimes_{A_j} N_j \right\}.$$

Then N is finite over A. Its subspace norm from $\prod_i N_i$ is canonical up to uniform equivalence, and the maps

$$\theta_i : A_i \otimes_A N \longrightarrow N_i, \qquad b \otimes (n_j)_j \longmapsto bn_i,$$

are isomorphisms with bounded inverses and recover the prescribed overlap identifications. The construction is functorial and gives an equivalence between finite A-modules and these finite-module gluing data. It also identifies N with the global sections of the sheaf obtained by gluing the $\widetilde{N}_{iV_i}$.

Proof. **Step 1: the matching Banach module.** Choose the defining rational quotient norms, so that $A \to A_i$ is contractive, and quotient canonical norms on the N_i. The module actions are contractive. The overlap modules are finite Banach modules and the α_{ij} are bounded by Theorem 1.18.3. Thus N is the kernel of a bounded A-linear map between finite products of Banach spaces. It is a Banach A-module with contractive action; finite generation is not used here. Uniformly equivalent choices of the finite local norms give uniformly equivalent norms on N.

Step 2: strictify the finite gluing diagram. Choose one free extension $L = K_r$ by Theorem 1.17.6 for the presentation of A and the finitely many covering radii. We may require L to be nontrivially valued: if K is trivial, include any additional radius different from 1 in the finite list. The extended cover is a unit-radius rational cover after the bounded comparisons of Proposition 2.26.2. Proposition 2.32.1 identifies its actual intersection algebras and the extended finite modules, including the overlap maps. The cocycle conditions persist. Lemma 2.37.1 gives a bounded identification

$$N_r \simeq \ker\left(\prod_i (N_i)_r \longrightarrow \prod_{i,j} (A_{ij} \otimes_{A_j} N_j)_r \right).$$

The strict local module sheaves glue to a sheaf $\mathcal{F}_r$ on the strict rational site. Explicitly, on a rational object W take the matching families of sections on $W \cap (V_i)_r$, using the prescribed overlap maps. The sheaf axiom follows by gluing in each local module sheaf and then checking the overlap identities after restriction to a cover. On a slice contained in $(V_i)_r$, projection to the ith component is an isomorphism: the other components are its restrictions transported by α_{ij}, and the cocycle identity is precisely their compatibility. Consequently $\mathcal{F}_r$ has finite local models, and its global sections are the displayed kernel. The strict Kiehl theorem, Theorem 2.28.12, now proves that N_r is finite over A_r and that its restriction maps recover $(N_i)_r$.

Step 3: descend finite generation and the restriction maps. Since L is nontrivially valued, the last clause of Theorem 1.18.9 applies to the Banach module N. It proves finite generation over A and equivalence of its matching norm with a canonical norm. The maps θ_i are therefore bounded maps between finite canonical A_i-modules. By Proposition 2.32.1, their free extensions are exactly the isomorphisms obtained in Step 2. The reflection assertion of Theorem 1.16.3 implies that each θ_i is an isomorphism with bounded inverse. Their overlap identities hold by the definition of N.

A morphism of gluing data maps matching families to matching families, which gives functoriality. Starting with a finite A-module M, the degree-zero equalizer in Theorem 2.32.2 identifies its matching module with M. Starting with gluing data, the maps θ_i identify their reconstruction with the original data, compatibly with morphisms. These are the two inverse natural isomorphisms. The same explicit sheaf gluing construction as in Step 2, now over K, and Lemma 2.37.2 identify its global sections with N. Empty components contribute zero modules. If X is empty, Theorem 1.20.1 gives $A = 0$ and every assertion is immediate. □

Theorem 2.37.4 (Kiehl equivalence on the arbitrary-radius rational site)**.** *On $(\mathcal{R}_X, \mathcal{O}_X^{\mathrm{rat}})$ the following conditions on a module sheaf $\mathcal{F}$ are equivalent: coherence in the sense of [StacksAlg, Tag 03DL, Definition 18.23.1(8)]; local finite presentation; possession of finite module models on some finite rational covering; and existence of a finite A-module M with $\mathcal{F} \simeq \widetilde{M}$. Thus*

$$\mathrm{Mod}^{\mathrm{fin}}(A) \;\simeq\; \mathrm{Coh}(\mathcal{O}_X^{\mathrm{rat}}), \qquad M \longmapsto \widetilde{M}, \qquad \mathcal{F} \longmapsto \mathcal{F}(X)$$

are inverse exact equivalences. Every coherent sheaf has a global finite free presentation. Its augmented Čech complex for every finite rational covering is norm-strict exact, with canonical norms on the section modules.

Proof. A coherent sheaf has finite local generators, and the kernels of the resulting maps from finite free sheaves are locally of finite type. Refining the finite cover supplies finite presentations. Each presentation on a slice is a matrix over A_V, since its columns are the images of the global basis sections. Lemma 2.37.2 identifies its cokernel sheaf with the sheaf of the finite matrix cokernel. Thus local finite presentation gives finite local module models. Their overlap sheaf isomorphisms give module isomorphisms by full faithfulness in the same lemma, with the required cocycle condition. Theorem 2.37.3 produces a global finite module M and compatible local isomorphisms with $\widetilde{M}$. Applying the sheaf equalizer on the covers $W \cap V_i$ glues these to an isomorphism on every object W.

Conversely, a finite A-module has a finite presentation because A is Noetherian by Theorem 1.18.3. Exactness of the module-sheaf functor gives a global finite free presentation of $\widetilde{M}$. For any V and any map $\mathcal{O}_V^t \to \widetilde{M}|_V$, full faithfulness identifies it with a map $A_V^t \to A_V \otimes_A M$. Its kernel is finite by Noetherianity of A_V, and exactness identifies the sheaf kernel with its associated finite-module sheaf. This proves coherence. The same lemma gives exactness and full faithfulness, and global sections recover M. The Čech assertion follows from Theorem 2.32.2. □

2.38 Affinoid coverings and coherent sheaves

We use the actual intersections of Corollary 2.36.10, the base-change identifications of Theorem 2.36.9, and the restriction maps of Lemma 2.36.5. All domains and intersection algebras in this section carry these identifications.

The rational refinement theorem allows the preceding module theory to be used on all affinoid domains. We retain the complete valued field K and its original absolute value; presentation radii are arbitrary positive real numbers. Domains are those of Definition 2.1.1, whose complete-extension test property has now been proved in Theorem 2.36.7.

Theorem 2.38.1 (Tate acyclicity for affinoid coverings)**.** *Let A be a generalized K-affinoid algebra, M a finite A-module, and $(V_i)_{i\in I}$ a finite affinoid-domain covering of $X = \mathcal{M}(A)$. Write A_J for the actual affinoid algebra of $V_J = \bigcap_{j\in J} V_j$. With canonical norms on the finite section modules, the augmented complex*

$$0 \longrightarrow M \longrightarrow \prod_i A_i \otimes_A M \longrightarrow \prod_{i<j} A_{ij} \otimes_A M \longrightarrow \cdots$$

is norm-strict exact. The all-tuples complex has the same property. Ordinary finite-module tensors in these formulas agree boundedly with their separated completed tensors.

Proof. Theorem 2.36.6 gives a finite original-A rational covering of each V_i. Combining them gives a finite rational covering $\mathcal{W}$ of X, with a chosen containing V_i for each member. Its augmented complex is norm-strict exact by Theorem 2.32.2.

For a nonempty tuple of coarse indices put $Y = V_{i_0} \cap \cdots \cap V_{i_p}$. Corollary 2.36.10 gives its actual iterated completed pushout A_Y and the canonical domain restrictions. If Y is empty, $A_Y = 0$ by the nonemptiness criterion of Theorem 1.20.1; the mixed intersections are then also represented by zero algebras. Otherwise use Theorem 2.36.9 for the whole domain X represented by $A \to A$, with $L = K$ and $C = A_Y$. It identifies each rational member W of $\mathcal{W}$ over Y with the rational domain defined by its pulled-back tuple, with actual algebra $A_Y \widehat{\otimes}_A A_W$. The restrictions cover Y, since the inverse images of a cover cover the source. Apply Theorem 2.32.2 over A_Y to this covering and the finite module $M_Y = A_Y \otimes_A M$. The canonical restriction maps and joint algebras agree with those of the mixed intersections by the same base-change and intersection identifications. Proposition 1.18.7 and the ordinary map $b\otimes(c\otimes m) \mapsto bc\otimes m$ give the bounded coefficient identifications, compatible with the augmentation and each restriction. Thus the ordered restricted complex is norm-strict exact on every coarse intersection, with the zero terms included.

For $X \neq \varnothing$, apply Proposition 2.21.5(1) to the coarse and rational families, using on each actual intersection D the finite canonical module $N(D) = A_D \otimes_A M$ and the canonical bounded restriction maps. Each rational member keeps its chosen containing V_i from the first paragraph. The restricted complexes are those just identified, including the zero complexes for empty intersections. This transfers norm-strict exactness and gives the all-tuples assertion. Proposition 1.18.7 gives the final ordinary-to-completed comparison. If X is empty, Theorem 1.20.1 gives $A = 0$ and hence $M = 0$, which includes the empty covering. □

Proposition 2.38.2 (Flatness of affinoid-domain restriction)**.** *Every affinoid-domain map $A \to B$ is flat as a homomorphism of ordinary rings. A finite affinoid-domain covering induces a faithfully flat map $A \to \prod_i A_{V_i}$. For a short exact sequence of finite canonical A-modules, restriction to B is norm-strict exact.*

Proof. Cover $\mathcal{M}(B)$ by the original-A rational domains R_j of Theorem 2.36.6, and put $S = \prod_j A_{R_j}$. Lemma 2.36.5 makes the same family rational over B, with the same algebras. By Proposition 2.32.3, $B \to S$ is faithfully flat and S is flat over A. For an injection $N' \to N$ of arbitrary A-modules, let Q be the kernel of $B \otimes_A N' \to B \otimes_A N$. Flatness over B identifies $S \otimes_B Q$ with the corresponding kernel after tensoring with S over A; that kernel is zero. Faithfulness gives $Q = 0$, proving flatness of B over A. The zero algebra handles an empty domain.

For a finite covering, refine each member by these rational covers. Their product S is faithfully flat over A and is an algebra over $B' = \prod_i A_{V_i}$. The latter is flat over A by the result just proved. If $B' \otimes_A N = 0$, then $S \otimes_A N = 0$, so $N = 0$; thus B' is faithfully flat. Finally the finite restricted modules carry canonical norms, and every finite-module map is norm-strict by Theorem 1.18.3. Algebraic exactness therefore gives the last assertion, with the completed-tensor interpretation supplied by Proposition 1.18.7. □

Definition 2.38.3 (The finite affinoid-domain site)**.** For $X = \mathcal{M}(A)$, let $\mathcal{D}_X$ be the category of all affinoid domains in X, with inclusions as arrows. Give it finite set-theoretic coverings by its objects, including the empty covering of the empty domain. Choose representing algebras A_V, with the unique bounded restriction maps, and define

$$\mathcal{O}_X^{\mathrm{aff}}(V) = A_V, \qquad \widetilde{M}^{\mathrm{aff}}(V) = A_V \otimes_A M \quad (M \text{ finite over } A).$$

Theorem 2.38.4 (Structure sheaf and rational comparison)**.** *The finite coverings define a pretopology on $\mathcal{D}_X$. The displayed assignments are a ring sheaf and module sheaves. The functor $M \mapsto \widetilde{M}^{\mathrm{aff}}$ is exact and fully faithful, with $\Gamma(X, \widetilde{M}^{\mathrm{aff}}) = M$. Inclusion of the arbitrary-radius rational site induces an equivalence of ringed topoi*

$$(\mathcal{R}_X, \mathcal{O}_X^{\mathrm{rat}}) \simeq (\mathcal{D}_X, \mathcal{O}_X^{\mathrm{aff}}),$$

identifying the sheaves associated to the same finite module.

Proof. Affinoid-domain intersections and restrictions exist and are compatible. Finite covers are stable under restriction and composition, proving the pretopology axioms. A finite cover of an object V is a finite affinoid-domain cover over A_V. Apply Theorem 2.38.1 to $A_V \otimes_A M$; tensor associativity gives exactly the equalizer for the displayed presheaf. This proves the sheaf axiom, and $M = A$ gives the ring sheaf. Restriction flatness proves exactness. Evaluation at X gives M. A module-sheaf map is determined on V by $b \otimes m \mapsto b \otimes u(m)$, where u is its map at X; conversely this formula defines a sheaf map for every A-linear u. Thus the functor is fully faithful.

For the comparison apply [StacksAlg, Tag 03A0, Lemma 7.29.1]. The inclusion is fully faithful and preserves covers and finite intersections, so restriction of sheaves is continuous. Every object of $\mathcal{D}_X$ has a finite covering by original-A rational objects, by Theorem 2.36.6. For a cover of a rational object by affinoid domains, refine each member by such a cover; their finite union is a rational refinement. This verifies cocontinuity as well as the covering condition in the comparison lemma. Local fullness and faithfulness follow from full faithfulness of the inclusion. On rational objects the representing algebras and restrictions agree by uniqueness, so the resulting equivalence identifies the ring sheaves and the finite-module sheaves as stated. □

Theorem 2.38.5 (Kiehl equivalence and effective affinoid gluing)**.** *Finite A-modules and coherent $\mathcal{O}_X^{\mathrm{aff}}$-modules are equivalent through $M \mapsto \widetilde{M}^{\mathrm{aff}}$ and global sections. The equivalence is exact, and coherence is equivalent to finite presentation. Every coherent sheaf has a global finite free presentation and norm-strict exact augmented Čech complexes for all finite affinoid-domain coverings. Finite modules with compatible gluing isomorphisms on such a covering descend effectively to a finite A-module, given by their matching kernel.*

Proof. The comparison just proved satisfies the cocontinuous comparison hypotheses of [StacksAlg, Tag 03DM, Lemma 18.23.2], so it preserves coherence and finite presentation. Theorem 2.37.4 identifies the coherent sheaves on the rational site with finite A-modules. Under comparison their extensions are exactly $\widetilde{M}^{\mathrm{aff}}$; hence the asserted equivalence, exactness and inverse follow from Theorem 2.38.4. A finite presentation over the Noetherian ring A gives the global presentation. Apply Theorem 2.38.1 to obtain the Čech assertion.

For the gluing statement, attach to each local finite module its sheaf on the corresponding domain slice. On an arbitrary object take matching sec-

tions on its intersections with the covering members. Local sheaf gluing and the cocycle identity give a sheaf whose restriction to each member is the prescribed local sheaf. Thus it is locally finitely presented and hence coherent. The equivalence gives a finite global module, and the sheaf equalizer identifies that module with the matching kernel. The identifications on each member recover the given gluing maps by construction. All section-module comparisons and inverses are bounded by Theorem 1.18.3. □

Corollary 2.38.6 (Affinoid Theorem B)**.** *For every coherent module $\mathcal{F}$ on $\mathcal{D}_X$ and every object V, one has $H^q(V, \mathcal{F}) = 0$ for $q > 0$. The same holds on $\mathcal{R}_X$. These are the right derived functors of sections in the corresponding categories of abelian sheaves.*

Proof. Apply [StacksAlg, Tag 03F9, Lemma 21.10.9]. Take its basis to be all objects and its covering system to be all finite coverings of the chosen site. Intersections remain in the site. Every covering of every object has vanishing higher Čech cohomology, by Theorem 2.38.1 and the Kiehl identification; for the rational site use Lemma 2.37.2. The covering system is cofinal by the definition of the generated topology. These verify the hypotheses of the cited lemma, which gives the asserted derived vanishing. □

References. This reconstruction combines the internally proved rational Tate theorem, finite-module descent and Gerritzen–Grauert refinement. The classical affinoid theory is developed in [Ber90, §§2.1–2.2] and [BGR84, §9.4.3]; the site-theoretic comparisons used in the proofs are the explicitly indicated Stacks Project results.

2.39 The analytic G-site of an affinoid space

We now pass from compact affinoid pieces to arbitrary analytic domains. The covering condition is topological, but it allows a point to lie on the boundary of every member containing it. This is the reason for using quasi-nets. The construction follows the atlas language of [Ber93, §1.2]; its algebraic gluing has already been proved in the preceding section.

Definition 2.39.1 (Quasi-nets and analytic domains)**.** A family (V_i) of subsets of a topological space V is a quasi-net if, for every $x \in V$, some finite subfamily consists of sets containing x and has union containing a neighborhood of x in V. Let $X = \mathcal{M}(A)$. A subset $V \subseteq X$ is an analytic domain if it has a quasi-net covering by affinoid domains of X, with the induced topology on V. A family of analytic subdomains of V is an admissible covering if it is a quasi-net on V. The empty family covers the empty domain.

Lemma 2.39.2 (Refining quasi-nets)**.** *Quasi-nets restrict to subsets and are transitive under quasi-net refinements. A finite covering by closed subsets is a quasi-net. If a compact space has a quasi-net covering, some finite collection of its members covers the space.*

Proof. Restriction follows by intersecting a neighborhood and its finitely many covering members with the subset. For transitivity, fix x and choose finitely many V_i containing x whose union contains a neighborhood N of x. For each chosen i, select finitely many refining members containing x whose union contains a relative neighborhood N_i of x in V_i. Write $N_i \supseteq O_i \cap V_i$ with O_i a neighborhood of x in the ambient space. Then $N \cap \bigcap_i O_i$ is a neighborhood covered by the finitely many selected refining members, all of which contain x.

For a finite closed covering, delete the members not containing x; their union is closed and misses x, so the remaining members cover a neighborhood of x. Finally, the neighborhoods supplied by a quasi-net cover a compact space. A finite subcover of these neighborhoods uses only finitely many of the original members and proves the compactness assertion. □

Proposition 2.39.3 (Analytic domains and their coverings)**.** *Analytic domains in X are closed under finite intersections. Their admissible coverings define a pretopology, denoted X_G. Every analytic domain is locally compact and Hausdorff. A subset is a compact analytic domain exactly when it is a finite union of affinoid domains of X. Every ordinary open subset of X is an analytic domain. If V is affinoid, the analytic domains and coverings inside V are precisely those obtained by applying Definition 2.39.1 to $\mathcal{M}(A_V)$.*

Proof. Take affinoid quasi-net coverings (U_i) of U and (V_j) of V. At $x \in U \cap V$, choose the finite subfamilies containing x and covering relative neighborhoods. The finitely many intersections $U_i \cap V_j$ contain x and cover a neighborhood

in $U \cap V$. They are affinoid domains by Corollary 2.36.10; hence $U \cap V$ is analytic. The same argument, together with Lemma 2.39.2, proves stability of admissible covers under pullback and composition, and the identity cover is admissible.

An affinoid quasi-net gives every point a compact neighborhood, namely a finite union of affinoid pieces. These pieces are compact and closed in the Hausdorff space X, so every analytic domain is locally compact and Hausdorff. If the domain is compact, the last assertion of Lemma 2.39.2 gives a finite affinoid cover. Conversely a finite union of affinoid domains is compact, and its finite closed covering is a quasi-net by that lemma.

For an open set O and $x \in O$, the evaluation topology gives finitely many strict upper or lower bounds on values of elements of A whose common locus contains x and is contained in O. Choose weak bounds strictly between the value at x and each prescribed strict bound. They define finitely many Weierstrass or Laurent domains; their intersection is rational by Proposition 2.31.3, is contained in O, and contains a neighborhood of x. These rational neighborhoods form a quasi-net covering of O. Finally, affinoid domains of X contained in V are affinoid over A_V, and affinoid domains over A_V are affinoid over A, by restriction and transitivity in Lemma 2.36.5. The induced topologies and quasi-net conditions coincide. This proves the last assertion. □

Theorem 2.39.4 (Comparison with the full analytic G-site)**.** *The inclusions of the finite affinoid-domain site and the arbitrary-radius rational site into X_G induce equivalences of topoi. The structure sheaf $\mathcal{O}_X^{\mathrm{aff}}$ therefore extends uniquely, up to its unique comparison isomorphism, to a ring sheaf $\mathcal{O}_{X_G}$. For an analytic domain V, its sections are compatible families of sections on affinoid domains contained in V. On an affinoid object V they equal A_V, with the original restriction homomorphisms.*

Proof. A finite affinoid-domain cover is a finite closed cover, so it is admissible by Lemma 2.39.2. Conversely, take an admissible cover (V_i) of an affinoid object V. Choose an affinoid quasi-net covering of each V_i. Transitivity makes their combined family a quasi-net of V, and compactness gives a finite affinoid refinement of the original cover. Every object of X_G is covered by affinoid objects by definition. Intersections of such objects remain affinoid.

These statements verify the comparison hypotheses of [StacksAlg, Tag 03A0, Lemma 7.29.1]: full faithfulness of the inclusion, continuity, coconti-

nuity by the finite refinement just constructed, and covering of every target object. Thus restriction gives an equivalence of topoi. Combining it with Theorem 2.38.4 gives the rational comparison. The inverse equivalence extends the ring sheaf. Concretely, its value on V is the compatible-family limit over all affinoid domains contained in V. Indeed, for an affinoid cover, compatibility on all intersections is the sheaf equalizer condition; refining two such covers compares them on their affinoid intersections. On an affinoid V the family has final section datum at V itself, recovering A_V. These identifications determine the restriction maps and the unique comparison isomorphism. □

Theorem 2.39.5 (Banach sections on compact analytic domains)**.** *On compact analytic domains, $\mathcal{O}_{X_G}$ is a sheaf of Banach K-algebras with bounded restrictions for finite domain coverings. For a finite affinoid covering $V = \bigcup_i V_i$, its Banach algebra of sections is*

$$\mathcal{O}_{X_G}(V) = \ker\left(\prod_i A_{V_i} \rightrightarrows \prod_{i,j} A_{V_i \cap V_j}\right),$$

with the induced maximum norm. Different finite affinoid covers and admissible norms give canonical bounded isomorphisms and uniformly equivalent section norms.

Proof. Proposition 2.39.3 supplies a finite affinoid cover. The preceding theorem identifies the section ring with the displayed matching kernel. The difference of the two restriction maps is bounded, so its kernel is closed in a finite product of Banach algebras and is complete. Multiplication is componentwise and is continuous.

For two covers (V_i) and (W_j), their intersections give a common finite affinoid cover. Restriction from the first matching kernel to this refinement is bounded. On each V_i, Theorem 2.38.1 applied to $(V_i \cap W_j)_j$ identifies A_{V_i} with its matching kernel by a bounded inverse. Taking the maximum of the finitely many inverse bounds shows that the map between the global matching kernels has bounded inverse. The same argument for (W_j) proves uniform equivalence and cover independence. Changing finitely many initial norms to equivalent ones does not affect this conclusion.

For compact domains $W \subseteq V$, choose finite affinoid covers of both. Restrict a matching family on V to all pairwise intersections of these covers,

and on each affinoid member of the cover of W use the same bounded gluing inverse. There are only finitely many restrictions and inverses, proving that $\mathcal{O}(V) \to \mathcal{O}(W)$ is bounded. Every finite covering of a compact domain by compact analytic domains has finite affinoid refinements. The sheaf equalizer is therefore the one already established on X_G, with its bounded Banach realization. □

Theorem 2.39.6 (Kiehl and Theorem B on the analytic G-site)**.** *The functors*

$$\mathrm{Mod}^{\mathrm{fin}}(A) \;\rightleftarrows\; \mathrm{Coh}(X_G), \qquad M \longmapsto \widetilde{M}, \qquad \mathcal{F} \longmapsto \Gamma(X, \mathcal{F})$$

are inverse exact equivalences, where $\widetilde{M}(V) = A_V \otimes_A M$ on affinoid domains. Every coherent sheaf has a global finite free presentation. For every affinoid domain V and every $q > 0$,

$$H^q(V_G, \mathcal{F}|_{V_G}) = 0.$$

The cohomology is derived sheaf cohomology on the full analytic G-site.

Proof. Theorem 2.39.4 is the cocontinuous ringed-site comparison of [StacksAlg, Tag 03A0], so [StacksAlg, Tag 03DM] preserves coherence and finite presentation. Theorem 2.38.5 therefore gives the equivalence and the global presentation. The equivalence preserves evaluation on each common affinoid object. On the slice at V, Proposition 2.39.3 identifies the site with V_G. The corresponding equivalence of abelian sheaf categories is exact and has an exact inverse. It preserves injective objects: apply the inverse equivalence to a monomorphism and use the defining extension property of an injective. Thus an injective resolution may be transported across the equivalence, and its section complexes agree. Their cohomology groups agree as well. Corollary 2.38.6 now gives the asserted vanishing. □

2.40 Analytic spaces from affinoid nets

Affinoid domains also serve as charts when no single chart is a neighborhood of a point. We use the net construction of [Ber93, §§1.1–1.4], with the overlap arguments expressed through the affinoid gluing results proved above. All affinoid algebras in this section may have arbitrary positive real radii.

Definition 2.40.1 (An affinoid atlas)**.** Let X be a locally Hausdorff topological space. A net τ of compact Hausdorff subsets is a quasi-net such that $\tau|_{U\cap V} = \{W \in \tau : W \subseteq U \cap V\}$ is a quasi-net of $U \cap V$ for every $U, V \in \tau$. An affinoid atlas consists of such a net, affinoid algebras A_V, homeomorphisms $V \simeq \mathcal{M}(A_V)$, and bounded homomorphisms $A_V \to A_U$ for inclusions $U \subseteq V$ in τ. The latter identify U with an affinoid domain of V, and are compatible with composition.

Lemma 2.40.2 (Topology and bounded gluing)**.** *A quasi-net determines the topology: a subset $O \subseteq X$ is open if and only if $O \cap V$ is relatively open for every member V. Let $Z = \mathcal{M}(B)$ and let $Z_1, \dots, Z_m$ be an affinoid-domain cover. Compatible bounded homomorphisms $C \to B_{Z_i}$ from an affinoid algebra C glue uniquely to a bounded homomorphism $C \to B$. Their induced maps on spectra glue to the induced map on Z.*

Proof. Suppose all $O \cap V$ are open and take $x \in O$. Choose $V_1, \dots, V_m$ containing x whose union contains a neighborhood N of x. Write $O \cap V_i = O_i \cap V_i$ near x, with O_i open in X and containing x. An open neighborhood of x contained in $N \cap \bigcap_i O_i$ lies in O. This proves the topological assertion. For bounded gluing, the degree-zero statement of Theorem 2.38.1 identifies B, by a bounded isomorphism, with the matching kernel of the finite diagram of the B_{Z_i}. The given homomorphisms take values in that kernel. Composition with its bounded inverse gives the required map; addition, multiplication and the unit can be checked after the injective restriction to the product. Evaluation on each Z_i proves the assertion about spectra. □

Proposition 2.40.3 (Transport and saturation of charts)**.** *If $U, V \in \tau$ and W is an affinoid domain of U contained in V, then its affinoid algebra has a canonical bounded A_V-algebra structure representing the same subset as an affinoid domain of V. These structures are transitive. Consequently the family $\bar{\tau}$ of affinoid domains in members of τ is a net with a unique extending affinoid atlas.*

Proof. Choose finitely many $C_i \in \tau|_{U\cap V}$ whose intersections with W cover W. This is possible by the quasi-net condition and compactness of W. Put $D_i = W \cap C_i$, using the intersection algebra inside U. Each D_i is an affinoid domain of W and of C_i, hence of V by transitivity. The maps $A_V \to A_{C_i} \to A_{D_i}$ agree on $D_i \cap D_j$: cover this compact intersection by sets $W \cap E$ with $E \in$

$\tau|_{C_i \cap C_j}$. On these sets the two maps agree by the original atlas functoriality. Affinoid Tate injectivity on the finite refinement proves their agreement on the intersection. Lemma 2.40.2 now gives $A_V \to A_W$, whose spectrum map is the inclusion of W.

We verify the domain property. Let T be affinoid and let $T \to V$ have image in W. The inverse images T_i of the domains $D_i \subseteq V$ form a finite affinoid cover of T. Each gives a map $T_i \to D_i \to W$. On $T_i \cap T_j$, refine by inverse images of the common atlas pieces $E \subseteq C_i \cap C_j$ used above. These pieces cover because the common point image lies in $D_i \cap D_j$. On every refined piece, both maps to U are the map through $E \to U$ prescribed by the original atlas. They therefore agree, and their factorizations through the original affinoid domain $W \subseteq U$ agree by its universal property. Tate gluing yields a bounded map $T \to W$. Its composite with $W \to V$ is the given map, as can be checked on the T_i. Any other factorization restricts uniquely to $T_i \to D_i$, since D_i is an affinoid domain both of W and of V; hence it is equal to the constructed factorization.

The same universal property proves independence of the finite cover and transitivity under inclusions. For the net assertion take $W \subseteq U$ and $W' \subseteq V$ in $\bar{\tau}$ and a point $x \in W \cap W'$. Choose finitely many $C_i \in \tau|_{U \cap V}$ containing x whose union contains a relative neighborhood of x. The domains $W \cap W' \cap C_i$, formed inside C_i, belong to $\bar{\tau}$, contain x, and cover a neighborhood in $W \cap W'$. This proves the net condition. The established transport gives all inclusion maps in $\bar{\tau}$ and proves their uniqueness. □

Definition 2.40.4 (Analytic domains and the analytic site)**.** An analytic domain $Y \subseteq X$ is a subset for which $\bar{\tau}|_Y$ is a quasi-net, equipped with the induced topology and the restricted atlas. A covering by analytic domains is admissible if it is a quasi-net. The associated site is denoted X_G.

Proposition 2.40.5 (Local calculus of analytic domains)**.** *Analytic domains are stable under finite intersections and under passage to analytic domains inside them. Admissible coverings define a pretopology. Every open subset is an analytic domain. Every point has a compact Hausdorff analytic neighborhood contained in any prescribed open Hausdorff neighborhood. On a chart $V \in \bar{\tau}$, the domain structure and site are precisely those of $\mathcal{M}(A_V)$ in Proposition 2.39.3.*

Proof. For intersections, intersect the finitely many chart members supplying the quasi-net condition at a point, and refine each resulting chart intersection by the net $\bar{\tau}$. Restriction and transitivity of quasi-nets, proved in Lemma 2.39.2, give the assertions about subdomains, pullbacks of covers, and composition of covers. Chart transport in Proposition 2.40.3 identifies each local refinement with an affinoid domain in the chart under consideration; this proves the assertion about its site.

Let $x \in O$ with O open. Choose finitely many charts V_i containing x whose union contains a neighborhood of x. Inside each V_i, the rational-neighborhood construction in Proposition 2.39.3 gives an affinoid domain $W_i \subseteq O \cap V_i$ containing a relative neighborhood of x. Their union contains a neighborhood of x in O. Thus O is analytic. If O is Hausdorff, the same union $D = \bigcup_i W_i$ is compact Hausdorff, is an analytic domain by its finite closed covering, and is a neighborhood of x in X. □

Theorem 2.40.6 (The structure sheaf of an atlas)**.** *There is a canonical sheaf of rings $\mathcal{O}_{X_G}$ whose restriction to every chart is its affinoid structure sheaf. Explicitly, for an analytic domain Y,*

$$\mathcal{O}_{X_G}(Y) = \{(s_V)_{V \in \bar{\tau}, V \subseteq Y} : s_V \in A_V, \quad s_V|_W = s_W \text{ whenever } W \subseteq V\}.$$

Given any chart quasi-net $\sigma \subseteq \bar{\tau}$, the family $\bar{\sigma}$ of all affinoid domains in members of σ is a net. Replacing τ by this net gives the same analytic domains, coverings and ring sheaf, with canonical comparison isomorphisms.

Proof. The displayed expression defines a ring presheaf. Given compatible sections on an admissible covering (Y_i) of Y, fix a chart $V \subseteq Y$. The intersections $V \cap Y_i$ are analytic domains of V and form an admissible covering. Refining by affinoid domains and using compactness yields a finite affinoid cover of V subordinate to the Y_i. The given sections agree on its overlaps, so Theorem 2.38.1 gives a unique element of A_V. It is independent of the chosen finite refinement, since two refinements have a common affinoid refinement. The elements obtained for different V restrict compatibly by the same uniqueness. This proves existence and uniqueness of the glued section. On a chart V, the family is determined by its member s_V, so the sheaf recovers A_V and all its restrictions.

For a chart quasi-net $\sigma \subseteq \bar{\tau}$, chart transitivity gives $\bar{\sigma} \subseteq \bar{\tau}$. For two members of $\bar{\sigma}$, refine their intersection by the net $\bar{\tau}$. Each refining chart is

contained in a member of σ and is an affinoid domain there by chart transport, hence belongs to $\bar{\sigma}$. Thus $\bar{\sigma}$ is a net. Each chart $V \in \bar{\tau}$ has an admissible cover by members of $\bar{\sigma}$: restrict σ to V by intersections, then refine each intersection by chart domains. Quasi-net transitivity proves equality of the analytic domains and coverings. The just-proved gluing construction supplies the inverse to restriction of compatible section families. Its local uniqueness makes all comparison isomorphisms transitive. □

Definition 2.40.7 (Locally bounded morphisms)**.** For two affinoid atlases, a morphism consists of a continuous map $f : X \to X'$ and, for every pair of charts $U \in \bar{\tau}$, $V \in \bar{\tau}'$ with $f(U) \subseteq V$, a bounded homomorphism $f_{U,V} : A_V \to A_U$ inducing $f|_U$. These maps commute with restrictions in both charts. We require that the charts U admitting such a target chart V form a quasi-net of X. An analytic space is an affinoid atlas considered up to the canonical identifications under chart refinement of Theorem 2.40.6.

Theorem 2.40.8 (Local extension and gluing of morphisms)**.** *Locally bounded morphisms form a category independent of atlas refinement. It is enough to specify compatible bounded affinoid maps on a chart quasi-net. Morphisms glue uniquely over admissible coverings. The inverse image of an analytic domain is analytic. The inclusion of an analytic domain $Y \subseteq X$ has the universal property*

$$\mathrm{Hom}(Z, Y) = \{f \in \mathrm{Hom}(Z, X) : f(|Z|) \subseteq Y\}.$$

Proof. Here compatibility of a collection of local maps means that, after refining a source overlap by common affinoid charts and refining the common target image by affinoid charts in the target overlap, the induced bounded homomorphisms coincide. Such refinements exist: target overlaps have their chart net, and an affinoid map pulls back each chart domain by Theorem 2.36.9. Pulling back a quasi-net along a continuous map preserves the quasi-net condition on its inverse image. Intersecting source chart nets then gives the required common source refinements.

Suppose the local maps have been specified, and take arbitrary charts U, V with $f(U) \subseteq V$. Intersect U with the source charts on which the maps are given. On each piece, refine the target overlap between its original target chart and V, then pull back those refinements. These are affinoid domains of U and form a quasi-net of U. Compactness gives a finite subcover. The

resulting maps from A_V agree on all overlaps by the prescribed compatibility, so Lemma 2.40.2 gives $f_{U,V} : A_V \to A_U$. Any second finite refinement gives the same map after their common refinement. This proves existence, uniqueness and restriction compatibility of all maps required in Definition 2.40.7. The topological part glues by Lemma 2.40.2.

The inverse image assertion is local on a presenting source chart and a target chart. Cover the target domain by chart domains and pull them back using the affinoid-domain base-change theorem. The resulting quasi-net of affinoid domains proves the assertion. For a composite $g \circ f$, refine source charts by inverse images of presenting charts for g, compose the bounded algebra maps there, and use the preceding extension construction. Common refinement and associativity of algebra homomorphisms prove associativity. The identity homomorphisms give identities. This construction also shows that replacing either atlas by a chart quasi-net does not change the morphism set.

For gluing along an admissible covering, present each local morphism on chart pieces contained in the relevant covering member. Their union is a quasi-net by transitivity. Agreement on pairwise intersections is exactly the compatibility used above, so extension gives a unique global morphism. Finally, if $f(|Z|) \subseteq Y$, pull back the chart net of Y. The local maps factor through its chart domains by the affinoid-domain universal property and glue to $Z \to Y$. Local uniqueness proves the displayed universal property. □

Corollary 2.40.9 (Intrinsic affinoid domains)**.** *The analytic domains isomorphic to affinoid spaces form a canonical maximal affinoid atlas. If two such domains satisfy $U \subseteq V$, then U is an affinoid domain of V in the original affinoid-algebra sense. Their algebras are $\mathcal{O}_{X_G}(U)$, and all restriction maps are bounded. The analytic site and structure sheaf constructed from this maximal atlas equal those already constructed.*

Proof. The inclusion $U \to V$ is a morphism between affinoid spaces and hence is induced by a bounded homomorphism. Its factorization property for affinoid tests follows from the analytic-domain universal property in Theorem 2.40.8; it is therefore an affinoid-domain map. Every original chart belongs to this family. The intersections of any two members are analytic domains and have affinoid chart quasi-nets, so the family is a net. Sections on each member are its affinoid algebra by the structure-sheaf theorem applied after refinement. Finally, every new chart already has an admissible

covering by old chart pieces. The refinement argument of Theorem 2.40.6 gives equality of the sites and ring sheaves. □

Theorem 2.40.10 (Gluing analytic spaces)**.** *Let (X_i) be analytic spaces with analytic domains $X_{ij} \subseteq X_i$ and isomorphisms $v_{ij} : X_{ij} \simeq X_{ji}$ satisfying the identity and cocycle conditions. Their analytic gluing exists and is unique in each of the following cases:*

1. *all X_{ij} are open in X_i;*
2. *each X_i is Hausdorff, every X_{ij} is closed in X_i, and for each i only finitely many X_{ij} are nonempty.*

The images of the X_i are analytic domains and form an admissible cover. They are open in the first case. In the second case they are closed and locally finite, and the glued space is Hausdorff. The gluing has the usual mapping universal property.

Proof. Form the quotient X of the disjoint union $S = \coprod_i X_i$ by the prescribed identifications. In the open case, the saturation of an open subset of one X_i is open in every X_j, so the quotient map is open and each X_i embeds as an open subspace. Hence X is locally Hausdorff.

In the second case, the saturation of a closed subset of S meets each X_i in a finite union of closed sets. Thus the quotient map q is closed, and each X_i embeds as a closed subspace. Each equivalence class is finite, since all its representatives meet any one of its representatives' components. Distinct classes are disjoint finite subsets of the Hausdorff space S, so they have disjoint open neighborhoods P, Q in S. The complements of $q(S \setminus P)$ and $q(S \setminus Q)$ are disjoint open neighborhoods of the corresponding quotient points. Thus X is Hausdorff. Fix i and let J_i be the finite set of indices meeting X_i. The union of the images of X_j for $j \notin J_i$ is closed: on each component its inverse image is either the entire component or a finite union of closed overlaps. Its complement is an open neighborhood of the image of X_i meeting only the finitely many components indexed by J_i. This proves local finiteness. A locally finite closed cover is a quasi-net, by applying the finite closed-cover assertion locally.

Take all images of affinoid chart domains in the X_i. Their family is a quasi-net by transitivity. On a pairwise intersection, the given analytic-domain isomorphism identifies the chart nets and their bounded restriction maps;

refining inside the overlap gives the net axiom. If a chart from X_i is contained in a chart from X_j, transport it through $X_{ij} \simeq X_{ji}$. It is an affinoid analytic domain in X_j, so Corollary 2.40.9 supplies the required bounded affinoid-domain inclusion map. The cocycle identity gives composition compatibility. This is an affinoid atlas on X. It identifies each X_i with its analytic-domain image. Theorem 2.40.8 gives the mapping universal property and uniqueness. □

Theorem 2.40.11 (Fibre products of analytic spaces)**.** *The category of analytic spaces over K has fibre products. If $Y \to X \leftarrow Z$ are morphisms, the product $P = Y \times_X Z$ has an affinoid chart net consisting of*

$$V \times_U W = \mathcal{M}\big(A_V \widehat{\otimes}_{A_U} A_W\big),$$

where U, V, W are affinoid analytic domains of X, Y, Z and both V, W map into U. Each such product is the analytic domain $p_Y^{-1}(V) \cap p_Z^{-1}(W)$ in P. The canonical map $|P| \to |Y| \times_{|X|} |Z|$ is surjective. These constructions are compatible with restricting any of the three spaces to analytic domains.

Proof. Step 1: compact Hausdorff spaces. Suppose first that X, Y, Z are compact Hausdorff. Choose a finite affinoid cover (U_a) of X. Its inverse images in Y and Z are compact analytic domains: they are analytic by Theorem 2.40.8 and closed because U_a is compact in the Hausdorff space X. Choose finite affinoid covers (V_{ai}) and (W_{aj}) of these inverse images. Form the finitely many affinoid products $P_{aij} = V_{ai} \times_{U_a} W_{aj}$ using Proposition 2.33.8.

For two triples $\alpha = (a, i, j)$ and $\beta = (b, i', j')$, define $P_{\alpha\beta} \subseteq P_\alpha$ by requiring its projections to lie in $V_{ai} \cap V_{bi'}$ and $W_{aj} \cap W_{bj'}$. It is a compact analytic domain, since these are closed analytic domains of the Hausdorff spaces Y, Z. To compare it with $P_{\beta\alpha}$, cover $U_a \cap U_b$ by finitely many affinoid domains R_l. Cover the inverse images of each R_l in $V_{ai} \cap V_{bi'}$ and $W_{aj} \cap W_{bj'}$ by finite affinoid families (E_{lm}) and (F_{ln}). The spaces

$$Q_{lmn} = E_{lm} \times_{R_l} F_{ln}$$

are affinoid domains in both P_α and P_β, by repeated use of Theorem 2.36.9. They cover the two designated overlaps: for a point of either overlap, first choose R_l containing its base point and then pieces containing its two projections. Their affinoid algebras are the same completed tensor products on both sides. On two such pieces, refine the base and both source intersections

by common affinoid domains; associativity of the bounded pushout, Corollary 2.33.9, identifies the resulting tensor products. Thus the identifications agree on overlaps and give $P_{\alpha\beta} \simeq P_{\beta\alpha}$ by Theorem 2.40.8. The same argument on triple intersections proves the cocycle condition.

The finite closed gluing of Theorem 2.40.10 now produces a compact Hausdorff analytic space P. Its projections glue. Given any analytic T with compatible maps to Y, Z, pull back the finite covers just used and refine by affinoid charts of T. On each resulting affinoid piece the bounded-pushout property gives the unique map to an appropriate P_α. On overlaps these maps agree after the common refinements Q_{lmn}, again by the bounded-pushout property. They therefore glue uniquely. This proves the product universal property in the compact Hausdorff case, with arbitrary analytic test spaces T.

Step 2: a compact Hausdorff base. Keep X compact Hausdorff and allow arbitrary Y, Z. By Proposition 2.40.5, cover Y, Z by open domains O_i, O'_j contained respectively in compact Hausdorff analytic domains D_i, D'_j. Step 1 constructs $D_i \times_X D'_j$. Restrict it to the open domain where the projections lie in O_i, O'_j. The result represents the product of these two open domains over X, by the analytic-domain universal property. For two index pairs, the corresponding open subdomains over $O_i \cap O_{i'}$ and $O'_j \cap O'_{j'}$ have the same universal property. Their unique comparison isomorphisms satisfy the cocycle condition. Open gluing constructs $Y \times_X Z$. The universal property follows by restricting a test space to the inverse images of these open domains and gluing its local maps.

Step 3: an arbitrary base. Cover X by open domains S_l contained in compact Hausdorff analytic domains C_l. Put $Y_l = f^{-1}(S_l)$ and $Z_l = g^{-1}(S_l)$. Step 2 constructs $Y_l \times_{C_l} Z_l$. Because both maps have image in S_l, its universal property is also that of $Y_l \times_{S_l} Z_l$. The products over two base opens agree on the open subdomains over their intersection. Open gluing therefore gives the required P for arbitrary X, Y, Z.

Finally take affinoid U, V, W as in the statement. The analytic domain $p_Y^{-1}(V) \cap p_Z^{-1}(W)$ has the same universal property as the affinoid bounded pushout $V \times_U W$, since two maps into U agree whenever their composites into X agree. Thus they are canonically isomorphic. The construction shows that these domains form a chart quasi-net, and the same observation proves compatibility with arbitrary analytic-domain restriction. Surjectivity on un-

derlying fibre-product sets is local on these charts and follows from Proposition 2.33.14. □

Theorem 2.40.12 (Arbitrary complete ground-field extension)**.** *Let L/K be any complete valued extension, with its specified isometric embedding. There is a scalar-extension functor $X \mapsto X_L$ on analytic spaces, characterized locally by*

$$\mathcal{M}(A)_L = \mathcal{M}(A\widehat{\otimes}_K L).$$

The canonical continuous map $\pi : X_L \to X$ is surjective. For every analytic domain $Y \subseteq X$, its inverse image is canonically Y_L. Scalar extension commutes with fibre products and with successive complete field extensions.

Proof. We first make the local mapping characterization explicit. A bounded map over K from an L-analytic space to a K-analytic space is defined by the chart condition in Definition 2.40.7, using bounded K-homomorphisms from the target K-affinoid algebra to the source L-affinoid algebra. The proof of Theorem 2.40.8 applies to these maps as well: all pullbacks of chart domains use Theorem 2.36.7 and Theorem 2.36.9. For an affinoid target, the bounded-pushout property gives

$$\mathrm{Hom}_L(T, \mathcal{M}(A\widehat{\otimes}_K L)) \simeq \mathrm{Hom}_{K,\mathrm{loc}}(T, \mathcal{M}(A)),$$

first on affinoid charts of T and then by gluing.

If X is compact Hausdorff, take a finite affinoid cover (U_i). Each intersection $U_i \cap U_j$ is a compact analytic domain and has a finite affinoid cover. Its inverse image in $(U_i)_L$ is therefore a finite union of the corresponding scalar-extended affinoid domains, hence a compact closed analytic domain. The analogous inverse image in $(U_j)_L$ is canonically isomorphic to it: identify the scalar extensions of common affinoid pieces, and use the completed base-change identity on their intersections. These isomorphisms satisfy the cocycle condition by local uniqueness. Finite closed gluing produces X_L. The displayed local mapping characterization glues to

$$\mathrm{Hom}_L(T, X_L) \simeq \mathrm{Hom}_{K,\mathrm{loc}}(T, X).$$

For general X, cover it by open domains O_i contained in compact Hausdorff analytic domains D_i. Construct $(D_i)_L$ and restrict to the open inverse image of O_i. On the inverse images of overlaps, the same mapping characterization gives unique comparison isomorphisms satisfying the cocycle condition.

Open gluing constructs X_L and proves the displayed characterization for all X. It also makes $X \mapsto X_L$ functorial.

The inverse image of an analytic domain is analytic by the mixed-base version of the local pullback argument. Its mapping characterization is that of Y_L, so the two spaces are canonically isomorphic. In particular, the inverse image of any affinoid chart is its affinoid scalar extension. Theorem 2.33.13 now proves surjectivity of π. For successive extensions, the canonical bounded isomorphism of the iterated completed tensor products in Corollary 2.33.9 gives the local comparison, and uniqueness glues it. The same corollary identifies the scalar extension of each affinoid product chart with the product of the extended charts. Since such charts form a quasi-net by Theorem 2.40.11, these local identifications glue and prove compatibility with fibre products. □

Theorem 2.40.13 (Points and fibres on general analytic spaces)**.** *For a point x of an analytic space, the completed residue fields computed in affinoid domains containing x identify canonically and isometrically with a field $\mathcal{H}(x)$. For a morphism $f : Y \to X$, the fibre at x is a canonical $\mathcal{H}(x)$-analytic space Y_x. On affinoid domains $V \subseteq Y$, $U \subseteq X$ with $f(V) \subseteq U$ and $x \in U$, it has the affinoid chart*

$$V_x = \mathcal{M}\big(A_V \widehat{\otimes}_{A_U} \mathcal{H}(x)\big).$$

The canonical map $Y_x \to Y$ identifies its underlying topological space with $f^{-1}(x)$. It preserves the completed residue field at every point.

Proof. Any two affinoid domains containing x have a common affinoid domain containing x, by the net condition. Corollary 2.33.15 identifies the fields along each inclusion. The isomorphisms satisfy the cocycle condition because they agree on evaluations, so they define $F = \mathcal{H}(x)$. A morphism induces an isometric embedding of these fields, checked on any presenting affinoid chart.

Evaluation on affinoid domains containing x defines a bounded map over K from the F-affinoid point $\mathcal{M}(F)$ to X. By Theorem 2.40.12, it corresponds to an F-analytic point map $i_x : \mathcal{M}(F) \to X_F$. Define

$$Y_x = Y_F \times_{X_F, i_x} \mathcal{M}(F).$$

Theorem 2.40.11 and the affinoid pushout identity give the displayed chart formula. Theorem 2.33.11 identifies each V_x homeomorphically with $V \cap f^{-1}(x)$ and preserves the completed residue fields. The charts V_x form a

quasi-net of Y_x, while the corresponding subsets of $f^{-1}(x)$ are the restriction of a presenting chart quasi-net of Y. These local homeomorphisms agree on overlaps. The topology criterion of Lemma 2.40.2, applied on both sides, makes their glued bijection a homeomorphism. The field identifications glue by the same restriction compatibility. □

2.41 Relative differentials and Jacobian presentations

We construct analytic differentials from convergent presentations and then express the Jacobian condition through a finite matrix. This is the presentation-theoretic part of the approach in [Duc18, §§5.1–5.2]. All finite modules below carry their canonical complete norms.

Lemma 2.41.1 (Inner rational neighborhoods)**.** *Let $x \in X = \mathcal{M}(A)$ and let O be an open neighborhood of x. There is a rational domain $U \subseteq O$ which is a neighborhood of x and for which $x \in \mathrm{Int}(U/X)$. It can be chosen as a finite intersection of Weierstrass and Laurent domains whose defining inequalities are all strict at x.*

Proof. The evaluation topology gives finitely many open conditions $|f_i| < a_i$ and $|g_j| > b_j$ whose common locus contains x and is contained in O. Choose radii $|f_i(x)| < r_i < a_i$ and $b_j < s_j < |g_j(x)|$, omitting lower conditions with no effective restriction. The domain

$$U = \{|f_i| \le r_i,\ |g_j| \ge s_j\}, \qquad A_U = A\{r^{-1}T, sS\}/(T_i - f_i,\ g_jS_j - 1).$$

The generalized rational-domain theorem identifies this actual quotient with the domain algebra. The joint quotient is norm-admissible, and its variables satisfy $|T_i(x)| < r_i$ and $|S_j(x)| < s_j^{-1}$. It therefore witnesses the claimed relative interior. The same strict inequalities define an open neighborhood of x contained in U. □

Lemma 2.41.2 (Lifting rational domains through a closed immersion)**.** *Let $B = R/I$ be a norm-admissible quotient of affinoid algebras. Every rational domain $W \subseteq \mathcal{M}(B)$ is the inverse image of a rational domain $V \subseteq \mathcal{M}(R)$. Moreover $B_W = R_V/IR_V$ as Banach algebras.*

Proof. Write $W = \{|b_i| \le \rho_i |b_0|\}$, with $(b_0, \dots, b_m) = B$. Choose lifts $g_i \in R$ and coefficients $a_i \in R$ such that $h = 1 - \sum_i a_i g_i$ belongs to I. For any $\tau > 0$, let V be defined by

$$|g_i| \le \rho_i |g_0| \quad (1 \le i \le m), \qquad |h| \le \tau |g_0|.$$

The denominator and all numerators generate the unit ideal, because $1 = h + \sum_i a_i g_i$. On $\mathcal{M}(B)$ the final numerator vanishes and the other conditions become precisely those defining W. The base-change formula of Theorem 2.36.9 gives the displayed quotient; the ideal IR_V is finitely generated and closed. □

Theorem 2.41.3 (Bounded universal differentials)**.** *Let $A \to B$ be a bounded affinoid homomorphism and choose a norm-admissible relative presentation*

$$P = A\{r^{-1}T\} \twoheadrightarrow B = P/I, \qquad I = (f_1, \dots, f_m).$$

The finite B-module

$$\Omega^{\mathrm{an}}_{B/A} = \left(\bigoplus_{j=1}^{n} B\, dT_j \right) \Big/ \left(\sum_{j=1}^{n} \frac{\partial f_i}{\partial T_j}\, dT_j : 1 \le i \le m \right)$$

has a bounded A-derivation $d : B \to \Omega^{\mathrm{an}}_{B/A}$ which represents bounded A-derivations from B to every complete Banach B-module whose action $B \times N \to N$ is bounded. In particular this applies to finite B-modules, and its canonical pair $(\Omega^{\mathrm{an}}_{B/A}, d)$ is independent of the presentation.

Proof. Termwise differentiation is bounded on P: since every integer has absolute value at most one, $\|\partial F/\partial T_j\| \le r_j^{-1}\|F\|$. For a complete Banach B-module N with bounded action and elements $v_1, \dots, v_n \in N$, the formula

$$D_v(F) = \sum_j \left(\frac{\partial F}{\partial T_j} \bmod I \right) v_j$$

defines a bounded A-derivation. The Leibniz rule holds first for polynomials and then for series by continuity. Conversely, polynomial density shows that every bounded A-derivation on P is this one with $v_j = D(T_j)$. It kills I exactly when $D_v(f_i) = 0$ for every i: the Leibniz rule and the fact that I acts trivially on N then give $D_v(\sum h_i f_i) = \sum h_i D_v(f_i) = 0$.

A bounded map killing I induces a bounded map on B because the quotient is norm-admissible. The relation module in the displayed definition is closed by Theorem 1.18.2. Thus the quotient is a finite complete module. Every B-linear map from it to N is bounded: take controlled coefficient lifts along a finite free presentation, and bound the images of the finitely many generators using the bounded action on N. We have therefore proved the natural bijection

$$\mathrm{Hom}_B(\Omega^{\mathrm{an}}_{B/A}, N) \simeq \mathrm{Der}^{\mathrm{bd}}_A(B, N).$$

Taking N to be the two differential modules constructed from two presentations gives mutually inverse maps commuting with d. Uniqueness also proves compatibility with every further presentation change. □

Proposition 2.41.4 (Base change and transitivity of differentials)**.** *Let L/K be complete and let $A \to C$ be a bounded map to an L-affinoid algebra. Put $B' = B\widehat{\otimes}_A C$. Then*

$$B' \otimes_B \Omega^{\mathrm{an}}_{B/A} \simeq \Omega^{\mathrm{an}}_{B'/C}.$$

For bounded affinoid maps $A \to B \to D$ there is a right-exact sequence of finite D-modules

$$D \otimes_B \Omega^{\mathrm{an}}_{B/A} \longrightarrow \Omega^{\mathrm{an}}_{D/A} \longrightarrow \Omega^{\mathrm{an}}_{D/B} \longrightarrow 0.$$

Proof. Proposition 2.33.8 gives $B' = C\{r^{-1}T\}/(f_1, \dots, f_m)$. The Jacobian matrix of the base-changed equations is the image of the original matrix. Tensoring a finite cokernel is right exact, so Theorem 2.41.3 proves the first formula. These are ordinary tensor products of finite modules: Proposition 1.18.7 shows that completion does not change them.

For transitivity let Q be the cokernel of the first map. It is finite and complete. A D-linear map $Q \to N$ is the same as a bounded A-derivation $D \to N$ that vanishes on B, hence the same as a bounded B-derivation. The universal property identifies Q with $\Omega^{\mathrm{an}}_{D/B}$ and identifies the quotient map with the canonical differential map. This proves the exact sequence. □

Theorem 2.41.5 (Differentials on analytic domains)**.** *If C is the algebra of an affinoid domain in $\mathcal{M}(B)$, then for every bounded affinoid map $A \to B$ the natural map is an isomorphism*

$$C \otimes_B \Omega^{\mathrm{an}}_{B/A} \xrightarrow{\sim} \Omega^{\mathrm{an}}_{C/A}.$$

In particular $\Omega^{\mathrm{an}}_{C/B} = 0$. *For an arbitrary analytic morphism* $f : Y \to X$, *these modules determine a coherent sheaf* $\Omega_{Y/X}$ *on* Y_G, *compatible with restriction to analytic domains and with every analytic base change over a complete ground-field extension.*

Proof. First suppose $C = B\{\rho^{-1}S\}/(gS_i - f_i)$ is rational. The unit-ideal condition makes g invertible in C. For a finite C-module N, every bounded A-derivation $D : B \to N$ has a unique possible extension to C, namely

$$D(S_i) = g^{-1}\big(D(f_i) - S_i D(g)\big).$$

These values do give an extension. On a series $\sum_\nu b_\nu S^\nu$, differentiate coefficients using D and variables using the displayed values. The coefficientwise derivative term converges because D is bounded; the variable term converges by the weighted derivative estimate used above and the uniform power bounds on the images of the S_i. Both estimates are uniform in the series norm. The resulting bounded derivation kills $gS_i - f_i$ and hence their ideal, so it descends through the norm-admissible quotient. Thus restriction induces a natural bijection on derivations, which proves the differential isomorphism by the universal property. Taking $A = B$ proves the vanishing.

For a general domain, Theorem 2.36.6 supplies a finite cover by domains rational in $\mathcal{M}(B)$. They are also rational in $\mathcal{M}(C)$: pull back their defining inequalities and use the domain universal property. On each covering algebra C_i, the rational result identifies both sides of the proposed map with $\Omega^{\mathrm{an}}_{C_i/A}$. The finite affinoid cover is faithfully flat on finite modules, by Theorem 2.38.1 and Proposition 2.38.2; applying it to the kernel and cokernel proves that the original map is an isomorphism.

For sheaf gluing, choose affinoid chart pairs $V \to U$ of f. The finite modules just constructed restrict compatibly under source-chart refinement. If $U' \subseteq U$ is an affinoid domain and $V \to U$ factors through U', then $V = V \times_U U'$ by the domain property; Proposition 2.41.4 gives the same differential module with base U'. Thus target-chart refinement is compatible as well. Common affinoid refinements and uniqueness of the universal derivation give the cocycle identities. Coherent-module gluing from Theorem 2.40.6 and the affinoid Kiehl equivalence produces the coherent sheaf. The base-change formula holds on every affinoid chart pair by Proposition 2.41.4; these canonical isomorphisms glue. □

Proposition 2.41.6 (The conormal descriptions)**.** *For an affinoid morphism $A \to B$, let J be the kernel of multiplication $B\widehat{\otimes}_A B \to B$. Then J/J^2 is canonically isomorphic to $\Omega^{\mathrm{an}}_{B/A}$, with db corresponding to $1 \otimes b - b \otimes 1$. If $W \hookrightarrow V$ is a closed immersion over an analytic space X, with coherent ideal $\mathcal{I}$, there is an exact sequence*

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{d} \Omega_{V/X}|_W \longrightarrow \Omega_{W/X} \longrightarrow 0.$$

In particular $\Omega_{\mathbb{A}^{n,\mathrm{an}}_X/X}$ is free with basis $dT_1, \ldots, dT_n$, and the same basis restricts to every analytic domain of this relative affine space.

Proof. Choose relative topological generators $b_1, \ldots, b_n$ of B. The ideal generated by $\delta b_i = 1 \otimes b_i - b_i \otimes 1$ is closed. In its quotient the two bounded maps from B agree on A and on the b_i, hence on all of B by polynomial density. Multiplication and the first-factor map then give inverse bounded maps between this quotient and B. Thus $J = (\delta b_1, \ldots, \delta b_n)$.

For a bounded A-derivation $D : B \to N$, the bounded bilinear rule $b \otimes c \mapsto bD(c)$ extends to the completed tensor product; call the extension F. The identity

$$F(uv) = \mu(u)F(v) + \mu(v)F(u)$$

holds on algebraic tensors and then everywhere by continuity. Hence F induces a B-linear map $J/J^2 \to N$ sending δb to $D(b)$. It is unique because the δb_i generate J/J^2. Conversely composition with $b \mapsto \delta b$ gives a bounded derivation. This is the same universal property as in Theorem 2.41.3, proving the first assertion. It also identifies the sheaf constructed above with the diagonal conormal sheaf.

For a quotient $C = B/I$, a derivation from B to a finite C-module factors through C precisely when it kills I. Since $D(ab) = aD(b)$ for $b \in I$, the obstruction is the induced map $I/I^2 \to N$. This gives the stated cokernel on affinoid chart pairs. The constructions commute with domain restriction, so the sequences glue on the analytic site. Finally relative Tate algebras have free differential module by their presentation with no equations. These free bases agree as the polydisc radii and base charts vary; relative affine space is their union, and Theorem 2.41.5 gives the assertion for its domains. □

Theorem 2.41.7 (The matrix criterion for a Jacobian presentation)**.** *Let V be an affinoid domain of $\mathbb{A}_X^{n,\mathrm{an}}$, and let $W \hookrightarrow V$ be the closed immersion defined by the ideal $(f_1, \dots, f_m)$. Write*

$$df_i = \sum_{j=1}^{n} J_{ij}\, dT_j.$$

For every $y \in W$,

$$\dim_{\mathcal{H}(y)}(\Omega_{W/X} \otimes \mathcal{H}(y)) = n - \operatorname{rank}_{\mathcal{H}(y)} J(y).$$

Consequently this presentation is Jacobian at y, with $m = n - \mathrm{rk}_y\, \Omega_{W/X}$, if and only if an $m \times m$ minor is nonzero at y. In that case m is the minimum number of generators of the ideal. There is an affinoid neighborhood $V' \subseteq V$ of y on which such a minor is invertible; on $W \cap V'$ the differential sheaf is free of rank $n - m$ and the restricted presentation is Jacobian at every point.

Proof. Proposition 2.41.6 gives a presentation $\mathcal{O}_W^m \xrightarrow{J} \mathcal{O}_W^n \to \Omega_{W/X} \to 0$. Tensoring with $\mathcal{H}(y)$ is right exact, so elementary linear algebra gives the rank formula. Full row rank is equivalent to the existence of the indicated nonzero minor. If the same ideal had $q < m$ generators, the same cokernel computation would give $\mathrm{rk}_y\, \Omega_{W/X} \geq n - q > n - m$, a contradiction.

Let Δ be a nonzero minor at y and choose $0 < c < |\Delta(y)|$. The Laurent domain $V' = \{|\Delta| \geq c\}$ is a neighborhood of y, and Δ is a unit on it. Permuting the coordinate columns, its invertible block expresses the first m basis vectors modulo the relations as linear combinations of the remaining $n-m$. Those remaining vectors are independent, since a relation among them would have zero first m coordinates and invertibility forces every relation coefficient to vanish. They form a free basis of the cokernel. The same invertible minor at every point proves the final assertion. If $m = 0$, take $\Delta = 1$ and $V' = V$. □

Theorem 2.41.8 (Domain locality and base change of quasi-smoothness)**.** *Use the Jacobian-presentation definition of quasi-smoothness in Section 2.9. Let $f : Y \to X$, let $y \in Y$, let D be an analytic domain of Y containing y, and let U be an analytic domain of X containing $f(D)$. Then f is quasi-smooth at y if and only if $D \to U$ is quasi-smooth there. Quasi-smoothness is preserved by every analytic base change, including complete ground-field extension. Analytic-domain inclusions are quasi-smooth and have zero relative differential sheaf.*

Proof. Start with a Jacobian presentation $W \hookrightarrow V \subseteq \mathbb{A}^{n,\mathrm{an}}_X$ at y, with m equations. To restrict the target, choose an affinoid domain V_0 in $V \cap \mathbb{A}^{n,\mathrm{an}}_U$ containing the image of y. Its inverse image W_0 is affinoid and closed in V_0, with the restricted m equations. The differential comparison of Theorem 2.41.5 identifies the relevant fibre ranks, so the same minor is nonzero at y.

To restrict the source, choose an affinoid domain $E \subseteq D \cap W_0$ containing y. Theorem 2.36.6 gives a finite cover of E by domains rational in W_0; choose a member E_0 containing y. Lemma 2.41.2 lifts E_0 to a rational domain $V_1 \subseteq V_0$ with $E_0 = W_0 \times_{V_0} V_1$. Thus $E_0 \hookrightarrow V_1 \subseteq \mathbb{A}^{n,\mathrm{an}}_U$ has the same equations and the same nonzero minor at y, and is the required presentation. Conversely a Jacobian presentation for $D \to U$ is also one for $Y \to X$: domain inclusions are transitive, $\mathbb{A}^n_U$ is a domain of $\mathbb{A}^n_X$, and the differential fibres agree.

For base change by $X' \to X$, let $y' \in Y \times_X X'$ lie over y. Pull back $W \hookrightarrow V$ and take an affinoid domain V' in $V \times_X X'$ containing the image of y'. The inverse image W' is an affinoid domain of $Y \times_X X'$ closed in V', and V' is a domain of $\mathbb{A}^n_{X'}$. The m pulled-back equations generate its ideal by the completed-pushout calculation. Proposition 2.41.4 identifies the differential fibres by extension from $\mathcal{H}(y)$ to $\mathcal{H}(y')$; the nonzero minor stays nonzero under this isometric field embedding. This proves base-change stability. The argument works verbatim over any complete extension. Finally an analytic-domain inclusion has locally the presentation $W = W \subseteq \mathbb{A}^0_X = X$, with no equations, and its differential sheaf vanishes by Theorem 2.41.5. □

Theorem 2.41.9 (Composition of Jacobian presentations)**.** *Let $Z \xrightarrow{g} Y \xrightarrow{f} X$ be analytic morphisms and put $y = g(z)$. If g is quasi-smooth at z and f is quasi-smooth at y, then fg is quasi-smooth at z. The differential fibre ranks satisfy*

$$\mathrm{rk}_z\, \Omega_{Z/X} = \mathrm{rk}_y\, \Omega_{Y/X} + \mathrm{rk}_z\, \Omega_{Z/Y}.$$

Proof. Write $d = \mathrm{rk}_y\, \Omega_{Y/X}$ and $e = \mathrm{rk}_z\, \Omega_{Z/Y}$. Choose $W \hookrightarrow V \subseteq \mathbb{A}^n_X$ at y with $n - d$ equations. By domain locality replace the part of Z under consideration by its inverse image over W, and choose a Jacobian presentation $T \hookrightarrow S \subseteq \mathbb{A}^m_W$ at z with $m - e$ equations.

Since S is affinoid, its m coordinate functions are bounded. Hence it lies in a closed relative polydisc P_W over W, the inverse image of a closed relative polydisc P over V. It is an affinoid domain of P_W by the domain

mapping property. Cover S by rational domains of P_W and choose one S_0 containing the image of z. Lemma 2.41.2 gives a rational domain $P_0 \subseteq P$ with $P_0 \cap P_W = S_0$. In particular P_0 is an affinoid domain of $\mathbb{A}_X^{n+m}$. Put $T_0 = T \times_S S_0$.

The ideal of S_0 in P_0 is generated by the original $n-d$ equations. The ideal of T_0 in S_0 is generated by the restricted $m-e$ equations; lift these through the surjection of affinoid algebras $\mathcal{O}(P_0) \to \mathcal{O}(S_0)$. The union of the two lists generates the ideal of T_0 in P_0. Thus the rank formula of Theorem 2.41.7 gives

$$\operatorname{rk}_z \Omega_{Z/X} = \operatorname{rk}_z \Omega_{T_0/X} \geq (n+m) - ((n-d) + (m-e)) = d + e.$$

The transitivity sequence of Proposition 2.41.4, tensored with $\mathcal{H}(z)$, gives the opposite inequality. The first term has dimension d, because its fibre is obtained from the fibre at y by a field extension, and the final term has dimension e. Hence equality holds. The immersion $T_0 \hookrightarrow P_0 \subseteq \mathbb{A}_X^{n+m}$ has exactly the required number of equations, so is a Jacobian presentation at z. □

Proposition 2.41.10 (Algebraic differentials under analytification)**.** *Let $Y \to X$ be a morphism of finite-type K-schemes. The pullback of the algebraic differential sheaf to Y^{an} is canonically isomorphic to $\Omega_{Y^{\mathrm{an}}/X^{\mathrm{an}}}$, compatibly with universal derivations. More concretely, if $S = R[T_1, \ldots, T_n]/(f_1, \ldots, f_m)$, $V = \mathcal{M}(A) \subseteq (\operatorname{Spec} R)^{\mathrm{an}}$, and $D = A\{r^{-1}T\}/(f_1, \ldots, f_m)$ is a relative radius piece, then*

$$D \otimes_S \Omega_{S/R} \simeq \Omega^{\mathrm{an}}_{D/A}.$$

Proof. For polynomial rings the Leibniz rule shows directly that an R-derivation is determined arbitrarily by its values on the T_j. It descends to S precisely when it kills the f_i. Thus $\Omega_{S/R}$ is the cokernel of the polynomial Jacobian matrix $(\partial f_i/\partial T_j)$. Tensoring this finite presentation with D gives exactly the presentation of Theorem 2.41.3, proving the formula. Both maps send the differential of a polynomial to the differential of its analytic image. Uniqueness makes them compatible with algebraic localization, changes of generators and analytic domain restriction. The relative radius pieces cover the inverse images of affine scheme charts by Proposition 2.33.27; hence these isomorphisms glue to the asserted sheaf isomorphism. □

2.42 Cohomology of analytic projective space

The standard finite affinoid cover of projective space makes its line-bundle cohomology an explicit calculation with convergent Laurent series. The monomial contraction used in the algebraic computation [StacksAlg, Tag 01XT] has norm at most one, and therefore acts on the analytic coefficient spaces as well.

Proposition 2.42.1 (Cohomology on a finite acyclic cover)**.** *Let X be an analytic space with an admissible finite cover $(U_i)_{0\le i\le n}$ such that all nonempty finite intersections are affinoid domains. For a coherent $\mathcal{O}_{X_G}$-module $\mathcal{F}$, the ordered Čech complex of this cover computes $H^q(X_G,\mathcal{F})$. In particular these groups vanish for $q>n$.*

Proof. The restriction of $\mathcal{F}$ to every intersection is coherent. Theorem 2.39.6 gives zero higher cohomology on that intersection. Thus the hypotheses of [StacksAlg, Tag 03F7, Lemma 21.10.7] hold for the covering on X_G, including repeated-index intersections, and the all-tuples Čech complex computes derived section cohomology. The ordered-to-all-tuples homotopy comparison used in Theorem 2.21.3 applies to this finite intersection coefficient diagram. It identifies the same groups with the ordered complex, which has no terms above degree n. □

Lemma 2.42.2 (Laurent coefficient normal form)**.** *For $0\le \ell\le s$, the quotient Banach algebra*

$$K\langle u_1,\dots,u_s,v_1,\dots,v_\ell\rangle/(u_1v_1-1,\dots,u_\ell v_\ell-1)$$

is isometrically the space of series $\sum_{\alpha\in\mathbf{Z}^\ell\times\mathbf{N}^{s-\ell}} c_\alpha u^\alpha$ whose coefficients tend to zero, with the maximum norm and Laurent-series multiplication.

Proof. Replace a monomial u^av^b by the monomial with exponent $(a_1-b_1,\dots,a_\ell-b_\ell,a_{\ell+1},\dots,a_s)$. The coefficient attached to any one exponent is a convergent sum: for each positive ε, the original Tate series has only finitely many coefficients of size at least ε. The same observation proves that the resulting coefficients tend to zero and that this reduction is contractive. Conversely represent a negative exponent by the corresponding power of v_i, using no simultaneous positive powers of u_i,v_i. This gives an isometric linear section of the reduction map.

For a monomial, its difference from its normal form is a sum of multiples of u_iv_i-1, by successively using $(u_iv_i)^d-1=(u_iv_i-1)\sum_{a=0}^{d-1}(u_iv_i)^a$. The multipliers have coefficient norms at most one. Apply this identity to each term of a Tate series. For any $\varepsilon>0$, only finitely many original terms have coefficients of size at least ε, and each of them produces finitely many terms in its multipliers. Consequently the multiplier series converge in the Tate algebra. This proves that the kernel is the actual ideal displayed above. The contractive reduction and its isometric section identify its quotient norm with the maximum coefficient norm. Laurent convolution is well-defined by the same finite-large-coefficient argument and agrees with polynomial multiplication on a dense subalgebra, which proves the algebra assertion. □

Theorem 2.42.3 (Twists on analytic projective space)**.** *Let $P=(\mathbf{P}^n_K)^{\mathrm{an}}$, and let $\mathcal{O}_P(m)$ be the line bundle obtained by analytifying the usual homogeneous transition functions. For every integer m, the canonical map*

$$H^q(\mathbf{P}^n_K,\mathcal{O}(m))\longrightarrow H^q(P_G,\mathcal{O}_P(m))$$

is an isomorphism for every $q\geq 0$, compatibly with complete extensions of K. If $n\geq 1$, the only possible nonzero groups have bases

$$H^0(P_G,\mathcal{O}_P(m))=\bigoplus_{\substack{\nu_j\geq 0\\ \sum_j\nu_j=m}} KX^\nu,$$

$$H^n(P_G,\mathcal{O}_P(m))=\bigoplus_{\substack{\nu_j<0\\ \sum_j\nu_j=m}} KX^\nu.$$

For $n=0$, $H^0=K$ for all m, and all higher groups are zero. In the standard finite affinoid Čech complex, the projection onto these finite-dimensional representatives and a homotopy on its complement can be chosen with norm at most one.

Proof. Use the finite affinoid covering $U_i=\{|X_j|\leq|X_i| \text{ for all } j\}$, constructed in Proposition 2.33.30. On $U_I=\bigcap_{i\in I}U_i$, fix $i\in I$. The ratios X_j/X_i have norm at most one, and the ratios with $j\in I$ are units with inverse of norm at most one. Lemma 2.42.2, using the frame X_i^m, identifies the section space of $\mathcal{O}_P(m)$ with the maximum-norm space of coefficient families tending to zero on

$$\Lambda_I(m)=\{\nu\in\mathbf{Z}^{n+1}:\textstyle\sum\nu_j=m,\ \nu_j\geq 0 \text{ if } j\notin I\}.$$

Restrictions are the inclusions of these coefficient spaces. Every U_I is affinoid, so Proposition 2.42.1 computes the required cohomology by this ordered complex.

For a fixed exponent vector ν, put $N(\nu) = \{j : \nu_j < 0\}$. Its coefficient occurs exactly in the components indexed by nonempty I containing $N(\nu)$. The differential is the alternating simplex differential. If $N(\nu)$ is nonempty and proper, choose the least index $k \notin N(\nu)$. The insertion operator $(hc)_I = (-1)^{\#\{i\in I: i<k\}} c_{I\cup\{k\}}$ when $k \notin I$, and zero otherwise, satisfies $dh + hd = 1$. The identity follows by cancelling the two terms deleting and inserting each index different from k; the term deleting k has positive sign and equals c_I. The condition on $N(\nu)$ is preserved on both sides because $k \notin N(\nu)$.

If $N(\nu)$ is empty, the full simplex complex has just the constant degree-zero class; the same insertion at $k = 0$ contracts the kernel of evaluation at its zeroth vertex. If $N(\nu)$ is the entire vertex set, only the degree-n component remains. All the insertion maps have norm at most one. Applied simultaneously to the coefficient vectors, they preserve convergence to zero and therefore define bounded maps on the completed analytic complex. Their homotopy identities hold first on finite Laurent sums and then by continuity.

The surviving exponent vectors are exactly the two finite sets in the statement. For $n = 0$, the single exponent is m and the complex has one term. The algebraic computation [StacksAlg, Tag 01XT, Lemma 30.8.1] uses the same coefficient complex with finite-support families. The natural map is its inclusion in the analytic complex. Since the surviving finite representatives and their projections are identical, it induces the asserted canonical isomorphism. The formulas have coefficients $0, 1, -1$; scalar extension therefore preserves all projections, representatives and comparison maps. □

Theorem 2.42.4 (Cohomology comparison for algebraic coherent sheaves)**.** *For every coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n_K$, analytification induces natural isomorphisms*

$$H^q(\mathbf{P}^n_K, \mathcal{F}) \;\simeq\; H^q((\mathbf{P}^n_K)^{\mathrm{an}}_G, \mathcal{F}^{\mathrm{an}}) \qquad (q \geq 0).$$

They respect connecting homomorphisms of short exact sequences.

Proof. Use the algebraic standard affine cover $D_+(X_i)$ and the analytic unit cover $U_i \subset D_+(X_i)^{\mathrm{an}}$. Restriction and scalar extension on each intersection give a map of their ordered Čech complexes. The algebraic complex computes cohomology by affine acyclicity, and the analytic complex does so

by Proposition 2.42.1. On every intersection, sections preserve short exact sequences of coherent sheaves: on the algebraic side this is the module description on an affine; on the analytic side it follows from Theorem 2.33.34 and affinoid Theorem B. The resulting comparison therefore commutes with connecting homomorphisms. These constructions give the usual cohomology map induced by analytification, since in degree zero they are restriction of sections and the acyclic resolutions give their derived maps.

We prove bijectivity by descending induction on q, simultaneously for all coherent $\mathcal{F}$. Both groups vanish for $q > n$, by [StacksAlg, Tag 01YS, Lemma 30.14.1(2)] and Proposition 2.42.1. Choose a surjection $\mathcal{L} \twoheadrightarrow \mathcal{F}$, where $\mathcal{L}$ is a finite direct sum of twists, using [StacksAlg, Tag 01YS, Lemma 30.14.1(1)], and let $\mathcal{R}$ be its coherent kernel. Analytification preserves this short exact sequence. In its two long exact sequences, the comparison maps for $H^q(\mathcal{L})$, $H^{q+1}(\mathcal{R})$, and $H^{q+1}(\mathcal{L})$ are isomorphisms, by Theorem 2.42.3 and the induction hypothesis.

Here is the required diagram chase. Given an analytic class a in degree q for $\mathcal{F}$, lift its connecting class to the algebraic $H^{q+1}(\mathcal{R})$. Its image in $H^{q+1}(\mathcal{L})$ is zero by injectivity there, so it comes from an algebraic class b of $\mathcal{F}$. The difference between a and the image of b comes from analytic $H^q(\mathcal{L})$, hence from algebraic $H^q(\mathcal{L})$. This proves surjectivity for every $\mathcal{F}$. Apply that surjectivity also to $\mathcal{R}$. If an algebraic class of $\mathcal{F}$ maps to zero analytically, injectivity in degree $q+1$ makes it the image of a class of $H^q(\mathcal{L})$. Analytically this latter class comes from $H^q(\mathcal{R})$. Lift that class by the surjectivity just proved and subtract it; injectivity for $\mathcal{L}$ then proves that the original class is zero. This completes the induction. □

Corollary 2.42.5 (Full faithfulness on projective space)**.** *For coherent algebraic sheaves $\mathcal{F}, \mathcal{G}$ on $\mathbf{P}^n_K$, the canonical map*

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Hom}(\mathcal{F}^{\mathrm{an}}, \mathcal{G}^{\mathrm{an}})$$

is bijective.

Proof. The sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is coherent: locally a finite presentation of $\mathcal{F}$ expresses it as the kernel of a map between finite sums of $\mathcal{G}$, over a Noetherian ring. Its analytification is $\mathcal{H}om(\mathcal{F}^{\mathrm{an}}, \mathcal{G}^{\mathrm{an}})$ by Theorem 2.33.34. Apply Theorem 2.42.4 in degree zero. Global sections of these Hom sheaves are precisely the stated sheaf morphisms, by the definition of the sheaf Hom. □

Proposition 2.42.6 (Complete base change for projective coherent cohomology)**.** *Let $P = (\mathbf{P}^n_K)^{\mathrm{an}}$, let $\mathcal{F}$ be coherent on P_G, and let L/K be any complete valued extension. For the coherent pullback $\mathcal{F}_L$ there are natural isomorphisms of finite-dimensional L-spaces*

$$L \otimes_K H^q(P_G, \mathcal{F}) \simeq H^q((P_L)_G, \mathcal{F}_L).$$

Proof. Use the standard finite unit cover and its Banach Čech complex $C^\bullet$. Put $Z^q = \ker d^q$ and $B^q = \operatorname{im} d^{q-1}$. By Theorem 2.33.38, every B^q is closed and Z^q/B^q is finite dimensional. Thus all these spaces are Banach. If K is nontrivially valued, Lemma 1.14.1 makes the surjections $C^q \to B^{q+1}$ norm-strict. If K is trivially valued, the coefficient algebras and their finite modules on this cover have the discrete $0/1$ quotient norms; the same surjections are norm-strict directly. The sequences

$$0 \longrightarrow Z^q \longrightarrow C^q \longrightarrow B^{q+1} \longrightarrow 0, \qquad 0 \longrightarrow B^q \longrightarrow Z^q \longrightarrow H^q \longrightarrow 0.$$

Apply Theorem 2.33.3 to both sequences. The completed tensor complex is the Čech complex on P_L: this follows on each intersection from Proposition 2.33.5 and finite-module base change, Proposition 1.18.7. The two exact sequences identify its cohomology with $H^q \mathbin{\widehat{\otimes}}_K L$. Since H^q is finite dimensional, its completed tensor product is its ordinary tensor product: choose a basis and use equivalence of finite-dimensional norms. The intersection identifications respect restrictions, so the resulting isomorphism is natural. □

Lemma 2.42.7 (Coherent modules on a closed analytic subspace)**.** *Let $i : Z \hookrightarrow X$ be a closed analytic subspace defined by a coherent ideal $\mathcal{I}$. Direct image identifies coherent modules on Z_G with coherent modules on X_G annihilated by $\mathcal{I}$, and is exact. If X has a finite affinoid cover with affinoid finite intersections, then*

$$H^q(Z_G, \mathcal{E}) = H^q(X_G, i_*\mathcal{E}).$$

Proof. On an affinoid domain $V = \mathcal{M}(A)$ the inverse image is $\mathcal{M}(A/J)$, where J is the finite ideal representing $\mathcal{I}|_V$. By the affinoid Kiehl equivalence, a coherent module on it is a finite A/J-module M. Restriction of scalars identifies such modules, with their morphisms and exact sequences, with finite A-modules killed by J. On a smaller affinoid domain with algebra A', both restrictions are $A' \otimes_A M$, viewed over A'/JA'. Thus these identifications agree on overlaps and glue uniquely. For the cohomology assertion, the inverse

images of the given affinoid cover and its intersections are affinoid, and the two Čech coefficient complexes are literally the same. Proposition 2.42.1 computes both derived cohomology groups from that complex. □

Theorem 2.42.8 (Global generation and GAGA on projective space)**.** *For every complete non-Archimedean field K, analytification is an equivalence*

$$\mathrm{Coh}(\mathbf{P}^n_K) \;\simeq\; \mathrm{Coh}((\mathbf{P}^n_K)^{\mathrm{an}}_G).$$

For every coherent analytic $\mathcal{F}$, the twist $\mathcal{F}(m)$ is generated by finitely many global sections for all sufficiently large m, and $H^q(P_G, \mathcal{F}(m)) = 0$ for every $q > 0$ and all sufficiently large m.

Proof. We prove the three assertions together by induction on n, for all complete fields simultaneously. For $n = 0$ they are the finite-module description of the one-point affinoid $\mathcal{M}(K)$ and its Theorem B. Suppose $n > 0$ and that the assertions hold in dimension $n - 1$.

Generation at a rational point. Fix a K-rational point x and a hyperplane H through it, defined by a nonzero linear form h. Set $\mathcal{C} = \ker(\mathcal{F}(-1) \xrightarrow{h} \mathcal{F})$ and $\mathcal{B} = \mathcal{F}/h\mathcal{F}(-1)$. Both are coherent and killed by the ideal of H. Lemma 2.42.7 identifies them with coherent sheaves on H^{an}. Writing $\mathcal{R}_m = h\mathcal{F}(m-1)$, we have short exact sequences

$$0 \longrightarrow \mathcal{C}(m) \longrightarrow \mathcal{F}(m-1) \longrightarrow \mathcal{R}_m \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{R}_m \longrightarrow \mathcal{F}(m) \longrightarrow \mathcal{B}(m) \longrightarrow 0.$$

By induction, $H^2(\mathcal{C}(m)) = H^1(\mathcal{B}(m)) = 0$ for large m. The cohomology groups here can be computed either on H^{an} or after direct image on P, by the preceding lemma. Their long exact sequences give surjections

$$H^1(\mathcal{F}(m-1)) \twoheadrightarrow H^1(\mathcal{R}_m) \twoheadrightarrow H^1(\mathcal{F}(m)).$$

All these spaces are finite dimensional by Theorem 2.33.38. The nonincreasing sequence $\dim_K H^1(\mathcal{F}(m))$ is eventually constant. For those m, both displayed surjections are isomorphisms. The second short exact sequence then gives a surjection $H^0(\mathcal{F}(m)) \twoheadrightarrow H^0(\mathcal{B}(m))$. By induction the latter sections generate $\mathcal{B}(m)$ for large m, so their lifts generate the fibre of $\mathcal{F}(m)$ at x.

All analytic points and a uniform twist. For an arbitrary $x \in P$, set $L = \mathcal{H}(x)$. The canonical L-rational point of P_L above x is defined by

the valued residue-field map. Apply the rational-point argument over L. By Proposition 2.42.6, all global sections upstairs are L-linear combinations of global sections downstairs. The fibre of the evaluation cokernel upstairs is exactly its downstairs fibre tensored with $L = \mathcal{H}(x)$. It is zero. Thus a sufficiently large twist is generated at every analytic point x, with the twist initially depending on x.

We justify the passage from fibres to open generation loci. For a finite module M over an affinoid algebra A, its fibre at a point with prime kernel $\mathfrak{p}$ vanishes exactly when $M_{\mathfrak{p}} = 0$: extension from $\kappa(\mathfrak{p})$ to $\mathcal{H}(x)$ is faithful, and the finite-module Nakayama lemma applies to $A_{\mathfrak{p}}$. This lemma follows directly from the determinant argument: if finitely many generators are linear combinations of themselves with coefficients in the maximal ideal, the determinant of one minus that coefficient matrix is a unit annihilating the module. If $M_{\mathfrak{p}} = 0$, clearing denominators of a finite generating list gives $a \notin \mathfrak{p}$ with $aM = 0$. Thus the zero-fibre locus is open.

The evaluation map uses a finite basis of $H^0(P_G, \mathcal{F}(m))$, so its cokernel is coherent. On each member of the finite closed unit cover its nonzero-fibre locus is a closed subset; the finite union is closed in P. Therefore the generation locus G_m is open in P. Moreover $G_m \subseteq G_{m+1}$: at any point choose X_i nonzero, and multiply generating sections by X_i, which is a unit after trivializing the twist there. The G_m cover the compact space P. A finite subcover and this monotonicity give one m_0 with $G_m = P$ for every $m \geq m_0$. The evaluation cokernel is then zero on each affinoid, since a nonzero finite module has a nonzero localization at some maximal ideal. This proves global generation.

Algebraization and vanishing. Global generation gives a surjection $\mathcal{O}_P(-a)^s \twoheadrightarrow \mathcal{F}$. Its kernel is coherent by the affinoid finite-module theorem. Apply global generation again to that kernel, obtaining a presentation

$$\mathcal{O}_P(-b)^t \xrightarrow{u} \mathcal{O}_P(-a)^s \longrightarrow \mathcal{F} \longrightarrow 0.$$

By Corollary 2.42.5, the map u is the analytification of a unique algebraic map between the corresponding sums of twists. Its algebraic cokernel $\mathcal{E}$ is coherent and satisfies $\mathcal{E}^{\mathrm{an}} \simeq \mathcal{F}$ by Theorem 2.33.34. This proves essential surjectivity; full faithfulness was already proved in that corollary. Finally, apply [StacksAlg, Tag 01YS, Lemma 30.14.1(4)] to $\mathcal{E}$, and transport the resulting vanishing by Theorem 2.42.4. This proves the remaining assertion and completes the simultaneous induction. □

Corollary 2.42.9 (GAGA for projective schemes)**.** *Let Z be a projective scheme over a complete non-Archimedean field K. Analytification induces an exact equivalence between coherent algebraic sheaves on Z and coherent sheaves on Z_G^{an}. For every coherent algebraic sheaf $\mathcal{E}$, the canonical maps*

$$H^q(Z,\mathcal{E}) \longrightarrow H^q(Z_G^{\mathrm{an}},\mathcal{E}^{\mathrm{an}})$$

are isomorphisms in every degree.

Proof. Choose a closed immersion $i : Z \hookrightarrow Q = \mathbf{P}^n_K$ with coherent ideal $\mathcal{I}$, and write $P = Q^{\mathrm{an}}$. On each affine chart, modules over the quotient ring are exactly modules over the original ring annihilated by the ideal. This identification, and its analogue for finite affinoid modules, commute with restriction. Thus algebraic closed pushforward and the analytic closed pushforward of Lemma 2.42.7 identify coherent sheaves on Z and Z_G^{an} with the respective ideal-annihilated subcategories on Q and P_G.

For an analytic coherent sheaf $\mathcal{F}$ on Z_G^{an}, Theorem 2.42.8 algebraizes $i_*^{\mathrm{an}}\mathcal{F}$ to a coherent $\mathcal{G}$ on Q. The multiplication map $\mathcal{I}\otimes\mathcal{G}\to\mathcal{G}$ becomes zero after analytification. Analytification commutes with this tensor product, as follows from finite presentations on affine and affinoid charts. Full faithfulness on Q therefore makes the algebraic multiplication map zero as well. Hence $\mathcal{G} = i_*\mathcal{E}$ for a coherent $\mathcal{E}$ on Z, and $\mathcal{E}^{\mathrm{an}} \simeq \mathcal{F}$. Full faithfulness restricts to these subcategories, and exactness is Theorem 2.33.34.

Finally, closed pushforward does not change the relevant cohomology. Analytically this is Lemma 2.42.7. Algebraically, use the standard finite affine cover of Q and its pullback to Z; both have affine intersections. By [StacksAlg, Tag 01XB] and [StacksAlg, Tag 03F7], their ordered Čech complexes compute cohomology. The two complexes are identical for $i_*\mathcal{E}$ and $\mathcal{E}$. The comparison maps commute with this identification because all maps are induced by restriction of the same sections. Theorem 2.42.4 on Q now gives the asserted comparisons on Z. □

References. The organization by cohomological comparison, generation by twists, and finite presentations follows [Ser56, §§13–17]. The analytic input is proved here by bounded Laurent contractions, compact restriction operators, and completed-field base change.

2.43 Relative compact operators and affinoid finiteness

A restriction to a smaller relative polydisc has finite-rank approximations over the coefficient algebra. To retain these approximations after passing to a closed module of matching sections, one needs a uniform integral bound. We establish that bound first. Throughout the first five results, K is nontrivially valued, $O = K^\circ$, and A is a strict K-affinoid algebra with a defining quotient norm. Write $A_0 = A_{\le 1}$. Proposition 3.14.7 identifies A_0 with a quotient of $O\langle X_1, \dots, X_n\rangle$. Banach module actions are taken to be contractive. An initially bounded action can be made contractive by the equivalent norm $\|m\|_* = \sup_{a \neq 0} \|am\|/\|a\|$.

Lemma 2.43.1 (Almost finite generation over the integral coefficient ring)**.** *Let E be a finite A_0-module and $N \subset E$ any submodule. For every real $0 < \alpha < 1$ there is a finite A_0-submodule $N_0 \subset N$ such that $aN \subset N_0$ for every $a \in O$ with $|a| \le \alpha$.*

Proof. It suffices to prove the assertion over $P_n = O\langle X_1, \dots, X_n\rangle$: take a finite free presentation above E and then pass to its image. We use the Gauss norm inherited from $K\langle X_1, \dots, X_n\rangle$. The coordinate changes in Lemma 1.11.3 preserve P_n. The norm bounds in Theorem 1.10.3 therefore give division within P_n by a normalized distinguished element.

First suppose that the valuation is discrete. We prove that P_n is Noetherian by induction on n, the case $n = 0$ following from the uniformizer description of ideals of O. For a nonzero ideal $J \subset P_n$, the norms of its nonzero elements have a maximum β. Choose $g \in J$ attaining it. After an integral isometric coordinate change and multiplication by a norm-one unit, write $g = cf$ with $|c| = \beta$ and f normalized distinguished of degree d in X_n. For $h \in J$, division gives $h = qf + r$ with $\|q\|, \|r\| \le \|h\| \le |c|$. Hence $q/c \in P_n$, so $r = h - (q/c)g \in J$. These remainders lie in $J \cap P_{n-1}[X_n]_{<d}$, a submodule of a finite free module over the Noetherian ring P_{n-1}. They have finitely many generators, which together with g generate J. This proves Noetherianity and the assertion in the discrete case.

Now suppose that the value group is dense in $\mathbf{R}_{>0}$. We record an elementary extension argument. In an exact sequence $0 \to E_1 \to E \to E_2 \to 0$, suppose that the assertion is known for the submodules $N \cap E_1$ and the image of N in E_2. Choose $b \in O$ with $\sqrt{\alpha} < |b| < 1$. Finite submodules

containing their respective b-multiples, together with lifts to N of the chosen quotient generators, generate a finite submodule of N containing b^2N. Since $|a| \leq \alpha < |b|^2$ implies $a/b^2 \in O$, it contains every required aN. Thus the assertion passes from these two submodules to N. This also proves passage from ideals to finite free modules by induction on the rank, and then to finite quotients.

We now induct on n, proving the assertion for all finite P_n-modules. When $n = 0$, for a nonzero ideal $J \subset O$ choose $g \in J$ with $|g| > \alpha \sup_{h\in J} |h|$; then $aJ \subset gO$ for all $|a| \leq \alpha$. The preceding rank and quotient argument finishes this case. Suppose the assertion is known over P_{n-1}, and let $J \subset P_n$ be a nonzero ideal. Put $\beta = \sup_{h\in J} \|h\|$ and choose b as above. Choose $g \in J$ with $\|g\| > |b|\beta$. Normalization and an integral isometric coordinate change give $g = cf$, with $|c| = \|g\|$ and f normalized distinguished, after absorbing a norm-one unit. The quotient $P_n/(f)$ is finite free over P_{n-1} by integral division.

For $h = fq \in J \cap (f)$, multiplicativity of the Gauss norm gives $\|q\| = \|h\| \leq \beta$. Therefore $bh = (bq/c)g \in (g)$, since $|b|\beta < |c|$. Thus $b(J \cap (f))$ lies in the single-generated submodule $(g) \subset J\cap(f)$. By induction, the b-multiple of the image of J in $P_n/(f)$ lies in a finite P_{n-1}-submodule of that image. Lift its generators to J and allow P_n-coefficients. The extension argument gives a finite submodule of J containing b^2J, hence every aJ with $|a| \leq \alpha$. Rank induction and passage to quotients complete the proof. □

Definition 2.43.2 (Relative complete continuity)**.** A bounded A-linear map $g : E \to F$ between Banach modules is completely continuous if it is an operator-norm limit of bounded A-linear maps h_i with finite A-module image. It is strictly completely continuous if these approximants can be chosen with a fixed $c \in O \setminus \{0\}$ such that $ch_i(E_{\leq 1})$ is contained in a finite A_0-submodule of $F_{\leq 1}$ for every i. The finite integral submodule may depend on i.

Theorem 2.43.3 (The relative Schwarz perturbation lemma)**.** *Let $f, g : E \to F$ be bounded A-linear maps of Banach modules, with f surjective and g completely continuous. Then $(f + g)(E)$ is closed in F, and the quotient is a finite A-module whose quotient norm is canonical.*

Proof. Lemma 1.14.1 gives a lift bound C for f. If a bounded map u satisfies $C\|u\| < 1$, successive lifts of the errors show that $f+u$ remains surjective with a uniform lift bound: starting with $y_0 = y$, choose $f(x_j) = y_j$, $\|x_j\| \leq C\|y_j\|$,

and put $y_{j+1} = y_j - (f+u)(x_j)$. The norms decrease geometrically and $\sum_j x_j$ is a lift of y. Choose a finite-image approximant h of g with $C\|g-h\| < 1$, and set $f' = f + (g-h)$. Thus f' is a norm-strict surjection.

Let $N = \ker h$. It is closed, and $Q = E/N$ is a complete finite A-module. Its norm is canonical: an algebraic finite generating set defines a bounded surjection $A^m \to Q$, which is norm-strict by the same open-mapping lemma. The image of $\ker f'$ in Q is an algebraic submodule of a finite module, hence closed by Theorem 1.11.5. Consequently $N + \ker f'$ is closed in E. Since f' is a quotient map, $f'(N)$ is closed in F, and $F/f'(N)$ is a finite module with its canonical norm. The image of $f' + h$ contains $f'(N)$; its image modulo that closed submodule is a submodule of the finite module $F/f'(N)$, hence closed and finite. Its inverse image is therefore closed in F, and its cokernel is finite with its canonical quotient norm. Since $f' + h = f + g$, this proves the assertion. □

Lemma 2.43.4 (Strict complete continuity into a closed target)**.** *Let $E = c_0(I, A)$ with its maximum norm and standard basis $(e_\lambda)_{\lambda\in I}$. Let $j : N \to F$ identify a Banach module with a closed submodule, with equivalent induced norm. If $g : E \to N$ is bounded and jg is strictly completely continuous, then g is strictly completely continuous.*

Proof. Use the induced norm on N. Multiplication of the maps by one fixed nonzero scalar, followed by division by that scalar at the end, reduces to the case that g is contractive and approximants $h : E \to F$ satisfy $h(E_{\le 1}) \subset L \subset F_{\le 1}$ for a finite A_0-module L. This rescaling also shows directly that equivalent norms and bounded pre- or postcomposition preserve strict complete continuity: multiply the fixed scalar by a sufficiently small scalar to absorb each uniform operator bound.

Fix $a \in O$ with $0 < |a| < 1$. Given $\varepsilon > 0$, choose such an h with $\|jg - h\| \le \varepsilon$. Apply Lemma 2.43.1 to $h(E_{\le 1}) \subset L$. It gives generators $y_1, \dots, y_s \in h(E_{\le 1})$ of a finite submodule containing $ah(E_{\le 1})$. Choose $x_j \in E_{\le 1}$ with $h(x_j) = y_j$ and put $z_j = g(x_j) \in N_{\le 1}$. Then $\|jz_j - y_j\| \le \varepsilon$. For each λ, choose coefficients $b_{j\lambda} \in A$ satisfying

$$h(e_\lambda) = \sum_j b_{j\lambda} y_j, \qquad \|b_{j\lambda}\| \le |a|^{-1}.$$

Define $g_0(e_\lambda) = \sum_j b_{j\lambda} z_j$. The uniform coefficient bound extends this rule to a bounded A-linear map on $c_0(I, A)$, with image in the finite module

generated by the z_j. The basis estimate gives $\|g - g_0\| \leq |a|^{-1}\varepsilon$. If $x = \sum_\lambda d_\lambda e_\lambda$ has norm at most one, the series $\sum_\lambda a d_\lambda b_{j\lambda}$ converge in A_0, so

$$ag_0(E_{\leq 1}) \subset \sum_j A_0 z_j \subset N_{\leq 1}.$$

The same fixed a works for every ε, proving strict complete continuity. □

Theorem 2.43.5 (Relative Schwarz with a presentation and a closed target)**.** *Let $f, g : E \to F$ be bounded A-linear maps, with f surjective. Suppose there are a bounded surjection $p : E^\flat \to E$ and a closed embedding $j : F \to F^\sharp$ of Banach modules such that jgp is strictly completely continuous. Then $(f + g)(E)$ is closed and its cokernel is a finite A-module with its canonical quotient norm.*

Proof. Every contractive Banach A-module M admits a bounded norm-strict surjection $q : c_0(I, A) \to M$. Take $I = M_{\leq 1}$ and send its basis element indexed by m to m. The map is contractive. For fixed $\pi \in K$ with $0 < |\pi| < 1$, any nonzero $x \in M$ can be divided by a power of π to have norm in $(|\pi|, 1]$; the corresponding single basis vector gives a lift of x of norm at most $|\pi|^{-1}\|x\|$.

Apply this to $M = E^\flat$. The composite $jgpq$ is strictly completely continuous by bounded precomposition. Lemma 2.43.4 makes gpq completely continuous into F. Theorem 2.43.3, applied to the surjection fpq and this map, proves the conclusion, because pq is surjective and hence does not change the image of $f + g$. □

Proposition 2.43.6 (Strictly compact relative restriction)**.** *Let B be a generalized K-affinoid algebra and let $u : A\langle T_1, \ldots, T_n\rangle \to B$ be bounded. Suppose that the spectral radius of each $t_i = u(T_i)$ is less than one. Then u, regarded as an A-linear map, is strictly completely continuous. The same holds for finite direct sums and for the restriction to any closed source submodule with its induced norm.*

Proof. Choose $0 < \theta < 1$ larger than all spectral radii. The spectral-radius formula supplies a constant C such that $\|t^\nu\| \leq C\theta^{|\nu|}$ for every multi-index. Replace the target norm by an equivalent norm making its A-action contractive. The degree-d truncation u_d has image in the finite A-module generated

by t^ν, $|\nu| < d$, and $\|u - u_d\| \le C'\theta^d$ for one fixed C'. Choose $c \in O \setminus \{0\}$ so small that $\|ct^\nu\| \le 1$ for every ν. Then

$$cu_d\big(A\langle T\rangle_{\le 1}\big) \subset \sum_{|\nu|<d} A_0\, ct^\nu \subset B_{\le 1}.$$

This proves the required uniform integral bound. For finitely many maps take a common small c; restriction of the approximants to a source submodule preserves the displayed bound and the operator convergence. □

Theorem 2.43.7 (Finiteness from finitely many inner relative presentations)**.** *Assume that K is nontrivially valued and A is strict. Let $A \to B$ be bounded, with B generalized affinoid. Suppose that $\mathcal{M}(B)$ has a finite affinoid-domain covering (V_i), and for each i there is a norm-admissible surjection $P_i = A\langle T_i\rangle \twoheadrightarrow B$ whose variables have spectral radius less than one on V_i. Then B is finite over A, and its given norm is equivalent to its canonical finite-A-module norm.*

Proof. Write $B_i = \mathcal{O}(V_i)$, and let $Z \subset \prod_i B_i$ be the kernel of the first Čech differential. Theorem 2.38.1 identifies B boundedly with Z, with bounded inverse. For comparison, take the outer covering consisting of one copy of $\mathcal{M}(B)$ for each i; its matching module is the diagonal $Z_0 \subset B^s$, also boundedly isomorphic to B. Restriction therefore induces a bounded isomorphism $r : Z_0 \to Z$.

The product presentation $q : \prod_i P_i \twoheadrightarrow B^s$ is norm-strict. Its inverse image $E^\flat = q^{-1}(Z_0)$ is closed, and $p = q|_{E^\flat} : E^\flat \twoheadrightarrow Z_0$ is norm-strict: any controlled lift of an element of Z_0 automatically belongs to this inverse image. With $j : Z \hookrightarrow \prod_i B_i$, the map jrp is the restriction of the product of the maps $P_i \to B_i$. Proposition 2.43.6 makes it strictly completely continuous. Apply Theorem 2.43.5 with $f = r$ and $g = -r$. The zero map then has finite cokernel Z, with its canonical norm. Transporting this norm through $B \simeq Z$ proves the theorem. □

Theorem 2.43.8 (Boundaryless affinoid morphisms are finite)**.** *Let K be any complete non-Archimedean field and $A \to B$ a bounded homomorphism of generalized K-affinoid algebras. The following conditions are equivalent:*

1. $\mathrm{Int}(\mathcal{M}(B)/\mathcal{M}(A)) = \mathcal{M}(B)$.

2. *B is a finite A-module and its given norm is equivalent to its canonical finite-module norm.*

Proof. The zero algebra is immediate. Suppose first that every point is interior. For $y \in \mathcal{M}(B)$ choose a witnessing norm-admissible relative presentation $A\{r_y^{-1}T_y\} \twoheadrightarrow B$. Choose radii s_{yj} with $|T_{yj}(y)| < s_{yj} < r_{yj}$. The Weierstrass domain

$$V_y = \{z \in \mathcal{M}(B) : |T_{yj}(z)| \leq s_{yj} \text{ for every } j\}$$

contains an ordinary neighborhood of y. Compactness gives a finite family of these domains covering $\mathcal{M}(B)$. Only finitely many presentation radii and witness radii now occur. Apply Theorem 1.17.6 with all these radii, adjoining one additional nonunit radius if K is trivially valued. We obtain a finite free-radius extension $F = K_u$, nontrivially valued, over which A_F is strict and all witness radii r_{yj} belong to $\sqrt{|F^\times|}$.

For each such radius choose an integer $m > 0$ and $c \in F^\times$ with $r_{yj}^m = |c|$. Adjoin a root of $X^m - c$, successively for the finitely many radii. Lemma 1.12.1 makes the resulting finite field extension L/F complete, and multiplicativity gives an element of absolute value r_{yj} in L. After this base change each witness presentation can be rescaled to unit radii. The inverse-image domains still cover by Theorem 2.36.9; on each such domain the rescaled variables have spectral radius at most $s_{yj}/r_{yj} < 1$. Relative presentations commute with this base change by Proposition 2.33.8. Thus Theorem 2.43.7 shows that B_L is finite over A_L.

We descend this finiteness in two steps. Choose an F-basis of L containing 1. Equivalence of finite-dimensional norms shows directly that, for every Banach F-space M, the completed tensor $M\widehat{\otimes}_F L$ is a finite direct sum of copies of M, with equivalent coefficient norms. Consequently A_L is finite over A_F, and the coefficient projection $L \to F$ taking 1 to 1 induces a bounded A_F-linear retraction $B_L \to B_F$. Hence B_F is a direct summand of the finite A_F-module B_L and is finite. Since F is nontrivially valued, its given Banach norm is canonical by the open-mapping argument. Theorem 1.18.9, applied to the original Banach A-module B, now gives finite generation and the canonical norm over the original field K. A bounded action can first be made contractive by the equivalent norm specified at the start of this section; the conclusion is unchanged.

Conversely, if B is finite with its canonical norm, Theorem 2.33.18 supplies one relative affinoid presentation whose generator radii strictly exceed their norms. It witnesses interiority at every point. □

References. The relative operator argument reconstructs the method of Schwarz and Kiehl in [Bosch14, §6.4, Theorems 3–4 and Lemmas 5–6]. The integral approximation, the closed-target passage, and the free-radius descent are proved above.

2.44 Composition and topological interior

Theorem 2.44.1 (The affinoid composition formula for interior)**.** *For composable affinoid morphisms $Z \xrightarrow{g} Y \xrightarrow{f} X$ over any complete non-Archimedean field,*

$$\operatorname{Int}(Z/X) = \operatorname{Int}(Z/Y) \cap g^{-1} \operatorname{Int}(Y/X).$$

Proof. The inclusion from right to left is the composition implication of Proposition 2.33.21. Write the algebras as $A \to B \to C$ and fix $z \in Z$, with images y and x. The completed residue-field embeddings identify the images of their graded reductions with graded subrings

$$R_A \subset R_B \subset R_C \subset \widetilde{\mathcal{H}(z)}^{\mathrm{gr}}.$$

If $z \in \operatorname{Int}(Z/X)$, the relative point criterion says that R_C is finite over R_A. It is then finite over R_B, using the same generating set. Each element of R_B is integral over R_A: multiplication by this element on the finite R_A-module R_C, which contains 1, gives a monic relation by the determinant argument. Furthermore R_B is finitely generated as an R_A-algebra, because the fixed absolute presentation result makes $\widetilde{B}^{\mathrm{gr}}$ finitely generated over $\widetilde{K}^{\mathrm{gr}}$, whose image is contained in R_A. Adjoining these finitely many integral generators successively shows that R_B is finite over R_A: the bounded powers below the respective monic degrees form a finite spanning family. Apply the reverse implication of the relative point criterion to $A \to B$ at y and to $B \to C$ at z. It gives $y \in \operatorname{Int}(Y/X)$ and $z \in \operatorname{Int}(Z/Y)$, as required. □

Lemma 2.44.2 (The image of a finite flat affinoid map)**.** *Let $A \to B$ be finite, norm-admissible and flat between generalized affinoid algebras. Its image on Berkovich spectra is open and closed.*

Proof. The algebra A is Noetherian, so the finite module B is finitely presented. By [StacksAlg, Tag 00NX, Lemma 10.78.2(1),(8)], its rank function on $\operatorname{Spec} A$ is locally constant. Hence its positive-rank locus is open and closed. For $x \in \mathcal{M}(A)$ with kernel $\mathfrak{p}$, the analytic fibre has algebra $B \otimes_A \mathcal{H}(x)$, with its complete finite-dimensional norm, by Proposition 2.33.19 and Theorem 2.33.11. This algebra is nonzero exactly when $B \otimes_A \kappa(\mathfrak{p})$ has positive dimension, since $\kappa(\mathfrak{p}) \to \mathcal{H}(x)$ is a field embedding. A nonzero finite-dimensional algebra has a bounded character after a finite residue-field extension, so its spectrum is nonempty. Therefore the image is the inverse image of the positive-rank locus. Inverse images of $D(a)$ are $\{|a| > 0\}$, and inverse images of Zariski closed subsets are common zero loci of continuous evaluations. The inverse image in question is consequently open and closed. □

Theorem 2.44.3 (Interior of an affinoid domain)**.** *If U is an affinoid domain of an affinoid space X, then $\operatorname{Int}(U/X)$ is the ordinary topological interior of U in X.*

Proof. Suppose x belongs to the ordinary interior. Lemma 2.41.1 gives a rational domain $W \subset U$ containing a neighborhood of x and satisfying $x \in \operatorname{Int}(W/X)$. By Lemma 2.36.5, W is also an affinoid domain of U. The composition formula for $W \to U \to X$ then gives $x \in \operatorname{Int}(U/X)$.

Conversely, let $x \in \operatorname{Int}(U/X)$. The relative boundary $D = U \setminus \operatorname{Int}(U/X)$ is compact, by openness of affinoid relative interior. The inclusion $U \hookrightarrow X$ is a topological embedding, so its compact image D is closed in the Hausdorff space X and does not contain x. Choose a rational domain V that is a neighborhood of x and is contained in $X \setminus D$. The base change $U_V = U \cap V$ is an affinoid domain of V. Every point of U_V was interior in U/X, so the forward base-change inclusion in Proposition 2.33.21 shows that $U_V \to V$ is boundaryless. By Theorem 2.43.8, its algebra is finite and norm-admissible over that of V. It is flat by Proposition 2.38.2. Lemma 2.44.2 now says that $U \cap V$ is open in V. As V is a neighborhood of x in X, this provides an ordinary neighborhood of x contained in U. □

Definition 2.44.4 (Relative interior on analytic atlases)**.** For a morphism $f : Y \to X$ of general analytic spaces, a point y is interior if, for every affinoid domain $U \subset X$ containing $f(y)$, there is an affinoid domain $V \subset f^{-1}(U)$ which is a neighborhood of y in $f^{-1}(U)$ and satisfies $y \in \mathrm{Int}(V/U)$.

Proposition 2.44.5 (Affinoid compatibility and arbitrary analytic domains)**.** *Definition 2.44.4 agrees with the original affinoid definition. For any analytic domain $V \subset X$, with no goodness or global Hausdorff hypothesis, it gives*

$$\mathrm{Int}(V/X) = \mathrm{Int}_{\mathrm{top}}(V, X).$$

Proof. For an affinoid map, the forward base-change inclusion proves that an affinoid interior point satisfies the new definition, taking V to be the entire affinoid inverse image of each U. Conversely, take $U = X$ in the definition. A witnessing affinoid neighborhood $V \subset Y$ has $y \in \mathrm{Int}(V/X)$. Apply Proposition 2.44.6 first with the affinoid pair (V, X) and then with the whole pair (Y, X). Both tests compute the same general relative interior at y, so $y \in \mathrm{Int}(Y/X)$ in the original affinoid sense.

For a domain inclusion, an ordinary neighborhood of y contained in V meets each affinoid $U \ni y$ in an open neighborhood in U. Choose a strict-inequality rational neighborhood inside that intersection, using Lemma 2.41.1. It is the required witness. Conversely, if the definition holds, its witness W over any affinoid $U \ni y$ is an affinoid domain of U. Theorem 2.44.3 makes W, and hence $V \cap U$, a neighborhood of y in U.

To pass from these neighborhoods to X, use its analytic net. Finitely many affinoid domains $U_1, \ldots, U_s$ containing y have union containing a neighborhood of y. For each i choose an open subset $O_i \subset X$ with $y \in O_i$ and $O_i \cap U_i \subset V$. Intersect the finitely many O_i with an open neighborhood contained in $\bigcup_i U_i$. The resulting open neighborhood lies in V. This proves the assertion for arbitrary analytic domains and also shows that the construction is unchanged by saturated atlas refinements. □

Proposition 2.44.6 (Ordinary local testing for good spaces)**.** *Let $f : Y \to X$ be a morphism of good analytic spaces. Let X_0 be an affinoid neighborhood of $x = f(y)$ and $Y_0 \subset f^{-1}(X_0)$ an affinoid neighborhood of y in Y. Then*

$$y \in \mathrm{Int}(Y/X) \quad \Longleftrightarrow \quad y \in \mathrm{Int}(Y_0/X_0).$$

In particular, relative interior is open for morphisms of good spaces.

Proof. If y satisfies the general definition, take a witnessing affinoid neighborhood $V \subset f^{-1}(X_0)$ with $y \in \mathrm{Int}(V/X_0)$. Choose an affinoid neighborhood W of y inside $V \cap Y_0$, for example a rational neighborhood in Y_0 contained in this ordinary neighborhood. It is an affinoid domain of both V and Y_0. Since W is a neighborhood in V, the domain-interior theorem and the composition formula give $y \in \mathrm{Int}(W/X_0)$, and the same formula for $W \to Y_0 \to X_0$ gives the desired condition.

For the converse, fix any affinoid domain $U \subset X$ containing x. Choose a rational neighborhood $U' \subset U$ of x contained in the open set $U \cap \mathrm{Int}_{\mathrm{top}}(X_0, X)$. Then U' is also an affinoid domain of X_0. The inverse image $V' = Y_0 \times_{X_0} U'$ is affinoid and is a neighborhood of y in $f^{-1}(U)$. The base-change inclusion gives $y \in \mathrm{Int}(V'/U')$; the domain-interior theorem gives $x \in \mathrm{Int}(U'/U)$. Composition proves $y \in \mathrm{Int}(V'/U)$, which is the required witness. Finally, on the ordinary interiors of Y_0 and X_0, the displayed criterion identifies the relative interior with an open affinoid-interior locus. These neighborhoods cover Y, proving openness. □

Theorem 2.44.7 (Base-domain locality for affinoid morphisms)**.** *Let $f : Y \to X$ be an affinoid morphism, let (X_i) be a finite affinoid-domain covering of X, and put $Y_i = f^{-1}(X_i)$. Then*

$$\partial(Y/X) = \bigcup_i \partial(Y_i/X_i).$$

Proof. The forward base-change inclusion for interior proves that the union on the right is contained in the left side. Conversely, fix $y \in Y$ with image x, and suppose $y \in \mathrm{Int}(Y_i/X_i)$ for every i with $x \in X_i$. Write $F = \widetilde{\mathcal{H}(x)}^{\mathrm{gr}}$, $G = \widetilde{\mathcal{H}(y)}^{\mathrm{gr}}$, and denote by $R_X, R_Y, R_{X_i}, R_{Y_i}$ the images of the corresponding graded affinoid reductions in these fields. We regard F as a graded subfield of G.

Each Y_i is the actual inverse-image domain, and restriction to X_i and Y_i preserves the relevant completed residue fields. Take any graded valuation ring $V \subset G$ containing R_X. Its intersection with F is a graded valuation ring containing R_X. Proposition 3.19.6 applied to the cover of the germ (X, x) supplies an index i such that $V \cap F$ contains R_{X_i}. By the assumed interiority and the relative point criterion, R_{Y_i} is finite over R_{X_i}. A valuation ring containing the latter contains the former, by integral closedness on homogeneous elements. In particular V contains $R_Y \subset R_{Y_i}$. Thus every

graded valuation of G containing R_X also contains R_Y. Lemma 3.19.1 and finite generation of R_Y from the fixed absolute presentation imply that R_Y is finite over R_X. The reverse implication of the relative point criterion now gives $y \in \mathrm{Int}(Y/X)$, proving the reverse inclusion of the boundary sets. □

Proposition 2.44.8 (Jacobian presentations of algebraically smooth morphisms)**.** *Let $f : Y \to X$ be a smooth morphism of finite-type K-schemes. Its analytification admits a Jacobian presentation at every point and is boundaryless. The analytic differential rank is the algebraic relative differential rank at the corresponding scheme point.*

Proof. By [StacksAlg, Tag 01V7, Lemma 29.35.11], near each scheme point the morphism has a standard smooth presentation

$$S = R[T_1, \dots, T_n]/(f_1, \dots, f_c), \qquad \det\left(\frac{\partial f_i}{\partial T_j}\right)_{1 \le i,j \le c} \in S^\times.$$

The determinant condition is the definition in [StacksAlg, Tag 00T6]. Choose an affinoid neighborhood of the image point in the analytification of $\operatorname{Spec} R$, and choose finite radii for the coordinates strictly larger than their values at the selected analytic point. The relative analytification construction identifies the resulting affinoid source piece with the quotient of the corresponding relative polydisc by these same c equations. The determinant remains invertible in its quotient algebra. Theorem 2.41.7 therefore gives a Jacobian presentation, with differential rank $n - c$. The equality with algebraic rank follows directly from Proposition 2.41.10, or from the same invertible matrix. Proposition 2.33.27 supplies boundarylessness, with its all-target-domain interpretation. □

References. The composition and domain-interior assertions are classical; compare [TemIntro, 3.4.2.3 and 4.2.4.1–4.2.4.3]. Their proofs here use the internal graded innerness criterion and relative compact-operator finiteness.

2.45 Gluing across a Laurent circle

The affinoid-neighborhood criterion for analytic germs rests on a two-chart calculation. The estimates below use arbitrary positive radii and the actual quotient norm on the overlap.

Lemma 2.45.1 (Small Laurent coordinate changes)**.** *Let $r_1, \dots, r_n, p, q > 0$, with $p \le q$, and set*

$$E = K\{r^{-1}T, q^{-1}S, pS^{-1}\}.$$

Give E its Laurent Gauss norm. Suppose $H_i, H \in E$ satisfy, for a real number $0 < \varepsilon < 1$,

$$\|H_i - T_i\| \le \varepsilon r_i, \qquad H = S(1+u), \quad \|u\| \le \varepsilon.$$

Then $T_i \mapsto H_i$, $S \mapsto H$ defines an isometric K-algebra automorphism of E.

Proof. The coefficient normal form in Theorem 2.13.2, with coefficients in $K\{r^{-1}T\}$, identifies E isometrically with its weighted convergent Laurent series; the weights are those displayed below. The geometric series gives $(1+u)^{-1}$ and $\|(1+u)^{-1} - 1\| \le \varepsilon$. Consequently H and H^{-1} satisfy the defining radius bounds, so the substitution defines a bounded algebra endomorphism Φ. The Laurent monomials $T^\nu S^j$, $\nu \in \mathbb{N}^n$, $j \in \mathbb{Z}$, have weights

$$w_{\nu,j} = r^\nu \begin{cases} q^j & j \ge 0, \\ p^j & j < 0. \end{cases}$$

Expanding one factor at a time and using $|m| \le 1$ for integers m shows

$$\|\Phi(T^\nu S^j) - T^\nu S^j\| \le \varepsilon w_{\nu,j}.$$

For negative j use the already established estimate for $(1+u)^{-1} - 1$. Passing from finite sums to convergent Laurent series yields $\|\Phi - 1\| \le \varepsilon$. The operator series $\sum_{m \ge 0}(1-\Phi)^m$ is therefore a continuous inverse to Φ. A bijective algebra homomorphism has an algebra-homomorphic inverse. Finally, $\|\Phi(a) - a\| < \|a\|$ for $a \ne 0$ proves $\|\Phi(a)\| = \|a\|$. □

Theorem 2.45.2 (Two Laurent charts form an affinoid space)**.** *Let $X = Y \cup Z$ be an analytic space covered by two affinoid domains $Y = \mathcal{M}(B)$ and $Z = \mathcal{M}(C)$, with affinoid intersection $Y \cap Z = \mathcal{M}(D)$. Suppose $0 < p < t < q$, $r_i > 0$, and bounded norm-strict epimorphisms*

$$K\{r^{-1}T, t^{-1}S, pS^{-1}\} \longrightarrow B, \quad T_i \longmapsto f_i, \quad S \longmapsto f,$$
$$K\{r^{-1}T, q^{-1}S, tS^{-1}\} \longrightarrow C, \quad T_i \longmapsto g_i, \quad S \longmapsto g.$$

Assume $Y \cap Z = Y\{|f| \ge t\} = Z\{|g| \le t\}$. Give D the quotient norm induced by

$$K\{r^{-1}T, t^{-1}S, tS^{-1}\} \longrightarrow D, \qquad T_i \longmapsto f_i,\ S \longmapsto f.$$

If $\|f_i - g_i\|_D < r_i$ and $\|f - g\|_D < t$, then X is affinoid. More precisely, for every $\delta > 0$ there are an affinoid algebra A, an isomorphism $X \simeq \mathcal{M}(A)$, and a bounded norm-strict epimorphism

$$K\{r^{-1}T, q^{-1}S, pS^{-1}\} \longrightarrow A, \qquad T_i \longmapsto h_i,\ S \longmapsto h,$$

such that, for the quotient norms of the displayed presentations of B and C,

$$\|h_i - f_i\|_B,\ \|h_i - g_i\|_C < \|f_i - g_i\|_D + \delta,$$
$$\|h - f\|_B,\ \|h - g\|_C < \|f - g\|_D + \delta.$$

Proof. First suppose the two epimorphisms are isomorphisms. The restrictions identify B and C with subalgebras of D. In the f-coordinates the latter is the Laurent Gauss algebra of the circle $|S| = t$. The hypotheses and Lemma 2.45.1, applied with equal inner and outer radius t, identify its g-coordinates isometrically as well. Choose $\varepsilon < 1$ exceeding all ratios $\|f_i - g_i\|_D/r_i$ and $\|f - g\|_D/t$.

Write a Laurent series $a = \sum a_{\nu,j} f^\nu f^j$ on the circle, where $f^\nu = f_1^{\nu_1} \cdots f_n^{\nu_n}$. Retain its terms with $j > 0$ in the f-coordinates, and write the terms with $j \le 0$ in the g-coordinates. The first sum b_0 converges in B, the second sum c_0 converges in C, and

$$\|b_0\|_B, \|c_0\|_C \le \|a\|_D, \qquad \|a - b_0 - c_0\|_D \le \varepsilon \|a\|_D.$$

Iterating on the error gives $a = b + c$, where b is a series in strictly positive powers of f and c is a series in nonpositive powers of g, with

$$\|b\|_B, \|c\|_C \le \|a\|_D, \qquad \|b/f\|_B, \|c/g\|_C \le t^{-1}\|a\|_D.$$

The divided estimates hold term by term: in B the largest radius of f is t, whereas in C the smallest radius of g is t.

Apply this splitting to $f_i - g_i = b_i + c_i$ and $f - g = b + c$. The matching pairs

$$h_i = f_i - b_i = g_i + c_i, \qquad h = f - b = g + c$$

define sections on X. The inequalities $\|b/f\|_B < 1$ and $\|c/g\|_C < 1$ show that h is invertible and has the same modulus as f on Y and as g on Z. On each chart, the change to the h_i, h coordinates satisfies Lemma 2.45.1; hence it is an isomorphism of the corresponding Laurent Gauss algebra. Therefore Y and Z, with these coordinates, are the standard two Laurent domains $|S| \le t$ and $|S| \ge t$ in

$$\mathcal{M}\big(K\{r^{-1}T, q^{-1}S, pS^{-1}\}\big).$$

Their identifications agree on the overlap. Theorem 2.40.10 identifies their union with this affinoid space. In this free case the coordinate errors are bounded above by the corresponding overlap errors, with no added δ.

For the general case put $E = K\{r^{-1}T, t^{-1}S, tS^{-1}\}$. By the defining quotient norm on D, choose lifts $G_i, G \in E$ of g_i, g such that

$$\|T_i - G_i\|_E < \min\{r_i, \|f_i - g_i\|_D + \delta\},$$
$$\|S - G\|_E < \min\{t, \|f - g\|_D + \delta\}.$$

Because S has constant modulus t on E, the substitution $T_i \mapsto G_i$, $S \mapsto G$ is an isometric automorphism by the lemma. Let B' be the lower Laurent algebra in the T_i, S coordinates and C' the upper Laurent algebra in the G_i, G coordinates, both regarded as subalgebras of E. Their spectra glue along $\mathcal{M}(E)$ to an affinoid space $X' = \mathcal{M}(A')$ by the free case. There are compatible epimorphisms $B' \to B$, $C' \to C$, $E \to D$.

Let I_B, I_C, I_D be their kernels. The stated descriptions of the overlap and the quotient formula for rational localization give $I_B E = I_D = I_C E$. The ideals are finite modules, so they glue to a coherent ideal sheaf on X'. Theorem 2.38.5 identifies this sheaf with $\widetilde{I}$ for an ideal $I \subset A'$, and identifies its two restrictions with I_B, I_C. It follows that $\mathcal{M}(A'/I)$ has affinoid domains with algebras B, C and overlap D. Theorem 2.40.10 therefore identifies X with $\mathcal{M}(A'/I)$. Finally the free-case coordinates on A' descend to A'/I; quotient maps do not increase the displayed quotient norms. The chosen lifting estimates give all required bounds. □

References. The two-chart argument reconstructs [Tem00, Lemma 3.2] with arbitrary positive radii, as required in [Tem04, Theorem 5.1]. The proof uses the internal rational localization, finite-module descent, and analytic gluing results.

2.46 Relative interior on arbitrary analytic spaces

The reduction of a germ records all the affinoid directions through its distinguished point. The good-germ criterion now makes it possible to pass from the affinoid interior calculation to arbitrary analytic spaces.

Theorem 2.46.1 (The intrinsic interior criterion)**.** *Let $f : Y \to X$ be a morphism of analytic spaces, let $y \in Y$, and put $x = f(y)$. With the all-target-domain definition of relative interior,*

$$y \in \mathrm{Int}(Y/X) \quad \Longleftrightarrow \quad \mathscr{R}(Y, y) \longrightarrow \mathscr{R}(X, x) \text{ is proper.}$$

Proof. Suppose first that y is interior. Choose finitely many affinoid domains $X_i \subset X$, all containing x, whose union is a neighborhood of x. Write $Y_i = f^{-1}(X_i)$. By Definition 2.44.4, each Y_i has an affinoid neighborhood V_i of y with $y \in \mathrm{Int}(V_i/X_i)$. Proposition 3.19.14 says that the reduction of $(V_i, y) \to (X_i, x)$ is proper. Since V_i is an ordinary neighborhood in Y_i, it represents the same germ. Theorem 3.19.9 identifies these morphisms with the restrictions of the original reduction over the open cover $\mathscr{R}(X_i, x)$ of $\mathscr{R}(X, x)$. Target locality in Corollary 3.19.13 proves properness.

Conversely, suppose the reduction is proper and let $U \subset X$ be any affinoid domain containing x. Put $V = f^{-1}(U)$. The inverse-image formula for germ reductions and Corollary 3.19.13 show that $\mathscr{R}(V, y) \to \mathscr{R}(U, x)$ is proper. The latter target is affine. By the definition of properness for birational spaces, its source is the pullback of that affine object to $\widetilde{\mathcal{H}(y)}^{\mathrm{gr}}$, hence is affine. Theorem 3.19.19 therefore makes (V, y) a good germ. Choose an affinoid neighborhood W of y in V. Its reduction agrees with that of V, so Proposition 3.19.14 gives $y \in \mathrm{Int}(W/U)$. This is the witness required for the arbitrary target domain U. □

Theorem 2.46.2 (Openness of relative interior)**.** *For every morphism of analytic spaces, $\mathrm{Int}(Y/X)$ is open in the ordinary topology of Y.*

Proof. Fix an interior point y and write $x = f(y)$. Work inside Hausdorff open neighborhoods of x and y; these do not alter their germs or the criterion in Theorem 2.46.1. Choose finitely many affinoid domains X_i containing x whose union contains an open neighborhood N of x. As in the proof of that theorem, for $Y_i = f^{-1}(X_i)$ choose affinoid neighborhoods V_i of y in Y_i such that $y \in \mathrm{Int}(V_i/X_i)$. Affinoid interior is open. Consequently there is an open neighborhood Z_i of y in Y_i, contained in V_i, on which the reduction of $(Y_i, z) \to (X_i, f(z))$ is proper for every $z \in Z_i$. Here we first restrict to the ordinary interior of V_i in Y_i, so the two source germs agree.

For each i, choose an ordinary open neighborhood O_i of y in Y with $O_i \cap Y_i \subset Z_i$, and put $O = f^{-1}(N) \cap \bigcap_i O_i$. This is an open neighborhood of

y. For $z \in O$, put $x' = f(z)$. The members X_i containing x' cover the germ (X, x'): the other finitely many compact domains are closed in the chosen Hausdorff neighborhood and can be removed near x', while $N \subset \bigcup_i X_i$. For every remaining member, $z \in Z_i$, so the reduced morphism over $\mathscr{R}(X_i, x')$ is proper. Target locality proves properness of $\mathscr{R}(Y, z) \to \mathscr{R}(X, x')$. The intrinsic criterion yields $z \in \mathrm{Int}(Y/X)$. Thus O is contained in the relative interior. □

Theorem 2.46.3 (Domain locality and analytic base change)**.** *Let (X_i) be an admissible analytic-domain covering of X and put $Y_i = f^{-1}(X_i)$. Then*

$$\partial(Y/X) = \bigcup_i \partial(Y_i/X_i).$$

If $X' \to X$ is an analytic morphism over any complete valued extension L/K, $Y' = Y \times_X X'$, and $y' \in Y'$ maps to $y \in Y$, then

$$y \in \mathrm{Int}(Y/X) \quad \Longrightarrow \quad y' \in \mathrm{Int}(Y'/X').$$

Proof. At each $x \in X$, admissibility supplies a finite subdomain-germ covering subordinate to the given family. Theorem 3.19.9 turns it into an open covering of $\mathscr{R}(X, x)$ and identifies the reductions of the inverse-image germs with the corresponding inverse images. A reduction morphism is proper if and only if all these restrictions are proper, by Corollary 3.19.13. Theorem 2.46.1 gives the displayed boundary identity. For an arbitrary analytic base change over L, the same corollary preserves properness of the reduction. Applying the intrinsic criterion on Y' proves the second assertion. □

Corollary 2.46.4 (The universal good-space test)**.** *A morphism $f : Y \to X$ is boundaryless if and only if, for every complete valued extension L/K and every morphism $X' \to X$ from a good L-analytic space, the space $Y' = Y \times_X X'$ is good and the induced good-space morphism $Y' \to X'$ is boundaryless.*

Proof. Suppose f is boundaryless. The preceding theorem makes every point of Y' interior over X'. At $y' \in Y'$, the germ of X' at its image is good and has affine reduction. Its proper inverse image is affine, so Theorem 3.19.19 makes the source germ good. Every point thus has an affinoid neighborhood, proving that Y' is good. Conversely, apply the stated test with $L = K$ to every target affinoid domain $U \subset X$. Its inverse image is good and boundaryless over U, so every point admits an affinoid neighborhood with an inner presentation. Definition 2.44.4 gives that every point of Y is interior. □

References. These arguments reconstruct [Tem04, Lemma 5.5, Theorem 5.2 and Corollary 5.6]. The good-germ, graded product, and scalar-extension theorems used in the proof are established internally.

Forward interior over an affinoid target. Let $f : Y \to X$ be an analytic morphism with X affinoid. Let $X' \to X$ be an analytic morphism over a complete valued extension L/K. Put $Y' = Y \times_X X'$, and let $y' \in Y'$ map to $y \in Y$. If $y \in \mathrm{Int}(Y/X)$, then $y' \in \mathrm{Int}(Y'/X')$.

Proof. Use X itself as the target test in Definition 2.44.4. It gives an affinoid domain V of Y which is an ordinary neighbourhood of y, with $y \in \mathrm{Int}(V/X)$. For an arbitrary affinoid domain U' of X' containing the image of y', put $P' = Y' \times_{X'} U'$. The inverse image of V in P' is $V' = V \times_X U'$. The analytic-domain and affinoid product assertions of Theorems 2.40.11 and 2.40.12 identify this with the affinoid chart for the actual completed tensor product. The ordinary open neighbourhood of y contained in V pulls back to one of y' contained in V', so V' is an affinoid neighbourhood of y' in the test space P'.

The forward affinoid base-change inclusion in Proposition 2.33.21, applied to $V \to X \leftarrow U'$, puts y' in $\mathrm{Int}(V'/U')$. This works for every target test U', which is the required relative interior condition. □

2.47 Universal flatness of analytic domains

The affinoid stalk calculation in Theorem 2.33.17 already contains the analytic argument: finite tensor relations descend to sufficiently small affinoid neighborhoods. We now transfer that result to arbitrary domains and good-space tests.

Theorem 2.47.1 (All analytic-domain inclusions are analytically flat)**.** *Every analytic-domain inclusion $j : U \hookrightarrow X$ is analytically flat, over any complete non-Archimedean ground field. If U and X are good, the ordinary local homomorphism*

$$\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{U,x} \qquad (x \in U)$$

is faithfully flat. In this good case, after every base change from a good space over every complete valued extension, the resulting domain is good and the same faithful-flat assertion holds.

Proof. First assume U and X good. Choose an affinoid neighborhood X_0 of x in X, and then an affinoid neighborhood U_0 of x in U contained in X_0. By Corollary 2.40.9, U_0 is an affinoid domain of X_0. Ordinary neighborhoods do not change stalks, so Theorem 2.33.17 gives the claimed flatness. The homomorphism is local by Proposition 2.33.16 and residue-field compatibility.

For clarity, a flat local homomorphism $R \to S$ of nonzero local rings is faithful for the following elementary reason. If $N \neq 0$, a nonzero element of N generates a submodule R/I with I proper. Flatness embeds S/IS into $S \otimes_R N$, and IS is contained in the maximal ideal of S, so this tensor product is nonzero. Apply this to the two stalks.

Let $X' \to X$ be a morphism from a good L-analytic space, with L/K complete. Its inverse image U' is an analytic domain by Theorem 2.40.8 and the construction of analytic products. To verify that U' is good in the current good-source case, fix $u' \in U'$, with images u, x'. Choose affinoid neighborhoods X_0 of u in X, U_0 of u in U contained in X_0, and X_0' of x' in X' contained in the inverse image of the ordinary interior of X_0. Then

$$U_0' = U_0 \times_{X_0} X_0'$$

is an affinoid domain of X_0', by Theorem 2.36.9, and contains an ordinary neighborhood of u' in U': intersect the inverse images of the ordinary interiors of U_0 and X_0'. Thus U' is good. The first part, applied over L, proves faithful flatness on its ordinary stalks. This verifies every good base-change test in Definition 2.9.1.

For a general domain inclusion, choose an affinoid chart of the target and an affinoid chart of its inverse-image domain. The latter is an affinoid domain of the former. What was just proved supplies all universal good-space flatness tests for this local map. Such charts form the analytic nets on source and target. The general G-local definition of analytic flatness therefore proves the assertion without assuming that an arbitrary domain of a good space is itself good. □

Proposition 2.47.2 (Coherent restriction on ordinary stalks)**.** *Let $U \subset Y$ be an analytic domain, with U, Y good, and let $\mathcal{F}$ be coherent on Y_G. For every $y \in U$, the canonical map gives*

$$(\mathcal{F}|_{U_G})_y \simeq \mathcal{F}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{U,y}.$$

Proof. Choose affinoid neighborhoods Y_0 in Y and U_0 in U with $U_0 \subset Y_0$. By Theorem 2.39.6, $\mathcal{F}|_{Y_0}$ comes from a finite module M over $A = \mathcal{O}(Y_0)$. On

every affinoid domain $V \subset Y_0$ its sections are $M \otimes_A A_V$. Ordinary stalks are filtered limits over affinoid neighborhoods, by Proposition 2.33.16; the same proof applies to these module sheaves. Tensor products commute with filtered colimits, so the two stalks are respectively $M \otimes_A \mathcal{O}_{Y,y}$ and $M \otimes_A \mathcal{O}_{U,y}$. Associativity of tensor products gives the displayed isomorphism. □

Corollary 2.47.3 (Flatness detected on source domains)**.** *Let $f : Y \to X$ be a morphism of good spaces, let $\mathcal{F}$ be coherent on Y_G, and let $U \subset Y$ be a good analytic domain containing y. Then $\mathcal{F}$ is naively X-flat at y if and only if its restriction to U is. The same equivalence holds for analytic X-flatness.*

Proof. Put $A = \mathcal{O}_{X,f(y)}$, $R = \mathcal{O}_{Y,y}$, $S = \mathcal{O}_{U,y}$, and $M = \mathcal{F}_y$. By the preceding results $R \to S$ is faithfully flat and the restricted module is $M \otimes_R S$. If M is A-flat, tensoring any injective map of A-modules first with M and then with the flat R-module S preserves injectivity; hence $M \otimes_R S$ is A-flat. Conversely, for an injective A-linear map $N_1 \to N_2$, the kernel of $N_1 \otimes_A M \to N_2 \otimes_A M$ is an R-module. Its tensor product with S vanishes if $M \otimes_R S$ is A-flat, and faithfulness makes the kernel zero. This proves naive-flatness equivalence.

A fibre product of good spaces is good: at a product point choose affinoid neighborhoods of its three images, with the two source neighborhoods mapping into the target neighborhood. Their affinoid product contains the intersection of the inverse images of the two ordinary source interiors, hence is a neighborhood of the product point. After any good base change over a complete extension, the inverse image of U is therefore a good analytic domain in a good source, by Theorem 2.47.1. Coherent pullback commutes with domain restriction, as follows from its local finite-module presentations. Apply the naive equivalence at every point above y. These are exactly the tests defining analytic flatness. □

2.48 Flatness of relative polydiscs

The coefficient description of finite modules supplies the algebraic flatness of relative Tate algebras. Passing to ordinary analytic stalks requires two filtered systems of neighborhoods, but no Noetherianity assertion about their limit.

Proposition 2.48.1 (Flat relative Tate algebras)**.** *Let A be a generalized affinoid algebra over a complete non-Archimedean field, and let $r = (r_1, \dots, r_n)$ be positive real radii. Then $A \to B = A\{r^{-1}T\}$ is faithfully flat as a map of rings.*

Proof. For every finite A-module M, Theorem 1.18.6 identifies $B \otimes_A M$ with the convergent weighted coefficient module $C_r(\mathbf{N}^n, M)$, functorially in M. The same theorem preserves injections of finite modules. Since A is Noetherian by Theorem 1.18.2, every finitely generated ideal I is a finite module; applying the assertion to $I \hookrightarrow A$ proves that $I \otimes_A B \to B$ is injective. The finite-ideal criterion [StacksAlg, Tag 00HD] gives flatness.

The constant coefficient map $p_0 : B \to A$ is an A-linear retraction of the coefficient inclusion. For every A-module N, the map $N \to B \otimes_A N$, $m \mapsto 1 \otimes m$, is consequently split by $p_0 \otimes 1$. Thus tensoring with B detects nonzero modules, which proves faithfulness. Only the linear retraction is used here; multiplicativity is unnecessary. □

Theorem 2.48.2 (Relative polydisc flatness on ordinary stalks)**.** *Let $P = \mathcal{M}(A\{r^{-1}T\}) \to X = \mathcal{M}(A)$ be the relative closed polydisc, and let $y \mapsto x$. Then the local map $\mathcal{O}_{X,x} \to \mathcal{O}_{P,y}$ is faithfully flat.*

Proof. Write $R = \mathcal{O}_{X,x}$ and $S = \mathcal{O}_{P,y}$. Fix a rational domain V in X containing an ordinary neighborhood of x. By the bounded tensor universal property, the inverse image P_V has algebra $A_V\{r^{-1}T\}$. It contains an ordinary neighborhood of y. The affinoid neighborhoods W of y in P contained in P_V therefore form a cofinal system. Each is an affinoid domain of P_V, by the domain restriction calculus. The preceding proposition and Proposition 2.38.2 give flat maps

$$A_V \longrightarrow A_V\{r^{-1}T\} \longrightarrow A_W.$$

Their composite is flat because the two consecutive tensor functors preserve injections. Ordinary stalks are the filtered colimits of these A_W by Proposition 2.33.16. For a finitely generated ideal $J \subset A_V$, any element of $J \otimes_{A_V} S$ is a finite sum represented at one stage. If its image in S is zero, that image vanishes after shrinking W once more. Flatness of the new stage then makes the tensor itself zero. The finite-ideal criterion proves that S is flat over each fixed A_V.

Finally, take a finitely generated ideal $I \subset R$ and $z \in I \otimes_R S$ whose image in S is zero. Choose generators $a_1, \dots, a_m$ of I and write $z = \sum_i a_i \otimes s_i$ with $s_i \in S$. Lift the a_i to elements $\widetilde{a}_i$ of one A_V, and put $J = (\widetilde{a}_1, \dots, \widetilde{a}_m)$. Then $z_V = \sum_i \widetilde{a}_i \otimes s_i \in J \otimes_{A_V} S$ maps to z and has zero image in S. The flatness just proved gives $z_V = 0$, hence $z = 0$. Again [StacksAlg, Tag 00HD] gives R-flatness. Evaluations at x and y are compatible through the isometric map $\mathcal{H}(x) \hookrightarrow \mathcal{H}(y)$, so the map is local. A nonzero module contains a cyclic submodule R/I with I proper; its tensor S/IS is nonzero because IS lies in the maximal ideal of S. Flatness preserves the cyclic injection and therefore proves faithful flatness. □

Corollary 2.48.3 (Analytic flatness of polydisc and affine-space projections)**.** *Over a good analytic space X, a relative closed polydisc of any positive radii and relative analytic affine space are good, and their projections to X are analytically flat. The same holds for the restriction of either projection to a good analytic domain of its source. These assertions include every good base change over every complete valued extension of the ground field.*

Proof. At a polydisc point over x, choose an affinoid neighborhood of x. Its inverse image is an affinoid polydisc containing a neighborhood of the chosen point. Thus the total space is good, and the preceding theorem gives flatness of its ordinary local map. After a good base change over a complete extension, the bounded tensor universal property identifies the pullback with the polydisc of the same radii. Its ordinary stalk maps are flat by the same proof over that extension. This is analytic flatness.

Every point of relative analytic affine space has a closed polydisc neighborhood: take each radius strictly larger than the value of its coordinate, and an affinoid neighborhood in the base. The interior inequalities give an ordinary neighborhood, so stalks agree with those of that polydisc. The construction works after every complete-extension good base change, proving the affine-space assertion. Finally, Theorem 2.47.1 supplies flat ordinary domain-stalk maps after all such base changes, with good pulled-back domains. Composing them with the projection-stalk maps proves the last assertion. □

2.49 Isolated fibres and finite neighbourhoods

The finite criterion for affinoid morphisms also has a pointwise form. Its proof separates two compact sets: the relative analytic boundary, which detects

finiteness, and the topological boundary of the chosen source neighbourhood, which allows one to work on ordinary open sets.

Theorem 2.49.1 (The local finite criterion)**.** *Let $f : Y \to X$ be a morphism of good analytic spaces, and let $y \in Y$, $x = f(y)$. The following conditions are equivalent:*

1. *y is isolated in the analytic fibre Y_x and $y \in \mathrm{Int}(Y/X)$;*
2. *there are ordinary open neighbourhoods $W \subset Y$ of y and $N \subset X$ of x such that $f(W) \subset N$ and $W \to N$ is norm-admissible finite.*

When these conditions hold, W and N may be required to lie in any prescribed ordinary neighbourhoods of y and x.

Proof. Assume the first condition. By the topological identification of analytic fibres, there is an ordinary open neighbourhood of y meeting $f^{-1}(x)$ only in y. Choose an affinoid neighbourhood X_0 of x inside the prescribed target neighbourhood. Goodness supplies an affinoid neighbourhood V of y contained in this isolating source neighbourhood, in the prescribed source neighbourhood, and in $f^{-1}(\mathrm{Int}_{\mathrm{top}}(X_0, X))$. Thus $V_x = \{y\}$ as a set. Proposition 2.44.6 gives $y \in \mathrm{Int}(V/X_0)$.

The relative boundary $D = \partial(V/X_0)$ is closed in the compact space V. Its image in the Hausdorff affinoid space X_0 is therefore compact and closed. This image does not contain x: the only point of V_x is y, which is interior. Choose a rational neighbourhood U of x in $X_0 \backslash f(D)$ and inside $\mathrm{Int}_{\mathrm{top}}(X_0, X)$. The inverse image $V_U = V \times_{X_0} U$ is affinoid. Every one of its points comes from $\mathrm{Int}(V/X_0)$, so interior stability under base change makes $V_U \to U$ boundaryless. Theorem 2.43.8 now makes this morphism norm-admissible finite.

It remains to obtain ordinary open neighbourhoods. Since both V and U were neighbourhoods of the chosen points, V_U is a neighbourhood of y in Y. Set

$$E = V_U \setminus \mathrm{Int}_{\mathrm{top}}(V_U, Y).$$

This is a closed subset of the compact space V_U, and its image in U is closed. The image avoids x, because $(V_U)_x = \{y\}$ and $y \notin E$. Choose an ordinary open neighbourhood N of x in X, contained in $\mathrm{Int}_{\mathrm{top}}(U, X)$ and disjoint from $f(E)$. Put $W = V_U \cap f^{-1}(N)$. Then W is contained in $\mathrm{Int}_{\mathrm{top}}(V_U, Y)$, and hence

$$W = \mathrm{Int}_{\mathrm{top}}(V_U, Y) \cap f^{-1}(N)$$

is open in Y. The map $W \to N$ is the base change of the finite map $V_U \to U$, so it is finite by Proposition 2.33.19 and analytic-domain gluing. It contains y and has all the required neighbourhood restrictions.

Conversely, suppose the finite map is given on ordinary open neighbourhoods $W \to N$. Choose an affinoid neighbourhood $X_0 \subseteq N$ of x on a finite chart, so its full inverse image V in W is affinoid and $V \to X_0$ is norm-admissible finite. As X_0 is a neighbourhood of x and W is open in Y, V is a neighbourhood of y in Y. Theorem 2.43.8 gives $y \in \mathrm{Int}(V/X_0)$, and Proposition 2.44.6 gives $y \in \mathrm{Int}(Y/X)$. By Lemma 2.35.1, the finite fibre in W is discrete. Since W is open, y is isolated in the original fibre. □

Proposition 2.49.2 (An ordinary local criterion for étaleness)**.** *Let $f : Y \to X$ be a morphism of good analytic spaces. Suppose $y \in Y$ is isolated in its analytic fibre and belongs to* $\mathrm{Int}(Y/X)$*. Then f is étale at y if and only if the local map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is flat and $(\Omega^1_{Y/X})_y = 0$, where $x = f(y)$. In this case one can choose ordinary neighbourhoods on which f is finite locally free and has zero relative differential sheaf.*

Proof. The local finite criterion and Lemma 2.35.1 reduce the converse to a finite morphism $\mathcal{M}(B) \to \mathcal{M}(A)$ with fibre over x consisting only of y, where these affinoids are neighbourhoods of the original points. Put $R = \mathcal{O}_{X,x}$, $S = \mathcal{O}_{Y,y}$, and $\mathfrak{p} = \ker(A \to \mathcal{H}(x))$. The finite-branch stalk identity is

$$B \otimes_A R \simeq S.$$

The local map $A_{\mathfrak{p}} \to R$ is faithfully flat. Indeed, Proposition 2.33.16 writes R as a filtered colimit of rational-neighbourhood algebras, all flat over A by Proposition 2.38.2. It is consequently flat over $A_{\mathfrak{p}}$. The maximal ideal contracts to $\mathfrak{p}A_{\mathfrak{p}}$ by evaluation, so faithful flatness follows from [StacksAlg, Tag 00HP].

If S is R-flat, then $B_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$-flat. For any injection of $A_{\mathfrak{p}}$-modules, the kernel after tensoring with $B_{\mathfrak{p}}$ becomes zero after the faithfully flat scalar extension to R, because the extended module is S. Thus that kernel is zero. As B is finite over the Noetherian ring A, [StacksAlg, Tag 00NX] makes $B_{\mathfrak{p}}$ free of finite rank. Lift a basis and its inverse finite presentation to a localization A_a with $a \notin \mathfrak{p}$; clearing the finitely many kernel and cokernel generators after a further such localization gives $B_a \simeq A_a^m$.

Let $P = \Omega^1_{B/A}$, a finite B-module and hence a finite A-module. Here algebraic and analytic differentials agree. Indeed an algebraic A-derivation

from the finite A-module B into a finite B-module is A-linear and automatically bounded for the canonical finite-module topologies. Thus the two finite differential modules represent the same derivations, by Theorem 2.41.3. The finite-branch stalk argument, applied to a finite presentation of P, gives $P\otimes_A R \simeq (\Omega^1_{Y/X})_y$. The assumed vanishing and faithful flatness imply $P_{\mathfrak{p}} = 0$. After enlarging $a \notin \mathfrak{p}$, finite generation gives $P_a = 0$. On the ordinary open neighbourhood $\{|a| > 0\}$ of x, the finite morphism is therefore locally free and has zero relative differential sheaf. To verify the first assertion analytically, restrict to a rational neighbourhood where $|a|$ is bounded below. Its algebra contains a^{-1}, so tensoring $B_a \simeq A_a^m$ gives a free finite algebra module there. Its ordinary local rings are the factors in the finite-branch decomposition; those factors are direct summands of a free module after passage to the base stalk. They are thus flat. The same argument applies after every analytic base change, since finite-module base change is ordinary tensor product. This proves the claimed étaleness and analytic flatness.

The forward implication follows from the local finite flat definition of an étale morphism and the same finite-branch stalk identity. □

References. The isolated-fibre criterion follows the compact-boundary method of [Ber93, Proposition 3.1.4]. The proof above supplies the passage from affinoid pieces to ordinary open neighbourhoods explicitly. The second result combines this criterion with finite-branch separation and finite-module flatness.

2.50 Quasi-smoothness at interior points

Lemma 2.50.1 (Rank-zero quasi-smooth fibres)**.** *Let Z be a good analytic space over a complete non-Archimedean field F. If $Z \to \mathcal{M}(F)$ is quasi-smooth at z and $\operatorname{rk}_z \Omega_{Z/F} = 0$, then z has an ordinary open affinoid neighbourhood finite over F. In particular, z is isolated in Z.*

Proof. Choose an affinoid neighbourhood Z_0 of z. By domain locality of Jacobian presentations, take an affinoid domain $U = \mathcal{M}(C)$ contained in $\operatorname{Int}_{\mathrm{top}}(Z_0, Z)$, containing z, with a closed presentation in a domain of $\mathbb{A}^n_F$ by n equations whose square Jacobian determinant is invertible. Such a domain is obtained from a rank-zero presentation by the Laurent restriction in Theorem 2.41.7. All points of U have the same rank-zero Jacobian description.

Choose a nontrivially valued complete extension L/F such that C_L is strict, using Theorem 1.17.6 and, if necessary, one further free-radius extension. Let

$\mathfrak{m}$ be any maximal ideal of C_L. By Theorem 1.12.3, $E = C_L/\mathfrak{m}$ is finite over L with its extending complete absolute value. Evaluation gives an E-rational point $u \in U_E$. The pulled-back equations retain their invertible determinant. Proposition 2.35.13, with $n - r = 0$, identifies the completion of $S = \mathcal{O}_{U_E,u}$ with E. The ring S is Noetherian by Theorem 2.35.11. Since its completion is faithfully flat by [StacksAlg, Tag 00MC], its maximal ideal is zero: that ideal becomes zero in its completion, and faithful flatness detects the zero module. Thus S is a field.

The local map $(C_L)_{\mathfrak{m}} \to S$ is faithfully flat. Indeed $C_L \to C_E$ is finite free by finite-extension base change, and the rational-neighbourhood description of S expresses it as a filtered colimit of algebras flat over C_E. The composite is flat, and it is local after localization at $\mathfrak{m}$; use [StacksAlg, Tag 00HP]. A local ring with a faithfully flat local map to a field is itself a field: its maximal ideal extends to zero, so is zero. Thus $(C_L)_{\mathfrak{m}}$ is a field for every maximal $\mathfrak{m}$. Every prime of C_L is maximal, since a strict inclusion of a prime in a maximal ideal would remain a strict inclusion after localization there. Therefore $\dim C_L = 0$. Finite strict normalization and Proposition 3.21.1 give a finite injection from a Tate algebra on zero variables. Consequently C_L is finite-dimensional over L.

Every finite F-linearly independent family in C remains L-linearly independent in C_L: algebraic scalar extension preserves linear independence, and Theorem 2.33.2 embeds the algebraic tensor product in its completion. Thus $\dim_F C \leq \dim_L C_L < \infty$. The canonical finite-dimensional topology is the affinoid topology by Lemma 1.12.1, so $U \to \mathcal{M}(F)$ is norm-admissible finite. It is boundaryless by Theorem 2.43.8. The composition formula for $U \to Z_0 \to \mathcal{M}(F)$ implies $\mathrm{Int}(U/Z_0) = U$, and Theorem 2.44.3 makes U open in Z_0. Since $U \subset \mathrm{Int}_{\mathrm{top}}(Z_0, Z)$, it is open in Z. A finite-dimensional affinoid algebra has finitely many spectral points, a discrete set by Lemma 2.35.1. This proves the assertion. □

Proposition 2.50.2 (Finite étale maps are quasi-smooth)**.** *A norm-admissible finite flat affinoid morphism with zero relative differential module is analytically flat and has geometrically regular analytic fibres. Consequently it is quasi-smooth, and these assertions hold after every complete-field extension and analytic base change. The same conclusions hold locally for étale analytic morphisms.*

Proof. Finite-module base change preserves finite flatness. On ordinary local rings, the finite-branch product decomposition makes each source stalk a direct summand of the finite flat algebra tensored with the base stalk. These stalks are therefore flat. This reasoning applies after every analytic base change, proving analytic flatness. Algebraic and bounded analytic differentials agree for finite algebras, by the finite-module argument in Proposition 2.49.2; their vanishing is preserved by base change.

Consider an analytic fibre and any complete extension E of its ground field. Its algebra D is finite-dimensional over E and satisfies $\Omega^1_{D/E} = 0$. Fix a point v and put $L = \mathcal{H}(v)$, a finite complete extension of E. Lift v to an L-rational point u of $\mathcal{M}(D \otimes_E L)$. The local ring S at u is a finite-dimensional local L-algebra with residue field L, by the finite-branch stalk product. Its ordinary algebraic differential module over L is zero. If $\mathfrak{n}$ is its maximal ideal, the augmentation $S \to L$ identifies

$$\mathfrak{n}/\mathfrak{n}^2 \simeq \Omega^1_{S/L} \otimes_S L = 0.$$

For completeness, the isomorphism sends the class of b to db for $b \in \mathfrak{n}$. Its inverse is induced by the derivation $s \mapsto s - \overline{s}$ modulo $\mathfrak{n}^2$, with $\overline{s} \in L$ the residue; the product rule is immediate modulo $\mathfrak{n}^2$. The finite module $\mathfrak{n}$ satisfies $\mathfrak{n} = \mathfrak{n}^2$, so the determinant proof of Nakayama gives $\mathfrak{n} = 0$. Hence $S = L$ is a field. The scalar-stalk map from $\mathcal{O}_{\mathcal{M}(D),v}$ to S is faithfully flat by Theorem 2.35.4. As in the previous lemma, its source is a field as well, and is therefore regular. Since E and v were arbitrary, the fibres are geometrically regular. Theorem 2.35.14 proves quasi-smoothness. Ordinary-local restriction gives the final assertion. □

Theorem 2.50.3 (Smoothness and interior quasi-smoothness)**.** *For a morphism $f : Y \to X$ of good analytic spaces and a point $y \in Y$, f is smooth at y if and only if it is quasi-smooth at y and $y \in \mathrm{Int}(Y/X)$. Here smoothness is the ordinary-open local étale factorization through a relative analytic affine space.*

Proof. Suppose first that f is quasi-smooth and interior at y. The strengthened presentation in Theorem 2.35.14 supplies affinoid neighbourhoods Y_0 of y and X_0 of $x = f(y)$, together with a closed immersion $Y_0 \hookrightarrow D \subset \mathbb{A}^n_{X_0}$ whose ideal is generated by r functions and whose Jacobian has an invertible $r \times r$ minor near y. Apply the ordinary Laurent-neighbourhood restriction

to make the minor invertible throughout the chosen source. Permute coordinates so that it uses $T_1, \dots, T_r$, and put $d = n - r$. The remaining coordinates define

$$g : Y_0 \longrightarrow \mathbb{A}^d_{X_0}, \qquad g = (f, T_{r+1}, \dots, T_n).$$

Choose a closed relative polydisc $P \subset \mathbb{A}^d_{X_0}$ whose radii are strictly larger than the corresponding coordinate bounds on the affinoid D. Then $g(Y_0) \subset P$, and P is a neighbourhood of $g(y)$. The domain D lies in $\mathbb{A}^r_P$ and remains an affinoid domain there by the domain mapping property. The same r equations with their invertible square Jacobian make $Y_0 \to P$ quasi-smooth with zero relative differential sheaf.

Proposition 2.44.6 gives $y \in \mathrm{Int}(Y_0/X_0)$. The affinoid composition formula for $Y_0 \to P \to X_0$ implies $y \in \mathrm{Int}(Y_0/P)$. The analytic fibre of $Y_0 \to P$ through y is good and quasi-smooth of differential rank zero, by base change. Lemma 2.50.1 makes y isolated in that fibre. Theorem 2.35.14 gives the ordinary local flatness of $Y_0 \to P$. Proposition 2.49.2 now makes it étale near y. Shrink its ordinary finite neighbourhoods inside the ordinary interiors of Y_0 in Y and P in $\mathbb{A}^d_{X_0}$; the small-neighbourhood assertion of Theorem 2.49.1 permits this. Restrict also inside the ordinary interior of X_0 in X. We obtain the required ordinary-open étale factorization of f, proving smoothness.

Conversely, a local étale map is quasi-smooth by Proposition 2.50.2. A relative analytic affine-space projection is quasi-smooth by its presentation with no equations. The composition theorem for Jacobian presentations therefore makes every smooth morphism quasi-smooth. The finite local maps defining étaleness are boundaryless by the affinoid finite criterion. Relative affine-space projections are boundaryless because, at each point, one may choose coordinate radii strictly larger than the coordinate values. The composition inclusion for relative interior, applied on good affinoid neighbourhoods, and ordinary-local testing then make the composite boundaryless at y. □

References. The smoothness comparison is [Duc18, Corollary 5.4.8]. The proof here combines the strengthened Jacobian presentation, the completed rank-zero calculation and the local finite criterion; these are all proved internally.

2.51 Finite fibre projections and original coefficients

A finite normalization of a fibre can be moved to functions on the original space. The argument first preserves finiteness under a small perturbation, and then realizes the finitely many coefficients on an affinoid neighbourhood of the base point.

Lemma 2.51.1 (Small perturbations of a finite normalization)**.** *Let C be a strict F-affinoid algebra over a complete non-Archimedean field. Suppose that $u_1, \dots, u_d \in C$ define a bounded finite homomorphism $F\langle U_1, \dots, U_d\rangle \to C$. If $v_i \in C$ satisfy $\rho_C(v_i - u_i) < 1$ for every i, then $U_i \mapsto v_i$ also defines a bounded finite homomorphism.*

Proof. Boundedness of the original map gives $\rho_C(u_i) \le 1$ by applying it to powers. Hence $\rho_C(v_i) \le 1$. The pointwise power-boundedness theorem, Theorem 1.20.2, shows that the powers of each v_i are bounded. The finitely many commuting elements therefore define the stated bounded Tate-algebra homomorphism. In the canonical reduction of C, the images of v_i and u_i coincide. Thus the two homomorphisms from $\widetilde{F}[U_1, \dots, U_d]$ into $\widetilde{C}$ are identical. Theorem 3.16.5 applies to both maps: finiteness of the first reduction map proves finiteness of the second. □

Lemma 2.51.2 (Approximating fibre functions by original coefficients)**.** *Let $A \to B$ be a bounded homomorphism of generalized K-affinoid algebras, let $x \in \mathcal{M}(A)$, put $F = \mathcal{H}(x)$ and $C = B\widehat{\otimes}_A F$. For finitely many $c_1, \dots, c_m \in C$ and $\varepsilon > 0$, there are a rational neighbourhood U of x in $\mathcal{M}(A)$ and elements $b_1, \dots, b_m \in B\widehat{\otimes}_A A_U$ whose images in C differ from the c_i by norm less than ε. The domain U may be chosen inside any prescribed ordinary neighbourhood of x.*

Proof. The image of $B \otimes_A F$ is dense in its completed tensor product. Approximate each c_i first by a finite sum $\sum_j \beta_{ij} \otimes \lambda_{ij}$. By the definition of the completed residue field, $\operatorname{Frac}(A/\ker x)$ is dense in F. Approximate all the finitely many λ_{ij} by fractions $a_{ij}(x)/d_{ij}(x)$ with $a_{ij}, d_{ij} \in A$ and $d_{ij}(x) \neq 0$. Their errors can be chosen small enough that the finite sums have error less than ε, since the completed tensor seminorm is bounded by the product of the two factor norms.

Choose positive numbers $r_{ij} < |d_{ij}(x)|$. The finitely many Laurent conditions $|d_{ij}| \geq r_{ij}$ define a rational neighbourhood of x on which all denominators are invertible. Intersect with a rational neighbourhood inside the prescribed ordinary neighbourhood, using Lemma 2.41.1 and rational transitivity. On the resulting U, set $b_i = \sum_j \beta_{ij} a_{ij}/d_{ij}$. Completed base-change associativity identifies the fibre algebra over x with the original C, and the preceding estimates give the assertion. □

Proposition 2.51.3 (Strict fibres near an inner point)**.** *Let $f : Y \to X$ be a morphism of good K-analytic spaces, let $x = f(y)$, and assume $y \in \mathrm{Int}(Y/X)$ and $|\mathcal{H}(x)^\times| \neq \{1\}$. There are affinoid neighbourhoods X_0 of x and V of y, with $V \subset f^{-1}(X_0)$, such that V_x is strictly $\mathcal{H}(x)$-affinoid. The source neighbourhood may be chosen inside any prescribed ordinary neighbourhood of y.*

Proof. Choose affinoid neighbourhoods $Y_0 \to X_0$ within the prescribed neighbourhoods. Proposition 2.44.6 makes y an inner point of this affinoid morphism. There is therefore a norm-admissible relative presentation $B = A\{r^{-1}T\}/I$ with $|T_i(y)| < r_i$. Put $F = \mathcal{H}(x)$. Since F is nontrivially valued, $\sqrt{|F^\times|}$ is dense in $\mathbb{R}_{>0}$: the rational powers of any fixed element of absolute value different from one already form a dense subgroup. Choose

$$|T_i(y)| < s_i < r_i, \qquad s_i \in \sqrt{|F^\times|}.$$

The Weierstrass domain $V = \{|T_i| \leq s_i \text{ for all } i\}$ contains an ordinary neighbourhood of y. Relative presentations and base change identify its fibre with a quotient of $F\{s^{-1}T\}$. It remains to exhibit this last algebra as strict.

Write $s_i^{e_i} = |a_i|$ with $e_i > 0$ and $a_i \in F^\times$, and choose $c_i \in F^\times$ with $|c_i| \geq s_i$. Then

$$F\{s^{-1}T\} \simeq F\langle U_1, W_1, \ldots, U_m, W_m\rangle/(c_i^{e_i}U_i^{e_i} - a_i W_i)_{i=1}^m, \quad T_i \longleftrightarrow c_i U_i.$$

Indeed the map to the left sends U_i to T_i/c_i and W_i to $T_i^{e_i}/a_i$, both power bounded by one. Conversely the relation gives a uniform bound for $(c_iU_i)^n/s_i^n$: divide n by e_i and use the finitely many possible remainders. The generalized Tate universal property therefore defines the map to the right. The maps are inverse on the polynomial generators and then on the completed algebras by density. This also proves norm admissibility. Quotients of a strict affinoid are strict, proving the assertion. □

Theorem 2.51.4 (Finite projection from a strict fibre)**.** *Let $Y = \mathcal{M}(B) \to X = \mathcal{M}(A)$ be an affinoid morphism, let $y \in \mathrm{Int}(Y/X)$, and put $x = f(y)$ and $F = \mathcal{H}(x)$. Assume $C = B\widehat{\otimes}_A F$ is strict, and set $d = \dim C$. After replacing X by an affinoid neighbourhood of x, there are d analytic functions on Y defining an X-morphism $g : Y \to \mathbb{A}^{d,\mathrm{an}}_X$ which restricts to a finite morphism from an ordinary neighbourhood of y to an ordinary neighbourhood of $g(y)$. Consequently there are affinoid neighbourhoods V of y and T of $g(y)$, with T an affinoid domain of $\mathbb{A}^{d,\mathrm{an}}_X$, for which $V \to T$ is finite.*

Proof. Theorem 1.12.2 and Proposition 3.21.1 provide a bounded finite injection $F\langle U_1, \dots, U_d\rangle \hookrightarrow C$. Approximate its coordinate images, by Lemma 2.51.2, with images v_i of functions on $Y_U = Y \times_X U$, where U is an affinoid neighbourhood of x. Choose the errors of norm less than one. Lemma 2.51.1 makes the resulting homomorphism $F\langle U\rangle \to C$ finite. Replace X, Y by U, Y_U; these replacements are ordinary neighbourhoods of the points under consideration.

Let g be the morphism defined by the chosen functions. Its fibre over $g(y)$ is, near y, a fibre of the finite F-affinoid morphism just constructed, and is therefore a finite set. In particular y is isolated in this fibre. To verify interior, choose an affinoid polydisc over X containing $g(Y)$, with coordinate bounds strictly larger than the norms of the chosen functions. It gives a factorization $Y \to P \to X$. The affinoid composition formula, Theorem 2.44.1, implies $y \in \mathrm{Int}(Y/P)$ because $y \in \mathrm{Int}(Y/X)$. Since $g(y)$ is in the ordinary interior of P in affine space, ordinary local testing gives $y \in \mathrm{Int}(Y/\mathbb{A}^{d,\mathrm{an}}_X)$.

Theorem 2.49.1 now produces ordinary open neighbourhoods on which g is finite. Choose an affinoid neighbourhood T of $g(y)$ inside the target open neighbourhood. Its inverse image V is affinoid and finite over T, and contains an ordinary neighbourhood of y. This proves the final assertion. □

References. These coefficient and perturbation steps reconstruct the finite-projection mechanism underlying [Duc07, Theorem 4.6 and Corollary 4.7], using internal strict normalization and the local finite criterion.

2.52 Generic fibre tests for universal injectivity

The proof of universal injectivity uses generic points of ordinary analytic open sets. Such points can be obtained from the finite spectral boundary and domain flatness; no dimension formula is needed for this step.

Lemma 2.52.1 (Generic analytic points in ordinary open sets)**.** *Let C be an integral generalized F-affinoid algebra. Every nonempty ordinary open subset of $\mathcal{M}(C)$ contains a point z for which $C \to \mathcal{H}(z)$ is injective. More generally every prime ideal of a generalized affinoid algebra is the kernel of an analytic point.*

Proof. First let D be any nonzero integral affinoid algebra. By Theorem 3.12.3, choose a finite spectral boundary $S \subset \mathcal{M}(D)$. If every point of S had nonzero kernel, choose a nonzero element in each kernel and take their product a. The algebra D is integral, so $a \neq 0$, whereas a vanishes on the whole finite boundary. Thus $\rho_D(a) = 0$, contradicting Proposition 3.21.4. At least one point has zero kernel. Applied to $D = C/\mathfrak{p}$, this also proves the last assertion for every prime $\mathfrak{p}$.

Now choose a nonempty rational domain $V = \mathcal{M}(B)$ inside the given ordinary open set. The algebra B is nonzero and flat over C by Proposition 2.38.2. Let $\mathfrak{q}$ be a minimal prime of B. Its contraction to C is zero. Indeed, a nonzero $a \in C$ acts injectively on B by flatness and hence on the nonzero ring $B_{\mathfrak{q}}$. If $a \in \mathfrak{q}$, its image in $B_{\mathfrak{q}}$ is nilpotent, since the only prime ideal of that ring is its maximal ideal; a nilpotent element cannot act injectively on a nonzero ring. The preceding paragraph supplies a point of $\mathcal{M}(B/\mathfrak{q})$ with zero kernel. Its image in V has kernel $\mathfrak{q}$ on B and therefore zero kernel on C, as required. □

Lemma 2.52.2 (Regular affinoids and their components)**.** *Let $Z = \mathcal{M}(C)$ be a generalized affinoid space whose ordinary local rings are regular. Then C is a regular Noetherian ring and a finite product of integral affinoid algebras. The corresponding analytic components are open and closed. In each component, the analytic points with kernel equal to its minimal prime are ordinarily dense.*

Proof. For every prime $\mathfrak{p} \subset C$, the preceding lemma supplies $z \in Z$ with kernel $\mathfrak{p}$. The map $C_{\mathfrak{p}} \to \mathcal{O}_{Z,z}$ is faithfully flat: the ordinary stalk is a filtered colimit of flat rational-neighbourhood algebras by Proposition 2.33.16, and the map is local with maximal-ideal contraction $\mathfrak{p}$. Both rings are Noetherian. The regularity descent theorem, [StacksAlg, Tag 00OF], therefore makes $C_{\mathfrak{p}}$ regular.

By [StacksAlg, Tag 00NP], every such local ring is a domain. In particular C is reduced, and its distinct minimal primes are pairwise comaximal: a maximal ideal containing two of them would yield a local ring with two distinct minimal primes. The Chinese remainder theorem and the finite number

of minimal primes give $C \simeq \prod_i C/\mathfrak{p}_i$. The idempotents of this product give precisely the open-and-closed analytic decomposition. Each factor is integral, and Lemma 2.52.1 proves the density assertion. □

Lemma 2.52.3 (Fibrewise density after complete scalar extension)**.** *Let $Z = \mathcal{M}(C)$ be geometrically regular over a complete field F, let $S \subset Z$ be Zariski open, and suppose z belongs to the ordinary closure of S. For every complete extension E/F, every point $z' \in Z_E$ above z belongs to the ordinary closure of the inverse image S_E.*

Proof. Use the preceding lemma to select the integral factor C_0 containing z. It is open and closed, so S meets this factor. Write S as the complement of the zero-locus of an ideal and choose, on C_0, a nonzero element a of that ideal. Then $D(a) \subset S \cap \mathcal{M}(C_0)$. The homomorphism $C_0 \to (C_0)_E$ is flat by Theorem 2.33.6; multiplication by a therefore remains injective. Geometric regularity and the preceding lemma make $(C_0)_E$ a product of integral affinoid algebras. The element a is nonzero on each factor, since it acts injectively. The generic-point density lemma shows that $D(a)$ is ordinarily dense in each factor, and hence in their union. The point z' belongs to this union, proving the assertion. □

Lemma 2.52.4 (Lifting a fibrewise non-zero-divisor)**.** *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local homomorphism of Noetherian local rings. If multiplication by $a \in S$ is injective on $S/\mathfrak{m}S$, then it is injective on S.*

Proof. Flatness identifies the successive quotients with

$$\mathfrak{m}^j S/\mathfrak{m}^{j+1} S \simeq (\mathfrak{m}^j/\mathfrak{m}^{j+1}) \otimes_{R/\mathfrak{m}} (S/\mathfrak{m}S).$$

The first factor is a vector space, so multiplication by a is injective on every displayed quotient. If $ab = 0$, induction gives $b \in \mathfrak{m}^j S$ for every $j \geq 0$. Since the map is local, these submodules lie in $\mathfrak{n}^j S$. Corollary 2.15.1, applied to the local Noetherian ring S, gives $b = 0$. □

Theorem 2.52.5 (Universal injectivity from a generic fibre isomorphism)**.** *Let $T \to X$ be quasi-smooth between good analytic spaces, let $t \in T$ and $x = f(t)$, and let $u : L \to N$ be a morphism of coherent sheaves with L free of finite rank. Suppose that*

$$t \in \overline{\mathrm{Bij}(u_{T_x})}^{T_x},$$

where closure means ordinary topological closure. Then the following are equivalent:

1. *u is injective at every point over t after every good analytic base change over every complete extension of the ground field;*
2. *u is injective at t;*
3. $t \in \overline{\mathrm{Bij}(u) \cap T_x}^{T_x}$.

Proof. All assertions are ordinary-local near the points, so take $T = \mathcal{M}(B)$ and $X = \mathcal{M}(A)$ affinoid. Put $F = \mathcal{H}(x)$ and $C = B\widehat{\otimes}_A F$. By Theorem 2.35.14, the fibre is geometrically regular. Lemma 2.52.2 gives an integral component $C_0 = C/\mathfrak{q}$ containing t. Choose an analytic point z of this component whose kernel on C is $\mathfrak{q}$. If $\mathfrak{p}, \mathfrak{p}' \subset B$ are the kernels of t, z, then $\mathfrak{p}' \subset \mathfrak{p}$. The assumption says that u_C is an isomorphism at $\mathfrak{q}$: a Zariski-open subset of this integral component is ordinarily dense exactly when it is nonempty, by Lemma 2.52.1.

The ordinary stalk at any analytic point is faithfully flat over the localization of its affinoid algebra at the point's kernel, by the same rational-neighbourhood argument used in Lemma 2.52.2. Therefore injectivity and bijectivity of our finite-module maps can be tested at these algebraic localizations. Their bijectivity locus is Zariski open: its complement is the union of the supports of the finite kernel and cokernel.

Assume (2). The map $u_{B_\mathfrak{p}}$ is injective, hence so is $u_{B_{\mathfrak{p}'}}$. At z the fibre map is bijective, so the cokernel of u tensored with $\mathcal{H}(z)$ vanishes. Nakayama's lemma at $B_{\mathfrak{p}'}$ implies that this cokernel is zero there. Thus $u_{B_{\mathfrak{p}'}}$ is an isomorphism. Its Zariski-open bijectivity locus contains the generic point of the fibre component, hence is ordinarily dense in that component. It contains t in its closure, proving (3).

For the converse, both the hypothesis and (3) remain valid after a complete scalar extension at every point above t. Indeed Lemma 2.52.3 preserves the relevant fibre closures; scalar extension is flat on finite affinoid modules, so it identifies the fibre bijectivity locus with its pullback, and pullback of the total bijectivity locus is contained in the new one. Extend the ground field to $F = \mathcal{H}(x)$, choose the canonical F-rational lift of x and a compatible lift of t, and work there. Injectivity descends back by Theorem 2.35.4. We may therefore assume x is rational.

Write $\mathfrak{m} = \ker(A \to F)$. Then $C = B/\mathfrak{m}B$. The map $A_{\mathfrak{m}} \to B_{\mathfrak{p}}$ is flat. To see this directly from analytic flatness, the local ring $\mathcal{O}_{T,t}$ is flat over $\mathcal{O}_{X,x}$ and hence over $A_{\mathfrak{m}}$; it is faithfully flat over $B_{\mathfrak{p}}$. Tensoring kernels with this latter faithfully flat algebra descends the asserted flatness. The ring $B_{\mathfrak{p}}/\mathfrak{m}B_{\mathfrak{p}} = C_{\mathfrak{p}/\mathfrak{m}B}$ is a regular local domain. Each element of $B_{\mathfrak{p}} \setminus \mathfrak{p}'B_{\mathfrak{p}}$ has nonzero image in that domain. Lemma 2.52.4 shows that it is a non-zero-divisor in $B_{\mathfrak{p}}$. Hence

$$B_{\mathfrak{p}} \longrightarrow B_{\mathfrak{p}'} \quad \text{and} \quad L_{\mathfrak{p}} \longrightarrow L_{\mathfrak{p}'}$$

are injective. By (3), $u_{\mathfrak{p}'}$ is bijective. The commutative localization square then makes $u_{\mathfrak{p}}$ injective, and faithful flat comparison with the ordinary stalk proves (2).

Finally (1) implies (2) by the identity base change. Conversely assume (2), hence (3), and make any good analytic base change, over any complete extension. At an image x' the new fibre is the complete scalar extension of T_x to $\mathcal{H}(x')$. Lemma 2.52.3 preserves the initial fibre hypothesis and the closure in (3); bijectivity on the total space is preserved by pullback. Quasi-smoothness is stable under this base change by Theorem 2.41.8. The implication (3)⇒(2), already proved, therefore gives injectivity at each point above t. This is (1). □

Corollary 2.52.6 (Analytic flatness and the generic-fibre cokernel)**.** *In the setting of Theorem 2.52.5, put $P = \operatorname{Coker}(u)$. The sheaf N is analytically X-flat at t if and only if u is injective at t and P is analytically X-flat at t.*

Proof. Suppose first that N is analytically flat. Make an arbitrary good analytic base change over a complete field, and choose a point t' above t, with image x'. The fibre hypothesis persists by Lemma 2.52.3. Enlarge the coefficient field once more to a nontrivially valued complete extension E of $\mathcal{H}(t')$, and choose the E-rational point t'' induced by the embedding $\mathcal{H}(t') \hookrightarrow E$. Its image x'' is also E-rational. If $\mathcal{H}(t')$ was trivially valued, a free-radius extension supplies such an E.

Set $R = \mathcal{O}_{X'',x''}$ and $S = \mathcal{O}_{T'',t''}$, and let $\mathfrak{m}$ be the maximal ideal of R. By Lemma 2.35.12, the fibre stalk is $S/\mathfrak{m}S$. On its regular integral component the fibre map from the free sheaf L is injective: its kernel is a submodule of a free module over a domain and becomes zero over that domain's fraction field, because the fibre map is generically bijective. Thus the map of stalks

$L_{t''} \to N_{t''}$ is injective modulo $\mathfrak{m}$. The ring S is Noetherian, and $N_{t''}$ is flat over R by analytic flatness. Apply [StacksAlg, Tag 00ME], with its finite source module equal to $L_{t''}$ and its flat target module equal to $N_{t''}$. It gives both injectivity and R-flatness of the cokernel $P_{t''}$.

The two scalar-extension maps on the base and source ordinary local rings are faithfully flat by Theorem 2.35.4; coherent stalks pull back by ordinary tensor product. Injectivity therefore descends to t', and Lemma 2.35.6 descends the naive flatness of P to t'. Since the original test and t' were arbitrary, P is analytically flat at t. Taking the identity test also proves that u is injective there.

Conversely, suppose u is injective and P analytically flat at t. Theorem 2.52.5 makes u injective after every required base change at every point above t. At such a point the short exact sequence $0 \to L \to N \to P \to 0$ has outer terms flat over the base local ring: this holds for P by assumption, and for L because it is free over the quasi-smooth, hence analytically flat, source. The ordinary-module argument in the proof of Theorem 2.35.14, applied to this sequence of modules over the base ordinary local ring, shows that the middle term is flat. This holds after every test, proving analytic flatness of N. □

Definition 2.52.7 (An analytic dévissage)**.** Let $Y \to X$ be a morphism of good spaces, let $\mathcal{F}$ be coherent on Y, and let $y \in \operatorname{Supp}\mathcal{F}$ map to x. An X-dévissage of $\mathcal{F}$ at y consists of an affinoid ordinary neighbourhood V of y, a sequence $n_1 > \cdots > n_r \geq 0$, and the following finite list of data. Put $T_0 = V$, $t_0 = y$ and $Q_0 = \mathcal{F}|_V$. For $1 \leq i \leq r$ one specifies:

1. an affinoid domain T_i of a smooth X-space of pure relative dimension n_i, and $t_i \in T_i$ above x;
2. a finite X-morphism $\pi_i : \operatorname{Supp}(Q_{i-1}) \to T_i$ whose fibre over t_i is the singleton $\{t_{i-1}\}$;
3. a finite free sheaf L_i and a morphism $u_i : L_i \to N_i = (\pi_i)_* Q_{i-1}$ with coherent cokernel Q_i, such that $t_i \in \overline{\operatorname{Bij}((u_i)_{(T_i)_x})}^{(T_i)_x}$.

One requires $t_i \in \operatorname{Supp}(Q_i)$ for $i < r$ and $Q_r = 0$. The support carries its closed analytic structure defined by the annihilator; Q_{i-1} is regarded as a coherent sheaf on that closed subspace. All radii are allowed.

Theorem 2.52.8 (Flatness along a given dévissage)**.** *For an X-dévissage as above, $\mathcal{F}$ is analytically X-flat at y if and only if u_i is injective at t_i for every $1 \leq i \leq r$. When these conditions hold, every u_i is universally injective at t_i in the sense of Theorem 2.52.5, and u_r is an isomorphism at t_r.*

Proof. Restriction to the ordinary neighbourhood V preserves all pointwise flatness tests. Regarding a sheaf as a sheaf on its closed support also preserves these tests: the closed immersion is finite, its point fibre is a singleton, and Theorem 2.35.17 applies. The same theorem, now applied to π_i, gives

$$Q_{i-1} \text{ is analytically } X\text{-flat at } t_{i-1} \quad \Longleftrightarrow \quad N_i \text{ is analytically } X\text{-flat at } t_i.$$

The spaces T_i are quasi-smooth over X by smoothness and domain invariance. Corollary 2.52.6 therefore identifies the condition on the right with the conjunction of injectivity of u_i at t_i and analytic X-flatness of Q_i there. Start with the zero sheaf Q_r, which is flat after every base change, and iterate these equivalences backwards. This proves the stated equivalence. The universal injectivity follows separately for each arrow from Theorem 2.52.5. Finally $Q_r = 0$ makes u_r surjective; its injectivity gives the last assertion. □

References. The injectivity and cokernel results reconstruct [Duc18, Propositions 8.1.8 and 8.1.10(1)]. Generic-point density follows from the internal spectral boundary and flat affinoid domains. For the cokernel criterion, rational scalar lifting and the Stacks flat-map lemma give a direct proof of the universally flat formulation. The dévissage definition and its flatness test follow [Duc18, 8.2.4 and Theorem 8.3.4(2)].

2.53 Compactness and the separatedness of proper morphisms

Definition 2.53.1 (Compact analytic spaces and morphisms)**.** In the properness discussion, a compact analytic space or domain means a quasi-compact Hausdorff space; quasi-compact alone means the finite-subcover property. A morphism $f : Y \to X$ is compact if the inverse image of every compact analytic domain of X is compact in this sense. A proper morphism is a compact morphism with empty relative analytic boundary.

This makes explicit the compactness convention in Section 2.7. The separatedness assertion there follows from compactness together with boundarylessness, by the theorem below.

Lemma 2.53.2 (Closed immersions are target G-local)**.** *Let $h : W \to P$ be an analytic morphism and let (P_i) be an admissible covering of P by analytic domains. If every pullback $h_i : W \times_P P_i \to P_i$ is a closed immersion, then h is a closed immersion.*

Proof. The underlying map is injective: two points with the same image belong to a common inverse-image domain, where h_i is injective. Write its image as $S \subset P$. On each P_i the intersection $S \cap P_i$ is closed. Refine the covering by affinoid domains. The net topology, Lemma 2.40.2, tests a set's closedness on this chart quasi-net, so S is closed in P. The same test proves that $W \to S$ is a homeomorphism. Indeed the pullbacks of the chart quasi-net form a quasi-net of W, and intersecting it on the target with S gives a quasi-net of S. On every such pair the map is a homeomorphism.

Let $\mathcal{I}_i$ be the coherent ideal defining h_i. Choose an affinoid chart $U = \mathcal{M}(A)$ in P_i on which the inverse image is the actual quotient $\mathcal{M}(B)$, where $B = A/I$ and $I = \Gamma(U, \mathcal{I}_i)$. For any affinoid domain $V = \mathcal{M}(C)$ of U, the map $A \to C$ is a bounded domain restriction. By Corollary 1.18.4, I is finite and closed and the quotient norm on B is the canonical finite-A-module norm for the cyclic presentation $A \to B$. With canonical finite-module norms on the terms, Proposition 2.38.2 restricts $0 \to I \to A \to B \to 0$ to a norm-strict exact sequence over C. Its left map identifies $I \otimes_A C$ with the ideal IC in C, since its image consists of sums of products of images of elements of I with elements of C. The canonical identification $B \otimes_A C = C/IC$ is the algebraic quotient identity. The same corollary over C gives closedness of IC and the actual complete quotient norm; this is the canonical norm for the cyclic C-module presentation. Applying Proposition 2.33.19 with $L = K$ identifies $B\widehat{\otimes}_A C$ with this actual quotient by a bounded isomorphism with bounded inverse. If V or the quotient is empty, the corresponding zero algebra gives the same assertions directly.

The bounded pushout property of the completed affinoid tensor product, Proposition 2.33.8, represents the analytic inverse image over V: on each affinoid chart of a test space the compatible maps are exactly the compatible bounded algebra maps, and these identifications agree on refinements and glue. This uses the given quotient chart over U, not closed-immersion locality for the morphism being proved. On a common affinoid refinement of $P_i \cap P_j$, the two restricted quotients thus represent the same inverse image over the refinement. Their canonical isomorphism over C makes their kernels in C equal. Consequently the ideals glue to a coherent ideal $\mathcal{I} \subset \mathcal{O}_{P_G}$. One

may construct this sheaf directly as compatible families of local sections, as in Theorem 2.40.6; finite presentation and coherence are local on the given admissible covering.

Fix an arbitrary affinoid domain $Q = \mathcal{M}(A)$ in P. Theorem 2.39.6 identifies $\mathcal{I}|_Q$ with the finite ideal $I = \Gamma(Q, \mathcal{I}) \subset A$. By Corollary 1.18.4, A/I is an affinoid quotient with its actual quotient norm. The space $\mathcal{M}(A/I)$ and $W \times_P Q$ become the same closed subspace after an affinoid refinement subordinate to the P_i. Their quotient maps and their inverses agree on overlaps, so Theorem 2.40.8 glues those identifications. Hence h is represented on every such Q by the admissible quotient $A \twoheadrightarrow A/I$. Together with the closed topological embedding, this is a closed immersion. □

Lemma 2.53.3 (The diagonal for a good Hausdorff source)**.** *If X is affinoid and Y is good and Hausdorff, every analytic morphism $f : Y \to X$ is separated.*

Proof. Put $P = Y \times_X Y$, with projections p_1, p_2. The diagonal is a topological embedding onto its image because $p_1\Delta$ is the identity. We describe its closed-subspace structure on an ordinary open covering of P. Fix $t \in P$ and let $y_i = p_i(t)$.

If $y_1 \neq y_2$, choose disjoint ordinary open neighbourhoods N_1, N_2 in the Hausdorff space Y. The open neighbourhood $p_1^{-1}(N_1) \cap p_2^{-1}(N_2)$ of t is disjoint from the diagonal image; there its closed-subspace ideal is the unit ideal.

If $y_1 = y_2 = y$, goodness supplies an affinoid neighbourhood $V = \mathcal{M}(B)$ of y. Then $V \times_X V$ is an affinoid domain of P containing the open neighbourhood $p_1^{-1}(\mathrm{Int}_{\mathrm{top}}\, V) \cap p_2^{-1}(\mathrm{Int}_{\mathrm{top}}\, V)$ of t. For $X = \mathcal{M}(A)$, the diagonal over this domain is exactly the multiplication quotient

$$B \widehat{\otimes}_A B \longrightarrow B.$$

It is bounded and has the bounded section $b \mapsto b \otimes 1$, so it is norm-admissible. Its kernel is a finite closed ideal by Theorem 1.18.2. The bounded pushout property identifies its spectrum with the diagonal, including the analytic points with their residue-field data; no identification of the analytic product with an ordinary topological product is used.

The two cases cover every point of P by ordinary open neighbourhoods on which the diagonal is a closed subspace. Its image is consequently closed, and the ideals agree on overlaps since they are the kernels of the same restriction

map. Lemma 2.53.2, applied to this ordinary open covering, proves that the full diagonal is a closed immersion. Thus f is separated. □

Theorem 2.53.4 (Proper analytic morphisms are separated)**.** *A compact boundaryless morphism of arbitrary analytic spaces is separated.*

Proof. Choose an affinoid chart covering (X_i) of the target and put $Y_i = f^{-1}(X_i)$. Since each X_i is compact Hausdorff, compactness makes Y_i compact Hausdorff. For each $y \in Y_i$, test the boundarylessness of f on the actual affinoid domain X_i in Definition 2.44.4. It gives an affinoid neighbourhood of y in Y_i, so Y_i is good. Lemma 2.53.3 therefore makes $Y_i \to X_i$ separated.

Let $P = Y \times_X Y$. The domains $P_i = Y_i \times_{X_i} Y_i$ form an admissible covering of P: they are the inverse images of the target chart covering under $P \to X$, by Theorem 2.40.11 and analytic-domain base change. The pullback of the full diagonal to P_i is the closed diagonal of $Y_i \to X_i$. Lemma 2.53.2 glues these closed immersions over the whole product, proving the assertion. □

References. The convention and conclusion follow the formulation of [TemIntro, Definition 4.2.4.1 and Exercise 4.2.4.2(iv)]. The proof is internal: boundarylessness provides good inverse images, Hausdorffness separates distinct projected points, and coherent quotient ideals glue over the analytic site.

2.54 Simple roots and finite étale germs

Lemma 2.54.1 (A simple completed-residue root lifts analytically)**.** *Let X be good, $x \in X$, $R = \mathcal{O}_{X,x}$, and $F = \mathcal{H}(x)$. If $P \in R[T]$ and $\alpha \in F$ satisfy $P_x(\alpha) = 0$ and $P'_x(\alpha) \neq 0$, there is a unique germ $b \in R$ with $P(b) = 0$ and $b(x) = \alpha$. Consequently the residue field $\kappa(x)$ is separably algebraically closed in F.*

Proof. The image of R in F is dense. Indeed, on an affinoid neighborhood $\mathcal{M}(A)$ it contains $\operatorname{Frac}(A/\ker x)$: a denominator nonzero at x is invertible on a rational ordinary neighborhood. This fraction field is dense in F by the definition of the completed residue field. Choose $a \in R$ whose value is sufficiently close to α. Then $P'(a)$ is a unit of R. Write

$$\frac{P(a+Z)}{P'(a)} = c_0 + Z + \sum_{i=2}^{d} c_i Z^i.$$

One can choose $\rho > 0$ and $0 < q < 1$ so that

$$|\alpha - a(x)| < \rho, \qquad |c_0(x)| < q\rho, \qquad |c_i(x)|\rho^{i-1} < q \quad (2 \le i \le d).$$

To justify the simultaneous choice, first use the Taylor coefficients of P_x at its simple root α to choose a sufficiently small ρ, and then approximate α by $a(x)$ closely enough; all the finitely many normalized coefficients vary continuously and $c_0(x)$ tends to zero. Represent the coefficients on an ordinary affinoid neighborhood and impose the weak inequalities $|c_0| \le q\rho$ and $|c_i| \le q\rho^{1-i}$. They define a Weierstrass domain U that is still an ordinary neighborhood of x, because every inequality is strict at x. In its defining quotient Banach norm the coefficients satisfy the same weak bounds: each one is the image of the corresponding scaled Weierstrass variable.

In $B = \mathcal{O}(U)$ consider the closed ball $\|z\| \le \rho$ and the map $\Phi(z) = -c_0 - \sum_{i\ge 2} c_i z^i$. The displayed bounds give $\|\Phi(z)\| \le q\rho$ and

$$\|\Phi(z) - \Phi(w)\| \le q\|z - w\|.$$

Starting with $z_0 = 0$, the iterates satisfy $\|z_{n+1} - z_n\| \le q^n\rho$, so completeness gives a unique fixed point z in this ball. The element $b = a + z$ satisfies $P(b) = 0$. Evaluation at x commutes with the convergent iteration, and $\alpha - a(x)$ is a fixed point in the same ball over F; uniqueness gives $b(x) = \alpha$. If two germs b, b' have the stated properties, factor $P(b) - P(b') = (b - b')Q(b, b')$. The value of $Q(b, b')$ at x is $P'_x(\alpha) \neq 0$, so $Q(b, b')$ is a unit and $b = b'$. Finally, lift the coefficients of the separable minimal polynomial of any element of F algebraic separable over $\kappa(x)$ to R and apply the first assertion. Its root is the value of a germ, hence belongs to $\kappa(x)$. □

Lemma 2.54.2 (Separable extensions and dense coefficient fields)**.** *Let F be a complete valued field and $k_0 \subset F$ a dense subfield. Every finite separable extension E/F has a presentation $E \simeq F[T]/(Q)$ with $Q \in k_0[T]$ monic and separable.*

Proof. By [StacksAlg, Tag 030N], choose α with $E = F(\alpha)$, and let $P \in F[T]$ be its monic minimal polynomial of degree d. The powers $1, \alpha, \dots, \alpha^{d-1}$ form an F-basis. By Lemma 1.12.1, its coordinate maximum norm is equivalent to the absolute value of E. Hence, if β is sufficiently close to α, the coordinate matrix of $1, \beta, \dots, \beta^{d-1}$ relative to this basis has the form $I + M$ with $\|M\| < 1$. The convergent matrix series $\sum_{j\ge 0}(-M)^j$ makes it invertible, so these powers are also a basis.

Approximate the coefficients of P by elements of k_0, retaining the leading coefficient 1, to obtain Q. Since $P'(\alpha) \neq 0$, sufficiently close approximation makes $Q(\alpha)$ arbitrarily small and keeps $Q'(\alpha)$ nonzero. The contraction argument in the preceding lemma, now in the complete field E and centered at α, produces a root β of Q as close to α as desired. The basis conclusion gives $F(\beta) = E$. The degree of Q is d, so $F[T]/(Q) \to E$ is an isomorphism. In particular Q is irreducible and separable over F, and therefore is separable over k_0. □

Theorem 2.54.3 (Finite étale germs and completed residue fields)**.** *For a good analytic space X and $x \in X$, passage to the analytic fibre induces an equivalence between finite étale germs over (X, x) and finite étale $\mathcal{H}(x)$-algebras, with arrows reversed on the algebra side. In particular, any finite separable extension of $\mathcal{H}(x)$ is realized by a finite étale morphism over an ordinary neighborhood of x, with exactly one point above x.*

Proof. Put $R = \mathcal{O}_{X,x}$, $k_0 = \kappa(x)$ and $F = \mathcal{H}(x)$. Finite analytic maps near x correspond to finite R-algebras: existence is Theorem 2.35.20, and morphisms descend after shrinking because the algebras have finite presentations. Explicitly, represent the images of finitely many algebra generators on a common neighborhood and shrink again to make their finitely many relations hold. All the resulting maps are bounded by finite-module automatic boundedness. Equality of two maps is tested on the same finite list. Finite étaleness corresponds to finite flatness and the vanishing of relative differentials, by Proposition 2.49.2; these conditions spread from the germ by finite presentations and finite branch separation.

We describe maps explicitly. A finite étale R-algebra is a finite product of local algebras by Corollary 2.35.21. A local factor C is finite free over R, and its residue algebra is a finite separable field over k_0 by Stacks 02GL(1). Choose a primitive residue element, using [StacksAlg, Tag 030N], and lift it to $c \in C$. If the residue degree is d, the elements $1, c, \dots, c^{d-1}$ are an R-basis: Nakayama gives a surjection from R^d, and flatness makes its finite kernel vanish after reduction and hence vanish. The expression of c^d in this basis gives

$$C = R[T]/(P), \qquad P \text{ monic}, \qquad P'(c) \in C^{\times}.$$

For a second local factor D, realized as an ordinary local ring $\mathcal{O}_{Z,z}$, an F-algebra map $C \otimes_R F \to D \otimes_R F = \mathcal{H}(z)$ is determined by a simple root of P in $\mathcal{H}(z)$. The last equality follows from the finite étale fibre description in

Stacks 02GL(1) and the fact that this local germ has a single point over x; its finite reduced fibre algebra has one field factor. Lemma 2.54.1, applied on Z, lifts this root uniquely to D. Thus it lifts the given homomorphism uniquely. Products cause no difficulty: an algebra map into a local factor selects exactly one source factor, as its orthogonal idempotents must map to zero or one. This proves full faithfulness.

For essential surjectivity it suffices to realize a finite separable field E/F. The preceding lemma supplies a monic $Q \in k_0[T]$ with $F[T]/(Q) = E$. Lift its finitely many coefficients to R, obtaining $\widehat{Q} \in R[T]$, and set $C = R[T]/(\widehat{Q})$. The algebra C is finite free. Its derivative is a unit: modulo the maximal ideal of R, the coprime pair Q, Q' has a polynomial Bezout identity; lifting this identity shows that multiplication by $\widehat{Q}'$ is surjective modulo that maximal ideal, hence surjective on the finite module C by Nakayama. Thus C is finite étale. Realize it on an ordinary neighborhood by the first paragraph. Its fibre is exactly E, so there is one point above x with completed residue field E. Finite products realize all finite étale algebras. □

Corollary 2.54.4 (An étale map with unchanged completed residue field)**.** *Let $f : Y \to X$ be étale between good spaces, and let $y \mapsto x$. If $\mathcal{H}(x) \to \mathcal{H}(y)$ is an isomorphism, then f induces an isomorphism between ordinary neighborhoods of y and x.*

Proof. An étale morphism is locally finite at y by Proposition 2.49.2. Separate the branch at y by Lemma 2.35.1. Its finite étale germ has fibre $\mathcal{H}(x)$, so Theorem 2.54.3 identifies it with the identity germ. The inverse morphism and the two inverse equalities involve finitely many affinoid algebra generators; representing them and shrinking the ordinary neighborhoods makes these equalities hold as analytic morphisms. □

References. The construction reconstructs the good-space case of [Ber93, Theorem 3.4.1]. The simple-root argument supplies the valued-field step explicitly; finite-dimensional basis stability replaces a separate invocation of Krasner's lemma in the realization of finite separable extensions.

2.55 Product neighborhoods in smooth families

Lemma 2.55.1 (Product neighborhoods of free Gauss points)**.** *Let X be good, $x \in X$, and let $r = (r_1, \dots, r_m)$ be $\mathcal{H}(x)$-free, with $0 \leq m \leq n$. In $\mathbb{A}^{n,\mathrm{an}}_X$ let ξ be the fibre point that is Gauss of radii r in the first m coordinates and has its remaining coordinates zero. Every ordinary neighborhood W of ξ contains a product*

$$U \times_K D_1 \times_K \cdots \times_K D_n$$

that contains ξ, where U is an ordinary neighborhood of x, the D_i for $i \leq m$ are open annuli around radius r_i, and the remaining D_i are open discs centered at zero.

Proof. Take an ordinary affinoid neighborhood $X_0 = \mathcal{M}(A)$ of x. The topology on relative affine space is generated locally by finitely many conditions $|P_j| \in I_j$ with $P_j \in A[T_1, \dots, T_n]$ and open intervals $I_j \subset \mathbb{R}_{\geq 0}$. This follows directly on each closed polydisc from polynomial density in its Tate algebra: approximate a finite list of functions sufficiently closely to preserve the strict open inequalities at the point. Choose such conditions defining a neighborhood contained in W.

Write $T = (T', T'')$, separating the first m coordinates. In each specialization $P_j(T', 0)$, freeness makes the values of distinct nonzero monomials at ξ distinct. If the specialization is nonzero at ξ, it therefore has a unique dominant monomial. The finitely many strict inequalities expressing its dominance, and the condition that its value lies in I_j, persist when the coefficients vary over a sufficiently small ordinary neighborhood of x and the values $|T_i|$ vary in sufficiently small intervals around r_i. Every term divisible by a coordinate of T'' can simultaneously be made strictly smaller by taking those coordinates in sufficiently small zero-centered discs; there are only finitely many such terms. If the specialization vanishes at ξ, all its coefficients have value zero at x; bound them on a small ordinary base neighborhood and make every term have value in a sufficiently small interval $[0, \varepsilon)$ contained in I_j. Intersecting these finitely many coefficient neighborhoods and choosing the annulus and disc radii proves the assertion. □

Proposition 2.55.2 (A product patch from a rational fibre point)**.** *Let $Y \to X$ be smooth of pure relative dimension n between good spaces. If the fibre over x has a point y with $\mathcal{H}(y) = \mathcal{H}(x)$, then Y has a nonempty ordinary open subset, over an ordinary neighborhood U of x, that is X-isomorphic to $U \times_K D$, where D is an n-dimensional open polydisc. The subset can be*

chosen inside any prescribed ordinary neighborhood of y, but need not contain y.

Proof. Restrict to the prescribed neighborhood and take an étale coordinate map $Y \to \mathbb{A}_X^{n,\mathrm{an}}$ near y, as supplied by Theorem 2.50.3. If z is the image of y, the inclusions $\mathcal{H}(x) \subset \mathcal{H}(z) \subset \mathcal{H}(y)$ force equality. Corollary 2.54.4 therefore identifies an ordinary neighborhood of y with an ordinary open set W in relative affine space. The fibre W_x contains a point with coordinates in $F = \mathcal{H}(x)$. Since $\kappa(x)$ is dense in F, there is a point z' in this same fibre open set all of whose coordinates belong to $\kappa(x)$: only finitely many strict polynomial-value conditions need to be preserved. Represent these coordinates by germs on X, shrink to an ordinary neighborhood where they are defined, and translate the corresponding relative affine coordinates. The translated open set contains the fibre origin. Lemma 2.55.1, with $m = 0$, now supplies the desired product. Translating back and using the étale local isomorphism completes the proof. □

Lemma 2.55.3 (Separable rigid points in smooth spaces)**.** *Over a nontrivially valued complete field F, every nonempty ordinary open subset of a good smooth F-analytic space contains a point whose completed residue field is finite separable over F.*

Proof. First consider a nonempty ordinary open set in $\mathbb{A}_F^{n,\mathrm{an}}$. Finite polynomial-value inequalities describe a smaller nonempty open neighborhood of any chosen point. The group $\sqrt{|F^\times|}$ is dense in $\mathbb{R}_{>0}$: powers with rational exponents of any fixed value different from one already are dense. Choose all upper and lower bounds for these inequalities, and the ambient polydisc radii, in this group, strictly outside the values at the point. This produces a nonempty affinoid domain inside the open set. After a finite extension realizing the finitely many radii as absolute values, its presentation is strict. A maximal ideal of this strict algebra gives, by Theorem 1.12.3, a point whose coordinates are algebraic over F. Surjectivity of complete scalar extension ensures that the chosen domain remains nonempty.

Separable algebraic elements are dense among algebraic elements over F. To see the only nontrivial case, let a have inseparable monic minimal polynomial P of degree d. Then $P' = 0$. For arbitrarily small $\varepsilon \in F^\times$, the polynomial $P(T) + \varepsilon T$ is separable, as its derivative is the nonzero constant ε. In a finite splitting extension write its roots as $b_1, \ldots, b_d$. Evaluation at a

gives $\prod_i |a - b_i| = |\varepsilon a|$, so one root approaches a as $\varepsilon \to 0$; the case $a = 0$ is already separable. Lemma 1.12.1 supplies compatible absolute values in the finite splitting extensions. Approximating the finitely many coordinates simultaneously preserves the finitely many strict polynomial inequalities. Their joint field is finite separable over F. This proves the assertion for affine space.

Now take an étale coordinate chart of the given smooth open set. An étale map is ordinary-open. Indeed, near any point it is finite flat; separate the finite fibre into branches over a small ordinary target neighborhood. A source ordinary neighborhood contains the relevant branch after a further target shrinking, by compactness and the closed-map property. The finite flat branch has open-and-closed image by Lemma 2.44.2, so the image contains a neighborhood of the image point. Apply the affine-space assertion to this nonempty open image. Over the resulting finite separable point the étale fibre is a nonempty finite union of finite separable fields. Any point of it gives a finite separable extension of F, as required. □

Theorem 2.55.4 (Product patches after a finite étale base change)**.** *Let $Y \to X$ be smooth of pure relative dimension n between good spaces, let $x \in X$, and suppose $\mathcal{H}(x)$ is nontrivially valued. For every nonempty ordinary open subset $W \subset Y_x$, there exist a finite étale morphism $X' \to U$ over an ordinary neighborhood U of x, its unique point x' above x, and a nonempty ordinary open subset $V \subset Y \times_X X'$ such that $V_{x'}$ maps into W and*

$$V \simeq U' \times_K D$$

over X', for an ordinary neighborhood U' of x' and an n-dimensional open polydisc D.

Proof. The preceding lemma gives $y \in W$ with $E = \mathcal{H}(y)$ finite separable over $F = \mathcal{H}(x)$. By Theorem 2.54.3, realize E/F by a finite étale morphism $X' \to U$ with one point x' above x and $\mathcal{H}(x') = E$. In the fibre product the evaluation at y gives an E-rational point y' over x'. The ordinary topology of an analytic fibre is the subspace topology by Theorem 2.33.11 and its gluing form. Hence one can choose an ordinary neighborhood of y' whose intersection with the fibre maps into W. Restrict to that neighborhood and apply Proposition 2.55.2. Its product patch has precisely the required properties. □

References. These proofs reconstruct [Duc18, Lemma 6.2.1, the rational-point case of Lemma 6.2.2, and the nontrivially valued completed-residue-field case of Lemma 6.2.4].

Theorem 2.55.5 (A product patch from a free Gauss residue field)**.** *Let $Y \to X$ be smooth of pure relative dimension n between good analytic spaces over any complete field K. Suppose that $y \in Y_x$ has an isometric $\mathcal{H}(x)$-identification $\mathcal{H}(y) = \mathcal{H}(x)_r$, where $r = (r_1, \dots, r_m)$ is free and $0 \le m \le n$. Inside any prescribed ordinary neighborhood of y there is a nonempty ordinary open subset which, over X, is isomorphic to*

$$U \times_K D \times_K \Delta,$$

where U is an ordinary neighborhood of x, D is an m-dimensional open polyannulus, and Δ is an $(n-m)$-dimensional open polydisc.

Proof. Restrict to the prescribed neighborhood. The field $\kappa(y)$ is dense in $\mathcal{H}(y)$, so each free coordinate value of norm r_i can be approximated by an element of $\kappa(y)$ closely enough to preserve its norm. Represent these elements by analytic germs f_i near y; a denominator with nonzero value is a unit on a smaller ordinary neighborhood. Proposition 3.24.5, applied on the analytic fibre, makes $(df_1)(y), \dots, (df_m)(y)$ independent in the relative differential fibre. Fibre base change of analytic differentials identifies this with the corresponding subspace of $\Omega^{\mathrm{an}}_{Y/X} \otimes \mathcal{H}(y)$. Since the latter has dimension n and is generated by differentials of local analytic functions, add $f_{m+1}, \dots, f_n$ so that their differentials form a basis.

The map $g = (f_1, \dots, f_n) : Y \to \mathbb{A}^{n,\mathrm{an}}_X$ is quasi-smooth of relative dimension zero at y. Explicitly, take an ordinary Jacobian presentation for Y/X, lift the finitely many f_i to its ambient affinoid algebra using the surjective closed-immersion presentation, and adjoin the equations $Z_i - f_i$ for the graph of g. Independence of the displayed differential basis says exactly that the original relation rows together with these n new rows have full rank; Theorem 2.35.14 gives the assertion. Because $Y \to X$ is inner at y, the interior composition formula makes g inner there. Its fibre through y is isolated by Lemma 2.50.1; it is ordinarily flat by the Jacobian theorem and its relative differential module vanishes. Proposition 2.49.2 therefore makes g étale near y.

Put $z = g(y)$. The completed field $\mathcal{H}(z)$ contains $\mathcal{H}(x)$ and the first m coordinate values. Lemma 3.24.4 says that these generate all of $\mathcal{H}(y)$ as

a complete field, so $\mathcal{H}(z) = \mathcal{H}(y)$. Corollary 2.54.4 identifies an ordinary neighborhood of y with an ordinary open subset $W \subset \mathbb{A}_X^{n,\mathrm{an}}$. Let p be projection onto the first m coordinates and put $a = p(z)$. Freeness implies that a is the Gauss point of polyradius r over x, and z is rational over $\mathcal{H}(a)$.

The fibre open set W_a contains a rational point whose last $n-m$ coordinates lie in the dense subfield $\kappa(a) \subset \mathcal{H}(a)$, by preserving the finitely many polynomial-value conditions of a neighborhood of z. Extend these coordinates to germs on the good space $\mathbb{A}_X^{m,\mathrm{an}}$ and translate the last coordinates over an ordinary neighborhood of a. The translated W contains the relative fibre origin. Lemma 2.55.1 with no free coordinates, applied over this new base, supplies $U_1 \times_K \Delta \subset W$ with U_1 an ordinary neighborhood of a. Apply the same lemma to $\mathbb{A}_X^{m,\mathrm{an}} \to X$ at the free Gauss point a: inside U_1 it gives $U \times_K D$. Restrict the first product to this second one, and undo the coordinate translation and the local étale isomorphism. The result is the stated product patch. □

References. The last theorem reconstructs the full free-polyradius case of [Duc18, Lemma 6.2.2]; its valued-field and differential inputs are proved in Section 3.24.

Lemma 2.55.6 (Lifting a finite extension of a trivial residue field)**.** *Let X be good, let $x \in X$ have trivially valued $F = \mathcal{H}(x)$, and let E/F be finite. There are an ordinary neighborhood U of x and a finite flat morphism $X' \to U$ whose fibre at x is $\mathcal{M}(E)$. In particular it has a unique point x' over x, with $\mathcal{H}(x') = E$, and the closed fibre of $\operatorname{Spec}\mathcal{O}_{X',x'} \to \operatorname{Spec}\mathcal{O}_{X,x}$ is reduced. If E/F is separable, the morphism may be chosen finite étale.*

Proof. Put $R = \mathcal{O}_{X,x}$ and let $\mathfrak{m}$ be its maximal ideal. The injection $\kappa(x) \hookrightarrow F$ has dense image. As F is trivially valued and hence discrete, it is surjective, so $R/\mathfrak{m} = F$. Write the finite extension as a tower generated by finitely many elements. At each step lift the coefficients of the monic minimal polynomial from the preceding residue field to the preceding ring, starting with R. Successive monic quotients give a finite free R-algebra C with $C/\mathfrak{m}C = E$. Every maximal ideal of the integral finite algebra C lies over $\mathfrak{m}$, and this residue quotient is a field, so C is local.

Theorem 2.35.20 realizes C by a finite analytic map over an ordinary neighborhood of x. Its ordinary direct-image stalk is C and its analytic fibre algebra is $C \otimes_R F = E$. To obtain flatness on a neighborhood, lift a basis of the free R-module C to finitely many sections of the finite direct-image module.

The resulting map from a free coherent module is an isomorphism at x. Its coherent kernel and cokernel vanish on a common ordinary neighborhood, since finitely many local generators vanish after one shrink. On that neighborhood the direct-image algebra is finite free. The finite-module base-change theorem then gives finite flatness, and the residue quotient proves the closed-fibre assertion. If E/F is separable, one may instead use Theorem 2.54.3, which gives the required finite étale realization directly. □

Theorem 2.55.7 (Product patches over arbitrary completed residue fields)**.** *Let $Y \to X$ be smooth of pure relative dimension n between good analytic spaces over any complete field. Given $x \in X$ and a nonempty ordinary open subset $W \subset Y_x$, there are an ordinary neighborhood U of x, a finite flat map $X' \to U$ with unique point x' above x and reduced ordinary closed fibre there, and a nonempty ordinary open subset $V \subset Y \times_X X'$ such that $V_{x'}$ maps into W and*

$$V \simeq U' \times_K D \times_K \Delta$$

over X'. Here U' is an ordinary neighborhood of x', D is either absent or a one-dimensional open annulus, and Δ is an open polydisc making the total relative dimension n. If $\mathcal{H}(x)$ is nontrivially valued, D may be omitted and $X' \to U$ may be chosen étale. If the ground field has characteristic zero, the map may also be chosen étale when $\mathcal{H}(x)$ is trivially valued.

Proof. The nontrivially valued case is Theorem 2.55.4. Suppose that $F = \mathcal{H}(x)$ is trivially valued. Proposition 3.24.9, applied to the smooth open fibre subset W, gives a finite extension E/F and a point $z \in W_E$ whose completed residue field is E or E_r for some $0 < r < 1$. Lemma 2.55.6 realizes E by a finite flat $X' \to U$ with the prescribed unique point and reduced closed fibre. Its fibre product with Y has analytic fibre $Y_x \widehat{\otimes}_F E$, by associativity of complete base change, so it contains the actual point z.

Choose an ordinary neighborhood of z in the fibre product whose intersection with the fibre is contained in W_E, using the subspace topology of analytic fibres. In the rational case apply Proposition 2.55.2. In the other case r is free over the trivially valued E; apply Theorem 2.55.5 with $m = 1$. Independence of the free differential also implies $n \geq 1$ in this case. These applications give exactly the stated ordinary product patches. In characteristic zero the finite extension E/F is separable, so the last clause of the lifting lemma makes the base change étale. □

References. The finite-flat lifting and the rational-or-Gauss point construction complete the product-patch mechanism of [Duc18, Proposition 6.2.4] for arbitrary completed residue fields.

2.56 Ordinary germs along a generic analytic fibre

Lemma 2.56.1 (Normal affinoid components and the identity principle)**.** *Let $X = \mathcal{M}(A)$ be a generalized affinoid space whose ordinary local rings are normal. Then A is a finite product of normal domains. The spectra of these factors are exactly the connected components of X, and they are open and closed. On each component, an analytic function that vanishes on a nonempty ordinary open set is zero.*

Proof. For each prime $\mathfrak{p} \subset A$, Lemma 2.52.1 gives an analytic point with kernel $\mathfrak{p}$. Its ordinary local ring R is faithfully flat over $A_{\mathfrak{p}}$, by rational-domain flatness and the ordinary-neighborhood colimit. Since R is a normal local ring, it is a domain. Faithful flatness makes $A_{\mathfrak{p}} \to R$ injective, so $A_{\mathfrak{p}}$ is also a domain. If u/v in its fraction field is integral, normality of R gives $u \in vR$; faithful flatness contracts vR to $vA_{\mathfrak{p}}$, proving that $u/v \in A_{\mathfrak{p}}$. Thus every $A_{\mathfrak{p}}$ is normal.

The Noetherian ring A is reduced, and distinct minimal primes are pairwise comaximal: otherwise their localizations at a maximal ideal containing both would contradict the domain property of that local ring. The Chinese remainder theorem gives a finite product of normal domains. Its idempotents supply open-and-closed affinoid factors. An integral factor has connected analytic spectrum: a partition into two nonempty open-and-closed sets would glue the functions zero and one to a nontrivial idempotent in its global-section ring, which is that integral factor by the structure-sheaf theorem. Finally, a nonempty ordinary open subset of an integral affinoid contains a kernel-zero point by Lemma 2.52.1. A function vanishing there is therefore zero in the algebra. □

Proposition 2.56.2 (Spreading a closed fibre neighborhood in characteristic zero)**.** *Let K be complete of characteristic zero, let $Y \to X$ be a morphism of good K-analytic spaces smooth at y, and put $x = f(y)$. Suppose that $\mathcal{O}_{X,x}$ is Artinian and $\mathcal{H}(x)$ is nontrivially valued. If a Zariski-closed subset $Z \subset Y$ contains an ordinary neighborhood of y in Y_x, then Z contains an ordinary neighborhood of y in Y.*

Proof. The assertion concerns underlying sets. We first reduce the base near x. On an ordinary affinoid neighborhood $\mathcal{M}(A)$ put $\mathfrak{p} = \ker(x)$, $D = A_{\mathfrak{p}}$ and $R = \mathcal{O}_{X,x}$. The map $D \to R$ is faithfully flat, so its map on spectra is surjective. Since R is local Artinian, D has only one prime and is zero-dimensional. Its reduction is therefore a field: its unique prime is the nilradical, which is nilpotent because the ring is Noetherian.

Let $N \subset A$ be the nilradical. The ordinary local ring of X_{red} at x is R/NR, by coherent closed-subspace restriction. The regular homomorphism $D \to R$ of Theorem 2.35.29, after quotienting by the nilradical of D, shows that R/NR is geometrically regular over the field D_{red}. It is also local Artinian, being a quotient of R. A zero-dimensional regular local ring is a field, since its maximal ideal has zero-dimensional cotangent space and Nakayama makes that ideal zero. The base change $Y \times_X X_{\mathrm{red}} \to Y$ is a homeomorphism on underlying spaces, since it is cut out by a nilpotent ideal. It preserves smoothness and the analytic fibre. Thus we may replace the base by its reduction and assume R is a field.

By Corollary 2.35.32, shrink to an ordinary neighborhood of x on which X is normal. Smoothness and the same corollary make Y normal near y. Choose a normal affinoid ordinary neighborhood $Y_0 = \mathcal{M}(B)$ of y and select its component containing y. Lemma 2.56.1 makes its algebra a normal domain. Write $Z \cap Y_0$ as the common zero set of finitely many functions $h_1, \dots, h_s \in B$. It suffices to show that these functions vanish on some nonempty ordinary open subset of Y_0, for the identity principle will then make them zero on all of Y_0.

Work in an ordinary open neighborhood of y inside Y_0, small enough that the morphism is smooth of constant relative dimension there. The hypothesis supplies a nonempty fibre open subset contained in Z. Theorem 2.55.4 gives a finite étale map $X' \to U$ near x, one point x' above x, and a nonempty ordinary open product patch in $Y' = Y \times_X X'$ whose fibre at x' lies in Z'. Its base is an ordinary neighborhood of x' and its other factor is an open polydisc D. The local ring $R' = \mathcal{O}_{X',x'}$ is a field: finite branch separation identifies it with a local factor of a finite étale algebra over the field R. The space X' is normal near x' by quasi-smooth ascent.

Inside the base of the product patch choose a normal affinoid ordinary neighborhood $X'_0 = \mathcal{M}(A')$ of x' and retain its component through x'. Then A' is a domain, and $A' \to R' \hookrightarrow \mathcal{H}(x')$ is injective. Indeed $A' \to R'$ is a nonzero flat map from a domain, so multiplication by every nonzero element

remains injective. Restrict the pulled-back h_j to $X_0' \times_K D$. Exhausting D by nested closed polydiscs identifies each such function with a power series

$$h_j = \sum_I a_{I,j} T^I, \qquad a_{I,j} \in A'.$$

The coefficients agree on the exhaustion because Tate-series expansions are unique, and restrictions to smaller polydiscs preserve their coefficients. On the fibre at x' this function vanishes at every point. In particular it vanishes at every Gauss point of a sufficiently small closed polydisc in D, so every coefficient $a_{I,j}(x')$ is zero by the Gauss norm formula. Injectivity of $A' \to \mathcal{H}(x')$ gives $a_{I,j} = 0$ for all I, j. Thus every h_j vanishes on $X_0' \times_K D$, which contains a nonempty ordinary open subset of the original product patch because X_0' is an ordinary neighborhood of x'.

The map $Y' \to Y$ is étale over the chosen base neighborhood and is ordinary-open by the finite-branch argument in Lemma 2.55.3. Hence the image of that open subset is a nonempty ordinary open subset of Y_0 contained in Z. The identity principle proves $Z \supset Y_0$. Since Y_0 was an ordinary neighborhood of y, this is the required assertion. □

References. The proposition reconstructs [Duc18, Proposition 6.3.1] in characteristic zero with nontrivially valued completed residue field, using the internally established normal-locus and product-patch results. Its conclusion is topological containment of a Zariski-closed subset, not equality of nonreduced closed analytic subspaces.

Lemma 2.56.3 (Dimension of an affinoid domain in affine space)**.** *Let F be a complete non-Archimedean field and let $V = \mathcal{M}(C)$ be a generalized affinoid domain of $\mathbb{A}_F^{d,\mathrm{an}}$. Then $\dim C \leq d$.*

Proof. Fix a maximal ideal $\mathfrak{q}$ of C. Lemma 2.52.1 realizes it as the kernel of a point $z \in V$. Choose a nontrivially valued complete extension $E/\mathcal{H}(z)$, using $E = \mathcal{H}(z)$ when possible and otherwise adjoining a free radius. Evaluation defines an E-rational point $w \in V_E$ above z. Rational-domain flatness and scalar-stalk faithful flatness give faithfully flat local maps

$$C_{\mathfrak{q}} \longrightarrow \mathcal{O}_{V,z} \longrightarrow \mathcal{O}_{V_E,w}.$$

Since V_E is an analytic domain of affine d-space and w is rational, Lemma 2.35.9 identifies the last ring with the ordinary affine-space germ at w. After translating the coordinates it is the ring of convergent power series in

d variables of Lemma 2.35.8; its completion is $E[[T_1, \dots, T_d]]$. It is a regular local ring of dimension d: regularity follows by faithful-flat descent from that formal power-series ring, and its maximal ideal and cotangent space are generated freely to first order by the d coordinate germs. Finally, a chain of primes of $C_{\mathfrak{q}}$ lifts, successively downward from the maximal ideal, along the displayed flat local maps by [StacksAlg, Tag 00HS]. Its length is therefore at most d. This holds at every maximal ideal of C, and every prime chain is contained in one, proving the assertion. □

Lemma 2.56.4 (Cutting out a fibre component over the original field)**.** *Let $Y = \mathcal{M}(B) \to X = \mathcal{M}(A)$ be an affinoid morphism, let $x \in X$, and set $F = \mathcal{H}(x)$ and $C = B\widehat{\otimes}_A F$. If $e \in C$ is idempotent and $y \in Y_x$ satisfies $e(y) = 1$, then, after replacing X by an ordinary affinoid neighborhood of x, a Weierstrass domain W of the resulting Y is an ordinary neighborhood of y and satisfies $W_x = \mathcal{M}(eC)$. In particular a strict fibre remains strict after this operation.*

Proof. Choose $0 < \delta < \varepsilon < 1$. Lemma 2.51.2 supplies, after the stated base restriction, a function b on Y whose image in C differs from e by norm less than δ. Define W by $|b - 1| \le \varepsilon$. At y the inequality is strict, so W is an ordinary neighborhood there. On $\mathcal{M}(eC)$ one has $|b - 1| < \delta$, whereas on $\mathcal{M}((1 - e)C)$ one has $|b - 1| = 1$. Thus its fibre has exactly the first factor as underlying domain. Its domain algebra is also that factor: on eC the Weierstrass condition is automatic by the norm bound, while on $(1-e)C$ the element $b-1$ has inverse of spectral norm one, so the condition $|b-1| \le \varepsilon < 1$ gives the zero algebra. Equivalently, the affinoid-domain universal property identifies W_x with the open-and-closed affinoid factor $\mathcal{M}(eC)$. This factor is a quotient of C, so it is strict when C is strict. □

Lemma 2.56.5 (Dimension lowering by a nonzero ordinary ideal)**.** *Let K have characteristic zero, let X be good, and let V be an ordinary affinoid neighborhood of a point t in $\mathbb{A}^{d,\mathrm{an}}_X$. Put $x = f(t)$, assume that $\mathcal{O}_{X,x}$ is a field and that $F = \mathcal{H}(x)$ is nontrivially valued, and let $0 \neq I \subsetneq \mathcal{O}_{V,t}$ be an ideal. The closed-subspace germ defined by I has an ordinary affinoid representative Z whose fibre Z_x is strict and has dimension at most $d-1$. The maps $Z \to X$ are inner at t, and their ordinary local rings are*

$$\mathcal{O}_{Z,t} = \mathcal{O}_{V,t}/I, \qquad \mathcal{O}_{Z_x,t} = \mathcal{O}_{V_x,t}/I\mathcal{O}_{V_x,t}.$$

Proof. The ordinary local ring is Noetherian, so finitely many generators of I extend to functions on an ordinary affinoid neighborhood of t. Choose a nonzero germ $h \in I$ and represent it there as well. Its image in the fibre germ is nonzero. Indeed, otherwise its zero set would contain an ordinary fibre neighborhood; Proposition 2.56.2, applied to the smooth map from the ordinary interior of V to X, would make h zero on an ordinary neighborhood of t, contrary to its choice.

Proposition 2.51.3 now gives an ordinary affinoid neighborhood with strict fibre algebra C. This fibre is an affinoid domain of $\mathbb{A}_F^{d,\mathrm{an}}$. Its ordinary local rings are regular by Theorem 2.35.23. Lemma 2.52.2 therefore writes C as a finite product of regular domains. Use Lemma 2.56.4 to retain the factor C_0 containing t, over the original field and without changing the ordinary germs. The image of h in C_0 is nonzero, since its fibre germ was nonzero. Lemma 2.56.3 gives $\dim C_0 \leq d$.

Cut out Z by the chosen generators of I. Its fibre algebra is the strict quotient C_0/I_C, where I_C contains the nonzero image of h. Every chain of primes containing I_C can be extended strictly at its lower end by the zero prime of the domain C_0. Hence $\dim(C_0/I_C) \leq d-1$. Coherent closed-subspace restriction gives the two displayed local-ring identifications. A closed immersion is finite and inner; composition with the inner map from the chosen ordinary neighborhood to X proves the final assertion. □

Theorem 2.56.6 (Regularity along an inner generic fibre in characteristic zero)**.** *Let K be a complete field of characteristic zero and let $f : Y \to X$ be a morphism of good analytic spaces. Let $y \in \mathrm{Int}(Y/X)$ and put $x = f(y)$. If $\mathcal{O}_{X,x}$ is a field and $\mathcal{H}(x)$ is nontrivially valued, then the canonical local homomorphism*

$$\mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{Y_x,y}$$

is regular: it is flat and all its algebraic prime fibres are geometrically regular.

Proof. We induct on an upper bound n for the dimension of a strict affinoid fibre of an ordinary affinoid representative of the source germ. Such a bound exists by Proposition 2.51.3. The induction assertion concerns every morphism satisfying the stated hypotheses that has such a representative with fibre dimension at most n. Thus later changes of representatives do not require a comparison with a minimal local-dimension invariant.

For such a representative let d be its strict fibre dimension, so $d \leq n$. Theorem 2.51.4 gives, on ordinary neighborhoods, a finite map to an affinoid

ordinary neighborhood V of t in $\mathbb{A}^{d,\mathrm{an}}_X$. Finite branch separation, followed by shrinking V, selects the branch containing y and makes y the only point above t. Put

$$R_0 = \mathcal{O}_{V,t}, \quad T_0 = \mathcal{O}_{V_x,t}, \quad S = \mathcal{O}_{Y,y}, \quad T = \mathcal{O}_{Y_x,y}.$$

The finite direct-image stalk formulas identify S with a finite R_0-algebra and give $T = S \otimes_{R_0} T_0$. To check the latter identity, apply finite affinoid base change and then take ordinary stalks. The unique point above t is still the unique point in the corresponding analytic fibre, so both direct-image stalk products have just one factor. It is enough to prove that $R_0 \to T_0$ is regular: flatness is preserved by this base change, and the fibre at a prime of S is a field extension of the corresponding geometrically regular fibre of $R_0 \to T_0$.

Both R_0 and T_0 are regular local rings. For R_0, use the smooth affine-space projection on the ordinary interior of V, Theorem 2.35.30, and regularity of the base field $\mathcal{O}_{X,x}$. For T_0, use Theorem 2.35.23. The regularity ascent used here is [StacksAlg, Tag 031E]. If R_0 is a field, its map to the regular ring T_0 is flat and geometrically regular by characteristic zero and Stacks 0381; this finishes that case. In particular it proves the induction when $n = 0$, since then $d = 0$ and V is an ordinary neighborhood of x in X, so $R_0 = \mathcal{O}_{X,x}$ is a field.

Suppose now that R_0 is not a field, and write $\mathfrak{m}$ for its maximal ideal. Every power $\mathfrak{m}^q$, $q \geq 1$, is nonzero because a regular local ring is a domain by Stacks 00NP. Lemma 2.56.5, applied to $I = \mathfrak{m}^q$, represents the resulting closed-subspace germ by an inner affinoid with strict fibre dimension at most $d - 1 \leq n - 1$. The induction hypothesis makes

$$R_0/\mathfrak{m}^q \longrightarrow T_0/\mathfrak{m}^q T_0$$

regular, and in particular flat. The rings R_0, T_0 are Noetherian and $\mathfrak{m}T_0$ lies in the maximal ideal of T_0. Apply the adic flatness criterion [StacksAlg, Tag 0523], with module T_0 over itself, to all these quotients. It follows that $R_0 \to T_0$ is flat.

It remains to examine its algebraic prime fibres. At the zero prime, the fibre is a localization of the regular ring T_0, hence regular and, by characteristic zero and Stacks 0381, geometrically regular. If $\mathfrak{p} \neq 0$ is a prime of R_0, apply the dimension-lowering lemma to $I = \mathfrak{p}$. Induction makes $R_0/\mathfrak{p} \to T_0/\mathfrak{p}T_0$ regular. Its fibre at the zero prime is exactly $T_0 \otimes_{R_0} \kappa(\mathfrak{p})$, and is geometrically regular. This proves regularity of $R_0 \to T_0$ and then of $S \to T$, completing the induction. □

Corollary 2.56.7 (Ordinary closed fibres and analytic fibres)**.** *Let K have characteristic zero, let $f : Y \to X$ be a morphism of good analytic spaces, and let $y \in \mathrm{Int}(Y/X)$ with $x = f(y)$ and nontrivially valued $\mathcal{H}(x)$. Put $R = \mathcal{O}_{X,x}$, $S = \mathcal{O}_{Y,y}$ and let $\mathfrak{m}$ be the maximal ideal of R. Then*

$$S/\mathfrak{m}S \longrightarrow \mathcal{O}_{Y_x,y}$$

is a regular faithfully flat local homomorphism.

Proof. The Noetherian ideal $\mathfrak{m}$ has finitely many generators. Represent them on an ordinary affinoid neighborhood of x and let X_0 be the closed analytic subspace they define. It contains x, and coherent closed-subspace restriction gives $\mathcal{O}_{X_0,x} = R/\mathfrak{m}$. Set $Y_0 = Y \times_X X_0$. The corresponding point has ordinary local ring $S/\mathfrak{m}S$, while its analytic fibre is unchanged because all the chosen generators vanish in $\mathcal{H}(x)$. Interior is preserved by base change. Theorem 2.56.6 applies to $Y_0 \to X_0$, giving the claimed regular map. It is local and flat between nonzero local rings, hence faithfully flat. □

References. The induction follows the mechanism of [Duc18, Theorem 6.3.3]. The finite-projection input is reconstructed with the global dimension of a strict fibre, and the intervening component cut and dimension bound are proved above. The adic flatness input is the exact algebraic statement [StacksAlg, Tag 0523].

Lemma 2.56.8 (A field germ at an auxiliary Gauss point)**.** *Let X be good over a complete field of characteristic zero, let $\mathcal{O}_{X,x}$ be a field, and suppose that $F = \mathcal{H}(x)$ is trivially valued. Choose $0 < r < 1$ and let x^* be the radius-r Gauss point above x in $X^* = X \times_K \mathbb{A}^{1,\mathrm{an}}_K$. Then $\mathcal{H}(x^*) = F_r$ is nontrivially valued and $\mathcal{O}_{X^*,x^*}$ is a field.*

Proof. The completed residue field is F_r by the Gauss evaluation and the completed-residue-field universal property. To prove the local-ring assertion, take a germ h with $h(x^*) = 0$. Lemma 2.55.1 gives an ordinary product neighborhood $U \times_K D$ of x^* on which h is represented, with D an open annulus containing radius r. Since $\mathcal{O}_{X,x}$ is a field, it is normal. Corollary 2.35.32 and Lemma 2.56.1 give a normal integral affinoid ordinary neighborhood $\mathcal{M}(A) \subset U$ of x. The map $A \to \mathcal{O}_{X,x} \hookrightarrow F$ is injective: it is a nonzero flat map from a domain followed by the residue-field embedding.

Choose a closed annulus $D_0 \subset D$ with r strictly between its two radii. The restriction of h to $\mathcal{M}(A) \times_K D_0$ has a unique convergent Laurent expansion $\sum_{j \in \mathbf{Z}} a_j T^j$, with $a_j \in A$. This is the weighted Laurent description of the

annulus algebra; uniqueness is preserved under restriction to smaller annuli. At the Gauss point x^* its value is

$$|h(x^*)| = \max_j |a_j(x)| r^j.$$

Thus every $a_j(x)$ is zero. Injectivity of $A \to F$ makes all a_j zero simultaneously, so h vanishes on this product, which is an ordinary neighborhood of x^*. The maximal ideal of $\mathcal{O}_{X^*,x^*}$ is exactly the kernel of evaluation in its completed residue field. We have proved that kernel zero, so this local ring is a field. □

Theorem 2.56.9 (Generic-fibre comparison for all characteristic-zero residue fields)**.** *Let K be any complete field of characteristic zero and let $f : Y \to X$ be a morphism of good analytic spaces. If $y \in \mathrm{Int}(Y/X)$ and $\mathcal{O}_{X,f(y)}$ is a field, then*

$$\mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{Y_{f(y)},y}$$

is a regular local homomorphism.

Proof. Put $x = f(y)$ and $F = \mathcal{H}(x)$. If F is nontrivially valued, use Theorem 2.56.6. Suppose that F is trivially valued. Form $X^* = X \times_K \mathbb{A}^{1,\mathrm{an}}_K$ and its point x^* as in Lemma 2.56.8; its ordinary local ring is a field and its completed residue field is F_r. Put $Y^* = Y \times_X X^* = Y \times_K \mathbb{A}^{1,\mathrm{an}}_K$, and choose the radius-$r$ Gauss point y^* above y. Here the Gauss seminorm on $\mathcal{H}(y)[T]$ is multiplicative for any radius; r need not be free over $\mathcal{H}(y)$. Its restriction to $F[T]$ is the chosen point x^*.

Interior survives the base change. Write

$$S = \mathcal{O}_{Y,y}, \quad S^* = \mathcal{O}_{Y^*,y^*}, \quad T = \mathcal{O}_{Y_x,y}, \quad T^* = \mathcal{O}_{Y^*_{x^*},y^*}.$$

The smooth projection $Y^* \to Y$ makes $S \to S^*$ regular by Theorem 2.35.30. The nontrivially valued comparison just proved makes $S^* \to T^*$ regular. Moreover $Y^*_{x^*} = (Y_x)_{F_r}$ by complete base-change associativity, so $T \to T^*$ is faithfully flat by Theorem 2.35.4. These are maps of Noetherian local rings and form the expected commutative square with $S \to T$.

We spell out composition and descent of regularity in this square. The composite $S \to T^*$ is flat. For a prime $\mathfrak{p} \subset S$ and a finite field extension $E/\kappa(\mathfrak{p})$, the algebra $S^* \otimes_S E$ is regular. Base change of the regular map $S^* \to T^*$ is flat with geometrically regular fibres; localization and [StacksAlg,

Tag 031E] make $T^* \otimes_S E$ regular. Thus $S \to T^*$ is regular. Faithful flatness of $T \to T^*$ descends S-flatness: tensor any injection of S-modules first with T and then with the faithfully flat T-module T^*, and detect its kernel. Finally

$$T \otimes_S E \longrightarrow T^* \otimes_S E$$

is faithfully flat and its target is regular. At every prime of the source choose a prime above it and apply regular-local-ring descent, [StacksAlg, Tag 00OF]. Hence every $T \otimes_S E$ is regular. This proves all geometrically regular prime fibres of $S \to T$, and completes the proof. □

Corollary 2.56.10 (Closed-fibre comparison in arbitrary characteristic-zero valuation)**.** *For a morphism $f : Y \to X$ of good analytic spaces over any complete characteristic-zero field and an inner point y with image x, put $R = \mathcal{O}_{X,x}$ and $S = \mathcal{O}_{Y,y}$. If $\mathfrak{m}$ is the maximal ideal of R, then*

$$S/\mathfrak{m}S \longrightarrow \mathcal{O}_{Y_x,y}$$

is regular and faithfully flat.

Proof. Realize a finite generating family of $\mathfrak{m}$ on an ordinary base neighborhood and form the associated closed analytic base change. As in Corollary 2.56.7, the new ordinary base ring is the field $R/\mathfrak{m}$, the ordinary source ring is $S/\mathfrak{m}S$, and the analytic fibre and the inner point are unchanged. Apply Theorem 2.56.9. Its regular map is local and flat, hence faithfully flat. □

2.57 Coherent flatness at an inner point

Finite projection replaces a coherent sheaf by a finite module on a relative affine-space domain. A basis at the generic point of an integral fibre then separates a free part from a quotient of smaller fibre dimension. We begin with the two module statements used in this induction.

Lemma 2.57.1 (Support under analytic pullback)**.** *Let $A \to B$ be a bounded homomorphism of generalized affinoid algebras, allowing a complete extension of the ground field for B, and let M be a finite A-module. For $z \in \mathcal{M}(B)$ with image $x \in \mathcal{M}(A)$,*

$$z \in \operatorname{Supp}(M \otimes_A B) \quad \Longleftrightarrow \quad x \in \operatorname{Supp}(M).$$

Consequently the support of the pullback of a coherent sheaf is the inverse image of its support. The closed support defined by the annihilator and its analytic base change have the same underlying set as the support of the pulled-back sheaf.

Proof. Put $\mathfrak{p} = \ker x$ and $\mathfrak{q} = \ker z$. A finite module over a local ring is zero exactly when its residue vector space is zero, by Nakayama's determinant argument. Hence x belongs to the support precisely when $M \otimes_A \kappa(\mathfrak{p}) \neq 0$. The bounded map supplies an injection $\kappa(\mathfrak{p}) \hookrightarrow \mathcal{H}(z)$, and extension of vector-space scalars is faithful. The canonical identifications

$$(M \otimes_A B) \otimes_B \mathcal{H}(z) \simeq M \otimes_A \mathcal{H}(z) \simeq (M \otimes_A \kappa(\mathfrak{p})) \otimes_{\kappa(\mathfrak{p})} \mathcal{H}(z)$$

therefore prove the equivalence. The same argument at $\mathfrak{q}$ identifies the left side with membership in the support over B. The internal finite-module correspondence identifies analytic pullback with the indicated ordinary tensor product of finite modules, by Theorem 2.39.6 and finite presentation. This gives the sheaf assertion on affinoid domains and hence globally. Finally the analytic base change of the annihilator-defined closed subspace has point set exactly the inverse image of its original point set, by the universal property of a closed immersion. That point set is the support just computed; no equality of the two annihilator ideals is asserted or needed. □

Lemma 2.57.2 (A generic basis lifted from original sections)**.** *Let $T = \mathcal{M}(B) \to X = \mathcal{M}(A)$ be affinoid, let $x \in X$, and suppose that $C = B\widehat{\otimes}_A\mathcal{H}(x)$ is a nonzero integral strict affinoid algebra. For every finite B-module N, there are a nonnegative integer r and a B-linear map $u : B^r \to N$ such that u_C is injective and is an isomorphism on a nonempty Zariski-open subset of $\mathcal{M}(C)$. If $Q = \operatorname{Coker}(u)$ and $d = \dim C$, then $Q_C = 0$ when $d = 0$, and otherwise*

$$\dim \operatorname{Supp}(Q_C) \leq d - 1 \qquad \text{whenever } Q_C \neq 0.$$

The bijectivity locus of u_C is ordinarily dense in the fibre. The columns of u may be chosen from any fixed finite generating family of N.

Proof. Let $E = \operatorname{Frac}(C)$ and $r = \dim_E(N_C \otimes_C E)$. The images of the given generators span this vector space. Choose r of them which form a basis and use the corresponding original elements of N to define u; if $r = 0$, use the zero module as source. The map u_C becomes an isomorphism over E. Its kernel is a submodule of the free module C^r and becomes zero over E. Since

C is a domain, a submodule of C^r has no nonzero element killed by a nonzero element of C. Thus that kernel is zero.

Right exactness gives $Q_C = \operatorname{Coker}(u_C)$. This is a finite torsion C-module. Kill a finite generating family by nonzero elements and multiply those elements, obtaining $a \in C \setminus \{0\}$ with $aQ_C = 0$; when $Q_C = 0$, take $a = 1$. Thus u_C is an isomorphism on $D(a)$, which is nonempty and ordinarily dense by Lemma 2.52.1. If $d = 0$, the Noetherian domain C is a field and every finite torsion module is zero. If $d > 0$ and $Q_C \neq 0$, every prime in its support contains a. Prepending the zero prime to a chain of primes containing a shows that its Krull dimension is at most $d-1$. The algebra $C/\operatorname{Ann}(Q_C)$ is strict; its analytic dimension equals its Krull dimension by Proposition 3.21.1 and strict finite normalization. This proves the asserted fibre-dimension bound. □

Lemma 2.57.3 (Lifting fibre injectivity at an inner point)**.** *Let K have characteristic zero, let $T \to X$ be a morphism of good analytic spaces, and let $t \in \operatorname{Int}(T/X)$ with image x and nontrivially valued $\mathcal{H}(x)$. Let $u : L \to N$ be a morphism of coherent sheaves. If N_t is flat over $\mathcal{O}_{X,x}$ and the induced map on the analytic fibre is injective at t, then u_t is injective and $\operatorname{Coker}(u)_t$ is flat over $\mathcal{O}_{X,x}$.*

Proof. Put $R = \mathcal{O}_{X,x}$, $S = \mathcal{O}_{T,t}$, and let $\mathfrak{m}$ be the maximal ideal of R. Coherent pullback identifies the fibre stalk map with $u_t \otimes_S \mathcal{O}_{T_x,t}$. Corollary 2.56.7 says that $S/\mathfrak{m}S \to \mathcal{O}_{T_x,t}$ is faithfully flat. Thus fibre injectivity descends to

$$L_t/\mathfrak{m}L_t \longrightarrow N_t/\mathfrak{m}N_t \quad \text{injective.}$$

The ring S is Noetherian, L_t is a finite S-module, and N_t is flat over R. The local flat-map criterion [StacksAlg, Tag 00ME], applied with these modules, gives both conclusions. Formation of the coherent cokernel commutes with ordinary stalks, identifying its module with the quotient in that criterion. □

Theorem 2.57.4 (Coherent flatness at an inner point in characteristic zero)**.** *Let K be a complete non-Archimedean field of characteristic zero, let $f : Y \to X$ be a morphism of good K-analytic spaces, and let $\mathcal{F}$ be coherent on Y_G. Suppose that $y \in \operatorname{Int}(Y/X)$ and that $\mathcal{H}(f(y))$ is nontrivially valued. Then $\mathcal{F}$ is naively X-flat at y if and only if it is analytically X-flat there. In particular this holds for every complete nontrivially valued characteristic-zero ground field and every relative inner point.*

Proof. Analytic flatness implies naive flatness by the identity test. For the converse, put $x = f(y)$ and $F = \mathcal{H}(x)$. If the coherent stalk is zero, it is zero on an ordinary neighborhood: finitely many generators vanish after a common shrink. Every pullback is then zero near every point above y, proving analytic flatness. Assume henceforth that y belongs to the support.

Step 1: the induction parameter. Proposition 2.51.3 supplies an ordinary affinoid representative with strict F-fibre. Let Z be the closed support of $\mathcal{F}$ on this representative. It is inner over X at y, since a closed immersion is finite and inner and interior is preserved under composition. Regard $\mathcal{F}$ as a coherent sheaf on Z. This changes neither kind of flatness, by Theorem 2.35.17. The algebra of Z_x is a strict quotient, and Lemma 2.57.1 identifies its point set with the support of $\mathcal{F}_x$. We prove the assertion by induction on an upper bound $n \geq 0$ for this strict fibre dimension. Restrictions to smaller ordinary affinoid neighborhoods cannot increase the analytic dimension of the support, by Corollary 3.22.8; on a strict fibre this is its Krull dimension.

Step 2: a finite projection with integral fibre target. Write $d = \dim Z_x \leq n$. Theorem 2.51.4 gives, on ordinary neighborhoods, a finite map from Z to an affinoid ordinary neighborhood T of t in $\mathbb{A}^{d,\mathrm{an}}_X$. Separate its finite branches and retain the one through y, so that the fibre over t is the singleton $\{y\}$. The point t is inner over X, since it is in the ordinary interior of T in relative affine space. Shrink T by Proposition 2.51.3 so that T_x is strict. Its ordinary local rings are regular by Theorem 2.35.23; Lemma 2.52.2 therefore decomposes its algebra into integral factors. Lemma 2.56.4 retains the factor through t by an ordinary domain over the original field. Pull the finite map back at each restriction. We obtain a finite map $\pi : Z \to T$ with singleton distinguished fibre, whose fibre algebra $C = \mathcal{O}(T_x)$ is integral and strict and satisfies $\dim C \leq d$ by Lemma 2.56.3. All distinguished ordinary germs remain unchanged.

Step 3: the quotient has smaller fibre support. Put $N = \pi_* \mathcal{F}$. It is coherent, and N_t is naively X-flat by Theorem 2.35.17. The finite-module description of N and Lemma 2.57.2 give a morphism $u : L \to N$ with L finite free, whose fibre map is injective and generically bijective on T_x. Set $Q = \mathrm{Coker}(u)$. Lemma 2.57.3 makes u_t injective and Q_t naively X-flat. The strict fibre support of Q is either empty or has dimension at most $\dim C - 1 \leq n - 1$. If $Q_t = 0$, the initial vanishing argument makes it analytically flat at t. Otherwise apply the induction hypothesis to Q on

its closed support in T: that support is still inner over X, and its fibre has the stated strictly smaller bound. Thus Q is analytically X-flat at t. This also covers the induction start $n = 0$, when the generic-basis lemma forces $Q_x = 0$ and Nakayama gives $Q_t = 0$.

Step 4: recover the original sheaf. The space T is quasi-smooth over X, and the bijectivity locus of u_x is ordinarily dense. Corollary 2.52.6, applied to the injectivity of u_t and the analytic flatness of Q_t, proves analytic X-flatness of N at t. The finite-pushforward equivalence then gives analytic flatness of $\mathcal{F}$ at y. That equivalence and the generic-fibre criterion include every good analytic base change over every complete extension; thus the conclusion is exactly the universal analytic-flatness condition. Finally a completed residue field over a nontrivially valued ground field is automatically nontrivially valued, proving the last sentence. □

References. The support statement is finite-module algebra. The generic-basis and dimension-induction steps reorganize the mechanism of analytic dévissage in [Duc18, §8.2 and Theorem 8.3.4]. The ordinary-to-analytic fibre comparison is proved internally in Corollary 2.56.7; the sole additional algebraic criterion is [StacksAlg, Tag 00ME].

2.58 Gauss lifting and flatness in trivial valuation

Lemma 2.58.1 (The ordinary germ at a free point of a trivial-field line)**.** *Let F be trivially valued and let $0 < r < 1$. At the radius-r Gauss point ξ_r of $\mathbb{A}^{1,\mathrm{an}}_F$, evaluation identifies the ordinary local ring with the underlying field of $F_r = F((T))$.*

Proof. Lemma 2.55.1, with base $\mathcal{M}(F)$, shows that annuli $a \le |T| \le b$ with $0 < a < r < b < 1$ form an ordinary affinoid neighborhood basis. Their analytic functions have unique Laurent expansions $\sum_{j\in\mathbf{Z}} c_j T^j$, with $|c_j|b^j \to 0$ as $j \to +\infty$ and $|c_j|a^j \to 0$ as $j \to -\infty$. This is the Laurent coefficient normal form of Theorem 2.13.2, applied to the annulus algebra. Since every nonzero coefficient has absolute value one, the positive coefficients are arbitrary and only finitely many negative coefficients can be nonzero. Thus the underlying algebra is exactly $F((T))$. Restriction between these annuli preserves every coefficient and is the identity on that field. The filtered colimit giving the ordinary stalk is therefore $F((T))$ as well. Gauss evaluation gives its original radius-r absolute value and is injective. □

Lemma 2.58.2 (Naive flatness lifts to an auxiliary Gauss point)**.** *Let $f : Y \to X$ be a morphism of good spaces over a complete characteristic-zero field, and let $y \in Y$ with image x have trivially valued $F = \mathcal{H}(x)$. Let $\mathcal{F}$ be coherent and naively X-flat at y. For $0 < r < 1$, form $X^* = X \times_K \mathbb{A}^{1,\mathrm{an}}_K$ and $Y^* = Y \times_K \mathbb{A}^{1,\mathrm{an}}_K$, and let x^*, y^* be their radius-r Gauss points above x, y. Then the pullback $\mathcal{F}^*$ is naively X^*-flat at y^*.*

Proof. Write $R = \mathcal{O}_{X,x}$, $S = \mathcal{O}_{Y,y}$, $R^* = \mathcal{O}_{X^*,x^*}$, $S^* = \mathcal{O}_{Y^*,y^*}$ and $M = \mathcal{F}_y$. The two affine-line projections are smooth, hence their ordinary local maps $R \to R^*$ and $S \to S^*$ are flat. Coherent pullback gives $M^* = \mathcal{F}^*_{y^*} = M \otimes_S S^*$. Since M is R-flat and S^* is S-flat, M^* is R-flat: tensoring an injection of R-modules with M gives an injection of S-modules, which remains injective after tensoring with S^*.

Let $\mathfrak{m}$ be the maximal ideal of R and put $I = \mathfrak{m}R^*$. Apply Corollary 2.56.10 to the inner smooth projection $X^* \to X$ at x^*. Its analytic fibre is $\mathbb{A}^{1,\mathrm{an}}_F$, so the preceding lemma gives a faithfully flat local map

$$R^*/I \longrightarrow F((T)).$$

Its source is a field. Indeed its maximal ideal extends to the zero ideal of the target field, and faithful flatness contracts that zero ideal to zero. Hence M^*/IM^* is flat over R^*/I.

There is also no relative Tor obstruction. Choose a free R-resolution of $R/\mathfrak{m}$. Since R^* is R-flat, its tensor with R^* is a free R^*-resolution of R^*/I. Tensoring that resolution with M^* is the same as tensoring the original resolution over R with M^*, and the latter has zero positive homology by R-flatness. Thus

$$\mathrm{Tor}_1^{R^*}(M^*, R^*/I) = 0.$$

The local rings R^*, S^* are Noetherian and M^* is finite over S^*. The local criterion [StacksAlg, Tag 00ML] now makes M^* flat over R^*, proving the assertion. □

Lemma 2.58.3 (Analytic flatness descends from a Gauss lift)**.** *Let $f : Y \to X$ be a morphism of good analytic spaces over any complete field, let $\mathcal{F}$ be coherent, and let $r > 0$. Form the simultaneous affine-line base change $Y^* \to X^*$ and let y^* be the radius-r Gauss point above y. Then $\mathcal{F}$ is analytically X-flat at y if and only if its pullback $\mathcal{F}^*$ is analytically X^*-flat at y^*.*

Proof. The forward implication is a base-change test in the definition. Conversely, fix an arbitrary complete extension L/K, a good L-analytic test $X_1 \to X$, and a point $y_1 \in Y_1 = Y \times_X X_1$ above y, with image x_1. Form $X_1^* = X_1 \times_L \mathbb{A}_L^{1,\mathrm{an}}$ and $Y_1^* = Y_1 \times_L \mathbb{A}_L^{1,\mathrm{an}}$. The radius-$r$ Gauss point y_1^* maps to y^*: the completed-residue embeddings are isometric and the coefficient-wise Gauss norm restricts to the same norm on every polynomial over the smaller field. Its base point x_1^* likewise maps to the chosen Gauss point in X^*.

The assumed analytic flatness at y^* therefore makes the pulled-back coherent stalk at y_1^* flat over $\mathcal{O}_{X_1^*,x_1^*}$. The horizontal maps in the local square

$$\begin{array}{ccc} \mathcal{O}_{X_1,x_1} & \longrightarrow & \mathcal{O}_{X_1^*,x_1^*} \\ \downarrow & & \downarrow \\ \mathcal{O}_{Y_1,y_1} & \longrightarrow & \mathcal{O}_{Y_1^*,y_1^*} \end{array}$$

are flat local maps by Theorem 2.48.3: take relative closed discs of radius greater than r, whose ordinary interiors contain the Gauss points and have the same ordinary stalks as the affine-line products. In particular the lower map is faithfully flat. Coherent pullback is ordinary tensor product on these stalks. Lemma 2.35.6 proves naive flatness at y_1. Since the complete extension, the good test and the point above y were arbitrary, this is analytic flatness at y. □

Theorem 2.58.4 (Inner coherent flatness in arbitrary characteristic-zero valuation)**.** *Let K be any complete field of characteristic zero, let $f : Y \to X$ be a morphism of good analytic spaces, and let $\mathcal{F}$ be coherent. At every $y \in \mathrm{Int}(Y/X)$, naive X-flatness of $\mathcal{F}$ is equivalent to analytic X-flatness.*

Proof. Only the implication from naive to analytic flatness requires proof. Put $x = f(y)$. If $\mathcal{H}(x)$ is nontrivially valued, this is Theorem 2.57.4. Otherwise choose $0 < r < 1$ and make the simultaneous affine-line base change. Lemma 2.58.2 gives naive flatness of the pullback at y^*. The point y^* is inner over X^* because interior is stable under base change, and $\mathcal{H}(x^*) = \mathcal{H}(x)_r$ is nontrivially valued. The same nontrivially valued theorem makes the pullback analytically flat at y^*. Lemma 2.58.3 then descends analytic flatness to y. □

References. These Gauss-point arguments extend the internally reconstructed characteristic-zero case of [Duc18, Theorem 8.3.4] to trivial valuations. The local

flatness criterion is the exact Stacks statement [StacksAlg, Tag 00ML]; the Tor comparison is proved by the explicit free-resolution calculation above.

2.59 Smooth ambient spaces for quasi-smooth germs

Lemma 2.59.1 (Residue fields of rank-zero quasi-smooth morphisms)**.** *Let $f : Y \to X$ be a morphism of good analytic spaces over any complete field. If f is quasi-smooth at y and $\mathrm{rk}_y \Omega_{Y/X} = 0$, then $\mathcal{H}(y)/\mathcal{H}(f(y))$ is finite separable.*

Proof. Put $x = f(y)$ and $F = \mathcal{H}(x)$. Quasi-smoothness and its differential rank survive passage to the analytic fibre. Lemma 2.50.1 therefore gives an ordinary open affinoid neighborhood of y in Y_x whose algebra is finite-dimensional over F. Its local factor D at y is the ordinary local ring of this finite space. By Theorem 2.35.14 that ring is regular. A zero-dimensional regular local ring is a field, so $D = E = \mathcal{H}(y)$, and E/F is finite.

For every finite field extension L/F, the extending absolute value makes L complete, and finite base change identifies the corresponding algebra with $E \otimes_F L$. All its local factors are regular by geometric regularity of the analytic fibre, hence fields. Thus $E \otimes_F L$ is reduced. If some $a \in E$ were inseparable over F, its minimal polynomial would acquire a repeated root over a finite splitting field L/F. The algebra $F[a] \otimes_F L$ would then contain a nonzero nilpotent. Its injection into $E \otimes_F L$, obtained by tensoring the injection $F[a] \hookrightarrow E$ with the field L, contradicts reducedness. Every element of E is therefore separable over F. □

Theorem 2.59.2 (An étale-domain factorization in rank zero)**.** *Let $f : Y \to X$ be a morphism of good analytic spaces, quasi-smooth of differential rank zero at y, and put $x = f(y)$. There are an ordinary affinoid neighborhood U of x, a finite étale morphism $T \to U$, and an ordinary affinoid neighborhood V of y such that $f|_V$ factors as an affinoid-domain inclusion $V \hookrightarrow T$ followed by $T \to U \hookrightarrow X$.*

Proof. Work on good affinoid ordinary neighborhoods on which the rank-zero Jacobian presentation holds. Write $F = \mathcal{H}(x)$ and $E = \mathcal{H}(y)$. By the preceding lemma E/F is finite separable. Theorem 2.54.3 realizes it

by a finite étale morphism $T \to U$ with a unique point t above x and a specified identification $\mathcal{H}(t) = E$ over F. In $Y \times_X T$ the multiplication map $E \otimes_F E \to E$ defines a point y' above (y, t) with completed residue field E. The projection $Y \times_X T \to Y$ is étale. Corollary 2.54.4 identifies its germ at y' with the germ of Y at y. Thus, after restricting ordinary neighborhoods, the second projection gives a lift $g : Y \to T$ whose completed-residue-field map at y is the identity of E. This lifted map is quasi-smooth of rank zero: before the local identification it is the base change of f, and the property is stable under such base change and ordinary restriction.

Consider the germ reductions of g. Both source and target germs are good, so their reductions are affine open subspaces of the same graded valuation space $\mathscr{P}_{\widetilde{K}^{\mathrm{gr}}}(\widetilde{E}^{\mathrm{gr}})$. Their structural maps are open embeddings, by Theorem 3.19.19 and the good-germ construction. Compatibility with the identity on $\widetilde{E}^{\mathrm{gr}}$ identifies $\mathscr{R}(Y, y) \to \mathscr{R}(T, t)$ with an inclusion of a nonempty quasi-compact affine open subspace. Theorem 3.19.8 supplies a subdomain germ (W, t) of (T, t) whose reduction is that open subspace. Its reduction is affine, so (W, t) is good by Theorem 3.19.19.

The lift factors through this actual subdomain germ. Indeed the inverse-image domain $Y \times_T W$ has the same reduction as (Y, y), by the inverse-image formula of Theorem 3.19.9. The order-reflecting domain classification therefore makes it the entire source germ. Restricting an ordinary neighborhood of y gives $g : Y \to W$. Its reduction is an isomorphism over the same graded field, hence proper. Theorem 2.46.1 makes y an interior point of this map. Also $Y \to W$ remains quasi-smooth of rank zero, since it is the preceding map base-changed by the domain inclusion $W \hookrightarrow T$.

The fibre point y is isolated by Lemma 2.50.1; the ordinary local map is flat and its relative differentials vanish by Theorem 2.35.14. Proposition 2.49.2 consequently makes $Y \to W$ étale near y. Its completed residue fields agree, because domain inclusions preserve completed residue fields. Corollary 2.54.4 now gives an isomorphism of ordinary neighborhoods in Y and W. Choose an affinoid ordinary neighborhood V of y within this isomorphism. Its image is an affinoid domain of W, and hence of T by transitivity of analytic domains. All the choices were made over ordinary restrictions of U; the original $T \to U$ remains finite étale. This is the required factorization. □

Lemma 2.59.3 (Independent functions as relative coordinates)**.** *Let $Y \to X$ be quasi-smooth of differential rank d at a point y of good analytic spaces. If analytic germs $f_1, \dots, f_\ell$ have linearly independent differentials in $\Omega_{Y/X} \otimes \mathcal{H}(y)$, then $(f, f_1, \dots, f_\ell) : Y \to \mathbb{A}_X^{\ell,\mathrm{an}}$ is quasi-smooth of differential rank $d - \ell$ at y.*

Proof. Use the ordinary-neighborhood presentation in Theorem 2.35.14: after restrictions, Y is cut out by r functions $g_1, \dots, g_r$ in an affinoid domain D of $\mathbb{A}_X^n$, with independent Jacobian rows and $n - r = d$. Represent the f_i on this Y and lift them to the affinoid algebra of D, using the surjection defining the closed immersion. Choose an affinoid polydisc neighborhood P of their values in $\mathbb{A}_X^\ell$. The graph of the new map is cut out, in the inverse image of D inside $\mathbb{A}_P^n$, by

$$g_1, \dots, g_r, \quad f_1 - Z_1, \dots, f_\ell - Z_\ell.$$

The original r Jacobian rows together with the ℓ rows df_i are linearly independent at y: this is precisely independence of the latter classes modulo the relation rows. Some $(r+\ell)$-minor is therefore nonzero at y, and it is invertible on an ordinary Laurent neighborhood. The graph ideal is generated by the displayed functions, so the Jacobian criterion makes the graph quasi-smooth over P with rank $n - r - \ell = d - \ell$. As P is an ordinary neighborhood of the target point, this is the asserted property over $\mathbb{A}_X^\ell$. □

Theorem 2.59.4 (Smooth envelopes of good quasi-smooth germs)**.** *Let $f : Y \to X$ be a morphism of good analytic spaces over any complete field. It is quasi-smooth at y if and only if an ordinary affinoid neighborhood V of y is, over an ordinary neighborhood of $x = f(y)$, an affinoid domain of a smooth analytic X-space. If the differential rank at y is d, the smooth ambient space may be chosen of relative dimension d.*

Proof. Assume first that f is quasi-smooth at y. On an ordinary affinoid representative, the coordinate differentials generate the finite module $\Omega_{Y/X}$. Choose d analytic functions whose differentials form a basis after tensoring with $\mathcal{H}(y)$. Lemma 2.59.3 makes the resulting map $g : Y \to \mathbb{A}_X^{d,\mathrm{an}}$ quasi-smooth of rank zero. Theorem 2.59.2 gives an ordinary affinoid neighborhood V of y, an affinoid ordinary neighborhood U of $g(y)$ in $\mathbb{A}_X^d$, and a finite étale map $T \to U$ in which V is an affinoid domain. Replace U by its ordinary interior near $g(y)$, replace T by its inverse image, and shrink V within the

resulting source neighborhood. Then $T \to U \hookrightarrow \mathbb{A}^d_X \to X$ is smooth of relative dimension d, while V remains an affinoid domain of T.

Conversely, smooth maps are quasi-smooth by Theorem 2.50.3, and restriction to an analytic domain preserves the Jacobian presentation and quasi-smoothness. Thus $V \to X$ is quasi-smooth. Since V is an ordinary neighborhood of y in Y, the same holds for the original germ. The differential rank is unchanged by domain restriction, giving the dimension assertion. □

References. The coordinate lemma and the smooth-envelope theorem reconstruct [Duc18, Proposition 5.4.5 and Theorem 5.4.6]. The rank-zero factorization uses the germ-reduction mechanism of the analytic Zariski main theorem, but here finite separability allows the full completed residue field to be realized by an étale germ. All domain and finite-map steps are proved in the preceding chapters.

2.60 A graph at its exact convergence radius

Throughout this section K is complete and nontrivially valued. No restriction on its characteristic is imposed. The argument follows the geometric continuation proof of [Duc18, 4.4.2–4.4.6]; the curve local-ring step is derived directly from the strict ordinary comparison already established in this book.

Lemma 2.60.1 (Reduced strict domains and ordinary stalks)**.** *Let A be a reduced strict K-affinoid algebra. Every ordinary local ring of $\mathcal{M}(A)$ is reduced. Every strict affinoid domain in $\mathcal{M}(A)$ has a reduced affinoid algebra.*

Proof. Fix x, set $D = A_{\ker(x)}$ and $R = \mathcal{O}_{\mathcal{M}(A),x}$. Theorem 2.35.42 makes $D \to R$ a regular, faithfully flat homomorphism. The reduced Noetherian ring D embeds into the product of the fraction fields of its finitely many minimal-prime quotients. Tensoring this injection with the flat D-module R embeds R into the product of the corresponding algebraic fibres. These fibres are geometrically regular by the cited comparison, hence reduced: every regular local ring is a domain by [StacksAlg, Tag 00NP]. Therefore R is reduced.

Let $V = \mathcal{M}(B)$ be a strict affinoid domain and let $\mathfrak{m}$ be any maximal ideal of B. Theorem 1.12.3 realizes it as a rigid point v. Its residue field is finite over K, so its residue field in degree one is finite over $\widetilde{K}$ by Lemma 3.24.2. Its canonical reduction in $\widetilde{B}$ is consequently a closed point, and Theorem 3.25.1

makes v an absolute inner point of V. Theorem 2.44.1 implies that it is inner for $V \hookrightarrow \mathcal{M}(A)$. Proposition 2.44.5 therefore makes V an ordinary neighborhood of v in $\mathcal{M}(A)$. Thus $\mathcal{O}_{V,v}$ is the reduced local ring just considered. The map $B_{\mathfrak{m}} \to \mathcal{O}_{V,v}$ is faithfully flat by Proposition 2.33.16 and domain flatness. It is injective, so $B_{\mathfrak{m}}$ is reduced. This holds for all maximal ideals of B, and hence B is reduced. □

Lemma 2.60.2 (Nonrigid ordinary germs on strict curves)**.** *Let T be a reduced good strictly K-analytic space of dimension at most one. At every nonrigid point t, $\mathcal{O}_{T,t}$ is a field.*

Proof. Choose a reduced strict affinoid neighborhood of t. By Lemmas 2.35.25 and 2.60.1, every germ at t can be represented on a reduced strict affinoid neighborhood $V = \mathcal{M}(B)$. By Corollary 3.22.8, $\dim B \leq 1$. Its point kernel $\mathfrak{p} = \ker(t)$ cannot be maximal, since every strict affinoid maximal point has finite residue field over K, whereas t is not rigid. It is therefore a minimal prime: a strictly smaller prime and a strictly larger maximal prime would give a chain of length two. The localization $B_{\mathfrak{p}}$ is a reduced ring with exactly one prime, and hence is a field.

Let a germ h vanish at t, and represent it by $b \in B$ on such a neighborhood. Then $b \in \mathfrak{p}$ and $b/1 = 0$ in the field $B_{\mathfrak{p}}$. There exists $c \notin \mathfrak{p}$ with $cb = 0$. The germ of c is a unit, since $c(t) \neq 0$, so $h = 0$. Thus the maximal ideal of $\mathcal{O}_{T,t}$ is zero, as required. □

Lemma 2.60.3 (An outer collar at the Gauss point)**.** *Let $D = \{|T| \leq 1\} \subset \mathbb{A}^{1,\mathrm{an}}_K$ and let η be its Gauss point. For every ordinary open neighborhood U of η, there is $R > 1$ with $D_R = \{|T| \leq R\} \subset D \cup U$. If f is analytic on D and an analytic function on U agrees with it on $U \cap D$, then f extends analytically to some such D_R.*

Proof. Choose $R_0 > 1$ and work inside the closed disc D_{R_0}. Polynomials are uniformly dense in its affinoid algebra. The evaluation topology and this uniform approximation imply that finitely many strict inequalities involving polynomial absolute values define a neighborhood of η contained in $U \cap D_{R_0}$. For a nonzero polynomial $P = \sum_{i=0}^{m} a_i T^i$, put $M = \max_i |a_i|$ and choose the largest index j with $|a_j| = M$. For every $s > 1$ sufficiently close to one,

$$|a_i| s^i < M s^j \qquad (i \neq j).$$

Consequently every analytic point z with $|T(z)| = s$ has $|P(z)| = Ms^j$. This tends to $M = |P(\eta)|$ as $s \downarrow 1$, uniformly over all such z. Apply this observation to the finitely many polynomials in the chosen inequalities. For some $1 < R < R_0$, all points with $1 < |T| \le R$ belong to U, proving the first assertion.

The disc D is a pure one-dimensional strict affinoid, and its Shilov boundary is $\{\eta\}$. By Theorem 3.25.4, its absolute boundary is also $\{\eta\}$. The affine line is boundaryless over K: a disc of radius strictly greater than $|T(z)|$ is an inner presentation at each of its points z. Theorem 2.44.1 and Proposition 2.44.5 show that every point of $D \setminus \{\eta\}$ has an ordinary neighborhood in the affine line contained in D. Together with U this proves that the two analytic domains D and U form an admissible covering of $D \cup U$. The structure sheaf glues the given functions, and restriction to D_R gives the required extension. □

Theorem 2.60.4 (The ambient germ of an exact-radius graph)**.** *Let $f(T) = \sum_{n\ge 0} a_n T^n \in K\langle T\rangle$ have convergence radius exactly one. Put $D = \mathcal{M}(K\langle T\rangle)$, let η be its Gauss point, and set $x = (\eta, f(\eta))$ in $\mathbb{A}^{2,\mathrm{an}}_K$. Then $\mathcal{O}_{\mathbb{A}^{2,\mathrm{an}}_K,x}$ is a field.*

Proof. Step 1: a hypothetical vanishing germ gives a curve. Write p for the first projection, put $X = p^{-1}(D)$, and let $Z \subset X$ be the graph of f, with isomorphism $p|_Z : Z \simeq D$. Its point x has $\mathcal{H}(x) = \mathcal{H}(\eta)$: the evaluated first coordinate already has dense fraction field in $\mathcal{H}(\eta)$, and the second coordinate belongs to that field. Thus $d_K(x) = 1$, by Lemma 3.22.4, and x is not rigid.

Suppose a nonzero ordinary germ h at x vanishes there. Choose a strict affinoid neighborhood $V = \mathcal{M}(A)$ representing it. By Theorem 2.35.23 and Lemma 2.52.2, we may retain the open-and-closed integral component containing x. The element $h \in A$ remains nonzero. This domain has dimension at most two, by Corollary 3.22.8 applied inside a sufficiently large strict bidisc. Therefore the reduced closed subspace $W = \mathcal{M}(A/\sqrt{(h)})$ has dimension at most one: every prime containing h is nonzero, and a chain in that quotient can be preceded by the zero prime of A. It contains x, so its dimension at x is one by the Abhyankar inequality.

The pullback of h to the curve D vanishes at its nonrigid point η. By Lemma 2.60.2 its germ there is zero. The same is true of each generator of $\sqrt{(h)}$, since its suitable power is a multiple of h and the graph is reduced.

After shrinking V, we therefore have a closed immersion $Z \cap V \subset W \cap X$. The latter is a reduced strict domain of the curve W, by Lemma 2.60.1; both its ordinary local ring at x and that of Z are fields by the curve lemma. Their quotient map is an isomorphism. The defining coherent ideal consequently has zero stalk at x, so its finitely many local generators vanish on a smaller ordinary neighborhood in X. Restricting V to a strict ambient neighborhood with that intersection gives

$$W \cap X = Z \cap V$$

as closed analytic subspaces of $V \cap X$.

Step 2: the projection is locally finite of rank one. For every point w of W lying above η, the residue-field tower formula and the affinoid dimension bound give

$$1 + d_{\mathcal{H}(\eta)}(w) = d_K(w) \leq 1.$$

Here the tower formula is Lemma 3.22.2 and the last inequality is Theorem 3.22.7. The fibre W_η is a strict $\mathcal{H}(\eta)$-affinoid: in a strict presentation of W, scalar extension and the equation fixing its first coordinate give such a presentation. The same maximum formula makes it zero-dimensional. Theorem 1.12.2 and Proposition 3.21.1 then make its algebra finite-dimensional over $\mathcal{H}(\eta)$, so its analytic point set is finite and discrete by Lemma 2.35.1. In particular x is isolated in this fibre.

The projection $\mathbb{A}^{2,\mathrm{an}}_K \to \mathbb{A}^{1,\mathrm{an}}_K$ is boundaryless: near each point use a relative disc in the second coordinate with radius strictly larger than its value. The inclusion of V is inner at x because V is an ordinary neighborhood, and the closed immersion of W is finite and boundaryless. The forward composition rule applied on these affinoid neighborhoods therefore makes $p|_W$ inner at x. Theorem 2.49.1 gives ordinary neighborhoods on which $p|_W$ is finite. Separate the branch through x and shrink the target about η as in Lemma 2.35.1. The ordinary target ring at η is a field by the curve lemma. The finite direct image is therefore flat at that point; Lemma 2.35.16 makes it free of some positive rank m after shrinking. We obtain affinoid neighborhoods W_0 of x in W and U of η in the affine line for which $p : W_0 \to U$ is finite free of rank m.

Base change to $U \cap D$ gives a finite free map of the same rank from $W_0 \cap X$. This source is an analytic domain of Z, by the closed-subspace equality proved above. Under $Z \simeq D$ it maps isomorphically onto its image. A positive-rank finite free algebra has nonzero fibres, so the image is all of

$U \cap D$. Consequently this base change is an isomorphism and $m = 1$. A unital finite free algebra of rank one is the base algebra: its unit is a basis after every residue-field specialization, so the unit map is an isomorphism by Nakayama and finite presentation. Thus $p : W_0 \simeq U$.

Step 3: contradict the exact radius. The second coordinate of the inverse section $U \to W_0$ is an analytic function agreeing with f on $U \cap D$. Restrict to the ordinary interior of U at η and apply Lemma 2.60.3. This extends f to D_R for some $R > 1$. Restriction $K\{R^{-1}T\} \to K\langle T\rangle$ preserves the power-series coefficients, so such an extension implies $|a_n|R^n \to 0$, contrary to the convergence-radius hypothesis. Hence no nonzero germ at x can vanish there. The maximal ideal of the ordinary local ring is zero, proving the theorem. □

Corollary 2.60.5 (Domain base change can destroy naive flatness)**.** *Choose* $\pi \in K$ *with* $0 < |\pi| < 1$ *and set*

$$f(T) = \sum_{n \geq 0} \pi^{\lfloor \sqrt{n} \rfloor} T^n, \qquad \phi : D \longrightarrow \mathbb{A}^{2,\mathrm{an}}_K, \qquad t \longmapsto (t, f(t)).$$

The morphism ϕ *is naively flat at the unit Gauss point* η*. Its base change by the analytic domain* $X = \{|T_1| \leq 1\}$ *is not naively flat at that point.*

Proof. The coefficients tend to zero, whereas their nth-root absolute values tend to one. Thus $f \in K\langle T\rangle$ has convergence radius exactly one. Theorem 2.60.4 makes the ordinary target local ring at $x = \phi(\eta)$ a field. Every module over a field is flat, so ϕ is naively flat at η.

Because ϕ factors through the domain X, its base change to X is its closed graph immersion, with equation $g = T_2 - f(T_1)$. The ordinary local ring $R = \mathcal{O}_{X,x}$ is a domain by Theorem 2.35.23 and [StacksAlg, Tag 00NP], using an affinoid neighborhood of x in X. The germ g is nonzero: differentiation in the second coordinate gives $\partial g/\partial T_2 = 1$. This derivation on convergent series restricts to the germ by Theorem 2.41.3 and Theorem 2.41.5. It vanishes at x, so $R/(g)$ is nonzero. Tensoring the injective multiplication map $R \xrightarrow{g} R$ with $R/(g)$ gives the zero map on the nonzero module $R/(g)$. This quotient is therefore not R-flat. The graph local ring is exactly $R/(g)$ by the coherent closed-subspace restriction formula, proving the assertion. □

2.61 Flat comparison with analytic fibres

Lemma 2.61.1 (Finite flat maps are ordinary-open)**.** *A finite flat morphism between good analytic spaces is open for the ordinary topology.*

Proof. Let $f : Y \to X$ be finite flat, let $y \in O$ with O ordinary open in Y, and put $x = f(y)$. Lemma 2.35.1 gives disjoint clopen branches over an ordinary neighborhood U of x, with the branch through y contained in O: in its construction choose the separating neighborhood of y inside O before removing the compact image of the complement of the separating neighborhoods. Choose an affinoid ordinary neighborhood $V = \mathcal{M}(A)$ of x in U, and write $f^{-1}(V) = \mathcal{M}(B)$. Set $R = \mathcal{O}_{X,x}$. By the same lemma,

$$B \otimes_A R \simeq \prod_{f(y_i)=x} \mathcal{O}_{Y,y_i}$$

as R-algebras. The identity base change in the definition of analytic flatness gives R-flatness at each y_i. The finite product is therefore flat over R. The affinoid algebra A is Noetherian by Theorem 1.18.2, so the finite A-module B has a finite presentation. Tensoring the presentation with R gives a finite presentation of $B \otimes_A R$. Since R is local, [StacksAlg, Tag 00NX(1) and (5)] makes $B \otimes_A R$ free over R.

Fix such a finite A-presentation of B. Proposition 2.33.16 expresses R as the filtered colimit of the actual algebras A_W of rational ordinary neighborhoods of x in V. Proposition 2.33.19 identifies the finite inverse-image algebra there with $B_W = B \otimes_A A_W$. Represent a basis of $B \otimes_A R$ in one B_W, and represent the inverse map on the finitely many generators of the chosen presentation over the same A_W, after shrinking. The finitely many relation equations hold after a further shrinking, so the inverse data define an A_W-linear map. Both inverse identities on the two finite lists of generators also hold after shrinking. Thus B_W is free over A_W.

The branch through y gives an idempotent factor of B_W by Lemma 2.35.1. This factor is finite flat over A_W, with its canonical finite-module topology. Lemma 2.44.2 makes its image open and closed in W. The image contains x and is contained in $f(O)$. Intersecting it with the ordinary interior of W gives an ordinary neighborhood of x in X contained in $f(O)$. □

Proposition 2.61.2 (Spreading a closed neighborhood of a generic fibre)**.** *Let $Y \to X$ be a morphism of good analytic spaces over any complete field, smooth at y, and put $x = f(y)$. If $\mathcal{O}_{X,x}$ is Artinian, then every Zariski-closed subset of Y containing an ordinary neighborhood of y in Y_x contains an ordinary neighborhood of y in Y.*

Proof. First reduce the base. On an ordinary affinoid neighborhood $\mathcal{M}(A)$ of x, put $\mathfrak{p} = \ker(x)$, $D = A_{\mathfrak{p}}$ and $R = \mathcal{O}_{X,x}$. The faithfully flat map $D \to R$ and the Artinian hypothesis imply that D is zero-dimensional. Its reduction is a field. If N is the nilradical of A, Theorem 2.35.45, after quotienting by the nilradical of D, makes R/NR geometrically regular over that field. It is also local Artinian, hence is a field. The nilpotent closed base change $X_{\mathrm{red}} \to X$ and its pullback to Y are homeomorphisms and do not alter the analytic fibre. Thus it suffices to treat the case in which R is a field.

Corollary 2.35.47 gives a normal ordinary neighborhood of x. Normality ascends along the smooth map by smooth normality ascent. Choose an ordinary affinoid neighborhood Y_0 of y and its normal integral component through y, using Lemma 2.56.1. Write the closed subset there as $V(h_1, \ldots, h_s)$. The identity principle reduces the assertion to finding a nonempty ordinary open subset of Y_0 on which every h_j vanishes. Restrict inside the ordinary interior of Y_0 and the locus where the relative dimension of the smooth map is constant.

Theorem 2.55.7 applies to the nonempty ordinary fibre open set in the hypothesis. It gives a finite flat base change $X' \to U$, with a unique point x' above x and reduced ordinary closed fibre there, and a nonempty ordinary open product patch in $Y' = Y \times_X X'$. Its factors are an ordinary base neighborhood U' of x', an open annulus D or a point, and an open polydisc Δ; its fibre at x' lies in the given closed subset. The ordinary ring $R' = \mathcal{O}_{X',x'}$ is finite over the field R, by finite branch separation and the uniqueness of the fibre point. Its ordinary closed fibre is itself, and is reduced. Thus R' is a field. Apply Corollary 2.35.47 at x' and then Lemma 2.56.1 to choose a normal integral affinoid ordinary neighborhood $\mathcal{M}(A') \subset U'$. The map

$$A' \longrightarrow R' \longrightarrow \mathcal{H}(x')$$

is injective: its first map is flat and nonzero with integral source, and its second is the evaluation of a field germ.

Restrict each h_j to $\mathcal{M}(A') \times_K D \times_K \Delta$. The weighted Laurent and Tate expansions on closed subannuli and closed subpolydiscs give unique coefficients in A'. They agree under all restrictions, so we may write

$$h_j = \sum_{(a,I)\in \mathbf{Z}^m \times \mathbf{N}^{n-m}} c_{a,I,j} S^a T^I, \qquad m \in \{0,1\}.$$

The fibre function vanishes at every point of $D_{\mathcal{H}(x')} \times \Delta_{\mathcal{H}(x')}$. Evaluation at its Gauss points, with radii in the corresponding intervals, makes every coefficient $c_{a,I,j}(x')$ zero. The displayed injection therefore makes all coefficients zero in A' simultaneously. Every h_j vanishes on the whole product, and hence on a nonempty ordinary open subset of Y'. By Lemma 2.61.1, its image in Y_0 is nonempty and ordinary-open. The functions vanish on this image, and the identity principle makes them zero throughout Y_0. This proves the assertion. □

Lemma 2.61.3 (A nonzero ideal lowers the strict fibre dimension)**.** *Let X be good over any complete field and let V be an ordinary affinoid neighborhood of t in $\mathbb{A}^{d,\mathrm{an}}_X$. Put $x = f(t)$ and assume that $\mathcal{O}_{X,x}$ is a field and $F = \mathcal{H}(x)$ is nontrivially valued. For every nonzero proper ideal $I \subset \mathcal{O}_{V,t}$, its closed-subspace germ has an inner ordinary affinoid representative over X with strict fibre algebra of Krull dimension at most $d-1$. Its ordinary and fibre stalks are respectively*

$$\mathcal{O}_{V,t}/I, \qquad \mathcal{O}_{V_x,t}/I\mathcal{O}_{V_x,t}.$$

Proof. Represent a finite generating family of I and a nonzero $h \in I$ on an ordinary affinoid neighborhood. Its fibre germ is nonzero: otherwise its zero set contains an ordinary fibre neighborhood, and Proposition 2.61.2, applied on the ordinary smooth interior of V, would then give an ordinary neighborhood on which h vanishes at every point. By smooth normality ascent and Corollary 2.35.47, take a reduced ordinary affinoid neighborhood of t. The spectral-radius and nilradical formulas in Proposition 1.20.3 and Proposition 3.21.4 make the germ of h zero. Proposition 2.51.3 gives an ordinary representative with strict fibre C. This fibre is an affinoid domain of $\mathbb{A}^d_F$, so Theorem 2.35.23 and Lemma 2.52.2 make C a finite product of regular domains. Lemma 2.56.4 retains the factor C_0 through t over the original ground field. The image of h in C_0 is nonzero, and Lemma 2.56.3 gives $\dim C_0 \le d$.

The chosen generators define a closed subspace with strict fibre algebra C_0/I_C. Every prime chain in this quotient lifts to a chain containing the

nonzero ideal I_C in the domain C_0; adjoining its zero prime increases the length by one. Thus the quotient has dimension at most $d-1$. Finite closed-subspace base change gives the two stalk identifications. The closed immersion is inner, and the chosen ambient ordinary neighborhood is inner over X, so their composite is inner as well. □

Theorem 2.61.4 (Flatness along a nontrivially valued inner generic fibre)**.** *Let $f : Y \to X$ be a morphism of good analytic spaces over any complete field, let $y \in \mathrm{Int}(Y/X)$, and put $x = f(y)$. If $\mathcal{O}_{X,x}$ is a field and $\mathcal{H}(x)$ is nontrivially valued, then $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y_x,y}$ is faithfully flat.*

Proof. Induct on an upper bound n for the Krull dimension of a strict fibre algebra of an ordinary affinoid representative of the source germ. Such a representative exists by Proposition 2.51.3. For a representative of fibre dimension $d \le n$, Theorem 2.51.4 gives, on ordinary neighborhoods, a finite map to an ordinary affinoid neighborhood V of a point t in $\mathbb{A}^d_X$. Separate the finite branches and retain the one through y, so that y is the only point above t. Put

$$R_0 = \mathcal{O}_{V,t}, \quad T_0 = \mathcal{O}_{V_x,t}, \quad S = \mathcal{O}_{Y,y}, \quad T = \mathcal{O}_{Y_x,y}.$$

The finite direct-image stalk formulas give S finite over R_0 and $T = S \otimes_{R_0} T_0$. This is an ordinary tensor identity: take finite affinoid base change, where completed and ordinary tensor products agree, and then take ordinary stalks. The unique point over t means that the two finite stalk products each have one factor. Therefore flatness of $R_0 \to T_0$ suffices.

The map from the ordinary interior of V to X is smooth. Since the base ordinary ring is a field, smooth normality ascent makes R_0 a normal local domain. If it is a field, its map to T_0 is flat. This includes $n = 0$, for then $d = 0$ and $R_0 = \mathcal{O}_{X,x}$. Otherwise every power $\mathfrak{m}^q$ of its maximal ideal is nonzero. Lemma 2.61.3 represents the closed germ defined by $\mathfrak{m}^q$ by an inner affinoid with strict fibre dimension at most $d-1 \le n-1$. Induction gives flatness of

$$R_0/\mathfrak{m}^q \longrightarrow T_0/\mathfrak{m}^q T_0$$

for all $q \ge 1$. Both rings R_0, T_0 are Noetherian and the extended ideal lies in the maximal ideal of T_0. The adic criterion [StacksAlg, Tag 0523] makes T_0 flat over R_0. Base change gives flatness of $S \to T$. This is a local map of nonzero local rings, so it is faithfully flat. The argument applies to

every representative with strict fibre dimension at most n, completing the induction. □

Lemma 2.61.5 (A generic Gauss field germ in arbitrary characteristic)**.** *Let X be good over any complete field, with $\mathcal{O}_{X,x}$ a field and $F = \mathcal{H}(x)$ trivially valued. If $0 < r < 1$ and x^* is the radius-r Gauss point above x in $X \times_K \mathbb{A}^1_K$, then $\mathcal{H}(x^*) = F_r$ and $\mathcal{O}_{X\times_K \mathbb{A}^1_K, x^*}$ is a field.*

Proof. The Gauss completed-residue calculation gives $\mathcal{H}(x^*) = F_r$. To test the kernel of ordinary evaluation, represent a germ h on a product neighborhood $U \times D$ supplied by Lemma 2.55.1, with D an open annulus around radius r. Corollary 2.35.47 and Lemma 2.56.1 give a normal integral affinoid ordinary neighborhood $\mathcal{M}(A) \subset U$ of x. Its flat nonzero map to the field $\mathcal{O}_{X,x}$ and then to F is injective. On a closed subannulus about r, write $h = \sum_{j\in\mathbf{Z}} a_j T^j$ with $a_j \in A$. If $h(x^*) = 0$, the Gauss norm formula forces all $a_j(x) = 0$, so the single injection $A \hookrightarrow F$ makes every coefficient zero. Thus h is the zero germ. The kernel of ordinary evaluation is the maximal ideal, so that maximal ideal is zero and the ordinary local ring is a field. This coefficient argument uses no restriction on characteristic. □

Theorem 2.61.6 (Faithful flatness along every inner generic fibre)**.** *Let $f : Y \to X$ be a morphism of good analytic spaces over any complete field. If $y \in \mathrm{Int}(Y/X)$ and $\mathcal{O}_{X,f(y)}$ is a field, then*

$$\mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{Y_{f(y)},y}$$

is faithfully flat.

Proof. The case of nontrivially valued $\mathcal{H}(x)$, where $x = f(y)$, is Theorem 2.61.4. Otherwise choose an affinoid domain U which is an ordinary neighborhood of x. The test U in Definition 2.44.4 provides an affinoid domain V which is an ordinary neighborhood of y in $Y \times_X U$, with $y \in \mathrm{Int}(V/U)$. Since x is in the ordinary interior of U, the source and base ordinary germs and the analytic fibre germ are unchanged. Replace Y, X by V, U. Choose $0 < r < 1$ and form their simultaneous affine-line products $Y^* \to X^*$. These spaces are good by Theorem 2.48.3. Take the radius-r Gauss points y^*, x^*; the Gauss seminorms are compatible under the isometric inclusion of completed residue fields. Lemma 2.61.5 makes $\mathcal{O}_{X^*,x^*}$ a field and $\mathcal{H}(x^*)$ nontrivially valued. For every affinoid domain $D \subset X^*$ containing x^*,

the test inverse image $Y^* \times_{X^*} D = V \times_U D$ is affinoid by Theorem 2.40.11. Proposition 2.33.21, applied to the affinoid diagram $V \to U \leftarrow D$, puts y^* in the relative interior of that entire test over D. Thus $y^* \in \mathrm{Int}(Y^*/X^*)$ by the same definition, and the nontrivial case applies. With

$$S = \mathcal{O}_{Y,y}, \quad S^* = \mathcal{O}_{Y^*,y^*}, \quad T = \mathcal{O}_{Y_x,y}, \quad T^* = \mathcal{O}_{Y^*_{x^*},y^*},$$

the map $S^* \to T^*$ is flat. Around the Gauss point, the projection $Y^* \to Y$ is the projection from the ordinary interior of a relative closed disc of radius greater than r. Theorem 2.48.3 therefore makes $S \to S^*$ flat. Finally $Y^*_{x^*} = (Y_x)_{\mathcal{H}(x)_r}$, and Theorem 2.35.4 makes $T \to T^*$ faithfully flat. Thus the composite $S \to T^*$ is flat, and faithful-flat descent along $T \to T^*$ gives flatness of $S \to T$. All maps are local and the rings are nonzero, proving faithful flatness. □

Corollary 2.61.7 (Ordinary closed fibres map faithfully flat to analytic fibres)**.** *Let $f : Y \to X$ be a morphism of good analytic spaces over any complete field and let $y \in \mathrm{Int}(Y/X)$ with image x. Write $R = \mathcal{O}_{X,x}$, $S = \mathcal{O}_{Y,y}$ and let $\mathfrak{m}$ be the maximal ideal of R. Then the canonical local homomorphism*

$$S/\mathfrak{m}S \longrightarrow \mathcal{O}_{Y_x,y}$$

is faithfully flat.

Proof. Choose an ordinary affinoid neighborhood $U = \mathcal{M}(A)$ of x on which a finite generating family of $\mathfrak{m}$ is represented by $a_1, \dots, a_q$. Use U in Definition 2.44.4 to choose an ordinary affinoid neighborhood $V = \mathcal{M}(B)$ of y in $Y \times_X U$ with $y \in \mathrm{Int}(V/U)$. The ordinary stalks on U, V at these points are R, S. Put $J = (a_1, \dots, a_q)$, $U_0 = \mathcal{M}(A/J)$ and $V_0 = \mathcal{M}(B/JB)$, with their quotient norms. The finite quotient and base-change formulas in Propositions 2.33.18 and 2.33.19 identify V_0 with $V \times_U U_0$. Each closed immersion has just one point above its indicated point, since the point seminorm kills the defining ideal and factors through the quotient. Write these points as x_0, y_0. Lemma 2.35.1 gives their ordinary stalks $R/\mathfrak{m}$ and $S/\mathfrak{m}S$. The analytic fibre germ is unchanged: $\mathcal{H}(x_0) = \mathcal{H}(x)$ by quotient evaluation, and a_i has zero image in this field for every i, so the two affinoid fibre algebras identified by Theorem 2.33.11 coincide under finite base change. For each affinoid domain $D \subset U_0$ containing x_0, its test inverse image is the affinoid $V_0 \times_{U_0} D = V \times_U D$. The forward inclusion of Proposition 2.33.21 on $V \to U \leftarrow D$ makes y_0 interior over D in that entire test. Thus $y_0 \in \mathrm{Int}(V_0/U_0)$ by the same definition.

Apply Theorem 2.61.6 to $V_0 \to U_0$ at y_0, with the field stalk $R/\mathfrak{m}$. Under the stated stalk and fibre identifications its faithfully flat map is the canonical map in the assertion. □

Lemma 2.61.8 (Naive flatness and a trivial-residue Gauss lift)**.** *For a morphism $Y \to X$ of good analytic spaces over any complete field, let $\mathcal{F}$ be coherent and naively X-flat at y, and suppose $\mathcal{H}(x)$ is trivially valued for $x = f(y)$. Its pullback under the simultaneous affine-line base change is naively X^*-flat at the radius-r Gauss lift y^* for any $0 < r < 1$.*

Proof. Put $R = \mathcal{O}_{X,x}$, $S = \mathcal{O}_{Y,y}$, $R^* = \mathcal{O}_{X^*,x^*}$, $S^* = \mathcal{O}_{Y^*,y^*}$ and $M = \mathcal{F}_y$. Choose relative closed discs of radius greater than r around the two Gauss points. Their ordinary stalks agree with those of the affine-line products, so Theorem 2.48.3 makes R^* flat over R and S^* flat over S. Coherent pullback gives $M^* = M \otimes_S S^*$; tensoring injections twice shows that M^* is R-flat. Put $I = \mathfrak{m}_R R^*$. Represent a finite generating family of $\mathfrak{m}_R$ on an ordinary affinoid neighbourhood of x, and let X_0 be the closed quotient it defines. The ordinary quotient stalk is $\mathcal{O}_{X_0,x} = R/\mathfrak{m}_R$. Its completed residue field at x is still $\mathcal{H}(x)$, since all generators vanish at x. On the simultaneous affine-line space the pullback closed quotient is $X_0^* = X_0 \times_K \mathbb{A}^{1,\mathrm{an}}_K$. On the actual affinoid product charts its ideal is the extension of the original ideal, by Theorem 2.40.11 and Corollary 3.27.2. Since the generators evaluate to zero at x, their pullbacks also evaluate to zero at x^*. The Gauss formula on the quotient coefficients gives that same point in X_0^*. The local ring of this quotient at the radius-r Gauss point is R^*/I. These ordinary quotient stalk identities are the single-point finite stalk formula of Lemma 2.35.1, applied to the affinoid quotient maps with their quotient norms. By Lemma 2.61.5 on the good space X_0, this ordinary ring is a field. Thus I is proper and M^*/IM^* is R^*/I-flat.

Tensor a free R-resolution of $R/\mathfrak{m}_R$ with the flat algebra R^*. It is a free resolution of R^*/I, and its tensor with M^* has no positive homology because M^* is R-flat. Hence $\mathrm{Tor}_1^{R^*}(M^*, R^*/I) = 0$. The rings R^*, S^* are Noetherian and M^* is finite over S^*. The exact local criterion [StacksAlg, Tag 00ML] makes M^* flat over R^*, as required. □

References. The closed-neighborhood argument and the flat generic-fibre comparison reconstruct the mechanisms of [Duc18, Proposition 6.3.1 and Theorem 6.3.3]. The induction needs faithful flatness, and uses the exact adic criterion

[StacksAlg, Tag 0523]. The trivial-valuation passage is the auxiliary Gauss construction established above; its module version uses [StacksAlg, Tag 00ML].

2.62 Inner coherent flatness over arbitrary complete fields

Lemma 2.62.1 (Lifting analytic-fibre injectivity at an inner point)**.** *Let $T \to X$ be a morphism of good analytic spaces over any complete field, let $t \in \mathrm{Int}(T/X)$ have image x, and let $u : L \to N$ be a morphism of coherent sheaves. If N_t is flat over $\mathcal{O}_{X,x}$ and the analytic-fibre map is injective at t, then u_t is injective and $\mathrm{Coker}(u)_t$ is flat over $\mathcal{O}_{X,x}$.*

Proof. Write $R = \mathcal{O}_{X,x}$, $S = \mathcal{O}_{T,t}$ and $\mathfrak{m} = \mathfrak{m}_R$. Coherent pullback identifies the fibre map with $u_t \otimes_S \mathcal{O}_{T_x,t}$. Corollary 2.61.7 gives a faithfully flat local map $S/\mathfrak{m}S \to \mathcal{O}_{T_x,t}$. Its faithful flatness detects the kernel of the finite map $L_t/\mathfrak{m}L_t \to N_t/\mathfrak{m}N_t$, so this map is injective. The ring S is Noetherian, L_t is finite over S, and N_t is R-flat. Apply [StacksAlg, Tag 00ME]: it gives injectivity of u_t and R-flatness of its cokernel. Ordinary stalks commute with coherent cokernels, proving the statement. □

Theorem 2.62.2 (Inner flatness with nontrivially valued completed residue field)**.** *Let $f : Y \to X$ be a morphism of good analytic spaces over any complete field and let $\mathcal{F}$ be coherent on Y_G. At an inner point y, with $\mathcal{H}(f(y))$ nontrivially valued, naive X-flatness of $\mathcal{F}$ is equivalent to analytic X-flatness.*

Proof. The identity base-change test proves the analytic-to-naive implication. Assume that $\mathcal{F}_y$ is naively flat, and put $x = f(y)$ and $F = \mathcal{H}(x)$. If the stalk is zero, a finite set of local generators vanishes on one ordinary neighborhood of y. Every pullback then vanishes near every point above y, giving the conclusion. We may assume the stalk is nonzero.

Step 1: a strict fibre-support induction. Choose an ordinary affinoid representative with strict F-fibre, using Proposition 2.51.3. Replace its source by the closed support of $\mathcal{F}$, defined by its annihilator. The sheaf is a finite module over this closed subspace, and its pushforward is the original sheaf. Theorem 2.35.17 identifies both naive and analytic flatness under this replacement. The closed immersion is finite and inner, so the new source is

still inner over X at y. Its fibre is a strict affinoid quotient. Lemma 2.57.1 identifies its point set with the support of $\mathcal{F}_x$. Induct on an upper bound n for this strict fibre dimension, allowing every such representative with dimension at most n. By Corollary 3.22.8, later restrictions to ordinary affinoid domains cannot increase this analytic support dimension. For a strict affinoid fibre it equals the Krull dimension.

Step 2: a finite projection to an integral fibre. Write the chosen closed support as Z and put $d = \dim Z_x \leq n$. Theorem 2.51.4 gives a finite map from an ordinary neighborhood of y in Z to an ordinary affinoid neighborhood T of t in $\mathbb{A}_X^{d,\mathrm{an}}$. Finite branch separation retains the branch through y and, after shrinking the target, makes y its only point above t. Since t is in the ordinary interior of T in relative affine space, it is inner over X. Shrink by Proposition 2.51.3 to make T_x strict. This fibre is an affinoid domain of $\mathbb{A}_F^d$; its ordinary rings are regular by Theorem 2.35.23. Lemma 2.52.2 decomposes its algebra into integral factors. Lemma 2.56.4 retains the factor containing t by an actual ordinary domain over the original field. Pull the finite map back after each restriction. We obtain

$$\pi : Z \longrightarrow T = \mathcal{M}(B), \qquad C = B\widehat{\otimes}_A F$$

with singleton distinguished fibre, with C integral and strict, and with $\dim C \leq d$ by Lemma 2.56.3. Here $\mathcal{M}(A)$ is the chosen ordinary affinoid base neighborhood. The source and target ordinary germs at the distinguished points have not changed.

Step 3: an original generic basis and a smaller quotient. Put $N = \pi_*\mathcal{F}$ and represent it by a finite B-module. The singleton finite-pushforward formula makes N_t naively X-flat. Fix any finite generating family of this B-module. Its images span $N_C \otimes_C \mathrm{Frac}(C)$. Choose a subset which is a basis, allowing the empty subset when its rank is zero, and use the corresponding original generators to define $u : B^r \to N$. Lemma 2.57.2 proves that u_C is injective, has ordinarily dense bijectivity locus, and that its cokernel Q_C has support of dimension at most $\dim C - 1$ unless it is zero. Here injection follows because its kernel is a torsion submodule of the free module C^r over a domain. The dimension drop follows by annihilating the finite torsion cokernel by one nonzero element of C and prepending the zero prime to each chain in its support. Thus the generators and the map u are defined over the original affinoid algebra, not merely over the completed residue field.

Let $L = \mathcal{O}_T^r$ and $Q = \mathrm{Coker}(u : L \to N)$. Lemma 2.62.1 gives injectivity of u_t and naive X-flatness of Q_t. By support compatibility under fibre base

change, the closed support of Q has strict fibre dimension at most $\dim C - 1 \leq n - 1$ whenever its fibre is nonempty. That closed support is inner at t. If $Q_t \neq 0$, the induction hypothesis applied on this support makes Q analytically X-flat at t; if $Q_t = 0$, local vanishing gives the same conclusion. When $n = 0$, the integral zero-dimensional fibre algebra C is a field, so $Q_C = 0$. Tensoring at the distinguished point gives $Q_t \otimes_{\mathcal{O}_{T,t}} \mathcal{H}(t) = 0$, and Nakayama gives $Q_t = 0$. This supplies the induction start.

Step 4: all analytic tests. The target $T \to X$ is quasi-smooth, being an analytic-domain restriction of relative affine space. The map u_x is bijective on an ordinarily dense subset of its fibre and u_t is injective. Corollary 2.52.6 therefore upgrades the analytic flatness of Q to analytic flatness of N. Explicitly, its universal-injectivity statement preserves $L \to N$ at every point above t under every good-space base change over every complete extension. The pulled-back cokernel is the pullback of Q by right exactness. Both that cokernel and the pullback of the free sheaf L are flat over the new base, so their extension, the pullback of N, is flat. This verifies all tests, not just those over the original field. The analytic version of Theorem 2.35.17 then gives analytic flatness of $\mathcal{F}$ at y, because the distinguished finite fibre has one point; it includes all later fibre points after arbitrary tests. The induction is complete. □

Theorem 2.62.3 (Naive and analytic flatness at every inner point)**.** *Let K be any complete non-Archimedean field, let $f : Y \to X$ be a morphism of good K-analytic spaces, and let $\mathcal{F}$ be coherent on Y_G. For every $y \in \mathrm{Int}(Y/X)$, the following are equivalent: $\mathcal{F}_y$ is flat over $\mathcal{O}_{X,f(y)}$; and $\mathcal{F}$ is analytically X-flat at y, tested after every good analytic base change over every complete valued extension of K.*

Proof. Only the naive-to-analytic implication requires proof. Put $x = f(y)$. If $\mathcal{H}(x)$ is nontrivially valued, use Theorem 2.62.2. Otherwise choose $0 < r < 1$ and form the simultaneous affine-line product $Y^* \to X^*$ with the radius-r Gauss points y^*, x^*. Lemma 2.61.8 proves naive flatness of the coherent pullback at y^*. The point remains inner by analytic base-change stability of interior, and $\mathcal{H}(x^*) = \mathcal{H}(x)_r$ is nontrivially valued. The preceding theorem makes this pullback analytically flat. Finally Lemma 2.58.3 gives analytic flatness downstairs. Its proof tests an arbitrary complete extension and good analytic base change and chooses the same-radius Gauss point above each test point; the two smooth projection maps on ordinary stalks are flat, and

the source one is faithfully flat. Thus the descent covers exactly all tests in the stated condition. □

References. This reconstructs the inner-point flatness theorem [Duc18, Theorem 8.3.4]. It reuses the internal support and original-generic-basis lemmas, and the all-characteristic ordinary-to-analytic fibre comparison. The dimension induction is written for strict fibre-support bounds; the final Gauss step handles trivially valued completed residue fields.

Closed-fibre injection on the ordinary projection chart. Let T be an affinoid domain of $\mathbb{A}_X^{d,\mathrm{an}}$, where X is a good analytic space over a complete field. Suppose t lies in the ordinary interior of T in relative affine space, with image $x \in X$, and suppose $\mathcal{H}(x)$ is nontrivially valued. In the notation of the proof, the canonical map

$$S/\mathfrak{m}_R S \longrightarrow \mathcal{O}_{T_x,t}$$

is injective.

Proof. Put $R = \mathcal{O}_{X,x}$, $S = \mathcal{O}_{T,t}$, and $\mathfrak{m} = \mathfrak{m}_R$. Realize finite generators of $\mathfrak{m}$ on an ordinary affinoid neighborhood of x, defining the closed base change $X_0 \to X$. The resulting base stalk is $R/\mathfrak{m}$. Over X_0, restrict to the ordinary open part of the relative affine space containing t, where the projection is smooth. The source stalk is $D = S/\mathfrak{m}S$, and the analytic fibre is unchanged. The ordinary finite-branch stalk formula of Lemma 2.35.1 identifies the quotient stalk on the base and on the source. The actual affinoid fibre quotient follows from Proposition 2.33.11 and the finite base-change formula of Proposition 2.33.19.

Let $h \in D$ have zero germ in $\mathcal{O}_{T_x,t}$. Represent h on an ordinary affinoid neighborhood of t in that smooth part. Its zero locus contains an ordinary neighborhood in the analytic fibre. Proposition 2.61.2 applies over the field stalk $R/\mathfrak{m}$ and gives pointwise vanishing on some ordinary neighborhood of t in the source. Take a reduced ordinary affinoid neighborhood inside it. Such a neighborhood exists by regularity of the actual relative affine-space map on ordinary local rings over the field stalk (Proposition 2.35.56), normality of the resulting regular local ring [StacksAlg, Tag 0567], and Corollary 2.35.47. On this reduced affinoid, pointwise vanishing gives spectral radius zero and hence $h = 0$, by Propositions 1.20.3 and 3.21.4. Therefore $D \to \mathcal{O}_{T_x,t}$ is injective. □

Inner coherent flatness in characteristic zero or at a rational base point. Let $f : Y \to X$ be a morphism of good K-analytic spaces, let $\mathcal{F}$ be coherent on Y_G, and let $y \in \mathrm{Int}(Y/X)$ have image x. Suppose $F = \mathcal{H}(x)$ is nontrivially valued and that either $\mathrm{char}\, K = 0$ or x is K-rational. Then $\mathcal{F}_y$ is flat over $\mathcal{O}_{X,x}$ if and only if $\mathcal{F}$ is analytically X-flat at y, with tests over every complete valued extension of K.

Proof. The identity test gives naive flatness from analytic flatness. Assume naive flatness. A zero coherent stalk vanishes on an ordinary neighborhood and remains zero under every pullback. Otherwise take an ordinary affinoid neighborhood with strict F-fibre by Proposition 2.51.3. Pass to the closed subspace cut out by the annihilator and use Theorem 2.35.17 to transport both flatness conditions. The immersion is finite and inner, and its fibre is a strict quotient. Lemma 2.57.1 identifies its fibre with the support on point sets. Induct on an upper bound n on the dimension of the strict fibre of such a representative. Ordinary shrinking cannot increase that support dimension by Corollary 3.22.8.

For the resulting space Z, put $d = \dim Z_x$. Theorem 2.51.4 and Lemma 2.35.1 give an ordinary finite projection $\pi : Z \to T = \mathcal{M}(B)$ with only y over $t = \pi(y)$; here t belongs to the ordinary interior of the target domain in $\mathbb{A}_X^{d,\mathrm{an}}$. We shrink the target to have strict fibre, and use Theorem 2.35.23 and Lemma 2.52.2 to split its fibre algebra into integral factors. Lemma 2.56.4 retains the factor at t by an ordinary restriction over K. Pull the finite map back along these restrictions. Write $C = B\widehat{\otimes}_A F$ for the resulting integral strict fibre algebra, where X has been restricted to an ordinary affinoid neighborhood $\mathcal{M}(A)$ of x. By Lemma 2.56.3, $\dim C \le d$. The point x and its germ have remained fixed.

The stalk of $N = \pi_*\mathcal{F}$ is naively X-flat by the singleton formula in Theorem 2.35.17. Represent N by a finite B-module and choose a subset of its original generators whose images form a basis over $\mathrm{Frac}(C)$. Use these elements to define $u : L = \mathcal{O}_T^r \to N$, including $r = 0$ when the rank is zero. Lemma 2.57.2 gives an injective u_x, an ordinarily dense fibre bijectivity locus and a cokernel Q_x whose support has dimension at most $\dim C - 1$ if nonempty.

Here put $R = \mathcal{O}_{X,x}$, $S = \mathcal{O}_{T,t}$ and $\mathfrak{m} = \mathfrak{m}_R$. The canonical homomorphism $D = S/\mathfrak{m}S \to E = \mathcal{O}_{T_x,t}$ is injective. In characteristic zero this follows from Corollary 2.56.10. In the rational case $F = K$, so K is nontrivially valued and Lemma 2.35.12 identifies this map with an isomorphism. Coherent pullback

identifies the map on the analytic fibre with $(u_t \bmod \mathfrak{m}) \otimes_D E$. A vector in the kernel of $D^r \to N_t/\mathfrak{m}N_t$ has zero image in E^r, since the analytic-fibre map is injective. The map $D^r \to E^r$ is injective because $D \to E$ is injective. Thus $u_t \bmod \mathfrak{m}$ is injective. The homomorphism $R \to S$ is local, S is Noetherian, L_t is finite over S, and N_t is flat over R. Apply [StacksAlg, Tag 00ME] to obtain injectivity of u_t and R-flatness of $Q_t = \operatorname{Coker}(u_t)$.

If $n = 0$, the integral zero-dimensional fibre algebra C is a field and $Q_x = 0$. Tensoring at t gives $Q_t \otimes_S \mathcal{H}(t) = 0$. The residue field of S embeds in $\mathcal{H}(t)$, so $Q_t/\mathfrak{m}_S Q_t = 0$, and [StacksAlg, Tag 00DV(2)] gives $Q_t = 0$. For $n > 0$ and $Q_t \neq 0$, the closed support of Q is inner at t, with strict F-fibre of dimension at most $n - 1$ by the support and dimension statements above. Apply induction on this closed support and the finite pushforward equivalence to obtain analytic X-flatness of Q at t. If $Q_t = 0$, local vanishing gives the same flatness.

The map $T \to X$ is quasi-smooth near t, and t lies in the ordinary closure of the fibre bijectivity locus. Apply the cokernel-to-target implication of Corollary 2.52.6. Its universal injectivity preserves u at every point over t in every good analytic base change over every complete valued extension. The pulled-back cokernel is the pullback of Q, and the free source and that cokernel are flat over the new base. Thus N is analytically X-flat at t. The analytic finite pushforward equivalence then proves the assertion for $\mathcal{F}$ at y. □

Chapter 3

Reduction, formal models and residual geometry

An analytic point can encode a variety over a residue field. Reduction makes this information visible and connects analytic functions with models. We distinguish two constructions from the start: canonical reduction uses the intrinsic spectral seminorm, whereas a formal model chooses an integral algebra. Their special fibres need not coincide. Graded reduction will then retain the sizes that ordinary reduction forgets.

3.1 Power-bounded functions and canonical reduction

Definition 3.1.1. For a Banach affinoid algebra A, an element is power-bounded if the sequence of its powers is bounded, and topologically nilpotent if that sequence tends to zero. Denote these subsets by A° and $A^{\circ\circ}$ respectively.

Proposition 3.1.2. *For an affinoid algebra one has*

$$A^\circ = \{a : \rho_A(a) \leq 1\}, \qquad A^{\circ\circ} = \{a : \rho_A(a) < 1\}.$$

The first is a subring, the second an ideal in it, and $\widetilde{A} = A^\circ/A^{\circ\circ}$ is reduced.

Proof. If $\rho_A(a) < 1$, Corollary 1.20.3 implies $\|a^m\| \to 0$. Conversely, if the powers tend to zero, some power has norm less than one and hence $\rho_A(a) < 1$.

A power-bounded element has spectral radius at most one. For the reverse inclusion use Theorem 1.20.2, which gives $\|a^m\| \leq C_a \rho_A(a)^m$ when $\rho_A(a) > 0$. If $\rho_A(a) = 0$, Proposition 3.21.4 makes a nilpotent. The ring and ideal assertions follow from the spectral seminorm inequalities. If $\widetilde{a}^m = 0$, then $\rho_A(a)^m < 1$, hence $\rho_A(a) < 1$ and $\widetilde{a} = 0$. □

3.2 Reduction and the residual directions of the disc

Let A° be the ring of power-bounded elements and $A^{\circ\circ}$ the ideal of topologically nilpotent elements. If $a \in A^\circ$, the boundedness of the sequence a^m and Lemma 1.6.1 imply $|a(x)| \leq 1$. If $a \in A^{\circ\circ}$, then $a^m \to 0$, which implies $|a(x)| < 1$. Thus evaluation induces a homomorphism

$$\widetilde{A} = A^\circ / A^{\circ\circ} \longrightarrow \widetilde{\mathcal{H}(x)}.$$

Its prime kernel defines $\mathrm{red}(x) \in \operatorname{Spec} \widetilde{A}$. When a formal model with ring of definition $A_0 \subset A^\circ$ is chosen, the same construction on A_0 gives specialization to its special fibre. The model reduction and the canonical reduction need not have identical coordinate rings.

Proposition 3.2.1. *For $\widetilde{f}_1, \dots, \widetilde{f}_m \in \widetilde{A}$ represented by $f_i \in A^\circ$, one has*

$$\mathrm{red}^{-1}\, V(\widetilde{f}_1, \dots, \widetilde{f}_m) = \{x : |f_i(x)| < 1 \text{ for every } i\}.$$

This set is ordinary open. In particular, when $\widetilde{A}$ is noetherian, reduction is anticontinuous: inverse images of Zariski closed subsets are open.

Proof. Membership in $V(\widetilde{f}_1, \dots, \widetilde{f}_m)$ says exactly that the reductions of all $f_i(x)$ vanish in the residual field. This is equivalent to the displayed strict inequalities. Each inequality defines an open set, and only finitely many occur. Noetherianity permits every closed subset to be expressed by finitely many equations. □

Example 3.2.2. For $A = K\langle T \rangle$ with its Gauss norm, multiplicativity gives $A^\circ = \{f : \|f\| \leq 1\}$ and $A^{\circ\circ} = \{f : \|f\| < 1\}$. A convergent series has only finitely many coefficients of size one, so reduction gives $\widetilde{A} = \widetilde{K}[T]$. At the Gauss point ζ_G, every nonzero reduced polynomial has value one. Its reduction ideal is therefore zero, and $\mathrm{red}(\zeta_G)$ is the generic point of the

residual affine line. Moreover, the residue field of the completed Gauss field is $\widetilde{K}(T)$: rational functions with nonzero reduction supply this field, and completion does not change residues because sufficiently close elements have the same residue.

If $a \in K^\circ$ lifts $\widetilde{a} \in \widetilde{K}$, the tube of the residual point $T = \widetilde{a}$ is $|T - a| < 1$. Thus different residual points specify different open directions adjacent to the Gauss point. Their union is not a replacement for the Gauss point itself. This is the simplest example of residual algebraic geometry carried by a single analytic point.

The same mechanism appears in a model metric. A formal line bundle has norm-one local frames, while its restriction to the special fibre may be nontrivial. Its degrees on residual curves enter the later comparison between PL positivity and residual nefness.

References. For the surrounding affinoid, reduction and graded-reduction theory, see [Gub13, Sections 3–6] and [CLD25, Chapters 10–12].

3.3 Shilov boundary and maximum values

Definition 3.3.1. A closed subset $S \subset \mathcal{M}(A)$ is a boundary for the algebra if every $|f|$, $f \in A$, attains its maximum on S. The Shilov boundary is the smallest such closed subset. Its existence is a theorem.

Theorem 3.3.2. *An affinoid space has a finite Shilov boundary. If A is strictly affinoid over a nontrivially valued field, $\widetilde{A}$ is a finite-type $\widetilde{K}$-algebra, the canonical reduction map is surjective, and each generic point of* $\operatorname{Spec} \widetilde{A}$ *has a unique analytic preimage. Those preimages are exactly the Shilov boundary.*

Proof. Corollary 3.12.5 proves finite Shilov boundaries for all radii. In the strict case, Theorem 3.16.3 proves finite generation of canonical reduction, and Theorem 3.16.16 realizes every prime ideal by an analytic point. Theorem 3.16.7 identifies the minimal primes with the Shilov points and proves uniqueness of each preimage. □

References. [TemIntro, 3.4.1.1–3.4.1.7].

3.4 Norms and analytic points

Let $V = \mathcal{M}(A)$ be a nonempty affinoid space over a complete non-Archimedean field. Its spectral seminorm is the supremum of the evaluations at analytic points:

$$\|f\|_{\mathrm{sp}} = \sup_{x \in V} |f(x)|.$$

The spectral seminorm is power-multiplicative. Multiplicativity on two unrelated elements is a stronger assertion, and is precisely what allows this seminorm to represent one analytic point.

Proposition 3.4.1. *The Shilov boundary of V consists of one point if and only if its spectral seminorm is multiplicative. When these conditions hold, the unique boundary point is the spectral seminorm.*

Proof. If the Shilov boundary is $\{\xi\}$, its maximum property gives the following identity for every element of A:

$$\|f\|_{\mathrm{sp}} = |f(\xi)|.$$

The right side is multiplicative. Conversely, a multiplicative spectral seminorm is a bounded multiplicative seminorm extending the absolute value of the ground field, hence a point ξ of V. It attains the supremum of every function. Thus $\{\xi\}$ is a boundary. Corollary 3.12.5 gives the smallest closed boundary, which must therefore be $\{\xi\}$.

□

Example 3.4.2 (Disc, circle and annulus)**.** The closed disc has one Shilov point, its Gauss point. A strict closed annulus $|a| \leq |T| \leq 1$, $0 < |a| < 1$, has the model

$$\mathfrak{A} = \operatorname{Spf} K^\circ\langle U, V\rangle/(UV - a), \quad U = T, \quad V = a/T.$$

Its special fibre has equation $uv = 0$. In this example it agrees with canonical reduction. The two irreducible components give the endpoint Gauss points of the annulus. For the radius-one circle, the algebra is $K\langle T, T^{-1}\rangle$ and the reduction is $\widetilde{K}[t, t^{-1}]$, which is integral; the two endpoints have coalesced into one Gauss point.

These assertions can also be checked directly from Laurent expansions: the spectral norm on the annulus is the maximum of the two endpoint Gauss norms, and neither endpoint can be omitted, since T and a/T attain their maxima at opposite ends.

Proposition 3.4.3 (A higher-dimensional boundary distinction)**.** *Let K be nontrivially valued and $X = \mathcal{M}(K\langle T, S\rangle)$. Its Shilov boundary consists of the Gauss point $\zeta_{1,1}$, but its analytic boundary contains every weighted Gauss point $\zeta_{1,\rho}$ with $\rho \in |K^\times|$, $0 < \rho < 1$.*

Proof. The unit Gauss norm is multiplicative, so Proposition 3.4.1 gives the Shilov assertion. Canonical reduction is $\widetilde{K}[t, s]$ by Proposition 3.9.2. At $\zeta_{1,\rho}$ its kernel in this ring is (s): positive powers of S become small, while nonzero polynomials in T retain norm one. The prime (s) is not maximal. Theorem 3.25.1 identifies the absolute interior with the inverse image of the closed points of canonical reduction, so this point lies in the analytic boundary. □

For a pure one-dimensional strict affinoid, Theorem 3.25.4 proves that the analytic and Shilov boundaries coincide: the nonclosed reduction points are precisely its generic points. The bidisc illustrates the distinction in higher dimension.

3.5 Formal models and their two fibres

In this section the valuation is nontrivial. Fix $\pi \in K^{\circ\circ} \setminus \{0\}$; powers of π define the topology of K°. We use admissible formal schemes locally of finite presentation over K°. An affine chart has the form

$$\mathfrak{X} = \operatorname{Spf} A_0, \qquad A_0 = K^\circ\langle T_1, \dots, T_m\rangle / I,$$

with the completed topology, finitely generated defining ideal, and no π-torsion. The last condition excludes vertical torsion; over a valuation ring it expresses the required flatness. Its generic and special fibres are

$$\mathfrak{X}_\eta = \mathcal{M}(A_0 \otimes_{K^\circ} K), \qquad \mathfrak{X}_s = \operatorname{Spec}(A_0 \otimes_{K^\circ} \widetilde{K}).$$

The equivalence with flatness follows directly from the valuation-ring structure. For every nonzero $b \in K^\circ$, some π^N belongs to bK°, so absence of

π-torsion implies absence of all K°-torsion. Every finitely generated ideal of K° is principal, and its tensor map to a torsion-free module is injective; the finite-ideal criterion [StacksAlg, Tag 00HD] gives flatness, while flatness preserves injectivity of multiplication by each nonzero scalar and therefore implies torsion-freeness.

The generic-fibre algebra is the corresponding affinoid algebra, with its complete quotient topology. Formal open charts glue along domains on the generic fibre, giving the generic-fibre functor. A formal model of an analytic space includes an identification of this generic fibre with the specified space.

References. [TemIntro, §5.2]; [GRW16, §§2–3].

Every $a \in A_0$ has absolute value at most one on the generic fibre. Evaluation at x therefore gives $A_0 \to \mathcal{H}(x)^\circ$, and reduction gives its specialization $\mathrm{red}_{\mathfrak{X}}(x) \in \mathfrak{X}_s$. If a locally closed stratum Z is specified in the special fibre, its tube is the subset $]Z[_{\mathfrak{X}} = \mathrm{red}_{\mathfrak{X}}^{-1}(Z)$ with its natural analytic-domain structure. On each formal affine chart, finite equations and inequations express it by strict inequalities and equalities of absolute values.

Example 3.5.1. For $\operatorname{Spf} K^\circ\langle T\rangle$, the generic fibre is the closed unit disc and the special fibre is the residual affine line. The tube of $\widetilde{a} \in \widetilde{K}$ is $|T - a| < 1$. For $UV = a$ as in Example 3.4.2, the tube of the node $u = v = 0$ is

$$|U| < 1, \quad |V| < 1 \quad \Longleftrightarrow \quad |a| < |T| < 1.$$

It is an open annulus, not a single analytic point. The generic points of the two components have endpoint Gauss points above them. Thus different sorts of strata have very different analytic fibres.

Remark 3.5.2. The special fibre of a chosen model can be nonreduced; canonical reduction is reduced by Proposition 3.1.2. Nor is the ring A° automatically an admissible algebra of finite presentation in every category under consideration. We therefore distinguish the canonical reduction $\operatorname{Spec} \widetilde{A}$ from an arbitrary model $\operatorname{Spf} A_0$. Whenever a canonical formal model is used in a later source theorem, its existence in the stated category is part of that theorem's hypotheses or construction.

Example 3.5.3 (A reduced algebra with two different norms)**.** Let K be discretely valued, with uniformizer π, and put $A = K\langle T\rangle/(T^2-\pi)$. Normalize its additive valuation by $v(\pi) = 1$. A root $b \in K$ would satisfy $2v(b) = 1$, impossible since $v(K^\times) = \mathbb{Z}$. Thus the quadratic polynomial is irreducible, and Weierstrass division identifies A with a quadratic field L. The quotient of the unit Gauss norm gives $\|T\|_{\mathrm{quot}} = 1$, whereas its spectral norm is $\rho_A(T) = |\pi|^{1/2} < 1$.

To verify the quotient norm, apply Theorem 1.3.1 to the polynomial $T^2-\pi$. Every lift of the class of T has remainder T, and the remainder operator is contractive. Its norm is therefore at least one; the lift T attains one. On the other hand $T^2 = \pi$ forces every multiplicative seminorm to take the value $|\pi|^{1/2}$ on T. This proves the spectral assertion.

The chosen model $K^\circ[T]/(T^2-\pi)$ has special fibre $\widetilde{K}[t]/(t^2)$, while canonical reduction of the field is its residue field, which is reduced. Hence reducedness of the generic algebra does not identify a quotient Gauss norm with its spectral norm, or identify a chosen special fibre with canonical reduction. Nilpotent multiplicity in a model can coexist with a reduced generic fibre.

3.6 Changing a model without changing its generic fibre

An admissible formal blow-up is obtained by blowing up an open coherent ideal and taking the corresponding formal completion, with the admissible torsion-free charts. Locally on the formal scheme, an open ideal contains a power of π. After passing to the generic fibre, that local power is a unit, so the centre disappears algebraically. The resulting map on analytic generic fibres is an isomorphism. This invariance is the basic reason that different models can describe the same analytic space.

Example 3.6.1. Blow up the ideal (T, a) on $\operatorname{Spf} K^\circ\langle T\rangle$, where $0 < |a| < 1$. The chart in which a generates the ideal has coordinate $U = T/a$ and generic domain $|T| \le |a|$. The chart in which T generates it has coordinate $V = a/T$ and generic domain $|a| \le |T| \le 1$. These two domains cover the original disc and meet in $|T| = |a|$. Their gluing is the identity on the original generic fibre, while the special fibre has acquired a new component.

Lemma 3.17.3 proves generic-fibre invariance for admissible formal blowups. For two quasi-paracompact admissible formal models with the same generic

fibre, Lemma 3.17.7 constructs a common formal modification. Such a common model permits the comparison of integral data defining model metrics.

References. [TemIntro, §§5.2–5.3].

3.7 Abhyankar dimension and Gauss fields

Definition 3.7.1. For an analytic point x, put

$$\begin{aligned} s_K(x) &= \operatorname{trdeg}_{\widetilde{K}} \widetilde{\mathcal{H}(x)}, \\ t_K(x) &= \dim_{\mathbb{Q}}\big((v(\mathcal{H}(x)^\times)/\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}\big), \\ d_K(x) &= s_K(x) + t_K(x). \end{aligned}$$

The analytic Abhyankar inequality says $d_K(x) \leq \dim_x X$. A point attaining equality is called Abhyankar in this local-dimensional sense. On a pure n-dimensional space, the full-dimensional Abhyankar points have $d_K(x) = n$.

Theorem 3.22.9 proves the inequality for arbitrary analytic spaces, using the affinoid dimension calculation and its compatibility with analytic domains.

References. For the general dimension theorem, see [CLD25, 11.2.3, especially (11.2.3.1)] and [Duc18, §1.4].

Neither $s_K(x)$ nor $t_K(x)$ is the dimension of the singleton $\{x\}$ as a topological space. They measure independent residual parameters and independent values. The following calculation proves the equality for the Gauss fields used throughout this book. The subsequent semistable chart is first treated as an explicit affinoid equation; its global skeletal interpretation is developed in Chapter 4.

3.8 Value-group rank and residual dimension at a Gauss point

A skeleton has a real dimension, but each of its points also has a completed residue field. These dimensions are related by the Abhyankar equality, not by identifying the residue field with the residue field of one formal stratum. A direct Gauss-norm calculation makes the difference visible.

Let K be complete with additive value group $\Gamma \subset \mathbb{R}$. On $K(Z_1, \dots, Z_n)$ use the weighted Gauss valuation with $v(Z_i) = w_i$. Write H_w for its completion and set

$$\Gamma_w = \Gamma + \sum_i \mathbb{Z} w_i, \qquad t = \dim_{\mathbb{Q}}\big((\Gamma_w/\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}\big),$$

$$L_w = \{m \in \mathbb{Z}^n : \langle m, w\rangle \in \Gamma\}.$$

Proposition 3.8.1. *The value group of H_w is Γ_w. The group L_w has rank $n - t$, and the residue field of H_w is a purely transcendental extension of $\widetilde{K}$ of degree $n - t$. In particular,*

$$\operatorname{trdeg}_{\widetilde{K}} \widetilde{H}_w + \dim_{\mathbb{Q}}\big((v(H_w^\times)/\Gamma) \otimes \mathbb{Q}\big) = n.$$

Proof. The value of a nonzero Laurent polynomial is the minimum of its finitely many term values, so it belongs to Γ_w. Quotients have the same property, and the constants and variables generate the whole group Γ_w. Completion adds no values: for a nonzero limit a, a sufficiently close approximation a_j has $|a - a_j| < |a|$, hence $|a_j| = |a|$. It adds no residue classes either, because an approximation with difference of norm less than one has the same residue.

The homomorphism $\mathbb{Z}^n \to \Gamma_w/\Gamma$, $m \mapsto \langle m, w\rangle$, is surjective with kernel L_w. Tensoring with $\mathbb{Q}$ gives rank $L_w = n - t$. Choose a basis $m_1, \dots, m_{n-t}$ of this free abelian group and constants $c_j \in K^\times$ with $v(c_j) = -\langle m_j, w\rangle$. The monomials $U_j = c_j Z^{m_j}$ have norm one.

Their residues are algebraically independent. A nonzero polynomial relation in their residues could be lifted to a Laurent polynomial in distinct exponent vectors $\sum_j a_j m_j$, with all nonzero leading coefficients of norm one. The Gauss norm of this polynomial is one, so its residue cannot be zero. Independence of the m_j ensures that the exponent vectors are distinct.

They generate the residue field as well. Let F/G be a rational function of norm one. Divide numerator and denominator by one term of G attaining its Gauss norm. Both resulting Laurent polynomials have norm one. Each surviving term in their reductions has exponent difference $m \in L_w$ and is a residue-field scalar times a Laurent monomial in the U_j. Terms of smaller norm disappear. The denominator reduction is nonzero by algebraic independence, so the residue of F/G belongs to $\widetilde{K}(\widetilde{U}_1, \dots, \widetilde{U}_{n-t})$. The completion observation proves the same assertion for H_w. □

3.8.1 The semistable simplex and its residue field

Consider the standard formal chart

$$\mathfrak{X} = \operatorname{Spf} K^\circ\langle X_0, \dots, X_r, T_1, \dots, T_{d-r}\rangle/(X_0 \cdots X_r - a), \qquad 0 < |a| < 1.$$

For $u_i \geq 0$ with $\sum_{i=0}^{r} u_i = v(a)$, eliminate $X_0 = a/(X_1 \cdots X_r)$ and take the Gauss valuation with weights $u_1, \dots, u_r$ on the Laurent variables and zero on the T_j. All original coordinates have norm at most one, so the valuation defines an analytic point of the generic fibre. Multiplicativity follows from the Gauss-norm argument; boundedness lets it extend from polynomials to the affinoid algebra.

These points form the canonical simplex. Their completed residue fields are the completions of the displayed Gauss rational fields: polynomials are dense in the affinoid algebra for a norm dominating the point seminorm, and rational quotients are then dense in the residue-field completion. Proposition 3.8.1 gives

$$s(x) = d - t(x), \qquad s(x) + t(x) = d.$$

Thus every point of the standard simplex is Abhyankar of the full analytic dimension d, although the simplex itself has real dimension only r. The remaining dimensions are carried by residue fields.

At a point in the relative interior of this real simplex, all X_i reduce to zero in this formal chart. Its reduction lies at the generic point of the stratum with free coordinates $T_1, \dots, T_{d-r}$, of dimension $d-r$. The completed residue field of the analytic point may have larger residual transcendence degree $d - t(x)$, because it also contains residues of norm-one Laurent quotients in the X_i. Those quotients need not be regular functions on the original formal chart. They can become visible after a suitable formal modification.

Proposition 3.26.1 supplies the integral normal form of this chart, its rational-domain presentation and the topology of the displayed simplex. It also proves the local analytic dimension used above and realizes any finite family of the indicated Laurent residues after an admissible formal modification.

Example 3.8.2 (Two points on the same annular skeleton)**.** Let the chart be $XY = a$, so $d = r = 1$, and take an interior skeletal point with $v(X) = u$. If u is rational over Γ, choose $m > 0$ and $c \in K^\times$ with $v(c) = mu$. Then X^m/c has norm one and transcendental residue, while the new value-group quotient has rational rank zero. Hence $(s, t) = (1, 0)$. If u is rationally independent

modulo Γ, no such monomial exists and $(s,t) = (0,1)$. Both points reduce to the same node in the original semistable model.

The calculation also clarifies why "generic point of a formal stratum", "Shilov boundary point of a tropical fibre", and "Abhyankar point" are related but different descriptions. The first depends on a model, the second on a tropicalization, and the third on an equality of valued-field dimensions. In regular-immersion arguments, it is the last equality that gives the local analytic dimension needed for the finite-flat comparison.

References. The standard semistable charts and skeletal valuations are those of [GRW16, Sections 3–4]. The Gauss-field calculation is proved here; general analytic dimension and invariance statements remain the geometric theorems cited in Chapter 2.

3.9 Graded reduction

Ordinary reduction records elements of size at most one modulo those of size less than one. To compare radii not supplied by K, it is useful to retain the same construction at every positive size.

Definition 3.9.1. For a valued field F, its graded reduction is

$$\widetilde{F}^{\mathrm{gr}} = \bigoplus_{r>0} F_{\leq r}/F_{<r}.$$

Multiplication sends degree r times degree s to degree rs. For an affinoid algebra, use the filtration defined by its spectral seminorm. A graded field means a graded ring in which every nonzero homogeneous element is invertible; its underlying ungraded ring need not be a field.

Proposition 3.9.2. *The graded reduction of the weighted Gauss polynomial algebra is the graded polynomial algebra over $\widetilde{K}^{\mathrm{gr}}$ with $\deg(\widetilde{T}_i) = r_i$. The same assertion holds for the generalized Tate algebra with its Gauss norm.*

Proof. A homogeneous leading class of size s is represented by the finitely many monomials $a_\nu T^\nu$ having $|a_\nu| r^\nu = s$. The remaining terms have strictly smaller norm. For a convergent series there are still only finitely many terms at the maximal size, and its tail is strictly smaller. This proves surjectivity. A nonzero homogeneous polynomial cannot map to zero, because its Gauss norm is exactly its specified size; this proves injectivity. Multiplication is compatible by Proposition 1.2.2. □

For the one-radius field K_r, where $r \notin \sqrt{|K^\times|}$, a nonzero Laurent sum has a single leading monomial. Consequently its graded reduction is $\widetilde{K}^{\mathrm{gr}}[t, t^{-1}]$ with $\deg t = r$. Its degree-one part is only $\widetilde{K}$, although the full graded ring records a new value. This is the algebraic counterpart of $(s, t) = (0, 1)$ in the annulus calculation.

The general theory uses homogeneous prime ideals, graded fraction fields and graded valuation rings. Many elementary arguments retain their form after all generators and localizations are required to be homogeneous. Their application to analytic germs, however, is a geometric theorem. Temkin's reduction of a germ organizes compatible residual data over modifications. In CLD25 these constructions support residual line bundles and the degree computations entering calibrations. We use the specified germ-reduction theorems where they are needed, rather than identifying a germ with the reduction of one chosen affinoid model.

References. [Tem04, §§1–3]; [TemIntro, §5.7]; [CLD25, Chapter 10].

3.10 Reduction of a germ

Ordinary reduction belongs to an affinoid algebra, or to a chosen formal model. A local construction at an analytic point must survive changing both. The appropriate residual object is a birational space. We describe it first in the strict, nontrivially valued setting and then indicate the graded extension.

Definition 3.10.1. For a field extension F/k, let $\mathrm{RZ}_k(F)$ be the set of valuation rings V with $k \subset V \subset F$ and fraction field F. Its basic open subsets have the form

$$U(f_1, \dots, f_m) = \{V : f_1, \dots, f_m \in V\}, \qquad f_i \in F.$$

If Z is an integral k-scheme with function field contained in F, a centre of V on Z is a point z such that $\mathcal{O}_{Z,z} \subset V$ and the maximal ideal of V contracts to that of $\mathcal{O}_{Z,z}$.

Proposition 3.10.2. *The space $\mathrm{RZ}_k(F)$ is quasi-compact and quasi-separated. Each displayed basic open is quasi-compact.*

Proof. We use product compactness in [StacksAlg, Tag 08ZU]. Encode a subset $V \subset F$ by its characteristic function in $\{0, 1\}^F$, with the product

of discrete topologies. The conditions that V contain k, be closed under addition, multiplication and negatives, and satisfy $a \in V$ or $a^{-1} \in V$ for every $a \neq 0$, are closed conditions on finitely many coordinates at a time. They characterize valuation subrings with fraction field F. Thus these subrings form a closed subset of the compact product.

Each $U(f_1, \dots, f_m)$ is clopen in this finer topology and therefore compact there. The Riemann–Zariski topology is coarser, so the same set is quasi-compact in it. These sets form a basis closed under finite intersections; hence the space is also quasi-separated. This proof does not assume that an arbitrary inverse limit of quasi-compact topological spaces is quasi-compact. □

References. Compare [TemIntro, 5.7.2.1–5.7.2.4]. The argument is the usual finite-condition compactness proof, included to make the topological input explicit.

A proper modification of Z does not change the valuations that have a centre: the valuative criterion lifts a centre to the modification, and separatedness makes the lift unique when the generic map is fixed. This explains why compatible centres on models, rather than one preferred scheme, are natural residual data.

Let x lie in a compact strictly analytic neighborhood U, with an admissible formal model $\mathfrak{U}$. Evaluation gives a pointed residual scheme

$$\operatorname{Spec} \widetilde{\mathcal{H}(x)} \longrightarrow \mathfrak{U}_s.$$

Its image has a closure Z in $\mathfrak{U}_s$. Replacing the model by a common admissible modification replaces this pointed scheme by a proper comparison inducing the identity on $\widetilde{\mathcal{H}(x)}$. Formally one localizes the category of pointed residual schemes at such proper maps. The corresponding object, or its realization as a birational space, is the reduction of the germ (X, x). Its independence of U and $\mathfrak{U}$ is Temkin's theorem.

The category in this ordinary-reduction construction is the strictly analytic category over a nontrivially valued field, as fixed at the start of the section. Its subdomain morphisms are strict subdomains. The graded construction in Theorems 3.19.9 and 3.19.10 treats arbitrary analytic domains and arbitrary complete ground fields.

Theorem 3.10.3 (Reduction of strict analytic germs)**.** *For strictly analytic spaces, germ reduction is functorial and independent of the model and neighborhood. Germ subdomains correspond to birational subspaces, compatibly with inclusions, intersections and finite unions. A germ is good precisely when its reduction is affine in the birational category. A morphism is separated at a point precisely when the corresponding reduction morphism is separated; the point lies in relative interior precisely when that reduction morphism is proper.*

Proof. The ordinary reduction, its functoriality, and its independence of strict charts and neighbourhoods are constructed in Theorem 3.19.22. Theorem 3.19.25 identifies it with the pointed residual scheme construction above. Its comparisons under common formal modification are proper by Proposition 3.19.24; Lemma 3.19.23 identifies the corresponding birational spaces.

The strict subdomain classification is part of Theorem 3.19.22. It preserves inclusions; intersections and finite unions are then identified by their respective greatest-lower-bound and least-upper-bound properties. The same theorem proves the three geometric criteria: affineness follows from Theorem 3.19.19 and degree-one valuation restriction; separatedness follows from Proposition 3.19.16; and relative interior follows from Theorem 2.46.1. These are internal results, and their strict/graded comparison is the pullback identity proved in Theorem 3.19.22. □

References. [TemIntro, 5.7.1.3, 5.7.2.10 and 5.7.4.1]; [Tem04].

The word affine here refers to a birational space admitting an affine scheme model in the stated category. It does not assert that the entire projective residual curve at a type II point is affine. This distinction is visible in the next example.

In particular, a pointed proper curve $\operatorname{Spec} F \to C$ has the proper comparison $C \to \operatorname{Spec} k$. The latter point, still pointed by F, is an affine scheme model of the same birational space. Scheme models in this category need not have function field equal to F.

Example 3.10.4. Let x be the unit Gauss point. For the closed unit disc, a residual model of its germ is $\operatorname{Spec} \widetilde{K}[t]$, pointed by the inclusion into $\widetilde{\mathcal{H}(x)} = \widetilde{K}(t)$. For $\mathbb{P}^{1,\mathrm{an}}$ at the same point, use the standard proper model; the residual model is $\mathbb{P}^1_{\widetilde{K}}$ with that function field. The map from the disc germ to the projective-line germ is represented by $\mathbb{A}^1_{\widetilde{K}} \hookrightarrow \mathbb{P}^1_{\widetilde{K}}$. It is not proper, reflecting

that the Gauss point lies on the relative boundary of the closed disc. The omitted point at infinity is the missing outward direction.

For arbitrary radii, replace the fields and reductions in this construction by graded ones. On an affinoid chart the starting map is $\widetilde{A}^{\mathrm{gr}} \to \widetilde{\mathcal{H}(x)}^{\mathrm{gr}}$. The reductions on good subdomains glue. The germ-subdomain, goodness and relative-interior statements have graded versions; see [TemIntro, 5.7.5.1]. This extension is necessary for the radius-r field of Section 1.8, whose ordinary residual field alone loses one analytic dimension.

3.10.1 Where a residual line bundle lives

For a formal line bundle $\mathfrak{L}$ on $\mathfrak{U}$, restrict $\mathfrak{L}_s$ to the closure Z above. After a common formal refinement, pull it back to the corresponding residual model. These restrictions are compatible under further pullback. They define line-bundle data on models of the germ reduction, rather than an arbitrary line bundle on the residue field itself. Intersection degrees are computed on a model where the relevant bundles and subvarieties are represented, and independence is then a projection-formula question.

This describes the formal-model origin of residual data. The extension to the PL metrics of CLD, and the comparison between their residual degrees and analytic curvature, require the constructions and theorems of [CLD25, Chapters 10, 17–18]. Chapter 20 retains these as separate inputs. In particular, identifying the residual object does not already prove a positivity theorem.

3.11 Formal line bundles and model metrics

Let $\mathfrak{X}$ be an admissible formal model of X and let $\mathfrak{L}$ be an invertible sheaf on it. Its generic fibre is an analytic line bundle L. On a formal open chart where $\mathfrak{L}$ has a frame e, prescribe $\|e(x)\| = 1$ on the associated generic domain. For a local section $s = fe$, set $\|s(x)\| = |f(x)|$.

Proposition 3.11.1. *These prescriptions define a continuous metric on L, independent of the formal frames. They commute with pullback of formal line bundles and remain unchanged when the model and line bundle are pulled back along an admissible formal blow-up.*

Proof. Two formal frames differ by a formal unit u. Both u and u^{-1} have absolute value at most one on the generic fibre, so $|u(x)| = 1$. Thus the prescribed norms agree on overlaps. Proposition 3.17.2 constructs the generic fibre from the affinoid domains of formal affine charts. These domains form its analytic net. On every such chart the norm of fe is the continuous function $|f|$, so the chartwise continuity criterion in Lemma 2.40.2 proves continuity of the glued metric. A pulled-back frame again has prescribed norm one, proving pullback compatibility. The generic-fibre isomorphism in Lemma 3.17.3 then proves invariance under admissible formal blowup. □

Model metrics on supplied compatible atlases. Let K be complete and nontrivially valued. Let $\mathfrak{X}$ be an admissible formal model with its identified analytic generic fibre X and supplied compatible formal-affine generic atlas. Let $\mathfrak{L}$ be an invertible sheaf with generic fibre L. Prescribing norm one for formal frames defines a continuous metric on L. For a formal morphism with a supplied compatible generic map this metric commutes with pullback. In particular it is unchanged under an admissible blowup with the compatible source and target atlases of generic invariance on supplied atlases.

Proof. On each formal affine the evaluation of an integral formal function lies in the valuation ring of the completed residue field, as in the affine specialization construction of Theorem 3.16.15. On each supplied formal affine, refine the trivializing formal opens by a finite cover of principal formal opens. The formal-topology clause of NAG-001944 gives the Noetherian topology on that affine; quasi-compactness and the principal basis give the finite refinement. Proposition 3.17.1 identifies the generic trace of each principal restriction with its actual closed rational domain on the given affine chart. The open traces on supplied atlases identify these domains with the full inverse images of the formal opens under specialization. They therefore give a finite covering of the actual generic affine chart. The transition maps on these domains are the supplied bounded maps. If another formal frame differs by a unit u, the evaluations of both u and u^{-1} have norm at most one there. Thus $|u| = 1$ on the overlap and the frame norms agree.

These trivializing domains refine the given analytic net by Lemma 2.39.2. On each one the norm of a section fe in the formal frame e is $|f|$. This is continuous, and the quasi-net topology test of Lemma 2.40.2 gives continuity of the glued metric. Equivalently, in any analytic trivialization its local norm function is tested on the intersections with this net.

The compatible generic pullback of a formal frame is the generic frame of the pulled-back formal sheaf. Its prescribed norm is again one, which proves the pullback equality on the trivializing domains and hence on the whole generic fibre. For the indicated blowup the supplied-atlas invariance identifies its actual generic map with an isomorphism. Under this identification the pullback equality gives the claimed invariance. □

For a model of $L^{\otimes m}$, taking the positive mth root gives the associated rational model metric on L. If L is the trivial bundle with chosen section 1, its weight $\varphi = -\log \|1\|$ is a model function. Changing the chosen trivialization of L changes this weight by the logarithm of an analytic unit; the metric itself remains an object on the line bundle.

Example 3.11.2. On the blow-up of (T, a) from Example 3.6.1, the pulled-back ideal becomes an invertible sheaf. Its generic fibre is canonically trivial because a is a unit over K. On the chart with generator a, the trivial section $1 = a^{-1}a$ has norm $|a|^{-1}$; on the chart with generator T, it has norm $|T|^{-1}$. Its weight is therefore $\log \max(|T|, |a|)$. With $t = -\log |T|$ and $c = -\log |a|$, this is $-\min(t, c)$. The dual line bundle gives $\min(t, c)$.

This example connects an integral modification with a corner of a PL function. To turn the corner into curvature, one still needs a differential calculus and its normalization. To turn a model line bundle into a positive metric, one needs numerical conditions on residual curves. Those are the additional assertions proved and compared in Chapters 15, 19 and 20. The construction here supplies their geometric input without assuming those conclusions.

3.12 Finite boundaries and the generic points of graded reduction

Finite normalization provides a direct route from the affinoid algebra theorems to the existence of a Shilov boundary. A finite algebra over a Tate algebra has finitely many points above its Gauss point. Minimal polynomials show that these points already detect every maximum. We first prove this assertion, then identify the minimal such boundary through graded reduction. All ground fields retain their given absolute values; trivial valuations and arbitrary positive affinoid radii are allowed.

Lemma 3.12.1 (Roots of a monic polynomial)**.** *Let H be a complete non-Archimedean field and let $P(Z) = Z^d + c_1 Z^{d-1} + \cdots + c_d$ be monic of positive degree. In a finite splitting field, endowed with the extending absolute value,*

$$\max_{P(\alpha)=0} |\alpha| = \max_{1\le i\le d} |c_i|^{1/i}.$$

Proof. A finite splitting field exists by [StacksAlg, Tag 09HU]; Lemma 1.12.1 gives its extending absolute value. Write R for the maximum on the right. If $|\alpha| > R$, the leading term α^d has strictly larger absolute value than every other term of $P(\alpha)$, so $P(\alpha) \neq 0$. Thus every root has size at most R. Conversely, if S is the maximum size of the roots, the elementary symmetric expressions for the coefficients give $|c_i| \le S^i$. This proves $R \le S$. The argument includes repeated roots and the polynomial Z^d. □

Proposition 3.12.2 (A boundary above a normalization Gauss point)**.** *Let B be a nonzero strict K-affinoid domain, and let $R = K\langle T_1, \ldots, T_d\rangle \hookrightarrow B$ be a bounded finite injection. Set $F = \mathrm{Frac}(R)$ and let H be the completion of F for the extending Gauss absolute value. The finite-dimensional H-algebra $D = B \otimes_R H$ is nonzero. Its finitely many maximal ideals, with the absolute values on their residue fields, give a finite set $S \subset \mathcal{M}(B)$, and*

$$\max_{x\in\mathcal{M}(B)} |b(x)| = \max_{s\in S} |b(s)| \qquad (b \in B).$$

Proof. The multiplicative Gauss norm extends to the fraction field by $|u/v| = \|u\|/\|v\|$. Its metric completion is again a valued field: if a nonzero Cauchy class is represented by u_n, then $|u_n|$ is eventually constant and positive, and $|u_n^{-1} - u_m^{-1}| = |u_n - u_m|/(|u_n||u_m|)$ proves convergence of the inverses. Put $E = B \otimes_R F$. This is a finite-dimensional F-domain, hence a field. Consequently $D = E \otimes_F H$ has positive finite dimension over H.

A finite-dimensional algebra has only finitely many maximal ideals. Indeed, any r distinct maximal ideals give, by the Chinese remainder theorem, a quotient which is a product of r nonzero fields, forcing $r \le \dim_H D$. Each residue field L is finite over H and has the absolute value of Lemma 1.12.1. The resulting maps $B \to L$ are bounded. To check this last assertion, choose finite R-module generators $b_1, \ldots, b_m$ of B. Proposition 1.14.4 identifies the given norm of B with its canonical finite-module norm. Thus every b has an expression $b = \sum a_i b_i$ with $\max_i \|a_i\| \le C\|b\|$, where C is independent of b.

Evaluation in L then gives $|b| \leq C\|b\| \max_i |b_i|$. These characters therefore define the asserted points.

Fix $b \in B$. By Theorem 1.13.1 and the minimal-polynomial argument in Lemma 1.13.3, the monic minimal polynomial of b over F belongs to $R[Z]$; write it as $Z^e + a_1 Z^{e-1} + \cdots + a_e$. Put $r = \max_i \|a_i\|^{1/i}$. Every analytic point x of B restricts to a bounded point of R, so $|a_i(x)| \leq \|a_i\|$. The dominant-term argument from the preceding lemma gives $|b(x)| \leq r$.

At the Gauss point the coefficients have sizes exactly $\|a_i\|$, so the same lemma supplies a root α of size r in a finite splitting field over H. The algebra $F[b] \otimes_F H = H[Z]/(Z^e + a_1 Z^{e-1} + \cdots + a_e)$ has the maximal ideal defined by this root. The extension

$$F[b] \otimes_F H \longrightarrow E \otimes_F H = D$$

is finite free of positive rank, since E is a finite-dimensional vector space over the field $F[b]$. It is in particular integral and faithfully flat. Lying over, [StacksAlg, Tag 00GQ], gives a maximal ideal of D above the chosen one. Its residue field contains $H(\alpha)$; uniqueness of the extending absolute value implies that the corresponding point $s \in S$ satisfies $|b(s)| = r$. This proves the maximum identity. □

Theorem 3.12.3 (Existence of a finite boundary)**.** *Every nonempty K-affinoid spectrum has a finite boundary.*

Proof. First suppose that A is strict. Its finitely many minimal primes $\mathfrak{p}_j$ exist by the Noetherian theorem (Theorem 1.11.5) and [StacksAlg, Tag 00FR]. Each quotient $B_j = A/\mathfrak{p}_j$ is a strict affinoid domain. Theorem 1.12.2 and Proposition 3.12.2 give a finite boundary of each $\mathcal{M}(B_j)$. Every point of $\mathcal{M}(A)$ kills some $\mathfrak{p}_j$: its prime kernel contains a minimal prime by [StacksAlg, Tag 00E0]. The induced seminorm on B_j is bounded, as is seen by taking the infimum over the representatives in the quotient norm. The union of the finite boundaries therefore detects every maximum on $\mathcal{M}(A)$.

For arbitrary radii, choose a free extension $L = K_r$ for which A_r is strict, as in Theorem 1.17.6. The strict case gives a finite boundary S_r upstairs. The spectral projection is surjective by Theorem 1.19.1. For every $a \in A$, its values upstairs are exactly the values of a at the projected points. It follows that the finite image of S_r is a boundary of $\mathcal{M}(A)$. □

Lemma 3.12.4 (A minimal finite boundary is the smallest closed boundary)**.** *Suppose that a compact space X of multiplicative seminorms on an algebra has continuous evaluation functions and has a finite boundary. Every inclusion-minimal finite boundary is contained in every closed boundary. In particular it is unique.*

Proof. Choose an inclusion-minimal finite boundary S and a point $\xi \in S$. Minimality supplies a with

$$m = |a(\xi)| > \max_{s \in S \setminus \{\xi\}} |a(s)| = q, \qquad m > 0,$$

where the maximum over the empty set is zero. Let C be any closed boundary and put $C_0 = \{y \in C : |a(y)| = m\}$, a nonempty compact set. For b in the algebra and $y \in C_0$, the boundary property of S gives

$$|b(y)| \leq \max_{s \in S} \big(|a(s)|/m\big)^n |b(s)| \xrightarrow[n \to \infty]{} |b(\xi)|.$$

If $|b(\xi)| > 0$, choose for each sufficiently large n a point $y_n \in C$ attaining the maximum of $|a^n b|$. The maximum equals $m^n |b(\xi)|$ for large n. Since $|a| \leq m$ and $|b|$ is bounded, we obtain $|a(y_n)| \to m$ and $|b(y_n)| \geq |b(\xi)|$. A convergent subnet has limit in C_0 and attains $|b(\xi)|$ there. For finitely many $b_1, \ldots, b_t$ with nonzero values at ξ, apply this conclusion to their product. The preceding upper bound for each factor forces all their values to equal the respective values at ξ at one common point of C_0. Factors vanishing at ξ vanish throughout C_0 by that same upper bound.

Thus the closed sets $\{y \in C_0 : |b(y)| = |b(\xi)|\}$, indexed by all b, have the finite intersection property. Compactness produces a point with all evaluations equal to those of ξ. Points are the seminorms themselves, so this point is ξ. Hence $\xi \in C$. This holds for every $\xi \in S$ and every closed boundary C, proving the assertion. □

Corollary 3.12.5 (The affinoid Shilov boundary)**.** *Every nonempty affinoid spectrum has a unique finite Shilov boundary. The Shilov boundary of the empty spectrum is empty.*

Proof. Apply Theorem 3.12.3 and Lemma 3.12.4, using the compact Hausdorff spectrum of Proposition 2.24.1. □

Proposition 3.12.6 (Power-bounded elements and spectral reduction)**.** *For every affinoid algebra A, power-bounded elements are exactly those with spectral radius at most one; topologically nilpotent elements are exactly those with spectral radius less than one. The ring $A^\circ/A^{\circ\circ}$ is reduced.*

Proof. The first equivalence is Theorem 1.20.2 with $s = 1$, together with Corollary 1.20.3. If $\rho(a) < 1$, choose s strictly between $\rho(a)$ and one; the same power bound gives $\|a^n\| \le Cs^n \to 0$. Conversely, if $a^n \to 0$, some power has norm less than one, whence $\rho(a) < 1$ by the spectral formula. The ultrametric and submultiplicative inequalities for ρ show that the first set is a ring and the second an ideal in it. Finally, $\widetilde{a}^n = 0$ implies $\rho(a)^n < 1$, hence $\rho(a) < 1$ and $\widetilde{a} = 0$. □

Theorem 3.12.7 (Generic points of graded reduction)**.** *Let A be a nonzero affinoid algebra, let Γ_A be its Shilov boundary, and let $\widetilde{A}^{\mathrm{gr}}$ be its graded spectral reduction. Evaluation at $\xi \in \Gamma_A$ induces a homogeneous prime*

$$P_\xi = \ker\bigl(\widetilde{A}^{\mathrm{gr}} \longrightarrow \widetilde{\mathcal{H}(\xi)}^{\mathrm{gr}}\bigr).$$

The ideals P_ξ are precisely the minimal homogeneous prime ideals of $\widetilde{A}^{\mathrm{gr}}$. They are distinct, their intersection is zero, and each has exactly one analytic preimage under reduction, namely ξ.

Proof. Evaluation preserves each filtered piece and its strictly smaller subspace, so it defines the displayed map. A graded valued-field reduction has no homogeneous zero divisors. Since the positive real grading group is totally ordered, the largest nonzero components of two finite graded sums have nonzero product; thus the graded ring is also an ordinary domain, and the kernel is a homogeneous prime. If a homogeneous class of degree r belongs to all P_ξ, its lift a satisfies $|a(\xi)| < r$ for all $\xi \in \Gamma_A$. The boundary is finite, so $\rho(a) = \max_{\Gamma_A} |a| < r$ and the class vanishes. Thus the intersection is zero.

For each ξ, minimality of the finite boundary (Corollary 3.12.5) supplies a_ξ whose maximum on Γ_A occurs only at ξ. Its initial class lies in every P_η for $\eta \neq \xi$, but not in P_ξ. Thus these primes are pairwise incomparable. Any homogeneous prime containing zero contains one of them: otherwise choose a homogeneous element of each P_ξ outside that prime and multiply the finitely many choices, contradicting their zero intersection. Pairwise incomparability then shows that these are exactly the minimal homogeneous primes.

It remains to identify the fibre. Suppose that an analytic point y has reduction kernel P_ξ, and choose $a = a_\xi$ as above with $m = |a(\xi)| > q = \max_{\eta\neq\xi} |a(\eta)|$. Since the initial class of a is not in P_ξ, we have $|a(y)| = m$. If $|b(\xi)| > 0$, then, for sufficiently large n, $\rho(a^n b) = m^n|b(\xi)|$. The initial class of $a^n b$ does not lie in P_ξ, so its value at y equals this radius. Dividing by m^n gives $|b(y)| = |b(\xi)|$. If $b(\xi) = 0$, the estimate $m^n|b(y)| \leq q^n \max_{\eta\neq\xi} |b(\eta)|$ for all n forces $b(y) = 0$. Thus all evaluations agree and $y = \xi$. □

References. The generic-point description is Temkin's [Tem04, Proposition 3.3(ii)–(iii)]. The proof here starts from the finite-normalization and minimal-polynomial results already established in the book; its finite-boundary argument supplies the existence input before applying graded reduction.

3.13 Graded reduction and coefficient extensions

Proposition 3.13.1 (The spectral norm of a coefficient expansion)**.** *Let A be an affinoid algebra and r a finite tuple of positive radii. For a series in $A\{r^{-1}T\}$,*

$$\rho\left(\sum_{\nu\in\mathbf{N}^m} a_\nu T^\nu\right) = \max_\nu \rho_A(a_\nu) r^\nu.$$

If r is free over K, the same formula, with indices in $\mathbf{Z}^m$, holds in $A_r = A\widehat{\otimes}_K K_r$.

Proof. At every spectral point of either coefficient algebra, restriction to A and the coordinate bounds give the upper bound on the right. The series tails tend uniformly to zero in the coefficient norm, so this estimate passes from polynomials to series. Conversely, for every $x \in \mathcal{M}(A)$ the Gauss seminorm on the polynomial algebra over $\mathcal{H}(x)$ extends by that same tail estimate and takes the value $\max_\nu |a_\nu(x)|r^\nu$. In the Laurent case this is precisely the section of Theorem 1.19.1. Taking the supremum over x and interchanging two suprema proves the formula, by Corollary 1.20.3. The coefficient weights tend to zero, so every positive supremum is a maximum; when all weights vanish, both sides are zero. □

Theorem 3.13.2 (Graded polynomial and free scalar extension)**.** *There are canonical graded algebra isomorphisms*

$$\widetilde{A}^{\mathrm{gr}}[t_1,\dots,t_m] \simeq \widetilde{A\{r^{-1}T\}}^{\mathrm{gr}}, \qquad \deg t_i = r_i.$$

When r is free over K, there are also canonical isomorphisms

$$\widetilde{A}^{\mathrm{gr}} \otimes_{\widetilde{K}^{\mathrm{gr}}} \widetilde{K_r}^{\mathrm{gr}} \simeq \widetilde{A}^{\mathrm{gr}}[t_1^{\pm 1},\dots,t_m^{\pm 1}] \simeq \widetilde{A_r}^{\mathrm{gr}}.$$

Proof. In a homogeneous class of degree s, only finitely many terms of a convergent series have spectral size s; all remaining terms together have spectral size strictly less than s. This follows because their coefficient norms tend to zero. Taking the initial classes of the maximizing coefficients therefore proves surjectivity of the displayed polynomial or Laurent map. Conversely, a nonzero homogeneous polynomial of degree s has a coefficient with nonzero class at its prescribed degree. The formula in Proposition 3.13.1 makes the spectral radius of any corresponding lift exactly s. Its image cannot vanish. This proves injectivity, and multiplication respects the initial classes by the filtered-algebra construction.

For K_r itself, the same coefficient calculation gives

$$\widetilde{K_r}^{\mathrm{gr}} = \widetilde{K}^{\mathrm{gr}}[t_1^{\pm 1},\dots,t_m^{\pm 1}].$$

The freeness condition is the one used to construct the complete field K_r. Tensoring this polynomial description with $\widetilde{A}^{\mathrm{gr}}$ gives the first isomorphism in the second line. □

References. These coefficient comparisons are the free-radius part of [Tem04, Proposition 3.1(i)]. The spectral coefficient formula and the actual free-extension presentations make the comparisons explicit.

3.14 Exact unit-ball generators and admissible formal algebras

For the formal constructions we need more than a bounded choice of generators: generators of norm one must give coefficient bounds with constant one. The following coefficient-ring argument supplies this strengthening of Theorem 1.11.4.

Lemma 3.14.1 (A coefficient ring with a uniform gap)**.** *Let K be a complete non-Archimedean field. For every sequence (a_i) in K° tending to zero, there is a closed subring $S \subset K^\circ$ containing all a_i, such that every element of S of absolute value one is a unit in S and there is a real $\theta < 1$ with $|s| \leq \theta$ for every $s \in S$ satisfying $|s| < 1$.*

Proof. Call a subring with the latter gap property a gap ring. The prime subring is a gap ring: in positive characteristic its induced absolute value is trivial, while in characteristic zero its elements of size less than one form a principal ideal of $\mathbf{Z}$. Localizing a gap ring at its elements of size one preserves the gap and makes its residue ring a field.

Suppose S_0 is such a localized gap ring and $|a| = 1$. If the residue of a is transcendental over $\widetilde{S}_0$, a polynomial $f \in S_0[Z]$ can have $|f(a)| < 1$ only when all its coefficients have size less than one, so the same gap works. If the residue is algebraic, lift its monic minimal polynomial to $g \in S_0[Z]$. Choose $\theta < 1$ bounding both $|g(a)|$ and the old gap. For $f \in S_0[Z]$ divide by the monic g, obtaining $f = qg + r$ with $\deg r < \deg g$. If $|f(a)| < 1$, then $|r(a)| < 1$; the minimality of the residual polynomial forces every coefficient of r to have size less than one. Hence $|f(a)| \leq \theta$. Thus adjoining one element of size one preserves the gap.

Only finitely many of the a_i have size one; adjoin these first. The remaining sequence is bounded by some $\varepsilon < 1$. In any polynomial in those remaining elements, every nonconstant monomial has size at most ε. Its constant term lies in the already constructed gap ring. Consequently the enlarged ring has gap bounded by the maximum of the old gap and ε. Localize once more at the size-one elements and take its closure in K. The gap survives closure. A size-one element of the closure differs by size less than one from a size-one element of the uncompleted ring; its inverse is obtained by a convergent geometric series. The resulting S has all the stated properties. □

Lemma 3.14.2 (Lifting a residual basis with a uniform gap)**.** *Let $V = c_0(I, K)$ with its maximum norm, where I is finite or countable, and let e_i be its coordinate basis. Let $S \subset K^\circ$ have the properties of Lemma 3.14.1. Suppose $y_j \in c_0(I, S)$ and that their residues form a basis of $\widetilde{V} = \widetilde{K}^{(I)}$. Then the y_j are an orthonormal basis of V.*

Proof. For a finite linear combination of the y_j, divide the coefficients by one of maximal absolute value. Residual linear independence then proves

that its norm is the maximum coefficient norm. Passing to convergent series gives an isometric map $c_0(J,K) \to V$ with closed image. The same argument identifies the closed S-span W_S of the y_j isometrically with $c_0(J,S)$.

The residue field k_S of S embeds in $\widetilde{K}$. The residual vectors of the y_j have finite support and coordinates in k_S. They form a basis already over k_S: linear independence descends immediately, and spanning descends by row reduction of the finite matrix involved in any expression for a coordinate vector. Thus, for every i, there is a finite S-linear combination z_i of the y_j whose residue equals that of e_i. The gap gives $\|e_i - z_i\| \le \theta$ and $\|z_i\| = 1$.

For $x = \sum_i a_i e_i \in c_0(I,S)$, put $P(x) = \sum_i a_i z_i \in W_S$. The series converges and satisfies $\|x - P(x)\| \le \theta\|x\|$. Repeating this construction on the error gives a convergent sum in W_S equal to x. Hence $c_0(I,S) = W_S$. In particular every e_i belongs to the closed K-span of the y_j. Since these coordinate vectors span a dense subspace of V, that closed span is all of V. □

Theorem 3.14.3 (Exact generators for Tate submodules)**.** *Put $T = K\langle T_1, \dots, T_n\rangle$ with its Gauss norm. For every T-submodule $N \subset T^s$, there are finitely many $v_1, \dots, v_q \in N$ of norm one such that every $v \in N$ has an expression*

$$v = \sum_{j=1}^{q} a_j v_j, \qquad a_j \in T, \qquad \max_j \|a_j\| \le \|v\|.$$

Consequently $N \cap (K^\circ\langle T\rangle)^s$ is generated by the same v_j over $K^\circ\langle T\rangle$.

Proof. The image $\overline{N}$ of the unit ball of N in $\widetilde{K}[T_1, \dots, T_n]^s$ is a submodule. The latter polynomial ring is Noetherian by [StacksAlg, Tag 00FN], so choose generators of $\overline{N}$ and lift them to norm-one vectors $v_1, \dots, v_q \in N$. From the monomial multiples $T^\nu v_j$ select a residual $\widetilde{K}$-basis of $\overline{N}$, and complete it to a basis of $\widetilde{K}[T]^s$ by adding some coordinate monomials $T^\nu e_i$. Denote these selected lifts by $(y_\beta)_{\beta\in J}$ and the first part by $J_0 \subset J$.

The coefficients of the finitely many v_j form a sequence tending to zero. Lemma 3.14.1 places them in a single gap ring S. All coefficients of every y_β belong to this same ring. Lemma 3.14.2 therefore shows that $(y_\beta)_{\beta\in J}$ is an orthonormal basis of T^s. Expand $v = \sum_\beta c_\beta y_\beta$ and let $v_0 = \sum_{\beta\in J_0} c_\beta y_\beta$. Grouping the latter series according to the chosen v_j gives $v_0 = \sum_j a_j v_j$ with $a_j \in T$ and $\|a_j\| \le \|v\|$. The coefficient sequences of each a_j tend to zero because $(c_\beta) \in c_0(J,K)$. Hence $v_0 \in N$.

If $v - v_0$ were nonzero, divide it by one of its largest coefficients in the orthonormal expansion. Its nonzero reduction would lie both in $\overline{N}$ and in the span of the complementary residual basis vectors. This is impossible. Thus $v = v_0$, which proves the assertion and the unit-ball consequence. □

References. The uniform-gap and orthonormal-basis method reconstructs [Bosch14, §2.3, Proposition 3, Theorem 6 and Corollary 10, pp. 25–29]. The explicit operator P records the uniform contraction in the basis lifting.

For the remainder of this section K is nontrivially valued, $O = K^\circ$, and $\pi \in O$ satisfies $0 < |\pi| < 1$. An algebra of topologically finite type means a quotient of $O\langle T_1, \dots, T_n\rangle$ with its quotient π-adic topology.

Theorem 3.14.4 (Torsion-free finite modules are finitely presented)**.** *Let A_0 be a topologically finite type O-algebra and let M be a finite A_0-module with no O-torsion. Then M is finitely presented over A_0. In particular, a topologically finite type O-algebra with no π-torsion is topologically finitely presented.*

Proof. Put $P = O\langle T_1, \dots, T_n\rangle$ and choose a surjection $P \twoheadrightarrow A_0$. Since M is finite over P, take a surjection $P^s \twoheadrightarrow M$ with kernel N. Inside $K\langle T\rangle^s = P^s[1/\pi]$, torsion-freeness gives

$$N = N[1/\pi] \cap P^s.$$

The equality $P[1/\pi] = K\langle T\rangle$ follows by bounding the coefficients of a restricted series. Apply Theorem 3.14.3 to $N[1/\pi]$. Its norm-one generators lie in N, and its coefficient bound shows that they generate N over P. Thus M is finitely presented over P. Their images also generate the kernel of $A_0^s \twoheadrightarrow M$, so M is finitely presented over A_0. Finally, in a real-valued rank-one field, π-torsion-freeness implies O-torsion-freeness: for $0 \neq a \in O$ choose m with $|\pi|^m \leq |a|$, so that $\pi^m \in aO$. Apply the result to $M = A_0$ as a quotient of P; its defining ideal is then finitely generated. □

Proposition 3.14.5 (Adic topology on finite formal modules)**.** *Let A_0 be topologically finite type over O, let M be a finite A_0-module, and let $N \subset M$ be any submodule. Its π-saturation*

$$N^{\mathrm{sat}} = \{m \in M : \pi^e m \in N \text{ for some } e \geq 0\}$$

is finitely generated. There is an $e \geq 0$ such that, for all $n \geq 0$,

$$\pi^{n+e} M \cap N \;\subset\; \pi^n N \;\subset\; \pi^n M \cap N.$$

Every finite A_0-module is π-adically complete and separated. In particular every submodule of a finite module is closed.

Proof. Choose $P = O\langle T\rangle \twoheadrightarrow A_0$ and a finite free P-module mapping onto M. Its composite map to M/N^{sat} has finitely generated kernel by the proof of Theorem 3.14.4, since the target is O-torsion-free. The image of this kernel is N^{sat}, which is therefore finite. A common exponent e sends its finite generators into N. If $x = \pi^{n+e}m \in N$, then $m \in N^{\mathrm{sat}}$ and $\pi^e m \in N$, giving the first inclusion.

To prove completeness of a finite module, present it as a quotient of P^s. A compatible sequence of classes modulo π^n has lifts $x_n \in P^s$ with $x_{n+1} - x_n \in \pi^n P^s$: recursively adjust a lift by an element of the presentation kernel. The x_n converge in the complete module P^s, yielding a lift of all the classes. For separatedness take $x \in \bigcap_n \pi^n M$ and apply the first inclusion to the cyclic submodule $A_0 x$. It gives $x \in \pi A_0 x$, so $x = \pi a x$ for some $a \in A_0$. The element $1 - \pi a$ is a unit: lift a to P and use the convergent geometric series there. Hence $x = 0$. Finally M/N is finite and separated, which says exactly that N is closed in M. □

Corollary 3.14.6 (Coherence of formal coefficient rings)**.** *Every topologically finitely presented O-algebra is coherent.*

Proof. For $P = O\langle T\rangle$, the image of a homomorphism $P^r \to P^s$ is finite and O-torsion-free, so its presentation kernel is finite by Theorem 3.14.4. Thus all such kernels are finite. Now write $A_0 = P/(g_1, \dots, g_t)$. Lift a matrix $A_0^r \to A_0^s$ to a matrix over P, and augment it by the st columns $g_j e_i$. The kernel of this augmented matrix is finite over P. Projection to the first r coordinates, followed by reduction modulo $(g_1, \dots, g_t)$, maps it onto the kernel of the original matrix. Hence that kernel is finite over A_0, which is the finite-matrix criterion for coherence. □

References. The formal-module arguments reconstruct the rank-one valuation-ring proof in [Bosch14, §7.3, Theorem 4, Corollaries 5–6, Lemma 7 and Proposition 8, pp. 163–166]. Their analytic input is the exact generator theorem proved above.

Proposition 3.14.7 (Affinoid generic fibres and integral presentations)**.** *Let A_0 be a topologically finite type O-algebra without π-torsion. Then $A = A_0[1/\pi]$ is naturally a strict K-affinoid algebra. A presentation $O\langle T\rangle \twoheadrightarrow A_0$ gives a complete quotient norm on A whose closed unit ball is precisely A_0. Conversely every strict K-affinoid algebra has such an admissible integral presentation. Moreover,*

$$A_0\langle U_1, \dots, U_m\rangle[1/\pi] \simeq A\langle U_1, \dots, U_m\rangle.$$

A homomorphism of admissible integral algebras induces a bounded homomorphism of these generic affinoid algebras.

Proof. Write $P = O\langle T\rangle$ and let J be the presentation kernel. Torsion-freeness gives $J = J[1/\pi] \cap P$. The orthonormal basis constructed in the proof of Theorem 3.14.3, applied to the submodule $J[1/\pi] \subset K\langle T\rangle$, splits that Banach vector space as the orthogonal sum of $J[1/\pi]$ and a closed complement. Every quotient class therefore has a lift attaining its quotient norm. In particular the quotient unit ball is $P/J = A_0$. The quotient is the affinoid algebra $A = A_0[1/\pi]$. This also proves the converse: starting with $K\langle T\rangle/I$, use $P/(I \cap P)$, which is torsion-free and hence topologically finitely presented by Theorem 3.14.4.

The π-adic topology on A_0 is the topology induced by this quotient norm. A restricted series with coefficients in A has bounded coefficients, so multiplication by a single power of π puts all of them in A_0; their convergence to zero is then equivalent to π-adic convergence. This proves the displayed identity. Norm-attaining coefficient lifts also show that $O\langle T, U\rangle \twoheadrightarrow A_0\langle U\rangle$ is surjective. The latter ring is torsion-free and topologically finite type, hence admissible. Finally a map sending A_0 into B_0 is bounded after inverting π: scaling by an appropriate power of π gives a bound with constant $|\pi|^{-1}$ in the two quotient norms. □

Lemma 3.14.8 (Completion of an algebraic quotient)**.** *For a ring R with its π-adic topology and an ideal $J \subset R$, the natural map $\widehat{R} \to \widehat{R/J}$ is surjective and its kernel is the closure of the image of J in $\widehat{R}$. If multiplication by π is injective on R/J, it is injective on $\widehat{R/J}$ as well, and*

$$(R/J)/\pi^n(R/J) \simeq \widehat{R/J}/\pi^n\widehat{R/J} \quad (n \geq 1).$$

Proof. Lift a compatible system in $R/(J + \pi^n R)$ recursively to $r_n \in R$. Compatibility allows one to alter r_{n+1} by an element of J so that $r_{n+1} - r_n \in \pi^n R$.

The resulting Cauchy system defines the required element of $\widehat{R}$. An element in the kernel can, modulo every π^n, be represented by an element of J, which is precisely the asserted closure condition. Put $C = R/J$. If $\pi x = 0$ in $\widehat{C}$, choose a representative $c \in C$ of its component modulo π^{n+1}. Then $\pi c \in \pi^{n+1}C$; cancellation in C yields $c \in \pi^n C$. Thus every component of x is zero. Finally a compatible system vanishing modulo π^n can be divided by π^n componentwise, using the same cancellation to verify compatibility. The kernel of $\widehat{C} \to C/\pi^n C$ is therefore $\pi^n \widehat{C}$, proving the last assertion. □

3.15 Admissible blowups on affine formal models

Theorem 3.15.1 (The charts of an admissible formal blowup)**.** *Let A_0 be an admissible O-algebra and let $I = (f_0, \dots, f_m) \subset A_0$ contain π^N for some N. For each i, set*

$$C_i = A_0[U_j : j \neq i]/(f_i U_j - f_j : j \neq i), \qquad B_i = C_i/\widehat{(\pi\text{-torsion})}.$$

The B_i are admissible O-algebras and are the affine charts of the π-adic completion of $\mathrm{Bl}_I(\operatorname{Spec} A_0)$*. The ideal IB_i is generated by the nonzerodivisor f_i. Equivalently,*

$$B_i \simeq \left(A_0\langle U_j : j \neq i\rangle/(f_i U_j - f_j : j \neq i)\right)/(\pi\text{-torsion}),$$

and in the parenthesized ring its π-torsion equals its f_i-torsion.

Proof. Write $R_i = A_0[U_j : j \neq i]$ and let L_i be the ideal of the displayed relations. Evaluation $U_j \mapsto f_j/f_i$ maps R_i onto the subring $D_i = A_0[I/f_i] \subset A_0[1/f_i]$. Its kernel is $L_i : f_i^{\infty}$: after localizing at f_i, the relations eliminate all the variables, and clearing denominators gives the assertion. Choose $a_j \in A_0$ with $\pi^N = \sum_j a_j f_j$. In C_i this becomes

$$\pi^N = f_i \left(a_i + \sum_{j \neq i} a_j U_j \right). \tag{$*$}$$

Consequently f_i-torsion in C_i is π-torsion. Conversely D_i, being a subring of $A_0[1/f_i]$, is π-torsion-free, so all π-torsion in C_i lies in that kernel. Thus $D_i = C_i/(\pi\text{-torsion})$. By Lemma 3.14.8, $B_i = \widehat{D}_i$ is π-torsion-free and is a quotient of $A_0\langle U_j : j \neq i\rangle$. It is topologically finite type, and Theorem 3.14.4

makes it topologically finitely presented. Equation $(*)$ shows that f_i is a nonzerodivisor on B_i and that $IB_i = (f_i)$.

The degree-zero part of the localization of the Rees algebra $\bigoplus_{d\geq 0} I^d t^d$ at $f_i t$ is precisely D_i: its elements are g/f_i^d with $g \in I^d$. Equality of two such fractions in $A_0[1/f_i]$ is the equality obtained in the homogeneous localization, because both are checked after multiplication by a power of f_i. These standard affine opens cover the Proj of the Rees algebra. Completing them gives the asserted blowup charts.

It remains to compare the two orders of completion and torsion removal. Put $C_i' = A_0\langle U\rangle/L_i A_0\langle U\rangle$. The polynomial π-torsion in C_i maps into the π-torsion τ of C_i'. The latter is finitely generated by Proposition 3.14.5, hence killed by one power of π; its induced topology is therefore discrete. Its image in B_i vanishes. Conversely the kernel of $C_i' \to B_i$ is the closure of the image of the polynomial torsion, by Lemma 3.14.8. The submodule τ is closed by Proposition 3.14.5, so this closure is contained in τ. The reverse inclusion holds because B_i is π-torsion-free. Thus this kernel equals τ. Its dense polynomial-torsion submodule equals τ because τ is discrete. Every element of τ is consequently killed by a power of f_i, as is already true polynomially. Equation $(*)$ gives the reverse torsion inclusion, completing the comparison. □

Theorem 3.15.2 (Generic-fibre invariance of an affine admissible blowup)**.** *In Theorem 3.15.1, let $A = A_0[1/\pi]$. There are canonical bounded isomorphisms*

$$B_i[1/\pi] \simeq A\langle U_j : j \neq i\rangle/(f_i U_j - f_j : j \neq i).$$

The associated spectra are the rational domains

$$V_i = \{x \in \mathcal{M}(A) : |f_j(x)| \leq |f_i(x)| \text{ for all } j\}.$$

They cover $\mathcal{M}(A)$, and the formal chart transitions induce exactly their rational-domain intersections. Thus the generic fibre of the completed blowup, glued from these charts, is canonically $\mathcal{M}(A)$.

Proof. The final presentation in Theorem 3.15.1 and Proposition 3.14.7 give the displayed algebra isomorphism. It is bounded in both directions: all coordinate images lie in the respective integral unit balls, and the finite affinoid quotient topologies agree. Equation $(*)$ in the preceding proof makes f_i invertible in the generic chart. Its inverse is $\pi^{-N}(a_i + \sum_{j\neq i} a_j U_j)$. Thus the

bounded universal property of this quotient is exactly that of the rational domain V_i, with $U_j = f_j/f_i$.

At every point of $\mathcal{M}(A)$ at least one f_i is nonzero, since $\pi^N \in I$. An index of maximal absolute value therefore supplies a chart containing that point. On the overlap of the ith and ℓth algebraic blowup charts, f_ℓ/f_i is invertible. Its formal completion has generic algebra obtained by adjoining its inverse with norm at most one. Since $|f_\ell/f_i| \leq 1$ already on V_i, this cuts out $|f_\ell/f_i| = 1$, precisely $V_i \cap V_\ell$. The inverse-coordinate substitutions agree on triple intersections. The rational-domain structure sheaf consequently identifies the gluing of all generic charts with $\mathcal{M}(A)$. □

Proposition 3.15.3 (The formal universal property)**.** *Let $\mathfrak{X} = \mathrm{Spf}(A_0)$ and let $\mathfrak{X}' \to \mathfrak{X}$ be the completed blowup above. Every morphism of adic formal O-schemes $h : \mathfrak{Y} \to \mathfrak{X}$ for which $I\mathcal{O}_{\mathfrak{Y}}$ is an invertible ideal factors uniquely through $\mathfrak{X}'$.*

Proof. Locally on $\mathfrak{Y}$, one of the f_i generates the invertible ideal: after choosing a local generator, the coefficients of the finitely many f_i generate the unit ideal, so one is a unit near the given point. On such an affine formal open $\mathrm{Spf}(B)$, the generator f_i is a nonzerodivisor. There are unique $u_j \in B$ with $f_i u_j = f_j$. The corresponding polynomial map $C_i \to B$ kills its f_i-torsion, hence factors through D_i. It extends uniquely to the adic completion B_i, since B is complete. The uniqueness of the fractions makes these maps agree on overlaps. Conversely any factorization must map this open into the chart where f_i generates the pulled-back ideal and must have the specified coordinates. This proves global uniqueness. □

References. The chart and universal-property statements are those of [Bosch14, §8.2, Propositions 6–9, pp. 186–189]. The proof above obtains the completion and torsion comparisons from exact unit-ball generators and the adic module topology, and gives the rational-domain identification explicitly.

3.16 Canonical reduction and finite morphisms

We now return to arbitrary complete ground fields, including trivially valued ones. Throughout this section the affinoid algebras are strict. The finite Gauss fibre used above also bounds the residue fields in which the canonical

reduction embeds. The allowed algebraic input, finiteness of normalization for polynomial rings over a field, then supplies its finite generation.

Lemma 3.16.1 (Residue fields of the normalization Gauss fibre)**.** *Let $R = K\langle T_1, \dots, T_d\rangle$, and let H be the completion of* $\mathrm{Frac}(R)$ *for the Gauss absolute value. Then $\widetilde{H} = \widetilde{K}(T_1, \dots, T_d)$. If L/H is finite with its extending absolute value, then $[\widetilde{L} : \widetilde{H}] \leq [L : H]$.*

Proof. An element of H° can be approximated to error less than one by a fraction f/g with $f, g \in R$ and $\|f\| \leq \|g\|$. The Gauss norm of a nonzero series belongs to $|K^\times|$; divide both f and g by a scalar of size $\|g\|$. Their reductions are polynomials, and the denominator has nonzero reduction. Thus every residue of H lies in $\widetilde{K}(T)$. Conversely, lift the numerator and denominator of a rational function over $\widetilde{K}$ to polynomials of Gauss norm at most one, with the denominator of norm one. Their quotient in H has the prescribed residue.

Lift a finite linearly independent family of elements of $\widetilde{L}$ to norm-one elements $u_1, \dots, u_q \in L$. If they satisfied a nonzero H-linear relation, divide its coefficients by a coefficient of maximal absolute value. Reduction would give a nonzero $\widetilde{H}$-linear relation among the chosen residues. Hence the u_i are linearly independent over H, so $q \leq [L : H]$. This bounds the dimension of the entire residue extension and proves the claim. □

Proposition 3.16.2 (Finite canonical reduction over a normalization)**.** *Let B be a strict affinoid domain with a bounded finite injection $R = K\langle T_1, \dots, T_d\rangle \hookrightarrow B$. Then the induced map $\widetilde{K}[T_1, \dots, T_d] = \widetilde{R} \to \widetilde{B}$ is finite.*

Proof. Use the finite extensions $L_1, \dots, L_s$ of the Gauss field H constructed in Proposition 3.12.2. They give an injective map

$$\widetilde{B} \longrightarrow \prod_{i=1}^{s} \widetilde{L}_i.$$

Indeed, a power-bounded element maps to zero in every residue field exactly when all its finitely many boundary values are less than one, which is exactly $\rho_B(b) < 1$. Put $k = \widetilde{K}$ and $S = k[T_1, \dots, T_d]$. For $b \in B^\circ$, its minimal polynomial over $\mathrm{Frac}(R)$ has coefficients in R, as in Lemma 1.13.3. The root calculation in Proposition 3.12.2 gives $\|c_j\|^{1/j} \leq \rho_B(b) \leq 1$ for each coefficient c_j. Reducing that monic equation shows that $\widetilde{b}$ is integral over S.

By Lemma 3.16.1, each $\widetilde{L}_i$ is a finite extension of $\mathrm{Frac}(S)$. The polynomial ring S is Nagata by [StacksAlg, Tag 0335, Proposition 10.162.16]; hence its integral closure S_i in $\widetilde{L}_i$ is finite over S. The preceding injection is S-linear and every coordinate is integral over S. Its image is therefore an S-submodule of the finite S-module $\prod_{i=1}^{s} S_i$. The Nagata property already gives Noetherianity of S by definition [StacksAlg, Tag 032R(2)]. The submodule assertion of [StacksAlg, Tag 00IK, Lemma 10.51.1] now makes $\widetilde{B}$ finite over S. □

Theorem 3.16.3 (Finite generation of strict canonical reduction)**.** *For every admissible strict presentation $K\langle X_1, \dots, X_n\rangle \twoheadrightarrow A$, the map*

$$\widetilde{K}[X_1, \dots, X_n] \longrightarrow \widetilde{A}$$

is finite. In particular $\widetilde{A}$ is a reduced finite type $\widetilde{K}$-algebra.

Proof. The zero algebra is immediate. Otherwise take the finitely many minimal primes $\mathfrak{p}_j$ of A. The canonical map

$$\widetilde{A} \longrightarrow \prod_j \widetilde{A/\mathfrak{p}_j}$$

is injective: if an element has spectral radius less than one on every component, the finite maximum of those radii is less than one on A. For each quotient, apply the presentation-compatible normalization theorem, Theorem 1.12.2. It gives a bounded finite injection from a Tate algebra R_j and a contractive lift $R_j \to K\langle X\rangle$. Proposition 3.16.2 makes $\widetilde{A/\mathfrak{p}_j}$ finite over $\widetilde{R}_j$, and therefore finite over $\widetilde{K}[X]$ via that lift. Their finite product is a finite module over the Noetherian ring $\widetilde{K}[X]$. The displayed injection makes $\widetilde{A}$ a submodule, so it too is finite. Reducedness was proved in Proposition 3.12.6. □

Lemma 3.16.4 (Simultaneous monic division with a small perturbation)**.** *Let R be a complete non-Archimedean Banach algebra. For $1 \le i \le m$, let $P_i \in R[Y_i]$ be monic of degree $d_i > 0$ with all lower coefficients of norm at most one, and let $G_i \in R\langle Y_1, \dots, Y_m\rangle$ satisfy $\|G_i\| < 1$. Then the quotient by the $P_i - G_i$ is generated over R by the monomials Y^ν with $0 \le \nu_i < d_i$. These generators admit coefficient lifts bounded by the norm of a representative.*

Proof. Successively divide by the univariate monic polynomials P_i, using Theorem 1.10.3. Since the coefficients of P_i lie in R, division in Y_i preserves the

degree bounds already obtained in the earlier variables. This gives bounded R-linear maps D_i and r, all of norm at most one, with

$$u = r(u) + \sum_i P_i D_i(u), \qquad r(u) \in \sum_{0 \le \nu_i < d_i} RY^\nu.$$

Put $E(u) = \sum_i G_i D_i(u)$ and $\theta = \max_i \|G_i\| < 1$. Then $\|E(u)\| \le \theta\|u\|$. Substituting $P_i = (P_i - G_i) + G_i$ and iterating yields the convergent identity

$$u = \sum_{n \ge 0} r(E^n u) + \sum_i (P_i - G_i) \sum_{n \ge 0} D_i(E^n u).$$

The first sum remains in the finite free monomial module and has coefficient norm at most $\|u\|$. This proves both assertions. □

Theorem 3.16.5 (The strict reduction criterion for finiteness)**.** *A bounded homomorphism $A \to B$ of strict K-affinoid algebras is finite if and only if the induced homomorphism $\widetilde{A} \to \widetilde{B}$ is finite.*

Proof. Suppose first that B is finite over A. For each minimal prime $\mathfrak{q}$ of B, the domain $B/\mathfrak{q}$ is finite over $A/(A \cap \mathfrak{q})$. Normalize the latter using a fixed strict presentation of A. The resulting normalizing Tate algebra $R_{\mathfrak{q}}$ lifts contractively to that presentation, and $B/\mathfrak{q}$ is finite over $R_{\mathfrak{q}}$. Proposition 3.16.2 shows that $\widetilde{B/\mathfrak{q}}$ is finite over $\widetilde{R}_{\mathfrak{q}}$, hence finite over $\widetilde{A}$. The injection $\widetilde{B} \hookrightarrow \prod_{\mathfrak{q}} \widetilde{B/\mathfrak{q}}$ and the Noetherian property of $\widetilde{A}$ (Theorem 3.16.3) prove this direction.

Conversely assume $\widetilde{B}$ is finite over $\widetilde{A}$. Choose the images $b_1, \dots, b_m$ of the coordinates in a strict presentation of B; they belong to B°. Each residue $\widetilde{b}_i$ satisfies a monic equation over $\widetilde{A}$: one may obtain it by applying the determinant trick to multiplication by $\widetilde{b}_i$ on a finite generating module. Lift its coefficients to $c_{ij} \in A^\circ$. Then

$$w_i = b_i^{d_i} + \sum_{j < d_i} c_{ij} b_i^j, \qquad \rho_B(w_i) < 1.$$

Append variables mapping to the finitely many power-bounded c_{ij} to a strict presentation of A. Bounded substitution gives an admissible surjection $R \twoheadrightarrow A$ from a Tate algebra for which these coefficients have lifts of norm at most one. The induced map $q : R\langle Y_1, \dots, Y_m\rangle \twoheadrightarrow B$ is admissible: the chosen presentation of B, using the Y_i alone, already supplies uniformly bounded

lifts. Let $W_i \in R[Y_i]$ be the lifted monic polynomial. Since w_i is topologically nilpotent, choose $e_i > 0$ and G_i of norm less than one with $q(G_i) = w_i^{e_i}$. Thus $W_i^{e_i} - G_i \in \ker q$.

Apply Lemma 3.16.4 to the univariate polynomials $W_i^{e_i}$. It shows that their perturbed quotient, and hence B, is finite over R. The action of R on B factors through A, so the same finite list of generators generates B as an A-module. □

References. This is the strict canonical-reduction criterion used in the proof of [Tem04, Proposition 3.1(iii)]. Its reconstruction here uses finite normalization, the finite Gauss fibre, the Stacks normalization theorem, and the explicit simultaneous division argument.

Lemma 3.16.6 (Spectral values in the strict case)**.** *For a strict K-affinoid algebra A, every nonzero spectral radius $\rho_A(a)$ belongs to $\sqrt{|K^\times|}$. The same holds for every nonzero value of a at a Shilov point.*

Proof. In the finite-boundary construction, each point is obtained from a finite extension L/H of a completed Gauss field. The value group of H is $|K^\times|$: a nonzero limit has the absolute value of any sufficiently accurate approximant, and fractions of Gauss series have values in $|K^\times|$. For $0 \neq u \in L$, a polynomial equation over H has at least two terms of maximal absolute value. Equality of their sizes gives $|u|^e \in |H^\times|$ for some nonzero integer e. Thus all nonzero values at the constructed finite boundary lie in $\sqrt{|K^\times|}$. Their finite maximum is $\rho_A(a)$. The minimal Shilov boundary is a subset of this boundary by Lemma 3.12.4, proving both assertions. □

Theorem 3.16.7 (Generic points of strict canonical reduction)**.** *For a strict affinoid algebra A, the reduction kernels*

$$P_\xi = \ker\bigl(\widetilde{A} \longrightarrow \widetilde{\mathcal{H}(\xi)}\bigr), \qquad \xi \in \Gamma_A,$$

are exactly the minimal prime ideals of $\widetilde{A}$. They are distinct, and each has the unique analytic preimage ξ.

Proof. These are prime ideals and their intersection is zero, by the finite boundary maximum formula. For each ξ, choose a_ξ peaking at ξ among Γ_A. If its maximum is m, Lemma 3.16.6 gives $e > 0$ and $c \in K^\times$ with $|c| = m^e$. Replacing a_ξ by a_ξ^e/c produces a power-bounded function with value one at

ξ and value strictly less than one at every other Shilov point. Its reduction is in every P_η except P_ξ. Thus these primes are pairwise incomparable. Their zero intersection implies that every prime contains at least one of them, by taking a product of elements outside that prime if none were contained. They are therefore exactly the minimal primes.

Suppose an analytic point y has reduction kernel P_ξ, and use the normalized peaking function $a = a_\xi$. Then $|a(y)| = 1$. If $|b(\xi)| > 0$, choose n large enough that $\rho(a^n b) = |b(\xi)|$, then choose $e > 0$ and $c \in K^\times$ with $|c| = |b(\xi)|^e$. The power-bounded element $a^{ne}b^e/c$ has value one at ξ; its reduction is outside P_ξ and hence has value one at y. It follows that $|b(y)| = |b(\xi)|$. If $b(\xi) = 0$, put $q = \max_{\eta \neq \xi} |a(\eta)| < 1$, with empty maxima equal to zero. The estimates $|b(y)| \leq q^n \max_{\eta \neq \xi} |b(\eta)|$ for all n give $b(y) = 0$. Thus y and ξ agree on every element of A, and are the same analytic point. □

3.16.1 Finite changes of affine formal models

Theorem 3.16.8 (Adjoining power-bounded functions)**.** *Let A_0 be an admissible K°-algebra and $A = A_0[1/\pi]$. If $f_1, \ldots, f_s \in A^\circ$, then $B_0 = A_0[f_1, \ldots, f_s] \subset A$ is a finite admissible A_0-algebra, with $B_0[1/\pi] = A$. There is a nonzero $c \in K^{\circ\circ}$ with $cf_i \in A_0$, and $\mathrm{Spf}(B_0) \to \mathrm{Spf}(A_0)$ is the admissible blowup of the open ideal $(c, cf_1, \ldots, cf_s)$.*

Proof. Choose an admissible presentation of A_0. By Proposition 3.14.7, A_0 is the closed unit ball for the induced quotient norm on A. The monic equations in Theorem 1.13.4 therefore show that each f_i is integral over A_0. Successively reducing powers by those equations makes B_0 a finite A_0-module. It is complete by Proposition 3.14.5. Sending restricted-series variables to the f_i is well defined, since all monomials in them belong to the bounded set B_0; this gives a topologically finite type presentation over A_0, and hence over K°. As a subring of A, B_0 is π-torsion free, and Theorem 3.14.4 makes it admissible.

Clear the finitely many denominators to choose c, and put $I = (c, cf_1, \ldots, cf_s)$. A power of π belongs to cA_0, so I is open. Its extension to B_0 is (c). Consider an admissible formal scheme $\mathfrak{Y}$ over $\mathrm{Spf}(A_0)$ on which I becomes invertible. Locally one of the displayed generators generates $I\mathcal{O}_\mathfrak{Y}$. If c does, every quotient $cf_j/c = f_j$ is a section. If cf_i does, the equation $c = (cf_i)d$ gives $d = f_i^{-1}$ on the generic algebra. The monic integral equation for f_i, multiplied by f_i^{1-r} when its degree is r, expresses

f_i as a polynomial in d with coefficients in A_0. Thus f_i is a unit in $\mathcal{O}_{\mathfrak{Y}}$ and c also generates $I\mathcal{O}_{\mathfrak{Y}}$. In either case all f_j extend uniquely. The extensions glue, since admissible formal rings inject into their generic algebras. They give the unique factorization through $\mathrm{Spf}(B_0)$.

The actual blowup is admissible by Theorem 3.15.1. Its universal property and the property just proved give inverse morphisms between it and $\mathrm{Spf}(B_0)$. This proves the last assertion. □

Corollary 3.16.9 (Lifting an affine analytic morphism)**.** *Let A_0, B_0 be admissible K°-algebras. Every bounded K-homomorphism $\varphi : B_0[1/\pi] \to A_0[1/\pi]$ extends to $B_0 \to A_0'$ after a finite admissible blowup $\mathrm{Spf}(A_0') \to \mathrm{Spf}(A_0)$. Two homomorphisms between admissible formal algebras that agree after inverting π are equal.*

Proof. Choose finitely many topological generators b_i of B_0. They are power bounded in its generic algebra. A bounded map takes power-bounded elements to power-bounded elements, since it takes the bounded set of their powers to a bounded set. Apply Theorem 3.16.8 to $f_i = \varphi(b_i)$. Evaluation of restricted series now gives $B_0 \to A_0'$. All defining relations vanish because they vanish in $A_0'[1/\pi]$ and A_0' has no π-torsion. The same injectivity proves the last assertion. □

Corollary 3.16.10 (A common affine formal model)**.** *Two admissible affine formal models of the same strict K-affinoid algebra are dominated by a common admissible affine model whose morphisms to both are finite admissible blowups.*

Proof. Regard the two models A_0, B_0 as subrings of their common generic algebra A. If $b_1, \dots, b_s$ topologically generate B_0, put $C_0 = A_0[b_1, \dots, b_s]$. Theorem 3.16.8 makes this a complete finite model containing B_0: any restricted series in the b_i converges in C_0 to its value in A. Thus $C_0 = A_0[B_0]$. Reversing the roles gives $D_0 = B_0[A_0]$ by the same argument. Both are the subring of A generated by A_0 and B_0, so $C_0 = D_0$. The cited theorem gives both required blowups. □

References. The finite-model modification is the construction in [Bosch14, 8.4, Lemma 6, pp. 208–209]. Here integrality, completeness and finite presentation are supplied by the proved unit-ball and formal-module statements, and the blowup identification follows from its explicit universal property.

3.16.2 Specialization to arbitrary reduction points

Lemma 3.16.11 (Integral finite normalization of a formal model)**.** *Let $A_0 \neq 0$ be an admissible K°-algebra. There is a finite injective continuous homomorphism $R_0 = K^\circ\langle Y_1, \dots, Y_d\rangle \hookrightarrow A_0$. The ring R_0 is an integrally closed domain.*

Proof. Use the presentation $K^\circ\langle X\rangle \twoheadrightarrow A_0$ and its generic-fibre presentation. The inductive proof of Theorem 1.12.2 retains integral coefficients. Here are the norm bounds that verify this assertion. The coordinate substitutions used there are isometries and therefore preserve the unit balls. For the normalized distinguished element, Weierstrass division sends an input of norm at most one to a quotient and a polynomial remainder whose coefficients have norm at most one. Consequently the images of $1, X_n, \dots, X_n^{s-1}$ generate the unit ball of the quotient over the image of $K^\circ\langle X_1, \dots, X_{n-1}\rangle$. Every element of A_0 admits a lift of norm at most one by Proposition 3.14.7, so this assertion applies to every element, not merely to a dense subring.

The intermediate quotient in the induction is again an admissible formal algebra: it is the quotient of a restricted-series algebra by the contraction of the generic kernel, hence is torsion free and topologically finite type; its finite presentation follows from Theorem 3.14.4. Apply the same argument to it and multiply the resulting finite generating lists. The base case is the full restricted-series ring. Thus the normalization injection from the cited theorem restricts to a finite injection $R_0 \hookrightarrow A_0$.

The fraction field of R_0 is that of $R = K\langle Y\rangle$, since inverting π gives R. An element integral over R_0 is integral over R and hence lies in R by Theorem 1.13.1. In its monic integral equation over R_0, a Gauss norm greater than one would make the leading term strictly dominant. Thus its Gauss norm is at most one, and it belongs to R_0. □

Lemma 3.16.12 (Lifting finitely generated residue fields)**.** *Let K be complete and let $l/\widetilde{K}$ be a finitely generated field extension. There is a complete valued extension L/K with a $\widetilde{K}$-embedding $l \hookrightarrow \widetilde{L}$.*

Proof. Choose a transcendence basis $t_1, \dots, t_e$ such that $l/\widetilde{K}(t_1, \dots, t_e)$ is finite. The completion of $\mathrm{Frac}(K\langle T_1, \dots, T_e\rangle)$ for the Gauss norm has residue field $\widetilde{K}(t_1, \dots, t_e)$ by Lemma 3.16.1. Express the remaining finite extension as a tower of simple extensions. At one step, lift the coefficients of the

monic irreducible residue polynomial to the valuation ring of the current complete field. A root in a finite extension has absolute value at most one, by Lemma 3.12.1. Its residue satisfies the chosen irreducible polynomial, so the resulting residue field contains the required simple extension. The extending absolute value and completeness are supplied by Lemma 1.12.1. Repeating the finitely many steps proves the assertion, including inseparable residue extensions. □

Lemma 3.16.13 (A zero-kernel point above a polydisc reduction point)**.** *Assume K is nontrivially valued. For every prime $\mathfrak{p} \subset \widetilde{K}[Y_1, \dots, Y_d]$, there is a bounded multiplicative norm on $K\langle Y\rangle$ whose reduction kernel is $\mathfrak{p}$.*

Proof. Put $l = \operatorname{Frac}(\widetilde{K}[Y]/\mathfrak{p})$ and realize it in $\widetilde{L}$ by Lemma 3.16.12. Choose $a_i \in L^\circ$ lifting the images of Y_i. In $L\langle U_1, \dots, U_d\rangle$ substitute $Y_i = a_i + \pi U_i$ and take the Gauss norm. The resulting homomorphism is injective: scalar extension $K\langle Y\rangle \hookrightarrow L\langle Y\rangle$ is coefficientwise injective, translation by the a_i is an isometric automorphism of the latter Tate algebra, and $Y_i \mapsto \pi U_i$ is coefficientwise injective. The map is contractive and the Gauss norm multiplicative. Modulo elements of norm less than one, the substitution is $Y_i \mapsto \widetilde{a}_i$, whose kernel is exactly $\mathfrak{p}$. This gives the required norm with zero kernel. □

Lemma 3.16.14 (Prescribing the center in an integral extension)**.** *Let R be an integrally closed domain with fraction field F, and let B be an integral domain finite over R. Suppose a real-valued non-Archimedean absolute value on F is at most one on R, with center $\mathfrak{p} = \{r \in R : |r| < 1\}$. For every prime $\mathfrak{q} \subset B$ above $\mathfrak{p}$, the absolute value extends to $\operatorname{Frac}(B)$ with center exactly $\mathfrak{q}$.*

Proof. Let $E = \operatorname{Frac}(B)$ and choose a finite normal extension N/F containing E, by taking a splitting field of finitely many minimal polynomials; see [StacksAlg, Tag 09HU]. Let C be the integral closure of R in N. There is a prime Q of C above $\mathfrak{q}$, since $B \subset C$ is integral and lying-over applies [StacksAlg, Tag 00GQ]. An extension of the absolute value to N exists without a completeness assumption on F: complete F to H, choose a maximal ideal of the nonzero finite H-algebra $N \otimes_F H$, and use Lemma 1.12.1 on its residue field. The resulting map from the field N is injective. Denote the extended absolute value by $|\cdot|_0$. Every element of C has norm at most one, by its monic equation over R. Thus it has a center Q_0 in C, also above $\mathfrak{p}$.

Put $G = \mathrm{Aut}_F(N)$. The fixed field N^G/F is purely inseparable by [StacksAlg, Tag 030M]. Every element $z \in C^G$ consequently has a power $z^{p^e} \in R$ in positive characteristic, and already lies in R in characteristic zero: the power lies in F and is integral over the normal ring R. It follows that Q and Q_0 have the same contraction to C^G, because membership of z is determined by membership of that power. By [StacksAlg, Tag 0BRI], some $\sigma \in G$ satisfies $\sigma(Q) = Q_0$. The absolute value $|z| = |\sigma(z)|_0$ then has center Q on C. Its restriction to E has center $\mathfrak{q}$ on B, as required. □

Theorem 3.16.15 (Surjectivity of formal specialization)**.** *Let A_0 be an admissible K°-algebra and $A = A_0[1/\pi]$. The map*

$$\mathrm{sp} : \mathcal{M}(A) \longrightarrow \mathrm{Spec}(A_0/K^{\circ\circ}A_0), \qquad x \longmapsto \{a \in A_0 : |a(x)| < 1\}/K^{\circ\circ}A_0$$

is well defined and surjective onto all prime ideals.

Proof. Every analytic point has value at most one on A_0, by Proposition 3.14.7 and Lemma 1.6.1. Its elements of value less than one form a prime ideal containing $K^{\circ\circ}A_0$, which proves well-definedness.

First suppose A_0 is a domain, and let $\mathfrak{q} \subset A_0$ represent a given special-fibre point. By Lemma 3.16.11, choose a finite injection $R_0 = K^\circ\langle Y\rangle \hookrightarrow A_0$. The contraction of $\mathfrak{q}$ defines a prime of $\widetilde{K}[Y]$. Lemma 3.16.13 gives a multiplicative norm on $K\langle Y\rangle$ centered there. Extend it to the fraction field of R_0 by quotients. Lemma 3.16.14 extends this absolute value to $\mathrm{Frac}(A_0)$ with center $\mathfrak{q}$. It is bounded on A: every element of A_0 is integral over R_0, hence has value at most one, and the quotient-unit-ball description gives contractivity on A. It therefore defines an analytic point specializing to $\mathfrak{q}$.

For general A_0, let $P_1, \ldots, P_t$ be the finitely many minimal primes of the Noetherian algebra A, and put $Q_i = P_i \cap A_0$. Their intersection is the nilradical of A_0, since $A_0 \hookrightarrow A$. A prime $\mathfrak{q}$ of A_0 therefore contains some Q_i: otherwise a product of elements of the Q_i outside $\mathfrak{q}$ gives a contradiction. The quotient A_0/Q_i is complete by Proposition 3.14.5, topologically finite type, and torsion free as a subring of A/P_i. It is admissible by Theorem 3.14.4. Apply the domain case to $\mathfrak{q}/Q_i$, then compose the resulting point with $A \to A/P_i$. The zero algebra has empty source and target. □

Theorem 3.16.16 (Surjectivity of strict canonical reduction)**.** *For every strict K-affinoid algebra, including a trivially valued ground field, the canonical reduction map $\mathcal{M}(A) \to \operatorname{Spec}(\widetilde{A})$ is surjective.*

Proof. Suppose first that K is nontrivially valued. Choose an admissible model $A_0 \subset A$ by Proposition 3.14.7. The algebra $\widetilde{A}$ is finitely generated by Theorem 3.16.3. Lift a finite generating list to A° and adjoin it to A_0. Theorem 3.16.8 gives a new admissible model B_0 for which $B_0 \to \widetilde{A}$ is surjective. A prime $\mathfrak{p} \subset \widetilde{A}$ pulls back to a special-fibre prime $\mathfrak{q}$ of B_0. Choose x with center $\mathfrak{q}$ by Theorem 3.16.15. For any $b \in A^\circ$, choose $a \in B_0$ with $b - a \in A^{\circ\circ}$. Then $|b(x)| < 1$ if and only if $|a(x)| < 1$, because $|b(x) - a(x)| < 1$. The reduction kernel of x is therefore exactly $\mathfrak{p}$.

If K is trivially valued, a strict Tate algebra is a polynomial algebra with the trivial norm, and its quotient norm is trivial. Thus $A^\circ = A$, and a sequence of powers tends to zero exactly when it is eventually zero. Consequently $A^{\circ\circ} = \sqrt{(0)}$. For a prime of $\widetilde{A} = A/\sqrt{(0)}$, give the corresponding prime quotient of A the trivial multiplicative norm. It is bounded for the quotient norm on A and has precisely the requested reduction kernel. □

Corollary 3.16.17 (Surjectivity of homogeneous graded reduction)**.** *For an arbitrary generalized K-affinoid algebra A, every homogeneous prime ideal of $\operatorname{gr} A$ is the reduction kernel of a point of $\mathcal{M}(A)$.*

Proof. First let A be strict and let $P \subset \operatorname{gr} A$ be homogeneous prime. Its degree-one part is a prime $P_1 \subset \widetilde{A}$; realize it by x using Theorem 3.16.16. If the initial class of a is nonzero of degree $r = \rho_A(a)$, Lemma 3.16.6 supplies $e > 0$ and $c \in K^\times$ with $|c| = r^e$. The class of a^e/c has degree one. Primality shows that the initial class of a belongs to P exactly when this degree-one class belongs to P_1, equivalently $|a(x)| < r$. This proves that the full graded kernel of x is P.

In general choose a free polyradius $\boldsymbol{r}$ making $A_{\boldsymbol{r}}$ strict. By Theorem 3.13.2, its graded reduction is the Laurent polynomial algebra $\operatorname{gr} A[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ with the stated degrees. Extend P to this Laurent polynomial algebra; the resulting homogeneous ideal is prime because $(\operatorname{gr} A)/P$ is a domain. The strict case gives a point of $\mathcal{M}(A_{\boldsymbol{r}})$ with that kernel. Its restriction to A has graded kernel P, because the Gauss extension is spectrally isometric on A. □

References. The norm-one normalization follows the proof of [Bosch14, 8.3, Proposition 8, pp. 200–202]. The additional independent Gauss perturbation and the prescribed-center argument above realize arbitrary prime ideals, as needed for the Berkovich and graded reduction maps; they also avoid a restriction to closed points of the special fibre.

3.17 Formal gluing and the Raynaud equivalence

In this section K is any complete nontrivially real-valued non-Archimedean field. Its value group need not be discrete. The analytic category is the category of strict analytic G-spaces: its affinoid charts are strict and its admissible open coverings are the strict affinoid coverings and their composites. A covering is of finite type if each member meets only finitely many others. Quasi-paracompact means admitting such a covering by quasi-compact admissible opens. The same definition, with ordinary open subsets, is used for formal schemes.

Proposition 3.17.1 (Principal formal opens and flat restriction)**.** *Let A_0 be admissible and $f \in A_0$. Write $A_0\{f^{-1}\}$ for the π-adic completion of $A_0[1/f]$. It is an admissible algebra,*

$$A_0\{f^{-1}\} = A_0\langle U\rangle/(fU-1), \qquad A_0\{f^{-1}\}/\pi^n = (A_0/\pi^n)[1/f].$$

The homomorphism $A_0 \to A_0\{f^{-1}\}$ is flat. On $\mathrm{Spf}(A_0)$, it is the restriction ring of the principal open $D(f)$. Its generic fibre is the strict rational domain $\{x : |f(x)| = 1\}$.

Proof. The polynomial quotient $A_0[U]/(fU-1)$ is $A_0[1/f]$ and is π-torsion free. Lemma 3.14.8 gives the completion and reduction formulas. The ideal generated by $fU-1$ in $A_0\langle U\rangle$ is closed by Proposition 3.14.5, so its completion has the displayed quotient presentation. It is torsion free by the same completion lemma and hence admissible.

Here is a direct flatness proof. Put $R = A_0[1/f]$ and $B_0 = \widehat{R}$. Let $I \subset A_0$ be a finitely generated ideal, choose a surjection $\alpha : A_0^s \to I$, and let $N = \ker\alpha$. The module N is finite by Corollary 3.14.6. Proposition 3.14.5 gives $e \geq 0$ with

$$I \cap \pi^{n+e}A_0 \subset \pi^n I \qquad (n \geq 0).$$

Algebraic localization preserves this containment. For $v \in \ker(B_0^s \xrightarrow{\alpha} B_0)$ choose $w_n \in R^s$ with $v - w_n \in \pi^{n+e}B_0^s$. The reduction formula shows $\alpha(w_n) \in$

$IR \cap \pi^{n+e}R \subset \pi^n IR$. Thus $\alpha(w_n) = \pi^n \alpha(u_n)$ for some $u_n \in R^s$, and $w_n - \pi^n u_n \in NR$. These vectors approach v. The image $NB_0 \subset B_0^s$ is closed by Proposition 3.14.5, so $v \in NB_0$. Hence $I \otimes_{A_0} B_0 \to B_0$ is injective. The finite-ideal flatness criterion [StacksAlg, Tag 00HD] proves flatness.

The formal structure sheaf is the inverse limit of the structure sheaves on $\operatorname{Spec}(A_0/\pi^n)$. Taking sections on $D(f)$ gives precisely the displayed inverse limit. After inverting π, the quotient presentation is $A\langle U\rangle/(fU-1)$, the domain $|f| \geq 1$. Since $f \in A_0$ has value at most one everywhere, this is the asserted equality domain. □

Proposition 3.17.2 (The global generic fibre and its faithfulness)**.** *The affine generic-fibre construction glues to a functor from admissible formal K°-schemes to strict analytic G-spaces. For a formal open $\mathfrak{U} \subset \mathfrak{X}$,*

$$\mathfrak{U}_\eta = \operatorname{sp}^{-1}(\mathfrak{U}).$$

The specialization map is surjective and functorial. The generic-fibre functor is faithful; it detects equality and covering of formal open subspaces. A quasi-paracompact formal scheme has a quasi-separated quasi-paracompact generic fibre.

Proof. On a principal formal open, the formula follows from Proposition 3.17.1 and $f \notin \operatorname{sp}(x)$ if and only if $|f(x)| = 1$. Principal opens form a basis, so it holds for every open. Finite intersections and refinements of these principal covers give the same rational restriction maps; hence the affinoid pieces glue, with the analytic structure sheaf of the affinoid G-site, and the resulting construction is independent of the chosen formal affine cover. Morphisms glue as well because their restrictions are induced by the given continuous algebra homomorphisms. Specialization is compatible with these maps by contraction of the ideals of elements of value less than one. Its surjectivity is local and follows from Theorem 3.16.15.

Two formal morphisms with equal generic fibres have equal underlying point maps, by specialization surjectivity and functoriality. We may therefore compare them on a common formal affine source and target. Their ring homomorphisms agree after inverting π, and hence agree before inverting it, as the target ring is torsion free. This proves faithfulness. The same surjectivity and the displayed formula detect formal open subsets and their covers.

For every $c \in K^{\circ\circ}$ some power c^n belongs to πK°, while $\pi \in K^{\circ\circ}$; consequently $\sqrt{\pi A_0} = \sqrt{K^{\circ\circ} A_0}$. Thus $\operatorname{Spec}(A_0/\pi A_0)$ and $\operatorname{Spec}(A_0/K^{\circ\circ} A_0)$ have the same underlying topological space, so the affine formal space is topologically Noetherian because its special fibre is a finite-type algebra over $\widetilde{K}$; no Noetherianity of $A_0/\pi A_0$ is needed. Intersections of quasi-compact formal opens are quasi-compact, and have finite principal-affine covers. Their generic fibres consequently have finite strict affinoid covers. A finite-type quasi-compact formal cover therefore induces a finite-type admissible cover of the same kind on the generic fibre; the intersection observation proves quasi-separatedness. □

Lemma 3.17.3 (Products of open ideals and global blowups)**.** *A coherent open ideal on an admissible formal scheme has an admissible formal blowup, obtained by gluing the affine constructions. It is compatible with restriction to formal opens, has the invertible-ideal universal property, and induces an isomorphism on generic fibres. Blowing up a finite product of coherent open ideals makes each factor invertible, so dominates the blowup of each factor. For a formal morphism $\mathfrak{X} \to \mathfrak{Y}$, every blowup of $\mathfrak{Y}$ is dominated over $\mathfrak{Y}$ by the blowup of its pulled-back ideal on $\mathfrak{X}$.*

Proof. The affine universal property in Proposition 3.15.3 identifies the constructions on overlapping formal open subsets: each is the universal space making the same restricted ideal invertible. The unique identifications satisfy the cocycle condition. Gluing locally ringed spaces [StacksAlg, Tag 01JB] gives the formal scheme, since being an admissible formal affine chart is a local property. Its generic-fibre invariance follows from Theorem 3.15.2 on each chart.

It remains to verify the elementary assertion about products. In a local ring let $IJ = (g)$ with g a nonzerodivisor. Write $g = \sum_i a_i b_i$ with $a_i \in I$, $b_i \in J$, and $a_i b_i = g c_i$. Since $\sum_i c_i = 1$, some c_i is a unit. For that index, a_i, b_i are nonzerodivisors, and for every $a \in I$ the relation $a b_i \in (g)$ gives $a \in (a_i)$ after cancelling b_i. Thus $I = (a_i)$, and likewise $J = (b_i)$. Apply this argument successively. An open ideal contains a power of π, so a generator of its invertible extension is a nonzerodivisor. The universal property now gives every stated domination. The pulled-back ideal is again open and finitely generated; the same universal property proves the base-change assertion. □

Generic invariance on supplied atlases. Let $b : \mathfrak{X}' \to \mathfrak{X}$ be the admissible blowup of a coherent open ideal. Use actual compatible formal-affine generic atlases, including the affine blowup charts and their principal refinements. Then the chartwise generic-fibre isomorphisms identify b_η with an isomorphism of these analytic G-spaces.

Indeed, the formal restriction and universal property in the preceding lemma identify the formal blowup on a formal open with the blowup of the restricted ideal. Refine overlaps of the supplied formal charts by principal formal opens. The generic trace of such an open $D(f)$ is the actual rational equality domain $|f| = 1$, by Proposition 3.17.1 and the equation $f \notin \mathrm{sp}(x)$ if and only if $|f(x)| = 1$. Specialization contracts along the given chart maps, so the inverse image of that trace under b_η is the trace of $b^{-1}(D(f))$. On an affine chart, Theorem 3.15.2 identifies the generic blowup charts and transitions with the actual rational domains and intersections of the original affinoid. On the common principal refinements the chart maps are consequently restrictions of the same chart isomorphism for the restricted formal blowup. Their inverses agree there as well. The two families glue as morphisms on the supplied admissible covers by Theorem 2.40.8. The two composites are identities on those covers, so are identities globally. This argument also transports the existing affine atlas to these compatible blowup charts, with the indicated restriction maps. It applies at each stage of an actual sequence of such blowups.

Open traces on supplied atlases. For an actual supplied formal-affine generic atlas, the preceding principal trace calculation gives the generic trace of a formal open by union on its charts. This is the specialization inverse image in the analytic G-site. Specialization is surjective on these charts by Theorem 3.16.15, hence surjective on the model. Inverse images therefore detect equality, coverage and disjointness of the formal opens and preserve their intersections.

Faithfulness for supplied generic arrows. For two given formal arrows with actual compatible generic charts, equality of their generic maps gives equality of their formal point maps by specialization surjectivity and contraction. At a formal source point, choose a common target affine neighborhood of its image and a source affine neighborhood mapping there under both arrows. The two pullback homomorphisms agree after inverting π. The coefficient ring of the source is π-torsion-free, so the homomorphisms agree there integrally. The arrows agree on this affine cover and hence on the source.

Lemma 3.17.4 (Extending an open ideal with controlled support)**.** *Let $\mathfrak{X}$ be a quasi-paracompact admissible formal scheme and $\mathfrak{U} \subset \mathfrak{X}$ a quasi-compact open. Every coherent open ideal $\mathcal{I}_{\mathfrak{U}}$ extends to a coherent open ideal $\mathcal{I}$ on $\mathfrak{X}$ that is the unit ideal on every open disjoint from $\mathfrak{U}$. Consequently a finite composition of admissible blowups on $\mathfrak{U}$ extends to such a composition on $\mathfrak{X}$, unchanged over opens disjoint from $\mathfrak{U}$.*

Proof. A finite-type quasi-compact open cover of $\mathfrak{X}$ contains a finite subfamily covering $\mathfrak{U}$. The union $\mathfrak{W}$ of these members and all members meeting them is quasi-compact and contains $\overline{\mathfrak{U}}$. Indeed, any member containing a point of this closure meets $\mathfrak{U}$ and hence one of the chosen finitely many members. The space $\mathfrak{W}$ is Noetherian topologically, by Proposition 3.17.2. In particular $\mathfrak{W} \setminus \overline{\mathfrak{U}}$ and $\mathfrak{U}$ are quasi-compact opens in it.

Choose N with $\pi^N \mathcal{O}_{\mathfrak{U}} \subset \mathcal{I}_{\mathfrak{U}}$. On the disjoint open union $\mathfrak{U} \cup (\mathfrak{W} \setminus \overline{\mathfrak{U}})$ of the scheme $\mathfrak{W}_N = (\mathfrak{W}, \mathcal{O}_{\mathfrak{W}}/\pi^N)$, prescribe the finite-type ideal $\mathcal{I}_{\mathfrak{U}}/\pi^N$ on the first part and the unit ideal on the second. Apply [StacksAlg, Tag 01PF] to extend this as a finite-type quasi-coherent ideal on $\mathfrak{W}_N$. Its inverse image in $\mathcal{O}_{\mathfrak{W}}$ is locally generated by lifts of its finite generators and π^N. Flatness of principal formal restrictions, proved in Proposition 3.17.1, identifies these ideals on overlaps; coherence follows from Corollary 3.14.6. Glue it with the unit ideal on $\mathfrak{X} \setminus \overline{\mathfrak{U}}$. The two prescriptions agree on their intersection and give the required $\mathcal{I}$.

For a composition of blowups, extend the first center, then the second center on the inverse image of $\mathfrak{U}$, and so on. At every stage the modification is an isomorphism on opens disjoint from that inverse image. Blowups of quasi-paracompact formal schemes remain quasi-paracompact: over each member of a finite-type cover their finitely many affine blowup charts form a finite refinement of the pulled-back cover, and nonintersecting original members remain disjoint. This justifies the induction. □

Proposition 3.17.5 (Formalizing a finite-type analytic covering)**.** *Use an actual compatible formal-affine generic atlas on $\mathfrak{X}$ and its principal traces on formal opens. On the blowups in the construction use its compatible affine blowup charts as in the supplied-atlas generic-invariance result above. Let $\mathfrak{X}$ be quasi-paracompact and admissible, and let (U_α) be a finite-type admissible covering of $\mathfrak{X}_\eta$ by quasi-compact strict analytic opens. There is an admissible blowup $\mathfrak{X}' \to \mathfrak{X}$ and a finite-type covering by quasi-compact formal opens $(\mathfrak{U}'_\alpha)$ with generic fibres exactly U_α.*

Proof. Use compatible supplied generic atlases throughout. Suppose first $\mathfrak{X} = \mathrm{Spf}(A_0)$. The cover is finite: quasi-compactness gives a finite subcover, and its members meet only finitely many members of the original finite-type cover. Every U_α has a finite strict affinoid cover; Theorem 2.27.5 then refines these affinoid domains by rational domains of $\mathcal{M}(A)$. Consider one such domain

$$V = \{x : |f_j(x)| \le |f_0(x)|,\ 1 \le j \le s\}, \qquad (f_0, \dots, f_s)A = A.$$

Multiply all f_j by the same power of π to put them in A_0. Clearing the coefficients in a unit-ideal relation shows that $I = (f_0, \dots, f_s)A_0$ contains a power of π. In its blowup, the formal open chart where f_0 generates the transformed ideal has generic fibre V, by Theorem 3.15.2. Blow up the product of the finitely many ideals obtained for the rational refinements. By the formal product property of Lemma 3.17.3, this dominates all their blowups. By generic invariance on the supplied charts, the inverse images of the selected charts have the prescribed rational generic fibres. Their union for each α gives $\mathfrak{U}'_\alpha$. These opens cover by detection of formal covers on the supplied atlas.

In general choose a finite-type formal affine cover $(\mathfrak{V}_i)$ and put $V_i = (\mathfrak{V}_i)_\eta$. Only finitely many U_α meet a fixed V_i: cover V_i by finitely many U_α and use finite type. The intersections $U_\alpha \cap V_i$ are quasi-compact by quasi-separatedness. The affine case gives an open coherent ideal I_i on $\mathfrak{V}_i$ realizing all these intersections. Extend it by Lemma 3.17.4, as the unit ideal away from $\overline{\mathfrak{V}_i}$. The product of the extended ideals is locally finite: on $\mathfrak{V}_j$ only those i with $\mathfrak{V}_i \cap \mathfrak{V}_j \neq \varnothing$ contribute. It is a coherent open ideal; blow it up.

Over $\mathfrak{V}_i$ the formal product property gives domination of the blowup of I_i. Use generic invariance on the supplied atlases for these blowups. Pull back the opens representing $U_\alpha \cap V_i$ and take their union over i to define $\mathfrak{U}'_\alpha$. Its generic fibre is U_α. Only finitely many V_i meet the quasi-compact U_α, by the same finite-type argument. Each local formal open is quasi-compact, since it lies in a quasi-compact blowup with Noetherian underlying space. Thus $\mathfrak{U}'_\alpha$ is quasi-compact. Detection of coverage and disjointness on the supplied atlas proves that these opens give the required finite-type cover. □

Theorem 3.17.6 (Lifting a global analytic morphism)**.** *Use actual compatible formal-affine generic atlases on $\mathfrak{X}$ and $\mathfrak{Y}$. On formal opens use their principal traces, and on successive blowups use the compatible affine blowup charts of the supplied-atlas generic-invariance result. Let $\mathfrak{X}, \mathfrak{Y}$ be quasi-paracompact admissible formal schemes. Every analytic morphism $f : \mathfrak{X}_\eta \to \mathfrak{Y}_\eta$ lifts to a formal morphism $f' : \mathfrak{X}' \to \mathfrak{Y}$ after a finite composition $s : \mathfrak{X}' \to \mathfrak{X}$ of admissible blowups, with $f'_\eta = f \circ s_\eta$. Two stages suffice.*

Proof. Use finite-type formal affine covers of source and target with their supplied compatible generic atlases. Pull back the target admissible cover and refine it on each affinoid source chart by finitely many strict affinoids. The resulting source cover is of finite type, and each member maps into a target affine generic chart. By cover realization on supplied atlases, one blowup makes it a formal open cover. Refine each quasi-compact formal member by finitely many formal affine opens. The resulting cover $(\mathfrak{W}_i)$ is still of finite type, and each $(\mathfrak{W}_i)_\eta$ maps into the generic fibre of a formal affine target chart $\mathfrak{V}_i$.

Corollary 3.16.9 gives on each $\mathfrak{W}_i$ a blowup and a lifting to $\mathfrak{V}_i$. Extend its center by Lemma 3.17.4, and blow up the locally finite product of the extended centers. This second stage dominates all the local blowups by the formal product property of Lemma 3.17.3. By generic invariance on the supplied atlases, the resulting morphisms to $\mathfrak{Y}$ have generic maps that are restrictions of f under the blowup identifications. On each overlap they agree by faithfulness for the supplied generic arrows. They glue to f'. □

Lemma 3.17.7 (A common modification of isomorphic generic fibres)**.** *Use actual compatible formal-affine generic atlases on the two models, with their principal traces on formal opens and the compatible affine blowup charts along the successive modifications. Let $\mathfrak{X}, \mathfrak{Y}$ be quasi-paracompact admissible formal schemes with a specified generic-fibre isomorphism. There is a quasi-paracompact admissible formal scheme $\mathfrak{Z}$ and morphisms to both, each a finite composition of admissible blowups, inducing that isomorphism.*

Proof. With the supplied compatible generic atlases, lift the isomorphism and its inverse by lifting on supplied atlases. Obtain modifications $s : \mathfrak{X}_1 \to \mathfrak{X}$, $t : \mathfrak{Y}_1 \to \mathfrak{Y}$ and maps $u : \mathfrak{X}_1 \to \mathfrak{Y}$, $v : \mathfrak{Y}_1 \to \mathfrak{X}$. Pull back the successive centers of t along u and blow them up in order. This gives $p : \mathfrak{X}_2 \to \mathfrak{X}_1$ and $a : \mathfrak{X}_2 \to \mathfrak{Y}_1$ with $ta = up$. Likewise pull back the centers of s along v,

obtaining $q : \mathfrak{Y}_2 \to \mathfrak{Y}_1$ and $b : \mathfrak{Y}_2 \to \mathfrak{X}_1$ with $sb = vq$. All s, t, p, q are finite composites of admissible blowups.

Identify the generic fibres at each stage by generic invariance on the supplied atlases. The respective composites have equal generic maps, so faithfulness for the supplied generic arrows gives $va = sp$ and $ub = tq$. The first equality shows that the successive centers of s pulled back along va are already invertible on $\mathfrak{X}_2$, since the map factors through s. The universal properties defining q therefore factor a through $\alpha : \mathfrak{X}_2 \to \mathfrak{Y}_2$. Similarly b factors through $\beta : \mathfrak{Y}_2 \to \mathfrak{X}_2$. The pullbacks of invertible ideals are invertible here: each contains a power of π, a nonzerodivisor on the admissible source. The generic maps of α, β are inverse, so the same faithfulness gives $\alpha\beta = 1$ and $\beta\alpha = 1$. Identify the spaces by this isomorphism. Their maps sp, tq are the required modifications. □

Generic fibres of quasi-paracompact formal schemes. Retain the standing complete nontrivially real-valued field K, its valuation ring O, and $\pi \in O$ with $0 < |\pi| < 1$. Let $\mathfrak{X}$ be an admissible formal O-scheme that admits a finite-type ordinary open cover by quasi-compact opens. Here finite type means that each member meets only finitely many others. The affine generic fibres glue canonically to a quasi-separated quasi-paracompact strict analytic G-space $\mathfrak{X}_\eta$, functorially for formal arrows between such schemes. Specialization is natural and surjective. For every formal open $\mathfrak{U} \subseteq \mathfrak{X}$, the generic fibre of $\mathfrak{U}$ identifies with the analytic domain $\mathrm{sp}_{\mathfrak{X}}^{-1}(\mathfrak{U})$ with its restricted strict charts and bounded restriction maps.

Proof. Refine each member of the finite-type ordinary open cover by finitely many formal affines $P_i = \mathrm{Spf}(A_i)$. The resulting family still has finite incidence. Each P_i is topologically Noetherian by the formal-topology clause of NAG-001944. In particular, every open subset of P_i has a finite principal refinement. We use the strict generic affinoid $X_i = \mathcal{M}(A_i[1/\pi])$ from Proposition 3.14.7, with the affine specialization map s_i.

We first produce the actual overlaps and their transitions. For a point of $P_i \cap P_j$, take a principal open $D_i(f)$ contained in that overlap and containing the point. Inside it take a principal open $D_j(g)$ of P_j containing the point. On $D_i(f)$, the reduction modulo π of the restriction of g is a/f^n for some $a \in A_i/\pi A_i$, by Proposition 3.17.1. Consequently $D_j(g) = D_i(f\widehat{a})$, where $\widehat{a} \in A_i$ lifts a. This is an equality of open subspaces of the same formal scheme with its restricted structure sheaf. Iterating the argument works on

a finite intersection of formal affines. A principal open of a shared principal piece is principal in each ambient formal affine, by the same localization formula. Since each formal overlap is quasi-compact, finitely many common principal pieces cover it.

Set $X_{ij} = s_i^{-1}(P_i \cap P_j)$ in X_i. The traces of the common principal pieces are the actual strict rational domains given by Proposition 3.17.1. They cover X_{ij} by the affine equality $s_i^{-1}(D(f)) = \{|f| = 1\}$. Thus X_{ij} is a compact closed analytic domain of the Hausdorff affinoid X_i, by Proposition 2.39.3. The finite rational traces are an admissible cover by Lemma 2.39.2. A shared affine formal open has the same coefficient ring viewed from either side. Its two generic identifications are bounded with bounded inverses by Proposition 3.14.7 and the principal restriction proposition. The local identifications agree on the common principal refinements because both restrict the same formal structure sheaf maps. Theorem 2.40.8 glues them and their inverses to $v_{ij} : X_{ij} \to X_{ji}$.

The maps v_{ij} preserve the formal centre by affine contraction. In particular $v_{ij}(X_{ij} \cap X_{ik}) = X_{ji} \cap X_{jk}$. On the common principal pieces of the actual triple overlap, $v_{jk} \circ v_{ij} = v_{ik}$ because all three maps are induced by the same restricted formal rings. This equality holds on the entire triple overlap by uniqueness of morphism gluing. If $P_i \cap P_j$ is empty, its trace is empty. The spaces X_i, their actual closed overlap domains, and the transitions therefore satisfy the second case of Theorem 2.40.10. Its gluing is Hausdorff, and the strict chart images form a locally finite admissible cover with finite incidence. This gives quasi-paracompactness. Intersections of quasi-compact analytic domains are analytic domains by Proposition 2.40.5 and are quasi-compact in this Hausdorff space. This gives quasi-separatedness.

For a second finite-type formal affine cover, its members and the P_i together have finite incidence. Indeed, a fixed quasi-compact formal affine is covered by finitely many members of the other cover. Every member of that other cover meeting the affine must meet one of these finitely many covering members. Construct the gluing for the union of the two covers. Both old chart families cover it, since a formal centre belongs to a chart in either family. Each of these subcovers is still locally finite and closed and therefore admissible. The bounded chart identities and their inverses glue by Theorem 2.40.8, giving the canonical comparisons. For three covers the comparisons compose by their identities on the common charts.

For any formal open $\mathfrak{U}$, the opens $\mathfrak{U} \cap P_i$ are quasi-compact and retain finite incidence. Refine each by finitely many principal pieces. Their generic affinoids form the actual restricted charts on the specialization inverse image. This identifies the generic fibre of $\mathfrak{U}$ with that domain, by the same gluing uniqueness. For a formal arrow, take a finite principal refinement on each source affine subordinate to the inverse images of the formal target affines. The bounded affine generic maps agree on overlaps, as can be checked after the actual principal restrictions of the source and target. They glue by Theorem 2.40.8. Identity and composition hold on the same refinements, hence globally. The maps s_i agree by contraction and glue as maps of sets. The contraction formula also proves naturality for the induced arrows. Each s_i is surjective by Theorem 3.16.15, so specialization is surjective. The empty cover gives the empty generic fibre. □

Theorem 3.17.8 (Existence of quasi-paracompact formal models)**.** *Every quasi-separated quasi-paracompact strict analytic G-space over K has a quasi-paracompact admissible formal K°-model.*

Proof companion for NAG-001950. Use the standing complete non-trivially real-valued field and the original quasi-separated quasi-paracompact strict analytic G-space X. The following verifies the gluing and stabilization steps in the original proof.

First refine each quasi-compact admissible member of the original finite-type cover by finitely many strict affinoid domains. The local strict affinoids form a quasi-net on the member, and compactness gives a finite point cover by Lemma 2.39.2. Each retained domain is closed on the member. Indeed its intersection with every affinoid test chart is quasi-compact by quasi-separatedness and hence closed in that Hausdorff test chart. Lemma 2.40.2 detects closedness on the quasi-net of test charts. The finite closed cover is thus admissible. Transitivity gives a finite-type strict affinoid cover of X. It is locally finite: a neighborhood contained in finitely many members can meet only members meeting at least one of that finite list. The cover has actual closed overlap domains with their inherited identifications. The closed-overlap case of Theorem 2.40.10, together with the quasi-net topology and the given chart maps, identifies its gluing with X and makes X Hausdorff.

Consequently a finite union of the compact domains used below is again an actual compact analytic domain. The members are closed in the union, so their finite cover there is a quasi-net. Refining by their affinoid charts

and using transitivity gives a quasi-net of actual affinoid domains of X on the union, which is the defining domain condition. The same test on each compact domain certifies its finite closed refinements as admissible. All the overlaps below carry these induced domain structures.

For the two-piece step, let V_1, V_2 be compact domains already equipped with quasi-paracompact formal models with the specified generic identifications. Their intersection W is a compact strict analytic domain, by the domain intersection rule and Hausdorffness. If it is empty, take the disjoint union of the models. Otherwise the cover (W, V_i) of V_i is admissible because it contains the entire domain. Apply the supplied-atlas scope of NAG-001947 on the two actual model atlases. The indicated blowups make W the generic fibre of one quasi-compact formal open in each modified model. The finite-incidence generic-fibre construction supplies their compatible generic atlases and the restricted identifications, so the supplied-atlas scope of NAG-001949 applies to the identity on W. It gives a common formal open model after the two actual finite sequences of modifications. Extend each entire sequence to its ambient model by Lemma 3.17.4. The restrictions to the successive inverse images of the original opens are the given sequences. The common modified models of W are therefore actual formal opens in the resulting ambient models.

Glue the two formal schemes along these ordinary opens. The underlying space is the topological gluing and the structure sheaf consists of sections with equal restrictions on the identified opens. Each point therefore retains an admissible formal affine neighborhood. The maps on the actual principal generic restrictions reconstruct precisely the given analytic identification of W. The finite-incidence construction and uniqueness of analytic gluing give the original analytic union $V_1 \cup V_2$ as the generic fibre. Starting with Proposition 3.14.7 on strict affinoids, induction over a finite admissible affinoid cover gives the compact case.

For the full space, take the locally finite intersection graph of the given finite-type compact cover. The unions for different components are disjoint and open-and-closed: each such union and its complement are locally finite unions of closed subsets. Work in one component and let U_n be the finite union of members at graph distance n from a chosen member. Each layer is a compact analytic domain. Nonneighboring layers are disjoint, and the family of layers is a locally finite closed admissible cover. At stage n keep a model of $V_n = \bigcup_{i \le n} U_i$ with quasi-compact formal opens representing each

layer. The intersection of U_{n+1} with V_n is exactly its intersection with U_n. Apply the two-piece construction to the terminal formal open for U_n and a compact model of U_{n+1}. Extend the sequence on the terminal side to the current ambient model via NAG-001946 and perform the prescribed ordinary formal open gluing. Take the inverse images of the earlier opens and the new layer's open as the new covering. The earlier opens remain quasi-compact since the inverse image of a quasi-compact formal open under each blowup has finite affine blowup covers over its finite affine charts.

If $i < n-1$, the open for U_i is disjoint from the terminal formal open. This follows from the disjointness detection on the actual generic atlases. Every extended center is the unit ideal there, so the sequence is the canonical identity over that open. For each fixed i the open consequently stabilizes after finitely many stages. The intersections and transition identifications of stabilized opens are read in one stage after both have stabilized. For a triple take a stage after all three have stabilized. The identifications are compatible under passage to later stages, and the triple cocycle holds there. Gluing the stabilized ordinary formal opens gives an admissible formal scheme. Their finite incidence follows from reflected disjointness for nonneighboring layers, and they are quasi-compact, so the result is quasi-paracompact. Its generic charts and restrictions are those of X on the actual closed admissible layer cover. Take disjoint unions over the components. The empty space has the empty formal model.

Proof. An affinoid space has a model by Proposition 3.14.7. Suppose two quasi-compact analytic opens U_1, U_2 have models $\mathfrak{U}_1, \mathfrak{U}_2$. Their intersection W is quasi-compact by quasi-separatedness. Apply cover realization on supplied atlases to the cover (W, U_i) of U_i: after a blowup $\mathfrak{U}_i' \to \mathfrak{U}_i$, a quasi-compact formal open $\mathfrak{W}_i' \subset \mathfrak{U}_i'$ represents W. By common modification on supplied atlases, the two models of W have a common modification $\mathfrak{W}''$. Extend its two sequences of centers to $\mathfrak{U}_i'$ by Lemma 3.17.4. The resulting models $\mathfrak{U}_i''$ have the same formal open $\mathfrak{W}''$ and glue along it. Their generic fibres glue along exactly W, giving $U_1 \cup U_2$. Induction on a finite affinoid cover proves existence for every quasi-compact space.

In general choose a finite-type cover by quasi-compact analytic opens. Its intersection graph is locally finite. Each graph component is countable, and distinct components give disjoint admissible open pieces, which can be treated separately. In one component fix a vertex, and let U_n be the union of the

finitely many covering members at graph distance n from it. Each U_n is quasi-compact, the U_n cover that piece, and $U_n \cap U_m = \varnothing$ whenever $|n-m| > 1$.

Inductively construct a model $\mathfrak{V}_n$ of $V_n = U_0 \cup \cdots \cup U_n$, containing formal opens representing each U_i, $i \leq n$. To add U_{n+1}, apply the two-piece gluing construction above to the current formal open representing U_n and a model of U_{n+1}. Their analytic intersection equals $V_n \cap U_{n+1}$. Extend the modifications on the U_n side to $\mathfrak{V}_n$, with support in the closure of that formal open, by Lemma 3.17.4. This leaves each formal open representing U_i, $i < n-1$, unchanged, because it is disjoint from the U_n open. Glue in the modified model of U_{n+1}. The preimages of the existing formal opens and the new open give the required cover of $\mathfrak{V}_{n+1}$.

For fixed i, its formal open is unchanged after finitely many stages. Write $\mathfrak{U}_i^{\infty}$ for this stabilized open. Overlap identifications are read at any sufficiently late common stage; stabilization makes them independent of the stage, and a common stage for three indices proves the cocycle identity. Glue these opens. Each $\mathfrak{U}_i^{\infty}$ is quasi-compact, having undergone only finitely many quasi-compact blowups, and meets only its two neighbors, since generic fibres detect disjointness. The glued model is therefore quasi-paracompact and has the desired generic fibre. Taking disjoint unions over graph components finishes the proof. □

Theorem 3.17.9 (Raynaud equivalence)**.** *Let $\mathcal{F}$ be the category of quasi-paracompact admissible formal K°-schemes, and let $\mathcal{S}$ be the class of admissible blowups. The generic-fibre functor induces an equivalence*

$$\mathcal{F}[\mathcal{S}^{-1}] \simeq \left\{ \begin{array}{c} \textit{quasi-separated quasi-paracompact} \\ \textit{strict analytic } G\textit{-spaces over } K \end{array} \right\}.$$

Proof companion for NAG-001951. Retain exactly the category $\mathcal{F}$ of quasi-paracompact admissible formal K°-schemes in the standing field setting, and the category $\mathcal{A}$ of quasi-separated quasi-paracompact strict analytic G-spaces. The finite-incidence construction gives the generic-fibre functor on every object and arrow of $\mathcal{F}$. It supplies the actual atlases needed by the supplied-atlas NAG-001945 scope to invert each admissible blowup. Inverting single blowups is equivalent to inverting their finite composites.

For a roof $\mathfrak{X} \xleftarrow{s} \mathfrak{P} \xrightarrow{f} \mathfrak{Y}$ with s a finite composite, the generic arrow is $f_\eta s_\eta^{-1}$. To compose with a roof $\mathfrak{Y} \xleftarrow{t} \mathfrak{Q} \xrightarrow{g} \mathfrak{Z}$, pull back the centers of the

successive stages of t along the current map from the current blowup of $\mathfrak{P}$. At each stage the formal-only NAG-001945 clause and the affine universal property give the blowup and the arrow to the corresponding stage of $\mathfrak{Q}$. Thus there are an actual finite composite $p : \mathfrak{P}' \to \mathfrak{P}$ and an arrow $a : \mathfrak{P}' \to \mathfrak{Q}$ with $ta = fp$. The composed roof has arrows sp and ga. Its generic arrow is the composite, since the generic denominator maps are isomorphisms.

For two roofs with the same generic arrow, identify their modified sources over the common generic source. Apply the supplied-atlas NAG-001949 scope to that specified identification. It gives a common formal source with finite composites to the two modified sources. On this refinement the arrows to $\mathfrak{X}$ agree by supplied-arrow faithfulness, since their generic arrows agree. The arrows to $\mathfrak{Y}$ also agree by the same faithfulness and the premise on the two roofs. Conversely a common refinement with both arrows equal gives equal generic arrows. The supplied-atlas NAG-001948 scope represents every generic arrow by a roof. Its proof uses the exact supplied-atlas NAG-001947 scope for the first covering blowup. These applications all receive the actual atlases from the finite-incidence construction.

Thus the roof classes are in bijection with the analytic arrows between the corresponding generic fibres. Composition calculated above is independent of choices and associative by this bijection, and the identity roof gives the identity. For a functor H inverting the admissible blowups, assign the roof the arrow $H(f)H(s)^{-1}$. The equality on common refinements makes this assignment well defined. The equality $ta = fp$ makes the assignment compatible with composition. Each roof is the composite of the image of its forward arrow with the inverse of the image of its denominator. The extension of H is therefore unique, and the same factorization proves the corresponding assertion on natural transformations. This is the localization universal property. The generic arrow bijection proves full faithfulness, and the exact NAG-001950 model construction proves essential surjectivity.

Proof. Every blowup is sent to an isomorphism by generic invariance on supplied atlases. Replacing $\mathcal{S}$ by its finite composites does not change the localization: a functor inverts one class if and only if it inverts the other. Use roofs $\mathfrak{X} \xleftarrow{s} \mathfrak{X}' \xrightarrow{f} \mathfrak{Y}$ with s such a finite composite. They can be composed by successively blowing up the pullbacks of the second roof's denominator centers on $\mathfrak{X}'$. The universal properties give the middle map, and the new source map remains a finite composite.

Two choices have a common refinement. Their modified sources have identified generic fibres, so common modification on supplied atlases gives modifications from a common source to both. The maps to the original source agree by generic-fibre faithfulness. If the roofs have the same analytic map, their target maps also agree on this refinement. Conversely, equality on a common refinement implies equality of analytic maps. lifting on supplied atlases ensures that every analytic map is represented by a roof. Thus roofs modulo common refinement are in bijection with the analytic morphisms between generic fibres. Taking common refinements of the two iterated pullback constructions proves associativity; the identity roof is the identity morphism.

This roof category has the localization universal property. If $H : \mathcal{F} \to \mathcal{D}$ inverts blowups, send the roof to $H(f)H(s)^{-1}$. Common refinements leave this expression unchanged, and the pullback construction proves compatibility with composition. This extension is unique, since every roof is the composite of its forward map and its denominator's inverse. Hence the category is $\mathcal{F}[\mathcal{S}^{-1}]$. The preceding bijection of morphisms proves full faithfulness, and Theorem 3.17.8 proves essential surjectivity. □

References. The gluing route is reconstructed from [Bosch14, 8.4, Theorem 3 and Lemmas 4–6, pp. 204–213]. All rank-one valuation rings are retained. Its geometric inputs are the proved rational refinement, affine blowup charts, specialization, and ideal extension. Finite composites suffice for the localization, so no replacement of a composite by a single blowup is used.

3.18 General graded finiteness and inner homomorphisms

Proposition 3.18.1 (Localization at the spectral maximum)**.** *Let A be affinoid, let $f \in A$, and suppose $r = \rho_A(f) > 0$. Set $B = A\{rf^{-1}\}$ and write $\bar{f}$ for the initial form of f. For $a \in A$,*

$$\rho_B(a) = \lim_{n \longrightarrow \infty} r^{-n} \rho_A(f^n a).$$

If this value is positive, the sequence is eventually constant. The canonical homomorphism is an isomorphism

$$(\widetilde{A}^{\mathrm{gr}})_{\bar{f}} \xrightarrow{\sim} \widetilde{B}^{\mathrm{gr}}.$$

Proof. Let Γ be the finite Shilov boundary from Corollary 3.12.5, and set $\Gamma_f = \{x \in \Gamma : |f(x)| = r\}$. This set is nonempty. The rational point theorem identifies $\mathcal{M}(B)$ with $\{|f| = r\} \subset \mathcal{M}(A)$. For every n its points satisfy

$$|a(x)| \le r^{-n}\rho_A(f^n a) = \max_{y\in\Gamma} \left(\frac{|f(y)|}{r}\right)^n |a(y)|.$$

The finite maximum on the right decreases to $m = \max_{\Gamma_f} |a|$. Conversely Γ_f consists of points of B, so $\rho_B(a) \ge m$. This proves the formula. If $m > 0$, every term with $|f(y)| < r$ is smaller than m for all sufficiently large n, giving eventual equality.

On B one has $|f| = r$, hence $\rho_B(f^j b) = r^j \rho_B(b)$ for all integers j. If a homogeneous class represented by a dies in $\widetilde{B}^{\mathrm{gr}}$, the limit formula gives $\rho_A(f^n a) < r^n \rho_A(a)$ for some n. Thus it is already killed by a power of $\bar{f}$, proving injectivity after localization. Every nonzero homogeneous class in $\widetilde{B}^{\mathrm{gr}}$ has a representative of the form a/f^j: the polynomial algebra $A[f^{-1}]$ is dense in B, so approximate a representative with error strictly below its positive spectral size. Increase j if necessary. Eventual equality gives $\rho_A(f^n a) = r^n \rho_B(a)$, and the initial class of $f^n a/f^{j+n}$ then comes from the displayed localization. This proves surjectivity. □

Lemma 3.18.2 (Orthogonal coordinates on one Gauss field)**.** *Let $K(t)_r$ be $K(t)$ with the Gauss absolute value of radius $r > 0$, and let $K(r)$ be its completion. It has an orthogonal K-basis (e_i), in the Banach sense, containing 1. The basis remains orthogonal for Gauss evaluation over every complete extension of K. The constant-coordinate projection is a bounded K-linear retraction $K(r) \to K$.*

Proof. First take $r = 1$. For each monic irreducible polynomial $\bar{q} \in \widetilde{K}[t]$, choose a monic lift $q \in K^\circ[t]$. Use the family

$$1, t, t^2, \dots, \qquad \frac{t^j}{q(t)^m} \quad (m \ge 1, \quad 0 \le j < \deg q).$$

All its members have norm one. Their reductions are the partial-fraction basis of $\widetilde{K}(t)$ over $\widetilde{K}$. For a finite linear combination, divide by a coefficient of largest norm. A nonzero reduced combination remains, so the norm is exactly the largest coefficient norm. The same argument works after any field extension: these rational functions remain linearly independent over the

extended residue field, as is seen by their distinct principal parts and their polynomial parts.

Their span is dense in $K(t)_1$. To check this, normalize the denominator of a rational function to have Gauss norm one. Factor its nonzero reduction into irreducible factors and lift the product, including its leading scalar, to a polynomial D. The denominator differs from D by a polynomial of norm strictly less than one. Its inverse is therefore a convergent geometric series with terms in the algebra generated by t and the chosen q^{-1}'s. Ordinary polynomial division and partial fractions put each term in the displayed span. Completion gives the asserted orthonormal basis.

If r is free over $\sqrt{|K^\times|}$, use the Laurent basis $(t^n)_{n\in\mathbf{Z}}$ from Lemma 1.16.1; every nonzero polynomial has one dominant monomial, so its inverse is a convergent Laurent geometric series. Otherwise choose the least $d > 0$ with $r^d = |c|$, $c \in K^\times$, and put $s = t^d/c$. Apply the radius-one construction to $K(s)$ and multiply its basis by $1, t, \dots, t^{d-1}$. The decomposition into powers of t modulo d shows both that this spans $K(t)$ over $K(s)$ and that its weighted Gauss norm is the maximum of the coordinate norms. After changing the coefficient field, the same assertion follows by clearing a common polynomial denominator in s: the remaining polynomials in t have disjoint exponent classes modulo d. This also proves the claimed extension-stability. In every case the coefficient of the basis vector 1 gives the stated retraction. □

Theorem 3.18.3 (Graded extension by arbitrary Gauss fields)**.** *Let A be affinoid and let L be an iterated Gauss-field extension of K, with finitely many arbitrary positive radii. There are canonical isomorphisms*

$$\widetilde{A}^{\mathrm{gr}} \otimes_{\widetilde{K}^{\mathrm{gr}}} \widetilde{L}^{\mathrm{gr}} \simeq \widetilde{A \widehat{\otimes}_K L}^{\,\mathrm{gr}}.$$

If (e_i) is the resulting orthogonal basis of L/K, then for every coefficient expansion,

$$\rho_{A_L}\left(\sum_i a_i e_i\right) = \max_i \rho_A(a_i)|e_i|.$$

Proof. At each stage of the iterated Gauss extension, the scalar algebra is the actual affinoid scalar extension of Proposition 2.33.5.

Let K be complete, $r > 0$, and $L = \widehat{K(t)}_r$. Use the orthogonal Banach basis $(e_i)_{i\in I}$ from Lemma 3.18.2 and put $w_i = |e_i|$. Here $c_0(I, w; A)$ consists of all families for which only finitely many $\|a_i\|w_i$ exceed any fixed positive

threshold. Let A carry its given admissible Banach algebra norm. At this step the scalar algebra is the actual separated completed projective product $A_L = A\widehat{\otimes}_K L$.

The coefficient functionals of the basis are bounded maps $\ell_i : L \to K$ of norm at most w_i^{-1}. On $A \otimes_K L$ they give

$$w_i \|(1 \otimes \ell_i)(z)\| \leq \|z\|_\pi.$$

Indeed for every finite representation $z = \sum_j a_j \otimes \lambda_j$ the ultrametric inequality bounds the left side by $\max_j \|a_j\||\lambda_j|$. Take the infimum over all such representations. For a finite coefficient family, the representation $\sum_i a_i \otimes e_i$ gives the reverse bound for $\|z\|_\pi$, hence equality with the weighted maximum. Finite basis expansions approximate all scalar factors of every finite tensor in the projective seminorm. Thus finite coefficient families are dense in the separated completion. They are also dense in the complete weighted c_0-space, so that

$$A_L \simeq c_0(I, w; A)$$

isometrically as Banach spaces, via the actual coefficient maps. This argument uses finite truncations for each family and places no countability condition on I.

For $x \in \mathcal{M}_K(A)$ set $F = \mathcal{H}(x)$ and $L_x = \widehat{F(t)}_r$. The Gauss absolute value is multiplicative on polynomials by Proposition 1.2.2, hence extends by ratios to the rational function field. The inclusion $K(t) \hookrightarrow F(t)$ is isometric for these Gauss norms and extends to $L \hookrightarrow L_x$. The map $a \otimes l \mapsto a(x)l$ is contractive for each finite tensor representation, by Lemma 1.6.1 and the isometric scalar inclusion. It therefore extends to a bounded L-algebra map $A_L \to L_x$ and gives a bounded point x_r of A_L. By the extension-stability in Lemma 3.18.2, the value of a finite coefficient family at this point is $\max_i |a_i(x)|w_i$. The same equality holds for every weighted c_0-family by truncation and contractivity.

Every bounded point of A_L restricts along the constant inclusion to a bounded point of A and has scalar values $|e_i| = w_i$. The ultrametric inequality and truncation bound its evaluation above by $\max_i \rho_A(a_i)w_i$. Conversely, for each i use the points x_r as x ranges through $\mathcal{M}_K(A)$ to bound the upper spectral seminorm from below by $\rho_A(a_i)w_i$. Consequently

$$\rho_{A_L}\Big(\sum_i a_i e_i\Big) = \max_i \rho_A(a_i)w_i.$$

Here $\rho_A(a_i) \le \|a_i\|$, so a positive supremum is a maximum. If all weights of the evaluated coefficients vanish the same inequalities give zero. The zero algebra gives both sides zero with the usual zero convention.

Assume the one-step coefficient and spectral formulas for the chosen positive Gauss radius, including the actual coefficient expansion on $A_L = A\widehat{\otimes}_K L$. Write L in the orthogonal Banach basis (e_i) of Lemma 3.18.2 and put $w_i = |e_i|$. For a submultiplicative spectral seminorm ρ, use the positive degree q quotient $\{\rho \le q\}/\{\rho < q\}$.

The bounded structural maps from A and L into A_L are filtered for these seminorms. For A this follows by restricting points and for L by their scalar absolute values. The same inequalities apply to the zero algebra. Hence multiplication of homogeneous representatives induces the canonical graded algebra homomorphism

$$\widetilde{A}^{\mathrm{gr}} \otimes_{\widetilde{K}^{\mathrm{gr}}} \widetilde{L}^{\mathrm{gr}} \longrightarrow \widetilde{A_L}^{\mathrm{gr}}.$$

For $z = \sum_i a_i e_i$ with $\rho_{A_L}(z) \le q$, only finitely many indices satisfy $\rho_A(a_i)w_i = q > 0$. Indeed $\rho_A(a_i) \le \|a_i\|$, and the coefficient weights tend to zero in the weighted c_0 sense. Among the remaining indices the supremum of the spectral weights is strictly below q: all but finitely many are smaller than $q/2$, and every retained nonleading weight is strictly below q. The spectral formula therefore identifies the degree q class of z with the finite sum of the classes $\mathrm{in}_{q/w_i}(a_i)\,\mathrm{in}_{w_i}(e_i)$ at equality indices. For a finite sum of homogeneous coefficients in degree q, choose representatives of size at most q/w_i. If some coefficient class is nonzero, the formula gives spectral size exactly q, so its image class is nonzero.

Applying the same argument to L with its orthogonal coefficient norm shows that the $\mathrm{in}_{w_i}(e_i)$ freely generate $\widetilde{L}^{\mathrm{gr}}$ over $\widetilde{K}^{\mathrm{gr}}$ as a graded module. The preceding surjectivity and injectivity in each degree prove that the displayed map is an isomorphism. Its multiplication is the one induced by multiplying actual lifts.

For iteration, let $K \subset L \subset L'$ be successive stages, and let (f_j) be the next orthogonal basis, of weights $v_j = |f_j|$. In the actual scalar algebra write $z = \sum_j b_j f_j$, $b_j = \sum_i a_{ij} e_i$. Each column is a weighted c_0-family and the numbers $(\sup_i \|a_{ij}\| w_i)v_j$ tend to zero. This is equivalent to the single weighted c_0 condition on $(\|a_{ij}\| w_i v_j)_{i,j}$: above any positive threshold only finitely many columns can occur and each contributes finitely many entries. Conversely the single condition implies both column conditions by testing at

half the given threshold. The same norm calculation with $A = K$ gives the product orthogonal basis $(e_i f_j)$ of L'/K.

The scalar-tower comparison of Corollary 2.33.9 identifies $(A_L)_{L'}$ with $A_{L'}$, compatibly with structural maps. On finite coefficient sums it sends the two-stage expressions to $\sum_{i,j} a_{ij}(e_i f_j)$. The weighted c_0 description and continuity give the same equality for limits. The two spectral formulas give

$$\rho_{A_{L'}}(z) = \max_j v_j \rho_{A_L}(b_j) = \max_{i,j} v_j w_i \, \rho_A(a_{ij}).$$

The graded maps compose by the usual algebraic tensor associator, since each is induced by multiplication of representatives. Iteration proves the claim for any finite tower, including the empty tower, whose comparison is the identity. □

Lemma 3.18.4 (Graded scalar descent and degree-one reduction)**.** *Every graded module over a graded field has a homogeneous basis. Scalar extension by a nonzero graded algebra over that field is faithful and exact. Finite generation of an algebra or a module descends under this scalar extension. If a graded k-algebra C is supported in the degree group of k, multiplication induces*

$$C_1 \otimes_{k_1} k \simeq C.$$

For algebras with this support condition, a graded map is finite precisely when its degree-one map is finite.

Proof. Choose a maximal linearly independent family of homogeneous elements. If its span is not the whole module, a nonzero homogeneous class in the quotient generates a copy of a shifted k; lifting it enlarges the family, a contradiction. This proves freeness and hence faithful exactness. If finitely many tensors generate a scalar extension, take the finitely many original coefficients appearing in them. Their cokernel becomes zero after faithful extension, so it was already zero. For algebra generation apply this argument to the quotient by the subalgebra generated by those coefficients, viewed as a graded vector space. Finally, in each nonempty degree g choose $u_g \in k_g^{\times}$. Multiplication by u_g identifies C_1 with C_g and proves the displayed isomorphism. A finite homogeneous set of module generators can be divided by the corresponding u_g's, reducing the last assertion to degree one. □

Lemma 3.18.5 (Finite admissible maps descend from Gauss fields)**.** *Let $A \to B$ be a bounded affinoid homomorphism and L/K a finite iterated Gauss-field extension. Then $A \to B$ is finite and admissible if and only if $A_L \to B_L$ is finite and admissible.*

Proof. In an orthogonal scalar basis, all completed tensor products are weighted coefficient c_0-spaces. A bounded admissible surjection $A^m \to B$ therefore remains such a surjection: lift each coefficient with a fixed common norm bound, which preserves coefficient decay. This proves the forward direction.

Conversely let $b'_1, \dots, b'_m$ generate B_L over A_L with a bounded lifting constant C. Approximate these generators by finite scalar sums b''_i so closely that $C \max_i \|b'_i - b''_i\| < 1$. Let M be a bound for the action $A_L \times B_L \to B_L$. In choosing the approximations require $MC \max_i \|b'_i - b''_i\| < 1$. They still generate with a bounded lifting constant. Indeed lift an element using the original generators, replace them by the approximations, and repeat on the resulting error. The errors decrease geometrically, and the coefficient corrections converge in A_L^m. Collect the finitely many elements $b_j \in B$ occurring in the finite scalar expansions of the b''_i. They generate B_L, again with a bounded lifting constant. For $b \in B$ write $b = \sum_j c_j b_j$, $c_j \in A_L$, with $\max \|c_j\| \le C'\|b\|$. Let $p_0 = p_{A,0} : A_L \to A$ be the constant-coordinate projection from Lemma 3.18.2. It is A-linear, has norm at most one, and fixes A. The corresponding projection $p_{B,0} : B_L \to B$ fixes B. In the final equality $b = \sum_j \varphi_L(c_j)(b_j \otimes 1)$, apply the B-valued constant-coordinate projection $p_{B,0}$. On each product it equals $\varphi(p_{A,0}(c_j))b_j$ by the coefficient formula, first on finite sums and then by continuity. Hence

$$b = \sum_j p_0(c_j) b_j, \qquad \max_j \|p_0(c_j)\| \le C'\|b\|.$$

This proves finite generation and admissibility over A. Only linearity of p_0 is used. □

Lemma 3.18.6 (A Gauss extension containing all spectral values)**.** *For finitely many K-affinoid algebras A_j, one can choose a finite iterated Gauss extension L/K such that every $(A_j)_L$ is strict and all its nonzero spectral values belong to $|L^\times|$. One may also require L to be nontrivially valued.*

Proof. First make the algebras strict by a common free-radius extension K_s/K. For each strict algebra, the proof of Theorem 3.12.3 gives a finite

boundary consisting of points whose residue fields are finite extensions of Gauss fields of strict polydiscs over K_s. The latter Gauss fields have value group $|K_s^\times|$. In a finite valued-field extension the value-group quotient is finite: vectors with values in distinct cosets are linearly independent over the base field, since a sum with a uniquely largest term cannot vanish. Its cardinality is therefore bounded by the field degree. Choose finitely many positive numbers representing generators of all these finite quotient groups. The spectral values of every original A_j lie in the group generated by $|K^\times|$, the entries of s, and these finitely many numbers, because they are attained on the specified finite boundaries.

Adjoin Gauss variables with precisely these radii, and, if necessary, one radius different from one. The resulting L has this generated value group and still makes all A_j strict. By Theorem 3.18.3, every positive spectral value on $(A_j)_L$ is the maximum of finitely many products of an original spectral value and a basis-vector norm. Both factors belong to $|L^\times|$, proving the assertion. □

Theorem 3.18.7 (The general graded finite-map criterion)**.** *A bounded homomorphism $A \to B$ of generalized affinoid algebras is finite and admissible if and only if*

$$\widetilde{A}^{\mathrm{gr}} \longrightarrow \widetilde{B}^{\mathrm{gr}}$$

is finite. In particular $\widetilde{A}^{\mathrm{gr}}$ is a finitely generated graded $\widetilde{K}^{\mathrm{gr}}$-algebra.

Proof. Choose L by Lemma 3.18.6. By Lemma 3.18.5, the analytic condition is equivalent to finite admissibility of $A_L \to B_L$. These algebras are strict over a nontrivially valued field, so Theorem 3.16.5 identifies this with finiteness of their canonical degree-one reductions. The support condition and Lemma 3.18.4 identify the latter with finiteness of the full graded map. By Theorem 3.18.3 that map is the scalar extension of the displayed map by $\widetilde{L}^{\mathrm{gr}}/\widetilde{K}^{\mathrm{gr}}$, and finite generation descends by Lemma 3.18.4. This proves both implications.

Apply the criterion to an admissible quotient presentation $K\{r^{-1}T\} \twoheadrightarrow A$. Its source reduction is the weighted polynomial algebra, by the first assertion of Theorem 3.13.2, proved using the coefficient formula of Proposition 3.13.1. The reduction of A is finite over the image of this finitely generated algebra, so is itself finitely generated. □

Finite graded reduction for a fixed absolute presentation. Let K be complete and let $P = K\{r^{-1}T\} \twoheadrightarrow B$ be an admissible absolute weighted Tate presentation. Then $\widetilde{B}^{\mathrm{gr}}$ is finite as a module over the image of $\widetilde{P}^{\mathrm{gr}} = \widetilde{K}^{\mathrm{gr}}[t_1, \dots, t_m]$, where $\deg(t_i) = r_i$ and t_i maps to the class of the image of T_i in degree r_i.

Proof. Apply Lemma 3.18.6 to the two actual presentations of P and B, giving the common nontrivially valued field L. The weighted Gauss norm of P is multiplicative by Lemma 1.2.2. It is itself a bounded point and bounds all bounded evaluations by Lemma 1.6.1, so it is the spectral norm. This argument applies to the actual weighted source P_L of Proposition 2.33.5. Consequently each $r_i = \rho_{P_L}(T_i)$ belongs to $|L^\times|$. Choose $c_i \in L^\times$ of absolute value r_i. The variables T_i/c_i identify P_L isometrically with a unit Tate algebra, and the map to B_L is the same admissible quotient in those coordinates.

For an actual upper affinoid C, its ordinary reduction is its degree-one spectral quotient. Bounded evaluation of all powers gives $\rho_C(b)^n \le \|b^n\|$. The positive-radius bound of Theorem 1.20.2 at radius one gives power boundedness when $\rho_C(b) \le 1$. If $\rho_C(b) < 1$, choose a positive s with $\rho_C(b) < s < 1$. The bound at s gives topological nilpotence, including when $\rho_C(b) = 0$. Conversely, a power of norm less than one bounds every evaluation strictly below one. These identifications use the original lifts.

Suppose $B_L \neq 0$. Use the actual finite family of minimal-prime quotients B_j in the strict case of Theorem 3.12.3. The product map of ordinary reductions $\widetilde{B_L} \to \prod_j \widetilde{B}_j$ is injective. Indeed, for a power-bounded lift with zero reduction in every component, the maximum of its finitely many component spectral sizes is strictly less than one. The degree-one comparison gives zero reduction upstairs. For the admissible strict quotient B_j of the same unit Tate source P_L, Theorem 1.12.2 gives the finite unit Tate normalizer of B_j and its contractive lift to P_L. Proposition 3.16.2 gives finiteness of $\widetilde{B}_j$ over the ordinary reduction of this normalizer. Its action factors through the ordinary residue polynomial ring $S = \widetilde{P_L}$ by the lift. Thus $\prod_j \widetilde{B}_j$ is finite over S. Equip $\widetilde{L}$ with the trivial valuation. Its unit Tate algebra is the polynomial ring S, so Theorem 1.11.5 gives finiteness of the actual S-submodule $\widetilde{B_L}$ of this product. The zero quotient has the same finiteness conclusion.

Both graded algebras over L have support in $|L^\times|$. The degree-one assertion of Lemma 3.18.4 gives module finiteness for the full graded map over L. The natural scalar comparison of Theorem 3.18.3 identifies that map with scalar extension of the given graded map over the graded base field. Finite-

ness descends by Lemma 3.18.4. Finally the absolute weighted Gauss formula identifies the graded reduction of P with the displayed polynomial ring. For each nonzero series, the terms of maximal weighted size form a finite sum and the remaining terms have strictly smaller norm. The Gauss maximum detects every nonzero homogeneous polynomial and the leading terms represent every homogeneous class. The induced map sends each variable to the stated initial class, proving the assertion.

Corollary 3.18.8 (Finite comparison maps for graded reductions)**.** *Let $A \to B, A \to C$ be bounded affinoid maps, and let L/K be any complete valued extension. The natural maps*

$$\widetilde{B}^{\mathrm{gr}} \otimes_{\widetilde{A}^{\mathrm{gr}}} \widetilde{C}^{\mathrm{gr}} \longrightarrow \widetilde{B \widehat{\otimes}_A C}^{\,\mathrm{gr}}, \qquad \widetilde{A}^{\mathrm{gr}} \otimes_{\widetilde{K}^{\mathrm{gr}}} \widetilde{L}^{\mathrm{gr}} \longrightarrow \widetilde{A_L}^{\,\mathrm{gr}}$$

are finite.

Proof. Present B and C by weighted Tate algebras over K. Their combined variables give an admissible quotient presentation of $D = B\widehat{\otimes}_A C$, by Proposition 2.33.8. The fixed absolute presentation result makes $\widetilde{D}^{\mathrm{gr}}$ finite over the image of the corresponding graded polynomial algebra. The image of each polynomial variable either vanishes or is the image of the initial form of its image in B or C. Indeed nonvanishing means that its spectral value has remained equal to its prescribed radius, so this equality already held in the appropriate factor. Consequently the image of the first comparison map contains the image of that polynomial algebra. The same finite module generators therefore work over this larger image.

For the second map, extend an admissible presentation $K\{r^{-1}T\} \twoheadrightarrow A$ to $L\{r^{-1}T\} \twoheadrightarrow A_L$. The same argument applies, with $\widetilde{L}^{\mathrm{gr}}$ as coefficient ring. Any nonzero initial coordinate after extension comes from its original initial coordinate in $\widetilde{A}^{\mathrm{gr}}$. □

Lemma 3.18.9 (Shrinking generator radii to spectral bounds)**.** *Let $A\{r^{-1}T\} \twoheadrightarrow B$ be an admissible affinoid presentation with $T_i \mapsto b_i$. If $0 < s_i \le r_i$ and $\rho_B(b_i) \le s_i$, substitution gives an admissible surjection $A\{s^{-1}T\} \twoheadrightarrow B$.*

Proof. Theorem 1.20.2 gives $\|b_i^n\| \le C_i s_i^n$ for all n. Multiplying these finitely many estimates proves bounded convergence of the substituted series. The original series space maps contractively into the smaller-radius series space,

and its composite with substitution is the original surjection. Thus surjectivity and the original bounded lifting estimate both persist. □

Theorem 3.18.10 (The graded innerness criterion)**.** *Let A be affinoid, let B be an A-affinoid algebra, and let $\varphi : B \to D$ be a bounded A-homomorphism to a commutative Banach A-algebra. Call φ inner over A if some admissible surjection $A\{r^{-1}T\} \twoheadrightarrow B$ has $\rho_D(\varphi(T_i)) < r_i$ for every coordinate. Then φ is inner over A if and only if*

$$\operatorname{im}(\widetilde{B}^{\mathrm{gr}} \to \widetilde{D}^{\mathrm{gr}}) \quad \textit{is finite over} \quad \operatorname{im}(\widetilde{A}^{\mathrm{gr}} \to \widetilde{D}^{\mathrm{gr}}).$$

Proof. For an inner presentation, Theorem 3.18.7 and the relative polynomial identity in the first assertion of Theorem 3.13.2 make $\widetilde{B}^{\mathrm{gr}}$ finite over $\widetilde{A}^{\mathrm{gr}}[T_1, \dots, T_n]$. All these coordinates vanish after mapping to $\widetilde{D}^{\mathrm{gr}}$, so the resulting image is finite over the image of $\widetilde{A}^{\mathrm{gr}}$.

Conversely take a relative presentation with generators b_j. By Lemma 3.18.9, reduce each radius to $s_j = \rho_B(b_j)$ when this is positive; when it is zero, choose any smaller positive radius. Write β_j for the corresponding initial coordinate, which is zero in the latter case. Finiteness of the image makes $\varphi(\beta_j)$ integral over the image of $\widetilde{A}^{\mathrm{gr}}$. Taking the homogeneous part of a monic equation gives one of degree $s_j^{d_j}$ with coefficient of $T^{d_j - i}$ of degree s_j^i. Lift these coefficients to elements $a_{ji} \in A$ with $\rho_A(a_{ji}) \le s_j^i$, and put

$$P_j(T) = T^{d_j} + \sum_{i=1}^{d_j} a_{ji} T^{d_j - i}, \qquad w_j = P_j(b_j), \qquad t_j = s_j^{d_j}.$$

Then $\rho_B(w_j) \le t_j$ and $\rho_D(\varphi(w_j)) < t_j$. For a zero initial coordinate one can simply take $P_j(T) = T$. Weighted power bounds define a bounded map $C = A\{t^{-1}U\} \to B$, $U_j \mapsto w_j$. In the graded reduction each β_j satisfies the monic equation $\overline{P}_j(\beta_j) - \bar{w}_j = 0$ over the image of $\widetilde{C}^{\mathrm{gr}}$, with $\bar{w}_j = 0$ if its degree has dropped. Moreover $\widetilde{B}^{\mathrm{gr}}$ is finite over $\widetilde{A}^{\mathrm{gr}}[\beta_1, \dots, \beta_n]$ by the general graded criterion applied to the shrunk presentation. Successively adjoining the integral β_j's proves that $\widetilde{C}^{\mathrm{gr}} \to \widetilde{B}^{\mathrm{gr}}$ is finite. The same criterion shows that $C \to B$ is finite and admissible.

Choose finite C-module generators $v_1, \dots, v_m$ of B and radii $R_i > \|v_i\|_B$. The map

$$A\{t^{-1}U, R^{-1}V\} \longrightarrow B, \qquad U_j \mapsto w_j, \quad V_i \mapsto v_i,$$

is bounded and surjective. It is admissible because a bounded C-module lift $b = \sum_i c_i v_i$ gives the lift $\sum_i c_i V_i$ with a fixed norm bound. The U-coordinates have strictly smaller spectral size in D by construction, and the V-coordinates do as well, since bounded homomorphisms decrease spectral radii. This is the required inner presentation. □

Forward implication for absolute evaluation. Let $X = \mathcal{M}(B)$ and $x \in \mathrm{Int}(X/\mathcal{M}(K))$. Then the image of $\widetilde{B}^{\mathrm{gr}} \longrightarrow \widetilde{\mathcal{H}(x)}^{\mathrm{gr}}$ is finite over the embedded graded field $\widetilde{K}^{\mathrm{gr}}$.

Proof. Definition 2.7.1 supplies an admissible absolute presentation with $|T_i(x)| < r_i$ for every coordinate. The isometric structure map $K \to \mathcal{H}(x)$ induces the embedded graded coefficient field. Evaluation in this actual complete valued field is filtered, since $|b(x)| \leq \rho_B(b)$. In degree r_i the class of the image of T_i maps to zero. By the fixed absolute presentation result, $\widetilde{B}^{\mathrm{gr}}$ has a finite list of module generators over the image of the initial-coordinate polynomial ring. Their images generate the evaluated image over $\widetilde{K}^{\mathrm{gr}}$, since all the initial coordinates vanish there.

Reverse finite admissibility for an absolute Tate algebra. Let $Q = K\{t^{-1}U\}$ be an absolute weighted Tate algebra and $\psi : Q \to B$ a bounded map to a generalized K-affinoid algebra. If $\widetilde{B}^{\mathrm{gr}}$ is finite over the image of $\widetilde{Q}^{\mathrm{gr}}$, then ψ is finite and admissible.

Proof. Choose the common scalar tower of Lemma 3.18.6 for the actual map ψ. The two extended algebras are strict over the nontrivially valued field L, and their graded supports lie in its value group. Theorem 3.18.3 identifies the extended graded map with the scalar extension of the original one, so it remains finite. Lemma 3.18.4 gives finiteness in degree one, which is the map on the strict canonical reductions. The reverse implication of Theorem 3.16.5 makes the actual map over L finite and admissible. Finally Lemma 3.18.5 descends finite admissibility to ψ.

Reverse implication for absolute evaluation. Take $A = K$, B a generalized K-affinoid algebra, $x \in \mathcal{M}(B)$, and $D = \mathcal{H}(x)$, with the evaluation map $B \to D$. If

$$S_x = \mathrm{im}\big(\widetilde{B}^{\mathrm{gr}} \longrightarrow \widetilde{\mathcal{H}(x)}^{\mathrm{gr}}\big)$$

is finite over $\widetilde{K}^{\mathrm{gr}}$, then $x \in \mathrm{Int}(\mathcal{M}(B)/\mathcal{M}(K))$.

Proof. Start with an admissible absolute presentation with generators b_j. By Lemma 3.18.9, use radii $s_j = \rho_B(b_j)$ when the spectral radius is positive and choose smaller positive radii when it is zero. Write β_j for the class of b_j in degree s_j; in the zero spectral radius case the class is zero. Finiteness of S_x gives for the image of each β_j a monic homogeneous equation of degree $s_j^{d_j}$ over $\widetilde{K}^{\mathrm{gr}}$. Lift the coefficient of degree s_j^i to $a_{ji} \in K$ with $|a_{ji}| \le s_j^i$, choosing zero for a zero coefficient. Put

$$P_j(T) = T^{d_j} + \sum_{i=1}^{d_j} a_{ji} T^{d_j - i}, \qquad w_j = P_j(b_j), \qquad t_j = s_j^{d_j}.$$

For $\beta_j = 0$, take $P_j(T) = T$. Then $\rho_B(w_j) \le t_j$ and $|w_j(x)| < t_j$. Bounded weighted substitution gives an actual map $Q = K\{t^{-1}U\} \to B$, $U_j \mapsto w_j$.

By the fixed absolute presentation result, $\widetilde{B}^{\mathrm{gr}}$ is finite over the graded algebra generated by the β_j and $\widetilde{K}^{\mathrm{gr}}$. Each β_j is integral over the image of $\widetilde{Q}^{\mathrm{gr}}$: its monic relation there is $\overline{P}_j(\beta_j) - \overline{w}_j = 0$, with the class $\overline{w}_j$ taken in degree t_j. Adjoining all the β_j and using these finite modules shows that $\widetilde{B}^{\mathrm{gr}}$ is finite over the image of $\widetilde{Q}^{\mathrm{gr}}$. By reverse finite admissibility for an absolute Tate algebra, the actual map $Q \to B$ is finite and admissible.

Choose module generators v_i giving an admissible finite Q-module presentation of B, and put a radius u_i on each new variable with $u_i > \rho_B(v_i)$. The map $Q\{u^{-1}V\} \to B$, $V_i \mapsto v_i$, is admissible: its linear polynomials in the V_i already give the admissible module presentation, after the harmless rescaling of the finite product norm by the positive u_i. The bounds at x for the U_j are $|w_j(x)| < t_j$, and those for the V_i follow from $|v_i(x)| \le \rho_B(v_i) < u_i$. This is an inner absolute presentation of B at x.

Relative innerness at an affinoid point. Let $A \to B$ be a bounded map of norm-admissible generalized affinoid algebras over the same complete non-Archimedean field K. For $y \in \mathcal{M}(B)$, let $E = \mathcal{H}(y)$, and for $D = A, B$ let R_D be the image of $\widetilde{D}^{\mathrm{gr}}$ in $\widetilde{E}^{\mathrm{gr}}$ under evaluation at y. Then $y \in \mathrm{Int}(\mathcal{M}(B)/\mathcal{M}(A))$ if and only if R_B is finite over R_A.

Proof. Fix an admissible absolute weighted Tate presentation $P = K\{r^{-1}T\} \twoheadrightarrow A$. In any admissible relative presentation $A\{s^{-1}X\} \twoheadrightarrow B$, adjoining X coefficientwise to $P \twoheadrightarrow A$ and composing gives an admissible absolute presentation $P\{s^{-1}X\} \twoheadrightarrow B$, by Proposition 1.17.3. If this relative presentation witnesses innerness at y, the initial classes of the X_j at the

assigned degrees s_j evaluate to zero in $\widetilde{E}^{\mathrm{gr}}$. The fixed absolute presentation result gives a finite module over the initial-coordinate polynomial image. Evaluating the finite generators, only the images of the coefficient field and the T_i are needed, and these belong to R_A. Thus R_B is finite over R_A.

For the converse, choose the coordinates b_j of a relative presentation and shrink to radii $s_j = \rho_B(b_j)$ when these are positive. For zero spectral radius choose any positive s_j at most the original radius. The presentation stays admissible by Lemma 3.18.9. Let β_j denote the class of b_j at degree s_j, allowing the zero class. If R_B is finite over R_A, the image of β_j satisfies a monic equation over R_A. Take its homogeneous component of degree $s_j^{d_j}$, where d_j is the monic degree. Lift its coefficient of degree s_j^i through $\widetilde{A}^{\mathrm{gr}}$ to $a_{ji} \in A$ with $\rho_A(a_{ji}) \le s_j^i$, using zero for a zero coefficient. Put

$$p_j(Z) = Z^{d_j} + \sum_{i=1}^{d_j} a_{ji} Z^{d_j - i}, \qquad t_j = s_j^{d_j}, \qquad w_j = p_j(b_j).$$

For zero evaluated class one may take $p_j(Z) = Z$. Then $\rho_B(w_j) \le t_j$ and $|w_j(y)| < t_j$. Positive-radius power bounds 1.20.2 give the bounded substitution $C = A\{t^{-1}U\} \to B$, with $U_j \mapsto w_j$.

In $\widetilde{B}^{\mathrm{gr}}$, each β_j is integral over the image of $\widetilde{C}^{\mathrm{gr}}$: its monic equation is $\overline{p}_j(\beta_j) - \overline{w}_j = 0$ in degree t_j, with $\overline{w}_j$ the image of the class of U_j. The fixed absolute presentation of B obtained above gives finiteness over the algebra generated by the image of $\widetilde{A}^{\mathrm{gr}}$ and the β_j. Indeed, if an old coefficient coordinate survives at its degree in B, spectral contraction shows it survives at that degree in A. Its class in B therefore comes from A. It follows by the monic equations that $\widetilde{B}^{\mathrm{gr}}$ is finite over the image of $\widetilde{C}^{\mathrm{gr}}$.

Apply coefficientwise presentation once more to obtain the admissible absolute presentation $Q = P\{t^{-1}U\} \twoheadrightarrow C$. The fixed absolute result makes $\widetilde{C}^{\mathrm{gr}}$ finite over the image of $\widetilde{Q}^{\mathrm{gr}}$. Transitivity, with both rings mapped to $\widetilde{B}^{\mathrm{gr}}$, makes $\widetilde{B}^{\mathrm{gr}}$ finite over the image of $\widetilde{Q}^{\mathrm{gr}}$. The selected reverse finite admissibility result applies to the actual absolute map $Q \to B$. Choose its finite module generators v_i with bounded coefficient lifts. Sending the coefficients through $Q \twoheadrightarrow C$ yields lifts with a uniform bound for the same generators over C. Choose radii $u_i > \rho_B(v_i)$, so weighted substitution defines $C\{u^{-1}V\} \to B$, $V_i \mapsto v_i$. Linear polynomials in V_i give bounded lifts: their norms are bounded by the common coefficient-lift bound times $\max_i u_i$. This is therefore an admissible relative presentation over A with coordinates w_j, v_i. Their values at y are strictly below t_j, u_i respectively, proving relative innerness. □

References. The graded comparison and innerness statements are [Tem04, Propositions 3.1 and 3.4]. The reconstruction uses the finite Shilov boundary for maximum localization, the partial-fraction Gauss basis for scalar descent, and the internally proved canonical finite-map criterion.

3.19 Graded valuation spaces and analytic germs

Gradings in this section are by the ordered group $\mathbf{R}_{>0}$. A graded valuation subring V of a graded field F is a graded subring such that, for every nonzero homogeneous $a \in F$, either a or a^{-1} belongs to V. Its homogeneous nonunits form its unique maximal homogeneous ideal. For a graded subfield $k \subset F$, write $\mathscr{P}_k(F)$ for these valuation subrings containing k. For homogeneous elements a_i, the sets

$$\mathscr{P}_k(F)\{a_1, \dots, a_n\} = \{V : a_1, \dots, a_n \in V\}$$

form a basis of its topology. For a graded subalgebra $R \subset F$, use $\mathscr{P}_k(F)\{R\}$ for the valuations containing R. A birational space with field F is a connected quasi-compact quasi-separated space with a local homeomorphism to $\mathscr{P}_k(F)$. It is affine if this map identifies it with $\mathscr{P}_k(F)\{R\}$ for a finitely generated graded k-subalgebra R.

Lemma 3.19.1 (Graded domination and the integral criterion)**.** *Every local graded subring of F is dominated by a graded valuation subring of F. For graded k-subalgebras $R \subset S \subset F$,*

$$\mathscr{P}_k(F)\{R\} = \mathscr{P}_k(F)\{S\} \quad \Longleftrightarrow \quad S \text{ is integral over } R.$$

If S is finitely generated over $k \subset R$, these conditions are also equivalent to S being finite over R.

Proof. A homogeneous prime of a ring graded by an ordered group is an ordinary prime: the extreme nonzero homogeneous terms in a product cannot cancel. Let C be a local graded subring with homogeneous maximal ideal $\mathfrak{m}$. Apply [StacksAlg, Tag 00IA] to the ordinary local domain $C_{\mathfrak{m}} \subset \mathrm{Frac}(F)$. Take the sum of the homogeneous parts of the resulting ordinary valuation ring. This is a graded valuation subring of F; its maximal homogeneous ideal contracts to $\mathfrak{m}$. This proves domination.

A graded valuation ring is integrally closed on homogeneous elements. If b were integral but outside it, b^{-1} would lie in its maximal homogeneous ideal.

Dividing a homogeneous monic equation by its leading power would then put 1 in that ideal. Conversely, if a homogeneous b is not integral over R, then b^{-1} is not a unit of $R[b^{-1}]$: an expression for its inverse would give a monic equation for b. By [StacksAlg, Tag 00E0(2)], choose an ordinary maximal ideal containing b^{-1}. Its homogeneous part is a homogeneous prime: a product of two homogeneous elements in that part has one factor in it. Localize at the homogeneous elements outside this prime and apply domination to the resulting graded local ring. The resulting valuation contains R and excludes b. Thus equality of the valuation sets detects integrality of all homogeneous elements of S. These elements generate S, so this is ordinary integrality as well. Finally, a finite set of algebra generators, each integral over R, gives a finite module by reducing their powers with their monic equations. □

Proposition 3.19.2 (Compactness of graded valuation spaces)**.** *For every graded subalgebra $R \subset F$, $\mathscr{P}_k(F)\{R\}$ is quasi-compact. In particular $\mathscr{P}_k(F)$ is quasi-compact and quasi-separated. Every nonempty open subset contains the point F, so these spaces are irreducible.*

Proof. Encode a graded subring by the membership functions of the homogeneous elements of F in the product of copies of $\{0, 1\}$. The conditions of containing R and k, being closed under products and sums in one degree, and containing a or a^{-1} are closed finite-coordinate conditions. The product is compact by [StacksAlg, Tag 08ZU]; it is Hausdorff since distinct points are separated by a coordinate cylinder. Hence its closed solution set is compact: add the open complement to any cover of that set and use the finite-subcover property. Its valuation topology is coarser, proving quasi-compactness. Finite intersections of the basic opens are of the same kind, hence quasi-separatedness. Finally F satisfies every membership condition, proving irreducibility. □

Lemma 3.19.3 (Laurent refinement in valuation spaces)**.** *Every finite covering of a subset $W \subset \mathscr{P}_k(F)$ by sets $W\{a_1, \dots, a_n\}$ admits a refinement by a Laurent covering $W\{b_1^{\epsilon_1}, \dots, b_m^{\epsilon_m}\}$, $\epsilon_i \in \{1, -1\}$, with homogeneous nonzero b_i.*

Proof. Write the given cover as $W_i = W\{a_{i1}, \dots, a_{in}\}$, removing zero entries and appending entries 1 so that $a_{in} = 1$ for every i. Let J consist of tuples

$j = (j_1, \dots, j_s)$ with at least one $j_i = n$, and set $c_j = \prod_i a_{ij_i}$. The rational sets

$$V_j = W\{c_{j'}/c_j : j' \in J\}$$

cover W, since finitely many nonzero homogeneous elements in a valuation field have a largest one for divisibility. Fix i with $j_i = n$ and a valuation in V_j. It belongs to some W_ℓ. If $\ell = i$ there is nothing to prove. Otherwise replacing j_ℓ by n shows that $a_{\ell j_\ell}^{-1}$ belongs to the valuation ring. It also contains $a_{\ell j_\ell}$, so this is a unit. Replacing j_i by h and j_ℓ by n now gives $a_{ih}/a_{\ell j_\ell}$ in the ring for every h. Thus $V_j \subset W_i$.

A rational covering defined by finitely many c_j has a Laurent refinement using all ratios $c_j/c_{j'}$. On a chosen Laurent cell these ratio conditions orient the complete finite graph. Contract the strongly connected components and choose a terminal component. Products along directed paths show that the ratio of every c_j to any chosen member of this terminal component belongs to all valuation rings in the cell; ratios inside a component are units. The cell is therefore contained in its rational chart. This proves the desired refinement. □

For an affinoid $X = \mathcal{M}(A)$ and $x \in X$, put

$$F_x = \widetilde{\mathcal{H}(x)}^{\mathrm{gr}}, \qquad R_{A,x} = \mathrm{im}(\widetilde{A}^{\mathrm{gr}} \to F_x), \qquad \mathscr{R}(X,x) = \mathscr{P}_{\widetilde{K}^{\mathrm{gr}}}(F_x)\{R_{A,x}\}.$$

Choose an admissible absolute weighted Tate presentation of A. By the fixed absolute presentation result, $\widetilde{A}^{\mathrm{gr}}$ is finite over the image of the graded polynomial algebra of that presentation. Evaluation in F_x passes this finiteness to the images of both algebras. Hence $R_{A,x}$ is a finitely generated graded $\widetilde{K}^{\mathrm{gr}}$-algebra, and $\mathscr{R}(X,x)$ is an affine birational space.

Proposition 3.19.4 (Affinoid neighbourhood invariance)**.** *If $Y = \mathcal{M}(B) \subset X$ is an affinoid domain containing x, its reduction is an open subspace of $\mathscr{R}(X,x)$. If Y is a neighbourhood of x in X, this open subspace equals $\mathscr{R}(X,x)$. Consequently $\mathscr{R}(X,x)$ depends only on the good analytic germ (X,x), with its actual completed residue field.*

Proof. Corollary 2.33.15 identifies the completed residue fields. Under this identification $R_{A,x} \subset R_{B,x} \subset F_x$. Choose an admissible absolute weighted Tate presentation of B. Apply the fixed absolute presentation result and evaluate in F_x, as for A. Finiteness over the evaluated polynomial image makes $R_{B,x}$ a finitely generated graded $\widetilde{K}^{\mathrm{gr}}$-algebra. Containment of homogeneous

generators is an open condition in $\mathscr{R}(X,x)$, and Proposition 3.19.2 makes this open quasi-compact.

If Y is a neighbourhood, the topology defined by function values gives a Laurent affinoid neighbourhood $W \subset Y$ of x cut out by finitely many inequalities $|f_i| \le r_i$ and $|g_j| \ge s_j$ that are all strict at x. Put $D = \mathcal{O}(W)$. Its presentation over A adjoins f_i at radius r_i and g_j^{-1} at radius s_j^{-1}. Choose an admissible absolute weighted Tate presentation of A. Proposition 1.17.3 combines it with the displayed relative presentation into an admissible absolute weighted Tate presentation of D. Apply the fixed absolute presentation result and evaluate in the common field F_x. The resulting finite module generators for $R_{D,x}$ are taken over the evaluated polynomial image. Each old variable has the same evaluated class in its assigned degree as the class from A. Each new variable has evaluated norm strictly less than its assigned radius, so its class evaluates to zero. Thus the coefficient image lies in $R_{A,x}$, and the same generators make $R_{D,x}$ finite over $R_{A,x}$. The intermediate $R_{B,x}$ is integral over $R_{A,x}$ and finitely generated as an algebra, hence finite. Lemma 3.19.1 gives equality of the valuation spaces. Any two affinoid neighbourhoods of the good germ contain such a common Laurent affinoid neighbourhood, with its domain structures identified by Corollary 2.40.9. The identifications are canonical and satisfy the cocycle condition. □

Lemma 3.19.5 (The reduction of an active Weierstrass inequality)**.** *For a good germ (X,x), let $f_1,\dots,f_m$ be analytic near x, with $r_i = |f_i(x)| > 0$, and write $\lambda_i \in F_x$ for their initial values. Then*

$$\mathscr{R}\big(X\{r^{-1}f\},x\big) = \mathscr{R}(X,x)\{\lambda_1,\dots,\lambda_m\}.$$

Every subdomain germ of a good germ has a finite cover by germs of this form. Every nonzero homogeneous element of F_x is the initial value of an invertible analytic function on some neighbourhood of x.

Proof. Take an affinoid representative A on which the functions are defined and put $D = A\{r^{-1}f\}$. Choose an admissible absolute weighted Tate presentation of A. Combine it with the actual quotient $A\{r^{-1}T\} \twoheadrightarrow D$ by Proposition 1.17.3 to obtain an absolute presentation of D. In the free-polyradius comparisons used for this presentation, Proposition 2.32.1 identifies the extended quotients naturally for the given bounded maps. Apply the fixed absolute presentation result and evaluate at x in the original field

F_x. The old variables give evaluated classes from $R_{A,x}$ in their assigned degrees. The new variable for f_i has assigned radius $r_i = |f_i(x)|$ and evaluates in that degree to λ_i. The resulting evaluated polynomial image is contained in $R_{A,x}[\lambda_1, \dots, \lambda_m] \subset R_{D,x}$. The finite module generators obtained by evaluation therefore make $R_{D,x}$ finite over $R_{A,x}[\lambda_1, \dots, \lambda_m]$. Lemma 3.19.1 gives the formula.

Theorem 2.36.6 gives a finite rational cover of any affinoid subdomain, and a subdomain germ is covered by finitely many affinoid subdomain germs. Discard rational members not containing x after shrinking the ambient neighbourhood. Every remaining denominator is nonzero at x, so becomes invertible after another shrinking. The rational inequalities then become Weierstrass inequalities. Those strict at x can be omitted on a neighbourhood, leaving precisely the displayed active inequalities. Finally $\operatorname{Frac}(A/\ker x)$ is dense in $\mathcal{H}(x)$. Approximate a lift of a nonzero homogeneous element with relative error less than one. It has the same initial value as a fraction a/b, with $b(x) \neq 0$ and $a(x) \neq 0$; on a smaller neighbourhood this fraction is analytic and invertible. □

Proposition 3.19.6 (Finite coverings and inverse images of germ reductions)**.** *Let (X, x) be a good germ and let (X_i, x) be finitely many good subdomain germs covering it. Then*

$$\mathscr{R}(X, x) = \bigcup_i \mathscr{R}(X_i, x).$$

For a morphism of good germs $(Y, y) \to (X, x)$ and a subdomain germ $X' \subset X$, reduction of the inverse image is the inverse image of $\mathscr{R}(X', x)$ under restriction of graded valuations. Intersections and finite unions of subdomain germs are consequently compatible with reduction. Here the reduction of a general subdomain germ of a good germ is the union of the reductions in any finite affinoid cover; this union is independent of that cover.

Proof. Use Lemma 3.19.5 to refine the cover and shrink an affinoid representative of X so that the members have the form

$$X_i = \{z : |f_{ij}(z)| \leq r_{ij},\ 1 \leq j \leq n\}, \qquad r_{ij} = |f_{ij}(x)|, \qquad f_{in} = 1.$$

All the functions may be made invertible near x. Let J be the tuples $a = (a_1, \dots, a_m)$ with at least one $a_i = n$, and put $g_a = \prod_i f_{ia_i}$, $s_a = \prod_i r_{ia_i}$. The rational domains

$$V_a = \{z : |g_{a'}(z)/g_a(z)| \leq s_{a'}/s_a \text{ for every } a' \in J\}$$

cover X, by choosing a largest $|g_a(z)|/s_a$. They refine the X_i. Indeed fix i with $a_i = n$ and $z \in V_a$, and choose ℓ with $z \in X_\ell$. If $\ell \neq i$, the tuples replacing a_ℓ by n first show $|f_{\ell a_\ell}(z)| = r_{\ell a_\ell}$. Replacing also a_i by j then gives $|f_{ij}(z)| \leq r_{ij}$ for every j. If $\ell = i$ this already holds. Thus $V_a \subset X_i$. Write $\gamma_a \in F_x$ for the initial value of g_a. Lemma 3.19.5 identifies the reductions of the V_a with

$$\mathscr{R}(X, x)\{\gamma_{a'}/\gamma_a : a' \in J\}.$$

These rational valuation sets cover $\mathscr{R}(X, x)$: any finite list of nonzero homogeneous elements admits a largest member in the divisibility preorder of a valuation ring. Since each lies in the reduction of the corresponding X_i, the covering assertion follows.

Define the reduction of a subdomain germ by the stated finite union. Each affinoid piece gives an open containment locus by Proposition 3.19.4, quasi-compact by Proposition 3.19.2. The finite union is thus a quasi-compact open. Two such covers have a common finite refinement by affinoid subdomain germs: refine their pairwise intersections. The proved covering assertion on each original affinoid piece identifies all three unions. Thus the definition is independent of the cover. For an active Weierstrass inequality, inverse image simply replaces its defining function by its pullback. The isometric embedding $\mathcal{H}(x) \hookrightarrow \mathcal{H}(y)$ preserves its initial value and degree. The displayed formula of Lemma 3.19.5 therefore proves the inverse-image assertion for each such inequality, and hence for finite intersections of them. Refine a general subdomain by these pieces and use the proved finite-cover assertion. Intersection is the particular case of inverse image by a subdomain inclusion; finite union follows from finite covers. □

Lemma 3.19.7 (A valuative inclusion gives a germ inclusion)**.** *Let V be an affinoid subdomain germ of a good germ (X, x), and let $W = X\{|g_1| \leq s_1, \dots, |g_m| \leq s_m\}$, where $s_j = |g_j(x)| > 0$. If $\mathscr{R}(V, x) \subseteq \mathscr{R}(W, x)$, then $V \subseteq W$ as subdomain germs.*

Proof. Write $V = \mathcal{M}(B)$ after choosing sufficiently small representatives. By Lemmas 3.19.1 and 3.19.5, each initial value μ_j of $g_j(x)$ is integral over $R_{B,x}$. Take a homogeneous monic equation and lift its coefficients to $a_{ji} \in B$ with $\rho_B(a_{ji}) \leq s_j^i$. The lifted polynomial satisfies

$$\left| g_j^{d_j} + \sum_{i=1}^{d_j} a_{ji} g_j^{d_j - i} \right| (x) < s_j^{d_j}.$$

This strict inequality persists in an open neighbourhood of x in V. There $|g_j| > s_j$ is impossible: its leading power would strictly dominate every other term, so the displayed absolute value would equal $|g_j|^{d_j} > s_j^{d_j}$. Intersect the finitely many neighbourhoods obtained this way. It lies in W, which is exactly the required germ inclusion. □

Theorem 3.19.8 (Subdomains of a good germ)**.** *Reduction gives an order-preserving bijection between subdomain germs of a good germ (X, x) and nonempty quasi-compact open subspaces of $\mathscr{R}(X, x)$. It preserves intersections, finite unions and inverse images.*

Proof. Every quasi-compact open is a finite union of basic opens $\mathscr{R}(X, x)\{\lambda_{i1}, \dots, \lambda_{in_i}\}$. Lift the nonzero homogeneous λ_{ij} to invertible analytic functions near x, by Lemma 3.19.5, and impose the corresponding inequalities with radii $|f_{ij}(x)|$. These finitely many closed domains in a common affinoid representative give a finite quasi-net on their union by Lemma 2.39.2. The reduction of that union is the prescribed one by Proposition 3.19.6. This proves surjectivity.

First suppose a subdomain germ $Y \subset X$ has full reduction. Refine Y by finitely many active Weierstrass germs Y_i, using Lemma 3.19.5. Their reductions cover $\mathscr{R}(X, x)$ and therefore admit a Laurent refinement by Lemma 3.19.3. Lift the finitely many homogeneous Laurent functions to invertible analytic functions near x. The resulting Laurent domains V_ϵ are actual joint domains by Theorem 2.31.1 and Proposition 2.31.3. Each opposite inequality is active at x, and Lemma 3.19.5 computes its reduction. The Laurent domains cover a neighbourhood of x in X. Each reduction is contained in the reduction of some Y_i, so Lemma 3.19.7 gives $V_\epsilon \subset Y_i$ as germs. There are only finitely many cells, so one common shrinking therefore makes their whole union lie in Y. Thus $Y = X$ as germs.

Now suppose $\mathscr{R}(Y, x) \subseteq \mathscr{R}(Z, x)$. Cover Y and Z by finitely many affinoid subdomain germs Y_i and Z_j. Proposition 3.19.6 shows that the reductions of the $Y_i \cap Z_j$, for fixed i, cover the reduction of Y_i. The preceding full-reduction assertion, applied inside the good germ Y_i, implies $Y_i \subset Z$. Thus $Y \subset Z$. This proves injectivity and order reflection. The compatibility assertions are already Proposition 3.19.6. □

Theorem 3.19.9 (The reduction functor on all analytic germs)**.** *There is a canonical functor from analytic germs over K to graded birational spaces over $\widetilde{K}^{\mathrm{gr}}$, extending $(X,x) \mapsto \mathscr{R}(X,x)$ for affinoid germs. It carries subdomain inclusions to open embeddings, finite subdomain coverings to open coverings, and commutes with intersections, finite unions and inverse images. Its field at (X,x) is $\widetilde{\mathcal{H}(x)}^{\mathrm{gr}}$.*

Proof. Use the domain chart quasi-nets of Proposition 2.40.5 and their restriction and refinement properties in Lemma 2.39.2. Choose finitely many affinoid subdomain germs (X_i,x) covering a neighbourhood of x. Refine each pairwise intersection by finitely many affinoid subdomain germs. By Proposition 3.19.4 each reduced piece of the overlap is open in the corresponding reduced chart, with the canonical identification of the completed residue field. Proposition 3.19.6 identifies the overlap with their finite union. Compute in either chart using the common finite refinement and those same field identifications. On triple intersections the maps agree, since each is the identity on the same field and is checked on the common affinoid refinement. Glue these open subspaces.

Proposition 3.19.2 makes each affine piece and each basic overlap piece quasi-compact, so the finite overlap unions are quasi-compact. The gluing is quasi-compact and quasi-separated and has a local homeomorphism to $\mathscr{P}_{\widetilde{K}^{\mathrm{gr}}}(F_x)$. Each affine piece is irreducible by the same proposition, and their overlap identifications identify the common trivial graded valuation F_x. Thus the gluing is connected. A refinement of the affinoid cover gives the same gluing by the finite-cover assertion of Proposition 3.19.6. Any two choices have a common refinement, proving independence of the atlas and of neighbourhood representatives.

By Theorem 2.40.8, a morphism of germs is represented after finite affinoid refinements by maps of good germs. On each such map use the isometric embedding of completed residue fields and restriction of valuations. These local maps agree on intersections by the inverse-image assertion of Proposition 3.19.6, hence glue. The same assertion proves all stated compatibilities; identities and compositions are checked on the common affinoid refinements. This also proves uniqueness of the extension. □

Theorem 3.19.10 (Subdomains of an arbitrary analytic germ)**.** *For every analytic germ (X, x), reduction is an order-preserving bijection between its subdomain germs and the nonempty quasi-compact open subspaces of its graded birational reduction. This includes arbitrary positive radii and trivially valued ground fields.*

Proof. Take a finite affinoid subdomain-germ cover (X_i, x). For a quasi-compact open W in the reduction, the intersections $W \cap \mathscr{R}(X_i, x)$ are quasi-compact by quasi-separatedness. Theorem 3.19.8 lifts them to subdomain germs $Y_i \subset X_i$. On overlaps the two lifts coincide, because their reductions coincide and the good-germ theorem reflects equality on every affinoid piece of the overlap. Hence their union is a subdomain germ of X with reduction W. This proves surjectivity. If two subdomain germs have equal reductions, intersect with each X_i and use the good-germ theorem to obtain equality there. The pieces cover the original germ, so the subdomains are equal. The same argument with inclusions proves order reflection. □

References. For the historical construction of graded valuation spaces and germ reductions, see [Tem04, Lemmas 1.4 and 2.1–2.4; Proposition 4.1; Theorem 4.5]. The argument here uses strict Laurent inequalities, fixed absolute weighted Tate presentations and evaluation in the graded residue field. The rational-product refinement makes the covering step explicit.

3.19.1 Products and separation of germ reductions

A morphism of birational spaces with graded field embedding $E \hookrightarrow F$ is a continuous map compatible with restriction $\mathscr{P}_k(F) \to \mathscr{P}_k(E)$. It is *separated*, respectively *proper*, if its canonical map

$$Q \longrightarrow P \times_{\mathscr{P}_k(E)} \mathscr{P}_k(F)$$

is injective, respectively bijective. This canonical map is always a local homeomorphism: use charts on which both structural maps are open embeddings into the same valuation space. Thus the two conditions mean an open embedding and a homeomorphism, respectively. A proper inverse image of an affine birational space is affine.

Proposition 3.19.11 (Reduction of a fibre-product germ)**.** *Let $T = Y \times_X Z$ and let t have images y, z, x. Put $F = \widetilde{\mathcal{H}(t)}^{\mathrm{gr}}$. Pull the three reductions back to $\mathscr{P}_{\widetilde{K}^{\mathrm{gr}}}(F)$, and denote them by Y_0, Z_0, X_0. Then*

$$\mathscr{R}(T,t) \simeq Y_0 \times_{X_0} Z_0.$$

Proof. First take $X = \mathcal{M}(A)$, $Y = \mathcal{M}(B)$ and $Z = \mathcal{M}(C)$. Write R_A, R_B, R_C, R_D for the images in F of the graded reductions, where $D = B\widehat{\otimes}_A C$. Corollary 3.18.8 says that R_D is finite over the graded subalgebra generated by R_B, R_C. Lemma 3.19.1 consequently gives

$$\mathscr{P}(F)\{R_D\} = \mathscr{P}(F)\{R_B, R_C\} = \mathscr{P}(F)\{R_B\} \times_{\mathscr{P}(F)\{R_A\}} \mathscr{P}(F)\{R_C\}.$$

These are identities of open subspaces with their structural maps, not just identities of points. In general choose finite affinoid subdomain-germ covers of the three spaces, refining the inverse image of the target cover. Their products cover the product germ by the analytic fibre-product construction. The displayed identifications agree on overlaps by the inverse-image compatibility in Theorem 3.19.9. They therefore glue to the asserted isomorphism. □

Proposition 3.19.12 (Complete scalar extension of a germ reduction)**.** *Let L/K be a complete valued extension and let $y \in X_L$ map to $x \in X$. Put $F = \widetilde{\mathcal{H}(y)}^{\mathrm{gr}}$. Then the reduction of (X_L, y) is the pullback of $\mathscr{R}(X,x)$ along*

$$\mathscr{P}_{\widetilde{L}^{\mathrm{gr}}}(F) \longrightarrow \mathscr{P}_{\widetilde{K}^{\mathrm{gr}}}(\widetilde{\mathcal{H}(x)}^{\mathrm{gr}}).$$

Proof. On an affinoid piece, the second finite comparison map of Corollary 3.18.8 identifies, up to a finite integral extension, the image of $\widetilde{A_L}^{\mathrm{gr}}$ in F with the subalgebra generated by the images of $\widetilde{A}^{\mathrm{gr}}$ and $\widetilde{L}^{\mathrm{gr}}$. Every valuation under consideration already contains the latter field. The integral criterion therefore gives exactly the displayed inverse image. Affinoid subdomain-germ covers and their overlaps commute with scalar extension, so Theorem 3.19.9 glues the identities. □

Corollary 3.19.13 (Proper reductions and base change)**.** *If the reduction of a morphism of analytic germs is proper, then its reduction after an analytic base change over any complete valued extension is proper. Properness of birational morphisms is local on an open covering of the target.*

Proof. A proper birational map is a homeomorphism to the specified pullback of its target. A further pullback is again a homeomorphism. For analytic products this is Proposition 3.19.11, and for change of ground field it is Proposition 3.19.12. Factor a general change into these two changes. For target locality, a local homeomorphism is bijective if its inverse images over every member of an open cover are bijective. This proves the final assertion. □

Proposition 3.19.14 (The interior criterion for good germs)**.** *For a morphism of good germs $(Y, y) \to (X, x)$, the point y is in the relative analytic interior if and only if $\mathscr{R}(Y, y) \to \mathscr{R}(X, x)$ is proper.*

Proof. Choose affinoid neighbourhood representatives $\mathcal{M}(B) \to \mathcal{M}(A)$. The ordinary local test identifies interiority here with interiority for the given good germs. In $F = \widetilde{\mathcal{H}(y)}^{\mathrm{gr}}$, let R_B and R_A be the images of the two graded algebras. The canonical birational map is the inclusion

$$\mathscr{P}(F)\{R_B\} \;\subseteq\; \mathscr{P}(F)\{R_A\}.$$

It is bijective exactly when R_B is finite over R_A, by Lemma 3.19.1 and the finite generation from the fixed absolute presentation. The relative point criterion, applied to $A \to B$ at y, identifies this with innerness at y. Shrinking the good representatives does not change their reductions by Proposition 3.19.4, so the criterion is a criterion on the original germs. □

Lemma 3.19.15 (Closed immersions and reductions)**.** *A closed immersion of analytic germs induces an isomorphism on reductions. Suppose a germ morphism factors as a closed immersion into a subdomain germ of its target. It is a closed immersion near its distinguished point if and only if its reduction is an isomorphism.*

Proof. For an admissible quotient $A \twoheadrightarrow B$ and a point of $\mathcal{M}(B)$, the completed residue fields on the two sides are identical. Compose an admissible absolute weighted Tate presentation of A with $A \twoheadrightarrow B$. The fixed absolute presentation result gives a finite module over the evaluated image of the graded polynomial algebra. By spectral contraction each coordinate surviving at its prescribed degree in B already survives at that degree in A. Thus the image of $\widetilde{B}^{\mathrm{gr}}$ is finite over the image of $\widetilde{A}^{\mathrm{gr}}$ in the same field. The valuation criterion then identifies the reductions. This proves the assertion on affinoid

pieces; finite subdomain covers prove it for closed immersions of arbitrary germs.

For the converse write the factorization as $(Y, y) \hookrightarrow (X', x) \hookrightarrow (X, x)$. The first part identifies $\mathscr{R}(Y, y)$ with $\mathscr{R}(X', x)$. Hence an isomorphism to $\mathscr{R}(X, x)$ means that the subdomain germ X' has full reduction. Theorem 3.19.10 gives $(X', x) = (X, x)$, so the factorization is a closed immersion on an ordinary neighbourhood. The forward direction follows from the first part. □

Proposition 3.19.16 (Separation of analytic germs)**.** *A morphism of analytic germs is separated if and only if its reduction is separated. In particular, an analytic germ whose reduction is affine admits a compact separated neighbourhood representative.*

Proof. The diagonal is locally a closed immersion into a subdomain. Choose finite affinoid subdomain-germ covers of source and target, with each source piece Y_i mapping to one target piece X_i. In $Z = Y \times_X Y$ the spaces $Z_i = Y_i \times_{X_i} Y_i$ are analytic domains. On Z_i the diagonal is given by the admissible quotient $B_i \widehat{\otimes}_{A_i} B_i \twoheadrightarrow B_i$; the bounded section $b \mapsto b \otimes 1$ proves admissibility. The finitely generated kernel ideals agree on overlaps: after a common affinoid refinement they are the kernels of the same restricted quotient. The finite-module equivalence of Theorem 2.39.6 glues these ideals to a closed subspace of $\bigcup_i Z_i$, whose charts are the Y_i. Thus the diagonal factors, as a germ, through this subdomain by a closed immersion.

Let $z = \Delta(y)$ and $F = \widetilde{\mathcal{H}(y)}^{\mathrm{gr}}$. Proposition 3.19.11 identifies $\mathscr{R}(Z, z)$ with the fibre product of two copies of $\mathscr{R}(Y, y)$ over $\mathscr{R}(X, x) \times_{\mathscr{P}(F_x)} \mathscr{P}(F)$. Its diagonal is an isomorphism precisely when the canonical map from $\mathscr{R}(Y, y)$ to that last space is injective. Lemma 3.19.15 therefore identifies separatedness of the reduction with the diagonal being a closed immersion on a neighbourhood of z.

We spell out the neighbourhood step. First restrict the target to a compact Hausdorff neighbourhood of x and the source to a compact Hausdorff neighbourhood of y mapping into it; these exist by Proposition 2.40.5. Their fibre product is compact Hausdorff by the construction in Theorem 2.40.11. Suppose the diagonal is a closed immersion over an open neighbourhood O of z. Its inverse image is an open neighbourhood of y. Choose a compact analytic neighbourhood Y' of y inside that inverse image. The diagonal of Y' is a closed immersion over the inverse image of O in $Y' \times_X Y'$. Its full

image is closed, because Y' is compact and this product is Hausdorff. The open complement of that image, together with the inverse image of O, covers the product; on the complement the closed subspace is empty. Thus the diagonal of Y' is a closed immersion everywhere. This proves separatedness of a representative, rather than merely a condition at one point of its diagonal. Conversely a separated representative has a closed diagonal, giving the reduction condition above. The last assertion follows since an affine birational space embeds into its valuation space. □

Lemma 3.19.17 (Affinoid intersections in a separated space)**.** *The intersection of two affinoid domains in a separated analytic space is affinoid.*

Proof. For domains $U, V \subset X$, their intersection represents $U \times_X V$ by the domain universal property. It is the inverse image of the closed diagonal of X in the affinoid space $U \times_K V$. A closed analytic subspace of an affinoid is affinoid: its coherent ideal is $\widetilde{I}$ for a finite ideal I by Theorem 2.39.6, and the space is represented by the admissible quotient A/I. Completed tensor products carry that quotient to its base-changed quotient. These observations prove the assertion. □

Lemma 3.19.18 (Preparing a two-sided Laurent neighbourhood)**.** *Suppose a separated germ (X, x) is covered by good subdomain germs Y, Z, and its reduction is affine. Suppose, for a nonzero homogeneous $\lambda \in F_x$, that*

$$\mathscr{R}(Y, x) = \mathscr{R}(X, x)\{\lambda\}, \qquad \mathscr{R}(Z, x) = \mathscr{R}(X, x)\{\lambda^{-1}\}.$$

Then one can choose affinoid neighbourhood representatives $Y = \mathcal{M}(B)$, $Z = \mathcal{M}(C)$, whose union is a neighbourhood of x, and elements and radii satisfying every hypothesis of Theorem 2.45.2.

Proof. Set $t = \deg(\lambda)$ and write $\mathscr{R}(X, x) = \mathscr{P}(F_x)\{\alpha_1, \dots, \alpha_m\}$, with homogeneous nonzero α_i of degree r_i. All shrinkings below are neighbourhood shrinkings inside Y and Z. Their union remains a neighbourhood in X: the two domains form a finite closed cover of a compact neighbourhood, and this is the quasi-net topology criterion of Lemma 2.40.2.

Step 1: an exact common annular overlap. Lift λ to invertible functions f on a neighbourhood in Y and g on a neighbourhood in Z. The cut $|f| \le t$ in Y has the same reduction as Y, and $|g^{-1}| \le t^{-1}$ in Z has the same reduction

as Z. By Theorem 3.19.8 these cuts are neighbourhoods of x. Choose affinoid neighbourhoods inside them, so

$$\rho_B(f) \le t, \qquad \rho_C(g^{-1}) \le t^{-1}.$$

Their intersection W is affinoid by Lemma 3.19.17. At x we have $|(f-g)(x)| < t$. Shrink the two neighbourhoods so that $\rho_W(f-g) < t$; continuity on the old intersection, and the relative-open-set definition, permit one common shrinking. Thus on W one has $|f| = |g| = t$, and

$$W \subset Y\{|f| \ge t\}, \qquad W \subset Z\{|g| \le t\}.$$

The three domain germs in this display have reduction $\mathscr{R}(X,x)\{\lambda,\lambda^{-1}\}$. The domain classification gives affinoid neighbourhoods $Y' \subset Y$, $Z' \subset Z$ with

$$Y' \cap Z = Y'\{|f| \ge t\}, \qquad Y \cap Z' = Z'\{|g| \le t\}.$$

Choose a strict Laurent neighbourhood W' of x inside $Y' \cap Z'$ in the old affinoid W. Write it using finitely many conditions $|u_i| \le a_i$ and $|v_j| \ge b_j$, strict at x. The first displayed overlap is a Weierstrass domain of Y' (its defining function is the analytic function f^{-1}), and the second is a Weierstrass domain of Z'. Restriction to a Weierstrass domain has dense image: truncate its Tate series in the adjoined functions, which are already functions on the source. Approximate u_i, v_j from each side, with errors respectively less than a_i, b_j on that overlap. The non-Archimedean inequality makes the resulting cuts have exactly W' as their intersections with the old overlap. They are neighbourhoods of x because the inequalities there are strict. Replacing Y, Z, W by these two cuts and W' gives the exact identities

$$W = Y\{|f| \ge t\} = Z\{|g| \le t\}, \qquad A = B\{tf^{-1}\} = C\{t^{-1}g\}.$$

Step 2: presentations with small extra coordinates. Lift each α_i to an invertible analytic function f_i near x in Y, and impose $|f_i| \le r_i$. Every such cut has full reduction in Y, since every valuation of $\mathscr{R}(Y,x)$ contains α_i. It is therefore a neighbourhood. Perform the first step again if needed to retain the exact overlap identities. Choose $p < t$ with $\rho_B(f^{-1}) < p^{-1}$, and set

$$D_B = K\{r_1^{-1}T_1, \dots, r_m^{-1}T_m, t^{-1}S, pS^{-1}\}.$$

Map T_i to f_i and S to f. Use the quotient presentations of the preceding Weierstrass cuts: the powers of f_i and f are bounded by constant multiples

of r_i^n and t^n, respectively. The strict inequality for f^{-1} gives its corresponding power bound with radius p^{-1}, either by the spectral-radius formula or Theorem 1.20.2. Thus this Tate substitution is a bounded homomorphism. The graded image of D_B in F_x contains $\widetilde{K}^{\mathrm{gr}}[\alpha_1, \dots, \alpha_m, \lambda]$: the displayed coordinates have exactly the prescribed initial values, and the inverse coordinate has zero initial value at its allowed radius $p^{-1} > t^{-1}$. Equality of the valuation sets for Y and this finitely generated subalgebra makes the image of $\widetilde{B}^{\mathrm{gr}}$ finite over it, by Lemma 3.19.1. It is therefore finite over the possibly larger graded image of D_B as well. Theorem 3.18.10, applied relatively to D_B, gives an admissible epimorphism

$$D_B\{a_1^{-1}U_1, \dots, a_l^{-1}U_l\} \twoheadrightarrow B, \qquad |u_j(x)| < a_j.$$

Give $A = B\{tf^{-1}\}$ the quotient norm from the corresponding boundary presentation in which $|S| = t$. By density of $C \to A$, choose $g_i, g' \in C$ with

$$\|f_i - g_i\|_A < r_i, \qquad \|f - g'\|_A < t.$$

On W one has $|g' - g| < t$. Restrict Z to $|g' - g| \le d < t$, where d is larger than the supremum on W; this cut contains W and is a neighbourhood of it. Then $|g'| = |g|$ throughout the new Z, so g' is invertible, has inverse bounded by t^{-1}, and defines the same overlap. Replace g by g'. Impose also $|g_i| \le r_i$. These cuts contain W and are neighbourhoods of x, since their reductions are full: the initial value of $g_i(x)$ is α_i. Choose $q > t$ and $q > \rho_C(g)$. Repeating the relative innerness argument with

$$D_C = K\{r_1^{-1}T_1, \dots, r_m^{-1}T_m, q^{-1}S, tS^{-1}\}.$$

Its graded image contains $\widetilde{K}^{\mathrm{gr}}[\alpha_1, \dots, \alpha_m, \lambda^{-1}]$. The same integral criterion gives an admissible epimorphism

$$D_C\{b_1^{-1}V_1, \dots, b_h^{-1}V_h\} \twoheadrightarrow C, \qquad |v_j(x)| < b_j.$$

Step 3: match all coordinates. Density of $C \to A$ and $B \to A$ gives elements $u_i' \in C$, $v_j' \in B$ with

$$\|u_i - u_i'\|_A < a_i, \qquad \|v_j - v_j'\|_A < b_j.$$

Cut Y by $|v_j'| \le b_j$ and Z by $|u_i'| \le a_i$. These cuts contain all of W, because the original v_j, u_i have the stated spectral bounds there; they are neighbourhoods of x because the values at x are strictly smaller than the bounds. Thus

the overlap and its exact Weierstrass descriptions are unchanged. The two presentations now have coordinate lists

$$(f_1,\dots,f_m,u_1,\dots,u_l,v'_1,\dots,v'_h),\qquad (g_1,\dots,g_m,u'_1,\dots,u'_l,v_1,\dots,v_h).$$

Use radii (r_i,a_i,b_j) for these lists and $p<t<q$ for the annulus coordinate. Appending the cut coordinates can only decrease the induced quotient norm on A, since every old lift is still a permitted lift of the same norm. Hence all strict difference estimates above remain valid in the new, actual overlap quotient norm. These are exactly the hypotheses of Theorem 2.45.2. □

Theorem 3.19.19 (Goodness is detected by graded reduction)**.** *An analytic germ over an arbitrary complete non-Archimedean field is good if and only if its graded birational reduction is affine.*

Proof. For a good germ choose an affinoid neighbourhood. The definition of its reduction and finite generation in Theorem 3.18.7 show that the reduction is affine.

Conversely suppose $R=\mathscr{R}(X,x)$ is affine. Proposition 3.19.16 permits a compact separated neighbourhood representative. Cover its germ by finitely many affinoid subdomain germs. Their reductions are affine opens in the affine space R. Express each as $R\{\lambda_1,\dots,\lambda_e\}$: each is affine in the common valuation space, so adjoining its finite generators to generators for R has this effect. Lemma 3.19.3 refines this cover by the Laurent cells

$$R\{\mu_1^{\epsilon_1},\dots,\mu_n^{\epsilon_n}\},\qquad \epsilon_i\in\{1,-1\}.$$

Each cell has an affinoid lift inside one original chart: lift the finitely many μ_i to invertible functions near x in that chart and impose the indicated Weierstrass inequalities, shrinking to an affinoid neighbourhood first. The reduction formula of Lemma 3.19.5 identifies its reduction with that cell. Theorem 3.19.10 says these lifts cover the germ.

Induct on n. For $n=0$ a single affinoid lift has full reduction and is a neighbourhood by the domain classification. For $n>0$, group the cells according to the sign of μ_1. Each group has affine reduction $R\{\mu_1^{\pm1}\}$ and a Laurent cover with $n-1$ remaining functions. The induction hypothesis makes its lifted germ good. Thus the original germ is the union of two good subdomain germs whose reductions are $R\{\mu_1\}$ and $R\{\mu_1^{-1}\}$. Lemma 3.19.18 supplies affinoid neighbourhood representatives with matched Laurent presentations.

Theorem 2.45.2 makes their union affinoid. This union is a neighbourhood of x, proving goodness. □

References. The diagonal criterion and the goodness theorem reconstruct [Tem04, Proposition 4.8 and Theorem 5.1] and the detailed neighbourhood preparation in [Tem00, Theorem 3.1]. The graded finite criterion, domain-germ classification and double Laurent gluing used in this proof are proved internally above.

3.19.2 Strict germs and ordinary formal-model reduction

Lemma 3.19.20 (Restriction to degree one)**.** *For an extension of graded fields $k \subset F$, the map*

$$\pi : \mathscr{P}_k(F) \longrightarrow \mathscr{P}_{k_1}(F_1), \qquad V \longmapsto V \cap F_1$$

is surjective. If Q_0 is an ordinary birational space with field F_1, its pullback Q along π is affine if and only if Q_0 is affine.

Proof. Let $V_0 \subset F_1$ be a valuation ring containing k_1. In the graded subring $C = kV_0 \subset F$, its degree-r part, for r in the support of k, is $c_r V_0$ for any nonzero $c_r \in k_r$. Distinct degrees are independent. Consequently $\mathfrak{m}_{V_0} C$ is proper and its contraction to V_0 is $\mathfrak{m}_{V_0}$. Choose an ordinary maximal ideal above it by [StacksAlg, Tag 00E0(2)], take its homogeneous part, and localize homogeneously. Lemma 3.19.1 gives a graded valuation ring V dominating this local ring. It contains k and contracts to V_0: if $a \in F_1 \setminus V_0$, then $a^{-1} \in \mathfrak{m}_{V_0}$ lies in the maximal ideal of V, so $a \notin V$. This proves surjectivity.

If $Q_0 = \mathscr{P}_{k_1}(F_1)\{b_1, \dots, b_m\}$, then $Q = \mathscr{P}_k(F)\{b_1, \dots, b_m\}$, proving one direction. Conversely suppose Q is affine. Surjectivity of π shows that the structural local homeomorphism of Q_0 is injective; hence Q_0 is an open subspace of $\mathscr{P}_{k_1}(F_1)$. Write $Q = \mathscr{P}_k(F)\{a_1, \dots, a_n\}$ with nonzero homogeneous a_i. Let H be the subgroup of degrees whose positive powers belong to the support of k. If $\deg(a_i) \notin H$, then $F_H[a_i^{-1}]$ is a polynomial graded algebra over the graded subfield F_H: distinct powers have distinct degree classes modulo H. Localize at the homogeneous prime (a_i^{-1}) and use graded domination. The resulting valuation contains F_H, hence F_1 and k, but excludes a_i. Its restriction is the trivial valuation F_1, which belongs to every nonempty open Q_0. This contradicts $Q = \pi^{-1}(Q_0)$. Thus $\deg(a_i)^{d_i} = \deg(c_i)$ for some $d_i > 0$ and nonzero homogeneous $c_i \in k$. Put $b_i = a_i^{d_i}/c_i \in F_1$. A valuation containing k contains a_i exactly when it contains b_i. Hence

$$Q = \pi^{-1}\big(\mathscr{P}_{k_1}(F_1)\{b_1, \dots, b_n\}\big).$$

Surjectivity gives the same expression for Q_0, proving that it is affine. □

Proposition 3.19.21 (Strict affinoid germs and model rings)**.** *Assume K is nontrivially valued. Let A be strict affinoid, A_0 an admissible affine model, and $x \in \mathcal{M}(A)$. Put $F = \widetilde{\mathcal{H}(x)}^{\mathrm{gr}}$ and let $R_0 \subset F_1$ be the image of $A_0/K^{\circ\circ}A_0$ under evaluation at x. Then*

$$\mathscr{R}(\mathcal{M}(A), x) = \pi^{-1}\big(\mathscr{P}_{\widetilde{K}}(F_1)\{R_0\}\big).$$

The ordinary birational space on the right is independent of A_0. It can also be computed using the image of the canonical reduction $\widetilde{A}$ in F_1.

Proof. The ring R_0 is a finitely generated $\widetilde{K}$-algebra: reduction of a restricted power series is a polynomial, and a finite topological presentation of A_0 supplies finitely many generators. Its image lies in the degree-one part of the image of $\widetilde{A}^{\mathrm{gr}}$, giving one inclusion of valuation sets.

Conversely let a graded valuation contain $\widetilde{K}^{\mathrm{gr}}$ and R_0. A nonzero homogeneous element of the image of $\widetilde{A}^{\mathrm{gr}}$ is the initial value of some $a \in A$ with $|a(x)| = \rho_A(a) = r > 0$. By Lemma 3.16.6, choose $d > 0$, $c \in K^\times$ with $r^d = |c|$. The element $b = a^d/c$ is power-bounded. The integral adjunction proved in Theorem 3.16.8 makes b integral over A_0. Reduce a monic equation at x: the initial value of b is integral over R_0 and therefore belongs to the valuation ring. It equals the dth power of the initial value of a, divided by the homogeneous unit given by c. The valuation ring thus also contains that initial value of a. This proves equality of the two sets.

The identical argument with all power-bounded elements in place of the model generators gives the canonical reduction formula. Finally Lemma 3.19.20 makes π surjective, so the ordinary valuation sets are identical for any two models. Their topologies are the induced valuation topology, giving the asserted independence. □

Theorem 3.19.22 (Ordinary reduction on the strict category)**.** *For a strictly K-analytic germ, with K nontrivially valued, there is a canonical ordinary birational reduction $\mathscr{R}_0(X, x)$ with field $\widetilde{\mathcal{H}(x)}$. It is functorial and independent of strict affinoid charts and neighbourhoods. Its nonempty quasi-compact open subspaces classify strict subdomain germs. It is related to the graded reduction by*

$$\mathscr{R}(X, x) = \mathscr{R}_0(X, x) \times_{\mathscr{P}_{\widetilde{K}}(\widetilde{\mathcal{H}(x)})} \mathscr{P}_{\widetilde{K}^{\mathrm{gr}}}(\widetilde{\mathcal{H}(x)}^{\mathrm{gr}}).$$

Goodness, separation of germ morphisms, and relative interiority are detected by affineness, separatedness, and properness of this ordinary reduction, respectively.

Proof. On each strict affinoid chart use the ordinary space in Proposition 3.19.21. Strict affinoid atlas intersections have common finite strict affinoid subdomain-germ refinements. The formula in that proposition identifies the pullbacks of the ordinary overlap spaces with the graded overlap spaces. Surjectivity of π in Lemma 3.19.20 reflects equality, inclusion and a finite covering of these open subspaces. Consequently the ordinary charts glue with exactly the same cocycle argument as Theorem 3.19.9, and their pullback is the displayed graded reduction. Common strict affinoid refinements give independence; valuation restriction gives functoriality. A neighbourhood shrinking has unchanged graded reduction, and hence unchanged ordinary reduction by the same surjectivity.

For the subdomain assertion work first in one strict affinoid chart. A quasicompact ordinary open is a finite union of basic opens defined by elements of $F_1 = \widetilde{\mathcal{H}(x)}$. Lift each nonzero element to an invertible analytic function of absolute value one at x, using the fraction approximation in Lemma 3.19.5. One may take strict rational neighbourhoods on which the numerator and denominator are invertible: strict rational neighbourhoods form a basis, since $\sqrt{|K^\times|}$ is dense in $\mathbf{R}_{>0}$. The cuts $|f| \leq 1$ are strict Weierstrass domains. Their finite unions lift the given ordinary open. Equality and inclusion of lifts follow by pulling back with π and applying Theorem 3.19.10. On a general strict germ perform the construction in finitely many strict charts; equality on overlaps glues the subdomain. This proves the stated classification.

The displayed pullback and Lemma 3.19.20 identify affineness of the two reductions. Apply Theorem 3.19.19. For a germ morphism the canonical map which defines ordinary birational separation or properness becomes, after pullback by the surjective degree-one restriction on the source field, the canonical map for the graded reductions. Injectivity and bijectivity are reflected by a surjective base change of sets. Hence ordinary and graded separation or properness are equivalent. Now use Proposition 3.19.16 and Theorem 2.46.1. This proves all three criteria. □

Strict ordinary reduction and neighborhood invariance. Let (X, x) be a germ in the strict analytic G-category over the standing complete nontrivially valued field. Put $F = \widetilde{\mathcal{H}(x)}^{\mathrm{gr}}$ and $F_0 = \widetilde{\mathcal{H}(x)}$. There is a canonical

ordinary birational reduction $\mathscr{R}_0(X,x)$ over the valuation space with field F_0. It is independent of the strict charts and of ordinary neighborhood representatives. Its pullback along degree-one restriction is the given graded reduction. On an actual strict affinoid chart $D = \mathcal{M}(A)$ containing x, with admissible affine model A_0, its chart is

$$O_D = \mathscr{P}_{\widetilde{K}}(F_0)\{\mathrm{im}(A_0/K^{\circ\circ}A_0 \to F_0)\}.$$

Use the strict chart net of the germ. Finitely many charts through x cover a neighborhood, and in the strict site each pairwise or triple intersection has such finite common strict refinements relative to that intersection. For example, choose the finitely many members through x by the quasi-net condition on the intersection. For a triple, apply the same condition a second time to each pairwise refining member and the third chart, then use transitivity of the quasi-net condition. In a prescribed ordinary open neighborhood, shrink the finitely many chart members through x by Lemma 2.35.25 inside each member. These are strict affinoid domains containing relative neighborhoods of x, so their union still contains a neighborhood of x. We use strict-site refinements throughout.

The completed residue fields of these nested chart domains are canonically the same by Corollary 2.33.15. Use an affine model of each actual strict algebra, which exists by Proposition 3.14.7. Put $P_0 = \mathscr{P}_{\widetilde{K}}(F_0)$, $P = \mathscr{P}_{\widetilde{K}^{\mathrm{gr}}}(F)$, and let $\pi_1 : P \to P_0$ be restriction of graded valuations. It is surjective by Lemma 3.19.20 and continuous by the defining containment opens. Proposition 3.19.21 gives the displayed ordinary chart, independently of the model, and identifies its inverse image with the actual graded chart.

For charts D_i, D_j, let O_{ij} be the finite union of the ordinary valuation opens of a common strict refinement of their intersection germ. Its inverse image under π_1 is exactly the graded overlap by Theorem 3.19.9. Surjectivity therefore puts O_{ij} in both ordinary charts and makes it independent of the chosen overlap refinement. Identify the two copies of O_{ij} by the identity on valuations in P_0. The graded triple intersection identities imply $O_{ij} \cap O_{ik} = O_{ij} \cap O_{jk}$ as subsets of P_0, again by surjectivity. The transition maps hence satisfy the cocycle. Glue these opens, with the topology tested on the open charts. The resulting space maps by a local homeomorphism to P_0. Pullback recovers the graded gluing: the pulled-back charts cover the pullback, and their identifications are the actual graded overlap identifications. Distinct chart sheets are kept in this gluing.

Each ordinary chart is a basic quasi-compact valuation open, since its residue algebra image is finitely generated as in Proposition 3.19.21. The quasi-compactness and irreducibility argument of Proposition 3.19.2 applies to the valuation spaces concentrated in degree one. The ordinary space is quasi-compact by its finite chart cover. Two basic opens in different charts intersect in the corresponding overlap, cut by their two containment conditions. This is a finite union of basic quasi-compact opens, since each overlap is such a union. Every quasi-compact open has a finite cover by chart basic opens, so intersections of quasi-compact opens are quasi-compact. The trivial valuation F_0 lies in every chart and every overlap, so the irreducible chart spaces glue to a connected space.

A common strict refinement of chart covers induces maps of the ordinary glued spaces over P_0. On a refining chart, the map is the inclusion of its ordinary valuation locus in the original chart. This inclusion and its agreement on overlaps follow from the corresponding graded maps and the surjectivity of π_1. After pullback the comparison is the canonical graded comparison of Theorem 3.19.9. A continuous map over P_0 between spaces locally homeomorphic to P_0 is itself a local homeomorphism: restrict around the source point and its image to two such charts, and use their common open image in P_0. If the pullback of this map along π_1 is an isomorphism, its map on each fibre over P_0 is bijective, since that fibre can be tested at a lift of its base point. It is thus a bijective local homeomorphism and hence an isomorphism. This proves independence of the strict covers. For ordinary neighborhood restriction, use the finite strict shrinkings above. The graded reductions are those of the same germ by Theorem 3.19.9, so the same argument gives the canonical ordinary comparison. The comparisons compose because they agree on common strict refinements.

Lemma 3.19.23 (Valuations and a proper pointed comparison)**.** *Let Z be an integral finite-type k-scheme equipped with a dominant map $\operatorname{Spec} F \to Z$. The spaces $\mathscr{P}_k(F)\{R\}$ for affine opens $\operatorname{Spec} R \subset Z$ glue to a birational space $\operatorname{Val}(Z, F)$. Its points are valuations of F/k together with a centre on Z. If $Z' \to Z$ is proper and the specified generic point lifts to $\operatorname{Spec} F \to Z'$, then the induced map*

$$\operatorname{Val}(Z', F) \longrightarrow \operatorname{Val}(Z, F)$$

is an isomorphism, with Z' replaced by the reduced closure of that generic point when necessary.

Proof. A valuation ring $V \subset F$ containing R has the unique centre $\mathfrak{m}_V \cap R$ on $\operatorname{Spec} R$. Indeed the elements outside that prime are units of V, so the map $R \to V$ factors through the required local homomorphism. Conversely a centre supplies this containment. For a principal open $D(h)$, the centre lies in $D(h)$ exactly when $h^{-1} \in V$. Thus restriction to scheme opens gives the usual basic open subspaces of the valuation space. Affine coverings of intersections therefore provide compatible gluing maps. On a nonseparated scheme the same valuation may have distinct centres; the gluing retains these as distinct points unless an overlap identifies them.

For a point (V, z) of $\operatorname{Val}(Z, F)$, apply the valuative criterion [StacksAlg, Tag 0BX5] to the specified generic map and $\operatorname{Spec} V \to Z$. It gives exactly one lift to Z'. Its image is contained in the closure of the specified generic point. The morphism factors through the reduced closure: a section of its defining radical ideal vanishes at the generic point of the integral scheme $\operatorname{Spec} V$, and hence vanishes identically. This proves bijectivity on centred valuations.

The map is a local homeomorphism. Around any lifted centre choose affine opens $\operatorname{Spec} R' \subset Z'$ and $\operatorname{Spec} R \subset Z$ with the first mapping into the second. The induced valuation map is the inclusion of the open $\mathscr{P}_k(F)\{R'\}$ into $\mathscr{P}_k(F)\{R\}$. Here the images of the coordinate rings in F are understood; they are finitely generated, so these are basic quasi-compact opens. A bijective local homeomorphism is a homeomorphism, and the field map is the identity. This proves the birational-space isomorphism. □

Proposition 3.19.24 (Special fibres of admissible modifications)**.** *Let K be nontrivially valued. An admissible formal blowup induces a locally projective morphism of special fibres over $\widetilde{K}$, and hence a proper morphism. A finite composite of admissible formal blowups induces a proper morphism as well. For a fixed point of the common generic fibre, the induced map between the reduced closures of its specializations is proper.*

Proof. Work on an affine formal open $\operatorname{Spf}(A_0)$, and write the open ideal as $I = (f_0, \dots, f_m)$. The algebraic blowup of $\operatorname{Spec} A_0$ in I is projective by [StacksAlg, Tag 02NS]. In the notation of Theorem 3.15.1, its affine chart is $D_i = A_0[I/f_i]$ and the formal chart is $B_i = \widehat{D}_i$. Both are π-torsion-free. Lemma 3.14.8 identifies $D_i/\pi D_i$ with $B_i/\pi B_i$. Since $\pi \in K^{\circ\circ}$, taking the further quotient by $K^{\circ\circ}/(\pi)$ gives

$$D_i/K^{\circ\circ} D_i \simeq B_i/K^{\circ\circ} B_i.$$

These identifications respect the fractional chart transitions. Thus the special fibre of the formal blowup is the special fibre of the algebraic blowup. A local closed embedding of the latter into a projective space remains a closed embedding after reduction. The special-fibre map is consequently locally projective and is proper by [StacksAlg, Tag 01WC]. Finite composites are proper by [StacksAlg, Tag 01W3].

Let Z' and Z be the reduced closures of the two specializations. The map $Z' \to \mathfrak{X}_s$ factors through Z: a function vanishing on Z vanishes at the specified generic point of Z', and therefore vanishes on the reduced integral Z'. Locally on Z, the scheme Z' is a closed subscheme of the base change of the preceding projective morphism. It is therefore locally projective over Z, hence proper by the same criterion. Applying this at each blowup gives the assertion for any finite admissible modification. □

Actual generic atlases for the compact common-model comparison. Let U be the quasi-compact quasi-separated strict neighborhood in the formal-model realization of a strict germ. Write the two admissible models as $\mathfrak{X}, \mathfrak{Y}$, with their actual formal-affine generic charts carrying the identified generic fibres to U by i_X, i_Y. Each model has a finite compatible formal affine atlas. Every formal open and every stage of an actual finite sequence of admissible blowups of either compact model has such an atlas, with actual generic maps for the formal arrows. The generic arrow of each admissible blowup is the canonical isomorphism on its rational charts.

Proof. For either given model, the actual affine generic charts cover U admissibly. Choose a finite point-cover by Lemma 2.39.2, retaining a finite germ-cover witness at x if one is specified. The given principal traces and affine specialization identify these charts with the full inverse images of their formal affines. Each affine specialization is surjective by Theorem 3.16.15. The same contraction on the overlaps glues these specializations, so the corresponding formal affines cover the model. This gives the required finite-incidence formal cover. The finite family of analytic charts is still admissible: each chart meets every test affinoid in a compact subset by quasi-separatedness, hence in a closed subset of that Hausdorff test chart. The quasi-net topology test of Lemma 2.40.2 gives closedness in U, and a finite closed cover is a quasi-net. The finite-incidence generic-fibre construction on this cover recovers the given generic fibre. On the actual common principal refinements the bounded chart identities agree with the supplied ones; the identities and their inverses

glue. These comparisons also preserve the marked point and its completed residue field.

For any compact model in the construction, a formal open is quasi-compact. Intersect it with each member of a finite affine cover and use the topological Noetherian clause of NAG-001944 on that affine. Each resulting intersection has a finite principal cover there. An admissible blowup of the compact model has a finite formal affine cover: restrict to each original affine and use the finitely many affine blowup charts for a finite generating list of the actual open ideal. The actual restrictions come from the formal existence and restriction clause of NAG-001945 and Theorem 3.15.1. This proves the compactness assertion by induction for finite composites. It includes the blowups of the successively pulled-back centres, since the actual pulled-back open ideals are finitely generated and contain a power of π. A finite product of the finitely many extended centres at a compact stage is again an open ideal. All of these statements precede any assertion about generic invariance.

Apply the finite-incidence construction to each resulting finite cover. On formal intersections it gives actual bounded principal identifications with their triple cocycle, and on formal arrows it gives actual analytic maps, checked on principal pieces subordinate to target affines. Thus for a blowup arrow the atlas on its source and the atlas on its target are compatible with the actual formal map. On an original affine, the actual blowup charts are the rational domains prescribed by Theorem 3.15.2. Their transition maps are the bounded rational restrictions. These are the same restrictions on the common principal pieces used in the finite-incidence construction. The supplied-atlas invariance of NAG-001945 therefore applies to this pair of atlases. It identifies the actual generic blowup arrow, an isomorphism, with the maps on these rational charts. This step supplies the isomorphism only after the underlying compact formal scheme and its compatible generic maps have been constructed.

□

Application to common modification. In cover realization use these atlases on the affine local blowups for the rational domains, on the product blowups and on the model obtained by extending the local centres. The formal opens produced have the actual restricted principal atlases. In morphism lifting use those same atlases at the cover-realization stage, on the

affine lifting blowups and on the second product blowup. The comparison of the formal lifts is on the actual generic restrictions of the given analytic arrow. In common modification the same construction applies to the lifts in each direction, the successive pulled-centre blowups and all the comparison arrows. Thus the hypotheses of the supplied-atlas NAG-001947–001949 proofs are supplied at each use. No application of model existence or localization is needed. Take the specified isomorphism in NAG-001949 to be $i_Y^{-1} i_X$. Its conclusion, with the generic blowup isomorphisms just verified, gives

$$i_X(b_X)_\eta = i_Y(b_Y)_\eta$$

as isomorphisms from a single actual generic fibre to U.

Pointed reduction of a supplied compact formal model. Let K be the standing complete nontrivially valued field. Let U be a quasi-compact quasi-separated strict analytic neighborhood of x, with a supplied admissible formal model $\mathfrak{U}$ and its compatible formal-affine generic atlas on U. Here compatibility includes the actual bounded restriction maps on the common principal traces of formal affine opens. Put $k = \widetilde{K}$, $F = \widetilde{\mathcal{H}(x)}^{\mathrm{gr}}$, $F_0 = \widetilde{\mathcal{H}(x)}$, and

$$P = \mathscr{P}_{\widetilde{K}^{\mathrm{gr}}}(F), \qquad P_0 = \mathscr{P}_k(F_0).$$

Let Z be the reduced closure in $\mathfrak{U}_s$ of the image of the specified residual evaluation $\operatorname{Spec} F_0 \to \mathfrak{U}_s$. Then there is a canonical isomorphism over P

$$\operatorname{Val}(Z, F_0) \times_{P_0} P \simeq \mathscr{R}(U, x).$$

With the strict ordinary reduction defined by the ordinary chart gluing, this comparison gives $\operatorname{Val}(Z, F_0) \simeq \mathscr{R}_0(U, x)$. If a common model with finite admissible blowup maps to two such models is supplied, assume its actual generic maps induce the same identification with U. Then the two proper pointed maps induce the ordinary chart comparisons over P_0.

Proof. On each formal affine $Q_i = \operatorname{Spf}(A_i)$, take the affine specialization from Theorem 3.16.15. These maps agree on the supplied principal trace identifications: a formal principal restriction inverts f, its generic trace is $|f| = 1$, and its residual point map is the restriction of the same evaluation. Thus the specialization sp on this supplied model is surjective and the generic chart $D_i = \mathcal{M}(A_i[1/\pi])$ is precisely the inverse image of Q_i. The latter assertion follows on every other chart by the principal trace description of

the formal overlaps. Choose finitely many D_i covering U by Lemma 2.39.2, and include the finitely many chart members through x whose union contains a neighborhood of x. Surjectivity now shows that the corresponding finite formal affines cover $\mathfrak{U}$. The selected chart members through x cover its germ. In particular $\mathfrak{U}$ is quasi-compact, with a finite-incidence affine cover.

The image of $\operatorname{Spec} F_0$ is $s = \operatorname{sp}(x)$, the generic point of Z. A formal chart meets Z on the special fibre exactly when it contains s, which is exactly when D_i contains x. On such a chart, the reduced closure is $\operatorname{Spec} R_i$, where $R_i = \operatorname{im}(A_i/K^{\circ\circ}A_i \to F_0)$: the kernel of evaluation in the field is prime and defines the reduced closure of the image point. These finitely many charts cover Z; hence Z is integral and of finite type over k.

For each pair of formal affines, their intersection has a finite cover by opens principal in both. In fact, around any point of the intersection take a principal open in the first affine and then a principal open of the second inside it. In the first principal open, the equation of the second has reduction modulo π of the form a/f^n, by Proposition 3.17.1. Clearing the denominator gives a principal open in the first affine as well. Each formal affine is topologically Noetherian by the formal-topology clause of NAG-001944, so finitely many such opens cover the intersection. Repeat the same refinement for a third affine. The formal restriction maps on these actual principal opens satisfy the triple cocycle.

Keep the pieces meeting Z, or equivalently containing s. On a piece principal in Q_i, with equation f, one has $\bar{f} \neq 0$ in R_i and the residual algebra image is $R_i[1/\bar{f}]$ in F_0. This again follows from the principal localization formula, first modulo π and then over k. Its analytic trace contains x, and the principal traces retained this way cover the corresponding analytic overlap germ. Indeed the entire overlap is covered by finitely many compact rational traces which are closed in the affinoid D_i. Their finite cover is admissible, and the members through x cover a relative neighborhood. The same argument applies to the triple overlap.

By Proposition 3.19.21, the actual graded reduction of the chart D_i at x is the inverse image of $\mathscr{P}_k(F_0)\{R_i\} \subseteq P_0$. The identical assertion for the retained principal trace uses $R_i[1/\bar{f}]$, with the same completed residue field by Corollary 2.33.15. Thus these are also the exact transition opens after pulling back to P. On the scheme side they are the valuation loci with centre in the actual principal pieces, by Lemma 3.19.23. On the analytic side the actual graded overlap is the union of the indicated graded opens by

Theorem 3.19.9. The two open gluings and their triple transitions agree over P. A valuation with distinct centres is identified only when the common open identifies the centres.

For the ordinary comparison, use the strict ordinary chart construction. Use these same charts and principal overlaps to compute the strict ordinary gluing. On every chart both maps are the identity of $\mathscr{P}_k(F_0)\{R_i\}$, and on overlaps both identify the identical principal valuation loci. Thus this ordinary comparison is compatible with further common refinements. It preserves the specified field F_0.

Finally, let b_X, b_Y be the supplied finite composites with common formal source $\mathfrak{W}$. Give $\mathfrak{W}$ its actual compatible formal-affine generic atlas, with both generic arrows induced by the given formal maps. On the compact models these atlases are supplied by the finite-incidence construction and the compact-model atlas. Apply the supplied-atlas invariance of NAG-001945 at each blowup stage. By the required equality of the generic identifications,

$$j = i_X(b_X)_\eta = i_Y(b_Y)_\eta \colon \mathfrak{W}_\eta \xrightarrow{\sim} U.$$

In particular $w = j^{-1}(x)$ is a single point above x on both legs, with the same canonically identified F_0.

The pointed maps on special fibres commute with evaluation in F_0: on compatible affine charts they are induced by the formal coefficient homomorphisms, followed by reduction of the same evaluation at w. Each maps the reduced pointed closure into the corresponding reduced pointed closure downstairs and is proper by Proposition 3.19.24. Apply Lemma 3.19.23 to each leg. Each resulting isomorphism keeps the valuation and sends its centre to its image.

To compare these isomorphisms with the ordinary chart gluing, refine the chart net on $\mathfrak{W}_\eta$ by the actual principal traces of a formal affine refinement subordinate simultaneously to the two inverse images of target affine covers. Such a refinement is obtained by taking principal opens inside the pairwise inverse images on each source affine. The formal topology clause gives finite principal refinements there, and the supplied trace construction gives their admissible generic covers. Keep the pieces through w; they cover a neighborhood in that net. Under the single isomorphism j they are a common strict affinoid refinement at x for the two generic atlases, with their actual bounded restriction maps. On every retained piece, both induced maps of valuation charts are the inclusions over the identity of the represented valuations of

F_0. Their centres contract along the corresponding formal maps. These are precisely the maps in the ordinary chart gluing. The comparison on overlaps and its compatibility with any further such supplied modification follow by composing these same maps on the common principal refinements. □

Theorem 3.19.25 (Formal-model realization of a strict germ)**.** *Let U be a quasi-compact quasi-separated strictly analytic neighbourhood of x over a nontrivially valued field K, and let $\mathfrak{U}$ be any admissible formal model of U. Write $F_0 = \widetilde{\mathcal{H}(x)}$, and let Z be the reduced closure in $\mathfrak{U}_s$ of the image of* $\operatorname{Spec} F_0 \to \mathfrak{U}_s$*. Then there is a canonical isomorphism*

$$\operatorname{Val}(Z, F_0) \simeq \mathscr{R}_0(U, x).$$

It is independent of the formal model and the neighbourhood. Common admissible modifications realize the comparison through proper pointed residual schemes, inducing the identity on F_0.

Proof. A formal model of U, in the standing convention of Formal models and their two fi is given with an identification of its analytic generic fibre with U. The generic fibre in that convention is the gluing of the actual formal-affine generic charts along their analytic domains. Transport this gluing to U by the specified identification. Its formal-affine generic charts thus give the compatible admissible chart family on U. On the formal principal opens the restriction maps are the bounded maps of Proposition 3.17.1; on intersections they are checked on common principal refinements. The underlying specialization maps on the affine charts are those of Theorem 3.16.15 and agree there by contraction. This uses the generic fibre of the given model as data. Its finite chart cover and the atlases for its modifications are obtained by the compact common-model atlas argument. The principal traces and contraction identify the full inverse images of the formal affine opens and the specialization maps on their overlaps.

Apply the pointed reduction of a compact formal model to this actual atlas and evaluation at x. It identifies the ordinary strict chart gluing with the valuation space of the reduced pointed closure Z, over the specified field F_0. The ordinary neighbourhood invariance identifies the ordinary chart gluings after shrinking representatives of the analytic germ at x. The identifications use the same completed residue field at x and compose on further common refinements.

For two models of the same U, use the compact-model atlas and common modification with the analytic isomorphism specified by their identifications with U. It supplies the compatible atlases on both legs and their common source, and the two actual generic maps from that source agree as isomorphisms to U. Thus the two legs lift x to the same point with the same F_0. The supplied-leg part of the pointed reduction gives proper maps of the reduced pointed closures. On valuations they keep the field and the valuation and contract the centre. By the common principal refinements they are the ordinary chart comparisons already used above, including after further such modifications. □

References. The ordinary realization reconstructs the formal-model description of germ reduction in [Tem00, Section 1.5]. The proof uses the internal strict/graded comparison, the internal Raynaud reconstruction, and only the displayed Stacks algebraic properness inputs.

3.20 Tubes, line bundles and elementary formal models

In this section K is complete and nontrivially valued, $O = K^\circ$, and $k = \widetilde{K}$. The following arguments make explicit the local constructions used in the examples of this chapter. Their formal and analytic gluing inputs have already been established.

Proposition 3.20.1 (Tubes of locally closed strata)**.** *Let $\mathfrak{X}$ be an admissible formal O-scheme, $X = \mathfrak{X}_\eta$, and $Z \subset \mathfrak{X}_s$ a locally closed subset. Then $]Z[_{\mathfrak{X}} = \mathrm{sp}^{-1}(Z)$ is an analytic domain of X, canonically with its induced analytic structure. On an affine formal chart it has a finite cover by sets of the form*

$$\{x : |f_i(x)| < 1 \ (1 \le i \le m), \quad |g(x)| = 1\}, \qquad f_i, g \in \mathcal{O}_{\mathfrak{X}}(\mathfrak{U}).$$

This construction commutes with intersections and inverse images under formal morphisms.

Proof. On $\mathfrak{U} = \mathrm{Spf}(A_0)$, every integral element has absolute value at most one by Proposition 3.14.7. Thus evaluation induces $A_0/K^{\circ\circ}A_0 \to \widetilde{\mathcal{H}(x)}$; equivalently the source is $A_0 \otimes_O k$. The special fibre is of finite type over k, hence Noetherian by [StacksAlg, Tag 00FN]. Write $Z \cap \mathfrak{U}_s$ as a finite union of

$V(\bar{f}_1, \dots, \bar{f}_m) \cap D(\bar{g})$, with lifts in A_0. The definition of specialization gives exactly the displayed absolute-value conditions.

The inequalities $|f_i| < 1$ define an ordinary open subset W of $\mathfrak{U}_\eta$. The locus $|g| = 1$ is the Laurent affinoid domain $|g| \geq 1$, because $|g| \leq 1$ everywhere. Their intersection is an analytic domain by Proposition 2.40.5. To verify the finite-union assertion, first take the common open subset W cut out by the equations for the closed part of Z, and write the open part as a finite union of principal opens. Each of the resulting domains is relatively closed in their union. At a point, omit the finitely many closed pieces not containing it, and take small rational neighborhoods inside the remaining pieces. They give the quasi-net required by Definition 2.40.4. Hence the union is an analytic domain with its induced topology.

The formal affine generic fibres form the analytic net of Proposition 3.17.2. The local domains just constructed therefore glue: their intersections agree as subsets, and analytic-domain structures are unique by Corollary 2.40.9 and chartwise domain calculus. The same uniqueness proves compatibility with intersections. Functoriality of specialization gives the asserted inverse-image identity, and domain base change supplies its analytic structure. The equations and the finite covers used above are local on formal affine charts; no global quasi-compactness of Z is assumed. □

Proposition 3.20.2 (The generic fibre of a formal line bundle)**.** *An invertible sheaf $\mathfrak{L}$ on an admissible formal scheme $\mathfrak{X}$ has a canonical invertible $\mathcal{O}_{X_G}$-module $\mathfrak{L}_\eta$ on $X = \mathfrak{X}_\eta$. This construction commutes with tensor products, duals and formal pullback. Its formal transition functions have absolute value one.*

Proof. Choose a formal affine open cover $\mathfrak{U}_i$ with frames e_i of $\mathfrak{L}$. On overlaps write $e_j = u_{ij} e_i$, where u_{ij} is a formal unit. By Proposition 3.17.2, the domains $U_i = (\mathfrak{U}_i)_\eta$ form an admissible analytic cover, and the functions u_{ij} restrict to analytic units on $U_i \cap U_j$. They satisfy $u_{ik} = u_{ij} u_{jk}$ on triple overlaps. For an analytic domain $V \subset X$, let $\mathcal{L}(V)$ consist of families of analytic coefficients a_i on $V \cap U_i$ satisfying $a_i = u_{ij} a_j$ on $V \cap U_i \cap U_j$. The structure-sheaf gluing theorem 2.40.6 makes this a sheaf of $\mathcal{O}_{X_G}$-modules. On U_i it is freely generated by e_i: a coefficient there determines every other local coefficient by the displayed transition relation. Thus it is an invertible sheaf.

A change of formal frames multiplies these coefficients by the corresponding formal units and gives a canonical isomorphism. Two covers have their common formal refinement; the isomorphisms just described agree there and hence globally. This proves independence. Products, inverses and pullbacks of the cocycles give respectively tensor products, duals and pullback, with the usual compatibility identities holding coefficientwise. Finally u_{ij} and u_{ij}^{-1} are integral on every formal affine subchart of the overlap, so both have absolute value at most one. It follows that $|u_{ij}| = 1$ everywhere on the analytic overlap. □

Lemma 3.20.3 (Canonical reduction of the standard annulus)**.** *Let $a \in K$ satisfy $0 < |a| < 1$, and put*

$$A_0 = O\langle U, V\rangle/(UV - a), \qquad A = K\langle U, V\rangle/(UV - a).$$

Then A_0 is admissible, $A_0 = A^\circ$, and

$$\widetilde{A} = A_0 \otimes_O k = k[u, v]/(uv).$$

The Shilov boundary consists of the Gauss points with $|U| = 1$ and $|U| = |a|$. Likewise the unit-circle algebra $K\langle T, T^{-1}\rangle$ has canonical reduction $k[t, t^{-1}]$ and a single Shilov point.

Proof. Apply Theorem 2.13.2 to the interval $[|a|, 1]$, rescaling its inverse coordinate by a. Every element of A has a unique expression

$$f = \sum_{n\geq 0} c_n U^n + \sum_{n\geq 1} d_n V^n, \qquad c_n \longrightarrow 0, \quad d_n \longrightarrow 0.$$

Its quotient norm is $N(f) = \max\{\max_n |c_n|, \max_{n\geq 1} |d_n|\}$. The same normal-form operators preserve integral coefficients: on a monomial one replaces $U^i V^j$ by $a^{\min(i,j)} U^{i-j}$ or $a^{\min(i,j)} V^{j-i}$; the corresponding quotient by $UV - a$ has integral coefficients. The operators are contractive after this rescaling, so the polynomial identities pass to restricted series. Hence A_0 embeds in A and is exactly the subset $N(f) \leq 1$. In particular it has no O-torsion and is admissible by Theorem 3.14.4.

At any analytic point, $|U|, |V| \leq 1$, so $|f| \leq N(f)$. At the two endpoint Gauss points, the respective values are

$$\max\{\max_n |c_n|, \max_{n\geq 1} |d_n||a|^n\}, \qquad \max\{\max_n |c_n||a|^n, \max_{n\geq 1} |d_n|\}.$$

Their maximum is $N(f)$. Thus $N = \rho_A$ and $A_0 = A^\circ$. A nonzero element with $N(f) < 1$ has a coefficient attaining this maximum, so it is an element of $K^{\circ\circ}A_0$; the converse is immediate. Therefore $A^{\circ\circ} = K^{\circ\circ}A_0$. Reducing the unique normal forms gives $k[u, v]/(uv)$. The endpoint pair is a boundary, and neither point can be omitted from this pair: U attains its maximum only at the outer endpoint of the pair, and V only at the inner one. Lemma 3.12.4 therefore makes the pair the Shilov boundary. For the unit circle the same normal form, now with interval $[1, 1]$, gives the multiplicative Laurent Gauss norm and reduction $k[t, t^{-1}]$. Proposition 3.4.1 gives its single Shilov point. □

Corollary 3.20.4 (The elementary formal blowup and its weight)**.** *For $0 < |a| < 1$, the admissible blowup of (T, a) on $\operatorname{Spf} O\langle T\rangle$ has generic domains $|T| \le |a|$ and $|a| \le |T| \le 1$. Its special fibre consists of the original affine line and one projective line meeting at a point. The invertible pulled-back ideal has canonically trivial generic fibre, and its model weight for the section 1 is* $\log \max(|T|, |a|)$.

Proof. The ideal is open since a divides a sufficiently large power of any fixed topologically nilpotent π. Theorem 3.15.1 gives the charts $O\langle U\rangle$, with $T = aU$, and $O\langle T, V\rangle/(TV - a)$, with $V = a/T$. Their formal overlap has U and V inverse units. The first special chart is $\mathbb{A}^1_k$ with coordinate u; the second is $tv = 0$, as in Lemma 3.20.3. Gluing the $t = 0$ component to the first chart by $u = v^{-1}$ gives $\mathbb{P}^1_k$. The $v = 0$ component remains the original affine line, meeting it at $t = v = 0$. Theorem 3.15.2 identifies the two generic domains and their gluing with the original disc. Proposition 3.20.2 constructs the generic line bundle of the invertible ideal, and Proposition 3.11.1 gives its model metric. On the first chart the ideal is generated by a, and on the second by T. Its generic ideal is the unit ideal because $a \in K^\times$. Thus $1 = a^{-1}a$ has norm $|a|^{-1}$ on the first chart and $1 = T^{-1}T$ has norm $|T|^{-1}$ on the second. Taking minus the logarithm gives the stated weight, and the formulas agree on the overlap. □

Corollary 3.20.5 (Disc and projective-line reductions at the Gauss point)**.** *At the unit Gauss point x, the ordinary germ reduction of the closed unit disc is represented by $\mathbb{A}^1_k$ pointed by $k(t)$; that of $\mathbb{P}^{1,\mathrm{an}}$ is represented by $\mathbb{P}^1_k$ pointed by the same field. The disc inclusion induces the usual open immersion of these models, which is not proper in the pointed birational category.*

Proof. The two formal affine lines $\operatorname{Spf} O\langle T\rangle$ and $\operatorname{Spf} O\langle S\rangle$ glue along the principal formal opens $TS = 1$. Their special fibre is $\mathbb{P}^1_k$. By Proposition 3.17.1, their generic fibres glue along $|T| = 1$; the result is $\mathbb{P}^{1,\mathrm{an}}$, since its two closed unit discs cover it, as proved in Proposition 2.33.30, and the coordinate transition is $S = T^{-1}$. The unit Gauss completed residue field has residue $k(t)$ by Proposition 3.8.1. Its specialization is the generic point of each of these residual models. The closure is therefore $\mathbb{A}^1_k$ for the disc and $\mathbb{P}^1_k$ for the projective line. Apply the fixed-model comparison to the whole disc and the whole projective line, each as a neighbourhood of x in its own space. The disc is the very same pointed T-chart in the projective-line model. The ordinary chart gluing therefore gives the inclusion via $\mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$. Finally the valuation ring $k[t^{-1}]_{(t^{-1})}$ in $k(t)$ has a centre on $\mathbb{P}^1_k$ but none on $\mathbb{A}^1_k$, since it does not contain t. Thus the corresponding valuation-space map is not surjective, and the inclusion is not proper in the pointed birational category. By Proposition 2.44.5, the relative interior of this analytic-domain inclusion is its ordinary interior in $\mathbb{P}^{1,\mathrm{an}}$. The Gauss seminorms of radii $r > 1$ on the affine T-line (Proposition 1.2.2) lie outside the disc and converge to x as $r \downarrow 1$, since the Gauss maximum of each polynomial is continuous in r. Thus x lies in the relative boundary of the inclusion. □

3.21 Dimension and the complete spectral norm

We complete two parts of the affinoid algebra theorem: the dimension in a finite normalization, and the equivalence of the spectral norm with the affinoid norm on a reduced algebra. For the second assertion the positive-characteristic argument passes through the completion of the Gauss fraction field.

Proposition 3.21.1 (Dimension of a strict normalization)**.** *For every complete non-Archimedean field K, $\dim K\langle T_1, \dots, T_n\rangle = n$. If $K\langle T_1, \dots, T_d\rangle \hookrightarrow A$ is a finite injection into a nonzero strict affinoid algebra, then $d = \dim A$.*

Proof. Put $R_n = K\langle T_1, \dots, T_n\rangle$. The coordinate ideals (0), (T_1), …, $(T_1, \dots, T_n)$ form a strictly increasing chain of primes: each quotient is a Tate algebra and its Gauss norm is multiplicative. Thus $\dim R_n \geq n$. For the reverse inequality induct on n, starting with the field $R_0 = K$. Given a nonzero prime $\mathfrak{p} \subset R_n$, choose $0 \neq f \in \mathfrak{p}$. Lemma 1.11.3 and Theorem 1.10.3 show, after an isometric coordinate change, that $R_n/(f)$ is finite over R_{n-1}.

The same holds for $R_n/\mathfrak{p}$. The dimension inequality for an integral map, [StacksAlg, Tag 00OJ], gives $\dim(R_n/\mathfrak{p}) \leq n-1$. Any prime chain in the domain R_n can be preceded by (0); after its first nonzero member its length is at most $n-1$. Its total length is therefore at most n. Finally, an integral injection preserves dimension by [StacksAlg, Tag 00OK]. Apply this to the displayed finite injection. □

Lemma 3.21.2 (Coordinates on a Banach space of countable type)**.** *Let V be a Banach K-space with a countable set whose linear span is dense. For every $0 < c < 1$ there is a finite or countable family (e_i) such that every element has a unique convergent expansion $\sum_i a_i e_i$ and*

$$c \max_i |a_i| \|e_i\| \;\leq\; \left\| \sum_i a_i e_i \right\| \;\leq\; \max_i |a_i| \|e_i\|.$$

In particular its coordinate functionals are bounded and separate points.

Proof. Enumerate a dense spanning set as $v_1, v_2, \ldots$, and choose $0 < c_j < 1$ with $\prod_{j \geq 1} c_j > c$. Process the v_j in order. If v_j lies in the span W of the vectors already chosen, add nothing. Otherwise W is finite dimensional, hence complete and closed by Lemma 1.12.1. Its distance $\delta = \inf_{w \in W} \|v_j - w\|$ is positive. Choose $w_j \in W$ with $\|v_j - w_j\| < \delta / c_j$, and adjoin $e = v_j - w_j$. For $w \in W$ and $a \in K$, the distance estimate and the ultrametric inequality give

$$\|w + ae\| \geq c_j \max\{\|w\|, |a| \|e\|\}.$$

Indeed, if $\|w\| > |a| \|e\|$ the left side equals $\|w\|$; otherwise it is at least $|a|\delta$. Induction proves the stated two-sided estimate for finite sums. The weighted coefficient space of Lemma 2.13.1 is complete, so the sum map extends to a bounded isomorphism onto a closed subspace of V. Its image contains every v_j and is therefore all of V. The lower estimate proves uniqueness and the bound $|\lambda_i(v)| \leq c^{-1} \|e_i\|^{-1} \|v\|$ for each coordinate. The zero space has the empty family. □

Theorem 3.21.3 (Separable completion of the Gauss fraction field)**.** *Let $R = K\langle T_1, \ldots, T_n \rangle$, let $F = \operatorname{Frac}(R)$ carry the Gauss absolute value, and let H be its completion. Then H/F is geometrically reduced. In particular, for every finite field extension E/F, the finite-dimensional H-algebra $E \otimes_F H$ is reduced.*

Proof. In characteristic zero every field extension is separable, hence geometrically reduced; this is the characteristic-zero case of [StacksAlg, Tags 030Z and 00I4]. Assume $\operatorname{char} K = p > 0$. Endow $K^{1/p}$ with $|a| = |a^p|^{1/p}$; it is complete, since taking pth powers identifies Cauchy sequences and their limits with those in K. Put $S_i = T_i^{1/p}$. Coefficientwise Frobenius identifies

$$R^{1/p} = K^{1/p}\langle S_1, \dots, S_n\rangle, \qquad F^{1/p} = \operatorname{Frac}(R^{1/p}).$$

We prove that every finite F-linearly independent family $u_1, \dots, u_m \in F^{1/p}$ remains linearly independent over H in $H^{1/p}$. Multiplying the family by a common nonzero element of R puts all its members in $R^{1/p}$: write $u_i^p = a_i/b_i$ with $a_i, b_i \in R$ and use $\prod_i b_i$. This multiplication preserves both kinds of independence.

Let K' be the closure, in $K^{1/p}$, of the K-algebra generated by the coefficients of the finitely many u_i. This algebra is a field: any nonzero element a has inverse a^{p-1}/a^p, with $a^p \in K$. Its closure is again a field, and the countably many monomials in its generators have dense K-linear span. Apply Lemma 3.21.2 to K'. Its bounded coordinate functionals $\lambda_j : K' \to K$, together with the unique decomposition of exponents modulo p, define bounded R-linear maps

$$\Lambda_{j,\mu} : K'\langle S\rangle \longrightarrow R, \qquad \sum_{\nu} a_\nu S^\nu \longmapsto \sum_{\eta \in \mathbf{N}^n} \lambda_j(a_{\mu+p\eta}) T^\eta, \quad 0 \le \mu_i < p.$$

The series on the right converges, its Gauss norm is at most $\|\lambda_j\|_{\mathrm{op}}$ times the norm of the original series, and the family of all $\Lambda_{j,\mu}$ separates points. R-linearity follows first for polynomial coefficients and then by continuity. The matrix whose columns are all the coordinates $\Lambda_{j,\mu}(u_i)$ has independent columns over F: a relation, after clearing its denominators in R, would give an element of $K'\langle S\rangle$ with all coordinates zero. Finite-dimensional linear algebra therefore supplies m rows with invertible $m \times m$ matrix over F. Its inverse and the bounds just obtained give a constant $C > 0$ such that

$$\max_i |b_i|_F \le C \left| \sum_i b_i u_i \right|_{F^{1/p}} \qquad (b_1, \dots, b_m \in F).$$

For coefficients in R this follows directly from the selected rows; for coefficients in F clear one common denominator and use multiplicativity of the Gauss norm. The field $H^{1/p}$ is complete by the same Frobenius argument used for $K^{1/p}$. Approximate any $h_i \in H$ by elements of F and pass to the

limit in the last inequality. It proves the asserted H-linear independence. Hence

$$H \otimes_F F^{1/p} \longrightarrow H^{1/p}$$

is injective and its source is reduced. [StacksAlg, Tag 030W, Lemma 10.44.2] now implies that H/F is geometrically reduced. Applying its definition to E/F proves the last assertion. □

References. The countable-coefficient argument follows the weak-stability proof for the Gauss fraction field in [BGR84, §5.3.1, pp. 212–213]. The formulation above records the needed geometrically reduced completion directly.

Proposition 3.21.4 (The spectral kernel and free scalar extension)**.** *For every generalized K-affinoid algebra A, the kernel of its spectral seminorm ρ_A is its nilradical. If A is reduced, every free-radius scalar extension A_r is reduced as well.*

Proof. For a strict affinoid domain B, use the notation of Proposition 3.12.2. Any maximal ideal of $D = E \otimes_F H$ gives a field map from the field E to its residue field, hence an injection. The resulting bounded point of B has zero kernel. For a general strict A, apply this observation to each quotient by a minimal prime. Thus any element vanishing at all analytic points belongs to every minimal prime and is nilpotent by [StacksAlg, Tag 00E0(4),(7)]. Conversely every multiplicative seminorm kills nilpotents.

For arbitrary radii choose a free strictification A_r. The Gauss section of Theorem 1.19.1 shows $\rho_{A_r}(a \otimes 1) = \rho_A(a)$. If this value is zero, the strict case makes $a \otimes 1$ nilpotent. The coefficient inclusion $A \hookrightarrow A_r$ is injective, so a is nilpotent. Finally, if A is reduced and $z = \sum_\nu a_\nu T^\nu \in A_r$ is nilpotent, evaluation at every Gauss-section point gives $\max_\nu |a_\nu(x)| r^\nu = 0$. Each a_ν therefore has spectral seminorm zero and is zero by the result just proved. Thus $z = 0$. □

Theorem 3.21.5 (Completeness of the reduced spectral norm)**.** *Let A be a reduced generalized K-affinoid algebra with its given norm-admissible affinoid norm. Then ρ_A is a complete norm, uniformly equivalent to the given norm. The statement holds for arbitrary positive radii and for a trivially valued ground field.*

Proof. First let A be a nonzero strict domain. Choose a finite normalization $R \hookrightarrow A$, put $F = \mathrm{Frac}(R)$, and let H be its Gauss completion. Write

$E = A \otimes_R F$. By Theorem 3.21.3, the finite-dimensional algebra $D = E \otimes_F H$ is reduced. Every prime of a finite-dimensional algebra over a field is maximal, and there are finitely many maximal ideals; the Chinese remainder theorem and the nilradical formula therefore give $D = \prod_{s=1}^{t} L_s$ with finite field extensions L_s/H. Its maximum absolute-value norm is complete and equivalent to every H-coordinate norm by Lemma 1.12.1.

Choose an F-basis $e_1, \dots, e_m$ of E. Since A is finite over R, one common nonzero denominator $b \in R$ makes all coordinates of bA belong to R. We obtain an injective R-linear map $j : A \to R^m$. The canonical finite-module theorem, and its compatibility with the affinoid norm in Proposition 1.14.4, imply

$$\|a\|_A \le C\|j(a)\|_{R^m} \le C' \max_s |a|_{L_s} \qquad (a \in A)$$

for constants independent of a. The second inequality is the finite-dimensional coordinate bound on D, including the fixed factor $|b|$. Proposition 3.12.2 identifies the last maximum with $\rho_A(a)$. The reverse inequality $\rho_A(a) \le \|a\|_A$ is the spectral-radius formula. This proves norm equivalence for a strict domain.

For a reduced strict A, the diagonal map into the finite product of its minimal-prime quotients is an injection of finite A-modules. It is norm-strict by Theorem 1.11.5. The domain case on the factors and the fact that every point kills a minimal prime give the same two-sided estimate for A. For arbitrary radii choose a free strictification A_r. It is reduced by Proposition 3.21.4. Apply the strict estimate upstairs and restrict to the isometric coefficient inclusion $A \hookrightarrow A_r$. The Gauss section again gives $\rho_{A_r}(a \otimes 1) = \rho_A(a)$, proving the desired estimate. Completeness and definiteness follow from equivalence with the given Banach norm. The zero algebra has the zero Banach space and causes no exception. □

References. This is the reduced affinoid norm theorem of [BGR84, §6.2.4, Theorem 1]. The proof above retains finite normalization and the Gauss fraction field, expressing the norm comparison through an explicit finite-module coordinate injection.

3.21.1 Dimension after complete scalar extension

The Krull dimension of a non-strict affinoid algebra need not be its analytic dimension. The following comparison makes the latter independent of the extension used to obtain a strict presentation.

Lemma 3.21.6 (Scalar invariance of strict affinoid dimension)**.** *Let A be a nonzero strict K-affinoid algebra and let L/K be any complete valued extension. Then A_L is strict and* $\dim A_L = \dim A$.

Proof. By Theorem 1.12.2, choose a finite injection $R = K\langle T_1, \dots, T_d\rangle \hookrightarrow A$. Proposition 3.21.1 gives $d = \dim A$. The finite R-module topology on A is its affinoid topology. Indeed finite generators give a bounded surjection $R^m \to A$; when K is nontrivially valued, Lemma 1.14.1 makes it norm-strict. When K is trivially valued, the defining strict quotient norms on R and A, and the finite-module quotient norm, take only the values zero and one, so they agree. The injection is now norm-strict: the image of the finite R-submodule R is closed with its canonical topology by Theorem 1.11.5. The strict exactness of completed scalar tensoring in Theorem 2.33.3 therefore gives an injection $R_L \hookrightarrow A_L$. The finite-module base-change theorem 2.33.19 shows that this is a finite map. The scalar presentation theorem 2.33.5 identifies R_L with $L\langle T_1, \dots, T_d\rangle$ and makes A_L strict. Applying Proposition 3.21.1 over L gives $\dim A_L = d$. □

Lemma 3.21.7 (Comparing a free strictification with any extension)**.** *Let s be a finite K-free polyradius and let K_s be its Gauss field. For every complete extension L/K, there is a complete field M with compatible isometric embeddings $L \hookrightarrow M$ and $K_s \hookrightarrow M$. One can take the completion of $L(T_1, \dots, T_m)$ for the Gauss absolute value of polyradius s.*

Proof. The weighted Gauss absolute value on a polynomial ring over a valued field is multiplicative by Proposition 1.2.2. It extends to the rational function field by taking quotients. The inclusion $K(T) \hookrightarrow L(T)$ is isometric because the coefficientwise maximum defining the Gauss value is unchanged. It therefore extends to an isometric map from the completion of $K(T)$ into M. This completion is K_s: the Laurent monomials are an orthogonal basis in the free-radius case, and a nonzero polynomial has a unique dominant monomial, so its reciprocal is a convergent Laurent geometric series, as in Lemma 1.16.1. The constant-field inclusion $L \hookrightarrow M$ is also isometric. Both maps restrict to the given inclusion of K, which proves compatibility. No independence of s over the larger value group of L is required. □

Theorem 3.21.8 (Intrinsic analytic dimension and scalar extension)**.** *Let A be a nonzero generalized K-affinoid algebra. For any complete extension L/K making A_L strict, the integer* $\dim A_L$ *is independent of L. Denote it by* $\dim_K \mathcal{M}(A)$*. For every complete extension E/K,*

$$\dim_E \mathcal{M}(A_E) = \dim_K \mathcal{M}(A).$$

Proof. Choose once a free strictification $F = K_s$ using Theorem 1.17.6, and set $d = \dim A_F$. Given another strictifying L, choose the common complete extension M of F and L in Lemma 3.21.7. Associativity of completed affinoid scalar extension 2.33.9 identifies $(A_F)_M$ and $(A_L)_M$ with A_M. Both A_F and A_L are strict, so Lemma 3.21.6 gives

$$d = \dim A_F = \dim A_M = \dim A_L.$$

This proves independence, including strictifying extensions which are not themselves affinoid over K. For arbitrary E/K, choose a complete extension N/E making $(A_E)_N$ strict. It is A_N by the same associativity, so its dimension is d by the independence just proved over K. The definition over E now gives the displayed equality. Nonvanishing under all scalar extensions follows from Theorem 2.33.6. □

References. This comparison implements the strictification definition in [TemIntro, §3.5] using the internal finite-normalization and completed-tensor theorems.

3.22 Abhyankar dimension and the dimension of analytic domains

The dimension of an affinoid can be recovered from its completed residue fields. Finite normalization proves this first in the strict case. A free Gauss extension then preserves the residual dimension of its canonical section, while no other point over the same point has larger relative residual dimension. This gives the general statement and, at the same time, the dimension comparison for domains and coverings.

Definition 3.22.1. For a valued extension E/F, put

$$\begin{aligned} s(E/F) &= \operatorname{trdeg}_{\widetilde{F}} \widetilde{E}, \\ t(E/F) &= \dim_{\mathbf{Q}}((v(E^\times)/v(F^\times)) \otimes_{\mathbf{Z}} \mathbf{Q}), \\ d(E/F) &= s(E/F) + t(E/F). \end{aligned}$$

The values are nonnegative integers or infinity. The invariant of an analytic point is $d_K(x) = d(\mathcal{H}(x)/K)$.

Lemma 3.22.2 (Towers, algebraic extensions, and completion)**.** *For a tower $K \subset F \subset E$ of valued fields,*

$$d(E/K) = d(E/F) + d(F/K).$$

Completion does not change the value group or the residue field. An algebraic valued extension has $s = t = 0$. Consequently algebraic extensions and completion do not change $d(-/K)$. No completeness assumption is needed for the tower or algebraic assertions.

Proof. The residue fields form a tower. Their transcendence degrees add: when both degrees are finite, choose a finite transcendence basis in each step. Their union is algebraically independent over the bottom field, by collecting coefficients of a polynomial in the second basis, and the whole extension is algebraic over that union. If one degree is infinite, arbitrarily long finite independent families already force the degree at the top to be infinite. For value groups the sequence

$$\begin{aligned} 0 \longrightarrow (v(F^\times)/v(K^\times)) \otimes \mathbf{Q} \longrightarrow (v(E^\times)/v(K^\times)) \otimes \mathbf{Q} \\ \longrightarrow (v(E^\times)/v(F^\times)) \otimes \mathbf{Q} \longrightarrow 0 \end{aligned}$$

is exact: clearing integer denominators verifies both injectivity and the middle kernel. Vector-space dimensions therefore add.

If $a \neq 0$ is in the completion, approximate it by a_j in the original field with $|a - a_j| < |a|$. Then $|a_j| = |a|$, so no value is added. A norm-one element has the same residue as an approximation whose error is less than one; hence no residue is added either.

Let a be nonzero and algebraic over F. In a polynomial relation $\sum c_i a^i = 0$, the largest nonzero term size is attained at least twice. Thus $(i - j)v(a) \in$

$v(F^\times)$ for two distinct indices, proving that the value-group quotient is torsion. For a norm-one algebraic element a, divide a nonzero polynomial relation by a coefficient of largest size. Its coefficients now have size at most one and not all their residues vanish. Reduction gives a nonzero polynomial relation for $\widetilde{a}$ over $\widetilde{F}$. Elements of size less than one reduce to zero. The residue extension is therefore algebraic. This proves the last assertion and completes the proof. □

Lemma 3.22.3 (The valued-field Abhyankar inequality)**.** *Let E/F be a valued extension with finite transcendence degree n. Then $d(E/F) \leq n$. The same bound holds after completion and after any subsequent algebraic valued extension.*

Proof. Choose norm-one elements $u_1, \ldots, u_a \in E$ with residues algebraically independent over $\widetilde{F}$, and choose $z_1, \ldots, z_b \in E^\times$ whose values are rationally independent modulo $v(F^\times)$. We show that the $a + b$ elements u_i, z_j are algebraically independent over F.

For every nonzero polynomial Q over F in the u_i,

$$|Q(u_1, \ldots, u_a)| = \max\{\, |c| : c \text{ is a coefficient of } Q \,\}.$$

Indeed, divide by a coefficient attaining the maximum and reduce; algebraic independence makes the resulting residue nonzero. Now write a nonzero polynomial in all variables as $\sum_\nu Q_\nu(u) z^\nu$, omitting zero coefficients. The sizes of its summands lie in pairwise distinct cosets of $|F^\times|$: equality for two exponents would give a nontrivial integer relation between the $v(z_j)$ modulo $v(F^\times)$. There is consequently a unique largest summand, and the sum is nonzero. Thus $a + b \leq n$. Taking all finite choices proves the inequality, including finiteness of both invariants. Lemma 3.22.2 proves the assertions about completion and algebraic extension. □

Lemma 3.22.4 (Residual dimension of a Gauss field)**.** *Let F be any complete valued field and let E be the completion of $F(T_1, \ldots, T_m)$ with its weighted Gauss absolute value, for arbitrary positive radii. Then $d(E/F) = m$.*

Proof. Consider one variable T, with $w = v(T)$. Its Gauss fraction field has value group $v(F^\times) + \mathbf{Z}w$, since the value of a polynomial is the minimum of its term values and quotients subtract these values. If no positive multiple of w lies in $v(F^\times)$, every nonzero polynomial has a unique term of largest size.

In a rational function of size one the quotient of the two leading terms has exponent zero; it differs from a scalar of norm one by an element of norm less than one. The residue field is therefore $\widetilde{F}$, and the value-group quotient has rational rank one.

Otherwise choose the least $q > 0$ with $qw = v(c)$ for some $c \in F^{\times}$, and put $U = T^q/c$. Its residue is transcendental over $\widetilde{F}$: a nonzero polynomial in U with coefficients of size at most one and at least one unit coefficient has Gauss norm one. Every residue of a rational function of norm one belongs to $\widetilde{F}(\widetilde{U})$. To see this, divide its numerator and denominator by a maximal term of the denominator. The terms that survive reduction have exponent differences divisible by q, by its minimality, and become scalar multiples of Laurent monomials in $\widetilde{U}$. The reduced denominator is nonzero by the just-proved transcendence. Thus the residue field is $\widetilde{F}(\widetilde{U})$, and the value-group quotient is torsion. In both cases $d = 1$. Completion changes neither invariant.

Adjoin the variables one at a time. The resulting completed field agrees with the completion for the simultaneous weighted Gauss norm. Indeed the norms agree on rational functions over F, and a rational function with finitely many coefficients in an intermediate completed field is approximated by replacing those coefficients by elements of its dense rational subfield. A sufficiently close denominator has the same positive norm, so its inverse also converges. This proves density at each step. The tower formula now gives $d(E/F) = m$, including $m = 0$. □

Proposition 3.22.5 (Residual dimension under free scalar extension)**.** *Let A be a generalized K-affinoid algebra, let r be a finite tuple of m radii free over K, and put $L = K_r$. For $x \in \mathcal{M}(A)$ and its Gauss lift $\sigma_r(x)$,*

$$d_L(\sigma_r(x)) = d_K(x).$$

For every $y \in \mathcal{M}_L(A_L)$ above x,

$$d_L(y) \leq d_K(x).$$

Proof. Write $F = \mathcal{H}_A(x)$ and $E = \mathcal{H}_{A_L}(y)$. The residue-field universal property embeds F and L isometrically in E. Denote the images of the m Gauss parameters by $t_1, \ldots, t_m$. The field E is the completion of $F(t_1, \ldots, t_m)$ with the induced absolute value. Here the density assertion is essential: finite Laurent polynomials with coefficients in A are dense in A_L by Theorem 1.16.2; bounded evaluation therefore puts the entire image of A_L in the closure of

$F(t_1, \dots, t_m)$. Conversely the completion defining E contains F, the parameters, and the closure of their fraction field. Inverting nonzero limits is continuous, so these two closures are the same complete valued field.

Lemma 3.22.3 gives $d(E/F) \leq m$. The free-polyradius field L is the completion of the Gauss rational field over K: the unique dominant term expands every rational inverse as a convergent Laurent geometric series. Thus Lemma 3.22.4 gives $d(L/K) = m$. The two tower formulas yield

$$m + d(E/L) = d(E/K) = d(F/K) + d(E/F).$$

If $d(F/K)$ is infinite, the asserted inequality is automatic; otherwise all terms are finite and cancellation of m proves it.

At the canonical point $\sigma_r(x)$ the norm of a finite Laurent polynomial is precisely its weighted Gauss norm over F, by Theorem 1.19.1. The same density argument identifies its completed residue field with the completion of $F(T_1, \dots, T_m)$ for that norm. In particular the parameters are algebraically independent here, even if their radii are no longer free over F. Lemma 3.22.4 gives $d(E/F) = m$, and the displayed tower equality proves $d(E/L) = d(F/K)$, with infinity allowed. □

Lemma 3.22.6 (Completed residue fields of a finite morphism)**.** *Let $A \to B$ be a bounded homomorphism of affinoid algebras with B finite as an A-module. If $y \in \mathcal{M}(B)$ lies above $x \in \mathcal{M}(A)$, then $\mathcal{H}(y)/\mathcal{H}(x)$ is finite. Hence $d_K(y) = d_K(x)$.*

Proof. Put $F = \mathcal{H}(x)$ and view it as a closed subfield of $\mathcal{H}(y)$. If $b_1, \dots, b_q$ generate B over A, the F-span of the evaluated B is generated by their images. This span is a finite-dimensional F-subalgebra and a domain. Multiplication by a nonzero element is an injective endomorphism of a finite-dimensional space, hence surjective, so this subalgebra is a field. It is complete and closed, by the finite-dimensional norm comparison in Lemma 1.12.1. It contains the evaluated fraction field of $B/\ker y$, which is dense in $\mathcal{H}(y)$. It is therefore all of $\mathcal{H}(y)$. The invariant equality follows from Lemma 3.22.2. □

Theorem 3.22.7 (Dimension as a maximum over analytic points)**.** *For every nonempty generalized affinoid space $X = \mathcal{M}(A)$ over any complete non-Archimedean field with its specified absolute value,*

$$\dim_K X = \max_{x \in X} d_K(x).$$

In particular every $d_K(x)$ is finite. This includes nonreduced algebras, trivial valuations, and arbitrary real radii.

Proof. **Step 1: the strict upper bound.** Suppose A is strict and fix x. Its kernel $\mathfrak{p}$ is prime, and $B = A/\mathfrak{p}$ is a strict integral affinoid algebra. The evaluated fraction fields of A and B at this point coincide, so their completed residue fields coincide. Choose a finite normalization

$$R = K\langle T_1, \dots, T_e \rangle \hookrightarrow B, \qquad e = \dim B \leq \dim A$$

using Theorem 1.12.2 and Proposition 3.21.1. Let z be the image of x in $\mathcal{M}(R)$. Lemma 3.22.6 gives $d_K(x) = d_K(z)$.

The field $\mathcal{H}_R(z)$ is the completion of $\mathrm{Frac}(K[T_1, \dots, T_e]/\mathfrak{q})$, where $\mathfrak{q}$ is the kernel on the polynomial subalgebra. Indeed polynomials are dense in R for its Gauss norm and evaluation at z is contractive. After approximating numerator and denominator, with denominator error smaller than its nonzero value, the same assertion holds for fractions. This proves the claimed density in the completed residue field. The polynomial fraction field has transcendence degree at most e. Lemma 3.22.3 therefore gives $d_K(z) \leq e \leq \dim A$. This argument uses the polynomial fraction field, not the generally much larger fraction field of the Tate algebra.

Step 2: a strict point attaining the bound. Choose a prime quotient $B = A/\mathfrak{p}$ with $\dim B = \dim A = d$. Such a quotient exists: a prime chain of maximal length starts at such a prime. The dimension is finite: a strict presentation $R_n \twoheadrightarrow A$ identifies every prime chain of A with a prime chain of R_n, whose dimension is n by Proposition 3.21.1. Normalize B finitely over R_d. Proposition 3.12.2 gives a point x above the unit Gauss point η of R_d. Polynomial density identifies $\mathcal{H}(\eta)$ with the completion of the unit Gauss field $K(T_1, \dots, T_d)$. Lemma 3.22.4 gives $d_K(\eta) = d$, and Lemma 3.22.6 gives $d_K(x) = d$. Its image in $\mathcal{M}(A)$ has the same residue field, proving attainment.

Step 3: arbitrary radii. Choose $L = K_r$ making A_L strict by Theorem 1.17.6. Its dimension is $d = \dim_K X$ by Theorem 3.21.8. For every $x \in X$, the Gauss-section equality in Proposition 3.22.5 and Step 1 give

$$d_K(x) = d_L(\sigma_r(x)) \leq d.$$

By Step 2 choose $y \in \mathcal{M}_L(A_L)$ with $d_L(y) = d$, and let x be its image. The other inequality in that same proposition gives $d \leq d_K(x)$. Combining both bounds gives $d_K(x) = d$. Every construction used the original absolute values. □

Corollary 3.22.8 (Dimension of domains and affinoid covers)**.** *If V is a nonempty affinoid domain of an affinoid space X, then $\dim_K V \leq \dim_K X$. If (V_i) is any cover of a nonempty affinoid X by nonempty affinoid domains, then*

$$\dim_K X = \sup_i \dim_K V_i.$$

A finite cover has a maximum in place of the supremum.

Proof. Corollary 2.33.15 identifies the completed residue fields of a point on V and on X. The preceding theorem therefore gives

$$\dim_K V = \max_{x \in V} d_K(x) \leq \max_{x \in X} d_K(x) = \dim_K X.$$

The union of the V_i is X, so taking the supremum of these equalities proves the covering formula. No flat-ring dimension inequality or boundary hypothesis is used. □

We give the empty analytic space dimension $-\infty$.

Theorem 3.22.9 (Local analytic dimension and the Abhyankar inequality)**.** *For any nonempty analytic space X, the supremum of the dimensions of its affinoid charts is independent of the chosen analytic atlas and equals*

$$\dim_K X = \sup_{x \in X} d_K(x).$$

It agrees with the affinoid dimension already defined. An analytic domain has dimension at most that of its ambient space. If $\dim_x X$ is the infimum of the dimensions of the analytic neighborhoods of x, then

$$d_K(x) \leq \dim_x X.$$

These conclusions require neither goodness nor Hausdorffness. At a good point the infimum can be taken over affinoid neighborhoods. At every point $\dim_x X$ is finite.

Proof. The completed residue field attached to a point is independent of its affinoid chart: on a common refinement this is Corollary 2.33.15, and the analytic-net construction identifies the maps on successive refinements. Thus $d_K(x)$ is an intrinsic point invariant. For any analytic atlas (X_i), Theorem 3.22.7 gives

$$\sup_i \dim_K X_i = \sup_i \sup_{x \in X_i} d_K(x) = \sup_{x \in X} d_K(x).$$

This proves atlas independence rather than assuming it in the definition. It also agrees with the old dimension on an affinoid chart, by the same theorem. For an analytic domain $U \subset X$, the residue-field invariance is checked on affinoid refinements of its domain atlas; it gives $\dim_K U = \sup_{x \in U} d_K(x) \leq \dim_K X$.

Every analytic neighborhood U of x contains that point, so $d_K(x) \leq \dim_K U$. Taking their infimum proves the desired inequality. If x is good, affinoid neighborhoods are cofinal among analytic neighborhoods: take one inside the ordinary open neighborhood contained in a given analytic neighborhood. Finally the net axiom gives finitely many affinoid charts X_i containing x whose union contains an ordinary open neighborhood W of x. Give W its open analytic structure. For every $w \in W$ one of these charts contains w, so the already proved formula gives

$$\dim_K W = \sup_{w \in W} d_K(w) \leq \max_i \dim_K X_i < \infty.$$

Thus W is an analytic neighborhood of finite dimension. This proves finiteness of $\dim_x X$ without a good-space assumption. □

References. The valued-field argument is the classical one formulated in [TemIntro, Exercise 2.3.3.2]. The strictification convention is [TemIntro, Definition 3.5.1.2]. All arguments needed here, including the Gauss-lift comparison and the analytic domain/atlas dimension comparison, have been given internally.

3.23 A compact analytic domain without affinoid neighbourhoods

The distinction between an analytic domain and a good analytic space already occurs in a polydisc. The obstruction is visible in the valuation space of punctured affine space.

Lemma 3.23.1 (The valuation space of punctured affine space)**.** *Let k be a field, let $n \geq 2$, and put $R = k[t_1, \ldots, t_n]$, $F = \mathrm{Frac}(R)$. Inside the space of valuation rings of F containing k, set*

$$Q = \bigcup_{i=1}^{n} \{V : R[t_i^{-1}] \subset V\}.$$

Then Q is not affine as a birational space. In fact, $\bigcap_{V \in Q} V = R$, but Q does not contain every valuation ring of F containing R.

Proof. Regard k as a complete trivially valued field. Its strict Tate algebra in n variables is exactly R: a convergent series has only finitely many nonzero coefficients. Theorem 1.13.1 therefore proves that R is a unique factorization domain. If $h \in F \setminus R$, write it as a reduced fraction and choose an irreducible factor P of its denominator. The exponent of P in a nonzero fraction defines a discrete valuation v_P of F; its valuation ring V_P contains R and does not contain h. The irreducible P cannot divide both t_1 and t_2. Hence some t_i is a unit of V_P, and $V_P \in Q$. Every member of Q contains R, so this proves

$$\bigcap_{V \in Q} V = R.$$

For a nonzero polynomial f, let $w(f)$ be the least total degree of a nonzero homogeneous component of f. The product of the lowest homogeneous components is nonzero, so $w(fg) = w(f) + w(g)$, and $w(f+g) \geq \min(w(f), w(g))$ whenever the sum is nonzero. Thus w extends by subtraction to a valuation of F. Its valuation ring V_0 contains R, while $w(t_i) = 1$ for every i. Consequently $V_0 \notin Q$.

If Q were affine, its inclusion in the fixed valuation space would identify it with $\{V : S \subset V\}$ for a finitely generated k-subalgebra $S \subset F$. Every element of S would belong to $\bigcap_{V \in Q} V = R$. But then $S \subset R \subset V_0$, which would force $V_0 \in Q$, a contradiction. □

Theorem 3.23.2 (A punctured closed polydisc is not good)**.** *Let K be any complete non-Archimedean field, possibly trivially valued, let $n \geq 2$, and let $r_1, \ldots, r_n > 0$. In*

$$X = \mathcal{M}(K\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\})$$

consider the subset

$$U = X \setminus \{z : |T_i(z)| < r_i \text{ for every } i\} = \bigcup_{i=1}^{n} U_i,$$

$$U_i = \{z \in X : |T_i(z)| = r_i\}.$$

Then U is a compact Hausdorff analytic domain of X. At the weighted Gauss point $x(\sum_\nu a_\nu T^\nu) = \max_\nu |a_\nu| r^\nu$, the space U has no affinoid neighbourhood.

Proof. Step 1: the domain structure. Each U_i is the Laurent domain $|T_i| \geq r_i$ in X; the opposite inequality already holds throughout X. All finite intersections are Laurent affinoid domains. Their finite cover of U is an analytic net: at a point $z \in U$, remove the finitely many closed members which do not contain z. The remaining relative open neighbourhood is covered by the members containing z. The domain and net constructions in Proposition 2.39.3 and Proposition 2.40.5 therefore give the analytic-domain structure on U. The displayed complement is open in X, so U is closed in the compact Hausdorff space X, hence compact Hausdorff.

Step 2: unit radii. Assume first that every $r_i = 1$, put $k = \widetilde{K}$, and let x be the unit Gauss point. Polynomials are dense in the Tate algebra. Approximating the numerator and denominator of a fraction, with error smaller than the nonzero denominator, shows that $\mathcal{H}(x)$ is the completion of $K(T_1, \ldots, T_n)$ for the Gauss absolute value. Every nonzero Gauss norm of a polynomial belongs to $|K^\times|$. For a norm-one fraction, divide its numerator and denominator by coefficients of their maximal norms. Reduction then identifies the residue field with $k(t_1, \ldots, t_n)$; completion has the same residue field by Lemma 3.22.2. The graded image of the Tate algebra in $F_x = \widetilde{\mathcal{H}(x)}^{\,\mathrm{gr}}$ is $\widetilde{K}^{\,\mathrm{gr}}[t_1, \ldots, t_n]$: only finitely many terms of a convergent series can have its maximal norm, and these give precisely its initial form.

All the U_i contain x. Near x, T_i is invertible; the additional active inequality defining U_i is $|T_i^{-1}| \leq 1$. By Lemma 3.19.5, its reduction adds t_i^{-1} to the graded image just computed. Gluing the domain germs by Theorem 3.19.9 and Theorem 3.19.10, the reduction of (U, x) is therefore the union of the graded valuation subspaces containing

$$\widetilde{K}^{\,\mathrm{gr}}[t_1, \ldots, t_n, t_i^{-1}], \qquad 1 \leq i \leq n.$$

It is exactly the pullback, by restriction to degree one, of the ordinary birational space Q in Lemma 3.23.1. That lemma says Q is not affine. Lemma 3.19.20 shows that its graded pullback is not affine either, including when

the valuation of K is trivial. Theorem C.1.114 now shows that x has no affinoid neighbourhood in U.

Step 3: arbitrary radii. Equip the rational function field $K(S_1, \dots, S_n)$ with the Gauss absolute value of polyradius r and let L be its completion. Multiplicativity and completion are the Gauss-field construction used in Lemma 3.18.2; no independence of the radii is required. The images c_i of S_i satisfy $|c_i| = r_i$. Scalar extension identifies

$$X_L = \mathcal{M}(L\{r^{-1}T\}) \simeq \mathcal{M}(L\{Z_1, \dots, Z_n\}), \qquad T_i = c_i Z_i.$$

The exact scalar presentation and domain base-change statements (2.33.5, 2.36.9, 2.40.12) identify $(U_i)_L$ with $|T_i| = r_i$, hence with $|Z_i| = 1$. Since the finite domain cover and its intersections base change, $U_L = \bigcup_i (U_i)_L$ is exactly the unit-radius punctured polydisc. Equivalently its underlying set is the inverse image of U: the projection preserves every number $|T_i|$, so it preserves the displayed complement condition as well.

Let y be its unit Gauss point in the Z-coordinates. For a polynomial with coefficients in K, evaluation at y is $\max_\nu |a_\nu c^\nu| = \max_\nu |a_\nu| r^\nu$; by continuity the same holds for every element of $K\{r^{-1}T\}$. Thus y lies over x. If x had an affinoid neighbourhood W in U, then W_L would be an affinoid domain of U_L. Moreover, an ordinary open neighbourhood of x contained in W has an open inverse image containing y and contained in W_L. Hence W_L would be an affinoid neighbourhood of y, contradicting Step 2 over L. This proves the assertion for every positive polyradius. □

References. The example is the classical one in [TemIntro, Example 4.2.1.4(i)]. Its proof here uses the internally proved germ-reduction criterion; the valuation obstruction and the arbitrary-radius base change are spelled out above.

3.24 Discrete residue fields and free Gauss coordinates

The elementary valued-field arguments in this section provide the field inputs for relative annulus–polydisc neighbourhoods. All absolute values retain their original normalization.

Lemma 3.24.1 (Uniform approximation implies density)**.** *Let E be a real-valued field and let $V \subset E$ be an additive subgroup. If some $0 < q < 1$ has the property that every $a \in E$ admits $b \in V$ with $|a - b| \leq q|a|$, then V is dense in E.*

Proof. Fix a. Start with $b_0 = 0$. Having chosen $b_j \in V$, choose $c_j \in V$ with $|a - b_j - c_j| \leq q|a - b_j|$ and put $b_{j+1} = b_j + c_j$. Induction gives $|a - b_j| \leq q^j|a|$, hence $b_j \to a$. □

Lemma 3.24.2 (Value groups and residues of a finite extension)**.** *For a finite valued extension E/F, both $e = [|E^\times| : |F^\times|]$ and $f = [\widetilde{E} : \widetilde{F}]$ are finite, and $ef \leq [E : F]$. If $|F^\times|$ is nontrivial cyclic, then $|E^\times|$ is nontrivial cyclic as well.*

Proof. Choose finitely many $a_i \in E^\times$ with distinct value classes modulo $|F^\times|$, and norm-one $u_j \in E$ with residues linearly independent over $\widetilde{F}$. The elements $a_i u_j$ are F-linearly independent. Indeed, in a purported relation first collect terms with fixed i. A nonzero sum $\sum_j c_j u_j$ has norm $\max_j |c_j|$: divide by a coefficient attaining the maximum and use residue independence. The nonzero collected sums for different i have distinct norms, so a largest term cannot cancel. This proves independence. Arbitrarily large finite choices are impossible in a finite dimensional F-space; hence e, f are finite and $ef \leq [E : F]$.

Write $|F^\times| = q^{\mathbf{Z}}$ with $0 < q < 1$. The finite quotient of value groups is killed by some integer $N > 0$, so $|E^\times| \subset q^{(1/N)\mathbf{Z}}$. A subgroup of this cyclic group is cyclic, and it contains $q^{\mathbf{Z}}$, so is nontrivial. □

Lemma 3.24.3 (Finiteness over a complete discrete subfield)**.** *Let E be a valued field whose value group is nontrivial cyclic. Let $F \subset E$ be complete and nontrivially valued. If $\widetilde{E}/\widetilde{F}$ is finite, then E/F is finite. In particular E is complete. No completeness of E is assumed in the hypotheses.*

Proof. The nontrivial subgroup $|F^\times|$ has finite index in the cyclic group $|E^\times|$. Choose representatives $a_1, \ldots, a_e \in E^\times$ of these value classes, and lift a residue-field basis to norm-one elements $u_1, \ldots, u_f \in E$. Let V be the F-span of the finitely many $a_i u_j$. For $z \neq 0$, choose i and $c \in F^\times$ with $|z| = |ca_i|$. Expand the residue of $z/(ca_i)$ in the chosen residue basis and lift its coefficients to F°. This gives $v \in V$ with $|z - v| < |z|$.

Write $|E^\times| = q^{\mathbf{Z}}$ with $0 < q < 1$. The last strict inequality implies $|z - v| \le q|z|$, uniformly in z. The preceding density lemma makes V dense in E. The restricted absolute value is a norm on the finite dimensional F-space V; Lemma 1.12.1 makes V complete, hence closed in the metric space E. Thus $V = E$. □

Lemma 3.24.4 (Coordinate changes in a free Gauss field)**.** *Let $r = (r_1, \dots, r_m)$ be free over a complete field K, and let $S_i \in K_r$ satisfy $|S_i| = r_i$. The isometric substitution $T_i \mapsto S_i$ is an automorphism of K_r. Thus the complete subfield generated by $K, S_1, \dots, S_m$ is all of K_r.*

Proof. The unique dominant term in the Laurent expansion of S_i has norm r_i. Freeness implies that it is $\alpha_i T_i$ with $\alpha_i \in K$ and $|\alpha_i| = 1$. Rescaling S_i by α_i^{-1} does not change the generated complete subfield and is itself an invertible coordinate scaling. We may therefore assume $|T_i - S_i| \le q r_i$ for one $q < 1$ and every i. The identity for inverse differences also gives $|T_i^{-1} - S_i^{-1}| \le q r_i^{-1}$.

For every Laurent monomial, telescoping products gives

$$|T^\nu - S^\nu| \le q r^\nu \qquad (\nu \in \mathbf{Z}^m).$$

The weighted Laurent description in Lemma 1.16.1 defines the substitution $\phi(\sum a_\nu T^\nu) = \sum a_\nu S^\nu$. It is an isometry: each nonzero finite sum has a unique largest term, and completion preserves the equality of norms. It is a field homomorphism by density of Laurent polynomials. The displayed estimate gives

$$|a - \phi(a)| \le q|a| \qquad (a \in K_r).$$

Hence its image is dense by Lemma 3.24.1. The image is also complete, since ϕ is an isometry of a complete field, and therefore closed. It is all of K_r. □

Proposition 3.24.5 (Independent differentials of free coordinates)**.** *Let Y be a K-analytic space and let $y \in Y$ with an isometric K-identification $\mathcal{H}(y) = K_r$, where $r = (r_1, \dots, r_m)$ is free over K. If analytic functions g_i near y satisfy $|g_i(y)| = r_i$, then*

$$(dg_1)(y), \dots, (dg_m)(y) \quad \text{in} \quad \Omega^{\mathrm{an}}_{Y/K} \otimes \mathcal{H}(y)$$

are linearly independent. In particular this applies to quasi-smooth spaces.

Proof. Choose an affinoid domain containing y on which the functions are represented, and write its algebra as A. By Theorem 2.41.5, its differential fibre is the asserted one. Set $S_i = g_i(y)$. The preceding lemma makes S_i coordinates of K_r. On a Laurent series in these coordinates the operator

$$D_j\left(\sum_{\nu\in\mathbf{Z}^m} a_\nu S^\nu\right) = \sum_\nu \nu_j a_\nu S^{\nu-e_j}$$

is bounded with norm at most r_j^{-1}, since every integer has absolute value at most one. The weighted coefficients still tend to zero, so the formula defines a continuous operator. The Leibniz identity on Laurent polynomials extends by continuity; hence D_j is a bounded K-derivation and $D_j(S_i) = \delta_{ij}$.

Compose D_j with the bounded evaluation $A \to \mathcal{H}(y) = K_r$. The resulting bounded derivation from A to the complete Banach A-module K_r is represented by Theorem 2.41.3. After tensoring with K_r, it gives a K_r-linear functional on the differential fibre taking $(dg_i)(y)$ to δ_{ij}. These m functionals form a left inverse to the map with columns $(dg_i)(y)$, proving independence. □

Lemma 3.24.6 (A discrete-residue point over a trivially valued field)**.** *Let F be a trivially valued field and let X be a nonempty boundaryless F-analytic space. There is $x \in X$ such that $\mathcal{H}(x)$ is either finite over F, or is finite over an isometrically embedded F_r for some $0 < r < 1$.*

Proof. Fix $0 < s < 1$. The field F_s is the Laurent-series field $F((T))$, with $|T| = s$: the weighted coefficient condition allows only finitely many negative powers and arbitrary nonnegative powers. In particular its value group is $s^{\mathbf{Z}}$ and its residue field is F. Theorem 2.40.12 gives surjectivity of complete scalar extension on analytic spaces, so X_{F_s} is nonempty. By Definition 2.44.4, each point of the boundaryless space X has an ordinary affinoid neighbourhood V on which it is interior over $\mathcal{M}(F)$. Indeed the only affinoid target domain containing its image is $\mathcal{M}(F)$ itself. For a point of X_{F_s} over that point, the scalar extension V_{F_s} is an ordinary affinoid neighbourhood, by the analytic-domain preimage assertion of Theorem 2.40.12. On the affinoid map $V \to \mathcal{M}(F)$, the forward base-change inclusion of Proposition 2.33.21 makes that lift interior in V_{F_s} over $\mathcal{M}(F_s)$. Since the new target is also a one-point spectrum, the same definition makes X_{F_s} boundaryless. The neighbourhoods just exhibited also make it good.

At any point of X_{F_s} apply Proposition 2.51.3 to its structure map. This gives a nonempty strict affinoid domain. Theorem 1.12.3 supplies a point

y whose completed residue field L is finite over F_s. Let x be its image in X and identify the complete field $E = \mathcal{H}(x)$ with its isometric image in L. Lemma 3.24.2 makes $|L^\times|$ nontrivial cyclic and $\widetilde{L}/F$ finite. Therefore $\widetilde{E}/F$ is finite and $|E^\times|$ is a subgroup of a cyclic group.

If E is trivially valued, it equals its residue field and is finite over F. Otherwise choose $\lambda \in E$ with $0 < r = |\lambda| < 1$. Since F is trivially valued, distinct powers of λ have distinct sizes; evaluation on $F[T, T^{-1}]$ is therefore isometric for the radius-r Gauss norm. Its completion is an isometric copy of F_r inside E, with residue field F. The group $|E^\times|$ is nontrivial cyclic and $\widetilde{E}/F$ is finite. Apply Lemma 3.24.3 to $F_r \subset E$. It gives the required finiteness. □

Lemma 3.24.7 (A discrete complete field with its coefficient field)**.** *Let E be a complete valued field with nontrivial cyclic value group. Suppose that $K \subset E$ is trivially valued and that reduction identifies K with $\widetilde{E}$. For a uniformizer $\tau \in E$, put $r = |\tau| \in (0,1)$. Then substitution defines an isometric isomorphism $K_r = K((T)) \xrightarrow{\sim} E$, sending T to τ.*

Proof. The distinct powers of τ have distinct norms, so substitution is an isometry on Laurent polynomials and extends isometrically to K_r. To prove surjectivity, take $a \in E^\circ$. Choose $c_0 \in K$ with the same residue as a. Since $|\tau|$ generates the value group below one, $a_1 = (a - c_0)/\tau$ still belongs to E°. Repeat, obtaining $c_j \in K$ and $a_{j+1} \in E^\circ$ with $a_j = c_j + \tau a_{j+1}$. Thus

$$\left| a - \sum_{j=0}^{N-1} c_j \tau^j \right| \leq r^N.$$

Completeness gives $a = \sum_{j \geq 0} c_j \tau^j$. Multiplication by a suitable power of τ handles every $a \in E$. Uniqueness follows from the norm of the first nonzero coefficient. □

Lemma 3.24.8 (Uniformization after a finite extension of constants)**.** *Let F be any field, endowed with the trivial absolute value, let $0 < s < 1$, and let $L/F_s = F((t))$ be finite separable. There are a finite extension K/F, a number $0 < r < 1$, and a surjective F-algebra homomorphism*

$$L \otimes_F K \twoheadrightarrow K_r$$

whose restriction to L is isometric for its given absolute value. The restriction to K is the constant-field embedding. Here the tensor product is the ordinary algebraic tensor product.

Proof. Step 1: algebraize the discrete valuation. The dense-coefficient lemma 2.54.2 supplies $Q \in F(t)[U]$ such that $L = F((t))[U]/(Q)$. Let $E = F(t)[U]/(Q)$, identified with its image in L. It is a field, since Q is irreducible even over $F((t))$. Moreover E is dense in L: approximate the finitely many coefficients of an expression in the basis $1, U, \dots, U^{\deg Q - 1}$.

Let B be the integral closure of $F[t]$ in E. It is finite over $F[t]$, by [StacksAlg, Tag 0335] and the definition of a Japanese domain, [StacksAlg, Tag 032F]. Thus $Y = \operatorname{Spec}(B)$ is a normal affine curve; its dimension is one by [StacksAlg, Tag 00OK]. Every $b \in B$ has $|b| \leq 1$, since it satisfies a monic equation whose coefficients have norm at most one. Hence $P = \{b \in B : |b| < 1\}$ is a prime over (t) and is maximal, with residue field finite over F. By [StacksAlg, Tag 0B8Y], B_P is a discrete valuation ring. Its units have norm one and its uniformizer has norm less than one. Consequently its valuation, scaled by the prescribed value $|t| = s$, is exactly the restriction of the given absolute value on L. In particular L is the completion of E at this closed point P.

Step 2: obtain a smooth rational center. Apply [StacksAlg, Tag 0GK5] to Y. After a finite purely inseparable extension F'/F, the normalization of $Y_{F'}$ is a finite disjoint union of curves smooth and geometrically irreducible over finite extensions of F'. The normalization is finite and surjective, by [StacksAlg, Tag 0BXR]; choose a closed point Q_0 above P on one component Z. Write F_0 for its field of definition in that statement and put $K = \kappa(Q_0)$, which is finite over F_0 by [StacksAlg, Tag 00FV], and hence finite over F. The base change Z_K is a smooth integral K-curve, and Q_0 gives a K-rational point Q on it.

The function field E' of Z_K is generated as a field by the images of E and K. Indeed normalization does not change the function field of the chosen irreducible component of $Y_{F'}$; this field is a quotient of $E \otimes_F F'$. After the indicated base change its function field is the corresponding quotient of $E \otimes_F K$. Since this tensor product is finite over E, its domain quotient is already a field. Thus E' is precisely the compositum of those two images.

Step 3: complete and check algebraic surjectivity. The local ring at the smooth rational point Q is a discrete valuation ring by [StacksAlg, Tag 0B8Y],

with residue field K. Choose a uniformizer τ and normalize its real absolute value so that the image of t still has value s. Let M be the completion of E' for this valuation. Completion changes neither the value group nor the residue field, as proved in Lemma 3.22.2. Therefore Lemma 3.24.7 identifies M with K_r, where $r = |\tau|$.

The valuation on E' restricts to the one centered at P on E: the local map $B_P \to \mathcal{O}_{Z_K,Q}$ sends units to units and the maximal ideal into the maximal ideal, and the normalization of $|t|$ fixes the scale. It consequently extends to an isometric embedding $L \hookrightarrow M$. Together with the constant inclusion $K \hookrightarrow M$ it gives $L \otimes_F K \to M$. Its image is a finite-dimensional L-algebra and a domain, hence a field: multiplication by a nonzero element is an injective endomorphism of a finite-dimensional vector space and is therefore surjective. By Lemma 1.12.1, this image is complete for its restricted absolute value and hence closed in M. It contains E', which is dense in M. The image is therefore all of $M = K_r$, proving the assertion for the algebraic tensor product itself. □

Proposition 3.24.9 (Gauss or rational points after finite constants)**.** *Let F be trivially valued, and let X be a nonempty smooth F-analytic space. There exist a finite extension K/F and $z \in X_K$ such that $\mathcal{H}(z)$ is isometrically K or K_r for some $0 < r < 1$. The image of z can be required to lie in any prescribed nonempty ordinary open subset of X.*

Proof. Replace X by the prescribed open subset. Fix $0 < s < 1$. Scalar extension is surjective, and smoothness is preserved by base change; hence X_{F_s} is nonempty and smooth. By Lemma 2.55.3, it has a point y for which $L = \mathcal{H}(y)$ is finite separable over F_s. Write x for its image in X and $E = \mathcal{H}(x) \subset L$. If E is trivially valued, it is finite over F: its residue field is a subfield of $\widetilde{L}$, which is finite over F by Lemma 3.24.2. Take $K = E$; the diagonal evaluation gives a K-rational point over x.

Suppose now that E is nontrivially valued. Lemma 3.24.8 gives a finite K/F and a quotient $L \otimes_F K \twoheadrightarrow M = K_{r_0}$ whose map on L is isometric. Let E_K be the image of $E \otimes_F K$ in M. This image is a finite-dimensional E-algebra and a domain, so is a field and is complete by Lemma 1.12.1. Its value group is a nontrivial subgroup of the cyclic group $|M^\times|$. Its residue field is exactly K, since it is a subfield of $\widetilde{M} = K$ and contains the constant field K. Lemma 3.24.7 therefore gives $E_K = K_r$ isometrically, for some $0 < r < 1$.

It remains to check that this field is an actual completed residue field on X_K. On an affinoid domain $\mathcal{M}(A)$ through x, compose evaluation with $A \otimes_F K \to E_K$. The map is bounded and extends to the completed tensor product: if $e_1, \dots, e_d$ is an F-basis of K, then the norm of $\sum a_i e_i$ in the target is at most $\max_i |a_i(x)|$, since the nonzero constants have norm one; this is bounded by a finite-module tensor norm. It defines a point z over x. The fraction field of its evaluation image contains the image of $\operatorname{Frac}(A/\ker x)$ and K. The first is dense in E; finite sums then show that the generated field is dense in E_K. Conversely E_K is already complete. Thus $\mathcal{H}(z) = E_K = K_r$ as asserted. □

References. The valued-field and point constructions reconstruct [Duc18, Lemmas 6.1.1–6.1.6]. The differential-independence argument uses bounded coordinate derivations directly, and thus avoids an extra pure-dimension comparison.

3.25 Absolute boundary and canonical reduction

Theorem 3.25.1 (Closed reduction points and absolute interior)**.** *Let A be a strict affinoid algebra over a complete non-Archimedean field K, with its specified absolute value. For $X = \mathcal{M}(A)$, the canonical reduction map satisfies*

$$\operatorname{Int}(X/K) = \operatorname{red}^{-1}\big((\operatorname{Spec} \widetilde{A})_{\text{closed}}\big).$$

Proof. Fix $x \in X$, put $k = \widetilde{K}$ and $R = \widetilde{A}$, and let S be the image of $\widetilde{A}^{\mathrm{gr}}$ in $\widetilde{\mathcal{H}(x)}^{\mathrm{gr}}$. Its degree-one part is $S_1 = R/\mathfrak{p}$, where $\mathfrak{p} = \operatorname{red}(x)$. The forward and reverse implications for absolute evaluation show that $A \to \mathcal{H}(x)$ is inner over K precisely when S is finite over $\widetilde{K}^{\mathrm{gr}}$. We show this is equivalent to S_1 being finite-dimensional over k.

A finite graded module has finitely many homogeneous generators. Every homogeneous component of the graded field $\widetilde{K}^{\mathrm{gr}}$ has dimension at most one over k. Thus finiteness of S implies finite dimension of S_1. Conversely, choose an admissible absolute weighted Tate presentation of A. The fixed absolute presentation result gives finitely many homogeneous module generators over the image of the initial-coordinate polynomial algebra. Taking their images in S together with the initial coordinates gives finitely many homogeneous algebra generators u_j over $\widetilde{K}^{\mathrm{gr}}$. Discard the zero generators. The degree r_j of each remaining generator is the spectral radius of a representative in A, since its initial form has nonzero image. Lemma 3.16.6 gives $e_j > 0$ and $c_j \in K^\times$

with $r_j^{e_j} = |c_j|$. Hence $u_j^{e_j}/\widetilde{c}_j \in S_1$. If S_1 is finite-dimensional over k, the algebra $\widetilde{K}^{\mathrm{gr}} S_1$ is finite over $\widetilde{K}^{\mathrm{gr}}$, and every u_j is integral over it. Adjoining these finitely many integral generators proves that S is finite over $\widetilde{K}^{\mathrm{gr}}$.

Theorem 3.16.3 makes R a finite-type k-algebra. By the Nullstellensatz [StacksAlg, Tag 00FV], $R/\mathfrak{p}$ is finite over k when $\mathfrak{p}$ is maximal. Conversely, a finite-dimensional domain over a field is a field: multiplication by any nonzero element is an injective linear endomorphism, hence surjective. Thus S_1 is finite-dimensional exactly when $\mathfrak{p}$ is a closed point. □

Lemma 3.25.2 (Dimension of strict canonical reduction)**.** *For every nonempty strict affinoid space $X = \mathcal{M}(A)$,*

$$\dim \widetilde{A} \le \dim_K X.$$

Proof. The ring $R = \widetilde{A}$ is a finite-type $k = \widetilde{K}$-algebra. For every prime $\mathfrak{p}$ of R, Theorem 3.16.16 gives a point x reducing to $\mathfrak{p}$. The inclusion $R/\mathfrak{p} \hookrightarrow \widetilde{\mathcal{H}(x)}$ extends to its fraction field. Thus

$$\dim(R/\mathfrak{p}) = \operatorname{trdeg}_k \operatorname{Frac}(R/\mathfrak{p}) \le \operatorname{trdeg}_k \widetilde{\mathcal{H}(x)} \le d_K(x) \le \dim_K X.$$

The first equality is [StacksAlg, Tag 00P0], and the last inequality is Theorem 3.22.7. Every finite chain of prime ideals of the Noetherian ring R lies over a minimal prime. Taking the maximum of the dimensions of these finitely many components gives the assertion. □

Lemma 3.25.3 (Isolated points of a strict affinoid space)**.** *An isolated point of a strict affinoid space is rigid and has local analytic dimension zero. In particular a pure one-dimensional strict affinoid space has no isolated points.*

Proof. Let $X = \mathcal{M}(A)$ and suppose $\{x\}$ is open. It is also closed because X is Hausdorff. Glue the constant function one on $\{x\}$ and zero on its complement using Theorem 2.40.6. The resulting global function is an idempotent $e \in A$. The direct factor $B = A/(1-e)$ is a strict affinoid algebra whose spectrum is $\{x\}$. Theorem 1.12.3 supplies a rigid point of its nonempty spectrum, necessarily x. Consequently $\mathcal{H}(x)/K$ is finite and $d_K(x) = 0$ by Lemma 3.24.2. Theorem 3.22.7 now gives $\dim_K \mathcal{M}(B) = 0$. Since this is an open neighborhood of x, its local dimension in X is zero. □

Theorem 3.25.4 (The boundary of a strict affinoid curve)**.** *Let X be a nonempty strict affinoid space of dimension at most one with no isolated points. Then its absolute analytic boundary equals its Shilov boundary. In particular this holds for every pure one-dimensional strict affinoid space.*

Proof. Write $X = \mathcal{M}(A)$ and $R = \widetilde{A}$. By Lemma 3.25.2, R has dimension at most one. Theorem 3.16.7 identifies its minimal primes with the Shilov points, each having a unique analytic preimage. A minimal prime $\mathfrak{p}$ cannot also be maximal. Indeed, choose finitely many generators $\bar{f}_i$ of $\mathfrak{p}$ and lifts $f_i \in A^\circ$. If $\mathfrak{p}$ were maximal, then $V(\mathfrak{p}) = \{\mathfrak{p}\}$ and

$$\mathrm{red}^{-1}(\mathfrak{p}) = \{y \in X : |f_i(y)| < 1 \text{ for every } i\}$$

would be an open singleton, contrary to the hypothesis. In a ring of dimension at most one, a nonmaximal prime is minimal: otherwise a strictly smaller minimal prime and a strictly larger maximal prime would form a chain of length two. Thus the nonclosed points of $\operatorname{Spec} R$ are exactly its generic points. Theorem 3.25.1 says that their inverse image is the analytic boundary, while Theorem 3.16.7 says that the same inverse image is the Shilov boundary. The last assertion follows from Lemma 3.25.3. □

Corollary 3.25.5 (Weighted Gauss points in the bidisc boundary)**.** *For $X = \mathcal{M}(K\langle T, S\rangle)$, the Shilov boundary is $\{\zeta_{1,1}\}$. For every real $0 < \rho < 1$, the weighted Gauss point $\zeta_{1,\rho}$ lies in the absolute analytic boundary of X.*

Proof. The unit Gauss norm is multiplicative and is the spectral norm, so its point alone is a boundary. The minimality assertion of Lemma 3.12.4 gives the Shilov statement. Reduction of the coefficient series gives $\widetilde{A} = \widetilde{K}[t, s]$: only finitely many coefficients of a restricted series can have norm one, and the others disappear modulo $A^{\circ\circ}$. At $\zeta_{1,\rho}$ the reduction kernel is (s). Indeed, a power-bounded series has value one there exactly when its part of S-degree zero has a coefficient of norm one. All positive S-degrees have norm at most $\rho < 1$. The prime (s) is not maximal, since its quotient is $\widetilde{K}[t]$. Apply Theorem 3.25.1. □

References. The classical strict interior criterion and the curve/bidisc distinction are explained in [TemIntro, 3.4.2.3(vi), 3.4.2.5]. The proofs here use the internal graded innerness, reduction and dimension theorems.

3.26 The standard monomial chart

Proposition 3.26.1 (The monomial simplex and its residual data)**.** *Let K be complete and nontrivially valued, let $0 < |a| < 1$, and let $0 \le r \le d$. Put*

$$A_0 = K^\circ\langle X_0, \dots, X_r, T_1, \dots, T_{d-r}\rangle/(X_0 \cdots X_r - a), \qquad A = A_0[1/a].$$

Then A_0 is admissible and $\mathcal{M}(A)$ is the rational domain $|a| \le |X_1 \cdots X_r|$ in the unit d-dimensional polydisc with coordinates $X_1, \dots, X_r, T$. The monomial points parametrized by

$$\Delta_a = \{(u_0, \dots, u_r) \in \mathbb{R}^{r+1}_{\ge 0} : u_0 + \cdots + u_r = v(a)\}$$

form a subspace homeomorphic to Δ_a. At every such point x_u, the completed residue field is the completion of $K(X_1, \dots, X_r, T)$ for its Gauss valuation of weights $u_1, \dots, u_r, 0, \dots, 0$, and

$$s_K(x_u) + t_K(x_u) = \dim_{x_u} \mathcal{M}(A) = d.$$

If all $u_i > 0$, specialization is the generic point of the stratum $X_0 = \cdots = X_r = 0$, of dimension $d-r$. Any finite family of Laurent monomials in the X_i with coefficients in K and value one at x_u becomes integral on a formal affine neighborhood of its specialization after an admissible formal modification. Its nonzero residues there are the original residues in $\widetilde{\mathcal{H}(x_u)}$.

Proof. Integral normal form. Write $M = X_0 \cdots X_r$. In a restricted series replace a monomial $X^\nu T^\mu$ by $a^q X^{\nu - q\mathbf{1}} T^\mu$, where $q = \min_i \nu_i$. There is a unique resulting series supported on exponent vectors whose minimum X-exponent is zero. Its coefficient at such a vector (λ, μ) is

$$\sum_{q \ge 0} a^q c_{\lambda + q\mathbf{1}, \mu}.$$

These sums converge and preserve integral coefficients. The identity

$$M^q - a^q = (M - a) \sum_{j=0}^{q-1} a^j M^{q-1-j}$$

gives a contractive division operator on polynomials. The coefficients of its value on a restricted series are $\sum_{q \ge 1} a^{q-1} c_{\nu + q\mathbf{1}, \mu}$; the finite-tail condition makes this a restricted series again. Thus the polynomial division identities

extend continuously, and their kernel is exactly $(M - a)$, not just its closure. The quotient is the torsion-free module of these normal forms. Hence A_0 is admissible by Theorem 3.14.4. Inverting a gives its complete strict affinoid algebra by Proposition 3.14.7. Eliminating $X_0 = a/(X_1 \cdots X_r)$ is exactly the rational-localization presentation asserted in the statement. For $r = 0$ the empty product is one and the domain is the whole d-polydisc.

Points and topology. At $u \in \Delta_a$, give X_i value u_i and each T_j value zero. After eliminating X_0, the Gauss norm is multiplicative by Proposition 1.2.2, and all original coordinates have norm at most one. It therefore extends to a bounded multiplicative seminorm on A. A polynomial becomes a finite Laurent polynomial after elimination; its value is a maximum of finitely many continuous functions of u. Restricted-series tails tend to zero uniformly on Δ_a, because all original coordinates have norm at most one. Thus the same continuity holds for every element of A. The inverse on the image is $x \mapsto (v(X_0(x)), \dots, v(X_r(x)))$; the coordinates are invertible in A, so these functions are continuous. This proves the claimed homeomorphism.

Residue fields and dimension. The Laurent polynomial algebra $K[X_1^{\pm 1}, \dots, X_r^{\pm 1}, T]$ is dense in A and its evaluation at x_u is injective. Indeed distinct normal-form monomials give distinct Laurent exponents, so their Gauss values cannot cancel. Passing to fractions and then completion identifies $\mathcal{H}(x_u)$ with the stated completed Gauss field; density is preserved on taking fractions because inversion is continuous at every nonzero element. Proposition 3.8.1 gives $d_K(x_u) = d$. The rational-domain description and Corollary 3.22.8 give $\dim \mathcal{M}(A) \le d$. The local Abhyankar inequality in Theorem 3.22.9 now gives

$$d = d_K(x_u) \le \dim_{x_u} \mathcal{M}(A) \le \dim \mathcal{M}(A) \le d.$$

If all $u_i > 0$, every X_i reduces to zero, whereas the reductions of the T_j are algebraically independent by the same Gauss calculation. The kernel of special-fibre evaluation is consequently $(x_0, \dots, x_r)$. The stratum is affine $(d - r)$-space, and its dimension is $d - r$ by [StacksAlg, Tag 00P0]. This proves the specialization assertion.

Making a residue visible on a model. Write one of the specified monomials as $h = f/g$ with $f, g \in A_0$ monomials having integral coefficients. This is possible after multiplication of numerator and denominator by a scalar. The denominator $g = cX^{\nu}$ divides a nonzero element of K°: choose $N \ge \max_i \nu_i$ and use $gX^{N\mathbf{1}-\nu} = ca^N$. Hence (f, g) is an open ideal. The g-chart of its ad-

missible blowup has f/g integral by Theorem 3.15.1, and contains the unique generic lift of x_u since $|f(x_u)| = |g(x_u)|$. For a finite family of monomials, blow up the product of these open ideals. Lemma 3.17.3 makes the resulting modification dominate all the individual blowups. The intersection of the inverse images of their g-charts is a formal open containing the specialization of the lifted point, by functoriality of specialization and Proposition 3.17.2. Take a formal affine neighborhood inside that intersection. All the chosen monomials are integral there, and their value one at the point makes their reductions nonzero. Generic-fibre invariance and evaluation identify them with the original residues. □

Lemma 3.26.2 (A spectral unit ball need not be an admissible model)**.** *Assume K is nontrivially valued and $r \notin \sqrt{|K^\times|}$. For the one-radius field K_r, its valuation ring K_r° is not topologically of finite type over K°.*

Proof. Choose $\pi \in K$ with $0 < |\pi| < 1$. Every element of K_r becomes power-bounded after multiplication by a power of π, so $K_r^\circ[1/\pi] = K_r$. If the torsion-free ring K_r° were topologically of finite type, Theorem 3.14.4 and Proposition 3.14.7 would make K_r a strict K-affinoid algebra. But its Laurent coordinate has spectral norm r, contrary to Lemma 3.16.6, which puts every nonzero spectral value of a strict affinoid algebra in $\sqrt{|K^\times|}$. □

3.27 Flatness over the ground field

Throughout this section K is a complete non-Archimedean field, with arbitrary characteristic and possibly trivial absolute value. Completed tensor products use the projective norm.

Lemma 3.27.1 (Banach coefficients over an affinoid algebra)**.** *Let C be a generalized K-affinoid algebra and let E be a Banach K-space. Put $T = E\widehat{\otimes}_K C$. For every finite canonical C-module N, there is a natural algebraic C-linear isomorphism*

$$T \otimes_C N \xrightarrow{\sim} E\widehat{\otimes}_K N, \qquad (e \otimes c) \otimes n \longmapsto e \otimes cn.$$

The algebraic C-module T is flat.

Proof. The algebra C is Noetherian by Theorem 1.18.2. Choose a finite presentation $C^a \xrightarrow{D} C^b \to N \to 0$, and put $R = \operatorname{im} D$. The image R is

closed; the maps $C^a \to R$ and $C^b \to N$ are norm-strict for the induced and canonical norms, by Theorem 1.18.3. Apply Theorem 2.33.3 to their kernel sequences. The finite coordinate injections and projections identify $E\widehat{\otimes}_K C^j$ with T^j. We obtain an exact sequence

$$T^a \xrightarrow{D} T^b \longrightarrow E\widehat{\otimes}_K N \longrightarrow 0.$$

By [StacksAlg, Tag 00DF], ordinary tensoring gives the same cokernel presentation for $T \otimes_C N$. The resulting isomorphism is the displayed map: the bounded action on N extends its pure-tensor formula in the first factor. This also proves naturality.

For a finitely generated ideal $I \subset C$, its induced norm is a canonical finite-module norm. Under the comparison just proved, the map $I \otimes_C T \to T$ is $E\widehat{\otimes}_K I \to E\widehat{\otimes}_K C$. It is injective by Theorem 2.33.2. The finite-ideal criterion [StacksAlg, Tag 00HD] proves flatness. □

Corollary 3.27.2 (Finite modules on an affinoid product)**.** *Let B, C be generalized K-affinoid algebras, let M be a finite canonical B-module, and put $D = B\widehat{\otimes}_K C$. There is a natural bounded D-linear isomorphism with bounded inverse*

$$D \otimes_B M \xrightarrow{\sim} M\widehat{\otimes}_K C, \qquad (b \otimes c) \otimes m \longmapsto bm \otimes c.$$

The left side has its canonical finite-D-module norm. This module is flat over C.

Proof. Proposition 2.33.8 makes D a generalized affinoid algebra. Apply Theorem 2.33.3 to a finite B-module presentation of M, split into its two norm-strict kernel sequences as in the preceding proof. It identifies $M\widehat{\otimes}_K C$ with the quotient of D^b by the image of the same presentation matrix on D^a. This is $D \otimes_B M$ as an algebraic module. The quotient is norm-strict, so its norm is equivalent to the canonical finite-D-module norm by Theorem 1.18.3. The formula on pure tensors identifies the D-actions and gives naturality. Lemma 3.27.1, applied to the Banach K-space M, gives C-flatness. □

Proposition 3.27.3 (Product flatness on ordinary stalks)**.** *Let Y, X be good K-analytic spaces, let $\mathcal{F}$ be coherent on Y_G, and put $P = Y \times_K X$. For $w \in P$ with image $x \in X$, the module $(p_Y^*\mathcal{F})_w$ is flat over $\mathcal{O}_{X,x}$.*

Proof. Write $y = p_Y(w)$. Choose ordinary affinoid neighborhoods $U = \mathcal{M}(B)$ of y and $V_0 = \mathcal{M}(C_0)$ of x. Their product is affinoid by Theorem 2.40.11 and contains the intersection of the inverse images of their ordinary interiors. Thus it is a neighborhood of w, and these products show that P is good. By Theorem 2.39.6, $\mathcal{F}|_{U_G}$ is represented by a finite canonical B-module M. Put $Q = (p_Y^*\mathcal{F})_w$.

First fix a rational domain $V \subset V_0$ containing an ordinary neighborhood of x. Write $C_V = \mathcal{O}(V)$ and $D_V = B\widehat{\otimes}_K C_V$. On $U \times_K V$ the pullback sheaf is represented by

$$Q_V = D_V \otimes_B M \simeq M\widehat{\otimes}_K C_V.$$

This follows from the finite presentation defining coherent pullback; Corollary 3.27.2 shows that Q_V is C_V-flat. Affinoid neighborhoods W of w contained in $U \times_K V$ are cofinal among its ordinary neighborhoods in P. Each W is an affinoid domain of this product, so its algebra A_W is D_V-flat by Proposition 2.38.2. The restricted module $Q_V \otimes_{D_V} A_W$ is therefore C_V-flat: tensoring an injection first with Q_V and then with A_W preserves injectivity.

The module version of Proposition 2.33.16, obtained from the same finite presentation, gives

$$Q = \varinjlim_W \bigl(Q_V \otimes_{D_V} A_W\bigr).$$

For a finitely generated ideal $J \subset C_V$, an element of $J \otimes_{C_V} Q$ is represented at one stage W. If its image in Q vanishes, that image vanishes after one further restriction. Flatness at the new stage makes the tensor zero there. The finite-ideal criterion proves that Q is flat over each fixed C_V.

Now put $R = \mathcal{O}_{X,x} = \varinjlim_V C_V$. Let $I = (a_1, \dots, a_m) \subset R$, and let $z = \sum_i a_i \otimes q_i \in I \otimes_R Q$ have zero image in Q. Lift the a_i to elements $\widetilde{a}_i$ of a single C_V, and put $J = (\widetilde{a}_1, \dots, \widetilde{a}_m) \subset C_V$. The coefficients q_i remain in Q. Then

$$z_V = \sum_i \widetilde{a}_i \otimes q_i \in J \otimes_{C_V} Q$$

maps to z and has zero image in Q. The C_V-flatness just proved gives $z_V = 0$, hence $z = 0$. A final application of [StacksAlg, Tag 00HD] proves R-flatness. □

Theorem 3.27.4 (Coherent sheaves are flat over the ground field)**.** *Let Y be a K-analytic space and let $\mathcal{F}$ be coherent on Y_G. Then $\mathcal{F}$ is analytically flat over $\mathcal{M}(K)$. More generally, for every K-analytic space X, the sheaf $p_Y^*\mathcal{F}$ on $Y \times_K X$ is analytically X-flat. In particular the projection $Y \times_K X \to X$ is analytically flat.*

Proof. First let Y, X be good. Let L/K be any complete valued extension, let X' be a good L-analytic space with a morphism to X, and consider any point of the base-changed product. By Theorem 2.40.12 and Corollary 2.33.9, the base change and its coherent pullback identify with

$$(Y \times_K X) \times_X X' \simeq Y_L \times_L X', \qquad p_{Y_L}^* \mathcal{F}_L.$$

The space Y_L is good: the inverse image of an ordinary affinoid neighborhood is affinoid and contains the inverse image of its ordinary interior. The sheaf $\mathcal{F}_L$ is coherent by its local finite presentations. Proposition 3.27.3, applied over L, proves naive flatness at every point of this product. These are all the good base-change tests, over all complete extensions, in Definition 2.9.1.

For general Y, X, choose affinoid analytic nets on both spaces. Their affinoid products form an analytic atlas of $Y \times_K X$ by Theorem 2.40.11. On every such product, the restricted sheaf is the pullback of the finite-module sheaf on the source chart. The good-space case supplies all the required flatness tests there. The G-local definition therefore gives the assertion on the general product. Taking $X = \mathcal{M}(K)$ proves the first statement; taking $\mathcal{F} = \mathcal{O}_Y$ proves the projection statement. □

Chapter 4

Analytic curves and skeleta

Curves make the previous constructions visible. Nested discs describe points, annuli carry real intervals, and reduction gives the directions issuing from a type II point. We prove the basic calculations and then state the model-theoretic results that globalize them. Throughout the curve classification and residual-curve discussion, K is algebraically closed, complete and nontrivially valued. The explicit annulus calculations themselves work over an arbitrary complete non-Archimedean field.

4.1 Disc seminorms and the four kinds of points

For $a \in K$ and $r \geq 0$, define a seminorm on $K[T]$ by

$$|P|_{a,r} = \max_j |c_j| r^j, \qquad P(T) = \sum_j c_j (T-a)^j,$$

where for $r = 0$ only the constant coefficient is retained. Proposition 1.2.2 proves multiplicativity for $r > 0$; for $r = 0$ this is evaluation at a. Write $\zeta_{a,r}$ for the resulting analytic point. If $|a - b| \leq r$, translating the polynomial in both directions gives $|P|_{a,r} = |P|_{b,r}$. Thus the disc, not its chosen centre, determines the point.

Theorem 4.1.1. *Every point of $\mathbb{A}^{1,\mathrm{an}}_K$ is either $\zeta_{a,r}$ for some a, r, or the limit of the seminorms of a decreasing sequence of closed discs with empty intersection. The four types are:*

1. *Type I: $r = 0$, an evaluation at $a \in K$.*

2. *Type II:* $r > 0$ *and* $r \in |K^\times|$.
3. *Type III:* $r > 0$ *and* $r \notin |K^\times|$.
4. *Type IV: the empty-intersection case.*

Proof. For a seminorm x, set $r = \inf_{a \in K} |T - a|_x$. Suppose the infimum is attained at a. If $|a - b| > r$, the ultrametric inequality with unequal sizes gives $|T - b|_x = |a - b|$. If $|a - b| \leq r$, the same inequality gives an upper bound r, while the definition of the infimum gives the reverse bound. Hence

$$|T - b|_x = \max(r, |a - b|).$$

Every polynomial splits into linear factors over K, so this identity and multiplicativity identify x with $\zeta_{a,r}$.

If the infimum is not attained, choose a_i such that $r_i = |T - a_i|_x$ decreases strictly to r. For $j > i$, one has $|a_i - a_j| = r_i$, so the discs $D_i = D(a_i, r_i)$ are nested and $r_i \in |K^\times|$. A common point b would give $|T - b|_x \leq r_i$ for every i, attaining the infimum, a contradiction. The radii cannot tend to zero, since completeness would then give a common limit of the centres.

For each $b \in K$, eventually $r_i < |T - b|_x$, so $|a_i - b| = |T - b|_x$ and $|T - b|_{a_i, r_i} = |T - b|_x$. Splitting a polynomial into its finitely many roots proves convergence to x. Conversely, a nested family of discs with empty intersection gives decreasing nonnegative values on each polynomial; their limits preserve products and the ultrametric inequality. A nonzero polynomial has all its roots outside a sufficiently small member of the family, so its value is then a nonzero constant. The resulting limit is a norm and cannot be a disc seminorm, by the empty-intersection property. □

A field is spherically complete when every nested family of closed balls has nonempty intersection. Over a complete field, empty nested families can be reduced to a sequence by choosing radii approaching their infimum. Thus spherical completeness excludes type IV points. Ordinary completeness alone only excludes nested empty families whose radii tend to zero.

References. [TemIntro, 2.3.3]; [CLD25, 11.1.13]. The proof above uses algebraic closedness in the factorization of polynomials.

4.1.1 What changes over the original ground field

For a complete field K_0 which is not algebraically closed, form $C = \widehat{\overline{K_0}}$ and use the scalar-extension projection $\mathbb{A}^{1,\mathrm{an}}_C \to \mathbb{A}^{1,\mathrm{an}}_{K_0}$. The classification above applies upstairs. A rigid point downstairs can have completed residue field $K_0[T]/(P)$ for an irreducible polynomial P, and need not be a K_0-rational point. If P is separable of degree d, its base change to C consists of d classical points, one for each root. Algebraically this is the product decomposition $C[T]/(P) \simeq C^d$.

Thus the phrases "classical point" and "evaluation at an element of K" coincide in the algebraically closed discussion, but not over a general ground field. Likewise residual directions over a non-algebraically-closed residue field carry degrees. These degrees must be restored when descending a geometric slope or measure formula. The completed-tensor-product construction in Section 2.6 gives the precise base change; a picture of discs over K_0 alone does not.

4.2 Residue fields and topology of the line

The point classification has a valued-field interpretation. For a type II point, translate and scale to the unit Gauss point; its residual field is $\widetilde{K}(t)$ and it adds no value-group rank. For a type III point, algebraic closedness makes $|K^\times|$ divisible, so its radius is rationally independent of the old value group. Proposition 3.8.1 gives the following table.

Type	$s_K(x)$	$t_K(x)$	Description
I	0	0	$\mathcal{H}(x) = K$
II	1	0	$\widetilde{\mathcal{H}(x)} \simeq \widetilde{K}(t)$
III	0	1	$v(\mathcal{H}(x)^\times) = \Gamma + \mathbb{Z}v(T-a)$
IV	0	0	immediate extension

For completeness, the last row can be checked directly. If x is represented by empty nested discs, each rational function has, on a sufficiently small disc, the same norm as its value at the centre. Moreover, after dividing by that central value its residue is one: for a linear factor $T-b$ outside that disc, the ratio $(T-b)/(a_i-b)$ differs from one by an element of norm less than one. Products and quotients give the assertion for rational functions. Completion

changes neither values nor residues, as proved in Proposition 3.8.1. Thus the extension is immediate.

The topology is the evaluation topology, not the metric topology of K extended by a few points. Increasing radii give continuous paths $r \mapsto \zeta_{a,r}$. Two discs join through the smallest disc containing both; for $D(a,r)$ and $D(b,s)$ its radius is $R = \max(r, s, |a-b|)$. Along a radial segment the natural length is a difference of logarithmic radii, so the distance between the two nonclassical disc points is

$$\ell(\zeta_{a,r}, \zeta_{b,s}) = \log(R/r) + \log(R/s).$$

A classical point is approached as a radius tends to zero and is at infinite distance in this logarithmic metric. Type IV endpoints are obtained by the nested-disc limit.

Example 4.2.1 (The two topologies need not agree)**.** At the unit Gauss point of $\mathbb{P}^{1,\mathrm{an}}$, the inward branches are $|T-a| < 1$, indexed by $\widetilde{a} \in \widetilde{K}$, and there is one outward branch. A basic ordinary neighborhood is specified by finitely many polynomial evaluations, so it contains every inward branch except possibly finitely many. Indeed a polynomial has constant Gauss size on each branch not containing a residual root of its normalized reduction.

Choose points at a fixed positive logarithmic distance from the Gauss point in distinct inward branches. They converge to the Gauss point in the Berkovich topology but not in the path metric. The path metric is useful on a fixed finite skeleton, yet it must not replace the ambient topology in statements about compact support or weak convergence of currents.

4.3 Annuli and a retraction with an explicit homotopy

Let $0 < r \le R$. The closed annulus $A[r,R]$ has affinoid algebra

$$K\{R^{-1}T, rT^{-1}\} = \left\{ \sum_{m\in\mathbb{Z}} a_m T^m : \begin{array}{ll} |a_m|R^m \to 0 & (m \to +\infty), \\ |a_m|r^m \to 0 & (m \to -\infty) \end{array} \right\}.$$

Its spectral norm is the maximum of its two endpoint Gauss norms. The skeleton is the subset $\Sigma = \{\zeta_{0,\rho} : r \le \rho \le R\}$. The function $t = -\log|T|$ identifies it with $[-\log R, -\log r]$.

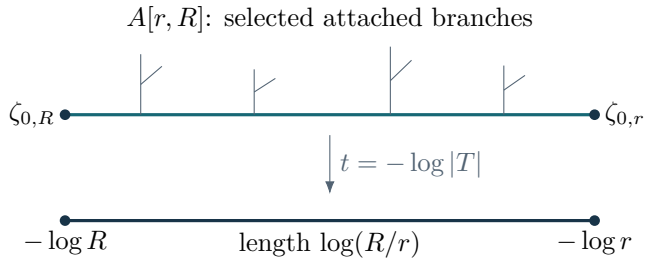


Figure 4.1: The highlighted Gauss points form the annular skeleton. The logarithmic coordinate maps it to the interval below and contracts the attached branches; only a few branches are drawn.

Proposition 4.3.1. *The map $\tau(x) = \zeta_{0,|T(x)|}$ is a continuous retraction onto Σ. There is a strong deformation retraction from the identity to τ which preserves $|T|$.*

Proof. Write $P(X+Z) = \sum_j P_j(X)Z^j$ for a polynomial, using its Taylor coefficients without division by factorials. For $u \in [0,1]$, define a seminorm by

$$|P|_{H(x,u)} = \max_j |P_j(x)|\,(u|T(x)|)^j.$$

This is evaluation at $T(x)+Z$ followed by the Gauss norm of radius $u|T(x)|$ over $\mathcal{H}(x)$, hence multiplicative. Its value on T is $|T(x)|$, so it defines a point of the same annulus. For $u = 0$ it is x. For $u = 1$, translation by the scalar $T(x)$ preserves the radius-$|T(x)|$ Gauss norm, since this scalar has exactly that size; hence the point is $\zeta_{0,|T(x)|}$.

For a skeletal point $x = \zeta_{0,\rho}$, each Taylor term has size at most $|P|_x$, while the $j = 0$ term already has that size. Thus the homotopy fixes Σ. Finally the displayed formula is continuous in (x,u) for every polynomial. Inversion of T is continuous on the annulus, and Laurent polynomial approximations converge uniformly in its affinoid norm. Therefore all analytic evaluations are continuous along the homotopy. This proves the strong deformation retraction. □

The same formula restricts to open annuli because it preserves $|T|$. This proof also explains what a retraction forgets: the seminorms of individual functions can change along a fibre of τ, even though the radius is fixed. A function factors through τ only when it is constant on those fibres.

4.4 Units on annuli and integral slopes

Proposition 4.4.1. *For a nonzero analytic function $f = \sum a_m T^m$ on a closed annulus, its skeletal function is*

$$F(t) = -\log|f(\zeta_{0,e^{-t}})| = \min_m\{v(a_m) + mt\}.$$

It is piecewise affine with integral slopes. If f is an analytic unit on the annulus, F is affine.

Proof. The formula is the Gauss norm. Uniform convergence of the Laurent series and the positive minimum of its Gauss norm on the compact radius interval show that only finitely many terms can attain the minimum. Thus F is the minimum of finitely many affine functions with integral slopes. In logarithmic radius $s = \log\rho$, the function $\log|f|$ is convex. If f has an inverse, multiplicativity gives $\log|f^{-1}| = -\log|f|$, and this negative is convex as well. Therefore $\log|f|$, and hence F, is affine. □

Example 4.4.2. For $f = T - b$ with $r < |b| < R$, one has $F(t) = \min(t, v(b))$. The slope drops by one when the radial path passes the radius of the zero. On the full analytic curve, a branch toward b is attached there; along that branch $-\log|T-b|$ increases with slope one. The additional branch restores the zero total outgoing slope at the nonclassical attachment point. Looking only at the interval would lose this contribution.

A monomial map $T \mapsto cT^m$ sends the skeletal coordinate to $mt + v(c)$ and multiplies lengths by $|m|$. Its injectivity on an interval does not make it unimodular: the tangent-lattice index is $|m|$. This distinction will reappear in tropical multiplicities and faithful tropicalization.

4.5 Residual curves and tangent directions

Let X be a smooth proper connected algebraic curve over K. At a type II point x, the field $\widetilde{\mathcal{H}(x)}$ is a one-variable function field over $\widetilde{K}$. Let C_x be its smooth projective curve. A tangent direction at x is a germ of a nontrivial path leaving x; equivalently it is a branch in a sufficiently small suitable neighborhood.

Theorem 4.5.1. *There is a canonical bijection $T_x X^{\mathrm{an}} \simeq C_x(\widetilde{K})$ at a type II point. A type III point has two directions, and a type IV point has one. In a semistable model containing x as a vertex, C_x is the smooth normalization of the residual component attached to x; nodes determine distinguished directions.*

Proof. Write $k = \widetilde{K}$. By [StacksAlg, Tag 0A26] the proper curve is projective. The empty-marking case of Theorem 4.38.2 gives a projective split nodal model with the specified generic fibre. By Theorem 4.13.2 the analytic generic fibre of its completion is X^{an}. Given a type II point x, apply Theorem 4.27.4 to $\{x\}$. On the resulting model Theorem 4.29.2 identifies the normalization of the component at x by its function field $\widetilde{\mathcal{H}(x)}$, and identifies its closed points with the intrinsic branches at x, independently of the model. The normal proper curve with this function field is its nonsingular projective model, which is smooth because k is perfect [StacksAlg, Tag 0BY3]. Its closed points are k-rational because k is algebraically closed. On a model containing x the cofinal punctured stars of Theorem 4.25.3 identify the two ends of a self-node with distinct normalization points.

For the remaining valences use the open disc and open annulus tubes of Theorem 4.23.3. Their complement consists of type II component points. Each tube is an open subspace of an analytic line in its chart, so it suffices to count local branches there. Use the disc classification of Theorem 4.1.1. At a type III point $x = \zeta_{a,r}$, $r \notin |K^\times|$. The circle $|T - a| = r$ consists of x: the nonzero monomials in each polynomial in $T - a$ have distinct norms, since $|K^\times|$ is divisible. Open annuli with radii straddling r are cofinal neighbourhoods. Indeed finitely many polynomial evaluations determine a basic neighbourhood, and after factoring those polynomials the absolute values of their linear factors near this circle are either constant or equal to $|T-a|$. Each punctured annulus has precisely two components, its inner and outer open annuli. Each is path connected by the radial homotopy of Proposition 4.3.1; shrinking preserves the two labels.

At a type IV point take nested closed discs with empty intersection representing x, passing to a subsequence with strictly decreasing radii. Writing the discs as $D(a_n, r_n)$, use the open discs $U_n = \{|T - a_{n+1}| < r_n\}$. Each U_n contains $D(a_{n+1}, r_{n+1})$ and lies inside $D(a_n, r_n)$. Thus these neighbourhoods of x are cofinal: eventually the discs exclude all the roots of any given finite list of nonzero polynomials, so those polynomial norms are constant on the disc. Each such open disc remains path connected on deleting x. Indeed any

two other points in it can be enlarged to the same closed disc strictly inside it, centred at its centre, with radius larger than both values of the centred coordinate. Along either enlargement the point is the starting point or a disc point with a centre in K: for a nested-disc point an enlargement exceeding its limiting radius contains a member of the defining sequence and is centred there. Thus neither path passes through x. These continuous enlargement paths are the disc homotopy of Proposition 4.21.2 in a disc coordinate, or equivalently the Gauss enlargement of Proposition 4.3.1. The punctured neighbourhoods are nonempty and give one branch.

It remains to check the model assertion when the supplied model $\mathfrak{X}$ is an arbitrary admissible semistable formal model of X^{an}. Return to a component vertex x on this model and write $\eta = \mathrm{sp}_{\mathfrak{X}}(x)$ for the generic point of its component, $C = \overline{\{\eta\}}_{\mathrm{red}}$, and $F = \widetilde{\mathcal{H}(x)}$. We use only that the special fibre is a reduced nodal curve and that the generic fibre is the given analytic curve. The following collars are taken on the split algebraic model realizing x, which we call the reference model.

On an affine formal chart containing η, the prime of the reduction defined by x is the minimal prime of C. Proposition 4.12.2 gives the unique point over η; Proposition 4.24.2 identifies the fraction field of the reduced chart component with F by evaluation. Restriction to a smaller chart gives the same map on the function field, so $k(C) = F$ canonically.

We verify completeness of the normalized component without a properness assumption on an individual affine chart. Apply Theorem 3.19.25 at x with the whole quasi-compact quasi-separated strictly analytic curve as neighbourhood, first with $\mathfrak{X}$ and then with the completed reference model. It identifies the two spaces of valuations with centre on the respective reduced components, over the space of valuations of F/k. Here the reduced closure in each application is exactly the indicated component. The reference component is proper, so every valuation of F/k has exactly one centre on it, by the valuative criterion for properness [StacksAlg, Tag 0BX5, (1) to (3)]. Consequently every such valuation has exactly one centre on C. Its normalization $N \to C$ is finite and birational [StacksAlg, Tag 0BXR, (1) and (3)], hence proper [StacksAlg, Tag 02OG, (1) to (2)], since C is locally noetherian. Lemma 3.19.23 applied to $N \to C$ thus gives exactly one centre on N as well.

For clarity, this centre comparison identifies N with the smooth projective curve C_x, over F. At a closed point of either normal curve the local ring is a DVR [StacksAlg, Tag 00PD, (5) to (1)]. A nontrivial valuation ring

of F/k dominating this DVR equals it: a uniformizer has positive value and every element of $F^\times$ is a unit times a power of that uniformizer. Each such valuation has a unique centre on each curve, and the centre cannot be generic. Thus the identification by valuations matches the closed points and their local rings inside F. Both curves are integral noetherian curves of finite type over k; their proper closed subsets are finite sets of closed points. This gives a homeomorphism, including the generic point. On any nonempty open of an integral scheme, the regular functions are the intersection of its local rings in its function field (the local representatives agree on overlaps). The homeomorphism therefore gives the unique isomorphism inducing the identity on F. Denote the normalization map by $\nu : C_x \to C$.

We now show that the branch indexed by $p \in C_x(k)$ actually specializes to $\nu(p)$ near x on $\mathfrak{X}$. Choose a finite formal affine cover of $\mathfrak{X}$, as in the proof of Theorem 3.19.25. The affinoid generic fibres U are compact in the Hausdorff curve. Near x we can discard the finitely many which miss x. For each remaining chart $\operatorname{Spf} A_0$, lift finitely many k-algebra generators of $A_0/K^{\circ\circ}A_0$ to $t_i \in A_0$. Write $\psi_i \in F$ for their images on the reduced component. Choose $h_i \in K(X)$ representing ψ_i at x, taking $h_i = 0$ if $\psi_i = 0$. For this choice, write each nonzero ψ_i on an affine open of the reference component as a quotient of residues of two functions on a reference algebraic model affine, with nonzero denominator at the component generic point. Lifting that quotient gives h_i of norm one at x. On a dense open of the reference component the numerator and denominator are regular and the denominator has nonzero reduction. There h_i is a regular model function after inverting the denominator. The functions h_i are analytic on an ordinary neighbourhood of x: its algebraic support is generic since a closed point of X is K-rational and cannot have type II. Continuity on U now gives

$$|t_i - h_i| < 1 \qquad \text{on } U \text{ near } x.$$

A single ordinary neighbourhood works on the corresponding chart intersections for all members of the finite list.

For a nonzero rational germ h of norm one at x, the reference branch collar with coordinate z, $|z| \to 1$, satisfies

$$|h(y)| = |z(y)|^{\operatorname{ord}_p(\widetilde{h}_x)} \quad \text{on a sufficiently thin full collar.}$$

Indeed h is a unit on a neighbourhood of x; cofinality of the reference stars puts the whole thin collar there. Theorem 4.24.5 gives the formula on the Gauss segment and Corollary 4.10.3 extends it to all points of the collar.

Fix p and a chart U. If $\nu(p)$ lies outside its special affine, some nonzero ψ_i has a pole at p: otherwise the coordinate ring of the affine piece of C, generated by the ψ_i, lies in $\mathcal{O}_{C_x,p}$. This would give the same valuation a centre in the affine piece, contrary to uniqueness. For this i, $|h_i| > 1$ on the thin collar, which excludes U there by $|t_i| \leq 1$ and $|t_i - h_i| < 1$. If $q = \nu(p)$ lies in the special affine, lift the values $\psi_i(p) \in k$ to $c_i \in K^\circ$. When $\psi_i - \psi_i(p)$ vanishes in F, $|h_i - c_i| < 1$ near x by continuity. Otherwise its order at p is positive, so the collar formula gives $|h_i - c_i| < 1$ on the thin collar. Consequently $|t_i - c_i| < 1$ there on U for every i. These inequalities cut out the tube of the single closed point q in this chart, since the reduced coordinate values of the generators determine the point. Finiteness of the cover gives the claim on a whole collar of the branch.

Finally the cofinal stars can respect these specializations simultaneously. Remove from C_x the finitely many reference node branches, the complements of the preimages of the special affines just used, and the complements of the dense smooth reference opens on which all h_i are regular model functions with the specified reductions. These exceptions are finite since each of the chosen opens contains the component generic point. For a remaining p, throughout its reference tube the reduction of h_i is its value $\psi_i(p)$: in a model affine $h_i - c_i$ reduces to zero at the specialization. Hence within the common ordinary neighbourhood from above, every point of that tube reduces on $\mathfrak{X}$ to $\nu(p)$, in whichever chart contains it. Take a reference star contained in this neighbourhood and in any prescribed neighbourhood of x, and thin its finitely many exceptional branches by the preceding collar argument. Theorem 4.25.3 supplies such stars with connected punctured pieces. Each piece is now in the formal tube of its specified point. Lemma 4.24.1 on the reduced nodal special fibre identifies the points above a node with its two geometric branches (the residue field is algebraically closed). At a self-node these are distinct points of this same C_x; their punctured-star pieces are disjoint even though both specialize to that node. □

For the Gauss point on $\mathbb{P}^1$, $C_x = \mathbb{P}^1_{\widetilde{K}}$: the finite residual points give the inward discs and the point at infinity gives the outward branch. On the closed unit disc, that outward branch is absent from the domain. On a general curve the residual curve can have positive genus. Replacing it by $\mathbb{P}^1$ would discard precisely the geometry that residual degree and harmonicity can detect.

4.6 Semistable vertex sets and formal models

Definition 4.6.1. A semistable vertex set of X^{an} is a finite set V of type II points such that $X^{\mathrm{an}} \setminus V$ is a disjoint union of open balls and finitely many open annuli. Its skeleton $\Sigma(X, V)$ is V together with the skeleta of those annuli. The length of an annular edge is its logarithmic modulus.

Theorem 4.6.2. *A smooth proper connected curve over the specified algebraically closed complete field has semistable vertex sets. Semistable formal models correspond to such vertex sets. A model dominates another exactly when its vertex set contains the other one. The associated finite metric graph is a skeleton of the analytic curve, and the local retractions glue to a deformation retraction onto it.*

Proof. Let $O = K^\circ$, $k = \widetilde{K}$, and choose $\pi \in O$ with $0 < |\pi| < 1$. Write C for the given smooth proper connected curve and $X = C^{\mathrm{an}}$. The curve is projective [StacksAlg, Tag 0A26]. Fix any admissible semistable formal model $\mathfrak{X}$ with its identification $\mathfrak{X}_\eta = X$. We will apply the correspondence for algebraic completions by proving that this very model is such a completion over X. Its special fibre $D = \mathfrak{X}_s$ is a reduced curve with at worst nodes. No pointed chart on $\mathfrak{X}$ is a hypothesis of the argument.

The formal model has a finite cover by formal affines: apply the finite chart argument in Theorem 3.19.25 to the whole compact strictly analytic curve. Their generic fibres form an analytic covering with the neighbourhood property, and specialization is surjective. Hence D is of finite type over k. For every irreducible component its generic point has a unique analytic lift by Proposition 4.12.2; this is a type II point by Proposition 4.24.2. The formal-model clause of Theorem 4.5.1 identifies the normalization of that component with its smooth projective residual curve. Thus the disjoint union P of the normalizations is proper over k, and $P \to D$ is finite surjective by [StacksAlg, Tag 0BXR] (or Lemma 4.24.1). In particular D is proper. Indeed, after any base change the preimage of a closed subset of D under this surjection has the same image in the base as the closed subset, so D is universally closed. The finite morphism $P \times_k P \to D \times_k D$ maps the closed diagonal of P onto the image of the diagonal of D. The latter image is closed. The diagonal is a closed immersion locally around its image, on $U \times_k U$ for affine opens $U \subset D$; it is therefore a closed immersion. This proves separatedness as well.

Put $R_n = O/\pi^n O$ and let D_n be the n-th scheme truncation of $\mathfrak{X}$, for $n \geq 1$. These form a cartesian tower, with $D_n \times_{R_n} k = D$. The ideal $K^{\circ\circ}/\pi^n O$ is nil since K has rank one. Its extension to any ring is nil, so passage to the closed fibre gives a homeomorphism also after any base change on R_n, for the base, the source and its fibre square. Universal closedness for D_n follows from that for D after these identifications. The diagonal image of D_n is likewise closed. The same affine diagonal argument over R_n gives separatedness.

On a formal affine $\operatorname{Spf} A_0$ admissibility gives a surjection $O\langle T_1, \ldots, T_m\rangle \to A_0$ and π-torsion-freeness. Every nonzero element of O divides some π^j, so A_0 is O-torsion-free, hence flat [StacksAlg, Tag 0539]. Thus $B = A_0/\pi^n A_0$ is flat over R_n. It is a quotient of $S = R_n[T_1, \ldots, T_m]$, because a restricted series reduces to a polynomial. To check finite presentation, apply [StacksAlg, Tag 05IK] on $\operatorname{Spec} S \to \operatorname{Spec} R_n$ to the cyclic S-module B. The polynomial morphism is finitely presented, the module sheaf is of finite type, and the base has only one point, so its weakly associated points are locally finite. Every stalk of this sheaf is R_n-flat: localization is a filtered colimit of copies of the flat module B. The theorem gives finite presentation on the entire affine space. Consequently the kernel of $S \to B$ is finitely generated, and B is a finitely presented R_n-algebra. The finite affine cover and separatedness now give properness, flatness and finite presentation of every D_n/R_n.

Proposition B.1.2 and Theorem B.1.3 apply to precisely this tower. They give a projective flat finitely presented O-scheme Y, at worst nodal of relative dimension one, together with compatible isomorphisms at all levels. On formal affines the rings and their restriction sheaves are the π-adic inverse limits of those truncations. We obtain an isomorphism $\widehat{Y}_\pi \simeq \mathfrak{X}$ and, by Theorem 4.13.2, the specified analytic isomorphism $(Y_K)^{\mathrm{an}} \simeq C^{\mathrm{an}}$.

This last identification comes from an isomorphism $Y_K \simeq C$. Here is the precise use of Theorem 2.34.8. For projective K-schemes L, M, embed M in projective space. An analytic map $L^{\mathrm{an}} \to M^{\mathrm{an}}$ pulls back the hyperplane line bundle and its generating sections. The theorem algebraizes the line bundle and the sections; the evaluation map is surjective because its coherent cokernel has zero analytification. They define a map to projective space [StacksAlg, Tag 01NE]. Its analytification is the original projective-space map, as one sees on each coordinate chart by taking ratios of the sections. Each homogeneous equation of M pulls back to an algebraic section of a tensor power whose analytification vanishes, hence to zero. The map therefore factors through M scheme-theoretically. Two such algebraic maps with the same analytification

have isomorphic bundles with the same ordered sections by full faithfulness, so are equal. Applying this argument to the isomorphism and its inverse gives the assertion, with the prescribed identification.

In particular Y_K is smooth. Theorem 4.18.2 gives split pointed charts at the finitely many special nodes: its finite extension is K, and smoothness of Y_K makes every smoothing parameter nonzero by Theorem 4.15.3. Smooth closed special points have residue field k, which is algebraically closed. Thus Y, with its generic identification, is a split nodal algebraic model of C. The formal isomorphism over X identifies the specialization maps and the component vertices.

Theorem 4.38.2, with no markings, supplies an initial model. Theorem 4.42.5 now applies to every model under consideration: it realizes every semistable vertex set, identifies the arrows inducing the identity on X with reverse inclusions, and identifies the graph with its original logarithmic edge lengths. The local disc and annulus retractions glue by Theorem 4.23.3; transport along the formal isomorphism gives the asserted deformation retraction. □

The special fibre explains the gluing. Tubes of smooth closed points are open balls; tubes of ordinary double points are open annuli; generic points of components lift to the vertices. The annular modulus comes from an equation $UV = a$, and the incidence of components gives the incidence of edges. This reconstructs a metric graph from a model, while Theorem 4.6.2 supplies the converse construction.

Adding a type II point in an annular edge subdivides the edge without changing its total length. Adding a vertex in a ball attaches a new tree segment. The skeleton therefore depends on the chosen model, although the metric on common paths is compatible. For punctured curves, one chooses a vertex set compatible with the finitely many punctures and adds the corresponding infinite rays. Over a non-algebraically-closed field, semistable reduction can require a finite extension; descent of models, weights and integrals must then be handled separately.

4.7 The slope formula and its sign

Let $f \in K(X)^\times$ and put $F = -\log|f|$ away from its zeros and poles. At a type II point x, choose $c \in K^\times$ with $|cf(x)| = 1$. The residue of cf is a

nonzero rational function $\widetilde{f}_x$ on C_x, defined up to a nonzero residual constant. Its divisor is therefore unambiguous.

Theorem 4.7.1 (Slope formula)**.** *For a direction v at a type II point, corresponding to $p_v \in C_x(\widetilde{K})$,*

$$d_v F(x) = \operatorname{ord}_{p_v}(\widetilde{f}_x).$$

Only finitely many slopes are nonzero, and their sum is zero. On a skeleton of the punctured curve containing all zeros and poles among its ends, F factors through the retraction and is piecewise affine with integral slopes. Toward a zero or pole a, with ray parameter $t = -\log|z|$ for a local parameter z, it has slope $\operatorname{ord}_a(f)$.

Open proof obligation (RC2). Prove equality of residual orders and analytic slopes in the stated scope; BPRStructure Theorem 5.15 supplies it.

Example 4.7.2. Take $f = T$ at the unit Gauss point of $\mathbb{P}^1$. Its residual function is t, with divisor $[0] - [\infty]$. The outgoing slopes of $-\log|T|$ are $+1$ toward the residual zero, -1 toward infinity, and zero in all other directions. For $\log|T|$, both signs reverse. The later identity $d'd''[\log|f|] = [\operatorname{div}(f)]$ uses the latter logarithm; graph Laplacian conventions must be matched to it, rather than inferred from the word "slope".

4.8 Higher-dimensional skeleta and tropical coordinates

The curve construction has a higher-dimensional local model, but existence of such a model is an additional hypothesis. Suppose a formal model is strictly semistable, with local ètale charts over

$$\operatorname{Spf} K^\circ\langle X_0, \dots, X_r, T_1, \dots, T_{d-r}\rangle/(X_0 \cdots X_r - a), \qquad 0 < |a| < 1.$$

Its standard skeleton has coordinates $u_i = -\log|X_i| \geq 0$ with $\sum_i u_i = v(a)$. The Gauss-point construction and residue calculation were given in Section 3.8. The simplex has real dimension r, while the generic fibre has analytic dimension d.

Definition 4.8.1 (Formal strictly semistable pair)**.** In the setting of [GRW16, Definition 3.7], take a connected quasi-compact admissible formal scheme with the local charts above, together with distinguished effective Cartier divisors H_i. Each H_i has irreducible support on the generic fibre and, in each chart, is either absent or defined by one of the free coordinates T_j. Their sum H gives a formal strictly semistable pair $(\mathfrak{X}, H)$. The irreducibility condition excludes self-intersections of an individual horizontal component.

When s horizontal coordinates vanish on the stratum under consideration, the corresponding canonical polyhedron is

$$\Delta(r,a) \times \mathbb{R}^s_{\geq 0}, \qquad \Delta(r,a) = \{(u_0, \dots, u_r) \geq 0 : \textstyle\sum_i u_i = v(a)\}.$$

The unbounded coordinates are $-\log|T_1|, \dots, -\log|T_s|$ on the complement of the horizontal divisor. The remaining free coordinates have Gauss radius one. Thus the polyhedron has dimension $r+s$, while the associated residual stratum has dimension $d-r-s$. Both enter a higher-dimensional slope formula.

Strata index the canonical polyhedra, and their incidence determines the gluing. Writing $X = \mathfrak{X}_\eta$, GRW constructs the skeleton in $X \setminus |H_\eta|$ and its canonical retraction; see [GRW16, §§3–5, especially Theorem 4.13]. This is an external geometric theorem, extending the explicit annulus construction. The availability of the local equation does not establish a global strictly semistable model for every variety. Alterations and extensions, when used to obtain models, require their own descent arguments.

Definition 4.8.2. On an analytic domain V, invertible analytic functions $f_1, \dots, f_m$ define an analytic moment map $f : V \to (\mathbb{G}_m^m)^{\mathrm{an}}$ and its tropicalization

$$f_{\mathrm{trop}}(x) = (-\log|f_1(x)|, \dots, -\log|f_m(x)|).$$

This map need not be injective or have image of the same dimension as V. Its coordinates present real functions; the analytic geometry supplies the weights and the admissible local dimensions.

We now pass from this analytic foundation to tropical geometry. Three constructions must be kept distinct: a skeleton attached to a model, the tropical image of a chosen moment map, and the tropical skeleton selected by Shilov points of tropical fibres. GRW relates them under precise hypotheses; the relations, degree formulas and faithful embeddings are developed in Chapter 6. Their differential applications start only after the lattice multiplicities and refinement rules of Chapter 5 have been established.

4.9 Finite Galois descent of the analytic topology

Fix a complete non-Archimedean field K with its given absolute value, and a finite Galois extension L/K with group G. Give L the unique extending absolute value of Lemma 1.12.1. We use the scalar extension constructed in Theorem 2.40.12. The action of G on X_L is semilinear over L and fixes its projection $p : X_L \to X$.

Lemma 4.9.1 (Galois action on completed fibres)**.** *For a K-analytic space X and $x \in X$, the fibre $p^{-1}(x)$ is a nonempty finite G-orbit. More precisely, putting $H = \mathcal{H}(x)$, there is a finite product decomposition*

$$H \widehat{\otimes}_K L = H \otimes_K L = \prod_{j=1}^{s} E_j, \qquad \sum_{j=1}^{s} [E_j : H] = [L : K],$$

where the E_j are finite field extensions of H, equipped with their unique extending absolute values. The factors correspond to the points y_j of the fibre, with $\mathcal{H}(y_j) \simeq E_j$. The G-action permutes the factors transitively and preserves their valued-field structures.

Proof. Choose a K-basis of L. Its coordinate maps are bounded by Lemma 1.12.1. Tensoring these finitely many maps with H compares the projective tensor norm with a coordinate maximum norm on $H^{[L:K]}$ in both directions. The latter is complete. Thus the algebraic tensor is already complete and is the displayed completed tensor.

For a K-affinoid algebra A, the same coordinate comparison identifies $A \widehat{\otimes}_K L$ with the finite free A-algebra $A \otimes_K L$ in its canonical finite-module topology. It is K-affinoid by Theorem 2.33.18. Every bounded K-point restricts on L to its unique extending absolute value, so its K-spectrum and L-spectrum agree. The affinoid fibre formula therefore applies to the projection and gives precisely $H \otimes_K L$. These identifications glue on the chart atlas.

By the primitive-element lemma, [StacksAlg, Tag 030N], write $L = K[\alpha]$ with separable minimal polynomial P. The coprimality of P and P' persists in $H[T]$ by the same polynomial Bezout identity. Factoring P into distinct monic irreducibles over H and applying the Chinese remainder construction gives the product of fields $E_j = H[T]/(P_j)$. The sum of their degrees is $\deg P = [L : K]$. Each field is complete and has precisely one extending

absolute value by Lemma 1.12.1. The fibre construction of Theorem 2.40.13 now gives the point and completed-residue-field assertions.

The fixed algebra is $(H \otimes_K L)^G = H$. Indeed, write an invariant element as $\sum_i h_i \otimes \ell_i$, with the finitely many h_i linearly independent over K. Applying each $\sigma - 1$ shows that every ℓ_i lies in $L^G = K$, where the last equality is [StacksAlg, Tag 09DW]. If the primitive idempotents of the product had two or more G-orbits, the sum of the idempotents in one orbit would be a nontrivial idempotent in this fixed field. This is impossible. Hence the action is transitive. An induced H-isomorphism between two factors preserves their absolute values by uniqueness, which completes the proof. □

Theorem 4.9.2 (The scalar-extension quotient)**.** *For every K-analytic space X, the scalar-extension projection is a continuous, open, closed surjection and induces a homeomorphism*

$$X_L/G \xrightarrow{\sim} X.$$

For every G-invariant subset $S \subseteq X_L$, its induced map $S/G \to p(S)$ is also a homeomorphism, with the subspace topologies. For every topological space T, the map $p \times \mathrm{id}_T$ is an open quotient map.

Proof. On an affinoid chart $V = \mathcal{M}(A)$, its inverse image is $V_L = \mathcal{M}_L(A \widehat{\otimes}_K L)$. The field automorphisms of L are isometries, so their coefficient actions extend to bounded algebra automorphisms and hence continuous maps of V_L. These actions agree on overlaps and give a continuous G-action on X_L. By Lemma 4.9.1, the projection is surjective and its fibres are exactly the orbits.

We verify closedness using the affinoid quasi-net atlas. Let C be closed in X_L. For each affinoid chart V, the set $C \cap V_L$ is closed in the compact space V_L; its image $p(C) \cap V$ is compact and thus closed in the Hausdorff space V. The quasi-net condition now makes $p(C)$ closed in X. To see this directly, for $x \notin p(C)$ choose finitely many chart members V_i containing x whose union contains a neighborhood N of x. Each relative open complement $V_i \setminus p(C)$ contains x, so it contains $O_i \cap V_i$ for an ambient open neighborhood O_i of x. The neighborhood $N \cap \bigcap_i O_i$ is disjoint from $p(C)$. This argument does not require X itself to be Hausdorff or good.

A closed continuous surjection is a quotient map. Since its fibres are the G-orbits, it gives the stated homeomorphism. It is open as well: for any

open $U \subset X_L$, the set $p^{-1}(p(U)) = \bigcup_{g \in G} gU$ is open, so $p(U)$ is open by the quotient property.

A G-invariant subset is saturated for p. If $F \subset S$ is relatively closed, write $F = C \cap S$ with C closed in X_L. Saturation gives $p(F) = p(C) \cap p(S)$, which is relatively closed. Thus $S \to p(S)$ is a quotient map with orbit fibres and yields the second homeomorphism. Finally, $p \times \mathrm{id}_T$ sends a basic open product $U \times W$ to the open product $p(U) \times W$. It is a continuous open surjection and therefore a quotient map. □

Corollary 4.9.3 (Descent of equivariant retractions)**.** *Let $S \subseteq X_L$ be G-invariant and let $H_L : X_L \times [0,1] \to X_L$ be a continuous G-equivariant strong deformation retraction onto S. There is a unique continuous strong deformation retraction $H : X \times [0,1] \to X$ onto $p(S)$ such that*

$$H(p(y), t) = p(H_L(y, t)).$$

Writing $r_L = H_L(-, 1)$ and $r = H(-, 1)$, a continuous function $F : X \to \mathbf{R}$ satisfies $F \circ p = (F \circ p) \circ r_L$ if and only if $F = F \circ r$.

Proof. Equivariance makes $p \circ H_L$ constant on every fibre of $p \times \mathrm{id}_{[0,1]}$. The quotient property in Theorem 4.9.2 produces a unique continuous H. Its value at $t = 0$ is the identity, its value at $t = 1$ lies in $p(S)$, and it fixes every point of $p(S)$ throughout the homotopy: choose a lift in S and use the same assertions for H_L. Finally $p \circ r_L = r \circ p$ and surjectivity of p prove the equivalence for F. □

For an analytic unit f on a K-analytic domain U, apply the corollary with $X = U$ and an equivariant strong deformation retraction of U_L. Its scalar extension satisfies $|f_L(y)| = |f(p(y))|$ by evaluation, so the last assertion applies to $F = -\log|f|$ on U. All absolute values here are the fixed original absolute value and its extensions; the logarithmic values are unchanged by descent.

4.10 Laurent coefficients and local slopes

Let K be a complete non-Archimedean field, with arbitrary characteristic and possibly trivial absolute value. For $0 < r \leq R$, put

$$A_{r,R} = K\{R^{-1}T, rT^{-1}\}, \qquad f = \sum_{n\in\mathbb{Z}} a_n T^n,$$
$$\|f\|_{r,R} = \max_{n\in\mathbb{Z}} |a_n| \max(r^n, R^n).$$

The coefficient description and its multiplication are supplied by Theorem 2.13.2 and Proposition 2.13.3. For $r \leq \rho \leq R$, write $\eta_\rho = \zeta_{0,\rho}$ and

$$M_f(\rho) = |f(\eta_\rho)| = \max_n |a_n|\rho^n, \qquad F_f(t) = -\log M_f(e^{-t}).$$

For $f \neq 0$, the maximum is positive and attained, and the set of maximizing exponents is finite. Denote its least and greatest elements by $m_f(\rho)$ and $M_f^{\exp}(\rho)$, respectively.

Lemma 4.10.1 (Extreme Laurent exponents)**.** *For $f, g \in A_{r,R} \setminus \{0\}$ and $r \leq \rho \leq R$,*

$$m_{fg}(\rho) = m_f(\rho) + m_g(\rho), \qquad M_{fg}^{\exp}(\rho) = M_f^{\exp}(\rho) + M_g^{\exp}(\rho).$$

If $r < \rho < R$ and $t_0 = -\log\rho$, the one-sided derivatives satisfy

$$F'_{f,+}(t_0) = m_f(\rho), \qquad F'_{f,-}(t_0) = M_f^{\exp}(\rho).$$

Proof. Write $f = \sum a_i T^i$ and $g = \sum b_j T^j$, and fix ρ. The weighted coefficient families tend to zero in both directions. In the coefficient of $T^{m_f+m_g}$, the product $a_{m_f}b_{m_g}$ is the unique summand of weighted size $M_f(\rho)M_g(\rho)$. Indeed, a summand of that size must use a maximizing exponent of each factor, and neither exponent can be smaller than the corresponding minimum. All other summands have strictly smaller size. Their sizes tend to zero, so their supremum is strictly smaller as well. The ultrametric inequality therefore gives a coefficient of that weighted size.

For a smaller exponent, every convolution summand has strictly smaller weighted size, and the same finite-tail argument applies. Gauss multiplicativity, Proposition 1.2.2, now gives the first formula. Choosing the greatest maximizing exponents gives the second formula.

By Proposition 4.4.1, F_f is the minimum of finitely many affine functions $v(a_n) + nt$ on the compact parameter interval. Near t_0, only those attaining

the minimum at t_0 can contribute. To the right their smallest slope is selected; to the left their greatest slope is selected. This proves both derivative formulas. □

Proposition 4.10.2 (Annular units). *An element $f \in A_{r,R}$ is a unit if and only if*

$$f = cT^m(1+h), \qquad c \in K^\times, \quad m \in \mathbb{Z}, \quad h \in A_{r,R}, \quad \|h\|_{r,R} < 1.$$

The exponent m is unique, and c may be taken to be the coefficient a_m of f. At every analytic point x of the annulus,

$$|f(x)| = |c|\,|T(x)|^m.$$

Proof. Suppose $fg = 1$. Lemma 4.10.1 gives, at every radius,

$$\left(M_f^{\exp} - m_f\right) + \left(M_g^{\exp} - m_g\right) = 0.$$

Both terms are nonnegative integers, so f has a unique maximizing exponent at every radius. Only finitely many exponents can maximize anywhere on $[r, R]$, as in the proof of Proposition 4.4.1. A unique maximizer is therefore locally constant in the radius. Connectedness of $[r, R]$ gives one exponent m throughout; the same conclusion is immediate if $r = R$.

Set $c = a_m$ and $h = f/(cT^m) - 1$. At either endpoint every coefficient term of h has size strictly less than one. The terms tend to zero, so their maximum is strictly less than one. The coefficient norm is the maximum of the two endpoint Gauss norms; hence $\|h\|_{r,R} < 1$. Conversely, Lemma 1.1.4 inverts $1 + h$, so the displayed form is a unit.

For any analytic point, Lemma 1.6.1 gives $|h(x)| < 1$, whence $|1+h(x)| = 1$. This proves the norm identity. Any displayed form has m as its unique maximizing exponent at every radius, which proves uniqueness. □

Corollary 4.10.3 (Units on standard open domains). *An analytic unit on a standard open annulus $r < |T| < R$, where $0 \le r < R \le \infty$, has*

$$|f(x)| = |c|\,|T(x)|^m \qquad (c \in K^\times,\ m \in \mathbb{Z}).$$

Thus its absolute value factors through the radial retraction. An analytic unit on a standard open or closed disc of positive radius with centre $a \in K$ has constant absolute value $|f(a)|$.

Proof. Restrict an annular unit and its inverse to closed subannuli of positive modulus. Proposition 4.10.2 applies to each restriction. When two radius intervals overlap in a nonempty open interval, the affine functions $\log|c| + m\log\rho$ agree there, so their slope and constant agree. Any two compact radius intervals lie in a common larger compact subinterval of (r, R); comparing with that subinterval gives the asserted global formula.

For a closed disc of radius $q > 0$, translate the centre to zero and write $f = \sum_{n\geq 0} a_n T^n$ and its inverse $g = \sum_{n\geq 0} b_n T^n$. Restriction to the radius-q circle identifies these coefficient series with elements of $A_{q,q}$. Their greatest maximizing exponents are nonnegative and sum to zero by Lemma 4.10.1. Both are zero. Consequently $a_0 \neq 0$ and $\|f/a_0 - 1\|_q < 1$, which gives $|f(x)| = |a_0|$ throughout the disc. Restricting an open-disc unit to each closed subdisc centred at a proves the open-disc assertion. □

Proposition 4.10.4 (Residual orders along Gauss-circle paths)**.** *Let $f = \sum a_n T^n \in A_{r,R} \setminus \{0\}$, let $r < \rho < R$ with $\rho \in |K^\times|$, and choose $b, c \in K^\times$ with $|b| = \rho$ and $|c| = M_f(\rho)$. Put $k = \widetilde{K}$ and*

$$d_n = c^{-1} a_n b^n, \qquad P_f(Z) = \sum_n \widetilde{d_n} Z^n \in k[Z, Z^{-1}].$$

Then P_f is a nonzero Laurent polynomial representing the residue of $c^{-1}f$ in $\widetilde{\mathcal{H}(\eta_\rho)} = k(\widetilde{T/b})$. For sufficiently small $s \geq 0$, consider the paths

$$\gamma_0(s) = \zeta_{0,\rho e^{-s}}, \qquad \gamma_\infty(s) = \zeta_{0,\rho e^{s}},$$
$$\gamma_\alpha(s) = \zeta_{bu,\rho e^{-s}} \quad \left(\alpha \in k^\times,\ u \in K^\circ,\ \widetilde{u} = \alpha\right).$$

They start at η_ρ, and

$$\left.\frac{d}{ds}\right|_{0+} \left(-\log|f(\gamma_\alpha(s))|\right) = \operatorname{ord}_\alpha(P_f) \qquad (\alpha \in k \cup \{\infty\}).$$

For $\alpha \in k^\times$ and $d = \operatorname{ord}_\alpha(P_f)$, the stronger identity

$$|f(\gamma_\alpha(s))| = |c| e^{-ds}$$

holds for all sufficiently small $s \geq 0$. The path germ is independent of the lift u. These statements also apply to a quotient of two nonzero elements of $A_{r,R}$ by subtracting their formulas. If k is algebraically closed, the sum of the displayed outgoing slopes is zero.

Proof. The coefficients d_n have size at most one, tend to zero in both directions, and at least one has size one. Hence only finitely many have nonzero reduction, and $P_f \neq 0$. Such a c exists because a maximizing term has size $|a_n b^n|$. Laurent polynomials are dense in $A_{r,R}$ for a norm dominating $|\cdot|_{\eta_\rho}$, so $\mathcal{H}(\eta_\rho)$ is the completion of the Gauss-valued field $K(T)$. In the coordinate $S = T/b$, Proposition 3.8.1 therefore identifies its residue field with $k(\widetilde{S})$. Laurent truncation leaves a tail of norm less than one, proving that the residue of $c^{-1}f$ is exactly P_f.

Its orders at zero and infinity are $m_f(\rho)$ and $-M_f^{\exp}(\rho)$. The corresponding path formulas follow from the two one-sided derivatives in Lemma 4.10.1; the outward path changes $t = -\log|T|$ to $t - s$.

Fix $\alpha \in k^\times$ and a lift u, so $|u| = 1$. For every $0 < q < 1$, substitution of $S = u + Z$ on the closed disc $|Z| \leq q$ gives

$$c^{-1} f\big(b(u+Z)\big) = \sum_{j \geq 0} \beta_j Z^j, \qquad \beta_j = \sum_{n \in \mathbb{Z}} d_n \binom{n}{j} u^{n-j}.$$

Here $\binom{n}{j}$ is the integer binomial coefficient, also for negative n. Its absolute value in K is at most one. The inner sum converges by Lemma 1.1.2. Moreover the double family of terms has norm at most $|d_n| q^j$; for each positive threshold only finitely many pairs exceed it. The same summability lemma therefore permits the interchange of sums. The negative-power expansion follows by the geometric inverse for $u + Z$, or equivalently by the integer binomial identity. Thus this is an equality of convergent series, and $|\beta_j| \leq 1$.

Reducing each coefficient identifies $\sum_j \widetilde{\beta}_j Z^j$ with the formal expansion of $P_f(\alpha + Z)$. If $d = \operatorname{ord}_\alpha(P_f)$, then $|\beta_d| = 1$ and $|\beta_j| < 1$ for $j < d$. Choose $0 < q_0 < 1$ so that

$$|\beta_j| < q_0^{d-j} \qquad (0 \leq j < d).$$

For $q_0 < q < 1$, the dth term has Gauss size q^d. Each lower term has smaller size by this choice, and each higher term has size at most q^{d+1}. Therefore the Gauss norm is exactly q^d. Taking $q = e^{-s}$ gives the asserted equality and derivative; at $s = 0$ the equality is $|f(\eta_\rho)| = |c|$.

If u' is another lift of α, then $|u - u'| < 1$. For q close enough to one, translation by $b(u - u')$ preserves the radius-ρq Gauss norm, so the two path germs agree. For a quotient, both numerator and denominator have positive Gauss value and the established formulas hold on a common short initial segment; subtracting logarithms proves the assertion. Finally, writing

$$P_f(Z) = Z^{m_f(\rho)} Q(Z), \qquad Q(0) \neq 0, \qquad \deg Q = M_f^{\exp}(\rho) - m_f(\rho).$$

If k is algebraically closed, the orders of Q at its nonzero roots sum to its degree. Adding the orders at zero and infinity gives zero. This proves the slope sum, and subtraction gives the same conclusion for quotients. □

All coefficient and Gauss-norm calculations above are unchanged by a complete valued extension of K that preserves its absolute value. At a radius already in $|K^{\times}|$, the initial Laurent polynomial is carried to the extended residue field coefficientwise, so the same path calculation applies there.

4.11 Implicit functions on entire tubes

The simple-root argument of Lemma 2.54.1 extends to a square system of restricted power series and can be made uniform on increasing affinoid domains. We use its contraction construction and the geometric inversion of Lemma 1.1.4, with the maximum norm for vectors and matrices.

Proposition 4.11.1 (A uniform implicit-function domain)**.** *Let A be a K-affinoid algebra over a complete non-Archimedean field, with an admissible submultiplicative Banach norm. Let $n \geq 1$, let $F = (F_1, \dots, F_n) \in A\langle Z_1, \dots, Z_n\rangle^n$ have coefficients of norm at most one, and fix $z_0 \in (K^{\circ})^n$. Write $J(Z) = (\partial F_i/\partial Z_j)$ and put*

$$e = F(z_0), \qquad J_0 = J(z_0), \qquad d = \det J_0,$$
$$B = A\langle Z_1, \dots, Z_n, Q\rangle/(F_1, \dots, F_n, Q \det J(Z) - 1),$$
$$p : \mathcal{M}(B) \longrightarrow \mathcal{M}(A).$$

Give the following subsets their analytic-domain structures:

$$W = \{x : |d(x)| = 1,\ \max_i |e_i(x)| < 1\},$$
$$Y = \{y \in p^{-1}(W) : \max_i |Z_i(y) - z_{0i}| < 1\}.$$

Then $p : Y \to W$ is an analytic isomorphism. Its inverse has Z-coordinate equal to the unique vector of analytic functions z on W satisfying $F(z) = 0$ and $\max_i |z_i(x) - z_{0i}| < 1$ at every $x \in W$.

Proof. Translation by z_0 is a contractive endomorphism of the unit Tate algebra. Indeed, expand each monomial by the integer binomial formula. Its coefficients have norm at most the norm of the original coefficient. For any

positive threshold, only finitely many original coefficients exceed that threshold, and each associated monomial has finitely many terms. The series criterion 1.1.2 therefore permits the sums to be interchanged and gives a restricted series with the same bound. Termwise differentiation is also bounded, since integer coefficients have absolute value at most one. Consequently

$$F(z_0+H)=e+J_0H+N(H),\qquad N_i(H)=\sum_{|\alpha|\geq 2}a_{i\alpha}H^{\alpha},\qquad \|a_{i\alpha}\|\leq 1.$$

The coefficients of each displayed series tend to zero. In particular all entries of J_0, its adjugate, and d have norm at most one.

The equation $|d|=1$ defines the Laurent affinoid domain $D=\{|d|\geq 1\}$, because $\|d\|\leq 1$. The set W is open in D, and Y is open in $p^{-1}(W)$; domain calculus and base change, Proposition 2.40.5, give their analytic structures. For $0<\delta<1$, put

$$W_\delta=\{x\in D:\max_i|e_i(x)|\leq\delta\},$$
$$A_\delta=A\langle D_1,\delta^{-1}E_1,\dots,\delta^{-1}E_n\rangle/(dD_1-1,E_1-e_1,\dots,E_n-e_n).$$

In the defining quotient norm, the original coefficients and $d^{-1}=D_1$ have norm at most one, $\|e\|\leq\delta$, and $\|J_0^{-1}\|\leq 1$ by the adjugate formula. These domains cover W admissibly: every point lies in the relative interior of some W_δ, by choosing $\max_i|e_i(x)|<\delta<1$.

For vectors h,h' with norms at most δ, telescoping each monomial gives

$$\|N(h)\|\leq\delta^2,\qquad \|N(h)-N(h')\|\leq\delta\|h-h'\|.$$

The same bounds pass to restricted series by convergence. Thus $\Phi(h)=-J_0^{-1}e-J_0^{-1}N(h)$ preserves the closed radius-δ ball in A_δ^n and has contraction constant $\delta<1$. The iteration from zero used in Lemma 2.54.1 gives a fixed point h_δ with $\|h_\delta\|\leq\delta$. Set $z_\delta=z_0+h_\delta$; then $F(z_\delta)=0$ and

$$J(z_\delta)=J_0(I+V_\delta),\qquad \|V_\delta\|\leq\delta<1.$$

The matrix geometric inverse gives $J(z_\delta)^{-1}$ of norm at most one. Its determinant therefore has norm at most one. Hence $Z\mapsto z_\delta$ and $Q\mapsto\det J(z_\delta)^{-1}$ define a bounded map $B\to A_\delta$ and an analytic section $s_\delta:W_\delta\to Y$.

For later use, the monomial telescoping also gives a matrix difference identity

$$F(z_0+h)-F(z_0+h')=(J_0+R(h,h'))(h-h'),$$
$$\|R(h,h')\|\leq\max(\|h\|,\|h'\|).$$

whenever both norms are less than one. Each entry of the remainder contains a monomial of positive degree in h, h'; the restricted-series tails converge, so this is an identity in the Banach algebra, not only at its analytic points.

If $0 < \delta_1 \le \delta_2 < 1$, the defining presentations give a contractive restriction $A_{\delta_2} \to A_{\delta_1}$. Apply the difference identity to h_{δ_1} and the restricted h_{δ_2}. The factor $J_0 + R = J_0(I + J_0^{-1}R)$ is invertible, since $\|J_0^{-1}R\| \le \delta_2 < 1$. Hence the roots agree. Analytic morphism gluing gives a section $s : W \to Y$.

We check the other inverse identity on analytic functions. The domains of Y defined by $p(y) \in W_\delta$ and $\max_i |Z_i(y) - z_{0i}| \le \gamma$, for $0 < \delta, \gamma < 1$, form an admissible affinoid cover. In their defining quotient norms, $\|Z - z_0\| \le \gamma$ and the pulled-back h_δ has norm at most δ. The difference identity becomes

$$0 = F(Z) - F(z_\delta) = (J_0 + R)(Z - z_\delta), \qquad \|J_0^{-1}R\| \le \max(\delta, \gamma) < 1.$$

Thus $Z = z_\delta$ in the affinoid algebra, and $Q \det J(Z) = 1$ identifies the Q coordinate as well. The base coordinates already agree, so $s \circ p$ is the identity as an analytic morphism. The argument preserves nilpotents.

Finally, let z' be another analytic solution with the stated strict bound. On W_δ, restrict further to the Weierstrass domains $\max_i |z'_i - z_{0i}| \le \gamma$, where $0 < \gamma < 1$. As δ and γ vary, these domains cover W admissibly. Their defining quotient norms make the map from A_δ contractive and give $\|z' - z_0\| \le \gamma$; thus all the restricted series substitutions converge. The same difference identity has an invertible matrix factor, since its error has norm at most $\max(\delta, \gamma) < 1$. It follows in each domain algebra that $z' = z_\delta$, and sheaf gluing proves uniqueness on W. □

Corollary 4.11.2 (Pointed Jacobian charts preserve entire tubes)**.** *Let K be nontrivially valued, put $O = K^\circ$ and $k = \widetilde{K}$, and let $\mathfrak{X} = \operatorname{Spf} A_0$ and $\mathfrak{Y} = \operatorname{Spf} B_0$ be admissible formal O-schemes with*

$$F = (F_1, \dots, F_n) \in A_0\langle Z_1, \dots, Z_n\rangle^n, \qquad J(Z) = (\partial F_i/\partial Z_j),$$
$$B_0 = A_0\langle Z_1, \dots, Z_n, Q\rangle/(F_1, \dots, F_n, Q \det J(Z) - 1).$$

Let $q \in \mathfrak{X}_s(k)$ and $z_0 \in O^n$ satisfy $\overline{F}(\overline{z}_0)(q) = 0$ and $\det \overline{J}(\overline{z}_0)(q) \neq 0$. Let $q' \in \mathfrak{Y}_s(k)$ be the point over q with $Z(q') = \overline{z}_0$ and $Q(q') = \det \overline{J}(\overline{z}_0)(q)^{-1}$. The generic-fibre map induces a canonical analytic isomorphism

$$]q'[_{\mathfrak{Y}} \xrightarrow{\sim}]q[_{\mathfrak{X}}.$$

The same conclusion holds for a finite composition of such pointed charts and for formal-open restrictions containing the chosen special-fibre points.

Proof. Proposition 3.14.7 gives the generic affinoid algebras and their quotient norms. The coefficients of each F_i have norm at most one and tend to zero in this norm, because they lie in A_0 and tend to zero π-adically. By Proposition 3.20.1, a point of $]q[_{\mathfrak{X}}$ satisfies $\max_i |F_i(z_0)| < 1$ and $|\det J(z_0)| = 1$. This tube therefore lies in W.

Over this base tube, specialization to q' is equivalent to $\max_i |Z_i - z_{0i}| < 1$. The base coordinates already specialize to q, these inequalities fix the residues of all Z_i, and $Q \det J(Z) = 1$ forces the specified residue of Q. These coordinates generate the special-fibre algebra and hence determine its specialization point. Thus

$$]q'[_{\mathfrak{Y}} = Y \cap p^{-1}(]q[_{\mathfrak{X}}).$$

Restricting the preceding isomorphism proves the assertion. Two lifts of $\overline{z}_0$ differ coordinatewise by elements of $K^{\circ\circ}$ and define the same strict inequalities; the unique inverse is consequently independent of the lift. A formal open containing q contains its entire tube in the generic fibre. Indeed, the principal-open specialization criterion applies to a principal neighborhood of q inside that formal open. Formal-open restrictions therefore preserve the isomorphism, and composition gives the final assertion. □

4.12 Reduced models and component points

Let K be complete and nontrivially valued, with its original absolute value. Put $O = K^{\circ}$, $\mathfrak{m} = K^{\circ\circ}$, and $k = \widetilde{K}$. Fix $0 < |\pi| < 1$.

Lemma 4.12.1 (A reduced special fibre determines the unit ball)**.** *Let A_0 be an admissible O-algebra, let $A = A_0[1/\pi]$, and assume that $A_0/\mathfrak{m}A_0$ is reduced. Then the quotient norm associated to any admissible integral presentation of A_0 is power-multiplicative and equals the spectral radius. Moreover,*

$$A^{\circ} = A_0, \qquad A^{\circ\circ} = \mathfrak{m}A_0, \qquad \widetilde{A} = A_0/\mathfrak{m}A_0.$$

In particular, both A and A_0 are reduced.

Proof. Use Proposition 3.14.7, including the norm-attaining lifts in its proof. It identifies A_0 with the quotient closed unit ball. For nonzero $a \in A$, its norm is attained by a restricted power-series lift, whose nonzero norm is the maximum of its coefficient absolute values. Thus $\|a\| \in |K^{\times}|$. It follows that

$$\mathfrak{m}A_0 = \{a \in A : \|a\| < 1\}.$$

One inclusion follows from the ultrametric inequality for finite sums. For the other, choose $c \in K$ with $|c| = \|a\| < 1$; then $a/c \in A_0$. The case $a = 0$ is immediate.

For any nonzero a, choose $c \in K^\times$ with $|c| = \|a\|$ and put $b = a/c$. Its class in $A_0/\mathfrak{m}A_0$ is nonzero. Reducedness implies that every positive power of this class is nonzero. Since $\|b^j\| \leq 1$, the displayed characterization of $\mathfrak{m}A_0$ gives $\|b^j\| = 1$ for all $j \geq 1$. Consequently

$$\|a^j\| = \|a\|^j, \qquad \rho_A(a) = \lim_{j\to\infty} \|a^j\|^{1/j} = \|a\|.$$

A nonzero nilpotent is incompatible with this identity. Power-bounded elements are exactly those of norm at most one, and topologically nilpotent elements are exactly those of norm less than one. This proves all three identifications. □

Proposition 4.12.2 (The analytic point of a residual component)**.** *Let $\mathfrak{X}$ be an admissible formal O-scheme with reduced special fibre, and let $X = \mathfrak{X}_\eta$. For each generic point η of an irreducible component of $\mathfrak{X}_s$, the fibre $\mathrm{sp}^{-1}(\eta)$ consists of a unique point $x_\eta \in X$. On each formal affine open, these points are exactly the Shilov boundary of its generic fibre. They agree under restriction to formal opens. If $\mathfrak{X}$ is quasi-compact, the resulting component-point set is finite.*

Proof. Choose a formal affine open $\mathfrak{U} = \operatorname{Spf} A_0$ containing η. The preceding lemma identifies its special fibre with strict canonical reduction. Theorem 3.16.7 therefore gives a unique point above each component generic point and identifies these points with the Shilov boundary of $\mathfrak{U}_\eta$.

The specialization and formal-open compatibility in Proposition 3.17.2 give $\mathrm{sp}^{-1}(\mathfrak{U}_s) = \mathfrak{U}_\eta$. Thus every global point above η already belongs to this affine generic fibre. This proves global uniqueness as well as existence. On two formal affine opens, restriction to an affine open in their intersection containing η shows that the constructed points agree.

Finally, a finite formal affine cover exists when $\mathfrak{X}$ is quasi-compact. By Corollary 3.12.5, the generic fibre of each member has a finite Shilov boundary. Every global component generic point lies in a member of the cover, so its analytic lift belongs to one of these finitely many finite boundaries. This proves finiteness. □

4.13 Algebraic models and formal completion

Let K be a complete nontrivially valued non-Archimedean field with its specified real-valued absolute value. Put $O = K^\circ$ and fix $\pi \in O$ with $0 < |\pi| < 1$. All completions below are π-adic. The value group need not be discrete.

Lemma 4.13.1 (Affine algebraic models)**.** *Let B be a finitely presented flat O-algebra, with a presentation*

$$B = O[T_1, \dots, T_n]/(f_1, \dots, f_s).$$

Its completion $\widehat{B}$ is an admissible O-algebra, and the natural maps give

$$\begin{aligned} \widehat{B} &= O\langle T_1, \dots, T_n\rangle/(f_1, \dots, f_s), \\ \widehat{B}[1/\pi] &= K\langle T_1, \dots, T_n\rangle/(f_1, \dots, f_s), \\ B/\pi^m B &\simeq \widehat{B}/\pi^m \widehat{B} \qquad (m \geq 1). \end{aligned}$$

The generic fibre of $\operatorname{Spf}\widehat{B}$ is canonically the analytic domain $|T_i| \leq 1$ in $(\operatorname{Spec} B[1/\pi])^{\mathrm{an}}$, with the displayed affinoid algebra. For $b \in B$, the completion of $B[1/b]$ is the principal formal-open algebra

$$\widehat{B[1/b]} \simeq \widehat{B}\langle U\rangle/(bU - 1).$$

Its generic fibre is the domain $|b| = 1$ in the preceding bounded-coordinate domain. These identifications respect the polynomial coordinate maps and their restrictions.

Proof. The completion of $O[T]$ is $P = O\langle T\rangle$: a compatible polynomial system modulo π^m gives one coefficient in the complete ring O at each monomial, and for every m only finitely many coefficients survive modulo π^m. Conversely such a restricted coefficient family gives the compatible polynomial system. The powers of π define the given topology on O.

Apply Lemma 3.14.8 to $O[T] \twoheadrightarrow B$. It makes $P \to \widehat{B}$ surjective, with kernel the closure of $(f_1, \dots, f_s)$ from $O[T]$. The ideal $(f_1, \dots, f_s)P$ is closed by Proposition 3.14.5. Polynomial approximation of the finitely many coefficients in a sum $\sum_i f_i g_i$ shows that this ideal is exactly the stated closure. This proves the restricted quotient presentation.

Flatness of B preserves injectivity of multiplication by π on O. Hence B has no π-torsion. Lemma 3.14.8 gives the same assertion for $\widehat{B}$ and gives

all the reduction isomorphisms. The displayed quotient is topologically of finite type; Theorem 3.14.4 makes it topologically finitely presented, so it is admissible. By Proposition 3.14.7, inverting π gives the displayed strict affinoid algebra. Theorem 2.33.23, with all radii equal to one, identifies it with the asserted analytic domain, including its analytic structure.

Localization preserves O-flatness. Apply the quotient calculation to $B[1/b] = B[U]/(bU - 1)$ to obtain the last presentation. Equivalently, its reductions modulo every π^m are $(B/\pi^m B)[1/b]$, so it is the principal open restriction of $\operatorname{Spf} \widehat{B}$. Proposition 3.17.1 identifies its generic domain by $|b| \geq 1$; every $b \in \widehat{B}$ already has absolute value at most one there, giving $|b| = 1$. All these maps agree on polynomial coordinates and their inverses, hence agree on the completions by continuity. □

Theorem 4.13.2 (Completion of a projective model)**.** *Let $\mathcal{C}$ be a projective, finitely presented, flat O-scheme, and let $\widehat{\mathcal{C}}$ be its π-adic formal completion. Then $\widehat{\mathcal{C}}$ is an admissible formal model of $(\mathcal{C}_K)^{\mathrm{an}}$: there is a canonical analytic isomorphism*

$$(\widehat{\mathcal{C}})_\eta \simeq (\mathcal{C}_K)^{\mathrm{an}}.$$

For a closed immersion $\mathcal{C} \hookrightarrow \mathbf{P}^N_O$, the generic fibres of the completed standard affine charts are the domains

$$V_j = \{x \in (\mathcal{C}_K)^{\mathrm{an}} : |X_i/X_j|(x) \leq 1 \text{ for every } i \neq j\}.$$

Their overlap maps are the usual coordinate-ratio maps.

Proof. Choose the indicated projective embedding and write $\mathcal{U}_j = \mathcal{C} \cap D_+(X_j) = \operatorname{Spec} B_j$. These are finitely presented flat affine O-schemes: flatness and local finite presentation restrict to open subschemes, and each $\mathcal{U}_j$ is affine and quasi-compact. Its coordinate algebra is generated by the ratios $T_i = X_i/X_j$. A finite-type presentation using these chosen generators has finitely generated kernel. Indeed, if $B_j = O[Z]/(F)$ is any finite presentation, express $Z = p(T)$ and $T = q(Z)$ in B_j; the finitely many relations $F(p(T))$ and $T - q(p(T))$ generate the kernel, by substituting in a polynomial relation.

By Lemma 4.13.1, each $\widehat{B}_j$ is admissible. On $\mathcal{U}_j \cap \mathcal{U}_\ell = D(X_\ell/X_j)$, the same lemma identifies the completed overlap with the principal formal open of $\operatorname{Spf} \widehat{B}_j$. The reduction formula modulo every π^m identifies these overlaps

with the restrictions of $\mathcal{C} \times_O O/\pi^m$. Their inverse limits therefore glue to exactly $\widehat{\mathcal{C}}$, proving its admissibility.

The generic fibre of the jth completed chart is, by that lemma and Theorem 2.33.26, the actual closed-unit-coordinate domain V_j of $(\mathcal{C}_K)^{\mathrm{an}}$, with the same quotient ideal. No reduction of that ideal is taken. The completed overlap adds the bound $|X_j/X_\ell| \le 1$ to $|X_\ell/X_j| \le 1$, hence gives the domain $|X_\ell/X_j| = 1$ inside V_j. This is exactly $V_j \cap V_\ell$; the remaining coordinates change by division by X_ℓ/X_j. Thus the coordinate identifications agree as bounded affinoid algebra homomorphisms on every overlap.

The domains V_j cover $(\mathcal{C}_K)^{\mathrm{an}}$: at any analytic point choose a homogeneous coordinate of maximal absolute value. This is precisely the projective affinoid covering in Proposition 2.33.30. Proposition 3.17.2 constructs the generic fibre by gluing the completed affine charts along the overlaps just identified. The same domains and the same transition maps consequently give the asserted analytic isomorphism. On every chart its map is induced by $B_j \to \widehat{B}_j[1/\pi]$, so these local maps define the natural comparison map to $(\mathcal{C}_K)^{\mathrm{an}}$. Its definition on affine restrictions is independent of the projective embedding, which proves canonicity. □

4.14 Stable models over valuation rings

A stable family of genus $g \ge 2$ is a proper, flat, finitely presented family of curves whose geometric fibres are connected nodal curves of genus g with ample dualizing sheaf. This is the stable-family convention of [StacksAlg, Tags 0E75, 0E76, 0E78]. A stable model of a smooth projective curve C/K over a valuation ring $R \subset K$ includes an identification of its generic fibre with C. We first obtain a finite extension of the fraction field by taking a closed point on a proper algebraic-space cover of the moduli stack.

Lemma 4.14.1 (Finite extensions in the valuative criterion)**.** *Let R be a valuation ring with fraction field K, let $\mathcal{X}$ be an algebraic stack proper over $\operatorname{Spec} R$, and let $x : \operatorname{Spec} K \to \mathcal{X}$ be an R-morphism. There is a finite field extension L/K such that, for every valuation ring $R' \subset L$ dominating R and having fraction field L, the object x_L extends to an object of $\mathcal{X}(R')$. Such valuation rings R' exist.*

Proof. The scheme $\operatorname{Spec} R$ is quasi-compact and quasi-separated, and $\mathcal{X} \to \operatorname{Spec} R$ is separated and of finite type. Chow's lemma for algebraic stacks,

[StacksAlg, Tag 0CQ8], gives an algebraic space P and a proper surjective morphism $P \to \mathcal{X}$. Proper morphisms compose by [StacksAlg, Tag 0CL7]; hence $P \to \operatorname{Spec} R$ is proper. The compatibility of properness for algebraic spaces and stacks is recorded in [StacksAlg, Tag 0CL4].

The fibre product $P_x = P \times_{\mathcal{X}} \operatorname{Spec} K$ is an algebraic space. Indeed, its map to $P \times_R \operatorname{Spec} K$ is the base change of the representable diagonal of $\mathcal{X}$; see [StacksAlg, Tag 04XS]. Properness and surjectivity survive this base change, so P_x is a nonempty proper algebraic space of finite type over K. Choose a nonempty affine open $W = \operatorname{Spec} B$ in an étale scheme presentation of P_x. The morphism $W \to \operatorname{Spec} K$ is locally of finite type and quasi-compact, so B is a finite-type K-algebra. A maximal ideal of B has residue field L finite over K by [StacksAlg, Tag 00FV]. The resulting point of $P_x(L)$ gives a point $p_L : \operatorname{Spec} L \to P$ and a specified isomorphism between its image in $\mathcal{X}(L)$ and x_L.

Fix any valuation ring $R' \subset L$ as in the statement. Apply the ordinary valuative criterion for the proper algebraic space P/R, [StacksAlg, Tag 0A40], to

$$\operatorname{Spec} L \longrightarrow P, \qquad \operatorname{Spec} R' \longrightarrow \operatorname{Spec} R.$$

It supplies a unique extension $\operatorname{Spec} R' \to P$ of p_L. Its composite with $P \to \mathcal{X}$, together with the preceding generic isomorphism, is the required extension of x_L. Finally, [StacksAlg, Tag 00IA] applied to the local subring $R \subset L$ supplies such an R'. Domination ensures $R' \cap K = R$: if $a \in K \setminus R$, then $a^{-1} \in \mathfrak{m}_R \subset \mathfrak{m}_{R'}$, which excludes $a \in R'$. □

Theorem 4.14.2 (Stable reduction over a valuation ring)**.** *Let R be a valuation ring with fraction field K, and let C/K be a smooth projective geometrically connected curve of genus $g \geq 2$. There is a finite field extension L/K such that, for every valuation ring $R' \subset L$ with fraction field L extending R, the curve C_L has a stable model $\mathcal{C}/R'$ which is a projective scheme, flat and of finite presentation over R'. If R has rank one, then R' has rank one.*

Proof. The curve C gives a K-object of the stack $\overline{\mathcal{M}}_{g,R}$ of stable curves of genus g, by [StacksAlg, Tags 0E75, 0E78]. This stack is proper over R: [StacksAlg, Tag 0E9C] proves properness over $\mathbf{Z}$, and [StacksAlg, Tag 0CL6] preserves it under base change. Lemma 4.14.1 therefore gives a finite extension L/K and, over each R' as stated, a stable family of algebraic spaces $f : \mathcal{C} \to \operatorname{Spec} R'$ whose generic fibre is identified with C_L. As an object of

the stack of curves, this family is proper, flat and of finite presentation by [StacksAlg, Tag 0D4Z].

We verify that this family is a projective scheme. Let s be the closed point of $\operatorname{Spec} R'$. Stability of $\mathcal{C}_s$ and [StacksAlg, Tag 0E74] give an open neighbourhood U of s on which $\omega_{\mathcal{C}/R'}$ is relatively ample. Every open neighbourhood of the closed point of a local spectrum is the whole spectrum: a basic open $D(a)$ containing that point has a a unit. Hence $U = \operatorname{Spec} R'$. By [StacksAlg, Tag 0D32], relative ampleness over the scheme $\operatorname{Spec} R'$ implies that $\mathcal{C}$ is a scheme. Properness and the ample invertible sheaf make f locally projective by [StacksAlg, Tag 0B5N]. Apply the same local-spectrum observation to the projective neighbourhood of s; thus f is projective.

For the last assertion, the value group of R has finite index in that of R' by [StacksAlg, Tag 0ASH]. Write the groups additively as $\Gamma \subset \Gamma'$. Every element of Γ' has a positive integer multiple in Γ. If $H' \subset \Gamma'$ is a nonzero convex subgroup, multiplying one of its nonzero elements shows that $H' \cap \Gamma$ is nonzero and convex in Γ. Rank one of Γ gives $H' \cap \Gamma = \Gamma$. For any positive $\gamma' \in \Gamma'$, some $n\gamma'$ lies in $\Gamma \subset H'$; convexity then gives $\gamma' \in H'$. Thus $H' = \Gamma'$, proving that Γ' has rank one. □

Corollary 4.14.3 (Stable models for the fixed absolute value)**.** *Let K be a complete non-Archimedean field with its given nontrivial absolute value, and let C/K be a smooth projective geometrically connected curve of genus at least two. There is a finite extension L/K, endowed with the unique extending absolute value, for which C_L has a projective stable model over L°. If $\operatorname{char} K = 0$, the extension L/K may be chosen Galois.*

Proof. Apply Theorem 4.14.2 to $R = K^\circ$. Lemma 1.12.1 supplies the unique extending absolute value on the finite extension L; its valuation ring L° dominates K°, and L is complete. We may therefore take $R' = L^\circ$ in that theorem.

If $\operatorname{char} K = 0$, take a finite normal closure M/K of L/K using [StacksAlg, Tag 09DT]. Every irreducible polynomial over a characteristic-zero field has nonzero derivative and is relatively prime to its derivative, so M/K is separable and hence Galois. Uniqueness of the extended absolute values identifies the restriction from M with the chosen value on L, giving $L^\circ \subset M^\circ$. Base change of the stable model to M° remains stable by [StacksAlg, Tag 0E76], and remains projective, flat and of finite presentation. Replacing L by M proves the assertion. □

Lemma 4.14.4 (Extension of an isomorphism of stable generic fibres)**.** *Let R be a valuation ring with fraction field K, and let $\mathcal{C}_1/R$ and $\mathcal{C}_2/R$ be stable families of the same genus $g \geq 2$. Every isomorphism $(\mathcal{C}_1)_K \simeq (\mathcal{C}_2)_K$ extends uniquely to an R-isomorphism $\mathcal{C}_1 \simeq \mathcal{C}_2$.*

Proof. The moduli stack $\overline{\mathcal{M}}_g$ is separated by [StacksAlg, Tag 0E9A]. By the definition of separatedness for stacks, [StacksAlg, Tag 04YW], its diagonal is proper. Pulling this diagonal back by the pair $(\mathcal{C}_1, \mathcal{C}_2)$ gives the algebraic space

$$I = \mathrm{Isom}_R(\mathcal{C}_1, \mathcal{C}_2) \longrightarrow \operatorname{Spec} R;$$

the identification with the isomorphism sheaf is the fibre-product description in [StacksAlg, Tag 04XS]. The morphism $I \to \operatorname{Spec} R$ is proper by base change. A specified generic isomorphism is a K-point of I. The proper algebraic-space valuative criterion, [StacksAlg, Tag 0A40], extends it uniquely to an R-point of I, which is precisely the required isomorphism. □

Corollary 4.14.5 (The semilinear action on a stable model)**.** *Let K be a complete non-Archimedean field, let L/K be finite Galois with group G, and give L its unique extending absolute value. Let C/K be a smooth projective geometrically connected curve of genus $g \geq 2$, and let $\mathcal{C}/L^\circ$ be a stable model of C_L with its generic-fibre identification. The canonical semilinear G-action on C_L extends uniquely to a semilinear G-action on $\mathcal{C}$. More explicitly, with $S = \operatorname{Spec} L^\circ$ and $b_\sigma = \operatorname{Spec}(\sigma^{-1}|_{L^\circ})$, there are unique automorphisms $\rho_\sigma : \mathcal{C} \to \mathcal{C}$ satisfying*

$$f \circ \rho_\sigma = b_\sigma \circ f, \qquad \rho_1 = \mathrm{id}, \qquad \rho_{\sigma\tau} = \rho_\sigma \circ \rho_\tau,$$

and inducing the canonical action on the generic fibre.

Proof. Uniqueness in Lemma 1.12.1 gives $|\sigma(a)| = |a|$ for every $a \in L$ and $\sigma \in G$, so G preserves L°. Our inverse convention gives $b_\sigma b_\tau = b_{\sigma\tau}$. The scheme $C_L = C \times_K \operatorname{Spec} L$ has the corresponding canonical left action induced by $\mathrm{id}_C \times \operatorname{Spec}(\sigma^{-1})$.

For a fixed σ, form $b_\sigma^*\mathcal{C} = \mathcal{C} \times_{S, b_\sigma} S$, with its structure map given by the second projection. Both this twist and $\mathcal{C}$ are stable families over S. The generic action gives an L-isomorphism $\mathcal{C}_L \to (b_\sigma^*\mathcal{C})_L$: its two components are the generic semilinear automorphism and the structure map to $\operatorname{Spec} L$. Lemma 4.14.4 extends it uniquely to an S-isomorphism $a_\sigma : \mathcal{C} \to b_\sigma^*\mathcal{C}$. Composing

with the first projection defines ρ_σ. Since b_σ is an automorphism, that projection is an isomorphism of schemes, and ρ_σ is an automorphism covering b_σ.

The maps $\rho_\sigma\rho_\tau$ and $\rho_{\sigma\tau}$ cover the same base map and agree on the generic fibre. Reinterpreting them as S-isomorphisms $\mathcal{C} \to b_{\sigma\tau}^*\mathcal{C}$, uniqueness in Lemma 4.14.4 makes them equal. The same argument gives $\rho_1 = \mathrm{id}$ and proves uniqueness of the action. □

4.15 Pointed nodal charts over the valuation ring

Let K be a complete non-Archimedean field with its given nontrivial absolute value. Write $O = K^\circ$, $\mathfrak{m} = K^{\circ\circ}$ and $k = O/\mathfrak{m}$. We turn the étale base neighbourhood in the nodal normal-form theorem into the spectrum of the valuation ring of a finite extension. The two field-theoretic ingredients are simple-root lifting and a lift of a finite separable residue extension.

Lemma 4.15.1 (Simple roots over a complete valued field)**.** *The valuation ring O is henselian. If $V \to \operatorname{Spec} O$ is* étale *and $v \in V$ lies over the closed point with $\kappa(v) = k$, then an affine open neighbourhood of v admits a section over $\operatorname{Spec} O$ through v. Its image is open in V.*

Proof. Let $F \in O[T]$ have a simple root $\bar{b} \in k$ after reduction, and choose a lift $b \in O$. Then $|F(b)| < 1$ and $|F'(b)| = 1$. Expanding at b gives

$$\frac{F(b+Z)}{F'(b)} = c_0 + Z + \sum_{i=2}^{d} c_i Z^i, \qquad |c_0| < 1, \quad |c_i| \le 1.$$

Choose a real number δ with $|c_0| < \delta < 1$. On the complete closed ball $|Z| \le \delta$, the map $\Phi(Z) = -c_0 - \sum_{i\ge 2} c_i Z^i$ takes values in the same ball and satisfies

$$|\Phi(Z) - \Phi(Z')| \le \delta\,|Z - Z'|.$$

Starting with $Z_0 = 0$, its iterates satisfy $|Z_{n+1} - Z_n| \le \delta^n |Z_1|$; the ultrametric inequality makes them Cauchy. Their limit Z is a fixed point. Thus $b+Z \in O$ lifts $\bar{b}$ and satisfies $F(b+Z) = 0$. This is the one-variable contraction used in the proof of Lemma 2.54.1, now applied to a residue root. The simple-root characterization in [StacksAlg, Tag 04GG] proves that O is henselian.

Choose an affine open $V_0 = \operatorname{Spec} B$ containing v. The map $O \to B$ is étale. Part (8) of [StacksAlg, Tag 04GG] supplies the O-algebra retraction $B \to O$

whose inverse image of $\mathfrak{m}$ is the prime of v. The corresponding section passes through v. A section of an unramified morphism is an open immersion by [StacksAlg, Tag 024T]; hence its image is open in V_0 and in V. □

Lemma 4.15.2 (Lifting a finite separable residue extension)**.** *Let l/k be a finite separable extension. There is a finite separable extension L/K, with the unique extending absolute value, and a k-isomorphism $l \simeq \widetilde{L}$ such that*

$$[L : K] = [\widetilde{L} : k] = [l : k], \qquad |L^\times| = |K^\times|.$$

Proof. By the primitive-element theorem, [StacksAlg, Tag 030N], write $l = k(\beta)$ with monic irreducible polynomial $p(T) \in k[T]$ of degree $d = [l : k]$. Lift its coefficients to a monic polynomial $P(T) \in O[T]$ of degree d. Choose a root α of P in an algebraic extension of K, set $L = K(\alpha)$, and use Lemma 1.12.1 to equip L with its complete extending absolute value. Then $[L : K] \leq d$.

If $|\alpha| > 1$, the leading term of $P(\alpha)$ strictly dominates every other term, which is impossible. Thus $\alpha \in L^\circ$, and $p(\overline{\alpha}) = 0$ defines an embedding $l \hookrightarrow \widetilde{L}$ sending β to $\overline{\alpha}$. The value-group and residue-degree inequality [StacksAlg, Tag 0ASH] gives

$$d \leq [\widetilde{L} : k] \leq [|L^\times| : |K^\times|]\,[\widetilde{L} : k] \leq [L : K] \leq d.$$

All inequalities are equalities. In particular the residue embedding is an isomorphism, the value-group index is one, and P is the minimal polynomial of α. Since p is separable, $p'(\overline{\alpha}) \neq 0$, so $P'(\alpha)$ is a unit of L°. The minimal polynomial is therefore separable. □

Theorem 4.15.3 (A rational pointed nodal chart after finite extension)**.** *Let $\mathcal{X} \to \operatorname{Spec} O$ be a morphism of schemes which is at-worst-nodal of relative dimension one, and let x be a node of its closed fibre. There are a finite separable extension L/K, an affine L°-scheme U,* étale *morphisms*

$$\mathcal{X}_{L^\circ} \longleftarrow U \longrightarrow W_a = \operatorname{Spec}\big(L^\circ[s,t]/(st - a)\big) \qquad (a \in L^{\circ\circ}),$$

and a $\widetilde{L}$-rational point u of the closed fibre of U such that its image x' in $\mathcal{X}_{\widetilde{L}}$ lies over x and its image in $(W_a)_{\widetilde{L}}$ is the origin $s = t = 0$. The point x' is a split node, with

$$\widehat{\mathcal{O}}_{\mathcal{X}_{\widetilde{L}},x'} \simeq \widetilde{L}[[s,t]]/(st).$$

If the generic fibre $\mathcal{X}_K$ is smooth over K, the parameter satisfies $a \neq 0$, and consequently $0 < |a| < 1$.

Proof. Apply the nodal normal-form theorem [StacksAlg, Tag 0CBY]. It gives étale maps $\mathcal{X} \leftarrow U_0$, $V \to \operatorname{Spec} O$ and

$$U_0 \longrightarrow V \times_{\operatorname{Spec} \mathbf{Z}[A]} \operatorname{Spec}\bigl(\mathbf{Z}[S,T,A]/(ST-A)\bigr),$$

and a point $u_0 \in U_0$ mapping to x. Let $v_0 \in V$ be its image. A nodal point of a curve has finite separable residue field over the ground field by [StacksAlg, Tag 0C4D]. The étale fibre description [StacksAlg, Tag 02GL] makes $\kappa(u_0)/\kappa(x)$ finite separable as well. Hence $l = \kappa(u_0)$ is finite separable over k.

Choose L/K by Lemma 4.15.2, with a specified identification $l = \widetilde{L}$. Base change the diagram to $O' = L^\circ$. The maps $\operatorname{Spec} \widetilde{L} \to U_0$ and $\operatorname{Spec} \widetilde{L} \to \operatorname{Spec} O'$ define a rational point u_1 of $(U_0)_{O'}$, whose image v_1 in $V_{O'}$ is rational. Replace $V_{O'}$ by an affine open containing v_1 and $(U_0)_{O'}$ by its inverse image. Since L is complete, Lemma 4.15.1 gives a section

$$j : \operatorname{Spec} O' \longrightarrow V_{O'}$$

through v_1. Pull back the diagram along j. Its image is open, so the resulting U_1 is open in $(U_0)_{O'}$; in particular $U_1 \to \mathcal{X}_{O'}$ remains étale. The other map becomes étale over $W_a = \operatorname{Spec}(O'[s,t]/(st-a))$, where $a \in O'$ is the pullback of the parameter A. These uses of composition and base change are [StacksAlg, Tags 02GN, 02GO]. The point u_1 belongs to this pullback because its image is the special point of the section. Choose an affine open neighbourhood $U \subset U_1$ of it and denote the point by u.

Let x' and w be the images of u in $\mathcal{X}_{\widetilde{L}}$ and $(W_a)_{\widetilde{L}}$. Both are rational: their residue fields are intermediate fields between $\widetilde{L}$ and $\kappa(u) = \widetilde{L}$. The point x' remains a node by the base-change invariance in [StacksAlg, Tag 0C56]. Applying the étale invariance of nodes, [StacksAlg, Tag 0C57], to the two maps from $U_{\widetilde{L}}$ shows that u and w are nodes. The curve $st - \overline{a} = 0$ is smooth wherever s or t is invertible; its only possible nodal point is therefore $s = t = 0$. Thus w is that origin and $\overline{a} = 0$.

The local rings at x', u and w of these special fibres are Noetherian, since the fibres are locally of finite type over $\widetilde{L}$. Each of the two étale local maps is flat, has the same residue field $\widetilde{L}$, and carries the source maximal ideal onto the target maximal ideal; flatness and the fibre assertion follow from [StacksAlg, Tags 02GS, 02GL]. Thus [StacksAlg, Tag 0AGX] identifies their completions. The completion at w is $\widetilde{L}[[s,t]]/(st)$, giving the displayed isomorphism for x'. The split-node criterion [StacksAlg, Tag 0CBW] completes this assertion.

Finally assume that $\mathcal{X}_K$ is smooth, and suppose $a = 0$. Then W_a has the section $s = t = 0$ over $\operatorname{Spec} O'$. The image of $U \to W_a$ is open by [StacksAlg, Tag 03WT] and contains the closed point of this section. Its inverse image on $\operatorname{Spec} O'$ is an open neighbourhood of the closed point, hence the whole local spectrum. In particular the image contains the generic point of the section, the node $(s,t) = (0,0)$ of $(W_0)_L$. Any point of U_L mapping to this node is itself a node by [StacksAlg, Tag 0C57]. On the other hand $U_L \to \mathcal{X}_L$ is étale and $\mathcal{X}_L$ is smooth, so U_L is smooth; preservation of smoothness under étale precomposition is included in [StacksAlg, Tag 036U]. This contradiction proves $a \neq 0$. Together with $a \in L^{\circ\circ}$ it gives $0 < |a| < 1$. □

4.16 Coordinate changes on annuli

Let K be a complete non-Archimedean field, with its specified absolute value. We use the Gauss skeleton, radial retraction and strong homotopy of Proposition 4.3.1. The parameter of that homotopy is denoted by $u \in [0,1]$.

Theorem 4.16.1 (Annulus isomorphisms and radial homotopies)**.** *Let $0 < r < R < \infty$ and $0 < r' < R' < \infty$, and let*

$$\phi : X = A_K[r,R] \xrightarrow{\sim} X' = A_K[r',R']$$

be an analytic isomorphism. Write T, S for the coordinates of X, X'. There are $c \in K^\times$, $\varepsilon \in \{1,-1\}$ and $h \in \mathcal{O}(X)$ with $\|h\|_{r,R} < 1$ such that

$$\phi^* S = cT^\varepsilon(1+h).$$

For every $r \le \rho \le R$,

$$\phi(\zeta_{0,\rho}) = \zeta'_{0,|c|\rho^\varepsilon}.$$

Thus ϕ restricts to an isometry of Gauss skeleta for their logarithmic metrics. In coordinates $t = -\log|T|$ and $t' = -\log|S|$, this isometry is $t' = \varepsilon t + v(c)$, where $v(c) = -\log|c|$. Moreover, for every $x \in X$ and $0 \le u \le 1$,

$$\phi\big(H_X(x,u)\big) = H_{X'}\big(\phi(x),u\big), \qquad \phi \circ \tau_X = \tau_{X'} \circ \phi.$$

Proof. Put $g = \phi^* S$. Proposition 4.10.2, applied to g and to the inverse isomorphism, gives

$$\begin{aligned} g &= cT^m(1+h), & \|h\|_{r,R} &< 1, \\ (\phi^{-1})^* T &= dS^n(1+k), & \|k\|_{r',R'} &< 1, \end{aligned}$$

with $c, d \in K^{\times}$ and $m, n \in \mathbb{Z}$. The norm identity in that proposition holds at every analytic point. Composing the two identities at a point of radius ρ gives

$$\rho = |d|\,|c|^{n}\rho^{mn} \qquad (r \le \rho \le R).$$

Comparing two distinct radii gives $mn = 1$. Hence $m = n = \varepsilon \in \{1, -1\}$. Surjectivity of ϕ and the existence of the Gauss point at every radius show that $\rho \mapsto |c|\rho^{\varepsilon}$ maps $[r, R]$ onto $[r', R']$.

We identify the image of each Gauss point. Set $q = \|h\|_{r,R} < 1$, $\sigma = |c|\rho^{\varepsilon}$, and take a nonzero Laurent polynomial $P(S) = \sum_j a_j S^j$. For every integer j, the geometric inverse and the integer binomial formula give

$$\left|(1+h)^{j} - 1\right|_{\zeta_{0,\rho}} \le q.$$

Consequently

$$\left| P(g) - \sum_{j} a_j c^j T^{\varepsilon j} \right|_{\zeta_{0,\rho}} \le q \max_j |a_j|\sigma^j.$$

The exponents εj are distinct. The Gauss norm of the Laurent polynomial on the right inside the difference is therefore $\max_j |a_j|\sigma^j > 0$, while the error has strictly smaller norm. Thus $|P(g)|_{\zeta_{0,\rho}} = \max_j |a_j|\sigma^j$. Laurent polynomials are dense in the target affinoid algebra, so bounded multiplicative seminorms agreeing on them agree on the entire algebra. This proves the asserted equality of actual analytic points.

For an arbitrary x, the norm formula gives $|S(\phi(x))| = |c|\,|T(x)|^{\varepsilon}$. Combining this with the Gauss-point equality proves $\phi\tau_X = \tau_{X'}\phi$. Taking negative logarithms gives the stated affine formula and its preservation of logarithmic distance.

It remains to compare the homotopies for $0 < u < 1$. Work in $F = \mathcal{H}(x)$ and put

$$\alpha = T(x), \quad \rho = |\alpha|, \quad \beta = g(x), \quad \sigma = |\beta| = |c|\rho^{\varepsilon}, \qquad \delta = u\rho.$$

For a Laurent series $h = \sum_{n\in\mathbb{Z}} h_n T^n$, its substitution on the disc $|Z| \le \delta < \rho$ is

$$h(\alpha + Z) = \sum_{j \ge 0} e_j Z^j, \qquad e_j = \sum_{n\in\mathbb{Z}} h_n \binom{n}{j} \alpha^{n-j}, \qquad |e_j|\rho^j \le q.$$

To check the convergence, the weighted terms are bounded by $|h_n|\rho^n u^j$. The first factor tends to zero as $|n| \to \infty$, and u^j tends to zero as $j \to \infty$. Above

any fixed positive threshold only finitely many pairs remain. Lemma 1.1.2 therefore permits the indicated sums and their interchange. The negative powers use the convergent geometric inverse of $\alpha + Z$. All integer binomial coefficients have absolute value at most one, which gives the displayed bound.

Expand $g = cT^{\varepsilon}(1 + h)$ at α as

$$g(\alpha + Z) = \beta + \sum_{j \geq 1} b_j Z^j.$$

The binomial coefficients of $(\alpha + Z)^{\varepsilon}$ have the same bound, so $|b_j|\rho^j \leq \sigma$ for $j \geq 1$. The first coefficient satisfies

$$b_1 = c\alpha^{\varepsilon - 1}\big(\varepsilon(1 + e_0) + \alpha e_1\big), \qquad |b_1| = \frac{\sigma}{\rho}.$$

Indeed $\varepsilon = \pm 1$ has absolute value one, and both e_0 and αe_1 have size at most $q < 1$. Hence, in the radius-δ Tate algebra over F,

$$g(\alpha + Z) - \beta = b_1 Z(1 + E(Z)), \qquad E(Z) = \sum_{j \geq 2} \frac{b_j}{b_1} Z^{j-1}, \qquad \|E\|_{\delta} \leq u < 1.$$

The bound follows termwise from $|b_j/b_1|\delta^{j-1} \leq u^{j-1}$; the same estimate proves convergence.

For a nonzero polynomial $Q(S) \in K[S]$, write $Q(\beta + W) = \sum_{j=0}^{N} q_j W^j$ over F, and put

$$M = \max_j |q_j|(u\sigma)^j > 0.$$

Replacing $W = b_1 Z(1 + E)$ by $b_1 Z$ changes the resulting polynomial by a series of radius-δ norm at most $uM < M$. The Gauss norm of $\sum_j q_j b_1^j Z^j$ is M. Consequently

$$\big\|Q\big(g(\alpha + Z)\big)\big\|_{\delta} = \max_j |q_j|(u\sigma)^j.$$

The left side is the evaluation of Q at $\phi(H_X(x, u))$. This follows from the construction of H_X and the bounded substitution $T \mapsto \alpha + Z$, whose coordinate and inverse have norms ρ and ρ^{-1}. The right side is the evaluation at $H_{X'}(\phi(x), u)$; the induced embedding $\mathcal{H}(\phi(x)) \hookrightarrow F$ is isometric and hence preserves the Gauss norm. Thus these two points agree on $K[S]$. Their common nonzero value on S gives agreement on Laurent polynomials, and density gives agreement on $\mathcal{O}(X')$. This proves the homotopy identity for $0 < u < 1$. At $u = 0$ it is the identity map, and at $u = 1$ it is the already proved compatibility of the radial retractions. □

Corollary 4.16.2 (The original absolute value and edge length)**.** *Isomorphic nondegenerate closed annuli over K have equal logarithmic moduli. For the nodal chart*

$$\operatorname{Spf} K^\circ\langle U, V\rangle/(UV - a), \qquad 0 < |a| < 1,$$

the radial edge has length $-\log|a|$, and this length and the preceding coordinate-change identities are preserved by complete isometric extensions of K.

Proof. The two logarithmic radius intervals are related by an affine map of slope ± 1, so $\log(R/r) = \log(R'/r')$. Lemma 3.20.3 and Proposition 3.26.1, with $r = d = 1$ in the latter's notation, identify the nodal generic fibre with $A_K[|a|, 1]$. Its length is therefore $-\log|a|$. A complete isometric field extension leaves the coefficients' absolute values, the Laurent coefficient norms and $|a|$ unchanged. Base change of an analytic isomorphism is an isomorphism by Theorem 2.40.12; reapplying the theorem over the extended field gives the asserted identities. □

4.17 Tubes in algebraic models

Let K be complete and nontrivially valued, with $O = K^\circ$, $k = \widetilde{K}$, and a fixed $\pi \in O$ satisfying $0 < |\pi| < 1$. For a flat finitely presented O-scheme $\mathcal{X}$, write $\widehat{\mathcal{X}}$ for its π-adic completion. All tubes below are taken in its formal generic fibre.

Proposition 4.17.1 (An algebraic pointed étale map preserves the tube)**.** *Let $f : \mathcal{Y} \to \mathcal{X}$ be an étale morphism between flat finitely presented O-schemes. Let $q' \in \mathcal{Y}_s(k)$ and $q = f(q') \in \mathcal{X}_s(k)$. The induced generic-fibre morphism restricts to an analytic isomorphism*

$$]q'[_{\widehat{\mathcal{Y}}} \xrightarrow{\sim}]q[_{\widehat{\mathcal{X}}}.$$

Proof. Lemma 4.13.1 makes these completions admissible, and is compatible with formal open restriction. A Zariski open containing the selected residual point retains its entire tube. We may therefore replace the source and target by affine neighbourhoods of q' and q.

By the square Jacobian presentation in [StacksAlg, Tag 02GU], after such shrinking the morphism is given by $R \to S$, where

$$S = R[Z_1, \dots, Z_n]/(F_1, \dots, F_n),$$
$$J(Z) = \left(\frac{\partial F_i}{\partial Z_j}\right)_{i,j}, \qquad \det J(q') \neq 0.$$

Localize once more at $\det J$, which is nonzero at q'. Adjoining its inverse as a variable Q gives the presentation $(F_1, \dots, F_n, Q \det J - 1)$. Completing the polynomial presentation by Lemma 4.13.1 gives

$$\widehat{S} = \widehat{R}\langle Z_1, \dots, Z_n, Q\rangle/(F_1, \dots, F_n, Q \det J - 1).$$

Here quotients are the separated completed quotients; their indicated finite relation ideals are closed by the admissible-presentation result. Choose lifts $z_{0i} \in O$ of the coordinates $Z_i(q') \in k$. They satisfy the hypotheses of Corollary 4.11.2, which supplies the required entire-tube isomorphism, including its analytic structure. The case of no variables is the identity case. All the preceding restrictions preserved the tubes, completing the proof. □

Corollary 4.17.2 (The tube of a rational smooth point)**.** *Let $\mathcal{X}$ be a flat finitely presented O-scheme. Suppose that $\mathcal{X} \to \operatorname{Spec} O$ is smooth of relative dimension d at $q \in \mathcal{X}_s(k)$. Then $]q[_{\widehat{\mathcal{X}}}$ is analytically isomorphic to the open unit d-dimensional polydisc over K.*

Proof. By [StacksAlg, Tag 054L], a Zariski neighbourhood $\mathcal{U}$ of q admits an étale morphism $\mathcal{U} \to \mathbb{A}^d_O$. Its image $\bar{a}$ is a k-rational point. Apply the preceding proposition. The completion of $\mathbb{A}^d_O$ is $\operatorname{Spf} O\langle T_1, \dots, T_d\rangle$; its generic fibre is the closed unit polydisc. If $a_i \in O$ lift the coordinates of $\bar{a}$, the specialization criterion identifies its $\bar{a}$-tube with $|T_i - a_i| < 1$ for every i. Translation by a gives the stated open unit polydisc. □

Corollary 4.17.3 (Transport of a split nodal chart)**.** *Let $0 < |a| < 1$ and put $\mathcal{Z}_a = \operatorname{Spec} O[U, V]/(UV - a)$, with special-fibre origin o. Suppose that there is a diagram*

$$(\mathcal{X}, q) \longleftarrow (\mathcal{U}, u) \longrightarrow (\mathcal{Z}_a, o)$$

of flat finitely presented O-schemes, with both arrows étale and all selected special-fibre points k-rational. Then $]q[_{\widehat{\mathcal{X}}}$ is an open annulus with coordinate T satisfying $|a| < |T| < 1$.

Proof. Apply Proposition 4.17.1 to both arrows. The formal completion of $\mathcal{Z}_a$ is the standard nodal chart of Proposition 3.26.1. Its origin tube is computed in Proposition 3.20.1: with $T = U$ and $V = a/T$, the two inequalities $|U| < 1$ and $|V| < 1$ become $|a| < |T| < 1$. Transport this already established analytic identification along the two tube isomorphisms. □

4.18 Simultaneous splitting of finitely many nodes

Let K be complete with its specified nontrivial non-Archimedean absolute value, and put $O = K^\circ$ and $k = \widetilde{K}$. Every finite extension below carries the unique absolute value extending this one.

Lemma 4.18.1 (Lifting a finite Galois residue extension)**.** *For every finite Galois extension l/k, there is a finite Galois extension L/K and a specified k-isomorphism $\widetilde{L} \simeq l$ such that*

$$[L : K] = [l : k], \qquad |L^\times| = |K^\times|,$$
$$\mathrm{Gal}(L/K) \xrightarrow{\sim} \mathrm{Gal}(l/k).$$

The last map is induced by reduction.

Proof. Use the construction in Lemma 4.15.2. Write $l = k(\beta)$, let $p \in k[T]$ be its monic minimal polynomial of degree d, and lift p to a monic $P \in O[T]$. That proof constructs $L = K(\alpha)$, with $P(\alpha) = 0$, $[L : K] = d$, $\widetilde{L} = l$, and $\overline{\alpha} = \beta$. It also proves that P is separable and that the value groups are equal. We retain these identifications.

Normality and separability of l/k give the d distinct roots $\beta_1, \dots, \beta_d$ of p in l. Apply the simple-root construction of Lemma 4.15.1 in the complete field L to each β_j. It gives a root $\alpha_j \in L^\circ$ of P with reduction β_j. Distinct reductions make these roots distinct. Thus P splits in L, and $L = K(\alpha)$ is its splitting field. By [StacksAlg, Tag 09HU], L/K is normal; it was already separable, so it is Galois.

Every K-automorphism of L preserves its absolute value by uniqueness in Lemma 1.12.1. It therefore induces a k-automorphism of l. Each $\alpha \mapsto \alpha_j$ defines a K-embedding of L into itself. Its image has degree d over K, so it is an automorphism. On the residue generator it induces $\beta \mapsto \beta_j$. These are precisely the d automorphisms of l/k, and a K-automorphism of L is determined by the image of α. The d constructed roots exhaust all roots of P

and have distinct reductions; thus an automorphism inducing the identity on l fixes α. Reduction is therefore a bijective homomorphism, as claimed. □

Theorem 4.18.2 (All lifts of a finite set of nodes)**.** *Let $\mathcal{X} \to \operatorname{Spec} O$ be finitely presented and at-worst-nodal of relative dimension one, and let $q_1, \dots, q_s$ be a finite set of nodes of $\mathcal{X}_k$. There is a finite Galois extension L/K such that every point q' of $\mathcal{X}_{\widetilde{L}}$ over one of the q_i is a $\widetilde{L}$-rational split node. Writing $O' = L^\circ$, each such point admits an affine O'-scheme U, a $\widetilde{L}$-rational point u, and pointed étale maps*

$$(\mathcal{X}_{O'}, q') \longleftarrow (U, u) \longrightarrow \big(\operatorname{Spec} O'[S,T]/(ST - a),\, \overline{0}\big), \qquad a \in L^{\circ\circ}.$$

Here $\overline{0}$ is the special-fibre origin, whose prime ideal is $(L^{\circ\circ}, S, T)$. If $\mathcal{X}_K$ is smooth, every resulting parameter satisfies $0 < |a| < 1$. One may take $|L^\times| = |K^\times|$ and $\operatorname{Gal}(L/K) \simeq \operatorname{Gal}(\widetilde{L}/k)$ by reduction.

Proof. If the node set is empty, take $L = K$. Otherwise, for each q_i, choose the original pointed normal-form diagram and point u_i used at the beginning of the proof of Theorem 4.15.3. Put $k_i = \kappa(q_i)$ and $l_i = \kappa(u_i)$. That proof shows that k_i/k and l_i/k_i are finite separable. Embed these fields compatibly into one separable closure of k. Choose primitive generators for the finitely many l_i/k by [StacksAlg, Tag 030N], and let l/k be the splitting field of their minimal polynomials. It is finite and normal by [StacksAlg, Tag 09HU]. Its generators are separable, so [StacksAlg, Tag 09HC] makes it separable as well. Thus l/k is finite Galois and contains every l_i and its normal closure.

Apply Lemma 4.18.1 to obtain L/K with $\widetilde{L} = l$. The points over q_i are the points of $\operatorname{Spec}(k_i \otimes_k l)$, and

$$k_i \otimes_k l \xrightarrow{\sim} \prod_{\sigma \in \operatorname{Hom}_k(k_i, l)} l, \qquad b \otimes c \longmapsto (\sigma(b)c)_\sigma.$$

Indeed, a primitive-element presentation writes the left side as $l[T]/p_i(T)$, where p_i splits into distinct linear factors. Evaluation at its roots is invertible by the Lagrange interpolation polynomials. Thus every point over q_i is rational and is indexed by an embedding $\sigma : k_i \hookrightarrow l$.

Fix such an embedding. By [StacksAlg, Tag 0BME, part (2)], it extends to a k-automorphism of l, since l/k is normal. Restriction to the chosen subfield l_i gives an embedding $\tau : l_i \hookrightarrow l$ extending σ. The map $\operatorname{Spec} l \to U_i$ defined

by τ gives a rational point of $(U_i)_{O'}$ lying over the selected point q' indexed by σ. Its image in the étale base neighbourhood $(V_i)_{O'}$ is rational as well.

Now reuse the henselian-section step of the proof of Theorem 4.15.3, with this rational point and the fixed field L. Lemma 4.15.1 supplies a section of $(V_i)_{O'} \to \operatorname{Spec} O'$ through the point. Its image is open. Pulling the original diagram back along that section and then taking an affine neighbourhood of the rational point gives exactly the two pointed étale maps displayed in the statement. The remaining local arguments of that theorem apply unchanged: they identify the image in the standard chart with its origin, give $a \in L^{\circ\circ}$, identify the completed special local ring at q' with $l[[S,T]]/(ST)$, and prove $a \neq 0$ when the generic fibre is smooth. No enlargement of L occurs in any of those steps.

There are only finitely many i and finitely many embeddings for each i. Applying the preceding construction to each one proves the assertion for every lift of every specified node, all over the same L. The value-group and Galois assertions were already supplied by the first lemma. □

4.19 Coordinate independence on open annuli

Corollary 4.19.1 (Open annuli and their radial homotopies)**.** *Let K be a complete non-Archimedean field with its specified absolute value. Let*

$$X = A_K(r,R), \qquad X' = A_K(r',R'),$$
$$0 \le r < R \le \infty, \qquad 0 \le r' < R' \le \infty$$

be standard open annuli with coordinates T, S. Their radial strong homotopies are obtained by restricting and gluing the closed-annulus homotopies of Proposition 4.3.1. For any analytic isomorphism $\phi : X \to X'$, there are $c \in K^\times$ and $\varepsilon \in \{1,-1\}$ such that

$$\begin{aligned} |S(\phi(x))| &= |c|\,|T(x)|^{\varepsilon} && (x \in X),\\ \phi(\zeta_{0,\rho}) &= \zeta'_{0,|c|\rho^{\varepsilon}} && (r < \rho < R),\\ \phi\big(H_X(x,u)\big) &= H_{X'}\big(\phi(x),u\big) && (x \in X,\ 0 \le u \le 1). \end{aligned}$$

In particular, ϕ preserves the radial retractions and induces a skeleton isometry $t' = \varepsilon t + v(c)$ for $t = -\log|T|$ and $v(c) = -\log|c|$. When both boundary radii are positive and finite, the logarithmic modulus $\log(R/r)$ is invariant under analytic isomorphism.

Proof. Every point of X belongs to the ordinary interior of some closed subannulus $A_K[a,b]$, with $r < a < b < R$. In the formula of Proposition 4.3.1, the homotopy keeps $|T|$ fixed at every parameter value. Its seminorm formula at a point depends only on that point and u, not on the chosen end radii. Consequently the homotopies on closed subannuli agree on overlaps. Their restrictions to the ordinary interiors give a continuous homotopy on $X \times [0,1]$. The initial map, final radial retraction, and fixed-point identity on the Gauss skeleton hold on each subannulus, hence on X. The same construction applies to X'.

Apply Corollary 4.10.3 to $\phi^* S$ and $(\phi^{-1})^* T$. It gives constants $c, d \in K^\times$ and exponents $m, n \in \mathbb{Z}$ with

$$|S(\phi(x))| = |c|\,|T(x)|^m, \qquad \rho = |d|\,|c|^n \rho^{mn} \quad (r < \rho < R).$$

The second identity may be evaluated at Gauss points of any two distinct radii. Thus $mn = 1$, so $m = n = \varepsilon \in \{1, -1\}$. Surjectivity and the existence of all Gauss radii show that $\rho \mapsto |c|\rho^\varepsilon$ maps (r, R) onto (r', R').

Fix $r < a < b < R$ and let $[a', b']$ be the increasing interval whose endpoints are $|c|a^\varepsilon$ and $|c|b^\varepsilon$. The norm identities for ϕ and its inverse identify the inverse image of $A_K[a', b'] \subset X'$ with $A_K[a,b]$ as a set. The universal property of analytic domains, proved in Theorem 2.40.8, restricts both morphisms and their inverse identities to these domains. Hence

$$\phi|_{A_K[a,b]} : A_K[a,b] \xrightarrow{\sim} A_K[a', b'].$$

Theorem 4.16.1 applies to this nondegenerate closed-annulus isomorphism. Its Gauss-point and homotopy identities are the restrictions of the asserted ones: both homotopies preserve their radii and remain in the respective closed subannuli. These subannuli cover X, proving all the identities. Negative logarithms give the skeleton isometry; its slope has absolute value one, so finite logarithmic interval lengths are preserved. □

4.20 The ends of residual tubes

Throughout this section K is complete and nontrivially valued, $O = K^\circ$ and $k = \widetilde{K}$. A curve model is a flat finitely presented O-scheme whose special fibre is a reduced pure one-dimensional scheme. Its formal generic fibre is assumed Hausdorff. We write v_η for the unique analytic point above a component generic point η, as in Proposition 4.12.2.

Lemma 4.20.1 (Boundary points of a closed-point tube)**.** *Let q be a closed point of the special fibre of a curve model $\mathcal{X}$, and put $D_q = \mathrm{sp}^{-1}(q)$ in $X = \widehat{\mathcal{X}}_\eta$. Then D_q is open in X, its closure C_q is compact, and*

$$C_q \setminus D_q \;\subseteq\; \{v_\eta : q \in \overline{\{\eta\}}\}.$$

Proof. Choose an affine open $\mathcal{U}$ containing q. Formal-open compatibility in Proposition 3.17.2 puts the entire tube D_q inside the compact affinoid $U = \widehat{\mathcal{U}}_\eta$. Since X is Hausdorff, U is closed in X, so C_q is a closed subset of U and is compact. On every formal affine chart, the inverse image of the closed residual point q, when it is present, is given by finitely many inequalities $|f_i| < 1$; use Proposition 3.20.1 and the Noetherian special fibre. These inverse images are relatively open. The affinoid net topology of Proposition 3.17.2 therefore makes D_q open in X.

For $x \in C_q$, write $p = \mathrm{sp}(x)$ as a prime of the affine special fibre. If $p \not\subseteq \mathfrak{m}_q$, choose $\bar{f} \in p \setminus \mathfrak{m}_q$ and an integral lift f. Then $|f(x)| < 1$, whereas $|f(y)| = 1$ for every $y \in D_q$. The relative open set $|f| < 1$ separates x from D_q, a contradiction. Thus $p \subseteq \mathfrak{m}_q$. In a pure one-dimensional Noetherian scheme the proper generalizations of a closed point are component generic points. Their analytic fibres are the singletons $\{v_\eta\}$ by Proposition 4.12.2. This proves the inclusion. □

Lemma 4.20.2 (Two components in a pointed nodal chart)**.** *Let $U_k \to \operatorname{Spec} k[s,t]/(st)$ be an* étale *morphism and let $u \in U_k(k)$ map to the origin. In $R = \mathcal{O}_{U_k,u}$, the ideals (s) and (t) are the two distinct minimal prime ideals. They identify the two distinct irreducible components of U_k through u. Under any* étale *map from U_k to a reduced curve, their generic points map to component generic points; these two images may coincide.*

Proof. The local map from the origin is flat, induces the identity on k, and extends the maximal ideal to the maximal ideal. Thus [StacksAlg, Tag 0AGX] identifies

$$\widehat{R} \simeq k[[s,t]]/(st).$$

The special-fibre ring R is Noetherian. By [StacksAlg, Tag 00MC], completion is faithfully flat, so $R/(s)$ and $R/(t)$ inject into $k[[t]]$ and $k[[s]]$, respectively. Consequently (s) and (t) are prime. Their extensions to $\widehat{R}$ are incomparable, so the original ideals are incomparable. Their intersection is

zero because it becomes zero in the displayed completion and $R \to \widehat{R}$ is injective. Since $st = 0$, every prime contains (s) or (t). These are therefore exactly the minimal primes. The prime correspondence for localization identifies them with distinct global components through u. Finally, an étale map is flat. Going down for a flat map, [StacksAlg, Tag 00HS], implies that the contraction of a minimal prime is minimal: any strictly smaller prime would lift below the chosen minimal prime. This proves the last assertion. □

Proposition 4.20.3 (Attaching the two ends of a nodal tube)**.** *Let $\mathcal{X}$ be a curve model and suppose given a pointed* étale *diagram*

$$(\mathcal{X}, q) \longleftarrow (U, u) \longrightarrow (W_a, \bar{0}), \qquad W_a = \operatorname{Spec} O[s,t]/(st-a), \quad 0 < |a| < 1,$$

where U is affine and the three selected special-fibre points are k-rational. In $Y = \widehat{U}_\eta$, put $D =]u[$ and $C = \overline{D}$. Let v_t, v_s be the component points of the branches $t = 0, s = 0$, respectively. Then

$$D \simeq A_K(|a|, 1), \qquad C = D \cup \{v_t, v_s\}, \qquad |s(v_t)| = 1, \quad |s(v_s)| = |a|.$$

Every net in D whose $|s|$ tends to 1 converges to v_t; every net whose $|s|$ tends to $|a|$ converges to v_s. For each neighborhood of either endpoint in C, a full radial tail of D is contained in that neighborhood. The image of C in X is $C_q = \overline{]q[}$; the only possible identification is that of v_t with v_s. The two endpoints map to the component points of the corresponding branches of q.

Proof. The entire-tube isomorphisms of Corollary 4.17.3 identify s with the annular coordinate. Write $U_s = \operatorname{Spec} S$ and $A = k[s,t]/(st)$. The given map $U \to W_a$ induces the étale morphism $\operatorname{Spec} S \to \operatorname{Spec} A$ by base change, by [StacksAlg, Tag 02GO]. In particular S is of finite type over k. The two branch fields are $F_t = k(s)$ and $F_s = k(t)$. The map $A \to F_t \times F_s$ is injective since $(s) \cap (t) = 0$ in A. Tensoring with S gives the injective map

$$S \longrightarrow (S \otimes_A F_t) \times (S \otimes_A F_s).$$

Indeed, for each prime $\mathfrak{q}$ of S over $\mathfrak{p}$ of A, the corresponding local map $A_\mathfrak{p} \to S_\mathfrak{q}$ is flat by [StacksAlg, Tag 02GU(1) and (2)]. Localizing the displayed map at $\mathfrak{q}$ thus gives an injection at every prime. By [StacksAlg, Tag 02GL(2)], each of the two affine fibres is a finite disjoint union of spectra of finite separable extensions of its branch field. Either fibre may be empty. This proves that S is reduced.

If $\mathfrak{q}$ is minimal in S, its contraction is minimal in A. Otherwise the contraction contains an element outside both minimal primes of A, by avoidance of these two primes. Such an element is a nonzerodivisor on S by the displayed injection. But its image in the nonzero field $S_{\mathfrak{q}}$ would be zero, a contradiction. For a minimal $\mathfrak{q}$ over the branch prime $\mathfrak{p}$, localize $S/\mathfrak{q}$ at the nonzero elements of $A/\mathfrak{p}$. The result is a nonzero domain quotient of the corresponding affine fibre algebra, hence a finite extension of that branch field. It is the fraction field of $S/\mathfrak{q}$. Its transcendence degree over k is one, so [StacksAlg, Tag 00P0] gives $\dim(S/\mathfrak{q}) = 1$. Thus the special fibre of U is a reduced pure one-dimensional curve. Lemma 4.20.2 gives exactly the two components through u. Lemma 4.20.1 gives $C \setminus D \subseteq \{v_t, v_s\}$. At v_t, the reduction of s is nonzero, so $|s(v_t)| = 1$; at v_s, the reduction of t is nonzero, so $|t(v_s)| = 1$ and $st = a$ gives $|s(v_s)| = |a|$.

The affinoid Y is compact Hausdorff. A net in D whose $|s|$ tends to 1 has cluster points in C, and continuity of $|s|$ forces every cluster point to be v_t. Compactness then implies convergence to v_t: otherwise a subnet outside a fixed neighborhood would have a cluster point there. The inner end is identical. Applying this to the Gauss points with radii tending to the two endpoints proves that both v_t and v_s belong to C. If N is a neighborhood of v_t in C, the compact set $C \setminus N$ has $\max |s| < 1$ after replacing N by an open subneighborhood. Thus a full outer tail lies in N. The minimum of $|s|$ on a compact complement similarly proves the inner-tail assertion.

By Lemma 4.20.2, the two residual generic points map to component generic points of $\mathcal{X}_s$. Functoriality of specialization and uniqueness of component points identify the images of v_t, v_s. The image of compact C is closed in Hausdorff X and contains $]q[$. Conversely, continuity and density of D in C put that image inside $\overline{]q[}$. The restriction $D \to]q[$ is an isomorphism, and an endpoint cannot map into $]q[$ because its specialization is generic. Consequently only the two endpoints can be identified. □

Corollary 4.20.4 (The complete nodal homotopy)**.** *Under the hypotheses of Proposition 4.20.3, the radial homotopy on $]q[$ extends uniquely by fixing the attached component points to a jointly continuous strong deformation retraction of C_q onto a closed edge of length $-\log|a|$. If the endpoint images coincide, this edge is a circle with two distinct incident half-edges at its vertex. The construction is independent of the chosen pointed nodal chart and annular coordinate.*

Proof. First work in $C = D \cup \{v_t, v_s\}$ of the proposition. The annulus homotopy of Proposition 4.3.1 preserves $|s|$. The full-tail assertion shows that fixing v_t, v_s extends this homotopy continuously at both endpoints, uniformly in its parameter. The resulting map $C \times [0,1] \to C$ is therefore jointly continuous. The Gauss radius map extends to a continuous injection of $[|a|, 1]$ into C, with endpoints v_s, v_t; compactness identifies its image with a closed interval. The extended homotopy fixes this interval and retracts onto it.

The map $C \to C_q$ is a compact-to-Hausdorff surjection, as is its product with $[0,1]$, so both are quotient maps. The only possible identified pair consists of the two fixed endpoints. The homotopy therefore descends to a continuous map on $C_q \times [0,1]$, with all strong-retraction identities unchanged. The quotient of the Gauss interval is an interval or, when its endpoints coincide, a circle. Its logarithmic length is $-\log|a|$ by Corollary 4.16.2. The full homotopy commutes with every open-annulus coordinate change by Theorem 4.19.1; continuity on the dense tube gives the same equality at the ends. This proves independence and retains both half-edges of a loop. □

4.21 Canonical contraction of a smooth tube

Lemma 4.21.1 (The compactification of a rational smooth tube)**.** *Let $q \in \mathcal{X}_s(k)$ be a point where a curve model $\mathcal{X}$ is smooth of relative dimension one. If η is the unique component generic point specializing to q, then*

$$D_q =]q[\simeq \{x : |T(x)| < 1\}, \qquad C_q = \overline{D_q} = D_q \cup \{v_\eta\}.$$

This closure is the one-point compactification of D_q. In each open-unit-disc coordinate T, the sets

$$\{v_\eta\} \cup \{x \in D_q : |T(x)| > r\}, \qquad 0 < r < 1,$$

form a neighborhood basis of v_η in C_q.

Proof. Corollary 4.17.2 gives the disc isomorphism. Smoothness makes the local ring of the special-fibre curve at q a regular one-dimensional local ring, hence a domain; equivalently its completed pointed étale coordinate ring is $k[[T]]$ and faithful flatness of completion gives a unique minimal prime. Thus exactly one component passes through q. Lemma 4.20.1 shows that the compact closure can add only v_η. The open unit disc is not compact:

the continuous function $|T|$ has supremum 1 and does not attain it, as witnessed by its Gauss points of radii $r < 1$. Therefore the closure must add v_η. A compact Hausdorff space obtained by adjoining one point to an open locally compact space is its one-point compactification: the complements of neighborhoods of the added point are compact subsets of the open space, and conversely every such compact subset is closed in the compactification.

For $r < 1$, the closed subdisc $|T| \le r$ is compact, so its complement in C_q is a neighborhood of v_η. Conversely, every compact subset of D_q has a maximum $|T| < 1$. These subdiscs are therefore cofinal among compact subsets, proving the asserted neighborhood basis. □

Theorem 4.21.2 (The intrinsic open-disc homotopy)**.** *Let D be the open unit disc over a complete nontrivially valued field K, and let $D^* = D \cup \{v\}$ be its one-point compactification. Write*

$$P(T+Z) = \sum_{j \ge 0} P_j(T) Z^j \qquad (P \in K[T]).$$

There is a jointly continuous strong deformation retraction $H_D : D^* \times [0,1] \to D^*$ onto $\{v\}$, characterized for $x \in D$ and $0 \le u < 1$ by

$$|P(H_D(x,u))| = \max_j |P_j(x)| u^j, \qquad H_D(x,1) = v, \qquad H_D(v,u) = v.$$

Every analytic isomorphism of open unit discs, including an isomorphism semilinear for an isometric field automorphism, commutes with this entire parametrized homotopy. The homotopy also commutes with projection under any complete isometric ground-field extension. In particular it gives a coordinate-independent contraction of C_q in Lemma 4.21.1.

Proof. For $u < 1$, evaluate at $T(x) + Z$ over $\mathcal{H}(x)$ and then take the radius-u Gauss seminorm. It is multiplicative and has

$$|T(H_D(x,u))| = \max\{|T(x)|, u\} < 1.$$

Thus it defines a point of D, and at $u = 0$ it gives x. On a neighborhood of any $(x,u) \in D \times [0,1)$, choose $r < 1$ strictly larger than both $|T(x)|$ and u. The displayed formula is continuous on polynomials and remains bounded on the closed radius-r disc; uniform polynomial approximation proves continuity for its affinoid algebra. This proves joint continuity on $D \times [0,1)$. At v, the preceding inequality preserves every tail $|T| > r$, uniformly in u. As $u \to 1$,

it also puts all images in $|T| > r$ for $u > r$, uniformly in x. The neighborhood basis of Lemma 4.21.1, whose one-point-compactification argument applies equally to D^*, proves joint continuity at every remaining point. The strong contraction identities follow from the construction.

We check coordinate independence explicitly. An analytic isomorphism is represented by a power series $g(T) = \sum_{n\geq 0} a_n T^n$ convergent on every closed disc of radius less than one. Such an expansion follows by restricting to those discs and using uniqueness of Tate coefficients on overlaps. Since $|g| < 1$ on D, evaluation at its Gauss points and passage of their radii to 1 give $|a_n| \leq 1$ for every n and $|a_0| < 1$. The inverse has the same coefficient bounds. The derivative identity for the two inverse series at 0 implies $|a_1| = 1$: both $|g'(0)|$ and the derivative of the inverse at a_0 are at most one and their product is one.

Put $F = \mathcal{H}(x)$, $\alpha = T(x)$ and $\beta = g(x)$. For $0 < u < 1$, expansion over F gives

$$g(\alpha + Z) = \beta + \sum_{j\geq 1} b_j Z^j, \qquad |b_1| = 1, \quad |b_j| \leq 1 \ (j \geq 2).$$

Indeed, $b_j = \sum_{n\geq j} a_n \binom{n}{j} \alpha^{n-j}$; these sums converge, and for $j = 1$ all terms after a_1 have norm at most $|\alpha| < 1$. On the radius-u disc,

$$g(\alpha + Z) - \beta = b_1 Z(1 + E(Z)), \qquad \|E\|_u \leq u < 1.$$

For a polynomial $Q(S)$ write $Q(\beta+W) = \sum_j q_j W^j$ and put $M = \max_j |q_j| u^j$. Replacing $W = b_1 Z(1 + E)$ by $b_1 Z$ changes its Gauss norm by an error at most $uM < M$ when $Q \neq 0$. The Gauss norm of $\sum_j q_j b_1^j Z^j$ is M. Thus

$$\big\|Q(g(\alpha + Z))\big\|_u = \max_j |q_j| u^j.$$

These are precisely the polynomial evaluations at $g(H_D(x, u))$ and $H_D(g(x), u)$. Points of the open disc are determined by polynomial evaluations, so the two points agree. The identities at $u = 0, 1$ follow from the endpoints and the extension of a disc isomorphism to its one-point compactification. An isometric automorphism of the coefficient field preserves all these formulas. Finally, under any complete isometric extension, the formulas agree on the original polynomials by the isometric embedding of completed residue fields. At the added endpoint the projection sends the endpoint to the endpoint; continuity follows also from its preservation of the radial tails. This proves all the claimed compatibilities. □

4.22 Smooth tubes with a separable residue field

Lemma 4.22.1 (The integral algebra of a residue-field lift)**.** *Let l/k be finite separable and let $L = K(\alpha)$ be the lift constructed in Lemma 4.15.2: the minimal polynomial $P \in O[T]$ of α is monic and reduces to the separable minimal polynomial of a generator of l/k. Put $d = [l : k]$. Then*

$$\left|\sum_{i=0}^{d-1} c_i \alpha^i\right| = \max_i |c_i|, \qquad L^\circ = \bigoplus_{i=0}^{d-1} O\alpha^i = O[T]/(P), \qquad L^{\circ\circ} = K^{\circ\circ} L^\circ.$$

For every flat finitely presented O-scheme $\mathcal{X}$, its formal completion and analytic generic fibre commute with this coefficient extension. On a completed affine algebra A_0 the corresponding formulas are

$$\begin{aligned} B_0 = A_0 \otimes_O L^\circ &= \bigoplus_{i=0}^{d-1} A_0 \alpha^i, \\ B_0[1/\pi] &= A \widehat{\otimes}_K L, \\ B_0/L^{\circ\circ} B_0 &= (A_0/K^{\circ\circ} A_0) \otimes_k l. \end{aligned}$$

If the original special fibre is reduced, so is the extended special fibre. Specialization commutes with the projection, and the inverse image of a closed-point tube is the disjoint union of the tubes of the points above that closed point.

Proof. For a nonzero coefficient vector, divide by a coefficient of largest absolute value. All normalized coefficients lie in O, and at least one has nonzero reduction. Linear independence of $1, \bar{\alpha}, \dots, \bar{\alpha}^{d-1}$ over k makes the reduction of this normalized sum nonzero, hence its absolute value is one. This proves the norm formula. It proves the two integral identifications as well, since membership in the closed or open unit ball is equivalent to that property for every coefficient.

A finite free direct sum commutes with every quotient by π^n and with the inverse limit defining completion. Thus for an affine algebraic chart, completing its base change gives exactly B_0; Lemma 4.13.1 makes this an admissible model. After inverting π, the finite coordinate comparison in the proof of Lemma 4.9.1, which uses only finiteness of L/K, identifies the algebraic tensor with the completed tensor. The displayed special-fibre formula follows from $L^{\circ\circ} = K^{\circ\circ} L^\circ$. These constructions respect affine restriction, so they glue.

For completeness, separability preserves the required reducedness directly. A reduced finite-type k-algebra R injects into the finite product of the fraction fields of its minimal-prime quotients. Tensoring with l, which is flat over k, preserves this injection. Each field tensor with $l = k[T]/(p)$ is reduced: p remains relatively prime to its derivative and hence factors into distinct irreducible factors. Thus $R \otimes_k l$ embeds in a product of reduced rings and is reduced. Finally, specialization is defined by the elements of absolute value less than one. The displayed algebra maps therefore make it commute with the projection. Taking inverse images of a singleton residual point gives the asserted tube decomposition. □

Proposition 4.22.2 (Descent of the smooth-tube contraction)**.** *Let q be a closed point where a curve model $\mathcal{X}/O$ is smooth of relative dimension one, and suppose $\kappa(q)/k$ is finite separable. Let η be its component generic point. Then*

$$\overline{]q[} =]q[\cup \{v_\eta\}.$$

This compact space admits a canonical jointly continuous strong deformation retraction onto v_η. After any finite Galois residue-field lift which splits q, it is the descent of the canonical open-disc contractions on the finitely many rational tubes. It is independent of that lift and is functorial for isomorphisms of curve models and isometric automorphisms of the ground field.

Proof. Take a finite Galois extension l/k containing the normal closure of $\kappa(q)/k$, and use Lemma 4.18.1 to obtain L/K. The separable polynomial factorization gives $\kappa(q) \otimes_k l \simeq \prod_{\iota:\kappa(q)\hookrightarrow l} l$. Consequently the points $q_1, \dots, q_m$ over q are rational smooth points of the extended model. The preceding lemma identifies the generic-fibre projection $p : X_L \to X$ and gives

$$p^{-1}(D_q) = \coprod_{j=1}^{m} D_{q_j}.$$

Each D_{q_j} has compact closure $C_j = D_{q_j} \cup \{v_j\}$ by Lemma 4.21.1. By flat going down ([StacksAlg, Tag 00HS]), each generic point above the unique component through q maps to a component generic point; its image is necessarily η. Thus $p(v_j) = v_\eta$. Continuity puts this point in $\overline{D_q}$, and Lemma 4.20.1 gives $C_q = D_q \cup \{v_\eta\}$.

The projection p is open by Theorem 4.9.2. Any continuous open surjection satisfies $p^{-1}(\overline{E}) = \overline{p^{-1}(E)}$: an open neighborhood upstairs maps to an

open neighborhood downstairs, which meets E whenever its centre lies in $\overline{E}$. Apply this to $E = D_q$ to obtain

$$p^{-1}(C_q) = \bigcup_{j=1}^{m} C_j.$$

The canonical homotopies of Theorem 4.21.2 agree on intersections of these closed sets, which consist only of fixed component points. Finite closed pasting gives a continuous homotopy on $p^{-1}(C_q)$. Its full coordinate and semilinear invariance makes it $\mathrm{Gal}(L/K)$-equivariant. The compact-to-Hausdorff surjection $p^{-1}(C_q) \to C_q$ remains a quotient map after product with $[0,1]$. Hence the homotopy descends, and its endpoint and fixed-point identities give the asserted strong contraction.

To compare two choices of lift, take a finite Galois extension of k containing both residue fields and lift it in the same way. The simple roots of each of the two defining polynomials lift in this common complete field; sending the original generators to these roots gives compatible isometric embeddings of both lifted fields. The ground-field compatibility in Theorem 4.21.2 makes the two homotopies equal after this common extension. Surjectivity of projection makes their descents equal. The same argument together with coordinate and semilinear invariance proves functoriality. □

4.23 Component stars and the global skeleton

Lemma 4.23.1 (Cofinitely many whole tubes in a vertex neighborhood)**.** *Let $\mathcal{X}$ be a curve model, let η be a component generic point of $\mathcal{X}_s$, and let $v = v_\eta$. For every ordinary open neighborhood N of v in $X = \widehat{\mathcal{X}}_\eta$, there are only finitely many closed points $q \in \overline{\{\eta\}}$ for which the whole tube $D_q = \mathrm{sp}^{-1}(q)$ is not contained in N.*

Proof. Take a finite affine cover of $\mathcal{X}$. Every member meeting $\overline{\{\eta\}}$ contains η, since an open set contains the generalizations of each of its points. Thus it suffices to prove the assertion inside a formal affine chart $\mathfrak{U} = \mathrm{Spf}\, A_0$ containing η. Write $A = A_0[1/\pi]$ and $R = A_0/\mathfrak{m}A_0$; let $P \subset R$ be the minimal prime of η.

Choose $\bar{g} \in R \setminus P$ which belongs to every other minimal prime of R. Such an element is obtained by multiplying, for each of the finitely many other minimal primes P', an element of $P' \setminus P$; if there are no others, take

$\bar{g} = 1$. The reduced ring $R_{\bar{g}}$ has the unique minimal prime $PR_{\bar{g}}$, hence that prime is zero and $R_{\bar{g}}$ is an integral domain. The complement of $D(\bar{g})$ on the component of η is finite: a nonzero regular function on a finite-type integral curve vanishes on a zero-dimensional Noetherian closed subset, which has finitely many points.

Lift $\bar{g}$ to $g \in A_0$ and use the principal formal open $B_0 = A_0\{g^{-1}\}$. Proposition 3.17.1 identifies its generic fibre with $|g| = 1$, and its special fibre is $\operatorname{Spec} R_{\bar{g}}$. Put $B = B_0[1/\pi]$. By Lemma 4.12.1, the spectral norm on B is attained by an integral presentation and its nonzero values belong to $|K^\times|$. By Proposition 4.12.2, this affinoid has the singleton Shilov boundary $\{v\}$. In particular, for $f \in A$, its restriction f_B satisfies

$$\|f_B\| = |f(v)|.$$

If this value is zero, $f_B = 0$ and the value of f is zero throughout every tube over $D(\bar{g})$. Otherwise choose $c \in K^\times$ with $|c| = |f(v)|$. Then $b = f_B/c$ belongs to B_0, and its reduction is a nonzero element of the integral curve ring $R_{\bar{g}}$. Outside its finitely many zeros, specialization gives $|b(x)| = 1$ for every point of the entire tube, and therefore

$$|f(x)| = |f(v)| \qquad (x \in D_q)$$

for all but finitely many closed q of this component in the chart. Here the whole tube lies in the chosen principal formal open by formal-open compatibility.

The topology of $\mathcal{M}(A)$ is generated by the evaluation functions $x \mapsto |f(x)|$. Choose a basic neighborhood of v inside $N \cap \mathcal{M}(A)$, using finitely many such functions. The preceding equality puts all but finitely many whole tubes in this basic neighborhood. Finally take the union of the finite exceptional sets over the chosen finite affine cover. □

Proposition 4.23.2 (A compact component star and its contraction)**.** *Let $\mathcal{X}$ be a curve model, and let S_η be the set of closed points on the component of η where $\mathcal{X}$ is smooth of relative dimension one. Assume that $\kappa(q)/k$ is separable for every $q \in S_\eta$. Then*

$$A_\eta = \{v_\eta\} \cup \bigcup_{q \in S_\eta} D_q$$

is a compact closed subspace of X. The canonical contractions of the completed tubes $C_q = D_q \cup \{v_\eta\}$ paste to a jointly continuous strong deformation retraction $H_\eta : A_\eta \times [0,1] \to A_\eta$ onto $\{v_\eta\}$.

Proof. Proposition 4.22.2 identifies the compact closure of every D_q with C_q and supplies its canonical contraction. Distinct C_q meet only at $v = v_\eta$, which their homotopies fix. Thus there is a well-defined pasted map H_η.

To prove compactness, take an open cover of A_η and a member containing v. It is the intersection with an open neighborhood of v in X. By Lemma 4.23.1, it contains all but finitely many whole D_q. Each of the finitely many remaining C_q is compact. Finite subcovers of those C_q, together with the chosen member, cover A_η. This proves compactness; closedness follows from the Hausdorff property of X.

Continuity away from v follows because every D_q is open in X. For continuity at v, let N be a relatively open neighborhood of v in A_η. There is a finite subset $E \subset S_\eta$ such that $C_q \subset N$ for $q \notin E$. For each $q \in E$, continuity on $C_q \times [0,1]$, together with compactness of $[0,1]$ and the equality $H_q(v,u) = v$, gives a relatively open neighborhood M_q of v in C_q such that

$$H_q(M_q \times [0,1]) \subset N.$$

Indeed, finitely many product neighborhoods of the compact set $\{v\} \times [0,1]$ suffice, and the intersection of their first factors is such an M_q. The finite union $F = \bigcup_{q \in E}(C_q \setminus M_q)$ is compact, closed in A_η, and does not contain v. Hence $M = A_\eta \setminus F$ is a neighborhood of v and

$$H_\eta(M \times [0,1]) \subset N.$$

This proves joint continuity uniformly in the homotopy parameter, despite the possibly infinite set S_η. The strong deformation retraction identities hold on every C_q and at v, so they hold globally. □

Theorem 4.23.3 (The skeleton of a split curve model)**.** *Let $\mathcal{X}$ be a curve model. Assume that every closed point of its special fibre is either a point where $\mathcal{X}$ is smooth of relative dimension one with finite separable residue field over k, or one of finitely many k-rational nodes q admitting a pointed diagram as in Proposition 4.20.3, with parameter $a_q \in O$, $0 < |a_q| < 1$. Then $X = \widehat{\mathcal{X}}_\eta$ has a canonical finite metric graph $\Sigma(\mathcal{X})$ with vertices v_η and one edge of length $-\log|a_q|$ for every node q. The ends of this edge attach to the component points of the two branches at q, allowing a self-loop. There is a jointly continuous strong deformation retraction*

$$H_{\mathcal{X}} : X \times [0,1] \longrightarrow X, \qquad H_{\mathcal{X}}(X,1) = \Sigma(\mathcal{X}).$$

On each nodal tube it is the radial annulus homotopy; on each completed smooth tube it is the canonical disc contraction of Proposition 4.22.2. The entire parametrized homotopy is independent of the pointed charts and commutes with model isomorphisms, including those semilinear for isometric ground-field automorphisms.

Proof. The vertex set $V = \{v_\eta\}$ is finite by Proposition 4.12.2. For every node q, let $I_q = [0, -\log|a_q|]$ and map its interior to the Gauss skeleton of D_q by the coordinate $-\log|s|$. Proposition 4.20.3 and Corollary 4.20.4 continuously extend this map to the two incident vertices, including when these coincide. Form a finite graph G from the disjoint union of V and the I_q by these endpoint identifications. Its map to X is continuous and injective: distinct interiors specialize to distinct nodes, and no interior point specializes to a component generic point. Since G is compact and X is Hausdorff, this is a homeomorphism onto its image, denoted $\Sigma(\mathcal{X})$. Give each edge its interval length. Corollary 4.16.2 makes the length independent of the pointed coordinate, using the original absolute value on K.

Every point of the special fibre is a component generic point or a closed point. Thus X is the finite union of the component stars A_η and the completed nodal tubes C_q. Proposition 4.23.2 makes the former compact, and Lemma 4.20.1 makes the latter compact. They are therefore closed in X. Their pairwise intersections consist only of incident vertices: the open tubes have distinct specializations, and their boundaries were identified in the endpoint propositions.

On each A_η use H_η, and on each C_q use Corollary 4.20.4. These homotopies fix all intersection vertices. The finite closed pasting lemma, applied to their products with $[0,1]$, gives a jointly continuous map $H_{\mathcal{X}}$. At time zero it is the identity; at time one it maps each star to its vertex and each nodal tube to its Gauss edge. It fixes all these vertices and edges at every time. These are exactly the strong deformation retraction identities.

Coordinate independence on smooth tubes is part of Proposition 4.22.2; on nodal tubes it is Corollary 4.19.1. A model isomorphism permutes residual components and closed points, sends their completed tubes to the corresponding completed tubes, and induces the indicated analytic or semilinear isomorphisms. Hence it commutes with every local homotopy, and therefore with their global pasting. □

Corollary 4.23.4 (Finite Galois descent of the model skeleton)**.** *Let L/K be a finite Galois extension of complete nontrivially valued fields, using the extension of the fixed absolute value on K, and let $G = \mathrm{Gal}(L/K)$. Suppose that a curve model $\mathcal{Y}$ over L° satisfies Theorem 4.23.3, its formal generic fibre is identified with X_L, and the natural semilinear G-action on X_L extends to $\mathcal{Y}$. Then the model homotopy descends to a jointly continuous strong deformation retraction of X onto the embedded finite graph $\Sigma(\mathcal{Y})/G$.*

Proof. The model homotopy is G-equivariant by the final assertion of Theorem 4.23.3. Theorem 4.9.2 identifies X_L/G with X and the quotient of its invariant skeleton with the skeleton's image in X. Corollary 4.9.3 then descends the entire homotopy and all strong retraction identities.

For completeness, this quotient is a finite graph. The G-action permutes vertices and edges; on an edge it is an isometry by annulus-coordinate invariance, hence, after identifying its interval, is either orientation-preserving or a reflection. Subdivide every edge at its midpoint. An element stabilizing a resulting edge cannot reverse it, because original vertices and new midpoints are distinct G-invariant classes. The quotient is therefore the finite graph whose vertices and edges are the corresponding orbits. Its edge lengths are the lengths of these subdivided intervals. Thus an inverted original edge has an image of half its length, whereas an edge with no inversion retains its total length after the unnecessary subdivision is removed. No field-degree rescaling is introduced. □

4.24 Residual functions and outgoing slopes

Let K be complete and nontrivially valued. We retain its given absolute value and write $O = K^\circ$, $\mathfrak{m} = K^{\circ\circ}$, and $k = O/\mathfrak{m}$.

Lemma 4.24.1 (Normalization points and model branches)**.** *Let D be a reduced finite-type curve over a field k, with at-worst-nodal singularities. Its normalization is the disjoint union of the normalizations C^ν of its irreducible components C. A smooth closed point q of D has one preimage on the normalization, on the unique component through q, with the same residue field. A split node q has precisely two preimages p_+, p_-, both with residue field k. The two points lie on the normalized components selected by the two local branches; they remain distinct if both branches belong to the same global component.*

For a pointed étale diagram with E affine

$$(D,q) \longleftarrow (E,u) \longrightarrow \big(\operatorname{Spec} k[s,t]/(st), o\big), \qquad \kappa(q) = \kappa(u) = k,$$

the normalization points over u identify with those over q and with the two origins of the normalized coordinate axes. Thus the branches $t = 0$ and $s = 0$ select unambiguously the two points of the normalization of D above q.

Proof. On a reduced affine chart, the total ring of fractions is the product of the fraction fields of the minimal-prime quotients. Its integral closure is the product of the integral closures in those fields; the product idempotents are integral. These descriptions localize and give the asserted component decomposition. Finiteness of normalization is [StacksAlg, Tag 0BXR]. At a smooth point the local ring is regular, and hence a normal discrete valuation ring by [StacksAlg, Tag 0B8Y]. The normalization is unchanged there. For a split node, the assertion about the two distinct rational normalization points is precisely [StacksAlg, Tag 0CBW].

Normalization commutes with the two étale maps by [StacksAlg, Tag 03GV]. To apply that relative-normalization statement, use the quasi-compact and quasi-separated map from the finite disjoint union of component generic points to the reduced curve. Its étale base change is a finite disjoint union of generic-point fields, so the relative normalizations are the ordinary ones. The fibre of each Cartesian normalization square at the rational point u is the two-point normalization fibre at its image. The normalization of $k[s,t]/(st)$ is $k[s] \times k[t]$: each factor is integrally closed, the product is finite over the original ring, and both rings have the same total ring of fractions. Its origins are the branches $t = 0$ and $s = 0$. This gives the specified identifications. □

Proposition 4.24.2 (The valued field of a component point)**.** *Let A_0 be an admissible O-algebra with reduced special fibre, let $A = A_0[1/\pi]$, and let $x = x_\eta$ be the component point corresponding to a minimal prime $P \subset \widetilde{A}$. The reduction map induces an isomorphism*

$$\operatorname{Frac}(\widetilde{A}/P) \xrightarrow{\sim} \widetilde{\mathcal{H}(x)}, \qquad |\mathcal{H}(x)^\times| = |K^\times|.$$

If η is the generic point of a residual curve component C, the left hand field is $k(C)$. In particular x has residual transcendence degree one and relative rational value-group rank zero.

Proof. By Proposition 4.12.2, the Shilov boundary consists of finitely many component points $x = x_1, x_2, \dots, x_r$, with distinct minimal reduction primes $P = P_1, P_2, \dots, P_r$. For each $j > 1$, choose $\overline{h}_j \in P_j \setminus P$ and lift their product to $h \in A_0$. If $r = 1$, take $h = 1$. Then

$$|h(x)| = 1, \qquad |h(x_j)| < 1 \quad (j > 1).$$

For $a \in A$ with $a(x) \neq 0$, choose N so large that $|a(x_j)h(x_j)^N| < |a(x)|$ for every $j > 1$. The boundary maximum formula gives $\rho_A(ah^N) = |a(x)|$. Lemma 4.12.1, including the norm-attaining lift used there, implies that this nonzero spectral value belongs to $|K^\times|$. Choose $c \in K^\times$ of this absolute value. Then $ah^N/c \in A_0$ and its reduction is outside P.

The homomorphism $\widetilde{A}/P \to \widetilde{\mathcal{H}(x)}$ is injective by the definition of P, so it extends to its fraction field. To prove surjectivity, first consider a/b with $a, b \in A$, $a(x)b(x) \neq 0$, and $|a(x)/b(x)| = 1$. Choose a single N large enough for both a and b in the preceding argument, and choose $c_a, c_b \in K^\times$ with $|c_a| = |a(x)|$ and $|c_b| = |b(x)|$. Both

$$a' = ah^N/c_a, \qquad b' = bh^N/c_b$$

belong to A_0 and have nonzero residues on C. Since $|c_a/c_b| = 1$, the residue of $a/b = (c_a/c_b)a'/b'$ lies in $\operatorname{Frac}(\widetilde{A}/P)$. Every unit of $\mathcal{H}(x)$ is within distance strictly less than one of such a fraction, because $\mathcal{H}(x)$ is the completion of $\operatorname{Frac}(A/\ker x)$. It therefore has the same residue as that fraction. This proves surjectivity.

The first paragraph gives the value-group assertion for fractions. A nonzero limit in a non-Archimedean valued field has the same absolute value as every sufficiently accurate approximant, so completion does not enlarge that value group. The final assertion follows when $\widetilde{A}/P$ has dimension one. □

Lemma 4.24.3 (Bounded expansions on pointed tubes)**.** *Let $\mathcal{U}$ be a flat finitely presented O-scheme with a rational special-fibre point u, and let $\mathcal{U} \to \mathbb{A}^1_O$ be étale at u, sending u to the special-fibre origin. Shrink $\mathcal{U}$ around u and write z for the target coordinate. Every integral function on its formal completion restricts to its tube as a power series*

$$g(z) = \sum_{j \geq 0} b_j z^j, \qquad b_j \in O, \quad |z| < 1.$$

Reducing the coefficients gives the expansion of $\overline{g}$ in the completed local ring $k[[\overline{z}]]$ of the special fibre at u.

For an étale pointed chart

$$(\mathcal{U}, u) \longrightarrow \big(\operatorname{Spec} O[s,t]/(st-a), o\big), \qquad 0 < |a| < 1,$$

with u, o rational, the corresponding assertion is that the restriction to $]u[$ is represented by

$$g = \sum_{i,j\geq 0} c_{ij} s^i t^j, \qquad c_{ij} \in O, \qquad |s| < 1, \quad |t| < 1, \quad st = a.$$

On the normalization branch $t = 0$, the reduction of g has completed expansion $\sum_{i\geq 0} \overline{c}_{i0} s^i$; on $s = 0$, it has expansion $\sum_{j\geq 0} \overline{c}_{0j} t^j$.

Proof. After a Zariski shrinking preserving the entire tube, the square Jacobian presentation of [StacksAlg, Tag 02GU] expresses the chart by a system $F(Z, W) = 0$ with coefficients in O, where $Z = z$ in the smooth case and $Z = (s, t)$ in the nodal case. Include the inverses of all localized functions as extra variables and equations. The Jacobian with respect to W is invertible at the selected residual point. In the nodal case take polynomial representatives of the coefficients before imposing $st = a$.

At $Z = 0$, the residual solution of $F(0, W) = 0$ has invertible Jacobian. Lemma 4.15.1 gives a solution $w_0 \in O^n$ lifting it. There is a unique formal solution $W(Z) \in O[[Z]]^n$ with constant term w_0. Indeed, after coefficients of degrees less than m have been chosen, the equations in degree m are linear equations with coefficient matrix $J_W F(0, w_0) \in \mathrm{GL}_n(O)$; solve them for every monomial of degree m. This recursive construction also proves uniqueness.

All coefficients lie in O, so these series converge on every closed polydisc of radii strictly less than one. Their values reduce to the prescribed point. The uniqueness in Corollary 4.11.2 identifies this analytic solution with the inverse of the pointed tube map. Choose a polynomial representative for each regular function in the presentation, including the inverse variables. Substitution gives its bounded-coefficient expansion. For an element of the π-adic completion, choose successively regular functions g_n such that $g = \sum_{n\geq 0} \pi^n g_n$. Expand each g_n in this way and sum the resulting series with the factors π^n. The coefficients converge in O, uniformly on every closed smaller polydisc. On the nodal tube the imposed relation $st = a$ ensures equality with the original function, regardless of the polynomial representatives used.

Reduction commutes with this construction. The reduced recursive solution is the inverse of the pointed étale map on completed special-fibre local

rings. In the smooth case this is $k[[z]]$. In the nodal case impose $st = 0$ and then restrict to either branch, giving the two stated expansions. Equivalently, these are the corresponding expansions on the normalization; normalization commutes with étale base change by [StacksAlg, Tag 03GV]. □

Lemma 4.24.4 (The first residual coefficient determines the slope)**.** *Let $g(z) = \sum_{j\geq 0} b_j z^j$ on $|z| < 1$, with $b_j \in O$, and suppose its reduction as a formal series is nonzero of order d. Then for all $\rho < 1$ sufficiently close to one,*

$$|g(\zeta_{0,\rho})| = \rho^d.$$

The same conclusion holds near the outer end of $|a| < |s| < 1$ for a bounded double series $g = \sum_{i,j\geq 0} c_{ij} s^i (a/s)^j$, if $\sum_i \overline{c}_{i0} s^i$ is nonzero of order d.

Proof. In the disc case, $|b_d| = 1$ and $|b_j| < 1$ for $j < d$. Choose $\rho_0 < 1$ so that $|b_j| < \rho_0^{d-j}$ for every $j < d$. For $\rho_0 < \rho < 1$, the dth term has Gauss size ρ^d, every lower term has smaller size, and every higher term has size at most ρ^{d+1}. The asserted norm follows.

For the annulus, regroup the convergent double series by the exponent of s. The summability criterion 1.1.2, applied on every closed subannulus, justifies the regrouping. Its Laurent coefficients are

$$b_n = \sum_{j\geq 0} c_{n+j,j} a^j \quad (n \geq 0), \qquad b_{-m} = a^m \sum_{i\geq 0} c_{i,i+m} a^i \quad (m \geq 1).$$

Consequently $b_n \in O$, $\overline{b}_n = \overline{c}_{n0}$ for $n \geq 0$, and $|b_{-m}| \leq |a|^m$. Choose ρ_0 as in the disc case and increase it, if necessary, so that $|a| < \rho_0^{d+1}$. For $\rho_0 < \rho < 1$ all negative terms satisfy

$$|b_{-m}|\rho^{-m} \leq (|a|/\rho)^m \leq |a|/\rho < \rho^d.$$

The positive terms are handled exactly as above, leaving the dth term as the unique maximal term. □

Theorem 4.24.5 (Residual order on a model branch)**.** *Let $\mathcal{X}$ be a flat finitely presented O-curve with reduced nodal special fibre and smooth generic fibre. Assume that its formal generic fibre X is Hausdorff. Let C be a special-fibre component, $x = x_C$ its component point, and $\nu : C^\nu \to C$ its normalization. Let $p \in C^\nu(k)$ lie over either a smooth point of $\mathcal{X}_s$, or a split node admitting a pointed chart as in Proposition 4.17.3. The pointed disc or annulus chart*

determines an outgoing Gauss segment $\gamma_p : [0,\epsilon) \to X$, *with* $\gamma_p(0) = x$ *and parameter* $\tau = -\log|z|$, *where* z *is the coordinate of the corresponding normalization branch and* $|z(x)| = 1$.

Let f *be an analytic germ at* x *with* $f(x) \neq 0$. *Choose* $c \in K^\times$ *such that* $|cf(x)| = 1$, *and let* $\widetilde{f}_x \in k(C)^\times$ *be the residue of* $cf(x)$ *under Proposition 4.24.2. For all sufficiently small* $\tau \geq 0$,

$$-\log|f(\gamma_p(\tau))| = -\log|f(x)| + \operatorname{ord}_p(\widetilde{f}_x)\,\tau.$$

In particular the outgoing slope equals $\operatorname{ord}_p(\widetilde{f}_x)$, *independently of the scalar* c *and of the pointed chart.*

Proof. We first treat an actual pointed chart on a formal affine neighborhood of the residual point. Its generic fibre is $\mathcal{M}(A)$, with reduced integral model A_0. The Gauss segment has endpoint x by Propositions 4.20.3 and 4.21.1. Choose $P, Q \in A$ with $Q(x) \neq 0$ such that the element $r(x) = P(x)/Q(x)$ of $\mathcal{H}(x)$ satisfies

$$|f(x) - r(x)| < |f(x)|.$$

This is possible because the fraction field of $A/\ker x$ is dense in $\mathcal{H}(x)$. The strict inequality persists on a neighborhood of x on which f and P/Q are analytic. The ultrametric inequality then gives $|f| = |r|$ there. Moreover $cf(x)$ and $cr(x)$ have the same residue. It therefore suffices to prove the formula for P/Q.

Apply the separator construction in the proof of Proposition 4.24.2 to P and Q, using the same function h and the same exponent N. For suitable $c_P, c_Q \in K^\times$, the functions

$$P' = Ph^N/c_P, \qquad Q' = Qh^N/c_Q$$

belong to A_0 and satisfy $|P'(x)| = |Q'(x)| = 1$. Their restrictions to C are nonzero regular functions near the selected residual point. Write d_P, d_Q for their orders at p on C^ν. By Lemmas 4.24.3 and 4.24.4, near the endpoint of the corresponding Gauss segment we have

$$|P'(\gamma_p(\tau))| = e^{-d_P\tau}, \qquad |Q'(\gamma_p(\tau))| = e^{-d_Q\tau}.$$

In the nodal case use the branch coordinate that has absolute value one at x; at the other end this exchanges s and $t = a/s$. Since $P/Q = (c_P/c_Q)P'/Q'$, subtracting the two logarithms proves the displayed formula. The quotient of the two reductions is $\widetilde{f}_x$ up to a nonzero constant, so its order is $d_P - d_Q$.

For a pointed étale diagram, pull the calculation back to its source $\mathcal{U}$. The component point selected by the chosen branch maps to x: its reduction is the corresponding generic point, whose analytic inverse image is unique by Proposition 4.12.2. Normalization commutes with this étale map by [StacksAlg, Tag 03GV]. On the normalized curves the local map at the rational branch point is therefore étale. Its fibre is a finite separable residue field by [StacksAlg, Tag 02GL], so the upstairs maximal ideal is generated by the downstairs maximal ideal. A uniformizer downstairs is consequently a uniformizer upstairs. Thus the order of the pulled-back residual function is exactly $\operatorname{ord}_p(\widetilde{f}_x)$. The pointed tube map is an analytic isomorphism, so the calculated norm is the original norm along γ_p.

Multiplying c by a scalar of absolute value one multiplies the residue by an element of $k^\times$, which has order zero. The right hand side is intrinsic to the normalized residual point; the same computation in any other pointed chart gives that same number. The logarithmic parameter is the annular length parameter of Proposition 4.19.1. □

Proposition 4.24.6 (Slopes at separable residual points)**.** *Keep the curve model and component of Theorem 4.24.5. Let p be a closed point of C^ν with $\kappa(p)/k$ separable. Choose a finite Galois residue-field lift L/K which makes the points above p rational and supplies pointed charts at any node under p. Let p' lie above p on the normalization of a component C' of the extended special fibre, let x' be its component point, and let $\gamma_{p'}$ be an outgoing Gauss segment at x'. Writing $\varpi : X_L \to X$ for projection, the projected segment $\gamma = \varpi \circ \gamma_{p'}$ is nonconstant and injective on a sufficiently short initial interval. For every analytic germ f at x with $f(x) \neq 0$,*

$$-\log|f(\gamma(\tau))| = -\log|f(x)| + \operatorname{ord}_p(\widetilde{f}_x)\,\tau \qquad (0 \leq \tau \ll 1).$$

The coefficient on the right is independent of L, of p', and of the pointed chart. The parameter uses the original absolute value on K, extended to L; no factor $[L:K]$ occurs in this slope.

Proof. Take a finite Galois residue extension containing the normal closure of $\kappa(p)/k$. If p lies over a node, enlarge it to include the finite residue extension used in Theorem 4.18.2. Its lift exists by Lemma 4.18.1. The algebraic and formal base changes agree with the analytic scalar extension by Lemma 4.22.1. Their special fibres are obtained by tensoring with $l = \widetilde{L}$, and remain reduced.

The finite separable morphism $\operatorname{Spec} l \to \operatorname{Spec} k$ is étale. By [StacksAlg, Tag 03GV], the normalization of the extended residual curve is the base change of the original normalization. The local map at p' over p is therefore étale. By [StacksAlg, Tag 02GL], its maximal ideal is generated by the image of the maximal ideal at p. Both local rings are discrete valuation rings, by [StacksAlg, Tag 0B8Y]. Consequently

$$\operatorname{ord}_{p'}(u|_{C'}) = \operatorname{ord}_p(u) \qquad (u \in k(C)^\times).$$

The generic point of C' maps to that of C, so specialization and the uniqueness of component points give $\varpi(x') = x$. The residue-field isomorphisms in Proposition 4.24.2 commute with these maps: they are induced by evaluating integral model functions and then taking residues. Thus the normalized residue of the pullback of f is the pullback of $\widetilde{f}_x$. Apply Theorem 4.24.5 at x', p' and use the displayed order equality. Evaluation of a pulled-back function has exactly its original absolute value. This proves the formula with the stated parameter and coefficient.

To see that the projected path is nonconstant and has no initial self-identifications, choose a uniformizer $u \in k(C)^\times$ at p. Write $u = \overline{P}/\overline{Q}$ in the fraction field of a residual affine chart, with both numerator and denominator nonzero on C, and lift them to integral functions P, Q. Then $r = P/Q$ is analytic near x, $|r(x)| = 1$, and its normalized residue is u. The already proved formula for r gives $|r(\gamma(\tau))| = e^{-\tau}$ on a sufficiently short interval. These values distinguish its parameters. □

Corollary 4.24.7 (Residual-degree weighted balancing)**.** *Assume that k is perfect and that the residual component C is proper over k. For a nonzero analytic germ f at its component point x, with $f(x) \neq 0$, index the model branches at x by the closed points p of C^ν and write $d_pF(x)$ for the slopes of $F = -\log|f|$ constructed above. Then only finitely many of these slopes are nonzero and*

$$\sum_{p \in C^\nu \text{ closed}} [\kappa(p) : k]\, d_pF(x) = 0.$$

If k is algebraically closed, every weight is one. For a general perfect residue field, the weights are the degrees over the original k, not over a splitting field.

Proof. The normalization is finite by [StacksAlg, Tag 0BXR]. Hence C^ν is a proper normal integral curve, with discrete valuation local rings at closed

points by [StacksAlg, Tag 0B8Y]. Perfection makes all its finite residue extensions separable, so Proposition 4.24.6 applies to every closed point and gives

$$d_p F(x) = \operatorname{ord}_p(\widetilde{f}_x).$$

A rational function on a Noetherian integral curve is a unit on a dense open subset, whose complement is a finite set of closed points. This proves the finiteness assertion. The degree-zero formula for its principal divisor is exactly [StacksAlg, Tag 02RU], applied over the original field k to the proper curve C^ν. It gives the stated weighted sum. □

References. The residual-curve formulation follows the organization of [BPR13, §5.13 and Theorem 5.15]. The analytic order comparison above is proved from bounded pointed-chart expansions and the internal annulus norm calculations. The algebraic balancing step uses the proved Stacks principal-divisor degree theorem, with the original residue-field degrees retained.

4.25 Residual branches as analytic directions

Lemma 4.25.1 (The radial germ of an open disc)**.** *Let D be an open unit disc with coordinate T, and let ζ_r, $0 < r < 1$, be its centered Gauss points. An analytic isomorphism between two such discs sends ζ_r to the centered radius-r Gauss point for all r sufficiently close to 1. On a sufficiently thin outer collar it also preserves the function $|T|$. These assertions hold for semilinear isometric isomorphisms and commute with complete isometric ground-field extension. The open disc and each collar $\{r < |T| < 1\}$ are path connected.*

Proof. First suppose that the ground valuation is nontrivial. Write an isomorphism as $g(T) = \sum a_n T^n$. The proof of Theorem 4.21.2 gives $|a_0| < 1$, $|a_1| = 1$, and $|a_n| \leq 1$ for $n \geq 2$. For $|a_0| < |T(x)| < 1$, the linear term uniquely dominates all others, so $|g(x)| = |T(x)|$. The same statement for the inverse makes the corresponding collars identical after increasing their lower radius if necessary. The homotopy identity gives

$$g(\zeta_r) = g(H_D(0, r)) = H_D(g(0), r) = \zeta_r \qquad (r > |a_0|).$$

The last equality is invariance of the radius-r Gauss seminorm under translation by an element of absolute value at most r. It follows directly by applying the ultrametric inequality first to translation and then to its inverse. This argument also gives $H_D(x, r) = \zeta_r$ whenever $|T(x)| \leq r$. Hence any two points

of D can be joined, using H_D up to a common time $r < 1$, to the same ζ_r. For a collar, the radial annulus homotopy joins each point to the Gauss point of the same radius, and the Gauss interval joins any two of these points inside the collar. These are continuous paths by Proposition 4.3.1. All formulas preserve the absolute value under semilinear isometries and agree on original polynomials after scalar extension, proving the remaining assertions.

It remains to treat a trivially valued K. Choose $0 < \eta < 1$. Then (η) is free over K, and Lemma 1.16.1 supplies a complete isometric extension $L = K_\eta$ with a Laurent parameter of absolute value η. The valuation on L is nontrivial. An isometric automorphism of K extends coefficient by coefficient to L, fixing that parameter. This extension preserves the weighted Laurent norm and its inverse is constructed in the same way.

For either given coordinate, take real radii $s_i \uparrow 1$, with $0 < s_i < 1$. The interiors of the closed discs of radii s_i cover the open disc. In the actual completed tensor presentation their algebras satisfy

$$K\langle s_i^{-1}T\rangle \mathbin{\widehat{\otimes}}_K L = L\langle s_i^{-1}T\rangle.$$

Indeed, the weighted Laurent coefficient description of Theorem 1.16.2 for the free radius η on the left is the weighted coefficient description for the same two indices on the right. In both descriptions the norm is the supremum with weights $s_i^n\eta^j$ and the weighted coefficient family tends to zero. The identifications commute with restriction to smaller discs and the radii are cofinal at every point with $|T| < 1$ over both fields.

Apply Theorem 2.40.12 to each open disc with its given coordinate. The scalar projections are continuous and surjective and commute with the given isomorphism and its extension. For a semilinear isomorphism, use the coefficient automorphism of L just described. Since projection restricts a seminorm to the original coefficients, the function $|T|$ pulls back to the same coordinate norm. The inverse image of each collar $q < |T| < 1$ is therefore the corresponding collar. The radius-r centered Gauss point over L projects to the radius-r centered Gauss point over K, by its formula on $K[T]$.

Apply the nontrivially valued case to the lifted isomorphism. It identifies the two collars and preserves their coordinate norms as soon as q bounds the absolute values at zero of the coordinate series of the isomorphism and its inverse. Both constants lie in K and have norm less than one, so they vanish in the trivially valued case. Surjectivity on each collar, and the commutative square with the scalar projections, give these assertions downstairs. Applying

the same square to the centered Gauss points gives their asserted images. For any two points of the disc or of a collar, choose their two lifts individually by surjectivity and join the lifts upstairs. The projection of that path joins the given endpoints. This uses only the surjectivity of the scalar projection. For any further complete isometric scalar extension the same projection identities on coordinate norms and on the Gauss seminorms follow by restriction of seminorms. These identities give scalar compatibility for the asserted collars and centered points. □

Proposition 4.25.2 (The canonical germ of a separable smooth tube)**.** *In the setting of Proposition 4.22.2, write $D_q =]q[$ and $C_q = D_q \cup \{v\}$. There is a canonical germ of an embedded Gauss ray approaching v from D_q, parametrized near v by $\ell = -\log r \geq 0$. There are also path-connected open collars $B_q(r) \subset D_q$, for $r < 1$ sufficiently close to 1, such that $\{v\} \cup B_q(r)$ form a neighborhood basis of v in C_q. The tube D_q is path connected. The germ and its parameter are independent of the finite residue-field lift and disc coordinates used to construct them.*

Proof. Use a finite Galois residue-field lift L/K splitting q, as in Proposition 4.22.2. Write $G = \mathrm{Gal}(L/K)$ and $p^{-1}(D_q) = \coprod_j D_j$, where D_j is an open unit disc, and $C_j = D_j \cup \{v_j\}$. The group G acts transitively on the residual points q_j: they are the embeddings of $\kappa(q)$ into the chosen normal residue extension, and residue-field Galois automorphisms lift by Lemma 4.18.1. Thus $p(D_j) = D_q$ and $p(C_j) = C_q$ for each j. In particular D_q is path connected by the preceding lemma.

Choose a coordinate T_j on every D_j. There are only finitely many transition isomorphisms induced by G. Lemma 4.25.1 therefore supplies a common $r_0 < 1$ such that, for all $r > r_0$, each transition preserves the entire collar $|T_j| > r$ and the centered Gauss point $\zeta_{j,r}$. Consequently

$$B_q(r) = p(\{|T_j| > r\}), \qquad \gamma_q(r) = p(\zeta_{j,r})$$

are independent of j. The union of the collars upstairs is G-invariant, and its complement after adjoining all v_j is the finite union of the compact closed subdiscs $|T_j| \leq r$. By the quotient identification of Theorem 4.9.2, $\{v\} \cup B_q(r)$ is open in C_q. Each $B_q(r)$ is a continuous image of an open annulus, hence path connected. For any neighborhood N of v in C_q, its inverse image contains all v_j. The rational tube neighborhood basis of Lemma 4.21.1 puts

all their sufficiently thin collars inside $p^{-1}(N)$. There are finitely many j, so one common r works. This proves the neighborhood-basis assertion.

Set $\gamma_q(1) = v$. The endpoint result makes $\gamma_q : (r_0, 1] \to C_q$ continuous. If two of its values with parameters greater than r_0 coincide, the orbit description of analytic fibres in Theorem 4.9.2 sends one chosen Gauss point to the other. Every such group element preserves its radius on these collars, so the two parameters agree. The function $|T_j|$ descends to a continuous function on $B_q(r_0)$ and extends by value 1 at v, by the collar neighborhood basis. Its restriction is the inverse parameter of γ_q, proving that the ray is embedded.

Changing the finitely many disc coordinates preserves these radii and rays after increasing r_0, by Lemma 4.25.1. Two residue-field lifts embed into a common such lift as in the proof of Proposition 4.22.2. Gauss points and collars commute with scalar projection, so the germs agree in that common comparison. The length parameter is $\ell = -\log r$ throughout; the residue extension does not alter the fixed absolute value. □

Theorem 4.25.3 (The residual branch star)**.** *Assume Theorem 4.23.3. Let C be a special-fibre component, let $v = v_C$, and let C^ν be its normalization. There is an ordinary open neighborhood W_v of v whose punctured connected components are in natural bijection with the closed points of C^ν. At a smooth point of the special fibre the corresponding component is its whole open tube. At a point above a node it is a thin annular collar of the selected node branch. A self-node contributes two distinct components.*

These components remain naturally indexed by the same closed points in a cofinal system of neighborhoods of v, obtained by making the finitely many nodal collars thinner and replacing finitely many smooth tubes by outer collars. Thus the germs of punctured-neighborhood components, or outgoing directions at v, identify canonically with the closed points of C^ν. Each has the coordinate- independent Gauss-ray representative described above.

Proof. Use the finite closed cover of X by the component stars and completed nodal tubes in the proof of Theorem 4.23.3. Include the whole star A_C. For every node incident to C, include a sufficiently thin open radial collar at each of its ends attached to v, together with v. For a self-node choose two disjoint collars inside its open annulus. Include no other vertices or component stars. Call the union W_v.

Its intersection with each component star is either the whole A_C or empty. Its intersection with a completed nodal tube is an open endpoint neighbor-

hood, the union of two disjoint endpoint neighborhoods after their common vertex is identified, or empty. These sets are relatively open by Proposition 4.20.3 and the compact quotient in Corollary 4.20.4. A subset whose intersections with every member of a finite closed cover are relatively open is open: its complement is a finite union of closed subsets. Hence W_v is open.

After deleting v, its pieces are precisely the smooth tubes on C and the chosen annular collars. Each piece is open, and its complement is the union of the other open pieces, so it is also closed in $W_v \setminus \{v\}$. Each is path connected by Proposition 4.25.2 and Lemma 4.25.1. They are therefore exactly the connected components. Lemma 4.24.1 identifies their labels with all closed points of C^ν, including two distinct normalization points for a self-node.

Finally let N be any neighborhood of v in X. Lemma 4.23.1 puts all but finitely many smooth tubes on C inside N. For each of the remaining tubes use the collar neighborhood basis in Proposition 4.25.2; for each incident nodal end use the full radial tails of Proposition 4.20.3. Shrinking these finitely many collars yields a neighborhood contained in $N \cap W_v$. It is open by the same finite-cover argument, since deleting a compact inner portion from finitely many smooth tubes is open on A_C. Its punctured pieces are again path connected, with unchanged labels. This proves cofinality and makes all transition maps on component labels the identity. The claimed direction correspondence and Gauss-ray representatives follow. □

4.26 Marked rays and rational functions

Proposition 4.26.1 (A punctured smooth tube)**.** *Let $\mathcal{X}/O$ be a curve model, smooth of relative dimension one at a rational special-fibre point q. Let $D =]q[$, let v be its component point, and let $a \in D(K)$. There is a pointed disc coordinate z with $z(a) = 0$, identifying D with $|z| < 1$. The ray*

$$R_a = \{v\} \cup \{\zeta_{0,\rho} : 0 < \rho < 1\}, \qquad \tau = -\log \rho \in [0, \infty)$$

is intrinsic to the pointed tube (D, a). The radial annulus homotopy on $D \setminus \{a\}$ extends by fixing v to a jointly continuous strong deformation retraction of $(D \cup \{v\}) \setminus \{a\}$ onto R_a.

Proof. Take the pointed smooth coordinate supplied by Proposition 4.17.2. Its value at a belongs to $K^{\circ\circ}$, so subtracting that value retains the unit-disc coordinate and puts a at zero. The complement is the standard open annulus

$0 < |z| < 1$. Corollary 4.19.1 supplies its radial homotopy. It preserves $|z|$ at every time. By Lemma 4.21.1, every net in D whose radii tend to one converges to v, uniformly over each outer radial tail. This proves joint continuity at all (v, u), $0 \le u \le 1$, after setting $H(v, u) = v$.

A change between two centered disc coordinates restricts to an isomorphism of $0 < |z| < 1$. The radius transformation in Corollary 4.19.1 must be $\rho \mapsto \rho$: it has the form $|c|\rho^{\pm 1}$ and maps $(0, 1)$ onto itself. That corollary identifies both the Gauss ray and the entire homotopy. The logarithmic parameter is therefore unchanged. □

Theorem 4.26.2 (The skeleton of a marked model)**.** *Let $\mathcal{X}$ satisfy Theorem 4.23.3. Choose finitely many distinct K-points $a_1, \dots, a_m$ whose specializations $q_1, \dots, q_m$ are distinct rational smooth points of the special fibre. Put $P = \{a_1, \dots, a_m\}$ and define*

$$\Sigma(\mathcal{X}, P) = \Sigma(\mathcal{X}) \cup \bigcup_{i=1}^{m} R_{a_i}.$$

This is the finite model graph with one infinite metric ray attached at the component point of each q_i. It is a strong deformation retract of $X \setminus P$, canonically relative to the marked model. Its retraction agrees with the model retraction outside the marked smooth tubes and with $y \mapsto \zeta_{0,|z_i(y)|}$ on their punctured interiors.

Proof. Let $D_i =]q_i[$ and $C_i = D_i \cup \{v_i\}$. Each C_i is closed in X by Lemma 4.21.1. The space $X \setminus P$ has the finite closed cover consisting of

$$X \setminus \bigcup_i D_i, \qquad C_i \setminus \{a_i\} \quad (1 \le i \le m).$$

On the first set restrict the model homotopy. That set is preserved because each unmarked smooth tube, each nodal tube, and every vertex are preserved. On $C_i \setminus \{a_i\}$ use Proposition 4.26.1. The intersections consist only of fixed vertices, so finite closed pasting, also after product with $[0, 1]$, gives a jointly continuous homotopy. Its initial map is the identity, its final image is the stated graph with rays, and it fixes that image pointwise. These are the strong retraction identities.

The coordinate $\tau = -\log |z_i|$ gives the ray its ordinary interval topology and length. As $\tau \downarrow 0$ the ray tends to v_i; as $\tau \to \infty$ it tends to a_i in

X, by the disc seminorm formula. Adjoin each a_i as the other endpoint of its compactified ray. The resulting finite compact gluing maps continuously and injectively into the Hausdorff space X, hence is homeomorphic to its image. Removing the a_i gives precisely the asserted finite graph with infinite rays. The local coordinate independence in Proposition 4.26.1 and the model homotopy's invariance prove the final assertion. □

Theorem 4.26.3 (Factorization and orders at the marked ends)**.** *Assume in addition that $\mathcal{X}$ is projective over O, that k is perfect, and that its generic fibre is a smooth integral curve X_K. Identify its analytic generic fibre with X_K^{an} by Proposition 4.13.2. Let $f \in K(X_K)^\times$, and suppose the support of $\mathrm{div}(f)$ is contained in the marked set P. Writing r_P for the retraction onto $\Sigma(\mathcal{X}, P)$, one has*

$$F = -\log|f| = F \circ r_P \qquad \text{on } X_K^{\mathrm{an}} \setminus P.$$

The restriction of F is affine with integral slope on each finite edge and on each marked ray. On the ray toward a_i, its outgoing slope is $\mathrm{ord}_{a_i}(f)$. At every component vertex it satisfies the residual-degree weighted balancing identity of Corollary 4.24.7.

Proof. The rational function is an algebraic unit off P, because the local rings of the smooth curve are discrete valuation rings, [StacksAlg, Tag 0B8Y]. Analytification therefore makes it an analytic unit on $X_K^{\mathrm{an}} \setminus P$. On every nodal tube, Corollary 4.10.3 gives $|f| = |c||z|^n$ with $n \in \mathbb{Z}$. Thus F factors through the annular retraction and is affine of integral slope on its edge.

On an unmarked smooth tube with rational residue point, the same corollary makes $|f|$ constant. For a separable nonrational residue point, split its residue field as in Proposition 4.22.2. The pullback of f has constant norm on each resulting disc, equal by endpoint continuity to its value at the corresponding component point. Those component points all map to the original vertex, so all these constants equal $|f(v)|$. Surjectivity of projection shows that $|f|$ has this constant value on the original tube.

Fix a marked tube. Its coordinate z_i can be chosen from a smooth algebraic chart and centered at a_i, as in Proposition 4.26.1. The generic-fibre chart is étale at a_i and has residue field K. By [StacksAlg, Tag 02GU(6)], z_i generates the maximal ideal of the curve's local ring at a_i. It is therefore a uniformizer. Put $m_i = \mathrm{ord}_{a_i}(f)$. The quotient $g_i = f/z_i^{m_i}$ is an algebraic unit near a_i; on

the remainder of its tube it is an analytic unit, because z_i has no other zero there and f has no other zero or pole there. These expressions agree on their overlap, so g_i is an analytic unit on the whole tube. Hence

$$|f(y)| = |g_i(a_i)|\,|z_i(y)|^{m_i}, \qquad F|_{R_{a_i}}(\tau) = m_i\tau - \log|g_i(a_i)|.$$

This proves both the factorization on the marked tube and the exact order at its end. The displayed local formulas cover $X \setminus P$ and agree at the component vertices by continuity, proving the global factorization. Each residual component is proper, since it is a closed component of the projective special fibre. Thus Corollary 4.24.7 applies at every vertex and proves the last assertion. □

4.27 Inserting vertices by explicit blowups

Let K be complete and nontrivially valued. We write $O = K^\circ$, $\mathfrak{m} = K^{\circ\circ}$ and $k = O/\mathfrak{m}$. A nodal curve model in this section is a flat finitely presented O-scheme with smooth generic fibre, reduced nodal special fibre, and Hausdorff formal generic fibre.

Lemma 4.27.1 (A symmetric subdivision of a nodal annulus)**.** *Let $b, \lambda \in \mathfrak{m} \setminus \{0\}$ with $|b| \le |\lambda|^2$, and put $c = b/\lambda^2 \in O$. The blowup of (λ, s, t) in $\operatorname{Spec} O[s,t]/(st-b)$ is covered by the three flat finitely presented charts*

generator	*chart ring*	*map to the original chart*
λ	$O[u,v]/(uv-c)$	$s = \lambda u, \quad t = \lambda v$
s	$O[s,w]/(sw-\lambda)$	$t = \lambda c w$
t	$O[t,w']/(tw'-\lambda)$	$s = \lambda c w'$.

Its formal completion has the same analytic generic fibre as the original model. On the original annular skeleton it inserts the radii $|s| = |\lambda|$ and $|s| = |b|/|\lambda|$. These are two new component points if $|b| < |\lambda|^2$, and one if $|b| = |\lambda|^2$. The resulting edge lengths are, respectively,

$$-\log|\lambda|, \quad \log\frac{|\lambda|^2}{|b|}, \quad -\log|\lambda|, \qquad \text{or} \qquad -\log|\lambda|, \quad -\log|\lambda|.$$

Proof. Set $A = O[s,t]/(st-b)$. Its normal form with monomials $1, s^i, t^j$ for $i, j > 0$ embeds A into $K[s, s^{-1}]$. The affine blowup chart for λ is

$A[s/\lambda, t/\lambda] \subset A[1/\lambda]$. It is exactly $O[u,v]/(uv-c)$: the displayed map is surjective onto that subring, and its injectivity follows from the same normal form, now with $u = s/\lambda$ and $v = c/u$.

For the s-chart, put $w = \lambda/s$. Inside $A[1/s]$ one has

$$t = \frac{b}{s} = \lambda c w, \qquad \frac{t}{s} = cw^2.$$

Consequently $A[\lambda/s, t/s]$ equals the subring generated by s, w, with sole relation $sw = \lambda$. The normal form again proves injectivity. Interchanging s, t gives the third chart. Each chart is a free O-module with the indicated normal-form basis, so is flat; its displayed presentation is finite. These computations identify the actual blowup subrings, not unsaturated presentations with extra vertical torsion. Their completions are the admissible charts of Theorem 3.15.1.

If $|c| < 1$, the middle special chart is $uv = 0$. The two outer special charts are $sw = 0$ and $tw' = 0$. On the first overlap $u = w^{-1}$; on the second $v = (w')^{-1}$. Thus the two new components are projective lines, joined to each other and to the two old branches in a chain. If $|c| = 1$, the middle special chart is $uv = \bar{c}$, and its two ends glue to the two outer affine pieces to give a single projective line between the old branches. The special fibres are reduced and nodal in both cases.

Theorem 3.15.2 identifies the completed generic charts with their rational domains. Lemma 3.20.3 then identifies the new component points with the stated Gauss radii. In the strict-inequality case the middle annulus has parameter c; the outer annuli have parameter λ. In the equality case the middle generic chart is a circle. The logarithmic length formulas follow, and their sum is $-\log|b|$. □

Lemma 4.27.2 (An intrinsic centre at a split node)**.** *Assume K is algebraically closed. Let q be a node of a nodal curve model $\mathcal{X}$. Choose a pointed* étale *chart with equation $st = b$, as in Theorem 4.15.3. For each $\rho \in |K^\times|$ with $|b| < \rho < 1$ there is an admissible blowup of $\widehat{\mathcal{X}}$, induced by an algebraic blowup supported at q, whose new nodal model contains the two annular radii ρ and $|b|/\rho$ as component points, counted only once if they coincide. All old component points are retained. A projective model remains projective.*

Proof. Shrink an algebraic neighborhood U of q so that q is its only nonsmooth special point. Set

$$\mathcal{J} = \mathrm{Fitt}_1\, \Omega_{U/O}.$$

The module of differentials is finitely presented, and $\mathcal{J}$ is a finite-type quasi-coherent ideal [StacksAlg, Tag 0C3E]. For an étale map, differentials pull back isomorphically [StacksAlg, Tag 04BX], and Fitting ideals commute with pullback [StacksAlg, Tag 0C3D]. The standard nodal presentation

$$O[s,t]/(st-b) \xrightarrow{(t,s)} \big(O[s,t]/(st-b)\big)^{\oplus 2} \longrightarrow \Omega_{(O[s,t]/(st-b))/O} \longrightarrow 0$$

therefore identifies the pullback of $\mathcal{J}$ with (s,t) in every chosen pointed nodal chart.

Choose $\lambda \in K^\times$ with $|\lambda| = \max(\rho, |b|/\rho) < 1$. Then $|b| \le |\lambda|^2$. On U use the ideal $\mathcal{I} = (\lambda) + \mathcal{J}$. It is the unit ideal on $U \setminus \{q\}$: on the generic fibre λ is invertible, and at a smooth special point the differential module is free of rank one, so $\mathcal{J}$ is the unit ideal. Hence $\mathcal{I}$ glues with the unit ideal on $\mathcal{X} \setminus \{q\}$. This gives a global finite-type ideal supported at q and containing (λ). A power of any fixed topologically nilpotent π lies in (λ), so its completed ideal is open.

Blowup commutes with flat, in particular étale, base change [StacksAlg, Tag 0805]. Thus its pullback to the auxiliary pointed chart is exactly the blowup of (λ, s, t) in Lemma 4.27.1. This use of the Fitting ideal makes the centre descend even when both local branches belong to the same global component. The chosen auxiliary neighborhood can be shrunk so its fibre over q is just the specified rational point. Its pullback over the exceptional fibre is consequently an isomorphism on that fibre. The new components and their attachment points are therefore precisely those computed in the standard charts.

The explicit charts are flat with reduced nodal special fibres and smooth generic fibres. These properties descend through the indicated étale neighborhoods; flatness is fpqc local on the source [StacksAlg, Tag 036K], and local finite presentation is fppf local on the source [StacksAlg, Tag 036N]. The blowup is quasi-compact and separated over the original finite-presentation model, hence the new model is finitely presented. Its formal generic fibre is unchanged by Lemma 3.17.3, so remains Hausdorff. Away from q the model is unchanged, including each old component generic point. The chart radii and lengths are those of Lemma 4.27.1. Finally the blowup is projective [StacksAlg, Tag 02NS]; if $\mathcal{X}$ is projective over O, the composite is projective by [StacksAlg, Tag 0C4P], since $\operatorname{Spec} O$ is quasi-compact and quasi-separated. □

Lemma 4.27.3 (Insertion inside a smooth residual tube)**.** *Let q be a rational smooth special point of a nodal curve model $\mathcal{X}$, and let z be the open-disc coordinate supplied by an algebraic smooth chart in Corollary 4.17.2. If $a, \lambda \in K$ satisfy*

$$|a| < 1, \qquad 0 < |\lambda| < 1,$$

then an algebraic blowup supported at q gives a new nodal model containing $\zeta_{a,|\lambda|}$ as a component point. It retains all old component points and adds one projective-line special component, attached at the old component through q. Its new edge has length $-\log|\lambda|$. Projectivity of $\mathcal{X}/O$ is preserved.

Proof. Translate the algebraic smooth coordinate by a lift of its residue value at q, so that its tube coordinate is z. Shrink its Zariski chart until q is the only special point with $\bar{z} = 0$; the fibre of an étale morphism over that point is finite on the chosen affine chart. The ideal $(\lambda, z - a)$ is then the unit ideal off q. Glue it with the unit ideal on $\mathcal{X} \setminus \{q\}$. It is finite type and its formal completion is open.

Flat base change of the algebraic blowup [StacksAlg, Tag 0805] reduces the calculation to Corollary 3.20.4, with $T = z - a$ and scale λ. The charts are

$$T = \lambda u, \quad O[u], \qquad \text{and} \qquad O[T, v]/(Tv - \lambda).$$

Their completions, exceptional projective line, reduced nodal special fibre and generic-fibre identification are exactly those proved there. Pullback to the chosen étale chart introduces no extra exceptional components: its fibre over the selected rational point is a single rational point after the preceding shrinking. The exceptional component point is the Gauss point of $|z-a| \leq |\lambda|$, and its attaching annulus is $|\lambda| < |z - a| < 1$. The old model is unchanged away from q. Flatness and finite presentation follow as in Lemma 4.27.2, while [StacksAlg, Tags 02NS and 0C4P] preserves projectivity. □

Theorem 4.27.4 (Realization of finitely many type II points)**.** *Assume K is algebraically closed, complete and nontrivially valued. Let $\mathcal{X}$ be a nodal curve model and put $X = \widehat{\mathcal{X}}_\eta$. For every finite set S of type II points of X, a finite composition of admissible blowups of $\widehat{\mathcal{X}}$, induced by algebraic blowups of $\mathcal{X}$, yields a nodal curve model whose component vertex set contains S and all the old component vertices. If $\mathcal{X}$ is projective over O, the new model is projective over O.*

Proof. By the tube decomposition in Theorem 4.23.3, a point not already a component vertex belongs to a smooth open-disc tube or to a nodal open-annulus tube. In either chart the exact classification of Theorem 4.1.1 represents a type II point as $\zeta_{a,r}$ with $r \in |K^\times|$.

In a smooth tube, membership in the open unit disc gives $|a| < 1$ and $r < 1$. Choose $|\lambda| = r$ and apply Lemma 4.27.3. In a nodal tube $|b| < |s| < 1$, first suppose that the point is radial, namely $\zeta_{0,\rho}$. Apply Lemma 4.27.2 at radius ρ.

For a nonradial point $\zeta_{a,r}$ in that annulus one has $r < |a|$; otherwise the disc is also centred at zero. Here $|b| < |a| < 1$. Insert the radial point of radius $|a|$ first. In the new component its residue class is a nonzero finite value of the scaled coordinate, hence is a smooth point rather than one of the two attaching nodes. The point $\zeta_{a,r}$ now lies in that smooth residual tube, so a second application of Lemma 4.27.3 inserts it. Thus each desired point is inserted in at most two steps, possibly with one extra reflected annular vertex. Applying this procedure successively to S terminates; every step preserves the previously obtained vertices and the stated model properties. □

Theorem 4.27.5 (A model adapted to finitely many marked points)**.** *Under the hypotheses of Theorem 4.27.4, let $P \subset X(K)$ be finite. There is a model obtained by finitely many of the preceding admissible blowups such that the points of P specialize to pairwise distinct smooth closed points of the special fibre. The construction can retain any finite set of previously specified type II vertices. It preserves projectivity.*

Proof. A K-rational analytic point cannot specialize to a component generic point, since Lemma 4.24.2 gives the latter a residual function field of transcendence degree one, whereas the rational point has completed residue field K. Thus every point of P lies in a closed-point tube. If a marked point belongs to a nodal tube $st = b$, insert the radial point with radius $|s(a)|$. Its reduction on the new component has nonzero finite scaled coordinate, so is smooth. Points already specializing to smooth points are unchanged by a blowup supported at a node. Repeating at most $|P|$ times puts every marked point in a smooth tube.

Suppose a smooth tube contains distinct marked points with coordinates $a_1, \dots, a_m$, where $m \geq 2$. Set

$$R = \max_{i,j} |a_i - a_j| \in |K^\times|, \qquad 0 < R < 1.$$

Insert $\zeta_{a_1,R}$. Every marked point lies in the closed disc $|z - a_1| \leq R$, so none lies in the new attaching annulus. On the exceptional affine line their specializations are the residues of $(a_i - a_1)/\lambda$, where $|\lambda| = R$. At least two of these residues differ: one is zero and another has absolute value one. The points are therefore partitioned into at least two nonempty groups, each lying in a distinct smooth residual tube. Repeat on any group of size greater than one. Each step increases the number of nonempty groups and never exceeds $|P|$, so the process terminates with distinct smooth specializations. All preceding blowups retain old vertices; insert any additional prescribed type II vertices first. □

4.28 Smooth tubes with arbitrary residue extensions

Lemma 4.28.1 (Scalar extension of algebraic formal models)**.** *Let L/K be a complete isometric valued extension and let $\mathcal{U}$ be a flat finitely presented O-scheme. Put $O' = L^\circ$, $l = \widetilde{L}$, and $\mathcal{U}' = \mathcal{U} \times_O O'$. Completion uses the same $\pi \in O$, $0 < |\pi| < 1$, on both sides. There are canonical identifications*

$$(\widehat{\mathcal{U}'}_\eta) \simeq (\widehat{\mathcal{U}}_\eta)_L, \qquad \mathcal{U}'_s = \mathcal{U}_s \times_k l,$$

compatible with specialization, formal-open restriction, and successive scalar extensions. In particular a geometrically reduced special fibre remains reduced after this operation.

Proof. Work first with $B = O[T_1, \dots, T_n]/(f_1, \dots, f_s)$. Base change preserves flatness and finite presentation. Lemma 4.13.1, applied over O and over O', gives generic algebras

$$A = K\langle T\rangle/(f_1, \dots, f_s), \qquad A' = L\langle T\rangle/(f_1, \dots, f_s).$$

Proposition 2.33.5 identifies A' with $A\widehat{\otimes}_K L$, with the actual extended relation ideal. Reduction of a restricted series modulo the maximal ideal leaves a polynomial: its coefficients tend to zero, so only finitely many have nonzero reduction. Conversely a restricted series with zero reduction has a largest coefficient absolute value less than one and belongs to the maximal ideal times the restricted series ring. Thus the special-fibre quotients are $k[T]/(\bar{f}_i)$ and $l[T]/(\bar{f}_i)$, which gives the second identification.

Principal affine restrictions have the common relation $bU - 1$ by Lemma 4.13.1; their generic domains are $|b| = 1$ on both sides. The local identifications therefore agree on restrictions and glue, using Theorem 2.40.12. Specialization commutes because its prime is defined by the integral functions with value less than one, and those values are unchanged by the isometric coefficient map. All comparisons agree on polynomial coordinates, so their composites for successive extensions agree as well. The last assertion is the definition of geometric reducedness. □

Lemma 4.28.2 (Lifting an arbitrary finite residue extension)**.** *For every finite extension l/k, there is a finite valued extension F/K with $\widetilde{F} \simeq l$ over k and*

$$[F : K] = [l : k], \qquad |F^{\times}| = |K^{\times}|.$$

If $\operatorname{char} K = 0$*, the extension F/K is separable.*

Proof. For a simple extension $l = k(\beta)$, use the construction in Lemma 4.15.2: lift the monic irreducible polynomial of β, adjoin a root, and apply the value-group and residue-degree inequality. Every step through the displayed chain of degree inequalities in that proof uses only irreducibility, so it gives the asserted residue field, degree, and value group for this simple extension. A general finite extension has a finite tower of simple extensions. Apply the construction successively, using completeness and the unique extending absolute value from Lemma 1.12.1 at every step. Degrees multiply and the value group does not change. Finally, in characteristic zero an irreducible polynomial has nonzero derivative relatively prime to it, so all finite extensions are separable. □

Lemma 4.28.3 (Galois rationalization of a residue-point support)**.** *Assume* $\operatorname{char} K = 0$*. For every finite extension κ/k, there exists a finite Galois extension N/K such that all points of* $\operatorname{Spec}(\kappa \otimes_k \widetilde{N})$ *are $\widetilde{N}$-rational and* $\operatorname{Gal}(N/K)$ *acts transitively on this finite set of points.*

Proof. By [StacksAlg, Tag 030K], write $\kappa/\kappa_s/k$ with κ_s/k separable and κ/κ_s purely inseparable. Choose a finite Galois extension l_s/k containing the normal closure of κ_s/k. There is a finite purely inseparable extension l/l_s containing every conjugate of κ over k: for finitely many generators β_i of κ, choose powers $\beta_i^{p^{e_i}} \in \kappa_s$, and adjoin to l_s the unique p^{e_i}-th roots of their images under all embeddings $\kappa_s \hookrightarrow l_s$. When $\operatorname{char} k = 0$, take $l = l_s$. The field

l therefore contains the image of every k-embedding of κ into an algebraic closure.

Lift l_s/k to a finite Galois valued extension E/K by Lemma 4.18.1. Over E, apply Lemma 4.28.2 to l/l_s, obtaining F/E with residue field l. Since K has characteristic zero, F/K is separable. Adjoining the finitely many roots of minimal polynomials of generators of F/K gives a finite normal extension N/K containing F, by [StacksAlg, Tag 09HU]. It is Galois, and $\widetilde{N}$ contains l.

Any residue field of the finite $\widetilde{N}$-algebra $\kappa \otimes_k \widetilde{N}$ embeds in an algebraic closure of $\widetilde{N}$. Its image is generated by $\widetilde{N}$ and a k-embedded copy of κ. The latter is already in l, so that residue field is $\widetilde{N}$. Thus the points are precisely the distinct k-embeddings of κ into $\widetilde{N}$. Restriction from $\mathrm{Gal}(N/K)$ onto $\mathrm{Gal}(E/K)$ is surjective by [StacksAlg, Tag 0BME]; the latter group induces $\mathrm{Gal}(l_s/k)$. It acts transitively on the embeddings of κ_s, and a purely inseparable extension has at most one extension of any embedding. All embeddings of κ exist in l, so this proves transitivity. □

Theorem 4.28.4 (Canonical smooth-tube contraction in characteristic zero)**.** *Assume* $\operatorname{char} K = 0$*, and let q be any closed point where a curve model $\mathcal{X}/O$ is smooth of relative dimension one. Let η be the unique component through q. Then*

$$C_q = \overline{]q[} =]q[\cup \{v_\eta\}$$

has a canonical jointly continuous strong deformation retraction onto v_η, functorial for model isomorphisms and isometric ground-field automorphisms. The tube is path connected and has the canonical Gauss-ray germ and path-connected collar neighborhood basis at v_η described in Proposition 4.25.2.

Proof. Choose an affine neighborhood $\mathcal{U}$ of q which is smooth of relative dimension one over O. Its special fibre is geometrically reduced by [StacksAlg, Tag 056T]; smoothness also gives a unique component through q. Apply Lemma 4.28.3 to $\kappa = \kappa(q)$. Lemma 4.28.1 identifies the formal generic fibre of $\mathcal{U}_{N^\circ}$ with U_N and its reduced smooth special fibre with $(\mathcal{U}_s)_{\widetilde{N}}$. The finitely many points q_j above q are rational smooth points. Thus their tubes D_j are open unit discs and their compact closures are $C_j = D_j \cup \{v_j\}$ by Lemma 4.21.1.

The projection $p : U_N \to U$ is the open Galois quotient of Theorem 4.9.2. Specialization gives $p^{-1}(D_q) = \coprod_j D_j$. Flat going down, [StacksAlg, Tag

00HS], maps all v_j to v_η. Lemma 4.20.1 therefore gives the displayed closure, and openness gives

$$p^{-1}(C_q) = \bigcup_j C_j.$$

The finite closed pasting and compact-product quotient argument in Proposition 4.22.2 now applies verbatim to the canonical disc homotopies: its topological proof requires exactly this equality, rational disc tubes upstairs, and full semilinear coordinate invariance. These have just been verified. It gives the desired jointly continuous contraction.

Two choices of N embed over K into a common finite Galois extension, obtained by taking a splitting field of their defining polynomials. Under further extension, a rational residual point remains a single rational point, and its open-disc tube is its analytic scalar extension by Lemma 4.28.1. The full ground-field compatibility in Theorem 4.21.2 makes the two constructions identical upstairs; surjectivity makes their descents equal. Shrinking $\mathcal{U}$ does not change the whole tube or its component endpoint, and the same comparison proves independence of $\mathcal{U}$, functoriality, and semilinear invariance.

Finally, the Galois group acts transitively on the q_j by Lemma 4.28.3. All the collar and ray arguments of Proposition 4.25.2 use only this transitivity, finitely many disc-coordinate transitions, and the topological quotient. They therefore give path connectedness, the cofinal connected collars, and the canonical embedded ray with parameter $-\log r$ in the present setting as well. □

Corollary 4.28.5 (Global skeletons in characteristic zero)**.** *Assume* $\operatorname{char} K = 0$. *In Theorem 4.23.3, the hypothesis that the smooth closed points have separable residue fields can be omitted. The same conclusion holds for the branch-star correspondence of Theorem 4.25.3. Equivariant homotopies of such models descend under the finite Galois hypotheses of Corollary 4.23.4.*

Proof. The proof of Proposition 4.23.2 uses separability only to obtain the completed smooth-tube contractions. Replace that input by Theorem 4.28.4. The cofinite whole-tube estimate is unchanged, so its proof gives the same compact stars and uniform continuity in time. Finite closed pasting then proves the global skeleton theorem with the same nodal intervals and edge lengths. For branch stars, replace the connected collar input by the same theorem; every topological argument and every normalization label is unchanged.

Model functoriality makes the homotopy equivariant, so Corollary 4.9.3 descends it. The finite-graph quotient argument is purely topological and again applies without change. □

4.29 Intrinsic residual curves and metric directions

For a point x admitting the cofinal branch stars of Theorem 4.25.3, define its set of outgoing directions by

$$T_x X = \varprojlim_{U \ni x} \pi_0(U \setminus \{x\}),$$

where U ranges through ordinary open neighborhoods, ordered by shrinking. The cofinal stars make this inverse system constant on their residual-point labels. In particular this definition depends only on the analytic topology.

Lemma 4.29.1 (Comparison of annular collars at one direction)**.** *Suppose two nodal models of the same analytic curve satisfy Theorem 4.25.3, have x as a component vertex, and $d \in T_x X$ corresponds in each model to a rational point of the normalized residual component. Their outgoing Gauss-ray germs and parameters $\ell = -\log r$ coincide. More precisely, if z, w are their outward-normalized collar coordinates, there is $r_0 < 1$ such that the two collars of inner radius $r > r_0$ are the same analytic domain and $|z| = |w|$ there.*

Proof. Write $A_z(a) = \{a < |z| < 1\}$ and similarly for w. The cofinality in Theorem 4.25.3 allows alternating refinements of the two stars. Since both branches represent d, we obtain numbers $a_0, a_1, b_0 < 1$ with

$$A_w(b_0) \subset A_z(a_0), \qquad A_z(a_1) \subset A_w(b_0).$$

Moreover, sufficiently thin collars of either model lie in any prescribed collar of the other. To see this directly, make that collar the selected component of a smaller star, and refine this star using the other model.

The analytic function z is invertible on $A_w(b_0)$. Corollary 4.10.3 gives

$$|z(y)| = c\,|w(y)|^n \quad (y \in A_w(b_0)), \qquad c > 0, \quad n \in \mathbf{Z}.$$

As the w-Gauss radius tends to 1, the point approaches x through every sufficiently thin z-collar. Hence $|z| \to 1$, so $c = 1$. Since both coordinate

absolute values lie strictly between zero and one, $n > 0$. Applying the same argument to w on $A_z(a_1)$ gives $|w| = |z|^m$ with $m > 0$. On this nonempty collar, composition gives $|z| = |z|^{mn}$; therefore $mn = 1$ and $m = n = 1$.

Choose $r_0 > \max(a_0, a_1, b_0)$. For $r > r_0$ the first inclusion and $|z| = |w|$ give $A_w(r) \subset A_z(r)$; the second gives the converse. The analytic-domain universal property identifies their analytic structures. The identity between these two standard annulus presentations is now an analytic isomorphism preserving radius. Corollary 4.19.1 identifies the Gauss points and their radial homotopies. In particular the ray germs and the parameter $-\log r$ agree, with the original absolute value unchanged. □

Theorem 4.29.2 (The canonical residual curve at a component vertex)**.** *Let K be algebraically closed, complete and nontrivially valued. Suppose that x is a component vertex of a proper nodal curve model of an analytic curve X. There is a normal proper k-curve C_x, canonically characterized by*

$$k(C_x) = \widetilde{\mathcal{H}(x)}.$$

For every such model, C_x is canonically the normalization of its component corresponding to x. The bijection between T_xX and the closed points of C_x, the outgoing Gauss-ray germ in each direction, and its parameter $\ell = -\log r$ are independent of the model.

If f is an analytic germ at x with $f(x) \neq 0$, choose $c \in K^\times$ with $|cf(x)| = 1$ and put $\varphi = \widetilde{cf}_x \in k(C_x)^\times$. Then for the direction corresponding to $p \in C_x$ one has, for all sufficiently small $\ell \geq 0$,

$$-\log|f(\gamma_p(\ell))| = -\log|f(x)| + \operatorname{ord}_p(\varphi)\ell.$$

Only finitely many of these slopes are nonzero, and their sum is zero.

Proof. Let C be the special component corresponding to x. Lemma 4.24.2 identifies its function field with $\widetilde{\mathcal{H}(x)}$ and gives $|\mathcal{H}(x)^\times| = |K^\times|$. The normalization C^ν is finite over C by [StacksAlg, Tag 0BXR], hence is proper by [StacksAlg, Tags 01WM, 01W3]. Here C is a closed component of the proper special fibre; properness is preserved by base change and closed immersion [StacksAlg, Tags 01W4, 01W5]. The equivalence between normal proper curves and their function fields [StacksAlg, Tag 0BY1] identifies any two such normalizations by the unique isomorphism inducing the identity of $\widetilde{\mathcal{H}(x)}$. This defines C_x.

The direction set itself is intrinsic by the cofinal description preceding the lemma. Every closed residual point is rational, since k is algebraically closed. Lemma 4.29.1 therefore identifies the ray germs and their parameters for the same intrinsic direction in any two models. It remains to check that the associated residual points correspond under the just constructed isomorphism of normal proper curves.

Every $\psi \in \widetilde{\mathcal{H}(x)}^{\times}$ is the residue of a norm-one analytic unit germ at x. Indeed, the proof of Lemma 4.24.2 represents it by a fraction P/Q of affinoid functions with $P(x)Q(x) \neq 0$; on a small neighborhood Q, and also P, are invertible by Proposition 2.2.4. Apply Theorem 4.24.5 to this germ in the two models. The rays and length parameters coincide, so the two residual-point valuations take the same value on every ψ. Thus their discrete valuation rings in the common function field coincide. They have the same centre on the identified proper curve: the two maps from the spectrum of this valuation ring agree at its generic point, and the closed diagonal of the separated curve makes them equal. Hence the residual points coincide.

The asserted formula is now Theorem 4.24.5 in any one model. Finite support and the sum-zero assertion are Theorem 4.24.7; here all closed-point degrees are one. Multiplying the normalizing constant c by a norm-one scalar multiplies φ by a nonzero constant and does not change any order. □

Corollary 4.29.3 (Principal slopes at arbitrary type II points)**.** *Let K be algebraically closed, complete and nontrivially valued. Let Y be a smooth integral projective K-curve possessing a projective nodal O-model, and put $X = Y^{\mathrm{an}}$. Every type II point $x \in X$ has the intrinsic residual curve, metric directions and local slope formula of Theorem 4.29.2. For every $f \in K(Y)^{\times}$, there is a finite marked skeleton of $X \setminus |\operatorname{div}(f)|$ such that $-\log|f|$ factors through its retraction, is integral affine on every open edge and ray, and has outgoing sum zero at every finite vertex. On a ray ending at $a \in |\operatorname{div}(f)|$ its slope towards a is $\operatorname{ord}_a(f)$. Any prescribed finite set of type II points can be included among the vertices.*

Proof. The generic-fibre comparison of Proposition 4.13.2 identifies the formal generic fibre with X. Theorem 4.27.4 realizes any chosen type II point, or any finite set of them, as component vertices of a projective nodal model. Theorem 4.29.2 applies at each such vertex and proves that the resulting objects do not depend on that choice.

The support $P = |\operatorname{div}(f)|$ is finite and consists of K-rational points. Apply Theorem 4.27.5 while retaining the prescribed vertices. The marked skeleton and every stated norm, affine and slope identity then follow from Theorem 4.26.2 and Theorem 4.26.3. □

4.30 Models in genus zero and over a discrete valuation ring

Proposition 4.30.1 (An explicit genus-zero model)**.** *Let K be complete and nontrivially valued of characteristic zero, with the fixed absolute value, and let C/K be a smooth projective geometrically connected curve of genus zero. There is a finite Galois extension L/K and an isomorphism $C_L \simeq \mathbf{P}^1_L$. The smooth projective model $\mathbf{P}^1_{L^\circ}$ gives a one-vertex skeleton and a jointly continuous strong deformation retraction of $(C_L)^{\mathrm{an}}$ onto that vertex. The value group of K may be nondiscrete.*

Proof. A nonempty affine open of C has a closed point with finite residue field over K, by [StacksAlg, Tag 00FV]. Take a finite normal closure of this field over K. Characteristic zero makes the resulting L/K Galois, and C_L has an L-rational point P. Smoothness makes P an effective Cartier divisor by [StacksAlg, Tag 0B8Y]. Thus $\mathcal{O}(P)$ has degree one. Flat cohomology base change [StacksAlg, Tag 02KH] gives $H^1(C_L, \mathcal{O}) = H^1(C, \mathcal{O}) \otimes_K L = 0$. The characterization [StacksAlg, Tag 0C6U] yields $C_L \simeq \mathbf{P}^1_L$.

The two standard affine charts of $\mathbf{P}^1_{L^\circ}$ are finitely presented and flat; the special fibre is the smooth integral curve $\mathbf{P}^1_{\widetilde{L}}$. Its completion has analytic generic fibre $(C_L)^{\mathrm{an}}$ by Theorem 4.13.2. Corollary 4.28.5 applies with no nodal edges and one component vertex, and gives the asserted homotopy. □

For the next three results, R is a complete discrete valuation ring with fraction field F, uniformizer ϖ, and residue field l. A *split nodal model* is a flat, proper, finitely presented relative curve with smooth generic fibre, whose special-fibre singularities are l-rational split nodes. Its other special-fibre points are smooth. At each node we include a rational pointed étale chart over R of the form supplied by Theorem 4.18.2. The word regular refers to its total space.

Lemma 4.30.2 (Equivariant resolution of a split nodal model)**.** *Let $\mathcal{Y}/R$ be a split nodal model. Repeatedly blow up the reduced union of all nonregular points of the total space. This process terminates with a regular split nodal model $\mathcal{Y}'/R$. It preserves projectivity and is functorial for semilinear isomorphisms preserving R. In particular every finite semilinear group action on $\mathcal{Y}$ lifts to $\mathcal{Y}'$.*

Proof. Nonregular points lie among the finitely many special nodes. By the pointed charts of Theorem 4.18.2, after absorbing a unit into one coordinate their local equation is $xy = \varpi^n$, $n \geq 1$, in an étale neighborhood with the same residue field. The completed local ring is $R[[x, y]]/(xy - \varpi^n)$ by [StacksAlg, Tag 0AGX]; it is regular precisely for $n = 1$. For $n \geq 2$, blow up (x, y, ϖ). The ϖ-chart has equation $x_1 y_1 = \varpi^{n-2}$, where $x = \varpi x_1$ and $y = \varpi y_1$. On the x-chart put $u = \varpi/x$; after removing the cancelled torsion the equation is $xu = \varpi$, with $y = x^{n-1}u^n$. The y-chart is symmetric. These are the local charts computed in the proof of [StacksAlg, Tag 0CDE]; flat base change of blowups is [StacksAlg, Tag 0805].

All new special nodes are rational and split. The two outer charts are regular. In the middle chart, $n = 2$ gives a smooth chart, $n = 3$ a regular node, and $n \geq 4$ gives a unique remaining nonregular point with exponent $n-2$. Around each center choose an affine or étale neighborhood omitting the other finitely many centers. The other center ideals are units there, so blowup of the finite reduced union restricts to exactly the local calculation just made. Every round strictly decreases each surviving exponent; after finitely many rounds no nonregular point remains. The center ideal is of finite type on this Noetherian scheme, so blowups are projective by [StacksAlg, Tag 02NS]; composition preserves projectivity over the affine base by [StacksAlg, Tag 0C4P]. They leave the generic fibre unchanged. Finally the nonregular locus and its reduced ideal are invariant under every semilinear isomorphism. Such an isomorphism acts on all powers of that ideal, hence on its Rees algebra and its Proj. These induced maps commute with composition, giving the asserted functoriality. □

Lemma 4.30.3 (Separating marked sections)**.** *Let $\mathcal{Y}/R$ be a regular split nodal model and let $P_1, \ldots, P_m$ be distinct F-rational points of its generic fibre. Their closures are sections meeting the smooth locus. Blowing up, at each stage, the reduced union of points at which at least two marked sections meet terminates in a regular split nodal model on which the marked sections*

are disjoint. Projectivity is preserved. If a finite semilinear group acts on the model and permutes the marked generic points, it acts on every stage and on the final marked model.

Proof. Properness extends each P_i uniquely to a section by [StacksAlg, Tag 0BX5]. Its special point is l-rational. At a split node the regular completed local equation is $xy = \epsilon\varpi$, with ϵ a unit, by the preceding local computation. A section specializing to the node would give $x, y \in \varpi R$, contradicting $xy = \epsilon\varpi$. One may evaluate this completed equation because a section through the point induces a continuous local map to the complete ring R. Consequently every marked section meets the smooth locus.

At a collision point choose a smooth étale coordinate T reducing to zero. The completed local ring is $R[[T]]$, by the smooth pointed chart and [StacksAlg, Tag 0AGX]. A section through this point has $T = a_i \in \varpi R$. Distinct sections have distinct a_i: the rational smooth-tube isomorphism of Corollary 4.17.2 is injective on their generic analytic points. For two sections with the same special point set $c_{ij} = \mathrm{ord}_\varpi(a_i - a_j)$, and set $c_{ij} = 0$ if their special points differ. This nonnegative integer is coordinate independent. Indeed a change of smooth coordinate in $R[[T]]$ has the form $g(T) = b_0 + b_1 T + \cdots$, with $b_0 \in \varpi R$, $b_1 \in R^\times$, and $b_n \in R$. For $a_i, a_j \in \varpi R$ the quotient $(g(a_i) - g(a_j))/(a_i - a_j)$ is a unit.

Blow up the collision point, whose ideal is (ϖ, T). By [StacksAlg, Tag 0805] its two local charts are the pullbacks of

$$\operatorname{Spec} R[U], \quad T = \varpi U; \qquad \operatorname{Spec} R[T, V]/(TV - \varpi).$$

They are regular, and their special fibres are smooth or split nodal. Every marked section through the center lifts to the first chart, where $U = a_i/\varpi$. Existence and uniqueness also follow from the blowup universal property [StacksAlg, Tag 0806], since the pulled-back center ideal is (ϖ). Thus marked sections remain smooth. For every pair meeting at the center, the new contact number is exactly $c_{ij} - 1$; this includes $c_{ij} = 1$, when their new special points are distinct. All other contact numbers are unchanged. The finite sum $\sum_{i<j} c_{ij}$ strictly decreases whenever a collision remains. It proves termination with disjoint sections.

The collision locus is a finite set of rational smooth points, invariant under any automorphism permuting the marked sections. Blowup of its reduced union therefore lifts the group action by the Rees-algebra argument in Lemma 4.30.2. Projectivity follows again from [StacksAlg, Tags 02NS, 0C4P].

The generic marked points remain unchanged throughout, and uniqueness of their extensions keeps their labels and their permutation action compatible. □

Theorem 4.30.4 (Split marked models over a complete discrete field)**.** *Let K be a complete nontrivially discretely valued field of characteristic zero. Let C/K be a smooth projective geometrically connected curve, and let D be a finite reduced closed subscheme of C. There is a finite Galois extension L/K such that D_L is a disjoint union of L-rational points and C_L has a regular projective split nodal L°-model on which their sections are disjoint and smooth. If $g(C) > 0$, the model may be chosen with a semilinear $\mathrm{Gal}(L/K)$-action extending the given action on C_L and permuting these sections. All absolute values extend the specified one on K.*

Proof. After extension to an algebraic closure, smoothness makes all local rings regular, so distinct irreducible components cannot meet. Geometric connectedness therefore makes C geometrically integral. By [StacksAlg, Tag 04L2(4)], $H^0(C, \mathcal{O}_C) = K$. Apply the all-genus semistable reduction theorem [StacksAlg, Tag 0CDN]. Its extension of discrete valuation rings is the ring of integers for the unique extending absolute value: if ϖ_0 is a uniformizer of K° and its order in the extension ring is $e > 0$, the formula $|b|' = |\varpi_0|^{\mathrm{ord}(b)/e}$ extends the given absolute value. Uniqueness in Lemma 1.12.1 identifies it with the fixed extension. Finite extension preserves a cyclic nontrivial value group by Lemma 3.24.2.

Enlarge this field to a finite Galois extension L_0/K also splitting all residue fields of D. These fields are separable since K has characteristic zero. The base-changed semistable model remains nodal. Choose a minimal regular proper model $\mathcal{M}$ over L_0°, using [StacksAlg, Tag 0C2W]. It is nodal by [StacksAlg, Tag 0CDG], and it is projective by [StacksAlg, Tag 0C5P], applied to its regular two-dimensional total space over the Noetherian base $\operatorname{Spec} L_0^\circ$.

Use Theorem 4.18.2 on the finitely many nodes of $\mathcal{M}_s$, and take a finite Galois normal closure L/K of the resulting extension. Split rational node equations remain split rational node equations after this further base change. Thus $\mathcal{M} \times_{L_0^\circ} L^\circ$ is a projective split nodal model. Apply Lemma 4.30.2 and then Lemma 4.30.3. The result has all the claimed geometric properties.

Suppose $g(C) > 0$. The unique minimal model theorem [StacksAlg, Tag 0C6B] extends the generic semilinear $\mathrm{Gal}(L_0/K)$-action to $\mathcal{M}$: apply unique-

ness to $\mathcal{M}$ and each coefficient twist with the prescribed generic identification. The extensions are unique, so their compositions satisfy the group law. Since L_0/K and L/K are Galois, this gives a semilinear action of $\mathrm{Gal}(L/K)$ on $\mathcal{M} \times_{L_0^\circ} L^\circ$ by restriction on the first factor and its natural action on the second. Both subsequent constructions are equivariant by the two preceding lemmas. □

4.31 Geometric slopes over an imperfect residue field

Lemma 4.31.1 (Smooth normalized residual components)**.** *Let D be a reduced finite-type curve over a field k. Assume that D is smooth away from finitely many split k-rational nodes. Then its normalization D^ν is smooth over k. For every field extension l/k, D_l is reduced and the finite morphism*

$$(D^\nu)_l \longrightarrow D_l$$

is the normalization. The same statements hold for each irreducible component C and its normalization C^ν.

Proof. Normalization is finite by [StacksAlg, Tag 0BXR]. Away from the nodes it is an isomorphism, because the smooth local rings are normal. At a node, Lemma 4.24.1 identifies the two normalization points with the origins of the two axes through actual Cartesian normalization squares for the pointed étale charts. These points are rational, and their local rings on the normalized components are discrete valuation rings. By [StacksAlg, Tag 0B8Y], a normal curve is smooth at a closed point whose residue field is separable over k. Thus D^ν is smooth at these points as well. This proves smoothness everywhere, also when the two node branches belong to the same component.

Consequently $(D^\nu)_l$ is normal and reduced for every l/k, by [StacksAlg, Tag 056T]. On an affine open of D, write its finite normalization as $R \hookrightarrow S$. The inclusion is injective because R is reduced. Tensoring with the flat field extension l/k gives $R \otimes_k l \hookrightarrow S \otimes_k l$. The right side is reduced, so D_l is reduced. This argument applies separately to each C as well, since C^ν is a component of the smooth scheme D^ν.

The base-changed finite map is an isomorphism off the inverse image of the finite node set. This open set contains every generic point of D_l: flat going down [StacksAlg, Tag 00HS] sends each minimal prime to a minimal

prime of D. The same holds for C_l. Thus on each affine chart the two rings have the same total ring of fractions. The source ring is normal and integral over the target ring. An element of their common total fraction ring integral over the target is also integral over the source, hence belongs to the source; the reverse inclusion follows from finiteness. Therefore the source is exactly the integral closure, proving the asserted normalization comparison. □

Lemma 4.31.2 (Residual tensor lengths and orders)**.** *Let C be a smooth integral curve over an arbitrary field k, let p be a closed point, and write*

$$\kappa = \kappa(p), \qquad s = [\kappa : k]_{\mathrm{sep}}, \qquad e = [\kappa : k]_{\mathrm{insep}}.$$

Choose a field extension l/k containing the images of all k-embeddings of κ into an algebraic closure of k. Then p_l has exactly s distinct support points, all l-rational, and its local ring at each support point has length e. If p' is any such point of the smooth curve C_l, then for every $\varphi \in k(C)^\times$,

$$\mathrm{ord}_{p'}(\varphi) = e\ \mathrm{ord}_p(\varphi).$$

Here φ is pulled back to the component of C_l through p'.

Proof. Let κ_s be the maximal separable subextension; then κ/κ_s is purely inseparable by [StacksAlg, Tag 030K]. Since l contains all the embeddings of κ_s, the usual factorization of a separable minimal polynomial gives

$$\kappa \otimes_k l = \prod_{\sigma : \kappa_s \hookrightarrow l} \big(\kappa \otimes_{\kappa_s, \sigma} l\big).$$

Each factor has dimension e over l. Let $\widehat{\sigma} : \kappa \hookrightarrow l$ be the unique extension of σ. It gives a surjection from that factor to l. In positive characteristic, choose finitely many generators β_i of κ/κ_s and powers $\beta_i^{p^{n_i}} \in \kappa_s$. The kernel is generated by $\beta_i \otimes 1 - 1 \otimes \widehat{\sigma}(\beta_i)$, each of which is nilpotent. Thus this is the unique maximal ideal. In characteristic zero each factor is simply l. Every composition factor of the resulting Artin local ring is l, so its length is its l-dimension e. There are precisely s factors.

Put $R = \mathcal{O}_{C,p}$ and $R' = \mathcal{O}_{C_l,p'}$. Both rings are discrete valuation rings, since the two curves are smooth. If u is a uniformizer of R, then

$$R'/uR' = (\kappa \otimes_k l)_{p'}, \qquad \mathrm{ord}_{R'}(u) = \mathrm{length}_{R'}(R'/uR') = e.$$

The local ring map $R \to R'$ carries units to units and is injective by flatness. Writing $\varphi = u^m v$ with $v \in R^\times$ therefore gives $\mathrm{ord}_{p'}(\varphi) = em$ for every $m \in \mathbf{Z}$, as asserted. □

Theorem 4.31.3 (Geometric slope formula in characteristic zero)**.** *Let K be complete, nontrivially valued and of characteristic zero, and retain its original absolute value. Let $\mathcal{X}/O$ be a curve model satisfying Corollary 4.28.5. Let x be the component vertex associated with C. For a closed point $p \in C^{\nu}$, put*

$$e_p = [\kappa(p) : k]_{\mathrm{insep}}, \qquad s_p = [\kappa(p) : k]_{\mathrm{sep}}.$$

Use the geometric Gauss-ray germ γ_p supplied by that corollary, parameterized by $\ell = -\log r$ after a finite valued extension making the residual point rational. If f is an analytic germ at x with $f(x) \neq 0$, choose $c \in K^{\times}$ such that $|cf(x)| = 1$, and let

$$\varphi = \widetilde{cf}_x \in k(C)^{\times} = k(C^{\nu})^{\times}.$$

Then for all sufficiently small $\ell \geq 0$,

$$-\log|f(\gamma_p(\ell))| = -\log|f(x)| + e_p \operatorname{ord}_p(\varphi)\ell.$$

The formula and its parameter are independent of the chosen finite valued extension. In particular the geometric outgoing slope is the integer

$$\sigma_p(f) = e_p \operatorname{ord}_p(\varphi).$$

Proof. Lemma 4.24.2 gives the normalizing scalar c and the displayed residual function field. If p lies over a node, it is rational by Lemma 4.24.1; hence $e_p = 1$ and the assertion is Theorem 4.24.5 in its actual pointed node chart.

Otherwise p corresponds to a smooth point q of the special fibre, with $\kappa(p) = \kappa(q)$. Choose the finite Galois extension N/K in Lemma 4.28.3, with $l = \widetilde{N}$ containing all the residue embeddings. By Lemma 4.28.1 the algebraic model base change has special fibre $(\mathcal{X}_s)_l$, and its formal generic fibre is the analytic scalar extension. Lemma 4.31.1 shows that this special fibre is reduced and that its normalization is $(\mathcal{X}_s^{\nu})_l$.

Choose a rational point p' above p, and let x' be the component endpoint of its rational smooth tube. It maps to x, so $|cf(x')| = 1$. The component-residue identifications commute with pullback: on integral affine functions both are ordinary special-fibre reduction, and the fraction-field identifications extend these maps. Thus the residue of cf at x' is the pullback of φ to the component through p'. Theorem 4.24.5 applied upstairs gives a geometric slope $\operatorname{ord}_{p'}(\varphi)$. By Lemma 4.31.2, this equals $e_p \operatorname{ord}_p(\varphi)$.

Theorem 4.28.4 identifies the projection of the upstairs Gauss-ray germ with γ_p, using the same radius r and the same extending absolute value.

Evaluation of a function pulled back from X is unchanged by projection. The upstairs equality therefore gives exactly the displayed equality on X. The independence of extension and coordinate follows from the ray-germ comparison in that theorem. Multiplying c by a norm-one scalar changes φ by a nonzero constant and leaves all orders unchanged. □

Corollary 4.31.4 (Separable-degree balance)**.** *Under the hypotheses of Theorem 4.31.3, assume that C is proper over k. Only finitely many geometric slopes $\sigma_p(f)$ are nonzero, and*

$$\sum_{p \in C^\nu \text{ closed}} s_p \, \sigma_p(f) = 0.$$

Equivalently, the algebraically normalized slopes $\sigma_p(f)/e_p$ satisfy the residue-degree identity

$$\sum_{p \in C^\nu \text{ closed}} [\kappa(p) : k] \, \frac{\sigma_p(f)}{e_p} = 0.$$

When k is perfect, $e_p = 1$ and this is Theorem 4.24.7.

Proof. The finite normalization C^ν is proper by [StacksAlg, Tags 0BXR, 01WM, 01W3]. The principal divisor of φ has finite support. Its degree is zero by [StacksAlg, Tag 02RU], so

$$0 = \sum_p [\kappa(p) : k] \operatorname{ord}_p(\varphi) = \sum_p s_p e_p \operatorname{ord}_p(\varphi) = \sum_p s_p \sigma_p(f).$$

Both displayed assertions and finite support follow. □

4.32 Marked tubes and punctured skeleta

Lemma 4.32.1 (The punctured-disc homotopy)**.** *Let D be an open unit disc, let $P \in D(K)$, and adjoin its outer endpoint v as in Lemma 4.21.1. Choose a disc coordinate T with $T(P) = 0$. On $D^*_P = (D \setminus \{P\}) \cup \{v\}$ there is a canonical jointly continuous strong deformation retraction onto*

$$S_P = \{v\} \cup \{\zeta_{0,r} : 0 < r < 1\}.$$

The parameter $\ell = -\log r$, with $\ell(v) = 0$, identifies S_P with the metric ray $[0, \infty)$. The homotopy preserves $|T|$, is independent of the coordinate centered at P, and commutes with complete isometric scalar extension and semilinear disc isomorphisms carrying the marked point to the marked point.

Proof. Using Theorem 4.21.2, define

$$H_P(x,u) = H_D\big(x, u|T(x)|\big) \quad (x \neq v), \qquad H_P(v,u) = v.$$

The parameter on the right is less than one. The weighted Gauss formula gives $|T(H_P(x,u))| = |T(x)| > 0$. On $D \setminus \{P\}$ joint continuity is composition of continuous maps. At v the entire outer tail $|T| > r$ is preserved for every u, so the full-tail neighborhood basis gives joint continuity there, uniformly in u. At $u = 1$ the radius-$|T(x)|$ Gauss norm is unchanged by translation by $T(x)$, giving $H_P(x,1) = \zeta_{0,|T(x)|}$. The same formula fixes every centered Gauss point for every u, as in Proposition 4.3.1. It starts at x, and hence is the asserted strong retraction.

A centered change of coordinate has the form $g(T) = a_1T + a_2T^2 + \cdots$ with $|a_1| = 1$ and $|a_n| \leq 1$. Thus $|g(x)| = |T(x)|$ throughout the punctured disc. The full coordinate invariance of H_D now proves that H_P is coordinate independent, not just its value at $u = 1$. The same argument proves semilinear compatibility and scalar-extension compatibility. Finally the map $\ell \mapsto \zeta_{0,\exp(-\ell)}$ is continuous for $\ell > 0$ and extends to v at zero by the tube endpoint theorem. Its inverse is the continuous function $-\log|T|$, extended by zero at v. This proves the topological and metric ray identification. □

Theorem 4.32.2 (A split marked model gives a punctured skeleton)**.** *Let $\mathcal{C}/O$ be a projective curve model satisfying either the hypotheses of Theorem 4.23.3 (finite separable residue fields at smooth closed points) or those of Corollary 4.28.5 (characteristic-zero K, with arbitrary residue fields at smooth closed points). Let $P_1, \dots, P_m$ be K-rational points of its smooth generic fibre whose sections have distinct smooth special points $q_1, \dots, q_m$. Put*

$$U = (\mathcal{C}_K)^{\mathrm{an}} \setminus \{P_1, \dots, P_m\}.$$

Adjoin to the compact model skeleton one infinite ray for every P_i, contained in $]q_i[$ and attached to its component vertex. Their union $\Sigma(\mathcal{C}; P_1, \dots, P_m) \subset U$ is a finite metric graph with finitely many infinite rays and is a jointly continuous strong deformation retract of U. The entire homotopy and all edge lengths are invariant under isomorphisms of marked models, including semilinear isometric isomorphisms.

Proof. The formal completion has generic fibre $(\mathcal{C}_K)^{\mathrm{an}}$ by Theorem 4.13.2. Each q_i is rational because it is the special point of an O-section. Its tube is

a rational open disc by Corollary 4.17.2. Translate a disc coordinate by its value at P_i to put that point at zero. Use Lemma 4.32.1 on the completed tube with P_i removed.

We justify the global pasting using a finite closed cover. For every component vertex v, start with its compact star A_v from Proposition 4.23.2, using when necessary the arbitrary-residue extension of Corollary 4.28.5. Remove from A_v all the finitely many marked open tubes incident to it; the resulting A_v^0 is closed in A_v and thus compact and closed in U. The original star homotopy restricts to A_v^0. Add the finitely many punctured completed marked tubes $C_{q_i} \setminus \{P_i\}$, which are closed in U, and the finitely many completed nodal tubes. These sets cover U and their intersections consist of fixed vertices. Their local homotopies therefore paste to a jointly continuous map on $U \times [0,1]$ by the finite closed-cover pasting lemma. This argument retains the uniform control of all unmarked tubes already proved for each A_v.

At time one, unmarked stars map to their vertices, nodal tubes to their compact Gauss edges, and each marked punctured tube to its ray. Every point of this union is fixed at every time. It is therefore the strong retraction image. The graph topology is the one induced from U: its compact core and each of its finitely many rays are closed subspaces, and they meet only at their attached endpoints. On a ray this follows from Lemma 4.32.1; on the compact core it is Theorem 4.23.3. For a finite closed cover these subspace topologies determine the topology on their union. The length coordinate on each ray is $-\log|T|$, and the compact edge lengths are the previously established annular lengths. The local homotopy invariances give all the asserted functoriality. □

Corollary 4.32.3 (Punctured curves over the original discrete field)**.** *Let K, C, D be as in Theorem 4.30.4. After its finite Galois extension L/K, $(C_L \setminus D_L)^{\mathrm{an}}$ has a strong deformation retraction onto a finite metric graph with one infinite ray for each point of D_L. If $g(C) > 0$, the homotopy descends to $(C \setminus D)^{\mathrm{an}}$ over the original field and the original absolute value. Its image is the finite metric quotient graph, with one infinite ray for each closed point of D.*

Proof. Apply Theorem 4.32.2 to the split marked projective model just constructed. For $g(C) > 0$ that model has the compatible Galois action; the full marked homotopy is equivariant. Corollary 4.9.3, applied to the analytic

open space $(C \setminus D)^{\mathrm{an}}$, descends it. This descent is valid without compactness: Theorem 4.9.2 gives the open quotient property after product with $[0,1]$.

The quotient of the compact core is a finite metric graph by Corollary 4.23.4. A group element carrying one marked disc to another carries its center to the other center. Hence it preserves the radius on the entire ray, by Lemma 4.32.1. Thus stabilizers act trivially on each ray, and quotient rays are indexed by Galois orbits of D_L, equivalently by the closed points of D. All parameters are computed with the extending absolute value; no degree factor is introduced. □

4.33 Model-independent geometric directions in characteristic zero

Lemma 4.33.1 (The norm profile on an entire residual collar)**.** *Assume Theorem 4.31.3. For every residual direction p there are an open collar $B_p(r_0)$ and a continuous radius function*

$$R_p : B_p(r_0) \longrightarrow (r_0, 1), \qquad B_p(r) = \{y \in B_p(r_0) : R_p(y) > r\},$$

whose Gauss-ray representative has radius r. The germ of this radius function is independent of the rationalizing extension and coordinates. For every analytic germ f at the component vertex x with $f(x) \neq 0$, after increasing r_0 one has on all actual points of the collar

$$|f(y)| = |f(x)|\, R_p(y)^{\,e_p \operatorname{ord}_p(\widetilde{cf}_x)}, \qquad |cf(x)| = 1.$$

Proof. At a rational node branch use its standard annular coordinate. At a smooth residual point, the construction of Proposition 4.25.2, extended by Theorem 4.28.4, produces finitely many rational open discs after a finite Galois extension. Lemma 4.25.1 makes the finitely many semilinear transition maps preserve the entire radius functions on sufficiently thin collars. These functions therefore descend through the analytic Galois quotient to R_p. The same lemma and a common finite Galois extension compare two constructions. They also show that $B_p(r)$ is exactly the image of any upstairs radius-r outer collar.

By Proposition 2.2.4, f and f^{-1} are analytic on a neighborhood of x. Cofinality of the branch stars puts a sufficiently thin collar inside that neighborhood. On every rational annular collar upstairs, Corollary 4.10.3 therefore

gives an all-point identity $|f| = ar^n$. Along its Gauss ray, Theorem 4.31.3 identifies $a = |f(x)|$ and $n = e_p \operatorname{ord}_p(\widetilde{cf}_x)$. These constants are the same on every conjugate collar. Evaluation and the radius function descend, proving the identity on the entire collar downstairs. □

Theorem 4.33.2 (Intrinsic residual curve and geometric metric)**.** *Let K be complete, nontrivially valued and of characteristic zero. Suppose that x is a component vertex of a proper curve model satisfying Corollary 4.28.5. There is a canonical smooth proper residual curve C_x over k with function field $\widetilde{\mathcal{H}(x)}$. For every such model of the same analytic curve X having x as a component vertex, it identifies canonically with the normalization of the component corresponding to x.*

The correspondence between the intrinsic outgoing directions T_xX and the closed points of C_x is independent of the model. So are the geometric radius germ R_p, the Gauss-ray germ and its parameter $\ell = -\log r$ in every direction. For every analytic germ f nonzero at x, the intrinsic geometric slopes are

$$\sigma_p(f) = [\kappa(p) : k]_{\mathrm{insep}} \operatorname{ord}_p(\widetilde{cf}_x), \qquad |cf(x)| = 1,$$

and satisfy the finite sum

$$\sum_{p \in C_x \text{ closed}} [\kappa(p) : k]_{\mathrm{sep}} \, \sigma_p(f) = 0.$$

Proof. The component-residue identification Proposition 4.24.2 gives the function field. The finite normalization of a proper residual component is proper by [StacksAlg, Tags 0BXR, 01WM, 01W3]; Lemma 4.31.1 makes it smooth. The normal proper curve/function-field equivalence [StacksAlg, Tag 0BY1] identifies these curves uniquely by the identity of $\widetilde{\mathcal{H}(x)}$. Call the common curve C_x.

Use the inverse limit of connected components of punctured ordinary open neighborhoods to define T_xX. The cofinal branch stars in Corollary 4.28.5 identify this set, for either model, with its residual points. Fix one intrinsic direction, and let p, p' be its two labels on C_x. We first identify these labels without presupposing equality of the two metric parameters.

Every $\psi \in k(C_x)^\times$ lifts to a norm-one analytic unit germ g at x: the proof of Proposition 4.24.2 represents it by an affinoid fraction P/Q with $P(x)Q(x) \neq 0$, and Proposition 2.2.4 makes P, Q invertible near x. By Lemma 4.33.1, the sign of $-\log|g|$ on a sufficiently thin collar in the first model

is the sign of $\operatorname{ord}_p(\psi)$. Cofinality of the two branch-star systems puts every sufficiently thin collar in the second model inside that collar. Its Gauss ray therefore has the same sign, and its slope formula gives

$$\operatorname{sgn}\operatorname{ord}_p(\psi) = \operatorname{sgn}\operatorname{ord}_{p'}(\psi) \qquad (\psi \in k(C_x)^\times).$$

Thus the two valuation rings in the common function field are identical. Their centres on the separated proper curve C_x are equal, by the closed-diagonal argument in Theorem 4.29.2. Consequently $p = p'$, and their residue fields, separable degrees and inseparable degrees coincide.

Choose now a uniformizer $\psi \in k(C_x)^\times$ at this common point p, and lift it to a norm-one unit germ g. Write $e = [\kappa(p) : k]_{\mathrm{insep}}$. The all-point collar formula gives, on the intersection of sufficiently thin collars,

$$|g| = R_p^e = (R'_p)^e.$$

Hence $R_p = R'_p$ there. By alternating the two cofinal stars one can choose a collar of each contained in a fixed collar of the other. After increasing both inner radii beyond these bounds, the equality of radius functions gives equality of every thinner collar as a subset of X. Their analytic structures agree by the analytic-domain universal property. Thus the entire radius germ is intrinsic.

Finally choose a common finite Galois extension rationalizing the residual-point supports in both models. The inverse image of an agreed thin collar is, in either presentation, a finite disjoint union of standard open annuli. These annuli are path connected, so their decomposition is the intrinsic connected component decomposition. Corresponding components are therefore the same analytic domains. On each, the two annular coordinates have the same radius function, namely the pullback of R_p. Their identity is a radius-preserving annular isomorphism, and Corollary 4.19.1 identifies the Gauss points of every radius. Projection gives the same Gauss-ray germ on X, with the same parameter $-\log r$.

The slope formula is Theorem 4.31.3, now with all its geometric data intrinsic. The finite-support weighted identity is Corollary 4.31.4. All constructions use the given absolute value. □

4.34 Equivariant marking over a nondiscrete valuation ring

Throughout this section K is a complete nontrivially valued field, $O = K^\circ$, and $k = \widetilde{K}$. A split nodal model is a separated flat finitely presented O-scheme with smooth generic fibre and reduced nodal special fibre, whose nodes are k-rational and admit the actual rational pointed charts of Theorem 4.18.2. Its formal generic fibre is assumed Hausdorff. A marking is a finite set of distinct K-rational generic points. In the proper case each extends uniquely to an O-section by [StacksAlg, Tag 0BX5].

Lemma 4.34.1 (An intrinsic blowup for a nodal cluster)**.** *Let q be a node of a split nodal model $\mathcal{X}$, and let a nonempty finite set $P(q)$ of marked sections specialize to q. There is a finite-type ideal $\mathcal{I}_q$, equal to the unit ideal away from q and open after formal completion, such that its blowup is again a split nodal model, retains every old component point, and moves at least one member of $P(q)$ to a smooth special point. No member of $P(q)$ moves to an outer attaching node. The ideal depends only on the marked cluster and the model, so every semilinear isometric model isomorphism carrying one marked cluster to another carries their ideals to one another. Projectivity over O is preserved.*

Proof. Use a pointed chart $st = b$ at q, with $0 < |b| < 1$. The rational-pointed tube comparison identifies $]q[$ with $|b| < |s| < 1$. A section specializing to q lifts uniquely to the pointed auxiliary chart: its pullback along that chart is étale over the henselian ring O and has the specified rational closed point, so the henselian property of Lemma 4.15.1 supplies the lift. Write

$$a_P = s(P), \qquad d_P = t(P), \qquad a_P d_P = b, \qquad \delta_q = \max_{P \in P(q)} \max(|a_P|, |d_P|).$$

Then $0 < \delta_q < 1$, $\delta_q \in |K^\times|$, and $|b| \le \delta_q^2$. Choose $\lambda \in O$ with $|\lambda| = \delta_q$. Near q put

$$\mathcal{I}_q = (\lambda) + \operatorname{Fitt}_1 \Omega_{\mathcal{X}/O}.$$

Shrink this neighborhood to omit every other node. The construction and proof of Lemma 4.27.2 apply verbatim to these rational pointed charts. That proof uses algebraic closedness only to supply the rational split chart, which is a hypothesis here: finite generation, gluing with the unit ideal, étale descent of the charts, retention of old vertices, and projectivity are all proved over

the displayed valuation ring. Its local pullback is (λ, s, t), so Lemma 4.27.1 gives the actual three charts.

Every section in the cluster lies in the λ-chart: its coordinates there are $u = a_P/\lambda$, $v = d_P/\lambda$, both in O. At least one section has u or v a unit by the definition of δ_q. If $|b| = \delta_q^2$, the entire middle special chart $uv = \overline{b/\lambda^2}$ is smooth. If $|b| < \delta_q^2$, its only node is $u = v = 0$. Thus at least one marked section becomes smooth, and the only possible remaining nodal marked specialization is the middle node. The outer attaching nodes are outside the middle chart.

It remains to check that the centre is intrinsic. Between two normalized annular coordinates $A(|b|, 1)$ and $A(|b'|, 1)$, Corollary 4.19.1 gives $|b'| = |b|$ and either $|s'| = |s|$ or $|s'| = |b|/|s|$. It therefore preserves the unordered pair $(|a_P|, |d_P|)$ and hence δ_q. This also holds semilinearly, since the field automorphism is isometric. A principal ideal of O depends only on the absolute value of its nonzero generator, and the Fitting ideal is intrinsic. The glued ideal is consequently independent of all choices and has the asserted covariance. □

Lemma 4.34.2 (An intrinsic blowup for a smooth cluster)**.** *Let at least two distinct marked sections of a split nodal model $\mathcal{X}$ specialize to the same smooth point q. There is an intrinsic finite-type ideal supported at q whose blowup leaves all these sections smooth and splits their specializations into at least two nonempty classes. It has the same semilinear covariance, retention of old vertices, and projectivity properties as Lemma 4.34.1.*

Proof. The point q is rational, being the special point of a section. Choose an algebraic smooth coordinate T at q, and write $a_P = T(P)$ for the finitely many sections in its cluster. All a_P have the same residue. The tube coordinate is injective on its K-points by Corollary 4.17.2, so these values are pairwise distinct. Put

$$0 < \delta_q = \max_{P,Q} |a_P - a_Q| < 1, \qquad |\lambda| = \delta_q.$$

Choose one section S_0 in the cluster, and define

$$\mathcal{I}_q = (\lambda) + \mathcal{I}_{S_0}.$$

The section is contained in the smooth locus: both its generic and special points are smooth, and the spectrum of the rank-one valuation ring has just

those two points. Its image is an effective Cartier divisor. Indeed it is finite flat of rank one over O, and [StacksAlg, Tag 0B9D] applies in the relative smooth curve locus; away from the image its ideal is the unit ideal. Thus $\mathcal{I}_q$ is a finite-type quasi-coherent ideal. It is the unit ideal away from q, since λ is invertible on the generic fibre and S_0 has only the one special point. Its completion is open.

Near q, the ideal of every section S_P in the cluster is generated by $T - a_P$. To see this, choose a Cartier generator f_P of its ideal. The function $T - a_P$ belongs to that ideal, so equals $f_P h_P$. On the smooth special fibre both $\overline{f}_P$ and $\overline{T - a_P}$ are local parameters at q; the former assertion follows also from the relative Cartier statement in Tag 0B9D. Hence $h_P(q) \neq 0$, and h_P is a unit after shrinking. Since $a_P - a_Q \in \lambda O$ for every pair,

$$(\lambda, T - a_P) = (\lambda, T - a_Q).$$

These ideals are the unit ideal away from q. Consequently the global centre is independent of S_0. Changing smooth coordinates does not change δ_q: an open-unit-disc isomorphism has linear coefficient of norm one and all higher coefficients of norm at most one, so for $|a|, |a'| < 1$ its difference divided by $a - a'$ is a unit of norm one. This is the coefficient calculation in Theorem 4.21.2. Thus the centre is also independent of the coordinate, and is covariant under the indicated semilinear isomorphisms.

Apply Lemma 4.27.3 with centre a_{P_0} and scale λ, reusing its actual blowup charts. Every marked section belongs to the smooth λ-chart, with coordinate $U = (T - a_{P_0})/\lambda$. At least two of these coordinates have distinct residues, because one pairwise difference has norm exactly δ_q. All marked specializations therefore remain smooth, and the old cluster separates into at least two nonempty residue classes. □

Theorem 4.34.3 (Equivariant finite marked refinement)**.** *Let $\mathcal{X}/O$ be a proper split nodal model, and let P be a finite set of distinct K-rational points of its generic fibre. A finite sequence of algebraic blowups with open completed centres produces a split nodal model $\mathcal{X}'$ in which the marked sections have pairwise distinct smooth specializations. Every old component point is retained. If $\mathcal{X}$ is projective, so is $\mathcal{X}'$. If a finite group G acts semilinearly and isometrically on $\mathcal{X}$ and preserves P, the sequence can be chosen G-equivariant, with the given generic action unchanged.*

Proof. First treat all nodes containing marked specializations. For each such node take the intrinsic ideal of Lemma 4.34.1, and blow up the product of these finitely many ideals. In a neighborhood of one centre that omits the other centres, every other ideal is the unit ideal. Thus this simultaneous blowup has exactly the local charts already proved there. This argument uses a finite product of finite-type ideals, not an intersection. The number of marked sections meeting nodes strictly decreases: at each treated node at least one becomes smooth, while every previously smooth section is untouched. After at most $|P|$ rounds all marked sections are smooth.

Now treat every smooth point containing at least two marked specializations, again by blowing up the finite product of the ideals in Lemma 4.34.2. All marks stay smooth. The partition of P by specializations strictly refines whenever a collision is present, and different old classes never merge because their centres are distinct and each blowup is an isomorphism off its centre. A partition of a finite set has at most $|P|$ classes, so this second process also terminates after finitely many rounds. Each blowup preserves properness, split nodality, finite presentation, old component points, and projectivity when present, by the two preceding lemmas.

The group permutes the relevant marked clusters at every round and preserves their intrinsic ideals. It therefore preserves their product. An automorphism of a scheme preserving an ideal acts on its graded Rees algebra and hence on its relative Proj. The multiplication and cocycle identities are inherited from the original action on that algebra; semilinear base automorphisms are handled by the same graded construction. Thus every stage is G-equivariant. No integer-valued contact function or discreteness of the value group enters either termination argument. □

Corollary 4.34.4 (Marked curves of genus at least two)**.** *Let K be complete of characteristic zero with its given nontrivial non-Archimedean absolute value, let C/K be a smooth projective geometrically connected curve of genus at least two, and let D be a finite reduced closed subscheme of C. There is a finite Galois extension L/K and a projective split nodal model of C_L whose L-rational marked sections D_L have distinct smooth specializations, with the canonical Galois action extending to the model. Consequently $(C\backslash D)^{\mathrm{an}}$ admits a jointly continuous strong deformation retraction onto a finite metric graph with one infinite ray for every closed point of D, using the original absolute value.*

Proof. Corollary 4.14.3 supplies a projective stable model after a finite extension. Enlarge to a finite Galois extension splitting the finitely many residue fields of D. Characteristic zero makes those extensions separable, so the normal closure remains finite Galois. Apply Theorem 4.18.2 to the finitely many special nodes, and then take a finite Galois closure over K once more. Base change of the stable model remains stable and projective. Rational split pointed charts persist under this further extension. The new nodal fibre is still geometrically reduced.

The canonical Galois action extends to this stable model by Corollary 4.14.5. Apply Theorem 4.34.3 to the finite invariant set of now rational markings. The resulting marked model remains projective and Galois-compatible. The compact skeleton exists over the final field even when its residue field is imperfect, by Corollary 4.28.5. Theorem 4.32.2 supplies the entire equivariant punctured homotopy. Corollary 4.9.3 descends it to the original analytic curve; the open-quotient product property of Theorem 4.9.2 applies to this noncompact space. The quotient-ray and length calculation is precisely the one in Corollary 4.32.3, whose descent argument uses only the equivariant marked model, not the discreteness used there to construct it. □

4.35 Equivariant genus-zero models

Lemma 4.35.1 (Gauss discs and fractional linear maps)**.** *Let F be a complete nontrivially valued field. For $a \in F$ and $r > 0$, let $\zeta_{a,r}$ be the Gauss seminorm with centre a and radius r. The set*

$$\mathcal{B}_F = \{\zeta_{a,r} : a \in F,\ r > 0\}$$

has the following tree metric: if $x = \zeta_{a,r}$, $y = \zeta_{b,s}$, and $R = \max(r, s, |a - b|)$, then

$$d(x, y) = 2 \log R - \log r - \log s.$$

Their interval is obtained by increasing the first radius from r to R and the second from s to R, with the common endpoint $\zeta_{a,R} = \zeta_{b,R}$ identified. Every finite hull is a finite compact metric tree, embedded with its metric topology in $(\mathbf{P}^1_F)^{\mathrm{an}}$. Fractional linear maps and semilinear maps induced by isometric field automorphisms preserve $\mathcal{B}_F$, its metric, and its intervals. They carry points with radius in $|F^\times|$ to points of that same kind.

Proof. The weighted Gauss formula gives

$$\zeta_{a,r} = \zeta_{b,s} \iff r = s \text{ and } |a-b| \le r.$$

For the forward implication evaluate $T-a$ and $T-b$. For the converse expand a polynomial about either centre: translation by an element of norm at most r is contractive for the radius-r Gauss norm, as is its inverse. Define $\zeta_{a,r} \le \zeta_{b,s}$ by $r \le s$ and $|a-b| \le s$. The displayed equality shows that this is well defined and that the least common upper bound is the point of radius R above.

For finitely many such points choose one upper bound ζ_{0,R_0}. Their upward intervals to that point intersect in terminal intervals, beginning at their pairwise least upper bounds. They therefore form a finite tree: after subdividing at those finitely many upper bounds, each added interval meets the preceding tree in a terminal interval and cannot create a circuit. Remove any terminal part beyond the last join to obtain the hull. Assign length $\log(v/u)$ to a radial interval $u \le r \le v$. The unique path just described has the stated length, proving the metric formula and tree assertion. Each radial map is continuous by the Gauss formula, and is injective because evaluation of its centred coordinate recovers r. The finite tree maps continuously and injectively into the Hausdorff analytic projective line. Compactness then identifies its metric and subspace topologies.

Translations and nonzero scalings plainly preserve the formula. Inversion $I(T) = 1/T$ satisfies

$$I(\zeta_{a,r}) = \begin{cases} \zeta_{a^{-1},\, r/|a|^2}, & r < |a|, \\ \zeta_{0,\, r^{-1}}, & r \ge |a|. \end{cases}$$

In the second case $\zeta_{a,r} = \zeta_{0,r}$, and reversing the coefficients of $T^n P(1/T)$ proves the formula. For $r < |a|$, write $T = a+U$ and expand $(a+U)^{-1} - a^{-1}$. Its linear term has norm $r/|a|^2$, and each higher term has strictly smaller norm on the radius-r Gauss algebra. This substitution is contractive into the algebra of radius $r/|a|^2$ in the inverse coordinate; the inverse substitution has the same property. Applying both inequalities to each polynomial proves equality of the Gauss norms and the formula.

On a radial interval the first formula preserves logarithmic lengths until $r = |a|$; the second reverses the logarithmic parameter afterwards. If both parts occur, their images meet only at $\zeta_{0,|a|^{-1}}$, since their smaller discs have centres separated by $|a|^{-1}$. Thus no backtracking occurs, and inversion maps

every interval isometrically to the interval between its endpoints. Every fractional linear map is a composition of affine maps and inversion, so it preserves the metric and the intervals. An isometric field automorphism sends $\zeta_{a,r}$ to $\zeta_{\sigma(a),r}$, which has the same properties. The displayed formulas also preserve radii in the value group. □

Lemma 4.35.2 (A fixed Gauss point after a finite extension)**.** *Suppose a finite group acts on* $(\mathbf{P}^1_F)^{\mathrm{an}}$ *by semilinear fractional linear maps with isometric coefficient automorphisms. The hull of the orbit of* $\zeta_{0,1}$ *has a unique metric centre* c *and the group fixes* c*. One may write* $c = \zeta_{a,r}$ *with* $a \in F$ *and* $r^2 \in |F^\times|$*. After adjoining a square root of an element of norm* r^2*, the point is a rational-radius Gauss point over that finite extension.*

Proof. The preceding lemma gives a finite invariant compact metric tree T. Its leaves belong to the original finite orbit, since the hull has no extra terminal interval. Choose diameter endpoints A, B, let the diameter be d, and let c be the midpoint of $[A, B]$. For any $z \in T$, let y be its attaching point on $[A, B]$, put $t = d(A, y)$ and $h = d(y, z)$. The diameter inequalities give $h \le \min(t, d - t)$. Hence

$$d(c, z) = |t - d/2| + h \le d/2.$$

For $x \neq c$, the tree paths give $\max(d(x, A), d(x, B)) = d/2 + d(x, c) > d/2$. Thus c is the unique point minimizing the maximum distance to T, so every isometry of T fixes it.

Write $A = \zeta_{\alpha,u}$ and $B = \zeta_{\beta,v}$, where $u, v \in |F^\times|$, and put $R = \max(u, v, |\alpha - \beta|)$. All these positive numbers belong to $|F^\times|$. If c is on the radial part from A to the join, then $c = \zeta_{\alpha,r}$ with $r^2 = R^2 u/v$; on the other part it has centre β and $r^2 = R^2 v/u$. If the orbit is a singleton, simply take $c = \zeta_{0,1}$. Choose $\gamma \in F^\times$ with $|\gamma| = r^2$. In the finite extension obtained by adjoining $\sqrt{\gamma}$, the uniquely extended absolute value gives $|\sqrt{\gamma}| = r$. All formulas for the action and for Gauss points commute with scalar extension, so the lifted point remains fixed under the extended action whenever that action is specified. □

Lemma 4.35.3 (The integral stabilizer of the unit Gauss point)**.** *An element of* $\mathrm{PGL}_2(F)$ *fixes* $\zeta_{0,1}$ *if and only if it is represented by a matrix in* $\mathrm{GL}_2(F^\circ)$*. The same criterion holds for a semilinear fractional linear transformation with isometric coefficient automorphism. Such a transformation extends uniquely to a semilinear automorphism of* $\mathbf{P}^1_{F^\circ}$*.*

Proof. A matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{GL}_2(F^\circ)$ reduces to an invertible matrix over the residue field. Its denominator $cT+d$ has Gauss norm one. If a polynomial P has Gauss norm one, its homogenized substitution has integral coefficients and nonzero reduction, because the reduced fractional transformation is an automorphism of the rational function field. Thus $P((aT+b)/(cT+d))$ again has Gauss norm one. Scaling an arbitrary nonzero polynomial by one of its maximal coefficients proves that M fixes the Gauss point.

Conversely rescale a representing matrix so that the largest entry norm is one, and write $\Delta = ad - bc$. Suppose first $\max(|c|, |d|) = 1$. If necessary precompose by inversion, which fixes $\zeta_{0,1}$, so that $|d| = 1$. Then

$$\frac{aT+b}{cT+d} = \frac{b}{d} + \frac{\Delta}{d^2}\frac{T}{1+(c/d)T}.$$

The last fractional transformation is represented by an integral matrix of determinant one and fixes the Gauss point by the first part. Hence the image of $\zeta_{0,1}$ is $\zeta_{b/d,|\Delta|}$. It equals $\zeta_{0,1}$ only when $|\Delta| = 1$. If $\max(|c|, |d|) < 1$, the numerator row has maximum norm one and $|\Delta| < 1$. Postcomposing by inversion interchanges the two rows, reducing to the previous case and showing that the image cannot be the Gauss point. Thus a fixing matrix has $|\Delta| = 1$, as required.

An isometric coefficient automorphism fixes $\zeta_{0,1}$, so it does not change the criterion. An integral invertible matrix defines an automorphism of $\mathbf{P}^1_{F^\circ}$ by its homogeneous linear coordinates. Two normalized representatives of the same projective matrix differ by a unit and define the same map. Uniqueness of an extension follows either on the two standard affine charts or by separation and density of the generic fibre. In particular, generic group identities hold for the extended maps. □

Theorem 4.35.4 (A Galois-compatible genus-zero model)**.** *Let K be a complete nontrivially valued field of characteristic zero, let C/K be a smooth projective geometrically connected genus-zero curve, and let $D \subset C$ be finite and reduced. There is a finite Galois extension M/K, with its uniquely extended absolute value, and an identification $C_M \simeq \mathbf{P}^1_M$ for which the canonical semilinear Galois action extends to $\mathbf{P}^1_{M^\circ}$. After equivariant admissible algebraic blowups, this model can be chosen split nodal and projective, with all points of D_M rational and specializing to distinct smooth points.*

Proof. Proposition 4.30.1 gives a finite splitting field for C. Enlarge it to a finite Galois extension L/K also splitting the residue fields of D, and choose $C_L \simeq \mathbf{P}^1_L$. In this coordinate the canonical Galois action is by semilinear fractional linear transformations. For completeness, any linear automorphism of the projective line has this form: the pullback of the affine coordinate has exactly one simple pole, at the rational inverse image of infinity; a rational function with that pole divisor is a ratio of linear polynomials. Composing a semilinear map with the inverse coefficient automorphism reduces it to this case.

Apply Lemma 4.35.2 to this finite action. It gives a fixed point $c = \zeta_{a,r}$, with $a \in L$ and $r^2 \in |L^\times|$. Choose $\gamma \in L^\times$ with $|\gamma| = r^2$, and let M/K be a finite Galois extension containing $L(\sqrt{\gamma})$. Such a finite Galois closure exists because the characteristic is zero. Set $\lambda = \sqrt{\gamma}$ in M. The canonical lift $c_M = \zeta_{a,r}$ is fixed by $\mathrm{Gal}(M/K)$: an automorphism restricts to one of $\mathrm{Gal}(L/K)$, and its matrix has coefficients in L; the explicit Gauss-disc formulas of Lemma 4.35.1 commute with this extension. Automorphisms trivial on L fix both the centre and the radius.

Replace the coordinate by $U = (T-a)/\lambda$. The fixed point becomes the unit Gauss point. Lemma 4.35.3 extends every semilinear Galois transformation to $\mathbf{P}^1_{M^\circ}$, and its uniqueness gives the group identities. This is a projective smooth model with geometrically integral special fibre. The finite rational marking set remains invariant. Apply Theorem 4.34.3 to obtain the asserted projective split marked model. □

Corollary 4.35.5 (Genus-zero retraction over the original field)**.**

For K, C, D as in Theorem 4.35.4, $(C \setminus D)^{\mathrm{an}}$ admits a jointly continuous strong deformation retraction onto a finite metric graph with one infinite ray for each closed point of D. If D is empty, the image can be a single point. All metrics use the original absolute value on K.

Proof. Apply Theorem 4.32.2 to the Galois-compatible marked model just constructed. Its smooth residual tubes are covered by Corollary 4.28.5, including imperfect residue fields. The entire homotopy is equivariant, so Corollary 4.9.3 descends it. The finite metric quotient and the ray-orbit calculation are those already established in Corollary 4.32.3. When D is empty no marked refinement is needed: the good model $\mathbf{P}^1_{M^\circ}$ has its single fixed Gauss vertex as skeleton. □

4.36 An explicit genus-one semistable model

Lemma 4.36.1 (Short Weierstrass equations and their parameter)**.** *Let F be a field of characteristic zero and let (E, O_E) be a smooth projective geometrically connected genus-one curve with an F-rational point. It has a pointed plane equation*

$$y^2 = x^3 + Ax + B, \qquad O_E = (0 : 1 : 0), \qquad 4A^3 + 27B^2 \neq 0.$$

The value

$$j = 1728\,\frac{4A^3}{4A^3 + 27B^2}$$

is independent of the pointed short equation. Two such equations with the same j become pointed isomorphic after a finite extension of F.

Proof. Geometric connectedness and smoothness imply geometric integrality, and [StacksAlg, Tag 04L2(4)] gives $H^0(E, \mathcal{O}_E) = F$. The curve is regular and hence Gorenstein by [StacksAlg, Tags 056S and 0DWG]. The Riemann–Roch and duality formulas [StacksAlg, Tag 0BS6] give $\deg \omega_E = 0$ and

$$h^0(E, \mathcal{O}_E(nO_E)) = n \qquad (n \geq 1).$$

Indeed the dual term has degree $-n$ and no nonzero section. A nonzero section of an invertible sheaf injects $\mathcal{O}_E$ with a finite-length quotient, so its degree is the nonnegative length of that quotient; this excludes negative degree. Choose functions x, y with exact pole orders two and three at O_E and no other poles. The six functions $1, x, y, x^2, xy, x^3$ have distinct pole orders and form a basis of $H^0(E, \mathcal{O}_E(6O_E))$. Consequently y^2 is their linear combination, with a nonzero x^3 coefficient. Scaling both x, y by that coefficient makes it one. This gives

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

The linear system $|3O_E|$, with basis $1, x, y$, is very ample by [StacksAlg, Tag 0H2V]. Its image is therefore the indicated plane cubic. Indeed the image is an integral plane curve, and its homogeneous ideal is principal by [StacksAlg, Tag 0BYB]. If its homogeneous generator has degree e, the genus formula [StacksAlg, Tag 0BYD] gives $(e-1)(e-2)/2 = 1$, whence $e = 3$. The cubic already displayed lies in this ideal and is nonzero, so it is a scalar multiple of the generator. Completing the square in y and eliminating the x^2 term by translation gives the stated short equation. These coordinate changes fix $(0 : 1 : 0)$. The partial derivatives of the short equation show smoothness precisely when $4A^3 + 27B^2 \neq 0$.

A pointed isomorphism between short equations pulls back the second x to $\alpha x + r$ and the second y to $\beta y + sx + t$, since it preserves the pole-order filtration, with $\alpha\beta \neq 0$. Comparing the coefficients of xy, y, x^3, x^2 gives $s = t = r = 0$ and $\beta^2 = \alpha^3$. Thus with $u = \beta/\alpha$ one has $\alpha = u^2$, $\beta = u^3$, $A' = u^4 A$, and $B' = u^6 B$. This preserves j. Conversely, for $j \neq 0, 1728$ put $r = A'/A$, $s = B'/B$. Equality of j gives $r^3 = s^2$. After choosing u with $u^2 = s/r$, one has $u^4 = r$ and $u^6 = s$, giving the isomorphism. For $j = 0$ take a sixth root of B'/B; for $j = 1728$ take a fourth root of A'/A. All these extensions are finite. □

Lemma 4.36.2 (The integral plane-cubic family)**.** *Let $O = F^\circ$ for a complete nontrivially valued field. For coefficients $a_i \in O$, the projective cubic*

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$$

is projective, flat and finitely presented over O, and its section $(0 : 1 : 0)$ is relatively smooth. If its generic fibre is smooth and its geometric special fibre is smooth or nodal, it is a nodal model of that generic curve.

Proof. The homogeneous coordinate ring is a free $O[Y, Z]$-module with basis $1, X, X^2$, because the equation is monic of degree three in X. It is therefore O-flat. Localization at each homogeneous coordinate remains flat; its degree-zero submodule is a direct summand and hence flat. These degree-zero rings are the standard affine charts of the projective cubic. The single homogeneous equation gives finite presentation, and the closed embedding in $\mathbf{P}^2_O$ gives projectivity. In the affine chart $Y = 1$, the partial derivative with respect to Z is one at $(X, Z) = (0, 0)$, so the indicated section lies in the relative smooth locus. A nontrivial real-valued valuation ring has only generic and

closed points. The stated two fibre conditions thus verify the nodal-model hypotheses. □

Proposition 4.36.3 (The Legendre model when two is a unit)**.** *Let F be complete of characteristic zero with nontrivial absolute value and residue characteristic different from two. Every pointed smooth projective genus-one curve over F has, after a finite extension, a projective nodal model over the resulting valuation ring. Its special fibre is either smooth or an irreducible cubic with one ordinary double point.*

Proof. Use Lemma 4.36.1 and split its cubic as $(x-e_1)(x-e_2)(x-e_3)$ after a finite extension. The roots are distinct. Choose the pair labelled e_1, e_2 to have maximal pairwise distance. After scaling $x-e_1$ by e_2-e_1 and adjoining a square root for the corresponding y-scaling, the equation becomes

$$Y^2 = X(X-1)(X-\lambda), \qquad \lambda = \frac{e_3-e_1}{e_2-e_1}, \qquad |\lambda| \le 1, \quad |1-\lambda| \le 1.$$

At least one of the last two norms is one. If both are one, the three residual roots are distinct. Since two is a unit, the affine partial derivatives then have no common zero on the cubic. The point at infinity is smooth as in Lemma 4.36.2.

If $|\lambda| < 1$, the residual equation is $v^2 = u^2(u-1)$, with its sole singular point at the origin. There the derivative ideal is $(u(2-3u), 2v) = (u, v)$. The local hypersurface ring has depth one: u is a nonzerodivisor since the equation is not divisible by u, and its dimension is one. Its first differential Fitting ideal is the derivative ideal just computed. Its residue field is the base field, so [StacksAlg, Tag 0C4D(5)] proves that this point is a node. If $|1-\lambda| < 1$, translate $u = X-1$; the same computation uses $v^2 = u^2(u+1)$ and derivative $u(-2-3u)$. The quadratic terms have two distinct linear factors over a separable extension, because the characteristic is not two. The residual cubic is irreducible over the algebraic closure: in the first case $u-1$ is not a square in $\overline{k}(u)$, so $v^2 - u^2(u-1)$ is irreducible, and the second case is identical with $u+1$. The integral-family lemma now supplies the required model. □

Proposition 4.36.4 (The Deuring model in residue characteristic two)**.** *Let F be complete of characteristic zero with nontrivial absolute value and residue characteristic two. Every pointed smooth projective genus-one curve over F has, after a finite extension, a projective model of the form*

$$y^2 + axy + y = x^3, \qquad a \in O.$$

Its special fibre is smooth if $|j| \leq 1$ and has exactly one ordinary double point if $|j| > 1$. Here j is the parameter of Lemma 4.36.1, and O is the valuation ring after extension.

Proof. Put $b = a^3$. Completing the square and eliminating the quadratic term over the characteristic-zero field gives the short coefficients

$$A = -\frac{a(b-24)}{48}, \qquad B = \frac{b^2 - 36b + 216}{864}.$$

Expansion gives the polynomial identity

$$b(b-24)^3 - (b^2 - 36b + 216)^2 = 1728(b-27).$$

Hence the discriminant and parameter of this equation are

$$\Delta = -16(4A^3 + 27B^2) = b - 27, \qquad j = \frac{b(b-24)^3}{b-27}.$$

If $|j| \leq 1$, choose a root of the monic polynomial

$$P_j(B) = B(B-24)^3 - j(B-27).$$

It belongs to a finite extension. Every such root b has norm at most one: at a value greater than one the monic highest term would strictly dominate all other terms. Modulo the maximal ideal, residue characteristic two gives $\overline{P}_j(1) = 1$. Thus $\overline{b} \neq 1$ and $|b-27| = 1$. Adjoin a cube root a of b; then a is integral. The discriminant is nonzero and the displayed parameter equals the prescribed j. The short-equation lemma therefore supplies an isomorphism to the original pointed curve after one more finite extension.

If $|j| > 1$, put $\epsilon = j^{-1}$ and solve

$$b = 27 + \epsilon b(b-24)^3.$$

The right side is a contraction of the closed unit ball: the polynomial $B(B-24)^3$ has integral coefficients and is one-Lipschitz on that ball, while $|\epsilon| < 1$.

Iteration starting at 27 converges in F to a solution. It has $\overline{b} = 1$, so both b and $b - 24$ are units. The equation then gives

$$0 < |b - 27| = |\epsilon| < 1.$$

Adjoin a cube root a and then the finite short-equation isomorphism as above. In either case further extension preserves all the strict and unit norm conditions.

We verify the special fibre directly. At infinity it is smooth by Lemma 4.36.2. On the affine chart in characteristic two, let $\alpha = \overline{a}$. The partial derivatives are

$$F_x = \alpha y + x^2, \qquad F_y = \alpha x + 1.$$

If $\alpha = 0$, there is no singular point. If $\alpha \neq 0$, the only candidate is $x_0 = \alpha^{-1}$, $y_0 = \alpha^{-3}$. Substitution into the equation shows that it is singular exactly when $\alpha^3 = 1$. Thus the integral-j case is geometrically smooth. In the other case use $u = x - x_0$, $v = y - y_0$. The derivative ideal is

$$(\alpha v + u^2, \alpha u) = (u, v).$$

The local hypersurface has depth one since u is a nonzerodivisor: its equation modulo u is v^2, so the equation is not divisible by u. Its residue point is rational and its first differential Fitting ideal is the displayed maximal ideal. By [StacksAlg, Tag 0C4D(5)] it is a node. Equivalently, its quadratic part

$$v^2 + \alpha u v + \alpha^{-1} u^2,$$

which has two distinct factors after a finite separable residue extension because $\alpha \neq 0$. These calculations exhaust all geometric singular points. Thus the special fibre is geometrically reduced and at worst nodal. The integral-family lemma finishes the model construction. □

Theorem 4.36.5 (Marked genus-one models after finite extension)**.** *Let K be complete of characteristic zero with its given nontrivial non-Archimedean absolute value. Let C/K be a smooth projective geometrically connected genus-one curve, and let $D \subset C$ be finite and reduced. There is a finite Galois extension L/K and a projective split nodal model of C_L such that all points of D_L are rational and have distinct smooth specializations. Consequently $(C_L \setminus D_L)^{\mathrm{an}}$ has a jointly continuous strong deformation retraction onto a finite metric graph with one infinite ray for every point of D_L.*

Proof. A nonempty affine open of C has a closed point of finite residue degree by [StacksAlg, Tag 00FV]. After that finite extension the curve has a rational point. Flat cohomology base change [StacksAlg, Tag 02KH] preserves genus one; geometric integrality is preserved as well. Apply Proposition 4.36.3 or Proposition 4.36.4 according to the residue characteristic.

Take a finite Galois closure over K that also splits the finitely many residue fields of D. The constructed projective nodal model survives this base change: flatness and finite presentation are preserved, and the geometric special fibres remain nodal. Apply Theorem 4.18.2 to its finite node set, then take a further finite Galois closure over K. The actual rational split charts persist. Theorem 4.34.3, used without a group-action hypothesis, separates the finite rational marking set by admissible algebraic blowups. Projectivity is preserved throughout. Finally Theorem 4.32.2, using its characteristic-zero arbitrary-residue branch, gives the asserted homotopy over the final field. Every extension carries the unique absolute value extending the originally specified one. □

4.37 Equivariant genus-one models by doubling

We use stable reduction of an auxiliary nodal curve and then separate its two labelled generic components. The local normalization calculation works over a valuation ring without a discreteness assumption.

Lemma 4.37.1 (The stable double of a genus-one curve)**.** *Let K be a characteristic-zero field and let C/K be a smooth projective geometrically connected curve of genus one. There is a stable curve Y/K of genus $d+1 \geq 2$ whose normalization is the disjoint union of two labelled copies C_1, C_2 of C. The construction is defined over K; consequently every scalar-extension Galois action preserves the two labels.*

Proof. Choose a closed point $q \in C$ with residue field E finite over K, using [StacksAlg, Tag 00FV] on a nonempty affine open. Put $d = [E : K]$. The extension is separable, since K has characteristic zero. Glue the two copies of q by the identity on $\operatorname{Spec} E$. This is the pushout of

$$\operatorname{Spec} E \amalg \operatorname{Spec} E \longrightarrow C_1 \amalg C_2, \qquad \operatorname{Spec} E \amalg \operatorname{Spec} E \longrightarrow \operatorname{Spec} E.$$

The first map is a closed immersion and the second is finite. An affine neighborhood of q in each copy supplies the affine-neighborhood condition

of [StacksAlg, Tag 0E25], so the pushout is a scheme. It is separated and of finite type over K by [StacksAlg, Tags 0E26, 0E27]. The natural map $\nu : C_1 \amalg C_2 \to Y$ is finite and surjective: on the glued affine chart it is the inclusion

$$R = A \times_E A \longrightarrow A \oplus A, \qquad A \oplus A = R(1,0) + R(0,1).$$

Properness of $C_1 \amalg C_2$ and [StacksAlg, Tag 03GN] show that Y is proper. The fibre-product description commutes with field extensions, since tensoring over K is exact. Over an algebraic closure the construction therefore glues the two smooth connected copies at d distinct pairs of rational points. At each glued point the completed local ring is

$$\overline{K}[[u]] \times_{\overline{K}} \overline{K}[[v]] \simeq \overline{K}[[u,v]]/(uv),$$

by [StacksAlg, Tag 0E36] and the smooth rational-point completion calculation already used in Lemma 4.21.1; [StacksAlg, Tag 0CBW] identifies these as split nodes. Thus Y is geometrically connected and nodal. The finite map ν is an isomorphism off the glued points and has normal source, so it is the normalization: an element of the common total function ring integral over a target affine ring is also integral over its finite normal source ring, and hence belongs to the latter.

The ring description gives the exact sequence

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \nu_*(\mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2}) \xrightarrow{\mathrm{res}_1 - \mathrm{res}_2} \mathcal{O}_{\operatorname{Spec} E} \longrightarrow 0.$$

Here the last sheaf is extended by zero along the closed immersion. The cohomology of the finite pushforward equals that on $C_1 \amalg C_2$ by [StacksAlg, Tag 089W]; the same result and affine acyclicity, [StacksAlg, Tag 01XB], give vanishing of higher cohomology of the last sheaf. The smooth geometrically connected curve C is geometrically integral: its geometric local rings are normal domains by Tag 033C, so distinct geometric irreducible components cannot meet, and connectedness leaves just one component. Hence $H^0(C, \mathcal{O}_C) = K$ by [StacksAlg, Tag 04L2, part (4)], the cohomology sequence is

$$0 \longrightarrow H^0(Y, \mathcal{O}_Y) \longrightarrow K^2 \xrightarrow{(a,b) \mapsto a-b} E \longrightarrow H^1(Y, \mathcal{O}_Y) \longrightarrow K^2 \longrightarrow 0.$$

It follows that $H^0(Y, \mathcal{O}_Y) = K$ and $\dim_K H^1(Y, \mathcal{O}_Y) = d+1$. Every geometric irreducible component has genus one, so there is no rational tail or bridge. Hence Y is stable by [StacksAlg, Tag 0E75]. All maps and both component labels are defined over K, which proves the equivariance assertion. □

Lemma 4.37.2 (Normalizing a nodal chart over a valuation ring)**.** *Let O be a valuation ring with fraction field F, and let $a \in O$. Put $R_a = O[u,v]/(uv-a)$. The integral closure of R_a in the normalization of $(R_a)_F$ is*

$$R'_a = \begin{cases} R_a, & a \neq 0, \\ O[u] \oplus O[v], & a = 0. \end{cases}$$

The ring R'_a is normal, finite and finitely presented over R_a, and flat and finitely presented over O. Its geometric fibres over O are smooth or nodal curves.

Proof. A valuation ring is integrally closed in its fraction field: if $x \notin O$ satisfied a monic equation over O, divide it by its leading power of x; all other terms lie in the maximal ideal, whereas the constant term is one. For $a \neq 0$, substitution $v = a/u$ identifies R_a with the Laurent polynomials $\sum_n c_n u^n$ satisfying

$$c_n \in O \quad (n \geq 0), \qquad c_n \in a^{-n}O \quad (n < 0).$$

Indeed the monomials $1, u, u^2, \dots, v, v^2, \dots$ span the quotient and their Laurent images are linearly independent over O. Inside $F[u,u^{-1}]$ we therefore have

$$R_a = O[u,u^{-1}] \cap O[v,v^{-1}], \qquad v = a/u.$$

Both rings on the right are normal domains by [StacksAlg, Tags 030A, 00GY]; their intersection is integrally closed in the common fraction field $F(u)$, since a monic equation with coefficients in the intersection is a monic equation over each ring. Hence R_a is normal and is already integrally closed in its generic ring $F[u,u^{-1}]$. The displayed monomial basis also proves its O-flatness.

For $a = 0$, identify R_0 with the pairs in $O[u] \oplus O[v]$ having the same constant term. The larger ring is generated as an R_0-module by the two idempotents $(1,0), (0,1)$, and has the finite presentation

$$R'_0 = R_0[e]/\big(e^2 - e, (1-e)u, ev\big), \qquad e = (1,0).$$

It is normal, as a product of two normal polynomial rings. If an element of $F[u] \oplus F[v]$ is integral over R_0, each projection is integral over the corresponding polynomial ring over O, so belongs to that ring. This proves the asserted integral-closure identity. It is free as an O-module and has an explicit finite presentation over O. Finally the fibres of R_a are $uv = \overline{a}$, and those of R'_0 are two disjoint affine lines. These equations prove the geometric-fibre assertion. □

Proposition 4.37.3 (Separating the generic components of a nodal model)**.** *Let F be a complete nontrivially valued field, put $O = F^\circ$, and let $\mathcal{Y}/O$ be a projective flat finitely presented nodal curve family. Suppose that all special nodes admit the split pointed charts of Theorem 4.15.3 over O. Suppose that the normalization of the generic fibre is a disjoint union of finitely many smooth proper geometrically connected curves $C_1, \dots, C_m$. The relative normalization of $\mathcal{Y}$ in $\coprod_i C_i$ is a finite projective morphism $\mathcal{N} \to \mathcal{Y}$, and*

$$\mathcal{N} = \coprod_{i=1}^{m} \mathcal{C}_i, \qquad (\mathcal{C}_i)_F = C_i.$$

Each $\mathcal{C}_i/O$ is normal, projective, flat and of finite presentation, with reduced nodal special fibre and split special nodes. Every semilinear isometric automorphism of $\mathcal{Y}$ which preserves the labels of the generic normalization induces automorphisms of all $\mathcal{C}_i$, compatibly with composition.

Proof. Choose $\pi \in O$ with $0 < |\pi| < 1$. The rank-one property gives $O[1/\pi] = F$, so the generic fibre is the quasi-compact open $D(\pi)$. The normalization map $\coprod_i C_i \to \mathcal{Y}_F$ is finite by [StacksAlg, Tag 0BXR, part (1)], since $\mathcal{Y}_F$ is of finite type over F. Followed by this open immersion, it is quasi-compact and quasi-separated. Its relative normalization is therefore defined by the integral-closure algebra and has the universal property of [StacksAlg, Tag 035I].

Relative normalization commutes with smooth, hence with étale, base change by [StacksAlg, Tag 03GV]. On an actual pointed nodal chart $\mathcal{Y} \leftarrow E \to \operatorname{Spec} R_a$, the generic normalization is the pullback of that of $(R_a)_F$; one can apply the same result over F, or identify the two normal branches directly. Thus the pullback of $\mathcal{N}$ to E is

$$E \times_{\operatorname{Spec} R_a} \operatorname{Spec} R_a'.$$

Lemma 4.37.2 gives its finite presentation and finiteness over E. At smooth points the normalization changes nothing: a smooth algebra A over the normal ring O is normal by [StacksAlg, Tag 033C], and its O-flatness embeds $A[1/\pi]$ in its total quotient ring. The images of the finitely many nodal charts and the smooth locus cover $\mathcal{Y}$: they cover the closed fibre, and any omitted generic point would have a closed specialization by properness, whereas this covering is open. Choose finitely many affine charts in the smooth locus to obtain a quasi-compact surjective étale cover. Finiteness now descends by [StacksAlg, Tag 02LA].

On the induced étale cover of $\mathcal{N}$, each ring is smooth over one of the normal rings R'_a and is therefore normal by Tag 033C. Normality descends along faithfully flat local maps by [StacksAlg, Tag 033G]. Flatness and local finite presentation over O descend on this source cover by [StacksAlg, Tags 036K, 036N]. Since $\mathcal{N} \to \mathcal{Y}$ is finite and $\mathcal{Y}/O$ is projective, $\mathcal{N}/O$ is quasi-compact and separated; it is consequently of finite presentation. The same charts show that its geometric fibres are reduced and nodal of pure dimension one. Above an $a = 0$ node the two branches become smooth; above an $a \neq 0$ node the normalization is locally the identity, so the remaining special nodes are split. By [StacksAlg, Tags 0B3I, 0C4P], finiteness and the projectivity of $\mathcal{Y}/O$ also make $\mathcal{N}/O$ projective.

It remains to separate the labels. On any affine open $\operatorname{Spec} A \subset \mathcal{N}$, O-flatness makes π a nonzerodivisor, so $A[1/\pi]$ embeds in the total quotient ring of A. Each labelled generic idempotent satisfies $e_i^2 - e_i = 0$. A normal ring is integrally closed in its total quotient ring by [StacksAlg, Tag 034M]; hence $e_i \in A$. The extensions agree on overlaps because multiplication by π is injective on every stalk. They remain orthogonal and sum to one, and therefore give the stated clopen decomposition. Each piece inherits all the preceding geometric properties. Finally the relative integral-closure construction is functorial under semilinear isomorphisms; the generic label idempotents are preserved, so their unique extensions are preserved as well. This proves equivariance and the composition law. □

Lemma 4.37.4 (Stable extension of a nodal generic curve)**.** *Let K be a complete nontrivially valued characteristic-zero field, and let Y/K be a stable curve of genus $g \geq 2$. There is a finite Galois extension L/K, with the original absolute value uniquely extended, and a projective stable model $\mathcal{Y}/L^\circ$ carrying the canonical semilinear $\operatorname{Gal}(L/K)$-action on its generic fibre. After enlarging L finitely and Galois, all special nodes admit split pointed charts over L°.*

Proof. Apply Lemma 4.14.1 to the K-object Y of the proper stack $\overline{\mathcal{M}}_{g,K^\circ}$, using [StacksAlg, Tags 0E9C, 0CL6]. This gives a stable family after a finite extension. The projectivity argument in the proof of Theorem 4.14.2 applies without change: Tag 0E74 makes the dualizing line bundle relatively ample over an open neighborhood of the closed point of the local base; that neighborhood is the whole base. Tags 0D32 and 0B5N then give a projective

scheme. This argument uses stability of the fibres, not smoothness of the generic fibre.

Take a finite normal closure as in Corollary 4.14.3; characteristic zero makes it Galois, and uniqueness of the absolute-value extension preserves the specified valuation. Apply Lemma 4.14.4 to the semilinear twists of the model. Its statement already permits nodal generic fibres. The twist construction and uniqueness argument of Corollary 4.14.5 therefore supply the group action and its composition law. Finally use Theorem 4.18.2 and take a further normal closure. Stability persists under this base change; the same uniqueness argument supplies the action of the enlarged group. Parameters $a = 0$ are permitted in the pointed charts. □

Theorem 4.37.5 (Equivariant marked genus-one models)**.** *Let K be a complete nontrivially valued characteristic-zero field, let C/K be a smooth projective geometrically connected curve of genus one, and let D be a finite reduced closed subscheme of C, possibly empty. There is a finite Galois extension L/K and a projective split nodal model $\mathcal{C}/L^\circ$ of C_L with a semilinear $\mathrm{Gal}(L/K)$-action extending the generic action, such that the points of D_L are rational and have pairwise distinct smooth specializations. Consequently $(C \setminus D)^{\mathrm{an}}$ has a jointly continuous strong deformation retraction onto a finite graph with finitely many infinite rays, obtained from the marked model skeleton by finite Galois descent with the original absolute value.*

Proof. Form the stable double Y/K of Lemma 4.37.1 and apply Lemma 4.37.4. Its generic normalization is $C_L \amalg C_L$, with two Galois-invariant labels. Proposition 4.37.3 therefore supplies two projective split nodal models over L°, each preserved by the canonical semilinear Galois action. Choose the first labelled model. Its generic fibre is the original C_L with its specified action.

Enlarge L/K finitely and Galois so that all points of D_L are rational. Such an extension exists because the finite residue extensions of closed points are separable in characteristic zero. Base change of a split nodal family remains split nodal; its projectivity, flatness and finite presentation persist. The previously constructed action extends to the larger group through the restriction to the earlier normal extension and the action on coefficients. Properness extends each rational marked point uniquely to a section, as in Theorem 4.34.3. That theorem applies to this projective separated model and the

Galois-invariant set of marked sections, and yields the required equivariant model with distinct smooth marked specializations.

Apply Theorem 4.32.2, using its characteristic-zero, arbitrary-residue-field alternative. The entire homotopy is equivariant, so it descends by Corollary 4.9.3. For the noncompact punctured space, Theorem 4.9.2 gives the open quotient property after product with $[0,1]$. The finite graph quotient, after subdivision at any inverted-edge midpoint, is again a finite graph with finitely many rays. All field extensions carry the unique extending absolute value throughout. □

4.38 Marked models over an algebraically closed valued field

Lemma 4.38.1 (Doubling at a rational point in any characteristic)**.** *Let F be a field of any characteristic and let C/F be a smooth projective geometrically connected genus-one curve with a specified point $P \in C(F)$. Gluing two labelled copies of C at P gives a stable curve of genus two whose normalization is $C \amalg C$.*

Proof. Use the pushout and integral-closure construction in the proof of Lemma 4.37.1, with $E = F$ and $d = 1$. That proof used characteristic zero only to ensure separability of the chosen closed-point residue field. Here the residue extension is the identity. The same fibre-product local ring, finite normalization map, exact cohomology sequence, and no-rational-tail-or-bridge criterion therefore apply over F and after every field extension. The computed genus is $d + 1 = 2$. □

Theorem 4.38.2 (Split marked models and skeleta in every genus)**.** *Let K be algebraically closed, complete and nontrivially non-Archimedean valued, of any characteristic. Let C/K be a smooth projective connected curve, and let $D \subset C$ be a finite reduced closed subscheme, possibly empty. There is a projective flat finitely presented split nodal model $\mathcal{C}/K^\circ$ of C in which the marked sections have pairwise distinct smooth specializations. Its formal completion gives a finite metric skeleton, with one infinite ray for each point of D, and a jointly continuous strong deformation retraction of $(C \setminus D)^{\mathrm{an}}$ onto that skeleton.*

Proof. Every closed point of C is K-rational by [StacksAlg, Tag 00FV]. If $g(C) = 0$, choose one such point P. It is an effective Cartier divisor of degree one by [StacksAlg, Tag 0B8Y]. The projective-line characterization [StacksAlg, Tag 0C6U, implication (7) to (1)] gives $C \simeq \mathbf{P}^1_K$. Take the smooth projective model $\mathbf{P}^1_{K^\circ}$.

If $g(C) \geq 2$, Theorem 4.14.2 gives a projective stable model after a finite extension of K. Since K is algebraically closed, that extension is K itself. If $g(C) = 1$, choose a rational point and form the genus-two stable double of Lemma 4.38.1. Apply Lemma 4.14.1 to its object of the proper stable-curve stack. Again its finite extension is K itself. The projectivity subargument of Theorem 4.14.2, as isolated in Lemma 4.37.4, depends only on stability and the local valuation base. It gives a projective stable model $\mathcal{Y}/K^\circ$ of this nodal double.

In the positive-genus cases, apply Theorem 4.18.2 to the finitely many special nodes. Its finite extension is still K, so all nodes already have the required rational split pointed charts over K°. In genus one, Proposition 4.37.3, which is valid in every characteristic, separates the two labelled generic components of $\mathcal{Y}$. Choose either resulting projective split nodal model of C. This finishes model existence in all three genus cases.

Properness extends the finitely many rational points of D to sections. Apply Theorem 4.34.3 with the trivial group to make their specializations smooth and pairwise distinct. Finally the residue field k of K is algebraically closed: lift any monic polynomial over k to a monic polynomial over K° and take a root in K; a root of norm greater than one is excluded by strict domination of the leading term, so reduction gives a root of the original polynomial. Thus every smooth closed residual point has residue field k. The perfect-residue alternative of Theorem 4.32.2 supplies the stated skeleton and the entire jointly continuous homotopy. □

4.39 Pointed formal charts and the ends of tubes

Let K be complete and nontrivially valued, with its given absolute value. Write $O = K^\circ$, $\mathfrak{m} = K^{\circ\circ}$ and $k = O/\mathfrak{m}$, and choose $\pi \in O$ with $0 < |\pi| < 1$. All formal schemes in this section are admissible and topologically finitely presented over O. They carry the π-adic topology. For a special-fibre point q, write $]q[$ for its entire inverse image under specialization.

Proposition 4.39.1 (Pointed formal étale maps)**.** *Let $f : \mathfrak{U} \to \mathfrak{X}$ be an adic, topologically finitely presented, formally étale morphism. Suppose that $q' \in \mathfrak{U}_s(k)$ maps to $q \in \mathfrak{X}_s(k)$. Then the generic-fibre morphism restricts to an analytic isomorphism*

$$]q'[_{\mathfrak{U}} \xrightarrow{\sim}]q[_{\mathfrak{X}}.$$

Proof. A formal open containing a special point contains its whole tube, by Proposition 3.17.2. Choose affine formal neighbourhoods of q and q' so that the latter maps into the former. Write them as $\operatorname{Spf} A_0$ and $\operatorname{Spf} B_0$. Lemma 5.23.1 gives a square Jacobian presentation

$$B_0 \simeq A_0[Z_1, \dots, Z_m, Q]/\widehat{(F_1, \dots, F_m, Q \det J - 1)}, \qquad J = \left(\frac{\partial F_i}{\partial Z_j}\right),$$

with $F_i \in A_0[Z_1, \dots, Z_m]$. The lemma's presentation statement applies to every admissible base A_0. The completion and admissible-presentation results 3.14.8 and 3.14.7 identify this with the corresponding separated restricted-series quotient. Its coefficients have norm at most one in the generic quotient norm.

Choose $z_{0i} \in O$ lifting $Z_i(q') \in k$. Reduction gives $\overline{F}(\overline{z}_0)(q) = 0$ and $\det \overline{J}(\overline{z}_0)(q) \neq 0$. Thus Corollary 4.11.2 applies to the entire tube. Its inverse is the unique analytic solution with $|Z_i - z_{0i}| < 1$; it identifies the analytic structure as well as the points. The open restrictions made above retain both tubes, so this is the claimed map. □

Lemma 4.39.2 (The boundary of a formal closed-point tube)**.** *Let $\mathfrak{X}_s$ be a reduced pure one-dimensional scheme and suppose $X = \mathfrak{X}_\eta$ is Hausdorff. For a closed point $q \in \mathfrak{X}_s$, put $D_q =]q[$ and $C_q = \overline{D_q}$ in X. Then D_q is open, C_q is compact, and*

$$C_q \setminus D_q \subseteq \{v_\eta : q \in \overline{\{\eta\}},\ \eta \text{ a component generic point of } \mathfrak{X}_s\},$$

where v_η is the unique component point of Proposition 4.12.2.

Proof. Choose an affine formal open $\mathfrak{V}$ containing q. Its generic fibre V is compact, contains the entire tube, and is closed in Hausdorff X. Consequently C_q is compact and lies in V. On each affine formal chart containing q, the ideal of q in the special fibre has finitely many generators. Integral lifts $f_1, \dots, f_t$ express its tube by $|f_j| < 1$ for all j, by Proposition 3.20.1. The

same proposition and the affinoid net in Proposition 3.17.2 show that these relatively open pieces define an open subset of X.

For $x \in C_q$, let $\mathfrak{p}$ be its specialization prime in $\mathfrak{V}_s$, and let $\mathfrak{n}_q$ be the prime of q. If $\mathfrak{p} \not\subseteq \mathfrak{n}_q$, choose $\overline{f} \in \mathfrak{p} \setminus \mathfrak{n}_q$ and an integral lift f. Then $|f(x)| < 1$, while $|f(y)| = 1$ for every $y \in D_q$. The open condition $|f| < 1$ in V contradicts $x \in \overline{D_q}$. Hence $\mathfrak{p} \subseteq \mathfrak{n}_q$. The proper generalizations of a closed point of a pure one-dimensional Noetherian scheme are component generic points. Proposition 4.12.2 identifies each of their analytic fibres with its single point v_η. This gives the assertion. □

Proposition 4.39.3 (Formal smooth and nodal tube closures)**.** *Let $\mathfrak{X}_s$ be reduced and pure one-dimensional, and suppose $X = \mathfrak{X}_\eta$ is Hausdorff. Let $q \in \mathfrak{X}_s(k)$ admit a pointed diagram*

$$(\mathfrak{X}, q) \longleftarrow (\mathfrak{U}, u) \longrightarrow (\mathfrak{S}, o)$$

of adic, topologically finitely presented, formally étale maps, with $\mathfrak{U}$ affine and u rational over k. Assume $\mathfrak{S}$ is either

$$\begin{aligned} & \operatorname{Spf} O\langle T\rangle, && o = (\overline{T}), \\ \text{or}\quad & \operatorname{Spf} O\langle S, T\rangle/(ST - a), && o = (\overline{S}, \overline{T}), \quad 0 < |a| < 1. \end{aligned}$$

In the smooth case $]q[$ is an open unit disc. Its closure is the disc together with the component point above the unique residual component through q, and is its one-point compactification. In the nodal case $]q[$ is the annulus $|a| < |S| < 1$. Its two radial ends attach to the component points of the corresponding two branches through q. Its closure is obtained by adjoining these ends; the two component points may coincide. Every neighbourhood of an attached endpoint in this closure contains the full corresponding radial tail.

Proof. Reduction of the displayed Jacobian presentations modulo $\mathfrak{m}$ gives ordinary étale maps on special fibres. Proposition 4.39.1, applied to both arrows, identifies their entire selected tubes. On the standard chart, Proposition 3.20.1 computes them as $|T| < 1$ and $|a| < |S| < 1$, respectively.

First work in the compact Hausdorff generic fibre of $\mathfrak{U}$. Its special fibre is smooth or nodal near u. Shrinking about u preserves the tube and permits us to assume it is reduced and pure one-dimensional. In the smooth case exactly one component passes through u: the completion of the local special-fibre ring is $k[[T]]$ by [StacksAlg, Tag 0AGX], and the ring injects into this

domain by [StacksAlg, Tag 00MC]. In the nodal case Lemma 4.20.2 gives exactly two components through u, corresponding to $T = 0$ and $S = 0$. Lemma 4.39.2 restricts the boundary to their component points.

The open unit disc is not compact, since $|T|$ has supremum one and never takes that value. Thus its compact closure must add the unique possible boundary point. A compact Hausdorff space obtained from an open locally compact space by adding one point is its one-point compactification. Its endpoint neighbourhoods contain the complements of the closed discs $|T| \leq r$, $r < 1$: every compact subset of the open disc has $\max |T| < 1$.

For the nodal chart, write v_T for the component point of the branch $T = 0$, and v_S for that of $S = 0$. Their reductions give

$$|S(v_T)| = 1, \qquad |T(v_S)| = 1, \qquad |S(v_S)| = |a|.$$

A net of points of the tube with $|S| \to 1$ has a cluster point in its compact closure. Continuity of $|S|$ and the boundary inclusion force every cluster point to be v_T. It therefore converges to v_T. The same argument treats $|S| \to |a|$ and v_S. Applying this to Gauss points proves that both endpoints occur. If an open neighbourhood of v_T contained no full outer tail, its compact complement would contain a net with $|S| \to 1$, a contradiction. The inner tail is treated in the same way. Equivalently, the complement has maximum $|S| < 1$ or minimum $|S| > |a|$.

An étale map of special-fibre curves sends component generic points to component generic points, by the going-down argument in Lemma 4.20.2. All components through q occur this way: the local map at u over q is faithfully flat, and going down, [StacksAlg, Tag 00HS], lifts each minimal prime below the prime of u. Functoriality of specialization and uniqueness of component points identify the analytic images of the endpoints. The image of the compact tube closure in Hausdorff X is closed and contains $]q[$. Conversely every point in that image is in $\overline{]q[}$, by continuity and density of the tube in its closure. The map is an isomorphism on the open tube, and an endpoint cannot map into it because its specialization is generic. Thus the only possible identification is between the two nodal endpoints. The endpoint-tail assertions descend through this compact-to-Hausdorff quotient. □

Definition 4.39.4 (Semistable formal curve models with pointed charts)**.** Suppose K is algebraically closed and C/K is a smooth connected projective curve. A point of C^{an} is of *type II* when its completed residue field has residual transcendence degree one and relative rational value-group rank zero. A *pointedly semistable formal model* of C^{an} is a quasi-compact admissible formal O-scheme with an identification of its generic fibre with C^{an}, reduced pure one-dimensional special fibre, and a diagram as in Proposition 4.39.3 at each closed special-fibre point. Morphisms induce the identity on C^{an}. Its vertex set consists of its component points v_η.

Corollary 4.39.5 (The specialization partition of a formal curve)**.** *For a pointedly semistable formal model, its vertex set is a finite semistable set of type-II points. Its complement is the disjoint union of the smooth open-disc tubes and finitely many nodal open-annulus tubes. These are exactly its connected components. The boundary of each tube is exactly the set of component points of the residual components passing through its special point.*

Proof. The vertex set is finite by Proposition 4.12.2. Proposition 4.24.2, applied on a formal affine neighbourhood of each component generic point, gives residual transcendence degree one and relative rational value-group rank zero. Thus each vertex is of type II under the intrinsic convention just stated. Every point of the finite-type special-fibre curve is either closed or a component generic point. The tubes therefore give the asserted disjoint partition, and Proposition 4.39.3 identifies their analytic structures and boundaries. They are open in C^{an} and clopen in the complement of the vertex set. Each is connected, so they are precisely the connected components. The nodal closed points are finite. Put $D = \mathfrak{X}_s$. It is quasi-compact and locally of finite type over k, hence Noetherian by a finite affine cover and [StacksAlg, Tag 00FN]. For a point q of D, put

$$e(q) = \dim_{\kappa(q)}\big((\Omega_{D/k})_q \otimes_{\mathcal{O}_{D,q}} \kappa(q)\big).$$

On an affine chart a finite presentation of the coordinate algebra gives a finite presentation of $\Omega_{D/k}$ by differentiating the defining relations. With $n \geq 1$ generators of the module, the locus $e(q) \geq 2$ is cut out by the $(n-1)$-minors of the relation matrix. For $n = 1$ the size-zero minor is 1, and for $n = 0$ the locus is empty. Thus these tests define a closed subset $N \subset D$.

Reduce a supplied pointed diagram modulo $\mathfrak{m} = K^{\circ\circ}$. On the affine opens of this diagram both residual maps are of finite presentation and retain

the unique square-zero lifting property. Indeed the targets of the lifting test over k are discrete and annihilated by $\mathfrak{m}$, so a lift for the formal map factors through the special fibre. The two residual maps are therefore étale by [StacksAlg, Tag 00UR]. For either residual affine map $A \to B$, differentials over k pull back. To see this, for a B-module M and a k-derivation $d : A \to M$, give $B \oplus M$, with square-zero summand M, the A-algebra structure $a \mapsto (a|_B, d(a))$. The identity map to the quotient B has a unique A-algebra lift from B, whose second coordinate is the unique extending k-derivation. This gives $\Omega_{B/k} = B \otimes_A \Omega_{A/k}$. Dimensions of residue fibres are therefore unchanged along either map.

On $k[T]$ the relative differential module has one free generator. On $k[S,T]/(ST)$ the differential presentation has two generators and the single relation $T\,dS + S\,dT$, so its rank-at-least-two locus is the origin. The pointed diagrams thus identify the closed points of N with the nodal closed points. At a point u over a nodal closed q, the ideal (S,T) generates the maximal ideal of $\mathcal{O}_{\mathfrak{U}_s,u}$, by [StacksAlg, Tag 02GU(1) and (6)] for the right residual map. Consequently the inverse image of the rank locus in $\operatorname{Spec}\mathcal{O}_{\mathfrak{U}_s,u}$ consists only of its closed point. The left residual map $\mathcal{O}_{D,q} \to \mathcal{O}_{\mathfrak{U}_s,u}$ is a flat local map by [StacksAlg, Tag 02GU(1) and (2)]. Its map on spectra is surjective by [StacksAlg, Tag 00HQ(3) and (2)], since the closed point is hit. Pullback of the residue-rank test now shows that q has no proper generalization in N.

Every point of the closed Noetherian subspace N specializes to a closed point there: choose a minimal nonempty closed subset of its closure and a point of that subset. As N is closed in D, such a point is also closed in D. It follows that every point of N is closed. Each irreducible component then consists of its single generic point, and there are finitely many irreducible components. Thus only finitely many nodal annular tubes occur. □

4.40 Bounded sections and specialization

Let K be a complete nontrivially real-valued non-Archimedean field, with its specified absolute value. Write $O = K^\circ$, $\mathfrak{m} = K^{\circ\circ}$, and $k = O/\mathfrak{m}$, and choose $\pi \in O$ with $0 < |\pi| < 1$. All formal schemes below are π-adic. An admissible formal algebra is understood to be π-torsion-free and topologically finitely presented over O. For a formal open $\mathfrak{U} \subset \mathfrak{X}$, put $D_{\mathfrak{U}} = \mathrm{sp}^{-1}(\mathfrak{U})$, with its analytic G-domain structure.

Proposition 4.40.1 (Recovery from bounded analytic sections)**.** *Let $\mathfrak{X}$ be an admissible formal O-scheme with reduced special fibre and generic fibre X. The generic-fibre map identifies*

$$\Gamma(\mathfrak{U},\mathcal{O}_{\mathfrak{X}}) = \{f \in \Gamma(D_{\mathfrak{U}},\mathcal{O}_X) : |f(x)| \le 1 \text{ for all } x \in D_{\mathfrak{U}}\}$$

for every formal open $\mathfrak{U}$. These identifications commute with restriction. For each $n \ge 0$, the ideal $\pi^n\Gamma(\mathfrak{U},\mathcal{O}_{\mathfrak{X}})$ corresponds to the same condition with bound $|\pi|^n$.

Proof. On $\mathfrak{U} = \operatorname{Spf} A_0$, put $A = A_0[1/\pi]$. Lemma 4.12.1 gives $A_0 = A^\circ$ and identifies its quotient norm with the spectral norm. Corollary 1.20.3 then gives

$$A_0 = \{a \in A : \max_{x\in\mathcal{M}(A)} |a(x)| \le 1\}.$$

The map $A_0 \to A$ is injective by π-torsion-freeness.

For general $\mathfrak{U}$, cover it by formal affines $\mathfrak{U}_i$. Proposition 3.17.2 makes their generic fibres an admissible analytic G-cover of $D_{\mathfrak{U}}$. A bounded analytic section restricts to an integral section on each $\mathfrak{U}_i$ by the affine equality. On each affine subopen of $\mathfrak{U}_i \cap \mathfrak{U}_j$, the two integral sections agree after inverting π, hence agree before inverting it. They therefore glue in $\mathcal{O}_{\mathfrak{X}}$. Conversely, the restrictions of an integral section have norm at most one and their generic domains cover $D_{\mathfrak{U}}$. This proves the equality and its restriction compatibility. Applying it to f/π^n proves the last assertion. □

Corollary 4.40.2 (A model determined by its specialization map)**.** *Let $\mathfrak{X}_1$ and $\mathfrak{X}_2$ be admissible formal models of the same analytic space X, with reduced special fibres. Suppose there is a homeomorphism $h : |\mathfrak{X}_{1,s}| \to |\mathfrak{X}_{2,s}|$ such that $h \circ \mathrm{sp}_1 = \mathrm{sp}_2$. There is a unique formal isomorphism inducing h and the identity on X.*

Proof. The underlying formal topology equals the special-fibre topology: on an affine chart, rank one gives $\sqrt{\pi A_0} = \sqrt{\mathfrak{m}A_0}$, as in Proposition 3.17.2. Corresponding formal opens under h have the same generic domain. Proposition 4.40.1 identifies their rings of sections, restrictions, and π^n ideals. These identifications give an isomorphism of locally ringed spaces and recover the local adic topologies. Its generic fibre is the identity. Uniqueness follows from the faithfulness in Proposition 3.17.2. □

Proposition 4.40.3 (Uniqueness from vertices and actual tubes)**.** *Let X be Hausdorff, and let $\mathfrak{X}_1, \mathfrak{X}_2$ be admissible formal models of X. Suppose their special fibres are reduced pure curves of finite type over k. Suppose also that each closed special point has a connected open tube, and that the boundary of this tube consists exactly of the component vertices of the components through that point. If the two models have the same set $V \subset X$ of component vertices, there is a unique formal isomorphism between them inducing the identity on X.*

Proof. Proposition 4.12.2 identifies the generic special points with V, with singleton specialization fibres. All remaining special points are closed. Their tubes partition $X \setminus V$ into connected open subsets, each also closed in $X \setminus V$ because its complement is a union of the other tubes. Thus the closed special points in both models are labelled by the same connected components U of $X \setminus V$.

A point labelled U lies on the special component labelled v if and only if $v \in \overline{U} \setminus U$. This recovers all generalizations between special points. The topology of a finite-type pure curve is determined by these incidences: every closed subset is a union of some of its finitely many components and finitely many closed points on the remaining components. Indeed, a proper closed subset of an irreducible Noetherian curve has dimension zero and is finite. Hence the pointwise identification just constructed is a homeomorphism. It intertwines specialization on V and on every U. Corollary 4.40.2 applies. □

Lemma 4.40.4 (Vertices under a formal domination)**.** *Let $\mathfrak{X}_1, \mathfrak{X}_2$ be admissible formal models of an analytic space X, with reduced pure one-dimensional special fibres of finite type over k. If a formal morphism $f : \mathfrak{X}_1 \to \mathfrak{X}_2$ induces the identity on X, then*

$$V(\mathfrak{X}_2) \subset V(\mathfrak{X}_1).$$

Proof. Let $v \in V(\mathfrak{X}_2)$, with $\eta = \mathrm{sp}_2(v)$ a component generic point. Put $p = \mathrm{sp}_1(v)$. Functoriality of specialization gives $f_s(p) = \eta$. If p were closed, its residue field would be finite over k by [StacksAlg, Tag 00FV], whereas the residue field of η has transcendence degree one by [StacksAlg, Tag 0A21(6)]. The induced injective k-homomorphism $\kappa(\eta) \to \kappa(p)$ is impossible. Thus p is a component generic point, and the singleton-fibre assertion of Proposition 4.12.2 says that its component vertex is precisely v. □

4.41 The finite forests between prescribed vertices

For the rest of this section and the next, assume that K is algebraically closed. Let C/K be a smooth connected projective curve and put $X = C^{\mathrm{an}}$. A split nodal algebraic model means a projective, flat, finitely presented O-model of C with reduced nodal special fibre and the split pointed algebraic charts of Theorem 4.15.3. The smoothing parameters at its nodes are nonzero. Theorems 4.13.2 and 4.23.3 identify its completed generic fibre with X and supply its embedded finite graph and retraction. The space X is compact Hausdorff by Proposition 2.33.30.

Lemma 4.41.1 (Connectedness and the genus of the special fibre)**.** *The space X is connected. For every split nodal algebraic model $\mathcal{C}$ of C, its special fibre is connected and has arithmetic genus $g(C)$.*

Proof. Smoothness over the normal field K makes C normal by [StacksAlg, Tag 033C]. Its irreducible components cannot meet, so connectedness makes it integral. Its ring of global functions is a finite field extension of the algebraically closed field K by [StacksAlg, Tag 04L2(1)]. Together with Corollary 2.42.9, this gives

$$H^0(X_G, \mathcal{O}_X) = H^0(C, \mathcal{O}_C) = K.$$

A nontrivial clopen decomposition of X would give a nonconstant idempotent by gluing the functions zero and one on its two members. The open cover is an admissible analytic cover, so this contradicts the displayed equality. Hence X is connected.

Its continuous retraction onto $\Sigma(\mathcal{C})$ has connected image. The vertices and edges of this graph are the special components and their nodes, with their actual incidences. Thus the dual graph is connected, and so is the union of the special components. The fields K and k are algebraically closed; these are the two geometric fibres of the rank-one valuation base. They are reduced, connected curves. By [StacksAlg, Tag 0E6L], $R^1 f_* \mathcal{O}_{\mathcal{C}}$ is finite locally free and commutes with base change. Its rank is constant on $\operatorname{Spec} O$, and its generic rank is $g(C)$. This proves the genus assertion. □

Lemma 4.41.2 (Finite forests outside a semistable vertex set)**.** *Let $\mathcal{C}$ be a split nodal algebraic model with vertex set W, skeleton Σ, and retraction $r : X \to \Sigma$. Let $V \subset W$ be a semistable vertex set, and let U be a connected component of $X \setminus V$ meeting W. Then $S = \Sigma \cap U = r(U)$ is a nonempty finite tree with its boundary endpoints removed. It has one boundary end if U is an open ball and two if U is an open annulus. Each end has a unique limit in V; the two annular ends are counted separately when their limits coincide. Every special component indexed by $W \cap U$ is a smooth $\mathbf{P}^1_k$.*

Proof. We first record the topology of the complement of Σ. Its components are open balls B, and

$$\overline{B} = B \cup \{r(B)\}.$$

For a smooth residual tube this is Proposition 4.22.2. Inside a nodal tube the annular Gauss retraction leaves residue-disc components. In an annular coordinate each such disc is $|T - a| < |a|$, attached at $\zeta_{0,|a|}$. Its closure in a compact closed subannulus is the disc together with this Gauss point, by the Gauss seminorm formula and polynomial factorization. This subannulus is compact and hence closed in X, so the same is its closure in X.

The retraction stays in U. The set U is open in X and closed in $X \setminus V$, so its boundary lies in V. If $y \in U$ retracted to $v \in V$, then y would lie in a smooth residual W-tube attached to v: residue discs inside a nodal tube retract to interior edge points, not vertices. That smooth tube is open and closed in $X \setminus V$ and therefore would equal U. It contains no vertex of W, a contradiction. Consequently $r(U)$ avoids V. It is connected and contains every $w \in W \cap U$, so it lies in U. Since r fixes Σ, this proves $r(U) = \Sigma \cap U$. It is a connected part of a finite graph obtained by deleting vertices in V.

There are no circuits. Use an actual ball or annulus coordinate on U. Theorem 4.1.1 describes every type II or III point as a Gauss point $\zeta_{a,t}$. At a type II point, removal partitions the analytic projective line into the inward residue discs and the outward disc. To check exhaustiveness, if $|T-a|(x) = t$ and $x \neq \zeta_{a,t}$, there is b with $|b - a| \leq t$ and $|T - b|(x) < t$. Otherwise every linear factor would have its Gauss value at x, and factorization over K would give $x = \zeta_{a,t}$. At a type III point, the inward and outward pieces give the analogous two-sided separation. These are disjoint open sets and distinguish the two sides of a radial interval.

The Gauss interval between two points of U lies in U. In a ball its joining radius is still smaller than the outer radius. In an annulus the value of $|T|$

on the interval is between the two endpoint values, or is their common value. Every interior point of this interval separates its endpoints. Therefore every path between the endpoints contains the interval. An embedded compact arc must equal the interval, since an interval contained in an arc with the same endpoints is the whole arc. This also agrees with the explicit interval calculation of Lemma 4.35.1. A circuit in S would give two different arcs between two of its Gauss points. Thus S is a tree.

Radial ends have limits in V. For a ball $U = \{|T| < R\}$, the connected outer tails $E_t = \{t < |T| < R\}$ have nested nonempty compact connected closures in X. Their intersection is nonempty and connected. It misses U and is contained in its finite boundary V, hence is a singleton $\{v_U\}$. Every neighborhood of v_U contains a whole tail: otherwise the nested compact sets outside that neighborhood would have a common point. The closed disc $|T| \leq t$ is compact in U and closed in X. It follows that every boundary point lies in every $\overline{E}_t$, so $\partial_X U = \{v_U\}$. For an annulus, the same argument applies to each of its two radial ends. A compact middle subannulus shows that its boundary consists of the two resulting limits v_-, v_+; they may be equal.

Each radial end occurs exactly once in the finite tree. Choose a compact disc or middle annulus containing the finite set $W \cap U$. Every graph end of S leaves all compact subsets of U, because its endpoint in the compact graph Σ lies in V. Its final portion therefore approaches one coordinate end. The cut-point separation just proved forces this portion to contain a sufficiently short radial Gauss tail. Two different graph ends cannot approach the same coordinate end, since they would share a nonempty Gauss interval.

Suppose a coordinate end were unrepresented. Some sufficiently far tail E would then miss S. Indeed, if all tails met S, one of its finitely many edges would have points escaping toward that end; compactness of the closed edge would produce its endpoint in V and a graph end approaching the chosen coordinate end. The connected set E lies in one ball component B of $X \setminus \Sigma$. Because B meets U and avoids V, we have $B \subset U$. Also $r(B) \in r(U) \subset U$, so the compact closure $B \cup \{r(B)\}$ misses V. All shorter tails lie in B, contrary to the nonempty limiting set in V proved above. Every coordinate end is therefore represented. This gives exactly one or two ends.

The unwanted components are smooth rational curves. For $w \in W \cap U$, translation and scaling identify its type II Gauss field with the unit Gauss field. Its residue field is $k(t)$ by Proposition 3.8.1. Proposition 4.24.2 iden-

tifies this with the function field of the special component D_w. The normalization $D_w^\nu \to D_w$ is finite by [StacksAlg, Tag 0BXR]. It is therefore proper, and D_w^ν is proper over k by [StacksAlg, Tags 01WM, 01W3]. The normal proper curve equivalence [StacksAlg, Tag 0BY1] now identifies D_w^ν with $\mathbf{P}_k^1$. A self-node of D_w would give a loop based at w in Σ. Its interior is connected, avoids V, and meets the open set U near w, so the entire loop lies in U. This contradicts acyclicity. Thus D_w has no self-node. At a node shared with another component, its own branch is smooth, by the pointed branch calculation in Lemma 4.20.2. Hence D_w itself is smooth and equals its normalization, as asserted. □

4.42 Exact contraction of auxiliary components

Lemma 4.42.1 (Pinching smooth sections over a local ring)**.** *Let R be a local ring. Let A and B be flat finitely presented R-algebras with augmentations to R. Their fibre product is a flat finitely presented R-algebra and commutes with arbitrary base change. If A and B are affine neighborhoods of sections in the smooth loci of separated relative curves, they can be shrunk around those sections so that $A \times_R B$ is* étale *over $R[u,v]/(uv)$.*

Consequently, pinching smooth sections of two proper flat finitely presented nodal R-curves produces a proper flat finitely presented nodal R-curve, compatibly with arbitrary base change.

Proof. Translate finite algebra generators by their augmentation values and write $A = R[x_1,\dots,x_m]/I$ and $B = R[y_1,\dots,y_n]/J$. The relation ideals are finitely generated and have zero constant terms. There is an isomorphism

$$A \times_R B = R[x_1,\dots,x_m,y_1,\dots,y_n]/(I,J,(x_iy_j)_{i,j}).$$

Indeed, after removing cross products every polynomial is represented as $c + f(x) + g(y)$ modulo I, J, with f, g of zero constant term; its image is zero precisely when $c = 0$, $f \in I$, and $g \in J$. As an R-module this ring is $R \oplus \ker(A \to R) \oplus \ker(B \to R)$, hence is flat. The split exact sequence

$$0 \longrightarrow A \times_R B \longrightarrow A \oplus B \xrightarrow{\epsilon_A - \epsilon_B} R \longrightarrow 0$$

proves arbitrary-base-change compatibility. The ring $A \oplus B$ is generated as an $A \times_R B$-module by $(1,0)$ and $(0,1)$, and the corresponding finite morphism is surjective.

For the local chart assertion, choose actual étale coordinates at the special points of the sections using [StacksAlg, Tag 054L], and subtract their section values. The chosen affine neighborhoods contain the entire sections, since an open subset of a local spectrum containing its closed point is the whole spectrum. The zero fibre of an étale coordinate map is étale and separated over R; its section is open and closed by [StacksAlg, Tag 024T]. Remove its other components and take a principal affine neighborhood of the section's special point. We obtain

$$R[u] \longrightarrow A \text{ étale}, \quad A/uA = R, \qquad R[v] \longrightarrow B \text{ étale}, \quad B/vB = R.$$

The algebra $D = A \otimes_R B$ is étale, hence flat, over $R[u, v]$. Tensoring

$$0 \longrightarrow R[u,v]/(uv) \longrightarrow R[u,v]/(v) \oplus R[u,v]/(u) \longrightarrow R[u,v]/(u,v) \longrightarrow 0$$

with D identifies its kernel with $A \times_R B$. Thus $A \times_R B = D/(uv)$, étale over $R[u, v]/(uv)$.

The scheme pushout along the two closed sections exists by [StacksAlg, Tag 0E25]. Its affine-neighborhood hypothesis holds because every fibre of a section map has at most one point. Near the joined section its rings are precisely the fibre products above; elsewhere the old curves are unchanged. These charts prove flatness, local finite presentation, and nodality. The disjoint union of the original curves maps finitely and surjectively to the pushout. The pushout is separated by [StacksAlg, Tag 0E26] and universally closed by [StacksAlg, Tag 03GN], hence proper. Quasi-compactness and separatedness promote local finite presentation to finite presentation. The same charts and their localization maps commute with arbitrary base change, proving the last assertion. □

Lemma 4.42.2 (Protection by elliptic tails)**.** *Let $\mathcal{C}$ be a split nodal algebraic model with vertex set W, and let $V \subset W$ be a semistable vertex set of X. There is a proper flat finitely presented prestable family $\mathcal{Y}_0/O$ obtained by attaching three smooth elliptic tails at distinct smooth points on every component indexed by V. Its genus is $g(C)+3|V|$, its generic fibre is stable, and its stabilization $c : \mathcal{Y}_0 \to \mathcal{Y}$ contracts precisely the original special components indexed by $W \setminus V$. The stable family $\mathcal{Y}$ is projective over O.*

Proof. First choose a smooth projective elliptic O-curve $\mathcal{E}$ with a smooth section. If $\operatorname{char} k \neq 2$, take the projective completion of

$$y^2 = x(x-1)(x-\lambda), \qquad \overline{\lambda} \notin \{0, 1\}.$$

Its fibres are smooth by the distinct roots and the unit 2. If $\operatorname{char} k = 2$, take $y^2 + y = x^3$. The special fibre has y-derivative one. In mixed characteristic a singular affine point would have $x = 0$ and $y = -1/2$, which does not satisfy the equation; in equal characteristic the derivative is again one. The point at infinity is smooth in both equations. Lemma 4.36.2 supplies flatness, finite presentation, and projectivity. Each fibre is a smooth plane cubic, so its global functions are the constants and its genus is one by [StacksAlg, Tag 0BYD]. Use the infinity section.

Choose three distinct smooth k-points on each special component indexed by V, outside its nodes. Such points exist because a nonempty smooth open of a curve over the algebraically closed field k has infinitely many closed points: its closed points are dense and rational by [StacksAlg, Tag 00FV], and finitely many closed points cannot be dense in a curve. By Corollary 4.17.2, each chosen tube contains a K-point. The valuative criterion [StacksAlg, Tag 0A40] extends this point uniquely to an O-section; its specialization is the chosen point. These sections are disjoint and lie in the relative smooth locus: an intersection, or a closed complement of the smooth locus meeting a section, would also meet its special point. Attach a separate copy of $\mathcal{E}$ at each section by Lemma 4.42.1.

Both geometric fibres of $\mathcal{C}$ are connected of genus $g(C)$ by Lemma 4.41.1. On either fibre, over its algebraically closed ground field F, one elliptic attachment gives

$$0 \longrightarrow \mathcal{O}_{A \cup E} \longrightarrow \mathcal{O}_A \oplus \mathcal{O}_E \longrightarrow \mathcal{O}_p \longrightarrow 0.$$

The final map on global sections is the surjective difference $F \oplus F \to F$. Its cohomology sequence gives $H^0(A \cup E, \mathcal{O}) = F$ and $H^1(A \cup E, \mathcal{O}) = H^1(A, \mathcal{O}) \oplus H^1(E, \mathcal{O})$. Thus the fibres remain connected and their genus is $g(C) + 3|V|$. The set V is nonempty: otherwise connected compact X would itself be an open ball or annulus. Hence this genus is at least three. The connected nodal-fibre criterion [StacksAlg, Tags 0E6L, 0E6T] makes $\mathcal{Y}_0$ prestable.

Apply stabilization [StacksAlg, Tag 0E8A]. Its fibre maps contract rational tails and then rational bridges as described in [StacksAlg, Tag 0E7Q]. In a component U of $X \setminus V$ containing an unwanted vertex, Lemma 4.41.2 gives a finite tree of smooth rational components with one or two labelled boundary ends. In a one-ended tree, successively remove leaves away from the boundary end. These are rational tails and remove the whole tree. In a two-ended tree, first remove all leaves away from the unique path joining its

ends; then contract the remaining rational bridges. The two boundary ends remain distinct branch labels if they attach to the same component in V, and contracting the path creates a self-node on that component. Every unwanted vertex is thereby removed. At each stage every V-component retains its three auxiliary elliptic attachments, and every elliptic component retains its one attachment. The remaining components are stable. The uniqueness of stabilization proves that exactly these components survive.

In the generic fibre the original smooth curve has at least three attachments and each elliptic tail has one, so this fibre is already stable. Its stabilization is therefore its identity. Finally [StacksAlg, Tag 0E74] makes the relative dualizing line bundle of $\mathcal{Y}$ ample over a neighborhood of the closed point of $\operatorname{Spec} O$, hence over the whole base. By [StacksAlg, Tags 0D32, 0B5N], $\mathcal{Y}$ is a projective scheme over this local base. □

Lemma 4.42.3 (Lifting to a generic normalization)**.** *Let O be a rank-one valuation domain with fraction field K. Let $\mathcal{N} \to \mathcal{Y}$ be a finite relative normalization in a specified finite normalization of $\mathcal{Y}_K$. Let T be a normal integral flat O-scheme. A morphism $f : T \to \mathcal{Y}$ with a specified lift on T_K has a unique lift $T \to \mathcal{N}$. If the generic lift lands in one clopen generic-labelled piece of $\mathcal{N}$, then the whole lift lands in that piece. If T/O is proper and $\mathcal{N}/O$ is separated, the lift is proper.*

Proof. Choose π with $O[1/\pi] = K$. Flatness makes T_K dense in T. For $Q = \operatorname{Spec} R \subset \mathcal{Y}$, write $\mathcal{N}_Q = \operatorname{Spec} B$. The algebra B is integral over R. The specified generic lift pulls each $b \in B$ back to a rational function in $K(T)$. Its monic equation over R pulls back to a monic equation over every local ring of $f^{-1}(Q)$. Those rings are integrally closed in $K(T)$, so the rational function is regular at every point of $f^{-1}(Q)$. Its local representatives glue, yielding an R-homomorphism $B \to \Gamma(f^{-1}(Q), \mathcal{O}_T)$. These maps agree on overlaps because their generic restrictions agree and T_K is dense. They define the lift, and the same argument proves uniqueness. The inverse image of any other clopen labelled piece is open and misses the dense generic fibre, so is empty. For properness, factor the lift through its closed graph in $T \times_O \mathcal{N}$ and the proper projection to $\mathcal{N}$; this is also [StacksAlg, Tag 01W6]. □

Theorem 4.42.4 (Exact realization by a split nodal model)**.** *Let K be algebraically closed, complete and nontrivially valued, with its original absolute value. Let C/K be a smooth connected projective curve and let V be a semistable vertex set of C^{an}. There is a split nodal algebraic model $\mathcal{C}_V/O$ whose component vertices are exactly the original points of V. More precisely, from every split nodal algebraic model $\mathcal{C}_0$ with vertex set containing V, there is a proper morphism $\mathcal{C}_0 \to \mathcal{C}_V$ inducing the identity on C.*

Proof. Theorem 4.38.2, with empty marking, supplies a split nodal model, and Theorem 4.27.4 supplies one whose vertex set W contains V. Start with any such $\mathcal{C}_0$. Apply Lemma 4.42.2, obtaining $\mathcal{C}_0 \to \mathcal{Y}_0 \to \mathcal{Y}$. The generic normalization of the stable curve $\mathcal{Y}_K$ is the disjoint union of C and the labelled auxiliary elliptic curves. The special nodes of $\mathcal{Y}$ have the required split pointed charts: apply Theorem 4.18.2; its finite extension of the algebraically closed field K is K. Proposition 4.37.3 therefore gives a finite relative normalization

$$\mathcal{N} = \mathcal{C}_V \amalg \coprod_j \mathcal{E}_j \longrightarrow \mathcal{Y}.$$

The C-labelled piece $\mathcal{C}_V$ is projective, flat and finitely presented with split reduced nodal special fibre.

The source $\mathcal{C}_0$ is integral and normal. Flatness embeds the ring of each nonempty affine open into its generic ring, the ring of a nonempty open of the integral curve C; this proves integrality. Smooth points are normal over the normal valuation ring by [StacksAlg, Tag 033C]. At a special node, the smoothing parameter is nonzero, so Lemma 4.37.2 gives normality of its standard ring $O[u,v]/(uv-a)$. Normality ascends along the pointed étale chart and descends to the model local ring by [StacksAlg, Tags 033C, 033G]. These loci cover $\mathcal{C}_0$.

The generic restriction of $\mathcal{C}_0 \to \mathcal{Y}$ has its specified lift to the C-labelled generic normalization, since the generic stabilization is the identity. Lemma 4.42.3 gives a proper morphism $q : \mathcal{C}_0 \to \mathcal{C}_V$ inducing the identity on C.

By Lemma 4.42.2, the original surviving special components are precisely those indexed by V. At their generic points $\mathcal{Y}$ is relatively smooth, so relative normalization is the identity there. The local normalization formula of Lemma 4.37.2 introduces no extra vertical components: at $a \neq 0$ it is the identity, and at $a = 0$ it separates the two branches. Each auxiliary elliptic

model is normal by smoothness over the normal base and is integral by flatness and its integral generic fibre. The same normalization-lift lemma puts its surviving special component in its own generic-labelled piece $\mathcal{E}_j$. Thus these auxiliary components belong to none of the other pieces, and $\mathcal{C}_{V,s}$ has exactly the surviving original components.

For $v \in V$, let η_v be its original component generic point. The point $q_s(\eta_v)$ is the generic point of its surviving component in $\mathcal{C}_{V,s}$. Functoriality of specialization for the completed morphism gives

$$\mathrm{sp}_{\mathcal{C}_V}(v) = q_s(\eta_v).$$

The singleton-fibre theorem 4.12.2 identifies the new component vertex with the original point v. Thus the vertex set is exactly V as a subset of C^{an}. □

Theorem 4.42.5 (The vertex correspondence for algebraic completions)**.** *Let K and C be as in Theorem 4.42.4. Consider the formal completions of split nodal algebraic models of C, with their specified identifications of the generic fibre with $X = C^{\mathrm{an}}$. Their isomorphism classes over X correspond bijectively to the semistable vertex sets of X. For two such models $\mathfrak{C}_W$ and $\mathfrak{C}_V$, there is a formal morphism $\mathfrak{C}_W \to \mathfrak{C}_V$ inducing the identity on X if and only if $V \subset W$; when it exists, it is unique. The graph attached to $\mathfrak{C}_V$ is $\Sigma(X, V)$ with its original logarithmic edge lengths.*

Proof. The actual smooth and nodal tube theorems and Theorem 4.23.3 show that every such model has a semistable vertex set and the asserted graph. Theorem 4.42.4 gives every semistable vertex set. Its uniqueness as a formal model follows from Proposition 4.40.3, using Proposition 4.22.2 and Proposition 4.20.3 for the exact boundaries.

If $V \subset W$, apply the contraction in Theorem 4.42.4 to an algebraic model underlying $\mathfrak{C}_W$ and complete the resulting morphism. The target has vertices V, so the preceding uniqueness identifies it with $\mathfrak{C}_V$ over X. Conversely, Lemma 4.40.4 gives the required inclusion for any such formal morphism. Its uniqueness is generic-fibre faithfulness. Finally, the nodal tubes are exactly the annular components of $X \setminus V$; their Gauss intervals give $\Sigma(X, V)$, and Corollary 4.16.2 identifies their lengths with their logarithmic moduli. □

Corollary 4.42.6 (The correspondence for pointed semistable formal models)**.** *Let K be algebraically closed, complete and nontrivially valued, and let C/K be a smooth connected projective curve. The vertex-set functor is an equivalence between the pointedly semistable formal models of Definition 4.39.4 and the partially ordered set of semistable vertex sets, with a unique arrow $W \to V$ when $V \subset W$. Every such formal model is isomorphic over C^{an} to the completion of a split nodal algebraic model. Its skeleton and logarithmic edge lengths are those of its semistable vertex set.*

Proof. By Corollary 4.39.5, a pointedly semistable formal model has a semistable vertex set V and the exact tube boundaries required in Proposition 4.40.3. Theorem 4.42.4 constructs a split nodal algebraic model $\mathcal{C}_V$ with that vertex set. Its completion belongs to the pointed formal class. Indeed, complete the pointed algebraic smooth and nodal charts π-adically. Their rings are admissible by Lemma 4.13.1. The completed maps are topologically finitely presented. They are formally étale because, on a discrete target annihilated by a power of π, maps from a completed ring factor through its corresponding algebraic quotient, where the algebraic étale lifting property [StacksAlg, Tag 00UR] gives the unique lift.

Proposition 4.40.3 now identifies the given formal model with $\widehat{\mathcal{C}_V}$ over C^{an}. The existence and uniqueness of arrows are consequently Theorem 4.42.5; alternatively the necessary vertex inclusion is already Lemma 4.40.4. The skeletal statement follows from the same identification and the final assertion of that theorem. □

4.43 Finite unramified coefficients and integral sections

Let K be a complete nontrivially real-valued field, with its given absolute value. Put $O = K^{\circ}$, $\mathfrak{m} = K^{\circ\circ}$ and $k = \widetilde{K}$, and fix $\pi \in \mathfrak{m} \setminus \{0\}$. A *finite unramified extension* in this section is the lift L/K of a finite separable extension l/k constructed in Lemma 4.15.2. Thus $[L : K] = [l : k]$ and $|L^{\times}| = |K^{\times}|$.

Lemma 4.43.1 (Integral Galois descent for unramified coefficients)**.** *Suppose l/k is finite Galois, take its Galois lift L/K, and write $O' = L^{\circ}$ and $G = \mathrm{Gal}(L/K)$. Then O'/O is finite free and* étale*, and*

$$O' \otimes_O O' \xrightarrow{\sim} \prod_{g \in G} O', \qquad a \otimes b \longmapsto (a\, g(b))_g.$$

For every O-module M, the coefficient inclusion identifies

$$M = (M \otimes_O O')^G.$$

If A_0 is an admissible formal O-algebra, its coefficient extension is the complete admissible algebra $A_0 \otimes_O O'$. Its generic algebra is $A_0[1/\pi]\widehat{\otimes}_K L$.

Proof. Write $O' = O[\alpha] = O[T]/(P)$ as in Lemma 4.22.1, with $1, \alpha, \dots, \alpha^{d-1}$ an orthonormal O-basis. The polynomial P is monic, and $P'(\alpha)$ has nonzero reduction. For a square-zero ideal, a lift b of a root is corrected uniquely to a root by $b - P(b)/P'(b)$; successive square-zero quotients give the same assertion for a nilpotent ideal. Thus the finite presentation is formally étale, hence étale by [StacksAlg, Tag 00UR]. Lemma 4.18.1 gives all its roots $g(\alpha)$ in O', with pairwise different residues. Evaluation on these roots is an isomorphism: its matrix on $1, T, \dots, T^{d-1}$ has Vandermonde determinant $\prod_{g<h}(h(\alpha) - g(\alpha)) \in (O')^\times$. This proves the tensor-product assertion, as well as faithful flatness.

The fixed module assertion holds without dividing by d. Here is an explicit descent argument. Let $z = \sum_i m_i \otimes b_i \in M \otimes_O O'$ be invariant. Under the displayed product isomorphism, the two elements

$$\sum_i m_i \otimes b_i \otimes 1, \qquad \sum_i m_i \otimes 1 \otimes b_i \quad \text{in } M \otimes_O O' \otimes_O O'$$

have equal g-coordinates for every g, so are equal. Apply the O-linear coefficient map $\lambda : O' \to O$ which takes the coefficient of 1 in the indicated basis; it satisfies $\lambda(1) = 1$. Applying λ on the first coefficient factor gives $z = (\sum_i \lambda(b_i) m_i) \otimes 1$. Injectivity of $m \mapsto m \otimes 1$ follows from the same coefficient map. This proves the equality of modules.

A finite free sum commutes with π-adic completion. The extended ring is π-torsion-free, and its finite presentation follows by extending an admissible presentation of A_0. For the generic algebra put $A = A_0[1/\pi]$. Inverting π identifies $(A_0 \otimes_O O')[1/\pi]$ with $A \otimes_K L$ by the coefficient maps. In Proposition 2.33.19 take the finite algebra to be L over K and the target to be A. The actual absolute value on L is its finite-module maximum norm in the displayed basis. The resulting maximum norm on $A \otimes_K L$ and the defining quotient norm of $(A_0 \otimes_O O')[1/\pi]$ given by Proposition 3.14.7 have the same closed unit ball $A_0 \otimes_O O'$. Scaling by powers of π compares the norms in both directions with constant $|\pi|^{-1}$, giving the required bounded isomorphism with $A\widehat{\otimes}_K L$. All these identifications preserve the chosen inclusion of K and its absolute value. □

Proposition 4.43.2 (Bounded sections after an unramified extension)**.** *Keep the Galois extension and notation of Lemma 4.43.1. Let $\mathfrak{X}$ be an admissible formal O-scheme with reduced special fibre, let $\mathfrak{X}'$ be its coefficient extension to O', and let $X' = X_L$ be the generic fibre. For a formal open $\mathfrak{U} \subset \mathfrak{X}$, write $\mathfrak{U}'$ for its inverse image and D, D' for the corresponding generic domains. Then*

$$\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}) = \Gamma(\mathfrak{U}', \mathcal{O}_{\mathfrak{X}'})^G = \{f \in \Gamma(D', \mathcal{O}_{X'})^G : |f| \leq 1\}.$$

These equalities commute with restriction and identify the π^n ideals by the bound $|\pi|^n$.

Proof. On an affine formal open the first equality is Lemma 4.43.1 with $M = A_0$. Its special-fibre extension is $(A_0/\mathfrak{m}A_0) \otimes_k l$, which is reduced by the separable-extension argument in Lemma 4.22.1. Proposition 4.40.1 therefore identifies the integral sections upstairs with the displayed bounded analytic sections. The actual generic domain upstairs is D_L by the coefficient presentation and the generic-fibre gluing of Proposition 3.17.2.

For a general formal open, restrict an invariant integral section to the inverse images of a formal affine cover. The affine equalities descend it uniquely on each member. Their restrictions agree because the upstairs restrictions agree and coefficient extension is injective. The descended sections consequently glue. The converse and restriction compatibility follow from the same affine identifications. Apply the equality to f/π^n for the last assertion. In particular this descent recovers the actual integral sheaf and its adic topology. □

4.44 Auxiliary multisections and generic normalization

Lemma 4.44.1 (Smooth finite étale multisections)**.** *Let $\mathcal{C}/O$ be a proper flat finitely presented curve, and let q be a smooth closed point of $\mathcal{C}_s$ with $l = \kappa(q)$ finite separable over k. There is a closed immersion*

$$\operatorname{Spec} L^\circ \longrightarrow \mathcal{C}$$

through q, for a finite unramified lift L/K of l/k. Its image is contained in the relative smooth locus. Distinct choices of the special points give disjoint such multisections.

If $\mathcal{C}_s$ is nodal, a finite union of its irreducible components can be met by a finite disjoint union of these multisections, avoiding any prescribed finite set of smooth points and all nodes, with at least three geometric intersection points on each geometric component in that union and none on the other components.

Proof. Choose the unramified lift by Lemma 4.15.2. Its valuation ring has the monic presentation $O[T]/(P)$ of Lemma 4.22.1, with P' a unit at the lifted root. The square-zero correction in the proof of Lemma 4.43.1 does not require a Galois extension: it proves this presentation formally étale. Thus L°/O is finite étale by [StacksAlg, Tag 00UR]. After this coefficient extension, the map $\operatorname{Spec} l \to \mathcal{C}_s$ gives a rational smooth point of $\mathcal{C}_{L^\circ,s}$. Its open-disc tube contains an L-point by Corollary 4.17.2. The valuative criterion, [StacksAlg, Tag 0A40], extends that point to a section over L°, with the given specialization. Compose with the coefficient projection to obtain $i : \operatorname{Spec} L^\circ \to \mathcal{C}$.

Choose an affine neighborhood $\operatorname{Spec} A$ of q. Its inverse image contains the closed point of the local spectrum $\operatorname{Spec} L^\circ$, hence that whole spectrum. Choose $a \in A$ whose image in l is a primitive generator β over k. Put $d = [l : k]$. The elements $1, i^*(a), \dots, i^*(a)^{d-1}$ are an O-basis of L°: relative to the basis in Lemma 4.22.1, their change-of-basis matrix reduces to the change between two bases of l/k. Its determinant is a unit of O. Thus $A \to L^\circ$ is surjective. The morphism i is finite, since its graph is a closed subscheme of $\mathcal{C} \times_O \operatorname{Spec} L^\circ$ and projection from this product is finite. Consequently the closed immersion just proved on the neighborhood of its entire image is a closed immersion into $\mathcal{C}$. The inverse image of the smooth locus is an open neighborhood of the closed point and is therefore the whole source.

Two multisections with different special points cannot meet: their intersection is closed in a scheme finite over O; if nonempty, its proper image in the local spectrum contains the closed point, contradicting their different specializations.

It remains to choose enough separable points. Choose an algebraic closure of k by [StacksAlg, Tag 09GT] and work over the separable closure k^{sep} of k inside it. Every nonempty open in the smooth part of a geometric component contains a k^{sep}-point. Indeed, a smooth coordinate gives an étale map to the affine line ([StacksAlg, Tag 054L]); its image is open ([StacksAlg, Tag 03WT]). The infinite field k^{sep} has a rational point in this image, and the nonempty étale fibre has finite separable residue fields by [StacksAlg, Tag 02GL], hence

rational points. A purely inseparable extension does not split the underlying components: for such a field extension the prime over a given prime is unique, because membership in it is detected after taking sufficiently high p-powers. Thus this argument applies to each geometric component.

Choose three points on each required component, successively avoiding the finitely many previously chosen Galois orbits and the prescribed exceptional points. Each chosen point is defined over a finite separable extension of k, since the curve is of finite type. Its image is a closed separable point of the original curve. Take all the finitely many resulting closed points and lift them by the first part. Their geometric specializations have the required incidence and disjointness. □

Lemma 4.44.2 (Pinching along finite étale markings)**.** *Let Z be finite* étale *over O, and let $Z \hookrightarrow \mathcal{A}$, $Z \hookrightarrow \mathcal{B}$ be closed immersions into the smooth loci of proper flat finitely presented nodal O-curves. Their pushout is a proper flat finitely presented nodal O-curve. Its formation commutes with arbitrary base change on O.*

Proof. The pushout exists by [StacksAlg, Tag 0E25]: each fibre of either closed immersion contains at most one point, so the required affine-neighborhood condition holds. On compatible affine neighborhoods its ring is $A \times_C B$, where both maps onto the coordinate ring C of the corresponding open in Z are surjective. This ring C is flat over O. There is an exact sequence

$$0 \longrightarrow A \times_C B \longrightarrow A \oplus B \xrightarrow{a-b} C \longrightarrow 0.$$

Let $P = A \times_C B$ and $Q = A \oplus B$, so that $0 \to P \to Q \to C \to 0$ is the displayed sequence of ordinary O-modules. For an arbitrary O-algebra T, choose a surjection from a free O-module F onto its underlying module and write R for the kernel. If a class in $P \otimes_O T$ maps to zero, lift it to p_F in $P \otimes_O F$. The image of p_F in $Q \otimes_O F$ comes from an element q_R of $Q \otimes_O R$. The image of q_R in $C \otimes_O R$ maps to zero in $C \otimes_O F$, so is zero because C is flat. Lift q_R to $P \otimes_O R$. These lifts exist because tensoring a module quotient is the quotient of the tensor product by the image of the relations, by the defining balanced maps. Tensoring the inclusion $P \to Q$ with the free module F is a direct sum of copies of that inclusion. It follows that p_F is the image of the lift in $P \otimes_O R$, so the original class vanishes in $P \otimes_O T$. The same quotient property identifies the image with the kernel of $(A \otimes_O T) \oplus (B \otimes_O T) \to C \otimes_O T$. The resulting isomorphism with the fibre

product is an algebra map by its two coordinate maps. Thus these affine pushouts and their localizations commute with every base change.

Choose a finite Galois extension of k splitting the finite étale scheme Z_s, and take its unramified lift by Lemma 4.18.1. Its integral coefficient extension is faithfully flat by Lemma 4.43.1. Over its valuation ring O', every rational point of Z_s lifts to an open-and-closed section of the finite étale scheme $Z_{O'}$, by Lemma 4.15.1. These sections exhaust it: their complement is finite over O' and has empty special fibre, hence is empty. The base-changed pushout is therefore the successive pinching of disjoint smooth sections. Lemma 4.42.1 proves it flat, finitely presented and nodal. These properties descend along the finite faithfully flat cover: use [StacksAlg, Tags 02L0, 0C5E], the latter including flatness and relative dimension one.

Both old curves map by closed immersions into the pushout, as is also clear from the two surjective projections of $A \times_C B$. Their disjoint union is a finite surjective cover. The pushout is separated by [StacksAlg, Tag 0E26] and universally closed over O by [StacksAlg, Tag 03GN]. Together with the finite presentation just established, these properties give properness. The affine kernel calculation already proves the stated base-change compatibility. □

Proposition 4.44.3 (Generic normalization of a nodal family)**.** *Let $\mathcal{Y}/O$ be a projective flat finitely presented nodal curve family. Its relative normalization in the normalization of $\mathcal{Y}_K$ is finite over $\mathcal{Y}$, normal, projective, flat and finitely presented over O, with reduced nodal geometric fibres. A decomposition of the generic normalization into open-and-closed pieces extends uniquely to such a decomposition of this relative normalization. The construction is functorial for semilinear isometric isomorphisms, including permutations of the generic pieces.*

In particular a nodal O-model with smooth geometrically integral generic fibre is normal and integral.

Proof. The generic normalization is finite by [StacksAlg, Tag 0BXR]. Since $O[1/\pi] = K$, its composite with the generic open immersion is quasi-compact and quasi-separated. The relative integral-closure construction of [StacksAlg, Tag 035I] therefore defines $\nu : \mathcal{N} \to \mathcal{Y}$.

Theorem 4.18.2 supplies a finite unramified Galois extension L/K giving split pointed charts at every special node. Its valuation ring is finite étale and faithfully flat over O by Lemma 4.43.1. Relative normalization commutes

with this base change and with the pointed étale charts by [StacksAlg, Tag 03GV]. On a chart $R_a = O'[u,v]/(uv-a)$ its actual algebra is

$$R'_a = R_a \quad (a \neq 0), \qquad R'_0 = O'[u] \oplus O'[v],$$

by Lemma 4.37.2. At smooth points it is the identity, since a smooth algebra over the normal valuation ring is normal ([StacksAlg, Tag 033C]). The finitely many nodal charts and the smooth locus cover the family: an omitted generic point would have a specialization in the proper closed fibre, which these open sets cover. These formulas prove, on an actual étale cover, finiteness over $\mathcal{Y}_{O'}$, normality, and flat finite presentation over O', with reduced nodal geometric fibres. They require no choice of labels for the generic components.

Finiteness descends by [StacksAlg, Tag 02LA]. Normality descends on local rings by [StacksAlg, Tag 033G]; flatness and local finite presentation descend on the induced source cover by [StacksAlg, Tags 036K, 036N]. The finite morphism to projective $\mathcal{Y}$ gives quasi-compactness and separatedness, hence finite presentation. Nodality descends by [StacksAlg, Tag 0C5E]. Finally finiteness and projectivity of $\mathcal{Y}/O$ give projectivity of $\mathcal{N}/O$ by [StacksAlg, Tags 0B3I, 0C4P].

On every affine open of $\mathcal{N}$, flatness makes π a nonzerodivisor. A labelled generic idempotent is an element of the total quotient ring satisfying $e^2 - e = 0$. Normality and [StacksAlg, Tag 034M] put it in the original affine ring. These extensions agree on overlaps by π-torsion-freeness; they remain orthogonal and sum to one. This proves the asserted unique decomposition. Integral closure and these idempotents are carried to their counterparts by every indicated isomorphism, proving functoriality.

For the last assertion, a flat model of the integral generic curve is integral: each affine coordinate ring embeds in its generic coordinate ring, a domain. In the preceding local charts its smoothing parameter is nonzero, because the generic fibre is smooth. Thus the relative normalization is locally the identity after the faithfully flat cover. The original model is consequently normal as well. □

Lemma 4.44.4 (Proper contractions as ringed-space quotients)**.** *Let $q : T \to Y$ be a proper birational morphism of integral schemes, with Y normal. Then $q_*\mathcal{O}_T = \mathcal{O}_Y$. Every morphism $h : T \to Z$ which is constant on the set-theoretic fibres of q factors uniquely through a morphism $Y \to Z$.*

Proof. Let $U = \operatorname{Spec} A \subset Y$ be affine. A section $f \in \Gamma(q^{-1}U, \mathcal{O}_T)$ is a rational function in the common function field F. For every valuation ring $A \subset R \subset F$ with fraction field F, properness extends the generic map to T to a map $\operatorname{Spec} R \to q^{-1}U$, by [StacksAlg, Tag 0A40]. Therefore $f \in R$. The normal domain A is the intersection of these valuation rings by [StacksAlg, Tag 090P], so $f \in A$. The reverse inclusion is pullback. These equalities commute with restriction and prove the sheaf assertion.

The proper map q is surjective, since its closed image contains the generic point of Y. It is therefore a closed quotient map on underlying spaces. The map h induces a unique continuous map $\bar{h} : Y \to Z$. For an open $U \subset Z$, pullback by h, followed by the sheaf equality, gives

$$\mathcal{O}_Z(U) \longrightarrow \mathcal{O}_T(q^{-1}\bar{h}^{-1}U) = \mathcal{O}_Y(\bar{h}^{-1}U).$$

These maps respect restrictions. They are local on stalks: choose t above y; if the image in $\mathcal{O}_{Y,y}$ of an element of $\mathfrak{m}_{Z,\bar{h}(y)}$ were a unit, its image in $\mathcal{O}_{T,t}$ would be a unit, contrary to locality of h. Hence $\bar{h}$ is a morphism of schemes. Surjectivity and the injective sheaf pullback give uniqueness. □

4.45 Geometric vertices and marked forests

Lemma 4.45.1 (An algebraically closed comparison field)**.** *There is an isometric extension $K \subset \Omega$ with Ω complete and algebraically closed and with $\widetilde{\Omega}$ an algebraic closure of k.*

Proof. Choose an algebraic closure $\overline{K}$ by [StacksAlg, Tag 09GT]. The unique absolute values of its finite subextensions, supplied by Lemma 1.12.1, agree on intersections and give an absolute value on $\overline{K}$. Let Ω be its completion. Completion changes neither the value group nor the residue field: approximate a nonzero element with error smaller than its absolute value, or a unit with error smaller than one, respectively.

We verify algebraic closedness. Let $P(T) = T^n + \sum_{i<n} a_i T^i \in \Omega[T]$ be monic. Choose monic $P_j \in \overline{K}[T]$ with coefficients converging to those of P. Their coefficients are uniformly bounded, so all their roots have absolute value at most one fixed $R \geq 1$: beyond R, the leading term would strictly dominate all the others. Write ϵ_j for the maximum coefficient difference between P_j and P_{j+1} and choose a root α_1 of P_1. If α_j has been chosen, then

$$|P_{j+1}(\alpha_j)| \leq \epsilon_j R^{n-1}.$$

Factor P_{j+1} over $\overline{K}$. At least one of its roots α_{j+1} satisfies $|\alpha_j - \alpha_{j+1}| \leq (\epsilon_j R^{n-1})^{1/n}$. These bounds tend to zero, so the ultrametric inequality makes (α_j) Cauchy. Its limit is a root of P. Induction on the degree now shows that Ω is algebraically closed.

The residue field of $\overline{K}$ is algebraic over k: each of its elements comes from a finite extension, whose residue degree is finite by [StacksAlg, Tag 0ASH]. It is algebraically closed, since lifting a monic residual polynomial and taking a root in $\overline{K}$ produces an integral root and hence a residual root. Completion retains that residue field, proving the assertion. □

Proposition 4.45.2 (The arithmetic of a model vertex)**.** *Let $\mathcal{C}/O$ be a projective flat finitely presented nodal model of a smooth projective curve. If v is the component point belonging to a special component E, then*

$$|\mathcal{H}(v)^\times| = |K^\times|, \qquad \widetilde{\mathcal{H}(v)} = k(E).$$

The field $k(E)/k$ is separably generated of transcendence degree one. For the comparison field Ω above, the entire inverse image of v in C_Ω^{an} consists of the component points of the geometric components above E. In particular it is finite.

Proof. The equalities are Proposition 4.24.2 on a completed affine neighborhood of the generic point of E. A nodal curve is smooth at its component generic points. A smooth étale coordinate, as in [StacksAlg, Tags 054L, 02GL], presents each function field as a finite separable extension of $k(t)$. This proves the separability and transcendence-degree assertions.

Lemma 4.28.1 identifies the actual base-changed model and its specialization map. The special fibre is $\mathcal{C}_s \times_k \widetilde{\Omega}$ and remains reduced. Every point over the generic point η of E is a component generic point. To check this, its fibre ring is $k(E) \otimes_k \widetilde{\Omega}$, an integral algebraic extension of the field $k(E)$ and hence zero-dimensional by [StacksAlg, Tag 00GS(3)]. Any prime properly below a prime over η would also lie over the minimal prime η, contradicting this zero-dimensionality. The points over η are therefore precisely the finitely many geometric component generic points. If an analytic point projects to v, its specialization lies over η, so it is one of their singleton component fibres by Proposition 4.12.2. Conversely each of those component points projects to v by functorial specialization and the same singleton-fibre statement. This proves the entire-fibre assertion. □

Definition 4.45.3 (Marked semistable models over the original field)**.** Let C/K be smooth, projective and geometrically connected. A *semistable algebraic model* of C means a projective flat finitely presented nodal O-curve with generic fibre identified with C and geometrically connected special fibre. Its special fibre is geometrically reduced. A *marking* is a finite étale O-scheme D embedded as a closed subscheme of the relative smooth locus. An empty marking is allowed. Write W for the component vertex set of the model. A morphism of marked models carries a specified isomorphism of its marking with the target marking, compatible with the curve morphism and with the specified generic identifications.

The *geometric marked dual graph* has one vertex for each component over $\overline{k}$, one edge for each geometric node, and one labelled leg for each point of $D_{\overline{k}}$. The two branches of a self-node give the two ends of its loop.

Fix a nonempty subset $V \subset W$, and let $\overline{V}$ denote all geometric vertices over its special components. For a connected component T of the subgraph induced by the vertices outside $\overline{V}$, let $b(T)$ count its incident edges leading to $\overline{V}$, with multiplicity, and let $m(T)$ count its marked legs.

Definition 4.45.4 (The marked forest condition)**.** The subset V satisfies the marked forest condition if every such T is a tree, all the geometric special components at its vertices are smooth $\mathbf{P}^1_{\overline{k}}$, and

$$b(T) + m(T) \leq 2.$$

Connectedness and $V \neq \varnothing$ give $b(T) \geq 1$. Thus a tree with one attaching branch carries at most one mark, and a tree with two attaching branches carries none. The two attaching branches may end on the same protected component.

Theorem 4.45.5 (Exact marked contraction over the original field)**.** *Let $(\mathcal{C}_0, D)$ be a marked semistable algebraic model of C, and let $\varnothing \neq V \subset W$ satisfy the marked forest condition. There is a marked semistable algebraic model $(\mathcal{C}_V, D_V)$ and a proper morphism*

$$q_V : \mathcal{C}_0 \longrightarrow \mathcal{C}_V$$

inducing the identity on C, with $D \xrightarrow{\sim} D_V$ and with component vertices exactly the original points of V in C^{an}.

On a geometric special fibre, the positive-dimensional fibres of q_V are exactly the trees T in Definition 4.45.4. A tree with $b(T) = 1$ maps to a smooth point, and a tree with $b(T) = 2$ maps to a node with the indicated attaching branches. Outside these fibres the special-fibre map is an isomorphism. Moreover $(q_V)_*\mathcal{O}_{\mathcal{C}_0} = \mathcal{O}_{\mathcal{C}_V}$.

Proof. Protect the desired components and markings. Lemma 4.44.1 gives a finite étale multisection A on the protected special components, disjoint from D and from the nodes, meeting each protected geometric component at least three times. There exists a smooth projective elliptic curve $\mathcal{E}/O$ with a section. If $\operatorname{char} k \neq 2$, use $y^2 = x(x-1)(x-\lambda)$ with $\overline{\lambda} \notin \{0, 1\}$; such a residue exists even when $k = \mathbf{F}_3$. If $\operatorname{char} k = 2$, use $y^2 + y = x^3$. The distinct-root and derivative calculations in the proof of Lemma 4.42.2 show smoothness of both fibres and of the infinity section. Lemma 4.36.2 gives flatness, finite presentation and projectivity, and [StacksAlg, Tag 0BYD] gives geometric genus one.

Put $Z = A \amalg D$. Attach $\mathcal{E} \times_O Z$ to $\mathcal{C}_0$ along Z and its infinity section. Lemma 4.44.2 produces a proper flat finitely presented nodal family $\mathcal{Y}_0/O$. On either geometric fibre, each attachment adds a smooth elliptic tail at one point. The exact sequence

$$0 \longrightarrow \mathcal{O}_{F \cup E} \longrightarrow \mathcal{O}_F \oplus \mathcal{O}_E \longrightarrow \mathcal{O}_p \longrightarrow 0$$

has a surjective difference map on constants. Its cohomology therefore preserves geometric connectedness and adds one to the genus. The original fibres have the same genus $g(C)$ by [StacksAlg, Tag 0E6L], applied to the proper flat family with geometrically reduced connected fibres. Consequently $\mathcal{Y}_0$ is prestable of genus $g(C) + \deg(Z/O) \geq 3$. Its generic fibre is stable: the original smooth component has at least three attachments, and each elliptic component has one.

Stabilize and separate the generic components. Apply [StacksAlg, Tag 0E8A] to obtain the stabilization $c : \mathcal{Y}_0 \to \mathcal{Y}$. Its generic map is an isomorphism. On the geometric special fibre it contracts rational tails and bridges as in [StacksAlg, Tag 0E7Q]. In a removed tree, regard its $b(T)$ attachments to protected components and its $m(T)$ elliptic marking tails as distinguished ends. There are one or two such ends. With one end, successively remove leaves away from it. With two ends, first remove the leaves off their unique joining path and then contract the rational bridges on that path. Every

removed vertex disappears. Every protected component keeps its three auxiliary elliptic attachments, and every elliptic component keeps its attachment; all these components are stable. Uniqueness of stabilization consequently identifies these as exactly the surviving components. In the two-ended case the ends retain separate branch labels even if they lie on the same protected component.

The stable family $\mathcal{Y}$ is projective over O: [StacksAlg, Tag 0E74] makes its dualizing line bundle ample over a neighborhood of the closed point, hence over the entire local base, and [StacksAlg, Tags 0D32, 0B5N] give projectivity. Apply Proposition 4.44.3 to $\mathcal{Y}$. Its generic normalization has the distinguished component C and the elliptic components attached along Z_K. Let $\mathcal{C}_V$ be its open-and-closed C-labelled piece. It is normal, projective, flat and finitely presented with nodal geometric fibres.

The source $\mathcal{C}_0$ is normal and integral by the same proposition. The morphism $\mathcal{C}_0 \to \mathcal{Y}_0 \to \mathcal{Y}$ has its specified generic lift to C. Lemma 4.42.3 therefore gives an actual proper morphism $q_V : \mathcal{C}_0 \to \mathcal{C}_V$ inducing the identity on C.

Identify the fibres, marks, and original vertices. The normalization calculation is the identity at smooth points and at nodes with nonzero smoothing parameter; at a node with zero parameter it separates the two generic branches. Hence it introduces no vertical component. Each surviving auxiliary or marking elliptic component belongs to its own generic-normalization piece: apply the same normalization-lift lemma to its smooth normal elliptic model after an unramified coefficient extension splitting Z. Thus $\mathcal{C}_V$ contains precisely the protected original components.

The geometric fibre maps of stabilization are compositions of the elementary rational-tail and rational-bridge contractions. They identify an unmarked one-ended tree with its smooth attaching point. A two-ended tree without marks gives a node joining its two original branches. For a one-ended tree with one mark, its second distinguished end is the marking elliptic tail. Stabilization joins that tail to the protected component, and generic normalization separates this horizontal node. The mark therefore lands at the resulting smooth point of $\mathcal{C}_V$. These descriptions also cover marks on protected components. Distinct marks remain distinct: each removed one-ended tree contains at most one, different such trees have different attaching points, and the original protected marks avoid those nodal attachments. They show that the claimed trees are exactly the positive-dimensional geometric fibres

and that the map is an isomorphism away from them. In particular the special fibre of $\mathcal{C}_V$ is geometrically connected.

The map $D \to \mathcal{C}_V$ is finite by the closed-graph argument, since D/O is finite. After a finite unramified cover splitting D, it is the union of the just identified disjoint smooth sections, hence a closed immersion. On an affine open of $\mathcal{C}_V$, its ring map is therefore surjective after the faithfully flat coefficient extension; the cokernel vanishes already over O. This proves the closed immersion. Its image is smooth because all its generic and geometric special points are smooth in the nodal family.

At the generic point η_v of any protected original component, stabilization is an isomorphism and generic normalization is the identity. Thus $q_V(\eta_v)$ is the generic point of the corresponding new component. Complete the projective models as in Theorem 4.13.2. Functoriality of specialization in Proposition 3.17.2 gives

$$\mathrm{sp}_{\mathcal{C}_V}(v) = q_V(\eta_v).$$

Proposition 4.12.2 identifies its singleton analytic fibre with the original point v. Every surviving component has just been accounted for, so the vertex set is exactly V. Finally apply Lemma 4.44.4 to the proper birational map q_V and its normal target. □

Proposition 4.45.6 (Necessity of the geometric marked forest condition)**.** *Let $q : (\mathcal{C}_0, D) \to (\mathcal{C}_1, D_1)$ be a morphism of marked semistable algebraic models inducing the identity on C and on the generic marking. Then $V(\mathcal{C}_1) \subset V(\mathcal{C}_0)$, and this subset satisfies the marked forest condition in $\mathcal{C}_0$.*

Proof. Complete q along π. Lemma 4.40.4 gives the vertex inclusion. Extend the original absolute value isometrically to the comparison field Ω of Lemma 4.45.1. Both base-changed models are actual projective flat nodal Ω°-models of C_Ω, and their analytic generic fibres are C_Ω^{an} by Lemma 4.28.1. Their nodes have split rational pointed charts: Theorem 4.18.2 applies, and a finite extension of the algebraically closed field Ω is Ω itself. Thus the algebraically closed model results apply to these models.

Proposition 4.45.2 identifies their vertex sets with the entire inverse images of the two original vertex sets. Put $\overline{W} = V(\mathcal{C}_{0,\Omega})$ and $\overline{V} = V(\mathcal{C}_{1,\Omega})$. For a component U of $C_\Omega^{\mathrm{an}} \setminus \overline{V}$ meeting $\overline{W}$, Lemma 4.41.2 shows that the source skeleton in U is a finite tree with one or two boundary ends, and all its component curves are smooth $\mathbf{P}^1_{\bar{k}}$. Its vertices and internal edges are exactly

one connected component T of the induced removed graph. Its boundary ends are exactly the $b(T)$ attaching branches, counted separately even when they end at the same protected component.

The set U is the actual tube of a closed point p of the target. For every removed component with generic point η, functorial specialization gives $q_s(\eta) = p$. Since p is closed, every point of that component, including all its marked specializations, also maps to p. If $b(T) = 1$, the target point p is smooth, and the closed immersion $D_{1,\overline{k}}$ has at most one geometric point over it. Thus $m(T) \leq 1$. If $b(T) = 2$, then p is nodal and the marking avoids it, so $m(T) = 0$. This proves the stated condition on every removed geometric tree. □

Proposition 4.45.7 (Algebraic uniqueness of an exact contraction)**.** *Let $q_i : \mathcal{C}_0 \to \mathcal{C}_i$, $i = 1, 2$, be morphisms of semistable algebraic models inducing the identity on C. If the targets have the same component vertices in C^{an}, there is a unique algebraic isomorphism $h : \mathcal{C}_1 \to \mathcal{C}_2$ with $q_2 = h \circ q_1$. It induces the identity on C.*

Proof. After extension to Ω, the target vertex sets are the same by Proposition 4.45.2. The split pointed charts supplied by Theorem 4.18.2 complete to the charts of Definition 4.39.4, as in the proof of Corollary 4.42.6. Corollary 4.39.5 gives the connected open tubes and their exact component-incidence boundaries. Their completed models are isomorphic over C_Ω^{an} by Proposition 4.40.3. Write h_s for the resulting special-fibre isomorphism. It intertwines their specialization maps. Functoriality of specialization and its surjectivity on the source, both in Proposition 3.17.2, give

$$h_s \circ (q_1)_{\overline{k}} = (q_2)_{\overline{k}}$$

on the underlying geometric special-fibre points. Consequently $(q_2)_{\overline{k}}$ is constant on every set-theoretic fibre of $(q_1)_{\overline{k}}$.

We descend this assertion about actual fibres. Every field extension is faithfully flat: its only base ideals are zero and the whole field, so [StacksAlg, Tag 00HD] gives flatness, and its nonzero closed fibre gives faithful flatness by [StacksAlg, Tag 00HP]. If $t_1, t_2 \in \mathcal{C}_{0,s}$ have the same image y under $(q_1)_s$, the tensor product $\kappa(t_1) \otimes_{\kappa(y)} \kappa(t_2)$ is nonzero by the same faithful-flatness criterion. By [StacksAlg, Tag 00E0(2)], a prime of this tensor product supplies a point of $\mathcal{C}_{0,s} \times_{\mathcal{C}_{1,s}} \mathcal{C}_{0,s}$ whose two projections are t_1, t_2. The extension

$\overline{k}/k$ is faithfully flat by the preceding argument. On an affine base change, its tensor functor on modules is the original coefficient tensor functor, so remains faithfully flat. Surjectivity on spectra, [StacksAlg, Tag 00HQ], lifts the chosen point to the base-changed fibre product. The lifted projections have the same $(q_1)_{\overline{k}}$-image, hence the same $(q_2)_{\overline{k}}$-image. Projecting back gives $(q_2)_s(t_1) = (q_2)_s(t_2)$. On the generic fibre q_1 and q_2 are identities. These two fibres exhaust the rank-one base, and their images cannot meet. Thus q_2 is constant on every fibre of q_1.

The morphisms q_i are proper: their graphs are closed in the products over O, and projection from those products is proper. Their targets are normal and integral by Proposition 4.44.3. Lemma 4.44.4 gives an actual algebraic factorization h. Interchanging the indices gives its inverse; uniqueness of factorization shows that the composites are the identity. This proves both the assertion and its uniqueness. □

Theorem 4.45.8 (The ordered category of marked contractions)**.** *Fix a marked semistable algebraic model $(\mathcal{C}_0, D)$. The category of its marked semistable algebraic contractions over C is equivalent to the following partially ordered set: its objects are the nonempty subsets V of $V(\mathcal{C}_0)$ satisfying the geometric marked forest condition, and there is one arrow $V' \to V$ precisely when $V \subset V'$. A contraction here includes its actual morphism from $\mathcal{C}_0$, and arrows commute with these morphisms.*

Proof. Existence is Theorem 4.45.5, necessity is Proposition 4.45.6, and uniqueness for each object is Proposition 4.45.7.

Suppose $V \subset V'$. On the geometric special fibre, every tree removed for V' lies in a tree removed for V: its vertices are still removed, and its connected edges remain in the induced removed graph. The fibre description in Theorem 4.45.5 therefore shows that q_V is constant on every geometric fibre of $q_{V'}$. The fibre-product argument in the proof of Proposition 4.45.7 descends this to the original special fibre. The generic assertion is immediate. Lemma 4.44.4 gives the unique factorization

$$\mathcal{C}_0 \xrightarrow{q_{V'}} \mathcal{C}_{V'} \longrightarrow \mathcal{C}_V.$$

It preserves the markings because its composite with $D \hookrightarrow \mathcal{C}_0$ does. Conversely, every such arrow gives $V \subset V'$ by Lemma 4.40.4. Composites are forced by uniqueness, giving the claimed equivalence. □

Corollary 4.45.9 (Semilinear group actions on exact contractions)**.** *Let a group G act on $(\mathcal{C}_0, D)$ by semilinear isometric automorphisms, and let V be invariant and satisfy the marked forest condition. There is a unique action of G on $(\mathcal{C}_V, D_V)$ making q_V equivariant. Its generic action is the prescribed action on C. Inclusions between invariant admissible vertex sets give equivariant contraction morphisms.*

Proof. Put $S = \operatorname{Spec} O$, $X_0 = \mathcal{C}_0$, $Y = \mathcal{C}_V$, and $q = q_V$. Write p_0 and p_Y for the structure maps. For $g \in G$, let τ_g be its automorphism of S, so that $p_0 g = \tau_g p_0$. Form the actual base change

$$Y_g = S \times_{\tau_g, S, p_Y} Y$$

with projections $p_g : Y_g \to S$ and $j_g : Y_g \to Y$. The second projection is an isomorphism of schemes, since τ_g is an isomorphism, and $p_Y j_g = \tau_g p_g$. The pair (p_0, qg) defines an O-morphism $h_g : X_0 \to Y_g$ with $j_g h_g = qg$.

To check constancy on fibres, let σ be the induced automorphism of k. Apply [StacksAlg, Tag 09GV] to $\overline{k}$ with the two k-structures given by the inclusion and its composite with σ. This gives an automorphism of $\overline{k}$ extending σ. With this choice, the geometric special fibre of h_g is identified, via the corresponding semilinear isomorphism of geometric target fibres, with the composite of the geometric action of g and the geometric contraction. The geometric action preserves the marked graph and permutes the removed trees because V is invariant. The geometric fibre description of Theorem 4.45.5 makes this composite constant on every set-theoretic fibre of the geometric contraction, including the singleton fibres. Apply the fibre-product descent argument in the proof of Proposition 4.45.7 to the k-morphisms q_s and $(h_g)_s$. It gives constancy on the actual special fibres of q_s. The generic map of q is an isomorphism. These two fibres exhaust the rank-one base, so h_g is constant on all set-theoretic fibres of q.

The contraction q is proper birational between integral schemes with normal target. By Lemma 4.44.4, there is a unique $\bar{h}_g : Y \to Y_g$ such that $\bar{h}_g q = h_g$. The two maps $p_g \bar{h}_g$ and p_Y to S agree after composition with q. Uniqueness in the same lemma makes them equal, so $\bar{h}_g$ is an O-morphism. Set $g_V = j_g \bar{h}_g$. Then $qg = g_V q$ and $p_Y g_V = \tau_g p_Y$. The generic action is the prescribed one because q is the specified identification on the generic fibre. Uniqueness of factorization through q gives $(gh)_V = g_V h_V$ and the identity element. In particular $(g^{-1})_V$ is the inverse of g_V. Since $q|_D : D \to D_V$ is the specified isomorphism, the equality $qg = g_V q$ gives the induced action

on the marking as a scheme. Finally, for an arrow of Theorem 4.45.8 from $Y' = \mathcal{C}_{V'}$ to Y, the two equivariance composites agree after precomposition with $q_{V'} : X_0 \to Y'$. Uniqueness of factorization through $q_{V'}$ makes them equal. □

4.46 Algebraic morphisms determined by original-field vertices

Let K be complete and nontrivially real-valued, with its specified absolute value. Put $O = K^\circ$, $\mathfrak{m} = K^{\circ\circ}$, and $k = \widetilde{K}$, and fix $0 \neq \pi \in \mathfrak{m}$. Let C/K be a smooth projective geometrically connected curve. Semistable algebraic models and finite étale smooth markings have the meaning of Definition 4.45.3. Every generic-fibre identification in this section is an identification with this fixed curve over the original field.

Lemma 4.46.1 (The actual closure of the generic graph)**.** *Let $\mathcal{X}_1, \mathcal{X}_2$ be projective flat finitely presented O-models of C. The schematic closure $\mathcal{G}$ of the generic diagonal in $\mathcal{X}_1 \times_O \mathcal{X}_2$ is integral, projective, flat and finitely presented over O. Its two projections*

$$p_i : \mathcal{G} \longrightarrow \mathcal{X}_i$$

are proper and induce the identity on C. If $\mathcal{X}_i$ is integral, p_i is birational and surjective.

For every complete isometric extension L/K, the completion of $\mathcal{G}_{L^\circ}$ is an admissible formal model of C_L^{an}. Its specialization is surjective onto every point of its special fibre and satisfies

$$\mathrm{sp}_{\mathcal{X}_{i,L^\circ}} = (p_i)_{\widetilde{L}} \circ \mathrm{sp}_{\mathcal{G}_{L^\circ}} .$$

Proof. On an affine open $\operatorname{Spec} B$ of the product, let $I \subset B[1/\pi]$ be the ideal of the generic diagonal. Its closure has ring

$$B/(I \cap B).$$

These ideals commute with localization and hence define a closed subscheme of the product. A nonempty such affine closure has coordinate ring contained in the coordinate ring of a nonempty open of the integral curve C. It is therefore a domain and has no O-torsion. The generic diagonal is dense, so

these affines give an integral scheme. Torsion-freeness over a valuation ring gives flatness by [StacksAlg, Tag 0539]. Each affine ring is of finite type over O, as a quotient of the coordinate ring of the product. It is consequently finitely presented by [StacksAlg, Tag 053E(1)]. The closure is projective as a closed subscheme of the projective product; in particular it is quasi-compact and separated. The local finite presentation just proved is therefore a finite presentation.

The projections are proper, being restrictions of the proper product projections to a closed subscheme. On the generic fibre they are the given identities. If a target is integral, their closed images contain its dense generic fibre, proving surjectivity and birationality.

Base change to L° preserves projectivity, flatness and finite presentation, and gives generic fibre C_L. Theorem 4.13.2 identifies the actual completed generic fibre with C_L^{an}. Lemma 4.28.1 identifies its special fibre and the completed projection maps with their coefficient extensions. Surjectivity and functoriality of specialization are Proposition 3.17.2. No reducedness or nodality of the graph's special fibre is needed for these assertions. □

Theorem 4.46.2 (Algebraic domination from vertex inclusion)**.** *For two semistable algebraic models $\mathcal{X}_1, \mathcal{X}_2$ of C, there is an O-morphism*

$$f : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$$

inducing the identity on C if and only if $V(\mathcal{X}_2) \subset V(\mathcal{X}_1)$ in C^{an}. When it exists, it is unique and proper, and $f_*\mathcal{O}_{\mathcal{X}_1} = \mathcal{O}_{\mathcal{X}_2}$.

Proof. A morphism of the indicated models is proper by its closed graph and the properness of the source over O. Its completion induces the analytic identity, so Lemma 4.40.4 proves the necessary vertex inclusion.

Suppose conversely that the inclusion holds. Form $\mathcal{G}$ and its two projections as in Lemma 4.46.1. Both targets are normal and integral by Proposition 4.44.3. We show that p_2 is constant on every set-theoretic fibre of p_1.

Choose the isometric comparison field Ω of Lemma 4.45.1, with residue field $\overline{k}$. The two base-changed target models are semistable algebraic models over Ω°. Their nodes have the split rational pointed charts of Theorem 4.18.2, since Ω is algebraically closed. Completing these charts gives the pointedly

semistable formal models of Definition 4.39.4. By Corollary 4.39.5, the fibres of their specialization maps are their singleton component vertices and the connected components of the complements of those vertex sets. Proposition 4.45.2 identifies their vertex sets with the entire inverse images of the original vertex sets, so

$$V(\mathcal{X}_{2,\Omega^\circ}) \subset V(\mathcal{X}_{1,\Omega^\circ}).$$

Consequently sp_2 is constant on every fibre of sp_1 on C_Ω^{an}. For a component generic point the latter fibre is a singleton. For a closed point it is a connected component of the complement of the larger vertex set. It avoids the smaller vertex set, and lies in one connected component of its complement, which is one fibre of sp_2.

Let $z_1, z_2 \in \mathcal{G}_{\overline{k}}$ have the same $(p_1)_{\overline{k}}$-image. The surjective specialization in Lemma 4.46.1 supplies $x_1, x_2 \in C_\Omega^{\mathrm{an}}$ with $\mathrm{sp}_{\mathcal{G}}(x_j) = z_j$. Functoriality gives $\mathrm{sp}_1(x_1) = \mathrm{sp}_1(x_2)$. The preceding constancy gives $\mathrm{sp}_2(x_1) = \mathrm{sp}_2(x_2)$, hence $(p_2)_{\overline{k}}(z_1) = (p_2)_{\overline{k}}(z_2)$.

Here is the descent of this actual fibre relation. A field extension is flat by [StacksAlg, Tag 00HD], since the only ideals of a field are zero and the whole field; its nonzero closed fibre makes it faithfully flat by [StacksAlg, Tag 00HP]. If $t_1, t_2 \in \mathcal{G}_s$ have the same p_1-image y, the tensor product $\kappa(t_1) \otimes_{\kappa(y)} \kappa(t_2)$ is therefore nonzero. A prime of it, supplied by [StacksAlg, Tag 00E0(2)], gives a point of $\mathcal{G}_s \times_{\mathcal{X}_{1,s}} \mathcal{G}_s$ with projections t_1, t_2. Faithful flatness of $\overline{k}/k$ and surjectivity on spectra, [StacksAlg, Tag 00HQ], lift this point to the base-changed fibre product. Its two projections have the same $(p_1)_{\overline{k}}$-image, and thus the same $(p_2)_{\overline{k}}$-image. Projection back shows $p_2(t_1) = p_2(t_2)$. On the generic fibre both projections are identities. The generic and special fibres exhaust the rank-one base, and their images cannot meet. This proves the required constancy on every fibre of p_1.

Lemma 4.44.4, applied to the proper birational map p_1 and its normal target, gives a unique algebraic factorization $p_2 = f \circ p_1$. It induces the identity on C. Any two morphisms $\mathcal{X}_1 \to \mathcal{X}_2$ inducing that identity are equal: their equalizer is closed because $\mathcal{X}_2/O$ is separated, and contains the schematically dense generic fibre of the flat integral source. Finally the same proper-quotient lemma applied to f gives $f_*\mathcal{O}_{\mathcal{X}_1} = \mathcal{O}_{\mathcal{X}_2}$. □

Corollary 4.46.3 (Algebraic uniqueness without a supplied common model)**.** *Two semistable algebraic models of C with the same component vertices are uniquely isomorphic by an isomorphism inducing the identity on C. The category of all semistable algebraic models of C, with morphisms inducing the identity on C, is equivalent to the partially ordered set of vertex sets that occur for such models, ordered by reverse inclusion.*

If both models carry finite étale *smooth markings identified with the same finite* étale *O-scheme D, with the same generic marking on C, every morphism in this category respects these identifications.*

Proof. The two vertex inclusions give morphisms in both directions by Theorem 4.46.2. Uniqueness in that theorem makes their composites identities. The same theorem identifies all morphism sets with the corresponding inclusion relation; composition agrees because the morphisms are unique.

For the marking assertion, compare the two maps from D to the target. Their generic restrictions agree. On each affine open of the finite étale scheme D, flatness makes its ring inject into its generic localization. The generic fibre of D is therefore schematically dense. The target is separated, so the equalizer of the two maps is closed and must be all of D. This proves compatibility with the specified marking isomorphisms. □

4.47 Unramified markings and common models

Definition 4.47.1 (Unramified marking data)**.** An *unramified marking datum* on C is a finite étale O-scheme D together with a closed immersion $D_K \hookrightarrow C$. It is not required to extend to a smooth marking on any model chosen in advance.

Lemma 4.47.2 (Descent of invariant ideals)**.** *Let L/K be the finite unramified Galois lift of a finite Galois extension of k, and put $O' = L^\circ$ and $G = \mathrm{Gal}(L/K)$. Let $\mathcal{X}$ be an O-scheme. Every G-invariant quasi-coherent ideal $\mathcal{J}$ of finite type on $\mathcal{X}_{O'}$, for the canonical coefficient action, is the extension of a unique quasi-coherent ideal $\mathcal{I}$ of finite type on $\mathcal{X}$. Its blowup is the coefficient extension of the blowup of $\mathcal{I}$. If $\mathcal{J}$ contains π^n, then $\mathcal{I}$ contains π^n.*

Proof. Use the basis $1, \alpha, \dots, \alpha^{d-1}$ of O'/O from Lemma 4.43.1. The matrix $(g(\alpha^i))_{g,i}$ is invertible over O', by its unit Vandermonde determinant. On an

affine open $\operatorname{Spec} A \subset \mathcal{X}$, put $A' = A \otimes_O O'$ and let $J \subset A'$ be the given invariant ideal. If $z = \sum_{i=0}^{d-1} a_i \alpha^i \in J$, then

$$g(z) = \sum_{i=0}^{d-1} a_i g(\alpha^i) \qquad (g \in G).$$

Inverting the matrix expresses each a_i as an O'-linear combination of the $g(z)$. Thus each a_i belongs to $I = J \cap A$, and $J = IA'$.

Choose finitely many generators of J. All their coefficients belong to I, and generate an ideal $I_0 \subset A$ with $I_0 A' = J$. The coefficient-of-1 map $A' \to A$ is A-linear and is the identity on A. Applying it to a finite expression for any $a \in J \cap A$ proves $a \in I_0$. Hence $I = I_0$ is finitely generated. The same coefficient map shows that extension of any ideal of A contracts back to itself. This proves uniqueness.

On a principal open of $\operatorname{Spec} A$, coefficient extraction commutes with localization. The ideals just constructed therefore agree on an affine basis of every overlap and glue to $\mathcal{I}$. The assertion about π^n follows by contraction. Finally O'/O is faithfully flat by Lemma 4.43.1, and [StacksAlg, Tag 0805] identifies the two blowups. This identification is induced by the coefficient map on their Rees algebras, and respects the canonical G-actions. □

Theorem 4.47.3 (Realizing an unramified marking on the original model)**.** *Let $\mathcal{X}$ be a semistable algebraic model of C, and let D be an unramified marking datum. A finite sequence of algebraic blowups supported in the special fibre gives a semistable algebraic model $\mathcal{X}' \to \mathcal{X}$ in which D_K extends to a closed immersion of D into the relative smooth locus. Every original component vertex is retained.*

Proof. Choose a finite unramified Galois extension L/K which splits the nodes of $\mathcal{X}$ and the finite étale scheme D_s. This can be done in the construction of Theorem 4.18.2: enlarge its chosen finite Galois residue extension to contain the splitting fields of the finitely many separable residue fields of D_s, and then take the lift of Lemma 4.18.1. The same pointed-chart construction works over this enlarged lift. Write $O' = L^\circ$ and $G = \operatorname{Gal}(L/K)$. This lift satisfies $[L : K] = [\widetilde{L} : k]$ and $|L^\times| = |K^\times|$, with the given absolute value. The residue fields in this choice are finite separable by [StacksAlg, Tag 02GL].

Every rational point of $D_{\widetilde{L}}$ extends to a section of $D_{O'}$ by Lemma 4.15.1. These sections are open and closed, since $D_{O'}$ is finite and étale. They are

disjoint: two distinct sections cannot meet generically, since their equalizer is open and closed in the connected local spectrum and their closed points differ. Their complement is finite over O' and has empty closed fibre. Its finite coordinate algebra M satisfies $M/\mathfrak{m}_{O'}M = 0$, so it is zero by [StacksAlg, Tag 00DV(2)]. Thus $D_{O'}$ is a disjoint union of copies of $\operatorname{Spec} O'$, and D_L gives a finite G-invariant set of distinct L-rational points of C_L.

Apply Theorem 4.34.3 to $\mathcal{X}_{O'}$ and this set. Its proof gives a finite sequence of blowups of G-invariant finite-type ideals. Its Hausdorff generic-fibre hypothesis follows from Theorem 4.13.2 and Proposition 2.33.30. Each ideal is the finite product of the intrinsic ideals of the nodal or smooth marked clusters. Their support is a finite set of closed special points, and they contain a power of the original π. At a nodal cluster, the actual charts are those of Lemma 4.27.1; at a smooth cluster they are those of Lemma 4.27.3. In particular these are actual flat nodal blowups over O', including when the value group is nondiscrete and when a node has both branches on the same component.

Descend the first ideal by Lemma 4.47.2 and blow it up over O. Its coefficient extension is the first prescribed blowup, with the same G-action. Repeat on the resulting original-field scheme. Induction descends the entire sequence. Every descended blowup is projective by [StacksAlg, Tag 02NS], and its composite with the original projective model is projective by [StacksAlg, Tag 0C4P]. Its generic fibre is unchanged because its centre contains a power of π. The nodal relative-curve property, including flatness and local finite presentation, descends from O' by [StacksAlg, Tag 0C5E]. Projectivity gives the required quasi-compactness and separatedness, hence finite presentation. The geometric special fibre remains connected: the displayed local charts replace a smooth point by an attached projective line, or a node by a chain of one or two projective lines. These replacements preserve connectedness after every residue-field extension. This proves that all descended stages are semistable algebraic models.

On the last coefficient extension, the marked sections give a G-invariant closed immersion $D_{O'} \hookrightarrow \mathcal{X}'_{O'}$. Its ideal is of finite type: locally at each marked point it is the ideal of a section in a smooth relative curve, as proved in Lemma 4.34.2; the sections are disjoint. Descend this ideal by Lemma 4.47.2, and write $Z \hookrightarrow \mathcal{X}'$ for the resulting closed subscheme. The morphism Z/O is finite by [StacksAlg, Tag 02LA], since its coefficient extension is $D_{O'}$. The equivariant isomorphism $Z_{O'} \simeq D_{O'}$ identifies their finite coordinate algebras.

Taking invariants and using Lemma 4.43.1 gives an O-isomorphism $Z \simeq D$, compatible with the generic marking. Its image is in the relative smooth locus. Indeed, the cover $\mathcal{X}'_{O'} \to \mathcal{X}'$ is étale. Around every marked point upstairs the structural map to $\operatorname{Spec} O'$ is smooth, and $\operatorname{Spec} O' \to \operatorname{Spec} O$ is étale. Smooth locality on the source, [StacksAlg, Tag 036U], first gives smoothness of the composite and then descends it along this cover at the marked image. Finally Theorem 4.46.2 applied to $\mathcal{X}' \to \mathcal{X}$ gives retention of all original component vertices. □

Lemma 4.47.4 (Three marked branches force a vertex)**.** *Assume the ground field is algebraically closed. Let $\mathcal{X}$ be a semistable algebraic model of a smooth projective curve, and let P be a finite set of rational points of its generic fibre whose sections have pairwise distinct smooth specializations. Suppose $v \in C^{\mathrm{an}}$ admits three pairwise disjoint connected open subsets B_1, B_2, B_3 with $v \notin \bigcup_i B_i$ such that*

$$\overline{B_i} = B_i \cup \{v\}, \qquad B_i \cap P \neq \varnothing.$$

Then v is a component vertex of $\mathcal{X}$.

Proof. Suppose v is not a vertex, and let U be its connected component in the complement of the model vertices. The split pointed charts of Theorem 4.18.2 complete to the pointed formal charts of Definition 4.39.4. Corollary 4.39.5 and Proposition 4.39.3 identify U with an actual open-disc or open-annulus tube. Its boundary has one point in the disc case and at most two in the annulus case. Every B_i meets U, since $v \in U \cap \overline{B_i}$ and U is open.

If B_i is not contained in U, it meets ∂U. Indeed, otherwise $B_i \cap U$ would be a nonempty proper subset both open and closed in the connected space B_i. The disjoint B_i meet distinct boundary points when they meet ∂U. Thus at most one of them is not contained in a disc tube, and at most two are not contained in an annular tube. In the first case U contains at least two members of P, contradicting their distinct specializations. In the second it contains at least one member of P, contradicting smoothness of its marked specialization. These contradictions prove the assertion. □

Theorem 4.47.5 (Common original-field semistable models)**.** *Any finite nonempty family of semistable algebraic models of C has a semistable algebraic model dominating all of them by morphisms inducing the identity on C. If the models are marked by the same unramified marking datum D, the common model and all the morphisms may be chosen to preserve that marking.*

Proof. First consider two models $\mathcal{X}$ and $\mathcal{Y}$. Lemma 4.44.1 gives a finite étale smooth marking A on $\mathcal{Y}$ with at least three geometric points on every geometric special component, all distinct and away from the nodes. If a marking D is already present, choose A disjoint from it. Apply Theorem 4.47.3 to $\mathcal{X}$ and the generic marking datum A, or $A \amalg D$ in the marked case. This gives a model $\mathcal{Z} \to \mathcal{X}$ where these markings have pairwise distinct smooth specializations.

Compare over the isometric algebraically closed field Ω of Lemma 4.45.1. At a component vertex w of $\mathcal{Y}_{\Omega^\circ}$, choose three of the auxiliary marked points on its geometric component. Their distinct smooth specializations give three disjoint open-disc tubes, each with closure equal to that disc together with w, by Proposition 4.39.3. They contain the chosen generic marked points. Lemma 4.47.4, applied to $\mathcal{Z}_{\Omega^\circ}$, makes w a component vertex there. Proposition 4.45.2 then gives the original-field inclusion $V(\mathcal{Y}) \subset V(\mathcal{Z})$. Theorem 4.46.2 produces the actual algebraic morphism $\mathcal{Z} \to \mathcal{Y}$. In the marked case both morphisms respect D by Corollary 4.46.3.

Induct on the finite number of models, replacing the first two by their common model. Each new common model dominates the preceding one, so composition retains every morphism already constructed, as well as the common marking. □

4.48 An intrinsic arithmetic condition on vertices

Fix the isometric algebraically closed comparison field Ω of Lemma 4.45.1, and write $p : C_\Omega^{\mathrm{an}} \to C^{\mathrm{an}}$ for the projection.

Definition 4.48.1 (Unramified branch certificates)**.** A point $v \in C^{\mathrm{an}}$ has an *unramified branch certificate* if there are a point $w \in p^{-1}(v)$, an unramified marking datum P, and three pairwise disjoint open analytic discs $B_1, B_2, B_3 \subset C_\Omega^{\mathrm{an}}$, with $w \notin \bigcup_i B_i$, such that

$$\overline{B_i} = B_i \cup \{w\}, \qquad B_i \cap P_K(\Omega) \neq \varnothing \quad (1 \le i \le 3).$$

Closures are taken in C_{Ω}^{an}.

Theorem 4.48.2 (Which original-field points occur as model vertices)**.** *Suppose C admits a semistable algebraic model over O. A point of C^{an} occurs as a component vertex of such a model if and only if it has an unramified branch certificate. Every finite set of points having such certificates is contained in the component vertex set of one original-field semistable algebraic model dominating any initially specified model. The certificate condition is independent of the chosen algebraically closed comparison field.*

For every such point v, its entire inverse image under p is finite and consists of type-II points. Moreover

$$|\mathcal{H}(v)^{\times}| = |K^{\times}|,$$

$\widetilde{\mathcal{H}(v)}/k$ *is separably generated of transcendence degree one.*

Proof. Suppose first that v is the vertex belonging to a special component E of a model $\mathcal{Y}$. Lemma 4.44.1 supplies an unramified marking P on $\mathcal{Y}$, meeting every geometric component over E at least three times in distinct smooth points. Choose one geometric component over E and its vertex w. Proposition 4.45.2 gives $p(w) = v$. The tubes of three of those marked specializations are disjoint open discs whose closures adjoin precisely w, by Proposition 4.39.3. They contain the corresponding points of $P_K(\Omega)$ and give the certificate.

Conversely, start with any specified semistable algebraic model $\mathcal{X}$. For a certificate (w, P, B_1, B_2, B_3), Theorem 4.47.3 gives a model $\mathcal{X}' \to \mathcal{X}$ on which P is a disjoint smooth marking. After coefficient extension to Ω, Lemma 4.47.4 makes w a component vertex of $\mathcal{X}'_{\Omega^\circ}$. By Proposition 4.45.2, its image v is an original component vertex of $\mathcal{X}'$. For finitely many certificates, repeat this construction one at a time. Each refinement retains all preceding vertices, so the final model contains every prescribed point and dominates the initial model.

Apply Proposition 4.45.2 to any model just obtained. It proves the displayed arithmetic equalities and identifies the entire inverse image with its finitely many geometric component vertices. These are type-II points by Corollary 4.39.5. The construction of a certificate from a model works for every comparison field, proving the asserted independence. □

Definition 4.48.3 (Admissible original-field vertex sets)**.** Fix an unramified marking datum D, possibly empty. A finite nonempty subset $V \subset C^{\mathrm{an}}$ is *admissible for* (C, D) if every point of V has an unramified branch certificate and $\overline{V} = p^{-1}(V)$ is a finite semistable vertex set of C_Ω^{an} with the following marking condition: every open-disc component of $C_\Omega^{\mathrm{an}} \setminus \overline{V}$ contains at most one point of $D_K(\Omega)$, and every open-annulus component contains none.

Theorem 4.48.4 (Intrinsic correspondence over the original field)**.** *Suppose C admits at least one semistable algebraic model over O, and fix an unramified marking datum D. The component vertex map induces a bijection between isomorphism classes of semistable algebraic models of C carrying D as a finite* étale *smooth marking and the finite nonempty vertex sets admissible for (C, D) in Definition 4.48.3. For two such models there is a morphism inducing the identity on C and respecting D precisely when the target vertex set is contained in the source vertex set; that morphism is unique. Thus this bijection is an equivalence with the partially ordered set under reverse inclusion. Admissibility is independent of the chosen comparison field.*

Proof. Let $\mathcal{Y}$ be such a marked model. Its vertices have certificates by Theorem 4.48.2. Proposition 4.45.2 identifies their entire inverse image with its geometric component vertices. After extension to Ω, the pointed charts and Corollary 4.39.5 make this a finite semistable vertex set. Distinct marked sections have distinct smooth specializations, so its open-disc tubes contain at most one marked point and its nodal annular tubes contain none. This proves admissibility.

Conversely, let V be admissible. By Theorem 4.48.2, choose an original-field semistable algebraic model whose vertex set contains V. Refine it by Theorem 4.47.3 to make D a disjoint smooth marking. Write $(\mathcal{X}, D)$ for the resulting marked model and W for its vertex set. It still has $V \subset W$. Over Ω, the entire inverse images satisfy $\overline{V} \subset \overline{W}$ by Proposition 4.45.2.

Consider a connected component T of the geometric dual subgraph on the vertices outside $\overline{V}$. Since $\overline{V}$ is semistable, Lemma 4.41.2 identifies T with the part of the source skeleton in one component U of $C_\Omega^{\mathrm{an}} \setminus \overline{V}$. It is a finite tree, and each of its geometric component curves itself is a smooth $\mathbf{P}^1_{\widetilde{k}}$, hence has no self-node. The number of its attaching branches is one if U is an open disc and two if U is an open annulus, with both branches counted even when they attach to the same protected component.

Every marking whose specialization is on a component of T belongs to U. Indeed, its smooth $\mathcal{X}$ tube is connected, avoids $\overline{V}$, and has that component vertex in its closure. Since this vertex lies in open U, the tube meets U and therefore lies in that connected component of the complement. The marking condition for V consequently bounds the number of marked legs on T by one in the disc case and by zero in the annular case. Thus T satisfies precisely the geometric marked forest condition of Definition 4.45.4.

Theorem 4.45.5 now gives an actual original-field marked model with component vertices exactly V. Its generic identification is the original one and its marking is isomorphic to the prescribed D. Corollary 4.46.3 proves uniqueness of the marked model, and Theorem 4.46.2, together with the marking assertion of that corollary, identifies all its morphisms. This proves the correspondence and its categorical assertion. Its characterization by original-field models also proves independence of the comparison field. □

Corollary 4.48.5 (Extension of semilinear actions)**.** *Let a group G act on C and an unramified marking datum D by compatible semilinear isometric automorphisms. If V is an invariant admissible vertex set and C admits a semistable algebraic model over O, the model corresponding to V carries a unique G-action inducing the prescribed generic action and preserving the marking. All morphisms arising from inclusions between invariant admissible vertex sets are equivariant.*

Proof. An isometric field automorphism preserves O. For $g \in G$, pull the model back by its base automorphism and use its prescribed generic curve isomorphism to identify the new generic fibre with C. This is again a semistable algebraic model. Functorial specialization identifies its vertices with the corresponding image of V, which is V. The pulled-back marking is identified with D by its given semilinear action. Corollary 4.46.3 therefore gives the unique semilinear model isomorphism extending g. Two composites which have the same generic action are equal by the schematic-density equalizer argument in the proof of Theorem 4.46.2. Thus these isomorphisms satisfy the group law and form the asserted action. The same uniqueness, applied to the two composites of a vertex-inclusion morphism with g, proves equivariance of every such morphism. □

4.49 Least semistable refinements over the original field

Keep the original field K, valuation ring O, curve C, and model category of Section 4.46. Fix the comparison field Ω of Lemma 4.45.1, and put $\overline{k} = \widetilde{\Omega}$. All vertex labels below are the actual analytic component points. In particular, retaining a label retains its point and its original absolute value.

Lemma 4.49.1 (Geometric components over one original component)**.** *Let Y be a scheme of finite type over a field k, and let $\overline{k}$ be an algebraic closure. The geometric irreducible components over any one irreducible component of Y form one orbit under $\mathrm{Aut}(\overline{k}/k)$. Consequently an invariant subset of the geometric components is the entire inverse image of a subset of the original components.*

Proof. Let $k^{\mathrm{sep}} \subset \overline{k}$ be the separable closure of k in $\overline{k}$. The assertion over k^{sep} is [StacksAlg, Tag 04KY]. In characteristic zero this is already the assertion. In characteristic $p > 0$, the remaining extension $\overline{k}/k^{\mathrm{sep}}$ is purely inseparable: an irreducible polynomial over k has the form $Q(T^{p^e})$ with Q separable, by successively removing zero derivatives. A root therefore has a p^e-th power separable over k.

For a k^{sep}-algebra A, write $B = A \otimes_{k^{\mathrm{sep}}} \overline{k}$. This is a faithfully flat extension: a field extension is flat by the finite-ideal criterion [StacksAlg, Tag 00HD], is faithfully flat by [StacksAlg, Tag 00HP], and its tensor functor on A-modules is the same coefficient tensor functor. Thus every prime of A has a prime above it by [StacksAlg, Tag 00HQ]. For every $b \in B$, some p-power b^{p^e} belongs to A: write b as a finite coefficient sum and choose e simultaneously for its finitely many coefficients. For a prime Q above P, one therefore has

$$b \in Q \quad \Longleftrightarrow \quad b^{p^e} \in P.$$

This determines Q uniquely and preserves inclusions in both directions. Hence minimal primes, and consequently irreducible components, correspond. These correspondences commute with affine restriction and with automorphisms of k^{sep}/k. Every such automorphism extends uniquely to $\overline{k}$: the image of a is the unique p^e-th root of the prescribed image of $a^{p^e} \in k^{\mathrm{sep}}$. The component orbit assertion over k^{sep} therefore gives the assertion over $\overline{k}$. The final statement follows by taking unions of these orbits. □

Definition 4.49.2 (Protected marked graphs and elementary pruning)**.** A *protected marked graph* is a finite connected multigraph G, with loops allowed, a finite set of labelled legs at its vertices, a subset R of its vertices, and a nonempty subset S of protected vertices. A loop contributes two to the degree $\deg_G(v)$; write $m(v)$ for the number of marked legs at v. A vertex is *deletable* if

$$v \notin S, \qquad v \in R, \qquad v \text{ has no loop}, \qquad \deg_G(v) + m(v) \leq 2.$$

Its degree is positive by connectedness and $S \neq \varnothing$. For a degree-one deletable vertex, remove its incident edge and transfer its zero or one marked leg to the neighbouring vertex. For a degree-two deletable vertex, remove it and replace its two edges by one edge between the other ends; this may create a loop. Such a vertex has no marked leg. Retain the vertex labels, the membership in R, and the protected labels at every other vertex. These operations are called *elementary pruning*.

Lemma 4.49.3 (Confluence and finite forests)**.** *Every sequence of elementary pruning terminates. Any sequence continued until no deletable vertex remains has the same labelled terminal graph, up to an incidence isomorphism fixing its retained vertices and its labelled legs.*

Suppose A is a subset of the unprotected vertices such that each connected component T of the induced subgraph on A is a tree, all its vertices belong to R, and

$$b(T) + m(T) \leq 2,$$

where $b(T)$ counts edges from T to its complement and $m(T)$ counts its marked legs. Contracting each such tree, transferring its possible mark in the one-ended case and joining its two ends in the two-ended case, is a sequence of elementary prunings. In particular it does not change the terminal graph.

Proof. Every pruning removes one vertex and retains the nonempty set S, so a sequence is finite. We verify a local commutation property. If two distinct deletable vertices are not adjacent, their operations commute. Their incident edges are disjoint; even when they have a common neighbour, the transferred labelled legs and the replacement edges give the same result in either order. Each vertex remains deletable after the other operation.

Suppose instead that the two vertices are adjacent. They have only one edge between them. Two parallel edges would exhaust both degrees, making

them the whole connected graph, contrary to the existence of a protected vertex. For the same reason they cannot both have degree one. If one is a leaf and the other has degree two, deleting the leaf leaves a leaf with at most one marked leg at the second vertex. Deleting the second vertex first joins the original leaf directly to the remaining neighbour. Both orders remove the pair and transfer the same possible leg to that neighbour. If both vertices have degree two, neither has a mark, and both orders suppress the same two-vertex path. Their other neighbours may coincide: both orders then give the same loop at that common neighbour. In all cases both second deletions remain permissible and the results agree.

Induct on the number of vertices to prove confluence. For two terminal sequences with the same first deletion, apply induction to the smaller graph. For different first deletions, the preceding commutation property gives a common graph after two deletions. Induction applied to each of the two one-deletion graphs identifies its terminal graph with the terminal graph of this common graph. Thus the two original terminal graphs agree. This also proves that their retained labels and transferred marked-leg labels agree.

For the forest assertion, each T has $b(T) \geq 1$, because the whole graph is connected and its complement contains S. A one-vertex tree is immediately deletable. If T has more than one vertex, it has at least two leaves: take the two ends of a longest simple path. Since its total number of external ends and marks is at most two, some leaf carries at most one of these external ends or marks. Its one internal edge therefore gives $\deg_G(v) + m(v) \leq 2$. Delete it. If it carries an external edge, that edge is joined to its internal neighbour; if it carries a mark, that mark is transferred to that neighbour. The remaining tree has the same external ends and marked legs, and its vertices still belong to R. Continue until the tree disappears. Its final operation is exactly the prescribed one-ended or two-ended contraction, including a loop when the two external ends have the same neighbour. Different connected components of A have no edge between them, so this procedure applies to all of them. Confluence proves the last assertion. □

Proposition 4.49.4 (The least contraction retaining specified vertices)**.** *Let $(\mathcal{X}, D)$ be a marked semistable algebraic model, with vertex set W, and let $\varnothing \neq S \subset W$. There is a marked semistable algebraic model $\mathcal{X}_S$ and a morphism $\mathcal{X} \to \mathcal{X}_S$ retaining S such that every marked contraction $\mathcal{X} \to \mathcal{Y}$ retaining S has a unique factorization*

$$\mathcal{X} \longrightarrow \mathcal{Y} \longrightarrow \mathcal{X}_S.$$

Its geometric marked graph is the terminal graph of Definition 4.49.2, with protected set equal to the entire inverse image of S and with R the vertices whose normalized residual component is $\mathbf{P}^1_{\overline{k}}$. The construction uses finitely many actual contractions over O.

Proof. Use the actual coefficient extension to Ω from Lemma 4.28.1 and the geometric marked dual graph of Definition 4.45.3. Its vertex set is the entire inverse image of W by Proposition 4.45.2. The normalized residual component at a retained geometric vertex is unchanged under a model morphism: it is the intrinsic normal proper curve of Theorem 4.29.2. This makes the decoration R independent of subsequent contractions.

A geometric component is itself a smooth projective line exactly when its normalization is a projective line and it has no self-node. Here a self-node is precisely a loop of the dual graph. Indeed the actual normalization squares of Lemma 4.24.1 separate exactly the two branches of each node. At a node joining different components, the selected component has just one smooth branch: in a pointed étale chart it is one of the two axes. Smooth locality on the source, [StacksAlg, Tag 036U], descends this assertion to that component. Away from the nodes normalization is already an isomorphism, as in Lemma 4.31.1. Thus a component without self-nodes is its smooth normalization; the assertion follows.

Let A be the set of all currently deletable geometric vertices. It is invariant under $\mathrm{Aut}(\overline{k}/k)$: the action preserves incidence, loops, the marked-leg set, the normalized component isomorphism class, and the protected set. Lemma 4.49.1 makes A the entire inverse image of a subset of original components.

Every vertex of the graph induced by A has degree at most two in the full graph. A connected component T of it cannot contain a cycle: all vertices of that cycle would already have full degree two and no edge leaving it. It would be the whole connected graph, contrary to the nonempty protected complement. Thus T is a tree. Put $a = |T|$. Counting its internal edges gives

$$b(T) + m(T) = \sum_{v \in T} \bigl(\deg_G(v) + m(v)\bigr) - 2(a-1) \le 2.$$

All its actual geometric components are smooth projective lines by the preceding paragraph. Therefore the complement of A satisfies the exact marked forest condition, and Theorem 4.45.5 constructs an actual original-O contraction deleting precisely A. Its fibre description gives exactly the graph

operations in Lemma 4.49.3, including the transfer of a mark and the creation of a self-node from two coincident ends. The whole marking remains the original finite étale O-scheme.

Repeat with all deletable vertices of the new model. Each nonempty round removes original components and retains S, so after finitely many rounds the process stops at a model $\mathcal{X}_S$ with no deletable geometric vertex. Each round is a forest contraction and hence decomposes combinatorially into elementary prunings by Lemma 4.49.3. That lemma identifies the resulting graph with its unique terminal graph. Single geometric deletions in this combinatorial comparison need not descend individually; every algebraic round used the entire invariant forest and the original-field contraction theorem.

Now let $q : \mathcal{X} \to \mathcal{Y}$ be any marked contraction retaining S. Proposition 4.45.6 says that its removed components satisfy the marked forest condition. The target has the graph of that forest contraction: the exact contraction of Theorem 4.45.5 to the same vertex set has the stated fibres and is identified with q by Proposition 4.45.7. It is consequently a sequence of elementary prunings on the geometric graph. Continuing to a terminal graph and applying confluence shows that every retained vertex of $\mathcal{X}_S$ is a vertex of $\mathcal{Y}$. The entire-fibre identification of Proposition 4.45.2 gives the same inclusion for original analytic vertices. Theorem 4.46.2 now supplies the actual map $\mathcal{Y} \to \mathcal{X}_S$, and Corollary 4.46.3 makes it marked. Uniqueness of the morphisms inducing the generic identity proves the factorization and its uniqueness. □

Theorem 4.49.5 (The least marked semistable hull)**.** *Suppose C admits a semistable algebraic model over O, and fix an unramified marking datum D. For every finite nonempty set $S \subset C^{\mathrm{an}}$ of points with unramified branch certificates, there is a least finite vertex set $H_D(S)$ admissible for (C, D) and containing S. Equivalently, there is a marked semistable algebraic model $\mathcal{M}_D(S)$ containing S such that every marked semistable algebraic model containing S has a unique marked morphism to $\mathcal{M}_D(S)$ inducing the identity on C.*

For $S \subset S'$ the resulting models have a unique compatible morphism $\mathcal{M}_D(S') \to \mathcal{M}_D(S)$. Any finite nonempty family of models marked by D has a least common dominating model: take S to be the union of their vertex sets. Any group of compatible semilinear isometric automorphisms of (C, D) preserving S extends uniquely to $\mathcal{M}_D(S)$.

Proof. Theorem 4.48.2 constructs an original-field semistable model containing S. Apply Theorem 4.47.3 to give it the prescribed smooth marking D, while retaining S; call the resulting model $\mathcal{X}$. Proposition 4.49.4 gives $\mathcal{M} = \mathcal{X}_S$.

Let $\mathcal{Y}$ be any other model marked by D and containing S. Theorem 4.47.5 gives an actual common marked semistable model $\mathcal{T}$ of $\mathcal{X}$ and $\mathcal{Y}$. Its geometric graph can be pruned to that of $\mathcal{X}$, and then to the terminal graph of $\mathcal{M}$, by Proposition 4.45.6 and Lemma 4.49.3. It can also be pruned to the graph of $\mathcal{Y}$. The protected geometric labels are the same entire inverse image of S throughout, by Proposition 4.45.2. Confluence therefore makes the terminal graph obtained from $\mathcal{Y}$ equal to that of $\mathcal{M}$, with the same retained analytic labels. In particular $V(\mathcal{M}) \subset V(\mathcal{Y})$. Theorem 4.46.2 and Corollary 4.46.3 give the unique marked map $\mathcal{Y} \to \mathcal{M}$.

The vertex set $H_D(S) = V(\mathcal{M})$ is admissible by Theorem 4.48.4. Conversely every admissible finite set containing S is the vertex set of a model in the preceding comparison. Thus it contains $H_D(S)$. This proves both the minimum and independence of the initially chosen model. It also proves that $\mathcal{M}$ is unique up to its unique generic-identity marked isomorphism.

If $S \subset S'$, the model for S' contains S, giving the stated unique morphism and all composition compatibilities. For a finite family of marked models, each vertex has a certificate by Theorem 4.48.2. Apply the construction to their finite nonempty union. Its model dominates every family member by vertex inclusion. Every common dominating model contains that union and therefore has a unique map to this one.

Finally a compatible semilinear isometry preserving S transports the class of marked models containing S to itself. The universal property identifies its transform of $\mathcal{M}$ with $\mathcal{M}$; equivalently, it preserves $H_D(S)$. Corollary 4.48.5 extends these isometries to the model and proves the group law and uniqueness. Every construction kept the original valuation ring and absolute value; Ω was used only for the geometric graph comparison. □

4.50 Integral Gauss charts and arithmetic branch certificates

Let K be complete and nontrivially valued for its specified real-valued absolute value. Put $O = K^\circ$, $k = \widetilde{K}$, and choose $0 \neq \pi \in K^{\circ\circ}$. For a polynomial $g \in O[T]$ with nonzero reduction, put

$$R_0 = O\langle T, Z\rangle/(g(T)Z - 1), \qquad R = R_0[1/\pi].$$

Proposition 3.17.1 identifies R_0 with the principal formal open of the integral unit disc. Its special fibre is $\bar{R} = k[t, 1/\bar{g}]$, and its generic fibre is the actual domain $|T| \leq 1$, $|g(T)| = 1$.

Definition 4.50.1 (Integral étale Gauss charts)**.** An integral étale Gauss chart is a nonzero admissible O-algebra A_0 which is finite projective as an R_0-module and is an étale R_0-algebra. Write $A = A_0[1/\pi]$, $U = \mathcal{M}(A)$ and $Y = \mathrm{Spec}(A_0/K^{\circ\circ}A_0)$. The following propositions use these notations for a fixed chart.

Proposition 4.50.2 (The actual Gauss fibres of an integral chart)**.** *The special fibre Y is a reduced pure curve. For each irreducible component E of Y, its function field $k(E)$ is finite separable over $k(t)$. Let η be the unit Gauss point of $\mathcal{M}(R)$ and let $H = \mathcal{H}(\eta)$ with its specified absolute value. The actual fibre of $U \to \mathcal{M}(R)$ over η consists exactly of the component points v_E. For every such point,*

$$[\mathcal{H}(v_E) : H] = [k(E) : k(t)],$$
$$\widetilde{\mathcal{H}(v_E)} = k(E), \qquad |\mathcal{H}(v_E)^\times| = |K^\times|.$$

More precisely, its actual integral fibre algebra satisfies

$$A_0 \otimes_{R_0} H^\circ \simeq \prod_E H_E^\circ,$$

where H_E/H is a finite unramified lift of $k(E)/k(t)$. After inverting π, this is the decomposition of the actual analytic fibre algebra into its completed residue fields.

Proof. Put $B = A_0/K^{\circ\circ}A_0$. It is finite projective and étale over the domain $\bar{R}$; étaleness commutes with base change by [StacksAlg, Tag 02GO]. As a direct summand of a finite free module, B injects into its localization

$$B \otimes_{\bar{R}} k(t) = \prod_{i=1}^{r} l_i,$$

where the $l_i/k(t)$ are finite separable fields by [StacksAlg, Tag 02GL]. Thus B is reduced. Let P_i be the kernel of its map to l_i. These prime ideals have intersection zero and are pairwise incomparable: localization gives the distinct factor primes in the displayed product. Every prime of B contains one of the P_i, because the product of these finitely many ideals is zero. They are consequently exactly the minimal primes. Moreover $\operatorname{Frac}(B/P_i) = l_i$: localization by the nonzero elements of $\bar{R}$ already gives l_i. By [StacksAlg, Tag 00P0], every component has dimension one. This proves the special-fibre assertions and identifies the index i with its component E.

The weighted Gauss calculation of Proposition 1.2.2 gives $|g(\eta)| = 1$. The field H is the completion of $K(T)$ for its unit Gauss norm. Indeed, polynomials and their fractions are dense in the completed residue field; adjoining $1/g$ does not change that dense rational subfield. A quotient of two polynomials has norm in $|K^\times|$. If it has norm one, dividing numerator and denominator by coefficients of their respective maximal sizes identifies its residue with a rational function of t. Conversely every rational function of t arises this way. Completion changes neither values nor residues by Lemma 3.22.2. Hence

$$\widetilde{H} = k(t), \qquad |H^\times| = |K^\times|.$$

Evaluation gives an O-homomorphism $R_0 \to H^\circ$.

Set $D_0 = A_0 \otimes_{R_0} H^\circ$. This algebra is finite projective and étale over H°. By [StacksAlg, Tag 00NX], its module is free, because H° is local. Its reduction is

$$D_0/H^{\circ\circ}D_0 = \prod_E k(E).$$

For each factor, Lemma 4.15.2 provides a complete finite extension H_E/H with residue field $k(E)$, unchanged value group, and degree $[k(E) : k(t)]$. Lemma 4.22.1 makes H_E° finite free over H° with that rank. The projection of the displayed product to $k(E)$ is a rational point of the special fibre of $\operatorname{Spec}(D_0 \otimes_{H^\circ} H_E^\circ)$. The henselian section lemma 4.15.1 gives a homomorphism $D_0 \to H_E^\circ$ reducing to that projection. Combining these homomorphisms gives

$$D_0 \longrightarrow \prod_E H_E^\circ.$$

Both sides are finite free of rank $\sum_E [k(E) : k(t)]$: on the left this rank is read by reduction of a basis. Its matrix has invertible reduction, so its determinant is a unit of the local ring H°. The adjugate matrix proves that it is an isomorphism.

Choose an R_0-linear splitting $A_0 \longrightarrow R_0^m \longrightarrow A_0$. The integral presentations of Proposition 3.14.7 identify A_0 and R_0 with the closed unit balls of their presentation norms. Both split maps preserve these unit balls. Scaling by powers of π makes both maps bounded after inversion, so the given norm on A is equivalent to its canonical finite-R-module quotient norm. Proposition 1.18.7 identifies

$$A\widehat{\otimes}_R H = A \otimes_R H = D_0[1/\pi] = \prod_E H_E.$$

Theorem 2.33.11 identifies this with the actual fibre over η, including its completed residue fields. Each factor has just its specified valued point, by uniqueness of the absolute value in Lemma 1.12.1. Its specialization in Y is the component generic point of E, by the reduction of the map just constructed. Proposition 4.12.2 identifies that point with v_E, and gives uniqueness above each component generic point. The degree, residue-field and value-group equalities follow from the construction of H_E. □

Lemma 4.50.3 (Unramified closed points from separable residual points)**.** *Suppose U is identified with an actual affinoid domain of the analytification of a smooth projective geometrically connected curve C/K. Every closed point $q \in Y$ with $l = \kappa(q)$ finite separable over k lifts to an O-homomorphism*

$$A_0 \longrightarrow L^\circ,$$

where L/K is a finite unramified lift of l/k. Its generic analytic point is the analytification of a closed point of C with residue field exactly L. Distinct residual points give distinct closed points of C. The disjoint union of the corresponding Spec L° *for any finite such list is an unramified marking datum of Definition 4.47.1.*

Proof. Choose L/K by Lemma 4.15.2. Its valuation ring has the monic presentation $O[\alpha] = O[S]/(P)$ and the orthonormal basis $1, \alpha, \dots, \alpha^{[l:k]-1}$ of Lemma 4.22.1. Since $P'(\alpha)$ is a unit, across a square-zero ideal the unique correction of a lifted root b is $b - P(b)/P'(b)$. This proves formal étaleness; the finite presentation is étale by [StacksAlg, Tag 00UR]. Thus L°/O is finite free and étale, with the original extending norm.

Choose $c \in L^\circ$ lifting the image of t at q. The value $g(c)$ is a unit because $\bar{g}(q) \neq 0$. Substitution $T \mapsto c$, $Z \mapsto g(c)^{-1}$ gives $R_0 \to L^\circ$. The finite étale algebra $A_0 \otimes_{R_0} L^\circ$ has the rational special point defined by q. Lemma 4.15.1

provides a section through it, and hence the stated map. It is continuous for the π-adic topology. Its generic map $A \to L$ is bounded by the integral-presentation norm of Proposition 3.14.7.

Let $x \in U \subset C^{\mathrm{an}}$ be the resulting point. On an affine algebraic neighbourhood of its image, the mapping property of Theorem 2.33.23 gives a homomorphism from its coordinate ring to L. Its image is a finite-dimensional domain over K, hence a field: the minimal polynomial of any nonzero element expresses its inverse as a polynomial in that element. Thus the corresponding point $p \in C$ is closed and $\kappa(p) \subseteq L$. Proposition 2.33.28 and completeness of finite extensions in Lemma 1.12.1 identify $\mathcal{H}(x)$ with $\kappa(p)$.

Choose $a \in A_0$ whose residue at q is a primitive generator β of l/k, using [StacksAlg, Tag 030N]. Write b for its value in L° and $e = [l : k]$. The change-of-basis matrix from $1, \alpha, \dots, \alpha^{e-1}$ to $1, b, \dots, b^{e-1}$ has invertible reduction, since both residue lists are bases of l/k. Its determinant is a unit, so $1, b, \dots, b^{e-1}$ is an O-basis of L°. Evaluation of analytic functions takes values in $\mathcal{H}(x)$, with the same values under its canonical embedding in L, by Proposition 1.6.2. Consequently $b \in \kappa(p)$ and $K(b) = L$, proving $\kappa(p) = L$.

Two different points q have different analytic lifts, because specialization is a function. A finite residue field over the complete field K has only its unique extending absolute value, so one closed algebraic point cannot give those two different analytic points. The resulting maps $\operatorname{Spec} L \to C$ are therefore distinct closed immersions. Their finite disjoint union is a closed immersion, while their integral coefficient schemes are finite étale over O. This is exactly the required marking datum. □

Theorem 4.50.4 (Integral Gauss charts give branch certificates)**.** *Suppose U is an actual affinoid domain of C^{an} as in Lemma 4.50.3. Every component point v_E of Y has an unramified branch certificate. If C admits a semistable algebraic model over the original O, then every v_E occurs as a component vertex of such a model. Finitely many points obtained from finitely many such integral charts can be retained simultaneously on one original-field semistable algebraic model.*

Proof. Fix the algebraically closed comparison field Ω of Lemma 4.45.1. Put

$$R_{\Omega,0} = \Omega^\circ\langle T, Z\rangle/(g(T)Z - 1), \qquad A_{\Omega,0} = A_0 \otimes_{R_0} R_{\Omega,0}.$$

The first algebra is admissible by Proposition 3.17.1. The second is finite projective over it, hence complete: an idempotent makes it a closed direct

summand of a finite free complete module. It is π-torsion-free and topologically of finite type, and is therefore admissible by Theorem 3.14.4. For the finite-type assertion, finite module generators give finitely many algebra generators; reducing monomials by their integral equations bounds all their products in a finite module, so restricted series converge. Étaleness is preserved by [StacksAlg, Tag 02GO].

The actual scalar presentation 2.33.5 identifies $R_{\Omega,0}[1/\pi] = R\widehat{\otimes}_K\Omega$. Finite tensor comparison 1.18.7 consequently identifies $A_{\Omega,0}[1/\pi] = A\widehat{\otimes}_K\Omega$. The domain base-change theorem 2.36.9 makes its spectrum the actual domain $U_\Omega \subset C_\Omega^{\mathrm{an}}$. Its special fibre is $Y_{\widetilde{\Omega}}$. Proposition 4.50.2, applied over Ω, makes it a reduced pure curve.

Choose a component E' of $Y_{\widetilde{\Omega}}$ above E. Such a component exists, and the component fibres are exactly those described in Lemma 4.49.1. Let w be its component point. Functoriality of specialization in Proposition 3.17.2 and uniqueness in Proposition 4.12.2 show that w maps to v_E.

Choose three closed separable points q_i of Y and geometric lifts q_i' lying on E', with the q_i pairwise distinct. Here is a construction over the original residue field. Inside $\widetilde{\Omega}$ take the separable closure k^{sep} of k. The purely inseparable step in Lemma 4.49.1 identifies the components over these two fields. In the component corresponding to E', take a nonempty open subset avoiding the other components and any finitely many previously selected Galois orbits. Its étale map to the principal open of $\mathbb{A}^1_{k^{\mathrm{sep}}}$ has open image by [StacksAlg, Tag 03WT]. The field k^{sep} is infinite: a finite field of cardinality q has the separable polynomial $X^q - X - 1$ without a root. Thus that image contains a rational point. Its nonempty étale fibre has a rational point by [StacksAlg, Tag 02GL]. It is defined over a finite separable extension of k, because the algebra has finitely many generators. Its image in Y is a closed separable point. Repeating this three times gives the required distinct q_i and q_i'.

Use Lemma 4.50.3 to form the unramified marking datum $P = \coprod_i \operatorname{Spec} L_i^\circ$. Each prescribed residue embedding $\widetilde{L}_i \to \widetilde{\Omega}$ lifts to an isometric K-embedding $L_i \to \Omega$. Indeed, in the monic presentation of Lemma 4.22.1, the prescribed residual root is simple. The henselian root construction 4.15.1 in Ω° lifts it to a root of the same polynomial. Irreducibility over K gives the embedding, and uniqueness in Lemma 1.12.1 makes it isometric. The resulting member of $P_K(\Omega)$ specializes to q_i'.

Let $B_i =]q_i'[$ in $\operatorname{Spf} A_{\Omega,0}$. The formal map to the principal open of the unit disc is adic, topologically finitely presented and formally étale. Composition with the formal open immersion into the disc has the same properties: unique lifting composes, and the relative presentation is obtained by adjoining Z with $g(T)Z = 1$ to a finite presentation. Completion gives the restricted-series presentation by Lemma 3.14.8. Translate the disc coordinate by a lift of $t(q_i')$. Proposition 4.39.3 now makes B_i an actual open unit disc whose closure in U_Ω is $B_i \cup \{w\}$. In particular the selected marked point lies in B_i, and the three tubes are disjoint.

These discs are open also in C_Ω^{an}. To verify this ordinary-topological assertion, fix $x \in B_i$ and an ambient affinoid neighbourhood W of x. The disc is an analytic domain and is absolutely boundaryless: in its disc coordinate, every point belongs to the strict interior of some smaller closed disc. Since W is an ambient ordinary neighbourhood, $B_i \cap W$ is an ordinary neighbourhood of x in B_i. Choose an affinoid neighbourhood V of x in B_i with $V \subset B_i \cap W$. Proposition 2.44.6 gives $x \in \operatorname{Int}(V/\Omega)$. The composition formula 2.44.1 for $V \to W \to \mathcal{M}(\Omega)$ then gives $x \in \operatorname{Int}(V/W)$. The domain-interior theorem 2.44.3 makes V a neighbourhood of x in W, and hence in C_Ω^{an}. This proves openness. Finally U_Ω is compact and closed in the Hausdorff curve C_Ω^{an}, by Propositions 2.24.1 and 2.33.29. Thus the same closure formula holds in the ambient curve. The data w, P, B_1, B_2, B_3 form the certificate of Definition 4.48.1.

If an original-field semistable model exists, the two remaining assertions are precisely the realization and finite simultaneous-insertion assertions of Theorem 4.48.2. □

4.51 Integral Gauss charts at component vertices

Keep the complete nontrivially real-valued field K, its specified absolute value, $O = K^\circ$, $k = \widetilde{K}$, and $0 \neq \pi \in K^{\circ\circ}$. All completions are π-adic.

Lemma 4.51.1 (Lifting residual finiteness)**.** *Let $R_0 \to A_0$ be a continuous homomorphism of admissible O-algebras. Suppose that A_0 is topologically of finite type over R_0 and that*

$$A_0/K^{\circ\circ}A_0 \quad \text{is finite over} \quad R_0/K^{\circ\circ}R_0.$$

Then A_0 is a finite R_0-module.

Proof. Choose a continuous surjection $P = R_0\langle Y_1, \dots, Y_s\rangle \twoheadrightarrow A_0$, and write y_i for the image of Y_i. Each $\bar{y}_i$ satisfies a monic polynomial over $\bar{R} = R_0/K^{\circ\circ}R_0$. Indeed, write multiplication by $\bar{y}_i$ on a finite list of module generators as a matrix over $\bar{R}$. Multiplication by the adjugate of $XI - M$ shows that its monic determinant annihilates every generator, hence also 1. Choose such a polynomial of positive degree d_i, and lift its coefficients to a monic $P_i(X) \in R_0[X]$. Then $P_i(y_i) \in K^{\circ\circ}A_0$.

Each of these finitely many memberships is a finite sum of elements of A_0 multiplied by scalars in $K^{\circ\circ}$. Choose one nonzero $\theta \in K^{\circ\circ}$ whose absolute value is at least those of all these scalars; if they all vanish, take $\theta = \pi$. Divisibility in O gives $P_i(y_i) = \theta q_i$ with $q_i \in A_0$. Lift q_i to $Q_i \in P$ and put

$$F_i = P_i(Y_i) - \theta Q_i.$$

These elements map to zero in A_0.

Give $R = R_0[1/\pi]$ the complete presentation norm whose closed unit ball is R_0, as in Proposition 3.14.7. On polynomials over R, division by the monic $P_i(Y_i)$ does not increase the coefficient norm: its lower coefficients belong to R_0. Perform these divisions in the order $Y_1, \dots, Y_s$. Their quotient and remainder operators are R-linear of norm at most one. Polynomial density and completeness extend them to $R\langle Y_1, \dots, Y_s\rangle$, and they preserve P. Thus there are continuous R_0-linear operators D_i and r such that

$$h = \sum_i P_i(Y_i)D_i(h) + r(h), \qquad r(h) \in L := \sum_{0 \le \alpha_i < d_i} R_0 Y^\alpha.$$

The displayed finite direct sum L is complete. Set $E(h) = \theta \sum_i Q_i D_i(h)$. Its norm is at most $|\theta|\,\|h\|$, so

$$h = \sum_i F_i D_i(h) + r(h) + E(h).$$

Iteration and convergence give

$$h = \sum_i F_i \left(\sum_{j \ge 0} D_i(E^j(h)) \right) + \sum_{j \ge 0} r(E^j(h)).$$

Both sums converge in the indicated integral modules: their terms have norms at most $|\theta|^j\|h\|$, and powers of θ eventually lie in every prescribed power of πO. The last sum belongs to L. Consequently the finitely many images of Y^α, $0 \le \alpha_i < d_i$, generate A_0 over R_0. □

Lemma 4.51.2 (Finite étale integral lifting)**.** *Let $R_0 \to A_0$ be as in Lemma 4.51.1. Suppose in addition that $A_0/\pi A_0$ is an* étale *$R_0/\pi R_0$-algebra. Then A_0 is finite projective as an R_0-module and is an* étale *R_0-algebra.*

Proof. The preceding lemma makes A_0 a finite R_0-module. It has no O-torsion, so Theorem 3.14.4 makes this module finitely presented. Its reduction modulo π is therefore a finitely presented module. It is flat by [StacksAlg, Tag 02GS], and hence finite projective by [StacksAlg, Tag 00NX]. Choose an idempotent matrix $\bar{e} \in M_m(R_0/\pi R_0)$ whose image is this module.

Lift $\bar{e}$ to a matrix e_0 over R_0. If $\delta = e^2 - e \in \pi M_m(R_0)$, then $(2e - I)^2 = I + 4\delta$ is invertible by its convergent geometric series. The Newton correction

$$e' = e - (2e - I)^{-1}(e^2 - e)$$

has error $e'^2 - e'$ divisible by δ^2. All matrices in this calculation commute, being convergent expressions in e. Starting at e_0 gives a convergent sequence with an idempotent limit $e \equiv \bar{e} \pmod{\pi}$. This works in every characteristic. Put $P = eR_0^m$. It is a complete separated direct summand of R_0^m.

Lift the images of the standard basis under the map $(R_0/\pi R_0)^m \to A_0/\pi A_0$ to A_0, and restrict the resulting map $R_0^m \to A_0$ to P. The map $\varphi : P \to A_0$ is an isomorphism modulo π. For $a \in A_0$, lift its class modulo π to $p_0 \in P$, write $a - \varphi(p_0) = \pi a_1$, and repeat. Completeness shows that $\sum_{n \geq 0} \pi^n p_n$ maps to a. Thus φ is surjective. If $\varphi(p) = 0$, the modulo-π isomorphism gives $p = \pi p_1$ in P. Since A_0 has no π-torsion, $\varphi(p_1) = 0$. Iteration puts p in $\bigcap_n \pi^n P = 0$. Thus $A_0 \simeq P$ is finite projective over R_0.

The algebra A_0 is finitely presented over R_0. For completeness, choose a finite module presentation with generators $b_1, \ldots, b_m$. Express 1 and all products $b_i b_j$ in these generators. The finite module relations, the relation for 1, and these product relations present A_0 as an algebra: they reduce every polynomial to a linear combination of the b_i, whose kernel is exactly the module relation space. It follows from the same polynomial presentation that Ω_{A_0/R_0} is a finite R_0-module and that its reduction modulo π is $\Omega_{(A_0/\pi A_0)/(R_0/\pi R_0)} = 0$. For every $a \in R_0$, $1 - \pi a$ is invertible by a geometric series. Thus π belongs to the Jacobson radical of R_0, and [StacksAlg, Tag 00DV] gives $\Omega_{A_0/R_0} = 0$. Finite projectivity implies flatness by [StacksAlg, Tag 00NX]; the flat, finitely presented, zero-differential criterion [StacksAlg, Tag 02GU] now proves ordinary étaleness. □

Theorem 4.51.3 (Integral Gauss-chart recognition of model vertices)**.** *Let C/K be smooth, projective and geometrically connected. Every component vertex of a semistable algebraic model of C over the original O, in the sense of Definition 4.45.3, is the component point of an integral* étale *Gauss chart whose generic fibre is an actual affinoid domain of C^{an}. The special fibre of this chart may be chosen integral.*

Consequently, if C admits one such original-field model, the following conditions on $v \in C^{\mathrm{an}}$ are equivalent:

1. *v is a component vertex of a semistable algebraic model over O;*
2. *v is a component point of an integral* étale *Gauss chart, as in Definition 4.50.1, whose generic fibre is an actual affinoid domain of C^{an};*
3. *v has an unramified branch certificate of Definition 4.48.1.*

Proof. Let $\mathcal{C}$ be the given model and let η_E be the generic point of its special component E. The special nodal curve is smooth at every component generic point. Flatness, finite presentation, and [StacksAlg, Tag 01V9] therefore make the family smooth at η_E. By [StacksAlg, Tag 054L] there is an affine open neighbourhood $\mathcal{V} = \operatorname{Spec} B$ of η_E with an étale coordinate map $\mathcal{V} \to \mathbf{A}^1_O$. The smooth family has relative dimension one, so the affine-space dimension in that lemma is one. Shrink $\mathcal{V}$ by a principal open so that its special fibre is integral. To do so, in the reduced finite-type k-algebra $B/K^{\circ\circ}B$, there are only finitely many minimal primes by [StacksAlg, Tags 00FO and 00FR]. Choose, for every minimal prime other than that of E, an element in that prime but not in the prime of E. Their product vanishes on all other components and is nonzero at η_E. Lift it to B and invert it. The remaining special fibre is a nonempty open of E. Its reduced coordinate ring has a unique minimal prime, which is zero by [StacksAlg, Tag 00E0], so it is a domain. The coordinate map remains étale.

Write $\bar{B} = B/K^{\circ\circ}B$ after this shrink, and write t for the residual coordinate. The map $\operatorname{Spec} \bar{B} \to \mathbf{A}^1_k$ is étale by [StacksAlg, Tag 02GO]; its image is nonempty open by [StacksAlg, Tag 03WT]. Hence $k[t] \to \bar{B}$ is injective. Its generic fibre is a nonzero domain étale of finite type over $k(t)$, so [StacksAlg, Tag 02GL] makes it a finite separable field. That field is $k(E)$.

Choose finitely many $k[t]$-algebra generators of $\bar{B}$. Each satisfies a monic polynomial over $k(t)$. Clearing the finitely many coefficient denominators gives one nonzero $\bar{g} \in k[t]$ for which

$$\bar{B}[1/\bar{g}] \quad \text{is finite over} \quad k[t, 1/\bar{g}].$$

Indeed, each chosen generator is now integral, and the monic relations reduce their powers to finitely many bounded-exponent monomials. Choose a lift $g \in O[T]$. The algebra $B[1/g(T)]$ remains finitely presented and flat over O, and is étale of finite presentation over $O[T, 1/g(T)]$.

Let A_0 be its actual π-adic completion, and set

$$R_0 = O\langle T, Z\rangle/(g(T)Z - 1).$$

Lemma 4.13.1 gives admissibility, the actual restricted presentation, and unchanged reductions modulo every π^n; the base completion is R_0 by Proposition 3.17.1. The relative algebraic presentation gives a continuous surjection $R_0\langle Y_1, \dots, Y_s\rangle \to A_0$. Because $\pi \in K^{\circ\circ}$, its reduction modulo $K^{\circ\circ}$ is exactly $\bar{B}[1/\bar{g}]$. Thus the residual-finiteness hypothesis of Lemma 4.51.1 holds. Its reduction modulo π is the base change of the ordinary étale algebraic coordinate map, and is étale by [StacksAlg, Tag 02GO]. Lemma 4.51.2 proves that A_0 is finite projective and étale over R_0. Its nonempty integral special fibre is the chosen open of E.

The completion of $\mathcal{C}$ is an actual formal model of C^{an} by Theorem 4.13.2. The completed affine open just constructed has generic fibre $U = \mathcal{M}(A_0[1/\pi])$, an actual affinoid domain of C^{an}; completion and restriction agree by Lemma 4.13.1 and Proposition 3.17.2. Its special generic point is still η_E. The component-point uniqueness and restriction statement of Proposition 4.12.2 therefore identify its unique component point with the original vertex v_E. This proves the first assertion.

It gives (1)⇒(2). Theorem 4.50.4 gives (2)⇒(3), and Theorem 4.48.2 gives (3)⇒(1) under the stated existence hypothesis. All constructions use O, its chosen π, and its specified absolute value throughout. □

4.52 A numerical vertex criterion over discretely valued fields

Except in statements explicitly requiring discreteness, K is any complete nontrivially real-valued field, with its specified absolute value, $O = K^{\circ}$ and

$k = \widetilde{K}$. Choose $0 \neq \pi \in K^{\circ\circ}$ for formal completions. Write η for the unit Gauss point of the analytic affine line and $H = \mathcal{H}(\eta)$. Thus H is the Gauss completion of $K(T)$, with residue field $k(t)$ and value group $|K^\times|$, as computed in Proposition 4.50.2.

Lemma 4.52.1 (An integral basis when the discrete value group is unchanged)**.** *Let H be a complete discretely valued field and let L/H be a finite extension with its extending absolute value. If $|L^\times| = |H^\times|$, then*

$$[L : H] = [\widetilde{L} : \widetilde{H}].$$

Every lift $b_1, \dots, b_f \in L^\circ$ of a $\widetilde{H}$-basis of $\widetilde{L}$ is an orthonormal H-basis of L and an H°-basis of L°. No separability assumption on the residue extension is needed.

Proof. Any finite residue-independent list has linearly independent lifts over H. In a nonzero relation, divide its coefficients by one having maximal absolute value and reduce. At least one coefficient has nonzero residue, contrary to residue independence. Consequently $f = [\widetilde{L} : \widetilde{H}]$ is finite and at most $[L : H]$: otherwise one could choose an independent residual list longer than that finite dimension. The same argument gives

$$\left| \sum_{j=1}^{f} c_j b_j \right| = \max_j |c_j| \qquad (c_j \in H).$$

Let ϖ be a uniformizer of H. The equality of value groups implies $L^{\circ\circ} = \varpi L^\circ$. For $a \in L^\circ$, express its residue in the given basis and lift the coefficients to H°. Subtracting the corresponding linear combination gives ϖa_1 with $a_1 \in L^\circ$. Repeat on $a_1, a_2, \dots$. For each basis vector the resulting coefficient is a convergent series $\sum_{n \geq 0} \varpi^n c_{n,j}$ in H°. The residual error has norm at most $|\varpi|^{n+1}$, so its limit is zero. This expresses a in $\sum_j H^\circ b_j$. Any element of L becomes integral after multiplication by a power of ϖ. Thus these f vectors span L, and the displayed norm identity proves both the degree equality and the asserted integral basis. □

Lemma 4.52.2 (Principal formal domains inside a Gauss neighbourhood)**.** *Let $g \in O[T]$ have nonzero reduction, and put*

$$D_g = \{x : |T(x)| \leq 1,\ |g(x)| = 1\}.$$

Every ordinary open neighbourhood N of η in D_g contains D_{gh} for some $h \in O[T]$ with nonzero reduction.

Proof. By Proposition 3.17.1, D_g is the actual principal formal generic domain, with its subspace topology in the closed unit disc. The evaluation topology of Proposition 1.6.3 therefore supplies finitely many functions $f_i \in K\langle T\rangle$ and an ordinary open neighbourhood N' of η in that disc, contained in the chosen relative neighbourhood after intersection with D_g, whose membership is ensured by keeping the values $|f_i|$ in specified open intervals containing $|f_i(\eta)|$. Zero functions impose no restriction and can be omitted.

For nonzero f_i, its Gauss norm is attained by a coefficient $a_i \in K^\times$. By the Gauss calculation and polynomial density in Proposition 1.2.2, choose $h_i \in O[T]$ with

$$\|a_i^{-1} f_i - h_i\| < 1, \qquad \|h_i\| = 1.$$

In particular $\bar{h}_i \neq 0$. Put $h = \prod_i h_i$, with the empty product equal to one. On D_{gh} every factor h_i has absolute value one: all have value at most one on the unit disc, and their product has value one. The strict approximation then gives $|f_i(x)| = |a_i| = |f_i(\eta)|$ throughout D_{gh}. Thus $D_{gh} \subset N' \cap D_g \subset N$. The reduction of gh is nonzero because $k[t]$ is a domain. □

Proposition 4.52.3 (An unramified Gauss germ gives an actual integral chart)**.** *Let X be a good K-analytic space, let $v \in X$, and suppose that an analytic morphism from an ordinary neighbourhood of v to the analytic affine line is* étale *at v and sends v to η. Put $L = \mathcal{H}(v)$, with the induced map $H \to L$. Suppose that L/H is finite, that $l = \widetilde{L}$ is finite separable over $k(t)$, and that*

$$[L : H] = [l : k(t)].$$

Then v belongs to an actual affinoid domain of X carrying an integral étale *Gauss chart. Its special fibre is integral and v is its unique component point.*

Proof. Choose a primitive element α of $l/k(t)$ by [StacksAlg, Tag 030N], and let $\bar{P} \in k(t)[S]$ be its monic minimal polynomial of degree f. Choose $g_0 \in O[T]$ with nonzero reduction so that all coefficients of $\bar{P}$ lie in $k[t, 1/\bar{g}_0]$.

Lift them to $O[T, 1/g_0]$, obtaining a monic polynomial P. On the free module $O[T, 1/g_0][S]/(P)$ with its monic remainder basis, let $\Delta \in O[T, 1/g_0]$ be the determinant of multiplication by P'. Its reduction is nonzero in $k(t)$, because $\bar{P}$ is separable. Write $\Delta = a(T)/g_0(T)^m$ with $a \in O[T]$; then $\bar{a} \neq 0$. Put $g = g_0 a$. The adjugate matrix makes P' a unit in

$$B_0^{\mathrm{alg}} = O[T, 1/g][S]/(P).$$

For an arbitrary square-zero ideal, a lifted root is corrected uniquely by $b \mapsto b - P(b)/P'(b)$. Thus this finite free algebra is formally étale, and is ordinarily étale by [StacksAlg, Tag 00UR].

Let

$$R_0 = O\langle T, Z\rangle/(g(T)Z - 1), \qquad B_0 = R_0[S]/(P).$$

The algebra B_0^{alg} is flat over O: coefficientwise injectivity in the ideal criterion [StacksAlg, Tag 00HD] makes $O[T]$ flat over O; localization is flat by [StacksAlg, Tag 00HT], and the final extension is finite free. The actual completion calculation in Lemma 4.13.1 identifies B_0 with the completion of B_0^{alg}, coefficient by coefficient in its finite free monic basis. It is admissible, finite free and étale over R_0. Its special algebra is finite free over $k[t, 1/\bar{g}]$ and injects into its generic fibre l, so it is a domain. Consequently Proposition 4.50.2 gives exactly one point w over η in $B = \mathcal{M}(B_0[1/\pi])$, with $[\mathcal{H}(w) : H] = f$ and residual extension $l/k(t)$. Here $0 \neq \pi \in K^{\circ\circ}$ is any choice used for the completion.

The residue element $\alpha \in \widetilde{L}$ is a simple root of $\bar{P}$. The complete-valuation henselian lemma 4.15.1 lifts it to a root of P in L°. It gives an H-homomorphism $\mathcal{H}(w) = H[S]/(P) \to L$. Both fields have degree f over H, so it is an isomorphism; it is isometric by Lemma 1.12.1.

Use the full algebraic generic cover

$$\operatorname{Spec} B_0^{\mathrm{alg}}[1/\pi] \longrightarrow \operatorname{Spec} K[T, 1/g].$$

Its analytification $\mathcal{B}$ is étale over the ordinary open $g \neq 0$ in the analytic line by Proposition 2.33.24 and Theorem 2.35.2. Theorem 2.33.26 identifies B with its entire inverse image of the affinoid domain D_g. In particular its entire fibre at η is $\{w\}$.

By the ordinary-open local definition of an analytic étale map, the map near v restricts to a finite flat map with zero relative differentials. Finite branch separation in Lemma 2.35.1 then gives a finite étale germ with fibre

field L. The completed-residue-field equivalence of Theorem 2.54.3 therefore gives an isomorphism over the analytic line between ordinary neighbourhoods of v in X and of w in $\mathcal{B}$. Write W for the latter neighbourhood.

By Proposition 1.6.3, the compact set $B \setminus W$ has closed image in the Hausdorff affinoid D_g. Its image avoids η because the full fibre is the singleton $w \in W$. Lemma 4.52.2 supplies a polynomial $h \in O[T]$ with nonzero reduction such that the domain D_{gh} avoids that image. Its full inverse image in B is therefore contained in W. It is the actual affinoid generic fibre of the principal formal restriction

$$B_0\langle U\rangle/(h(T)U - 1)$$

by Proposition 3.17.1. This restriction is finite projective and étale over the principal base chart for gh, by ordinary base change and [StacksAlg, Tag 02GO]. Its special algebra is the corresponding nonzero localization of the integral special algebra of B_0. The ordinary-neighbourhood isomorphism transports its actual affinoid domain into X. Its unique component point is transported from w to v, proving the assertion. □

Theorem 4.52.4 (Discrete original-field vertex recognition)**.** *Suppose K is complete and discretely valued, with its specified nontrivial absolute value. Let C/K be a smooth projective geometrically connected curve. Let $v \in C^{\mathrm{an}}$ satisfy*

$$|\mathcal{H}(v)^{\times}| = |K^{\times}|,$$
$$\widetilde{\mathcal{H}(v)}/k \text{ is separably generated of transcendence degree one.}$$

Then v is the unique component point of an actual integral étale *Gauss chart in C^{an}, and has an unramified branch certificate. If C admits one semistable algebraic model over the original O, these two displayed conditions are necessary and sufficient for v to occur as a component vertex of such a model.*

Proof. Put $L = \mathcal{H}(v)$ and $l = \widetilde{L}$. The scheme image of v is the generic point of C. Indeed, a nongeneric point on an integral curve is closed; its residue field is finite over K by [StacksAlg, Tag 00FV]. Its analytic completed residue field is that same finite complete field by Proposition 2.33.28 and Lemma 1.12.1, whose residual field is algebraic over k by Lemma 3.22.2. This contradicts the assumed residual transcendence degree. Thus $F = K(C)$ embeds densely in L, by the same analytification-valued-fibre proposition.

Choose $\bar{t} \in l$ as a separating transcendence basis, so that $l/k(\bar{t})$ is algebraic separable. Density supplies $t_0 \in F$ of norm one with residue $\bar{t}$. Choose a rational smooth coordinate $s \in F$ from a smooth affine chart at the generic point, using [StacksAlg, Tag 054L]. In its standard smooth Jacobian presentation, solving the invertible square subsystem expresses every differential in terms of ds; consequently $\Omega_{F/K}$ has basis ds. For a uniformizer ϖ of K, choose a sufficiently large integer N and set

$$t = t_0 + \varpi^N s.$$

Then $|\varpi^N s(v)| < 1$, so the residue of t is still $\bar{t}$. We can require $dt \neq 0$: in the one-dimensional space $\Omega_{F/K}$, at most one scalar among the distinct ϖ^N makes $dt_0 + \varpi^N ds$ vanish.

The rational function t is transcendental over K. The finitely generated field F/K has transcendence degree one, so it is a finite extension of $K(t)$: its finite field generators are algebraic over $K(t)$. The quotient of $\Omega_{F/K}$ obtained by imposing $dt = 0$ is $\Omega_{F/K(t)}$, as follows directly from the universal property of derivations. It is zero. On an affine open where t is regular, the map to the affine line is of finite presentation; this follows from [StacksAlg, Tag 00FO] and a finite polynomial presentation. At its generic point the local map is the field extension $K(t) \to F$, which is flat by [StacksAlg, Tag 00HD] because the only ideals of the base field are zero and itself. The criterion [StacksAlg, Tag 02GU] therefore makes this map étale at the generic point. This algebraic open analytifies to an ordinary open by Proposition 2.33.24. Theorem 2.35.2 gives an actual analytic étale map near v.

For a polynomial $Q \in K[T]$, division by a coefficient of maximal size and reduction at v give

$$|Q(t)| = \max_i |Q_i|,$$

since $\bar{t}$ is transcendental over k. Thus v maps to the unit Gauss point, and completion embeds the Gauss field H isometrically into L. The image of $F \otimes_{K(t)} H \to L$ is a finite-dimensional H-subalgebra of the field L, containing F. It is a field: multiplication by a nonzero element is injective on this finite-dimensional space and hence surjective. It is complete and therefore closed by Lemma 1.12.1. Since F is dense in L, this image is all of L. In particular L/H is finite.

The Gauss-field computation in Proposition 4.50.2 makes H discretely valued with value group $|K^\times|$ and residue field $k(\bar{t})$. The assumed value-group equality and Lemma 4.52.1 give

$$[L : H] = [l : k(\bar{t})].$$

In particular the right side is finite; by the choice of $\bar{t}$, this residual extension is separable. Proposition 4.52.3 constructs the actual integral chart in C^{an}. Theorem 4.50.4 supplies its unramified branch certificate.

When one original-field semistable model exists, realization follows from Theorem 4.48.2. Conversely, every component vertex of such a model has exactly the stated value group and separably generated residual function field by Lemma 4.45.2. The field, its absolute value and the uniformizer used in the proof have not been replaced by an extension or rescaling. □

4.53 An original-field rational Gauss degree criterion

Let K be complete and nontrivially valued for its specified real-valued absolute value, and put $O = K^\circ$ and $k = \widetilde{K}$. Let C/K be a smooth projective geometrically connected curve and put $F = K(C)$. Semistable algebraic models are those of Definition 4.45.3.

Theorem 4.53.1 (Rational Gauss degree criterion)**.** *Assume that C admits one semistable algebraic model over the original O. A point $v \in C^{\mathrm{an}}$ occurs as a component vertex of such a model if and only if there is a rational function $t \in F$, defined at v, satisfying the following conditions. Write $L = \mathcal{H}(v)$ and $l = \widetilde{L}$.*

1. *$|t(v)| = 1$, and its residue $\bar{t} \in l$ is transcendental over k.*
2. *$F/K(t)$ and $l/k(\bar{t})$ are finite separable field extensions.*
3. *If H_t is the completion of $K(t)$ for the norm induced by v, then the actual extension $H_t \to L$ is finite and*

$$[L : H_t] = [l : k(\bar{t})].$$

Here H_t is the unit Gauss completion for the coordinate t. The criterion applies without a discreteness or characteristic-zero assumption on K.

Proof. Suppose first that v is the component vertex of a model $\mathcal{C}$, corresponding to the generic point η_E of its special component E. Use the construction in Theorem 4.51.3, retaining its original algebraic coordinate. More explicitly, that construction chooses an actual affine Zariski open of $\mathcal{C}$ at η_E with an étale map to $\mathbf{A}^1_O$. It removes the other special components by a principal open, makes the residual map finite by a principal shrink of the affine-line base, and completes that same algebraic open. The coordinate T therefore restricts to a rational function $t \in F$; it is not a newly chosen analytic coordinate. The resulting completed algebra is finite projective and étale over

$$O\langle T, Z\rangle/(g(T)Z - 1), \qquad \bar{g} \neq 0,$$

its special fibre is an open of E, and its actual component point in C^{an} is the original v.

By Proposition 4.50.2, the point v lies above the unit Gauss point in coordinate t, its residual function field $l = k(E)$ is finite separable over $k(\bar{t})$, and its actual completed field degree is precisely this residual degree. In particular $|t(v)| = 1$ and $\bar{t}$ is transcendental. The retained algebraic coordinate is also étale on the generic fibre by [StacksAlg, Tag 02GO]. Localizing its integral generic coordinate ring by all nonzero polynomials in t gives a nonzero domain which is étale of finite type over $K(t)$. By [StacksAlg, Tag 02GL] it is a finite separable field, necessarily F. This proves all three conditions with the actual original-field rational function t.

Conversely, suppose the conditions hold. The scheme image of v is generic: otherwise it is a closed point of the integral curve, its residue field is finite over K by [StacksAlg, Tag 00FV], and Proposition 2.33.28 and Lemma 1.12.1 identify L with that finite complete field. Its residual field would be algebraic over k by Lemma 3.22.2, contrary to the transcendence of $\bar{t}$. Thus all of F embeds into L with its specified norm.

On an affine algebraic open where t is regular, the rational map to $\mathbf{A}^1_K$ has finite presentation: the coordinate rings are finite type over K and Noetherian by [StacksAlg, Tag 00FO]. At the generic point its local map is $K(t) \to F$. This field map is flat by the ideal criterion [StacksAlg, Tag 00HD], its maximal ideals are zero, and its residue extension is finite separable by hypothesis. The criterion [StacksAlg, Tag 02GU] therefore makes the map étale at the generic point. Shrink to that algebraic étale open. Proposition 2.33.24 and Theorem 2.35.2 give an actual analytic étale germ at v.

For any nonzero polynomial $Q \in K[T]$, divide by a coefficient of maximal absolute value and reduce at v. Transcendence of $\bar{t}$ gives

$$|Q(t)(v)| = \max_i |Q_i|.$$

Consequently this germ sends v to the unit Gauss point, and its actual completed-residue-field map is $H_t \to L$. Condition (3) and the finite separable residual extension in (2) are exactly the hypotheses of Proposition 4.52.3. It supplies an integral étale Gauss chart whose generic fibre is an actual affinoid domain of C^{an} and whose component point is v. The equivalence in Theorem 4.51.3, using the assumed existence of one original-field model, realizes v as a component vertex of such a model. Every norm and every field embedding in the degree identity is the one induced by the original point v. □

4.54 Unit degrees and original-field Gauss charts

All valued fields in this section are complete for their specified nontrivial real-valued non-Archimedean absolute values. For a valued field H, write H°, $H^{\circ\circ}$ and $\widetilde{H}$ for its valuation ring, maximal ideal and residue field. The trace is the trace of the finite-dimensional multiplication operator.

Definition 4.54.1 (Immediate extensions)**.** A valued extension L/H is *immediate* if $|L^\times| = |H^\times|$ and the residue inclusion $\widetilde{H} \to \widetilde{L}$ is an isomorphism.

Lemma 4.54.2 (A contractive normalized trace)**.** *Let L/H be a finite extension of degree n, with the extending absolute value. Then*

$$|\operatorname{Tr}_{L/H}(x)| \le |x| \qquad (x \in L).$$

If $|n| = 1$, the map

$$P = n^{-1} \operatorname{Tr}_{L/H} : L \longrightarrow H$$

is an H-linear retraction of norm at most one. In particular, a finite immediate extension of degree n with $|n| = 1$ is trivial.

Proof. Fix $x \in L$, and let $Q(T) = T^d + a_{d-1}T^{d-1} + \cdots + a_0$ be its minimal polynomial over H. Choose an algebraic closure of L by [StacksAlg, Tag 09GT] and let M/H be the finite extension generated by the roots of Q. Give M its extending absolute value using Lemma 1.12.1. Every root x_i of

Q, counted with its multiplicity, has absolute value $|x|$: the H-isomorphism $H(x) \to H(x_i)$ is isometric by uniqueness of the extending absolute value. Thus

$$|a_{d-1}| = \left|\sum_{i=1}^{d} x_i\right| \leq |x|.$$

Choose an $H(x)$-basis $e_1, \ldots, e_m$ of L. In the H-basis $x^j e_i$, $0 \leq j < d$, multiplication by x has m companion-matrix blocks, each of trace $-a_{d-1}$. Consequently

$$\mathrm{Tr}_{L/H}(x) = -m a_{d-1}.$$

The ultrametric inequality gives $|m| \leq 1$, proving the bound. For $h \in H$, its multiplication matrix is h times the identity, so $\mathrm{Tr}_{L/H}(h) = nh$. This proves the assertion about P when $|n| = 1$.

Suppose now that L/H is immediate and nontrivial. The retraction P has a nonzero kernel; take $0 \neq x \in \ker P$. Equality of value groups supplies $c \in H^\times$ with $|c| = |x|$. Since the residue fields agree, choose $b \in H^\circ$ with the same residue as x/c. Then $a = cb$ satisfies

$$|x - a| < |x| = |a|.$$

On the other hand, $P(x - a) = -a$, so contractivity gives $|a| \leq |x - a|$, a contradiction. □

Proposition 4.54.3 (Unchanged value group and a unit degree)**.** *Let L/H be a finite extension of degree n. Suppose that*

$$|L^\times| = |H^\times|, \qquad |n| = 1, \qquad \widetilde{L}/\widetilde{H} \text{ is separable.}$$

Then

$$[L : H] = [\widetilde{L} : \widetilde{H}].$$

More precisely, L is generated by a root of a monic polynomial over H° with irreducible separable reduction. Lifts of any $\widetilde{H}$-basis of $\widetilde{L}$ form an orthonormal H-basis of L and an H°-basis of L°.

Proof. The residue extension is finite. Indeed, lifts of any finite residue-independent list are linearly independent over H: normalize a proposed nonzero relation by a coefficient of maximal absolute value and reduce. Thus its degree f is at most n.

Choose a primitive residue element $\bar{b}$ by [StacksAlg, Tag 030N]. Lift its monic minimal polynomial to $Q \in H^\circ[T]$, of degree f. The simple-root construction in Lemma 4.15.1, applied to L, gives a root $b \in L^\circ$ with residue $\bar{b}$. The powers $1, b, \dots, b^{f-1}$ are residue-independent and hence linearly independent over H. Since $Q(b) = 0$, the intermediate field $E = H(b)$ has degree f. The same normalization of a linear combination proves

$$\left|\sum_{i=0}^{f-1} c_i b^i\right| = \max_i |c_i| \qquad (c_i \in H).$$

It follows that $E^\circ = \sum_i H^\circ b^i$, $\widetilde{E} = \widetilde{L}$, and $|E^\times| = |H^\times|$. The field E is complete by Lemma 1.12.1. Consequently L/E is immediate.

Put $m = [L : E]$, so $n = mf$. Both integers have absolute value at most one, and $|n| = 1$ forces $|m| = 1$. Lemma 4.54.2 now gives $L = E$. This proves the degree equality and the monic presentation. For any other lifted residue basis, the reduction of its change-of-basis matrix is invertible. Its determinant is a unit of H°, so the inverse matrix is integral by the adjugate formula. The same reduction argument gives the asserted norm identity and the two basis assertions. □

Lemma 4.54.4 (A separating rational Gauss germ)**.** *Let K be as above, put $k = \widetilde{K}$, and let C/K be a smooth projective geometrically connected curve. Suppose $v \in C^{\mathrm{an}}$ has residue field $l = \widetilde{\mathcal{H}(v)}$ separably generated of transcendence degree one over k. For every separating transcendental element $\bar{t} \in l$ there is a rational function $t \in K(C)$, defined at v, such that $|t(v)| = 1$, its residue is $\bar{t}$, and the induced map to the analytic affine line is* étale *on an ordinary neighbourhood of v. Moreover $K(C)/K(t)$ is finite separable, v maps to the unit Gauss point, and*

$$H_t \longrightarrow \mathcal{H}(v)$$

is finite, where H_t is the unit Gauss completion of $K(t)$. In particular $l/k(\bar{t})$ is finite separable.

Proof. The scheme image of v is the generic point. Otherwise it is a closed point on the curve, with residue field finite over K by [StacksAlg, Tag 00FV]. Proposition 2.33.28 and Lemma 1.12.1 then make $\mathcal{H}(v)/K$ finite. Its residue extension is algebraic by Lemma 3.22.2, a contradiction. The same analytification proposition identifies $K(C)$ with a dense valued subfield of $L = \mathcal{H}(v)$.

Choose $t_0 \in K(C)$ of norm one with residue $\bar{t}$. By [StacksAlg, Tag 054L], a smooth affine chart at the generic point has a rational étale coordinate s. Its invertible Jacobian subsystem expresses all differentials in terms of ds, and gives $\Omega_{K(C)/K} = K(C)\,ds$. Fix $0 \neq \pi \in K^{\circ\circ}$. For sufficiently large N,

$$t = t_0 + \pi^N s$$

still has residue $\bar{t}$. We can also require $dt \neq 0$: at most one of the distinct scalars π^N can make $dt_0 + \pi^N ds$ vanish.

The residue transcendence shows that t is transcendental over K. The finitely generated field $F = K(C)$ is algebraic over $K(t)$ and hence finite over it. The universal property of derivations gives $\Omega_{F/K(t)} = \Omega_{F/K}/F dt = 0$. On an affine open where t is regular, the associated map to the affine line is of finite presentation: present its algebra as a quotient of a polynomial algebra over $K[T]$ and apply the Noetherian assertion [StacksAlg, Tag 00FO] to its kernel. The generic local map is a field extension, flat by the ideal criterion [StacksAlg, Tag 00HD]. By parts (4), (1) and (6) of [StacksAlg, Tag 02GU], the map is étale at the generic point and $F/K(t)$ is finite separable. Its étale locus is an algebraic open containing the scheme image of v. Proposition 2.33.24 and Theorem 2.35.2 give the asserted ordinary analytic neighbourhood.

For a nonzero polynomial $Q \in K[T]$, divide by a coefficient of largest absolute value and reduce at v. The nonzero reduced polynomial cannot vanish at the transcendental $\bar{t}$, so

$$|Q(t)| = \max_i |Q_i|.$$

This is the unit Gauss norm. Completion embeds H_t isometrically in L. The image of $F \otimes_{K(t)} H_t \to L$ is a finite-dimensional H_t-subalgebra of a field. It is a field: multiplication by any nonzero element is an injective, hence surjective, endomorphism of that finite-dimensional vector space. By Lemma 1.12.1 it is complete and closed in L. It contains the dense field F, so it equals L. Finally, residue-independent lifts are linearly independent over H_t, as in Proposition 4.54.3. Thus $[l : k(\bar{t})] \leq [L : H_t] < \infty$; separability was part of the choice of the separating element. □

Theorem 4.54.5 (Numerical vertex recognition in residue characteristic zero)**.** *Let K be a complete nontrivially real-valued field with residue characteristic zero, and let C/K be a smooth projective geometrically connected curve. If $v \in C^{\mathrm{an}}$ satisfies*

$$|\mathcal{H}(v)^{\times}| = |K^{\times}|,$$
$$\widetilde{\mathcal{H}(v)}/\widetilde{K} \text{ is separably generated of transcendence degree one,}$$

then v is the unique component point of an actual integral étale *Gauss chart in C^{an} and has an unramified branch certificate.*

If C admits one semistable algebraic model, in the sense of Definition 4.45.3, over the original K°, the two displayed conditions characterize the points which occur as component vertices of such models. In this existence class, a finite nonempty set satisfying those conditions is realized exactly with a specified unramified marking datum precisely when its entire geometric inverse image satisfies the marked semistable-vertex condition of Definition 4.48.3.

For arbitrary residue characteristic, the chart and branch-certificate conclusions, and realization under original-model existence, hold at a point satisfying the displayed conditions if one separating rational Gauss germ supplied by Lemma 4.54.4 has completed degree $n = [\mathcal{H}(v) : H_t]$ with $|n| = 1$.

Proof. Choose the rational Gauss germ by Lemma 4.54.4. Put $L = \mathcal{H}(v)$ and $l = \widetilde{L}$. The Gauss calculation in Proposition 4.50.2 gives

$$|H_t^{\times}| = |K^{\times}|, \qquad \widetilde{H_t} = k(\bar{t}).$$

Thus L/H_t has unchanged value group and finite separable residue extension. In residue characteristic zero every positive integer has absolute value one. Proposition 4.54.3 therefore gives

$$[L : H_t] = [l : k(\bar{t})].$$

In arbitrary residue characteristic the stated hypothesis $|n| = 1$ gives exactly the same conclusion by that proposition. Proposition 4.52.3 supplies the actual integral chart, and Theorem 4.50.4 supplies its branch certificate.

Under original-model existence, Theorem 4.48.2 realizes the point. The converse follows from Proposition 4.45.2. For a finite nonempty set, the pointwise certificates and the entire geometric marked-complement condition are

precisely the hypotheses and conclusions of Theorem 4.48.4. All coordinates, completions and traces use the original absolute value; the argument does not require a discrete value group. □

4.55 An immediate extension and a nonrealizable curve point

For a complete valued field F, write $G_F(T)$ for the completion of $F(T)$ in its unit Gauss absolute value. All field embeddings below retain the specified absolute values.

Lemma 4.55.1 (Finite constant extensions of a Gauss field)**.** *Let L/K be a finite extension of complete real-valued fields. The canonical map*

$$G_K(T) \otimes_K L \longrightarrow G_L(T)$$

is an isomorphism of fields and finite-dimensional normed $G_K(T)$-algebras. In particular

$$[G_L(T) : G_K(T)] = [L : K].$$

No separability or discrete-valuation hypothesis is required.

Proof. Choose a K-basis $e_1, \dots, e_n$ of L. Lemma 1.12.1 gives constants $c, C > 0$ with

$$c \max_i |a_i| \le \left| \sum_i a_i e_i \right| \le C \max_i |a_i| \qquad (a_i \in K).$$

Apply these inequalities coefficient by coefficient to polynomials $P_i \in K[T]$ and then take the largest coefficient norm. The Gauss norms satisfy

$$c \max_i \|P_i\| \le \left\| \sum_i e_i P_i \right\| \le C \max_i \|P_i\|.$$

The Gauss norm is multiplicative by Proposition 1.2.2. Division by a common nonzero denominator in $K[T]$ therefore gives the same inequalities for any $P_i \in K(T)$. They also prove that the e_i are independent over $K(T)$. Their span is a finite-dimensional subalgebra of the field $L(T)$, so it is a field: multiplication by a nonzero element is an injective endomorphism and hence surjective. It contains L and T, and is consequently all of $L(T)$.

The displayed equivalent norms identify the completion of this vector space with $\bigoplus_i G_K(T)e_i$. Multiplication agrees on the dense subfield $L(T)$ and extends continuously, since the Gauss norm is multiplicative. By definition its completion is $G_L(T)$. This proves the asserted map is an isomorphism and computes its degree. □

Proposition 4.55.2 (An explicit immediate extension of degree p)**.** *Fix a prime p, a real number $0 < \rho < 1$, and symbols $q_0, q_1, \dots$ with $q_{m+1}^p = q_m$. Put*

$$K_m = \mathbf{F}_p((q_m)), \qquad |q_m| = \rho^{1/p^m}, \qquad K = \widehat{\bigcup_{m\geq 0} K_m}, \qquad q = q_0.$$

Then K is a complete field with a nondiscrete real-valued absolute value and

$$|K^\times| = \rho^{\mathbf{Z}[1/p]}, \qquad \widetilde{K} = \mathbf{F}_p.$$

The polynomial $P(S) = S^p - S - q^{-1}$ is irreducible over K. For a root α and $L = K(\alpha)$, the unique extending absolute value makes L/K a degree-p immediate extension.

Proof. In K_m, addition and multiplication are coefficientwise Laurent series operations. Every nonzero series has an invertible lowest term, and its inverse is obtained by a geometric series after factoring that term. Lowest nonzero terms prove multiplicativity of the indicated absolute value. A Cauchy sequence has eventually constant coefficients in every bounded exponent range and a common lower bound for its nonzero exponents, which proves completeness. Substitution $q_m = q_{m+1}^p$ gives compatible isometric embeddings. Their union has precisely the displayed value group and residue field. Completion retains both by Lemma 3.22.2.

Let

$$a_m = \sum_{i=1}^{m} q_i^{-1}, \qquad a_0 = 0.$$

Then $P(a_m) = -q_m^{-1}$. For $b \in K_m$, write $b = a_m + c$. If $|c| \leq 1$, the term $-q_m^{-1}$ strictly dominates $c^p - c$. If $|c| > 1$, its additive valuation is a negative integer multiple of $1/p^m$, so

$$v(c^p - c) = p\, v(c) \leq -p/p^m < -1/p^m.$$

Here $v(q) = 1$ is used only to name the exponents; the absolute value remains $|q| = \rho$. It follows in either case that

$$|P(b)| \geq \rho^{-1/p^m} > 1.$$

If P had a root $\beta \in K$, density would give $b \in K_m$ with $|b - \beta| < 1$. But then $P(b) = (b-\beta)^p - (b-\beta)$ would have norm less than one, a contradiction.

Choose a root α in an algebraic closure supplied by [StacksAlg, Tag 09GT]. All roots of P are the distinct elements $\alpha + c$, $c \in \mathbf{F}_p$. If the minimal polynomial of α had degree d with $0 < d < p$, its next-to-leading coefficient would be $-d\alpha - \sum_{i=1}^{d} c_i \in K$ for some $c_i \in \mathbf{F}_p$. Since $d \neq 0$ in $\mathbf{F}_p$, this would give $\alpha \in K$. Thus P is irreducible of degree p. Equip L with its unique complete extending absolute value by Lemma 1.12.1.

Write $\beta_m = \alpha - a_m$. Its equation is

$$\beta_m^p - \beta_m = q_m^{-1}.$$

The right side has norm greater than one. Hence $|\beta_m| = \rho^{-1/p^{m+1}}$. The norm-one element $y_m = q_{m+1}\beta_m$ satisfies

$$y_m^p - q_{m+1}^{p-1} y_m = 1.$$

Its residue therefore satisfies $\bar{y}_m^p = 1$, and equals one in the residue field of characteristic p.

Take $0 \neq x \in L$. Write it in the basis $1, \alpha, \ldots, \alpha^{p-1}$. Approximate its finitely many coefficients by elements of one K_m, obtaining $x_m = \sum_{j<p} d_j \alpha^j$ with $d_j \in K_m$ and $|x - x_m| < |x|$. Expanding $\alpha = a_m + \beta_m$ writes $x_m = \sum_{j<p} b_j \beta_m^j$ with $b_j \in K_m$. The nonzero terms have pairwise different valuations:

$$v(b_j \beta_m^j) = v(b_j) - j/p^{m+1}, \qquad v(b_j) \in p^{-m}\mathbf{Z}.$$

Indeed their numerators modulo p distinguish the indices $0 \leq j < p$. Let j index the unique term of largest norm, and put $h = b_j q_{m+1}^{-j} \in K^\times$. Then $|h| = |x_m| = |x|$, while the residue of x_m/h is $\bar{y}_m^j = 1$. The strict approximation gives the same residue for x/h. Every nonzero element of L consequently has a value already in K, and every norm-one element has residue in $\mathbf{F}_p$. This proves immediacy. $\square$

Theorem 4.55.3 (The weak numerical conditions do not characterize vertices)**.** *Use K, q, L and α from Proposition 4.55.2. Let $J = G_L(Y)$ and put $X = \alpha + qY$ in J. Restriction of this absolute value to $K(X)$ defines a point $v \in (\mathbf{P}^1_K)^{\mathrm{an}}$ satisfying*

$$\mathcal{H}(v) = G_L(Y),$$
$$|\mathcal{H}(v)^\times| = |K^\times|, \qquad \widetilde{\mathcal{H}(v)} = \mathbf{F}_p(y).$$

Nevertheless v is not a component vertex of any semistable algebraic model over the original K° in Definition 4.45.3.

The actual rational Gauss coordinate

$$u = \frac{X^p - X - q^{-1}}{q}$$

has $|u(v)| = 1$ and residue $-y$, and $K(X)/K(u)$ is finite separable. Its completed extension satisfies

$$[\mathcal{H}(v) : G_K(u)] = p, \qquad [\widetilde{\mathcal{H}(v)} : \mathbf{F}_p(\bar{u})] = 1.$$

Thus the obstruction occurs on a curve which already has the smooth original model $\mathbf{P}^1_{K^\circ}$.

Proof. The element X is transcendental over K, since Y is transcendental over L. Its induced multiplicative seminorm on $K[X]$ gives an affine analytic point by Theorem 2.33.23, and hence a point of the analytic projective line by Proposition 2.33.24. Proposition 2.33.28 identifies its completed residue field with the closure of $K(X)$ in J. Set

$$b = X^p - X - q^{-1} = q^p Y^p - qY.$$

The Gauss norm gives $|b| = |q| = \rho < 1$. The series $z = \sum_{i \geq 0} b^{p^i}$ converges in $\mathcal{H}(v)$, satisfies $z^p - z = -b$, and has norm at most ρ. Hence $X + z$ is a root of P. Its difference from α has norm at most $\rho < 1$, whereas every other root $\alpha + c$, $0 \neq c \in \mathbf{F}_p$, is at distance one from α. Therefore $X + z = \alpha$. It follows that $\mathcal{H}(v)$ contains L and $Y = (X - \alpha)/q$. Since $L(Y)$ is dense in J, it equals J. The unit Gauss calculation of Proposition 4.50.2, applied over L, now gives the displayed value group and residue field.

We have $u = q^{p-1}Y^p - Y$, so $|u| = 1$ and $\bar{u} = -y$. Reduction of a normalized nonzero polynomial in u shows that its induced norm is the unit Gauss norm, over both K and L. In $G_L(u)$ iterate

$$Y_0 = -u, \qquad Y_{r+1} = q^{p-1}Y_r^p - u.$$

All iterates have norm at most one. Their successive differences decrease to zero, since the map has Lipschitz constant at most $|q|^{p-1} < 1$ on that ball. The element $Y \in J$ is its fixed point, and the same estimate shows $Y_r \to Y$ in J. The complete subfield $G_L(u)$ is closed, so it contains Y. Consequently $J = G_L(u)$, and Lemma 4.55.1 gives $[J : G_K(u)] = [L : K] = p$. The residual degree is one because $\bar{u} = -y$. The equation $X^p - X - q^{-1} - qu = 0$ makes $K(X)/K(u)$ finite, and its derivative -1 makes the extension separable.

It remains to exclude all original models, not just this particular coordinate. Choose the comparison field Ω of Lemma 4.45.1, using an algebraic closure containing L. For every $c \in \mathbf{F}_p$ let w_c be the Gauss point over Ω of the disc

$$|X - (\alpha + c)| \leq |q|.$$

The map $\alpha \mapsto \alpha + c$ defines a K-automorphism of L, and is isometric by Lemma 1.12.1. Applied coefficientwise to any polynomial in $K[X]$, it shows that every w_c restricts to v. These points are distinct: at w_c, the function $X - (\alpha + c)$ has norm $|q|$, while at $w_{c'}$ for $c' \neq c$ it has norm one.

Suppose v were a model component vertex, with special component E. Proposition 4.45.2 gives $k(E) = \mathbf{F}_p(y)$ and identifies the entire geometric inverse image of v with the component generic points above E. That set has one element. Indeed,

$$\mathbf{F}_p(y) \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p = \overline{\mathbf{F}}_p(y)$$

is a field. To verify the equality, every finite coefficient extension $k'/\mathbf{F}_p$ stays linearly independent over $\mathbf{F}_p(y)$, by clearing polynomial denominators. The tensor with k' is therefore a finite-dimensional domain and hence a field; it contains the inverses of all nonzero polynomials with coefficients in k'. Taking the union over these finite coefficient fields proves the equality. The comparison-field residue is $\overline{\mathbf{F}}_p$ by Lemma 4.45.1. The p distinct points w_c above v contradict this singleton-fibre conclusion. Thus no such original model can have v as a component vertex. Finally, the standard model $\mathbf{P}^1_{K^\circ}$ has its two polynomial affine charts, flat by the coefficientwise ideal criterion [StacksAlg, Tag 00HD] and smooth with their displayed coordinates. It is projective and its special fibre is the geometrically integral projective line. It belongs to the stated model category. □

4.56 Semistable models after finite unramified extension

Let K be a complete nontrivially real-valued field with its specified absolute value, and put $O = K^\circ$ and $k = \widetilde{K}$. Finite unramified extensions mean the lifts of finite separable residue extensions constructed in Lemma 4.15.2. Their absolute values extend the original one. We use the projective flat finitely presented nodal models of Definition 4.45.3 and the unramified marking data of Definition 4.47.1.

Proposition 4.56.1 (Effective descent of an equivariant unramified model)**.** *Let L/K be the finite unramified Galois lift of a finite Galois residue extension, put $O' = L^\circ$, and write $G = \mathrm{Gal}(L/K)$. Let C/K be smooth, projective and geometrically connected, and let D be an unramified marking datum on C.*

Suppose a semistable algebraic model $\mathcal{Y}/O'$ of C_L, marked by $D_{O'}$, has a semilinear G-action whose generic action is the canonical coefficient action on C_L and whose action on the marking is the canonical action on $D_{O'}$. Then there is a marked semistable algebraic model $\mathcal{X}/O$ of (C, D) and a compatible G-equivariant isomorphism

$$\mathcal{X} \times_O O' \simeq \mathcal{Y}.$$

The projection $\mathcal{Y} \to \mathcal{X}$ is finite étale *of constant rank $[L : K]$. Descent is unique up to unique marked generic-identity isomorphism. Marked generic-identity morphisms between two such equivariant models descend uniquely if and only if they are G-equivariant.*

Proof. Lemma 4.43.1 gives an actual finite free étale and faithfully flat extension O'/O, of rank $d = [L : K]$, and the isomorphism

$$O' \otimes_O O' \simeq \prod_{g \in G} O', \qquad a \otimes b \longmapsto (a\, g(b))_g.$$

Under this decomposition, the semilinear isomorphisms of $\mathcal{Y}$ identify its two pullbacks to $\mathrm{Spec}(O' \otimes_O O')$. On the triple tensor product the cocycle condition is the group law. They therefore constitute a descent datum on the actual scheme $\mathcal{Y}$. Its structure morphism over O' is projective, so [StacksAlg, Tag 0CCJ] makes this datum effective. We obtain an O-scheme $\mathcal{X}$ and the displayed isomorphism, respecting the datum.

Fully faithful descent for schemes, [StacksAlg, Tag 02W0], descends the specified generic isomorphism with C_L and its inverse to an isomorphism $\mathcal{X}_K \simeq C$. It also descends the equivariant map $D_{O'} \to \mathcal{Y}$ to a map $D \to \mathcal{X}$. The latter is a closed immersion by [StacksAlg, Tag 02L6]. The morphism $\mathcal{X} \to \operatorname{Spec} O$ is of finite presentation and proper by [StacksAlg, Tags 02L0, 02L1]. It is flat and at-worst-nodal of relative dimension one by [StacksAlg, Tag 0C5E]. These descent assertions apply over the valuation ring itself, without a Noetherian hypothesis.

The special fibre is geometrically connected and reduced. Indeed, for any extension k'/k choose a common field E receiving k' and $l = \widetilde{L}$. Such a field is obtained from a maximal proper ideal of the nonzero finite-dimensional k'-algebra $k' \otimes_k l$; an ideal of maximal possible proper vector-space dimension is maximal. The base change $\mathcal{X}_{k'} \times_{k'} E$ is the base change of $\mathcal{Y}_l$ to E, and is connected and reduced. Projection onto $\mathcal{X}_{k'}$ is surjective: at any point its fibre algebra is the nonzero finite-dimensional tensor of its residue field with E, which has a field quotient by the same argument. Thus $\mathcal{X}_{k'}$ is connected. Its affine coordinate rings inject into their scalar extensions to E, so their nilpotents are zero. This proves both geometric assertions for every k'.

The marked points belong to the relative smooth locus. At a point z of D, choose a point z' of $D_{O'}$ above it. Differentials commute with this base change, as is seen from their universal property for derivations. Consequently

$$(\Omega_{\mathcal{X}/O} \otimes \kappa(z)) \otimes_{\kappa(z)} \kappa(z') \simeq \Omega_{\mathcal{Y}/O'} \otimes \kappa(z').$$

The vector space on the right has dimension one, because $\mathcal{Y}$ is smooth of relative dimension one along its marking. The vector space before scalar extension therefore also has dimension one. The fibre of $\mathcal{X}/O$ has local dimension one at z; its marking is finite over the base, so z is closed in that fibre. Flatness and the differential criterion [StacksAlg, Tag 01V9(4)] show that $\mathcal{X}/O$ is smooth at z.

It remains to recover projectivity. The projection $q : \mathcal{Y} \to \mathcal{X}$ is the base change of $\operatorname{Spec} O' \to \operatorname{Spec} O$. It is finite free of rank d on every affine open, surjective, and étale by [StacksAlg, Tag 02GO]. Because O' is affine and $\mathcal{Y}/O'$ is projective, $\mathcal{Y}$ has an ample invertible sheaf by [StacksAlg, Tag 0B45]. The finite locally free case of [StacksAlg, Tag 0BD4(1)] gives an ample invertible sheaf on $\mathcal{X}$. Together with properness over the affine base, [StacksAlg, Tag 0B45] now makes $\mathcal{X}/O$ projective. This norm argument uses the finite locally free rank d; it involves no division by d.

All the requirements of the stated model category are now satisfied. Finally [StacksAlg, Tag 02W0] gives uniqueness of descent and full faithfulness on the indicated morphisms. The generic and marking compatibilities can be tested after the same faithful base change, so descend with the morphisms. Two generic-identity morphisms between the descended models are equal: their equalizer is closed by separatedness, and contains the schematically dense generic fibre of the flat source. On an affine open, this density is the injection of its torsion-free coordinate ring into its generic localization. □

Theorem 4.56.2 (Unramified invariance of semistable-model existence)**.** *Let C/K be smooth, projective and geometrically connected, let D be an unramified marking datum, and let L/K be any finite unramified extension. Then (C, D) has a marked semistable algebraic model over the original O if and only if (C_L, D_{L°) has one over L°.*

In the Galois case, given any unmarked semistable algebraic model $\mathcal{Y}/L^\circ$, one may choose the descended model so that its base change to L° dominates every Galois conjugate of $\mathcal{Y}$ and carries the prescribed marking. No bound on the genus or assumption on the residue characteristic or discreteness of the valuation is required.

Proof. Base change of a projective model is projective by base change of its closed projective embedding. Finite presentation is preserved by the same finite algebra presentations, and nodality and flatness are preserved by [StacksAlg, Tag 0C5B]. The special fibre is the corresponding residue-field extension, as in Lemma 4.28.1; geometric connectedness and reducedness are preserved by their definitions. Closed immersions and finite étale markings are preserved by base change, the latter by [StacksAlg, Tag 02GO]; smoothness along them follows by base change of the standard smooth presentations in [StacksAlg, Tag 01V9]. This proves the forward implication.

First assume that L/K is Galois, with group G. Let $\mathcal{Y}/L^\circ$ be a semistable algebraic model, initially without a marking. In $(C_L)^{\mathrm{an}}$ put

$$S = \bigcup_{g \in G} gV(\mathcal{Y}).$$

This is a finite nonempty invariant set of component vertices of the conjugate models. Each has an unramified branch certificate by Theorem 4.48.2, applied over L, where the model $\mathcal{Y}$ already supplies the requisite existence hypothesis. Theorem 4.49.5 gives the least model $\mathcal{M}_{D_{O'}}(S)$ marked by $D_{O'}$,

where $O' = L^\circ$. Its vertex set contains each $gV(\mathcal{Y})$. Theorem 4.46.2 therefore gives actual generic-identity morphisms from it to all conjugate models.

The canonical semilinear action of G on $(C_L, D_{O'})$ preserves S. The equivariance assertion of the same least-hull theorem extends it uniquely to the actual algebraic model $\mathcal{M}_{D_{O'}}(S)$, with the group law and the specified action on the marking. Proposition 4.56.1 descends this model to the original O. This proves both the reverse implication and the domination assertion in the Galois case.

For a general finite unramified L/K, write $l = \widetilde{L}$. Choose a finite Galois extension l'/k containing l. Concretely, take a primitive element of the finite separable extension l/k by [StacksAlg, Tag 030N], form the splitting field of its separable minimal polynomial by [StacksAlg, Tag 09HU], and use [StacksAlg, Tag 0EXM] for its Galois property. The field is finite because it is generated by finitely many algebraic roots. Lemma 4.18.1 supplies its finite unramified Galois lift L'/K.

There is an actual isometric K-embedding $L \hookrightarrow L'$. Indeed, the construction of Lemma 4.15.2 presents $L = K(\alpha)$ by a monic minimal polynomial $P \in O[T]$ with separable irreducible reduction, and identifies $\bar{\alpha} \in l$ with its residue generator. Its image in l' is a simple root of $\bar{P}$. The simple-root construction in Lemma 4.15.1, applied inside the complete field L', lifts it to a root $\alpha' \in (L')^\circ$ of P. Sending α to α' embeds L in L'; uniqueness of the extending absolute value in Lemma 1.12.1 makes this embedding isometric. It consequently embeds the actual valuation rings.

Base change the given model over L° to $(L')^\circ$, using the properties verified in the first paragraph. Apply the Galois case to L'/K. The resulting marked model is over the original valuation ring with the original generic curve and marking. All absolute values throughout are the specified extending values; none has been rescaled. □

4.57 Geometric components obstructing original-field models

Let K be complete and nontrivially real-valued, with its fixed absolute value, and put $O = K^\circ$ and $k = \widetilde{K}$. Semistable algebraic models have the meaning of Definition 4.45.3. Choose the isometric algebraically closed comparison field Ω of Lemma 4.45.1; write $\overline{k} = \widetilde{\Omega}$. For a smooth projective geometrically

connected curve C/K, let $p : C_{\Omega}^{\mathrm{an}} \to C^{\mathrm{an}}$ be projection. A function field over $\overline{k}$ is called rational here if it is isomorphic over $\overline{k}$ to $\overline{k}(t)$.

Proposition 4.57.1 (A nonrational component cannot disappear)**.** *Let C/K be a smooth projective geometrically connected curve. Suppose $\mathcal{Y}/\Omega^{\circ}$ is a semistable algebraic model of C_{Ω}, and let w be the component vertex of a special component E whose function field $\overline{k}(E)$ is not rational. Then every semistable algebraic model $\mathcal{X}/O$ of C satisfies*

$$w \in V(\mathcal{X}_{\Omega^{\circ}}), \qquad p(w) \in V(\mathcal{X}).$$

Consequently, if C has such an original-field model, the point $v = p(w)$ must satisfy

$$|\mathcal{H}(v)^{\times}| = |K^{\times}|,$$
$$\widetilde{\mathcal{H}(v)}/k \text{ is separably generated of transcendence degree one.}$$

Failure of either condition at v rules out every original-field semistable algebraic model, including every such marked model.

Proof. Assume $\mathcal{X}$ exists. Its coefficient extension $\mathcal{X}_{\Omega^{\circ}}$ is a semistable algebraic model of the same curve as $\mathcal{Y}$, by Lemma 4.28.1 and preservation of the nodal relative-curve condition under base change, [StacksAlg, Tag 0C5B]. Apply Theorem 4.47.5 over the complete field Ω, with empty marking. It supplies an actual semistable model $\mathcal{T}/\Omega^{\circ}$ and generic-identity morphisms

$$\mathcal{T} \longrightarrow \mathcal{Y}, \qquad \mathcal{T} \longrightarrow \mathcal{X}_{\Omega^{\circ}}.$$

The first morphism and Proposition 4.45.6 show that $w \in V(\mathcal{T})$. Let F be its special component in $\mathcal{T}$. The actual residue-field identifications of Proposition 4.45.2, applied over Ω, give

$$\overline{k}(F) = \widetilde{\mathcal{H}(w)} = \overline{k}(E).$$

If w were absent from $V(\mathcal{X}_{\Omega^{\circ}})$, the necessity of the geometric forest condition for the second morphism would make F a smooth $\mathbf{P}^1_{\overline{k}}$. Its function field would be rational, contrary to the displayed equality. Hence w is a vertex of the coefficient extension of $\mathcal{X}$. The geometric vertex identification in Proposition 4.45.2 projects it to the original vertex $v = p(w)$ and gives the two asserted arithmetic conditions. Forgetting a marking retains a semistable algebraic model, so the same obstruction applies to every prescribed marking. □

Theorem 4.57.2 (A potential smooth model with an original-field obstruction)**.** *Assume* $\operatorname{char} k \notin \{2,3\}$*, and choose* $a \in K$ *with* $0 < |a| < 1$*. Let* C_a *be the projective curve*

$$y^2 z = x^3 + a z^3.$$

It is smooth and geometrically connected. If

$$|a|^{1/2} \notin |K^\times| \quad \text{or} \quad |a|^{1/3} \notin |K^\times|,$$

then C_a *has no semistable algebraic model over the original valuation ring* O*.*

Nevertheless, if $b \in \Omega$ *satisfies* $b^6 = a$ *and* $L = K(b)$*, the curve has the smooth projective model*

$$\mathcal{E}/L^\circ : \quad Y^2 Z = X^3 + Z^3, \qquad [x : y : z] = [b^2 X : b^3 Y : Z].$$

The unique component vertex of $\mathcal{E}_{\Omega^\circ}$ *is fixed by every isometric* K*-automorphism of* Ω*. In particular, for* $K = \mathbf{Q}_p$*,* $p \geq 5$*, with the specified absolute value* $|p| = p^{-1}$*, the choice* $a = p$ *gives the stated nonexistence over* $\mathbf{Z}_p$*.*

Proof. We first verify the actual model and its special curve. The integers 2 and 3 are units in O, and hence in L°. The two affine charts $Z \neq 0$ and $Y \neq 0$ cover $\mathcal{E}$: the equation rules out a projective point with $Y = Z = 0$. Their coordinate rings are

$$L^\circ[X, Y]/(Y^2 - X^3 - 1),$$
$$L^\circ[u, v]/(v - u^3 - v^3).$$

Each is free over the polynomial algebra in the first variable, since its equation is monic, up to a unit, in the second. They are therefore flat over L°. They have finite algebra presentations; the projective closed embedding makes the resulting morphism separated and quasi-compact, so it is finitely presented.

On the first chart, $Y^2 - X^3 = 1$ makes $D(X)$ and $D(Y)$ a cover; the corresponding partial derivatives $-3X^2$ and $2Y$ are units. On the second chart, a point with $v \neq 0$ belongs to the first chart, whereas a point with $v = 0$ has $1 - 3v^2$ invertible. These are actual standard smooth presentations by [StacksAlg, Tag 01V9(6)]. Thus $\mathcal{E}/L^\circ$ is smooth of relative dimension one.

Over any extension field of $\widetilde{L}$, the polynomial $X^3 + 1$ is not a square in the rational function field in X: its order at infinity is -3, whereas the order of a square is even. Therefore $Y^2 - X^3 - 1$ is irreducible. Its quotient

ring injects into the corresponding quadratic field over that rational function field, by its monic degree-two basis, so the affine chart is integral. The only point of its projective complement is $[0 : 1 : 0]$. Every fibre is smooth of relative dimension one; in particular this point is on a one-dimensional component and cannot be an additional component by itself. The affine chart is consequently dense, and the fibre is connected. It is geometrically reduced and regular by [StacksAlg, Tag 056T]. This verifies the exact semistable-model conditions, including geometric connectedness, over L°.

The same calculation with $a \neq 0$ over any extension of K proves that C_a is smooth and geometrically connected: the affine chart has equation $y^2 = x^3 + a$, with partial derivatives $2y$ and $-3x^2$, which cannot both vanish on it; the unique point at infinity has a unit partial derivative in the chart $y \neq 0$. Its affine polynomial is irreducible by the same odd-order calculation. The root b exists in the chosen algebraically closed Ω. The extension L/K has its actual degree $[K(b) : K] \leq 6$ and is complete with the extending absolute value by Lemma 1.12.1. The displayed coordinate change identifies the generic fibre of $\mathcal{E}$ with $C_{a,L}$, and $|b| = |a|^{1/6}$ for this unchanged extending value.

Write E for the special curve after extension to $\overline{k}$. Its function field is not rational. Here is a direct verification. On the affine chart the differential

$$\omega = \frac{dX}{2Y}$$

extends across a zero of Y as $dY/(3X^2)$, since $2Y\,dY = 3X^2\,dX$ and $X \neq 0$ there. Near infinity put $u = X/Y$ and $v = Z/Y$, so $v = u^3 + v^3$. The affine coordinates are u/v and $1/v$; differentiating the equation gives

$$(1 - 3v^2)\,dv = 3u^2\,du, \qquad \omega = \frac{v\,du - u\,dv}{2v} = -\frac{du}{1 - 3v^2}.$$

This is regular at infinity. The differential is nonzero: on the affine function field the derivation $\delta(X) = 2Y$, $\delta(Y) = 3X^2$ respects the equation and evaluates ω to 1.

If $\overline{k}(E)$ were $\overline{k}(t)$, [StacksAlg, Tag 0BY1], applied to the smooth projective curve E and $\mathbf{P}^1_{\overline{k}}$, would identify these curves. But a regular differential on $\mathbf{P}^1$ has the form $f(t)\,dt$ on its affine line, with $f \in \overline{k}[t]$; at $s = t^{-1}$ it is $-f(s^{-1})s^{-2}\,ds$, which is regular at $s = 0$ only when $f = 0$. The nonzero ω is a contradiction.

Complete $\mathcal{E}_{\Omega^\circ}$ along any fixed $0 \neq \pi \in K^{\circ\circ}$. Theorem 4.13.2 identifies its actual analytic generic fibre with $C^{\mathrm{an}}_{a,\Omega}$. Proposition 4.12.2 supplies the

unique component vertex w over the generic point of E. Put $v = p(w)$. The reduction identification in Proposition 4.24.2 shows that the affine functions X and Y have norm one at w, since both have nonzero residues at the generic point of E. Hence the original rational functions x/z and y/z satisfy

$$|(x/z)(v)| = |a|^{1/3}, \qquad |(y/z)(v)| = |a|^{1/2}.$$

They are nonzero elements of $\mathcal{H}(v)$, so either failure in the hypothesis makes its value group strictly larger than $|K^{\times}|$. Proposition 4.57.1 now rules out every original-field semistable model.

Finally, let σ be an isometric K-automorphism of Ω. Then $\sigma(b) = \zeta b$ with $\zeta^{6} = 1$, and $|\zeta| = 1$. The canonical semilinear action on $C_{a,\Omega}$ transforms the displayed integral coordinates by

$$X \longmapsto \zeta^{-2} X, \qquad Y \longmapsto \zeta^{-3} Y, \qquad Z \longmapsto Z.$$

These substitutions and their inverses preserve the equation over Ω°, so give an actual semilinear automorphism of the smooth model. Its unique special component, and thus its component vertex w, is preserved. For $K = \mathbf{Q}_p$ and $a = p$, the two displayed values are $p^{-1/3}$ and $p^{-1/2}$, neither of which lies in $|\mathbf{Q}_p^{\times}| = p^{\mathbf{Z}}$. Throughout, the absolute value was extended and never rescaled. □

Chapter 5

Tropicalization, multiplicities and balancing

5.1 Tropicalization of subvarieties of a torus

Throughout these foundations, K is a complete non-Archimedean field with a nontrivial absolute value. We write

$$v(a) = -\log|a|, \qquad v(0) = +\infty, \qquad \Gamma = v(K^{\times}) \subset \mathbb{R}.$$

Algebraic closure will be stated when needed; it is a standing hypothesis of GK17, not a consequence of completeness. The real polyhedral constructions below do not themselves involve a field. We retain the series normalization $\mathrm{d}'\mathrm{d}'' = \mathrm{d}'\mathrm{d}''$, without a factor 2π and without squaring a norm.

5.1.1 Cancellation and repeated minima

Suppose that $a_1 + \cdots + a_s = 0$, with each $a_i \neq 0$, in a non-Archimedean valued field. The smallest of the numbers $v(a_i)$ must occur at least twice. Indeed, if a_j had strictly smaller valuation than all the others, then

$$|a_j| > \max_{i \neq j} |a_i| \quad \text{but} \quad |a_j| = \Big|\sum_{i \neq j} a_i\Big| \leq \max_{i \neq j} |a_i|,$$

a contradiction. This observation also applies to evaluation at a Berkovich point: all the evaluated terms lie in its completed residue field $\mathscr{H}(x)$.

Let $T = \mathbb{G}_m^r = \operatorname{Spec} K[z_1^{\pm 1}, \dots, z_r^{\pm 1}]$. For a nonzero Laurent polynomial

$$f = \sum_{u \in \mathbb{Z}^r} a_u z^u, \qquad z^u = z_1^{u_1} \cdots z_r^{u_r},$$

where only finitely many a_u are nonzero, define

$$F_f(w) = \min_{a_u \neq 0} \big(v(a_u) + u_1 w_1 + \cdots + u_r w_r \big). \tag{5.1.1}$$

This is a continuous, concave, piecewise affine function on $\mathbb{R}^r$. Concavity follows because it is the minimum of affine functions. Its *tropical hypersurface* is the set where that minimum is attained for at least two distinct exponents. The word hypersurface denotes this set with suitable multiplicities, not the graph of F_f.

At an analytic point $x \in T^{\mathrm{an}}$, put

$$\operatorname{trop}(x) = \big(-\log |z_1(x)|, \dots, -\log |z_r(x)| \big).$$

If $f(x) = 0$, the preceding argument proves that $\operatorname{trop}(x)$ belongs to the tropical hypersurface of f. For arbitrary x, the ultrametric inequality gives only

$$-\log |f(x)| \;\geq\; F_f(\operatorname{trop}(x)). \tag{5.1.2}$$

Equality can fail because terms of equal size can cancel. We will obtain equality at the particular Gauss points of Section 6.1. It must not be used at every analytic point.

5.1.2 The tropical line

Example 5.1.1 (The MIN tropical line)**.** For $f = 1 + z_1 + z_2$, all three coefficients have valuation zero, so

$$F_f(w_1, w_2) = \min(0, w_1, w_2).$$

Two terms can be minimal in exactly the following three ways:

minimal terms	conditions	ray
$0,\ w_1$	$w_1 = 0,\ w_2 \geq 0$	$\mathbb{R}_{\geq 0}(0, 1)$
$0,\ w_2$	$w_2 = 0,\ w_1 \geq 0$	$\mathbb{R}_{\geq 0}(1, 0)$
$w_1,\ w_2$	$w_1 = w_2 \leq 0$	$\mathbb{R}_{\geq 0}(-1, -1)$.

Their union will be denoted by L. Each ray has weight one; the meaning of this weight is explained in the next section.

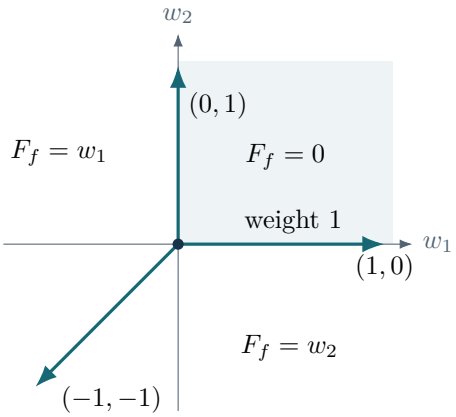


Figure 5.1: The tropical line for the MIN convention. The open regions record the unique smallest term; the thick rays record ties. Replacing MIN by MAX would reverse the picture.

This calculation is worth doing before memorizing a familiar tropical-line diagram. With the convention $v = -\log|\cdot|$, the rays point right, up, and southwest. On the first quadrant the constant term is smallest. On the left-hand region the z_1-term is smallest, and on the lower region the z_2-term is smallest.

Definition 5.1.2 (Tropicalization of a closed subscheme)**.** For a closed subscheme $X \subset T$, its set-theoretic tropicalization is

$$\mathrm{Trop}(X) = \mathrm{trop}(X^{\mathrm{an}}) \subset \mathbb{R}^r.$$

The scheme structure will enter through the weights, even though this underlying set does not detect nilpotents.

Theorem 5.1.3 (Polyhedrality and the hypersurface theorem)**.** *For a closed subscheme $X \subset T$, the set $\mathrm{Trop}(X)$ is a finite union of integral Γ-affine polyhedra. If X is pure of dimension d, this union can be made into a pure d-dimensional polyhedral complex. If $X = V(f)$ for a nonzero Laurent polynomial, then $\mathrm{Trop}(X)$ is exactly the locus of repeated minima of F_f.*

Here the definition uses all Berkovich points. It requires no assertion that a prescribed real valuation vector is realized by a K-rational point. Over an algebraically closed nontrivially valued extension K', one can instead take the

closure of the coordinatewise valuations of $X(K')$. The converse implication in the hypersurface assertion is a theorem about lifting tropical data, not a consequence of the ultrametric inequality alone.

References. [Gub13, Theorem 3.3, Proposition 3.8, Theorem 5.6 and Example 5.8]. The guide treats arbitrary valued ground fields and performs analytification over their completions. The elementary argument above proves the necessary cancellation condition.

For an ideal with several generators, intersecting the tropical hypersurfaces of those generators gives a *tropical prevariety*. It may be larger than $\operatorname{Trop}(X)$. The fundamental theorem uses the whole ideal, or a specially chosen finite tropical basis. A generating set for the ordinary ideal need not be a tropical basis.

Example 5.1.4 (Tropical prevarieties and translation)**.** Assume $|2| = 1$. The polynomials $f_1 = 1 + z_1 + z_2$ and $f_2 = 2 + z_1 + z_2$ have the same tropical polynomial $\min(0, w_1, w_2)$. Their tropical hypersurfaces therefore agree. Nevertheless, $f_2 - f_1 = 1$, so (f_1, f_2) is the unit ideal and $V(f_1, f_2)$ is empty.

More generally, for $abc \neq 0$, the tropical hypersurface of $a + bz_1 + cz_2$ is a translate of the standard tropical line. Subtracting $v(a)$ from the three term valuations gives

$$\min\big(0, w_1 + v(b) - v(a), w_2 + v(c) - v(a)\big).$$

Its vertex is $(v(a) - v(b), v(a) - v(c))$. Translation changes the vertex but preserves the primitive directions and the weights, which are one.

5.2 Lattices, polyhedra, multiplicities, and balancing

5.2.1 Character lattices and integral affine structures

The expression z^u is intrinsic to the torus even though its coordinate notation is not. Let

$$M = \operatorname{Hom}(T, \mathbb{G}_m) \simeq \mathbb{Z}^r, \qquad N = \operatorname{Hom}(M, \mathbb{Z}),$$
$$M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}, \qquad N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}.$$

Thus M is the character lattice and N its dual, the cocharacter lattice. Write χ^m for the character associated to $m \in M$. Tropicalization is characterized without coordinates by

$$\langle m, \operatorname{trop}(x) \rangle = -\log |\chi^m(x)|.$$

A torus homomorphism induces an integral linear map on cocharacter lattices and hence a real linear map on tropicalizations. Translation by a torus point adds a constant valuation vector. These are the origins of the integral affine changes of coordinates used throughout GK17.

Definition 5.2.1 (Integral affine data)**.** An integral Γ-affine function on $N_{\mathbb{R}}$ is a function $\langle m, w \rangle + c$, with $m \in M$ and $c \in \Gamma$. An integral Γ-affine polyhedron is an intersection of finitely many half-spaces

$$\{w : \langle m_i, w \rangle \geq c_i\}, \qquad m_i \in M,\ c_i \in \Gamma.$$

Allowing $c_i \in \mathbb{R}$ gives integral $\mathbb{R}$-affine polyhedra. A polytope is a bounded polyhedron. A face is the polyhedron itself or its intersection with a supporting hyperplane. The relative interior means interior in the smallest affine space containing the polyhedron.

The adjective integral controls slopes, not the coordinates of every vertex. For example, $2t \geq \lambda$ is an integral Γ-affine inequality when $\lambda \in \Gamma$; its endpoint $\lambda/2$ need not lie in Γ. In general, vertices may lie in the divisible hull of Γ. This distinction is relevant to strict analytic domains and to model metrics over a fixed ground field.

A finite polyhedral complex $\mathscr{C}$ contains every face of each of its cells, and the intersection of two cells is empty or a face of both. Its support is their union. It is pure d-dimensional if every maximal cell has dimension d. A subdivision cuts cells into smaller cells without changing the support. The underlying tropical cycle will not depend on such cuts.

For a cell σ, let L_σ be the vector space parallel to its affine span and put

$$N_\sigma = N \cap L_\sigma.$$

For a facet τ of σ, the quotient N_σ / N_τ is a free rank-one lattice. Its generator pointing from τ into σ is a *primitive normal class*. Choose a lift $n_{\sigma,\tau} \in N_\sigma$. Another lift differs by an element of N_τ.

Primitive refers to the lattice: a vector cannot be a positive integer multiple of another lattice vector on the same ray. It does not mean Euclidean

length one. For instance, $(1,2)$ is primitive, and the natural integral coordinate on its line is $s \mapsto s(1,2)$; its Euclidean length $\sqrt{5}$ plays no role in a tropical multiplicity.

References. [GK17, Appendix A.1–A.6 and Definition 1.1]; [Gub13, Appendix A.1–A.4].

5.2.2 Tropical multiplicities

The reason for attaching weights is already visible in one variable. The scheme $z^d = a$ has length d, while its tropicalization is the single point $v(a)/d$. That point must carry weight d to retain the algebraic degree. This remains true if the scheme is nonreduced, for example when the characteristic divides d.

In higher dimensions, the relevant algebra is an *initial degeneration*. Here is a construction that does not assume a chosen section of the valuation. For $w \in N_{\mathbb{R}}$, pass to a valued extension K'/K and choose $t \in T(K')$ with $\operatorname{trop}(t) = w$. Translate $X_{K'}$ by t^{-1}, take its schematic closure in the split torus over $(K')^\circ$, and take the special fibre. The result is denoted $\operatorname{in}_w(X)$. Different choices give equivalent schemes after residue-field extension and torus translation.

For a Laurent polynomial, this operation retains the terms attaining the minimum in (5.1.1), after rescaling them to have valuation zero, and reduces their coefficients. For an ideal, one takes the initial forms of *all* its elements. The fundamental theorem says

$$w \in \operatorname{Trop}(X) \quad \Longleftrightarrow \quad \operatorname{in}_w(X) \neq \varnothing.$$

In the Laurent coordinate ring, a monomial is a unit. Thus an initial ideal containing a monomial defines the empty scheme.

At a regular point of $\operatorname{Trop}(X)$, lying in the relative interior of a maximal cell, its tropical multiplicity is the sum of the multiplicities of the geometric irreducible components of $\operatorname{in}_w(X)$. An irreducible-component multiplicity means the length of the local ring at its generic point. Passing to an algebraically closed residue-field extension ensures that geometric components are counted. This integer is locally constant on the regular locus and therefore gives a weight m_σ to each maximal cell.

Example 5.2.2 (A binomial weight)**.** Let $q = dq_0 \in M$, with q_0 primitive and $d > 0$, and consider

$$X = V(\chi^q - a), \qquad a \in K^\times.$$

Its tropicalization is the hyperplane $\langle q, w\rangle = v(a)$, with weight d. Extend q_0 to a basis of M; in the resulting coordinates the equation is $z_1^d = a$. Its initial degeneration is a product of a zero-dimensional scheme of length d with a torus of dimension $r - 1$. Thus the weight is a scheme-theoretic multiplicity, including repeated roots, rather than a count of distinct points.

References. [Gub13, Definitions 5.1 and 13.4, Remark 5.7, Theorems 5.6 and 13.8].

5.2.3 Computing an initial ideal: why all relations matter

The phrase "keep the lowest-weight terms" first defines $\mathrm{in}_w(f)$ for a single polynomial. For $X = V(I)$, the geometric object is instead determined by

$$\mathrm{in}_w(I) = (\mathrm{in}_w(f) : f \in I),$$

with the coefficient normalization made over a suitable valued extension as above. A generating set of I need not be a tropical basis or a Gröbner basis at w.

Here is a calculation at $w = (0, 0)$. Assume the residue characteristic is not 2, choose $0 < |\pi| < 1$, and work in $K[x^{\pm 1}, y^{\pm 1}]$. Let

$$f_1 = x + y + 1, \qquad f_2 = x + y + 1 + \pi(y - 1), \qquad I = (f_1, f_2).$$

The initial forms of these two displayed generators both equal $x+y+1$. They define a line in the residue torus. But

$$\frac{f_2 - f_1}{\pi} = y - 1 \in I, \qquad f_1 - (y - 1) = x + 2 \in I.$$

Thus $I = (y - 1, x + 2)$, and its initial degeneration is the single residue-torus point $(-2, 1)$. Taking initials of only the two original generators loses a relation and even gives the wrong dimension.

The construction by translation and closure is useful precisely because it is intrinsic to the ideal. A coefficientwise computation is reliable when one knows a suitable basis, or proves directly that the proposed initial generators generate the entire initial ideal.

5.2.4 Generic weights and cell multiplicities

To attach the weight m_σ to a maximal cell σ, choose w in its relative interior in a subdivision on which the initial-degeneration type is controlled. This is a regular point of the tropical support: locally that support is a piece of an affine space of the expected dimension. Moving w inside that regular region does not change its multiplicity.

For balancing at a codimension-one face τ, there are several neighbouring maximal cells. Each carries the weight computed on its own relative interior. A point of τ is useful for studying the entire local star, but it is not substituted for all these interior points in the definition of their weights. The multiplicity function need not be described as one locally constant function across every singular face of the support.

5.2.5 The algebraic origin of balancing

For the tropical line, three primitive vectors sum to zero. For a hypersurface, the dual Newton subdivision often makes the balancing identity an elementary polygon calculation. Neither argument proves the theorem for an arbitrary higher-codimension ideal.

The general proof relates the star near a point to the tropicalization of an initial degeneration. After this reduction, the balancing relation is an intersection-theoretic degree identity. Roughly, a character defines a principal divisor on an appropriate toric compactification; its intersection degree with the relevant proper cycle is zero. The contributions along the toric boundary are the multiplicities times the primitive normal components. Their sum is therefore zero. Passing back through the initial degeneration recovers the balancing equation for the original tropicalization.

This is a proof mechanism, not a replacement for the compactification and multiplicity arguments in [Gub13, §13, Theorem 13.11]. It explains the content of the theorem: the algebraic multiplicities satisfy a compatibility that arbitrary weights on the same support would not satisfy. Once balancing is known, Stokes translates it into closedness of the integration current.

5.2.6 The balancing condition

Definition 5.2.3 (A balanced tropical cycle)**.** Give each maximal cell σ of a pure d-dimensional integral $\mathbb{R}$-affine complex an integer weight m_σ. The complex is balanced if, for every $(d-1)$-dimensional face τ,

$$\sum_{\sigma \supset \tau} m_\sigma n_{\sigma,\tau} \ \in \ L_\tau. \tag{5.2.1}$$

Equivalently, the sum is zero in $N_{\mathbb{R}}/L_\tau$. With integer weights it lies in N_τ. A tropical cycle identifies balanced weighted complexes that agree after subdivision; cells of zero weight do not contribute to its support.

For a curve, τ is a vertex and $L_\tau = 0$. Balancing says that the weighted sum of primitive outgoing direction vectors is zero. The tropical line satisfies

$$(1,0) + (0,1) + (-1,-1) = 0.$$

An arbitrary graph drawn with straight edges need not satisfy this equation. Conversely, a balanced weighted graph is a valid abstract tropical cycle even when no algebraic realization has been specified.

Theorem 5.2.4 (Balancing of algebraic tropicalizations)**.** *If X is a pure-dimensional closed subscheme of a split torus, its tropical multiplicities make* $\mathrm{Trop}(X)$ *a balanced tropical cycle of dimension* $\dim X$*. The resulting cycle is independent of the chosen polyhedral subdivision.*

Open proof obligation (RC2). The sole supplied proof source for the stated algebraic tropical balancing/subdivision claim is GubGuide Theorem 13.11 and Remark 13.12.

Later, integration over a constant-weight cycle will define a current δ_C. Balancing is precisely the cancellation of facet terms that makes this current d'- and d''-closed. This explains why the combinatorial condition belongs in a differential theory. GK17 also allows *smoothly varying* real weights, with a pointwise balancing equation. In that setting derivatives of the weights remain inside the cells: balancing alone does not imply that δ_C is closed [GK17, Proposition 2.16]. We will retain constant weights until that distinction has been explained.

Example 5.2.5 (Primitive directions and lattice index)**.** For the rays with primitive directions $(1,0),(0,1),(-2,-3)$, balancing is

$$(m_1 - 2m_3, m_2 - 3m_3) = (0,0).$$

The smallest positive integral weight vector is $(2,3,1)$. Replacing the primitive directions by Euclidean unit vectors changes their normalization and hence the coefficients in the balancing equation.

The cone generated by $(1,0)$ and $(1,2)$ is simplicial, since the two vectors are linearly independent. Their determinant is two, so the lattice they generate has index two in $\mathbb{Z}^2$. Simpliciality does not imply unimodularity.

Example 5.2.6 (A binomial of multiplicity two)**.** Let $a \in K^\times$. The tropical support of $z_1^6 z_2^4 - a$ is

$$6w_1 + 4w_2 = v(a).$$

The exponent difference is $2(3,2)$, with $(3,2)$ primitive. The weight is therefore two. Equivalently, use the primitive character $s = z_1^3 z_2^2$; the transverse equation is $s^2 = a$ and has length two. In residue characteristic two the initial transverse scheme can be nonreduced, but its length remains two. Counting distinct geometric roots would lose this multiplicity.

5.3 Initial degenerations and multiplicity calculations

5.3.1 Initial degenerations over valued extensions

Let $M \simeq \mathbb{Z}^r$, let $T = \operatorname{Spec} K[M]$, and let $Z \subset T$ have ideal I. Fix $w \in N_{\mathbb{R}}$. The valuation of a Laurent term $a_m \chi^m$ at weight w is

$$v(a_m) + \langle m, w \rangle.$$

If w is realized by a torus point p over a suitable valued extension, substitute $\chi^m = p^m \chi^m_{\text{new}}$. For each relation, divide by an element whose valuation is the minimum of its term valuations, then reduce. The terms attaining the minimum survive; the others vanish in the residue field. The ideal generated by all these reduced relations defines the initial degeneration.

This description explains why one sometimes extends the field before writing an initial ideal with ordinary residue coefficients. A weight in $N_{\mathbb{R}}$ need

not lie in the original value group. Graded reduction or the corresponding tilted algebra packages the same information without pretending that every real weight is realized by a point over the original field. The eventual tropical multiplicity must be invariant under the permitted extensions and choices.

5.3.2 A nonempty variety whose chosen equations give a larger prevariety

Consider the ideal in three Laurent variables

$$I = (x + y + 1,\ x + y + z).$$

The variety is nonempty: it is given by $x + y = -1$ and $z = 1$, inside the torus. At the weight $w = (-1, -1, 1)$, the two displayed generators both have their minimum attained by the terms x and y. Thus this weight belongs to the intersection of their two tropical hypersurfaces.

But subtracting the equations gives $z - 1 \in I$. At the same weight, its constant term has value 0 and its z term has value 1. Its initial form is the nonzero constant -1. Therefore the full initial ideal is the unit ideal, and the weight is not in $\mathrm{Trop}(Z)$.

This example isolates the issue without an empty algebraic variety: solving the minimum-tie condition for a chosen generating set does not necessarily impose the minimum-tie condition for every algebraic relation. The intersection of the chosen tropical hypersurfaces is a tropical prevariety; obtaining the actual tropical variety requires a tropical basis or another method controlling the full initial ideal.

5.3.3 A multiplicity calculation that keeps nilpotents

For the one-variable equation $x^m = a$, the tropical support is the point $w = v(a)/m$. After a suitable translation and residue extension, the initial equation is $z^m - \widetilde{b}$ with $\widetilde{b} \neq 0$. Its zero-dimensional length is m. If the residual characteristic divides m, the polynomial can be inseparable and some geometric roots can coincide. The sum of scheme-theoretic multiplicities is still m.

Thus a tropical multiplicity is not generally the number of distinct geometric roots of an initial equation. It counts lengths at generic components. This is why taking a reduced initial degeneration before measuring multiplicity can give the wrong answer.

In higher dimension, at a regular point of a maximal tropical cell, the initial degeneration has a torus symmetry parallel to the cell. Quotienting by that tangent torus leaves a zero-dimensional transverse scheme. Its length is the conceptual source of the cell weight. The algebraic construction makes this description invariant under the choices needed to realize the weight.

For the hypersurface defined scheme-theoretically by $(1 + x + y)^2$, the support of the tropicalization is unchanged but its generic multiplicity is two. This is a useful distinction between a reduced support and a cycle. If one first replaces the subscheme by its reduction, the cycle changes. The multiplicity is not inferred from the picture of the three rays.

5.3.4 Why the weight is constant only in the appropriate region

The initial degeneration can change when w crosses a Gröbner wall. On a sufficiently fine polyhedral subdivision, the relevant initial data are constant in the required sense on relative interiors. The tropical multiplicity is locally constant on the regular locus of a maximal cell. At a lower-dimensional face, several maximal cells meet and there is no single weight to obtain by pretending that the face is a generic point of one chosen cell.

When defining the weight attached to σ, choose w in its relative interior away from the lower-dimensional exceptional locus. When proving balancing at a face τ, choose a point in the relative interior of τ and examine the local star, with the weights already assigned to adjacent maximal cells. These are two different choices of point serving two different purposes.

5.3.5 A proof mechanism for balancing

Near a codimension-one face, quotient the local star by the tangent space of that face. The transverse geometry is one-dimensional. In the corresponding initial-degeneration picture, characters of the quotient torus give rational functions on a compactification of the relevant curves. A principal divisor has degree zero. Its orders at the boundary points are the pairings of the character with the primitive outgoing directions, weighted by the transverse multiplicities. Hence, for every quotient character m,

$$\sum_{\sigma \supset \tau} m_\sigma \langle m, u_{\sigma/\tau} \rangle = 0.$$

Characters separate vectors in the quotient real vector space, so the weighted normal sum vanishes. This explains the algebraic source of balancing. Making the argument general requires the initial-degeneration, compactification, and multiplicity theorems; the vector identity is its conclusion, not an extra combinatorial assumption about an algebraic tropicalization.

Example 5.3.1 (Valuation of a coefficient)**.** For $v(\pi) = \lambda > 0$, the tropical polynomial of $1 + x + \pi y$ is $\min(0, w_1, \lambda + w_2)$. Its repeated-minimum locus is the standard tropical line translated by $(0, -\lambda)$. Its vertex is $(0, -\lambda)$, and the outgoing primitive directions remain $(1, 0), (0, 1), (-1, -1)$, all with weight one. The coefficient valuation affects position, not the tangent lattices.

References. [Gub13, Sections 5–6, 10–13]; [MS15].

5.4 Corners and stable intersections

5.4.1 Corner divisors

On a weighted tropical curve, a continuous piecewise affine function φ has a divisor at its vertices:

$$\operatorname{div}_C(\varphi) = \sum_p \left(\sum_{e \ni p} m_e \, \partial_e \varphi(p) \right) [p]. \tag{5.4.1}$$

Here ∂_e is differentiation in the primitive lattice coordinate pointing away from p, and subdivision inserts vertices at all breakpoints. This is the outgoing-slope convention used throughout the series. On the line,

$$\operatorname{div}_{\mathbb{R}}(\max(0, t)) = [0], \qquad \operatorname{div}_{\mathbb{R}}(\min(0, t)) = -[0].$$

The coefficient is the right derivative minus the left derivative. These are the previously studied atomic-curvature calculations in a tropical divisor notation.

In higher dimension, a normal vector is defined only modulo a tangent vector. A formula must compensate for this ambiguity.

Definition 5.4.1 (The corner divisor)**.** Let C be a constant-weight tropical cycle and let φ be continuous and affine on every cell of a subdivision. Denote the linear part of its restriction to σ by ℓ_σ. The weight of $\mathrm{div}_C(\varphi) = \varphi \cdot C$ on a facet τ is

$$m_\tau(\varphi \cdot C) = \sum_{\sigma \supset \tau} m_\sigma \ell_\sigma(n_{\sigma,\tau}) - \ell_\tau \left(\sum_{\sigma \supset \tau} m_\sigma n_{\sigma,\tau} \right). \tag{5.4.2}$$

Integral slopes give integral divisor weights when the weights of C are integral. Real slopes are also allowed, with real divisor weights.

Balancing puts the vector in the second term in L_τ, where ℓ_τ is defined. Replacing a lift $n_{\sigma,\tau}$ by $n_{\sigma,\tau} + a$, with $a \in N_\tau$, changes the first term by $m_\sigma \ell_\sigma(a)$ and the second by $m_\sigma \ell_\tau(a)$. These agree because φ is continuous along τ. Hence the formula is independent of the lifts. If φ is the restriction of one ambient affine function, the two terms cancel completely.

The resulting divisor is again balanced and is independent of subdivision. This is a substantive compatibility theorem; it makes it possible to iterate corner operations. The same construction in GK17 applies to piecewise smooth functions and smooth weights, using derivatives at each point of the face.

References. [GK17, Definition 1.10, Remark 1.11 and Propositions 1.12–1.14]; Section 10.3.

There is a useful analytic dictionary. For constant-weight C,

$$\mathrm{d}'\mathrm{d}''[\varphi] = \delta_{\varphi \cdot C} \tag{5.4.3}$$

when φ is piecewise affine and the currents are taken on C. For a piecewise smooth function there is also the facewise Hessian:

$$\mathrm{d}'\mathrm{d}''[\varphi] = [\mathrm{d}'_P \mathrm{d}''_P \varphi] + \delta_{\varphi \cdot C}.$$

Thus differentiating each affine piece and obtaining zero does not compute the derivative of the whole current. The missing terms live on the interfaces.

References. [GK17, Corollaries 3.19–3.20].

Example 5.4.2 (The sign of a tropical polynomial)**.** The tropical polynomial F_f in (5.1.1) is concave. Consequently the effective tropical hypersurface is the corner divisor of $-F_f$, not of F_f. Near a general facet where two exponents differ by dq_0, with q_0 primitive, the slope jump of $-F_f$ in a transverse integral coordinate is $d > 0$. This recovers the binomial multiplicity.

In particular, for the tropical line,

$$u(w) = \max(0, -w_1, -w_2) = -F_f(w), \qquad \operatorname{div}_{\mathbb{R}^2}(u) = L, \qquad \mathrm{d}'\mathrm{d}''[u] = \delta_L.$$

This statement is on the real tropical chart. It does not replace the analytic function $-\log|f|$ by $F_f \circ \mathrm{trop}$ at points where cancellation occurs.

Example 5.4.3 (The correction term in the corner formula)**.** Subdivide $\mathbb{R}^2$ along $x = 0$, with both weights one, and use normal lifts $n_+ = (1, a)$ and $n_- = (-1, b)$. For $\varphi = 2x + 3y$, formula (5.4.2) gives

$$(2 + 3a) + (-2 + 3b) - 3(a + b) = 0.$$

For $\psi = \max(0, x) + 3y$, it gives

$$(1 + 3a) + 3b - 3(a + b) = 1.$$

Thus the affine function has no corner, whereas the second function has divisor $[x = 0]$. The correction along the common tangent space is exactly what removes the dependence on the chosen lifts.

Example 5.4.4 (Slopes on the tropical line)**.** Give the three rays outgoing slopes a, b, c, using their primitive parameters. The coefficient of the corner at the central vertex is $a + b + c$. For the restriction of an ambient affine function $\ell(w) = Aw_1 + Bw_2 + C$, these slopes are $B, A, -A - B$, so the sum is zero. For a general piecewise affine function they need not sum to zero. In particular, slopes $0, 0, -1$ give negative curvature at the vertex although the function is affine, hence convex, on every ray.

This calculation is the simplest reason the continuous facewise-convexity criterion in CLD12 requires correction. A condition on each maximal cell does not control the terms on their interfaces. Appendix A gives the exact source locator and the corrected interpretation.

5.4.2 Stable intersection of tropical cycles

Two coincident lines have a one-dimensional set-theoretic intersection, whereas the intersection of two codimension-one cycles in a plane should have dimension zero. Tropical geometry resolves this discrepancy by a displacement procedure.

For cycles $C, D \subset N_{\mathbb{R}}$, translate D by εv, with v generic and $\varepsilon > 0$ small, intersect transverse faces, and take the limiting cycle as $\varepsilon \to 0$. When cells σ, τ meet transversely in complementary dimensions, their contribution has multiplicity

$$m_\sigma m_\tau \, [N : N_\sigma + N_\tau]. \tag{5.4.4}$$

The index counts how far the two integral tangent lattices are from generating the ambient lattice. It is finite under the stated transversality condition. For two primitive lines in $\mathbb{Z}^2$, it is the absolute determinant of their direction vectors.

Theorem 5.4.5 (Stable intersection)**.** *For constant-weight integral $\mathbb{R}$-affine tropical cycles in $N_{\mathbb{R}}$, the displacement construction defines a cycle $C \cdot D$, independent of the sufficiently small generic displacement and of subdivision. It is bilinear, associative and commutative, its codimension is the sum of the codimensions, and its support is contained in $|C| \cap |D|$. The construction extends to real weights and to the smooth weights used by GK17.*

Independence of displacement uses balancing. It is not a general fact about arbitrary weighted polyhedral sets. The combinatorial intersection is defined before any attempt to multiply singular currents.

References. [GK17, Sections 1.3–1.4, especially formula (1.4.1)].

Example 5.4.6 (A transverse intersection of multiplicity two)**.** In $\mathbb{R}^2$, give each of the lines

$$H = \{x = 0\}, \qquad Q = \{x + 2y = 0\}$$

weight one. Their primitive tangent vectors are $(0, 1)$ and $(2, -1)$. They generate the sublattice of vectors with even first coordinate, of index two. Hence

$$H \cdot Q = 2[0].$$

The corresponding algebraic example is

$$z_1 = 1, \qquad z_1 z_2^2 = 1.$$

Its coordinate algebra is $K[z_2^{\pm 1}]/(z_2^2 - 1)$, of length two. In characteristic two the two roots merge into a nonreduced point; the intersection length and tropical index remain two. Both individual tropical lines have weight one because their defining exponent vectors $(1, 0)$ and $(1, 2)$ are primitive.

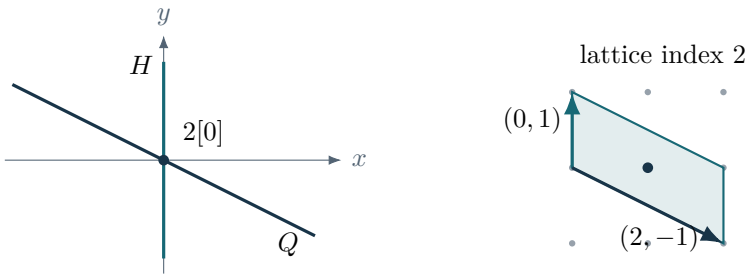


Figure 5.2: Transversality does not force multiplicity one. The two primitive tangent vectors span a fundamental parallelogram of lattice area two; the quotient lattice has two elements.

The same index will appear in products of curvature currents. With $u = \max(0, x)$ and $v = \max(0, x + 2y)$, the corner cycles are H and Q. Their product in the delta-calculus is therefore $2\delta_0$. This formula is justified by that calculus; it is not a definition of the product of two arbitrary distributions.

Example 5.4.7 (Two different self-intersections)**.** A straight line translated in a nonparallel direction becomes disjoint from itself, so its stable self-intersection is zero. The tropical line L behaves differently. Translate its vertex by $(\varepsilon, 2\varepsilon)$. There is one transverse intersection, at $(0, \varepsilon)$, between the original vertical ray and the translated southwest ray. Their primitive directions have determinant of absolute value one. Letting $\varepsilon \to 0$ gives

$$L \cdot L = [0].$$

Although both set-theoretic self-intersections contain the entire original curve, the stable answers differ.

Example 5.4.8 (Determinants and intersection multiplicities)**.** Let $p = (a, b)$ and $q = (c, d)$ be primitive integral covectors with $ad - bc \neq 0$. The weight-one lines $\langle p, w\rangle = 0$ and $\langle q, w\rangle = 0$ have primitive tangent vectors $(-b, a)$ and $(-d, c)$. Their stable intersection is the origin with weight

$$[\mathbb{Z}^2 : \mathbb{Z}(-b, a) + \mathbb{Z}(-d, c)] = |ad - bc|.$$

The torus homomorphism $(z_1, z_2) \mapsto (z_1^a z_2^b, z_1^c z_2^d)$ has the same degree. Indeed, its character-lattice map has cokernel of order $|ad - bc|$, so its finite kernel has that scheme-theoretic length. The formula remains valid in inseparable characteristic, where the kernel need not be reduced.

Example 5.4.9 (A nontrivial lattice index)**.** Let two weighted rational lines in $\mathbb{R}^2$ have primitive directions $u = (1, 0)$ and $v = (1, 2)$, with weights a and b. A generic small translate meets transversely, and the stable intersection multiplicity is

$$ab[\mathbb{Z}^2 : \mathbb{Z}u + \mathbb{Z}v] = ab|\det(u, v)| = 2ab.$$

The factor two is an index of integral tangent lattices. It is not the angle between the lines. If the two lines coincide, their set-theoretic intersection is one-dimensional, whereas the stable intersection is defined by a displacement construction and the expected codimension. This is why ordinary intersection of supports cannot be substituted for tropical intersection of cycles.

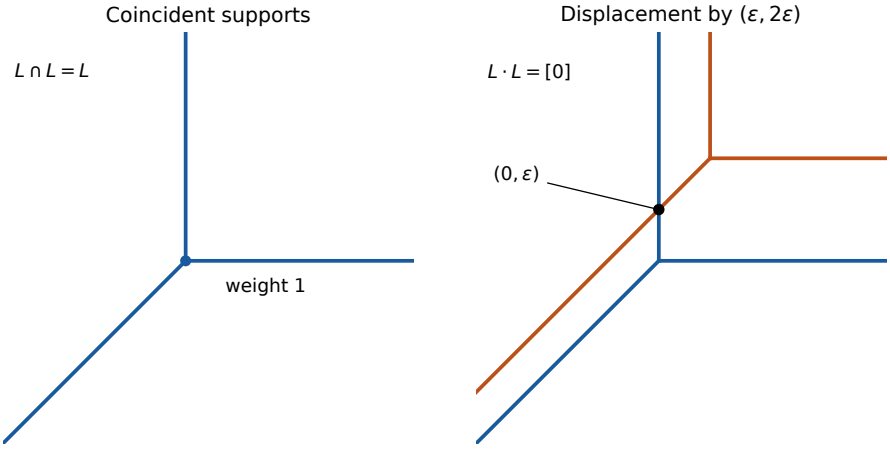


Figure 5.3: The set-theoretic self-intersection of a tropical line is the whole line. A generic displacement gives one transverse point of lattice multiplicity one, which tends to the vertex. The stable self-intersection is that weighted vertex.

5.5 Weighted Gauss points and the role of cancellation

The tropical image contains real vectors which need not be valuations of K-rational points. Weighted Gauss seminorms explain this already for a torus, without changing the ground field.

Proposition 5.5.1. *Let K be a complete non-Archimedean field, $M \simeq \mathbb{Z}^r$, and $w \in N_{\mathbb{R}}$. The rule*

$$|f|_w = \max_m |a_m| e^{-\langle m, w\rangle}, \qquad f = \sum_m a_m \chi^m \in K[M],$$

is a multiplicative norm extending that of K. It determines a point $s(w)$ of the analytified torus, and $\operatorname{trop}(s(w)) = w$. The map s is continuous.

Proof. Choose a basis of M and put $r_i = e^{-w_i}$. The Laurent-polynomial case of Proposition 1.2.2 gives multiplicativity with these radii.

The norm is nonzero on every nonzero polynomial and has the specified values on characters. These facts give the analytic point and its tropicalization. Continuity follows from the topology of analytification: evaluation of each Laurent polynomial along s is the maximum of finitely many continuous functions. □

At these points one has $-\log|f(s(w))| = F_f(w)$. At a general analytic point with the same tropicalization only the inequality $-\log|f(x)| \geq F_f(w)$ holds. Thus s is a section on the torus, but it is not a section on every subvariety: if f vanishes on a subvariety, the Gauss norm of a nonzero f remains nonzero. A point of the subvariety over w must carry cancellation information absent from the torus Gauss point.

5.6 Primitive lattices and the index in a pushforward

Lemma 5.6.1. *If $L \subset N_{\mathbb{R}}$ is a rational linear subspace, then $N_L = N \cap L$ is saturated in N. If $L_\tau \subset L_\sigma$ are rational subspaces with dimension difference one, N_σ / N_τ is free of rank one.*

Proof. If $kn \in N_L$ for a nonzero integer k and $n \in N$, then $n = k^{-1}(kn) \in L$, so $n \in N_L$. Thus the quotient has no torsion. The same argument inside N_σ proves that N_σ / N_τ is torsion free. Its rank is the dimension difference, since both subspaces are rational. A finitely generated torsion-free abelian group of rank one is infinite cyclic. □

Proposition 5.6.2. *Let $F : N \to N'$ be integral linear and injective on the span of a d-dimensional rational polyhedron σ. Let $\sigma' = F(\sigma)$ and*

$$e = [N'_{\sigma'} : F(N_\sigma)].$$

For a compactly supported (d, d)-superform α on the relative interior of σ' one has

$$\int_\sigma F^* \alpha = e \int_{\sigma'} \alpha.$$

Proof. Choose bases of the two tangent lattices. The resulting square integral matrix A has nonzero determinant and $|\det A| = e$. Pullback acts on both exterior factors, so the top-form coefficient is multiplied by $(\det A)^2$. The ordinary change of variables in the scalar integral contributes $|\det A|^{-1}$. Their product is e. The sign convention for top superforms is the same on both sides and does not introduce an orientation sign. □

For finitely many maximal cells mapping onto one image cell, summing this formula produces the weight $\sum_\sigma m_\sigma [N'_{\sigma'} : F(N_\sigma)]$. This explains the lattice factor in tropical pushforward and integration. Identifying that weighted pushforward with the tropicalization of an algebraic image, divided by the generic algebraic degree when appropriate, requires the Sturmfels–Tevelev multiplicity theorem. The change-of-variables calculation does not prove that algebraic theorem.

Example 5.6.3. For a nonzero integer d and $F(t) = dt$ on the tropical line, $F^*(a(y)d'y \wedge d''y) = d^2 a(dt)d't \wedge d''t$. Integration gives $|d| \int a(y)dy$. The same factor is the degree of the torus isogeny $z \mapsto z^d$. If the characteristic of the fibre field divides d, distinct geometric points need not count that degree; scheme lengths do. For an initial fibre this is the residue characteristic. An isogeny over a characteristic-zero ground field remains separable even when its degree is divisible by the residue characteristic.

5.7 Monomial maps and scheme-theoretic degree

The lattice indices in tropical intersection are algebraic degrees. The following calculation proves this for a general isogeny of split tori, including inseparable characteristic. It also distinguishes the multiplicity of an initial degeneration from the number of geometric points in a generic fibre.

Let M and M' be free abelian groups of rank r, and let $A : M' \hookrightarrow M$ be injective with finite cokernel. Over any field k, the inclusion of group algebras gives a morphism

$$f : T_M = \operatorname{Spec} k[M] \longrightarrow T_{M'} = \operatorname{Spec} k[M'], \qquad f^*\chi^{m'} = \chi^{Am'}.$$

Proposition 5.7.1. *The map f is finite faithfully flat of degree $d = [M : A(M')]$. Every fibre algebra has vector-space dimension d over the residue field of the target point.*

Proof. Choose one representative b from each coset in $M/A(M')$. Every monomial has a unique expression $\chi^b\chi^{Am'}$, so

$$k[M] = \bigoplus_{b \in M/A(M')} k[M']\chi^b$$

as a module over $k[M']$. Independence follows because distinct monomials form a k-basis of the group algebra. Hence this algebra is finite free of positive rank d, and thus faithfully flat. Tensoring with the residue field of a point of $T_{M'}$ produces a d-dimensional algebra over that field. It is finite, hence zero-dimensional, and its dimension is its scheme length with residue degrees counted. □

Proposition 5.7.2 (Smith normal form and inseparability)**.** *After integral changes of bases, f is the map*

$$(z_1, \dots, z_r) \longmapsto (z_1^{d_1}, \dots, z_r^{d_r}), \quad d_i > 0, \quad d_1 \mid d_2 \mid \cdots \mid d_r,$$

and $d = \prod_i d_i = |\det A|$. Over an algebraically closed field of characteristic $p > 0$, write $d_i = p^{a_i}e_i$ with $p \nmid e_i$. Each geometric fibre has $\prod_i e_i$ distinct points, each of length $p^{\sum_i a_i}$. In characteristic zero every geometric fibre is reduced with d points.

Proof. Smith normal form is the integer row-and-column reduction of a full-rank integral matrix to the displayed diagonal matrix. The basis changes are torus automorphisms, so they preserve the fibre schemes and degree. In one coordinate, after choosing a p^{a_i}th root b_i of the nonzero target coordinate,

$$z_i^{d_i} - b_i^{p^{a_i}} = (z_i^{e_i} - b_i)^{p^{a_i}}.$$

The polynomial $z_i^{e_i} - b_i$ has e_i distinct nonzero roots, since its derivative does not vanish at any root. The local quotient at each root is isomorphic to $k[t]/(t^{p^{a_i}})$ and has length p^{a_i}. Tensoring these one-coordinate algebras proves the count and the length in r coordinates. The same derivative argument with $a_i = 0$ applies in characteristic zero. □

This statement uses the characteristic of the field over which the fibre is formed. If K has characteristic zero and residue characteristic p, the isogeny $z \mapsto z^p$ has reduced geometric fibres over K, while its reduction can have a nonreduced fibre of length p. The total degree is p in both calculations; the source of multiplicity differs. Tropical multiplicity uses the appropriate initial degeneration over a residue field.

5.8 Binomial complete intersections and tropical weights

Return to a complete non-Archimedean field K. Let $m_1, \dots, m_r$ be linearly independent elements of $M \simeq \mathbb{Z}^r$, let $a_i \in K^\times$, and set

$$Z = V(\chi^{m_1} - a_1, \dots, \chi^{m_r} - a_r) \subset T_M.$$

Theorem 5.8.1. *The scheme Z is finite of length $d = [M : \sum_i \mathbb{Z}m_i]$. Its tropicalization is one point w_0, uniquely determined by*

$$\langle m_i, w_0 \rangle = v(a_i) \quad (1 \le i \le r).$$

This point has tropical multiplicity d. The conclusion includes nonreduced schemes and weights outside the original value group.

Proof. The equations define the fibre of the isogeny associated with the sublattice generated by the m_i, over the point $(a_1, \dots, a_r)$. Proposition 5.7.1 proves the length and nonemptiness. Every analytic point of Z satisfies the

displayed valuation equations. They have a unique real solution because the m_i form a real basis. A finite nonzero K-algebra has a maximal ideal with finite residue extension of K; the extended absolute value gives an analytic point. Thus the tropicalization is exactly $\{w_0\}$.

Choose a valued extension containing a solution t of the binomial system. Translation $z = ty$ changes the equations to $\chi^{m_i}(y) = 1$. The closure of this kernel in the torus over the valuation ring is finite free of rank d: the same coset-basis proof works over a ring, and imposing the target coordinates equal to one preserves that free basis. Its generic fibre is the translated Z, and it has no valuation-ring torsion. Hence it is the schematic closure. Its special fibre has dimension d as a vector space over the residue field. After algebraic closure of that residue field, the sum of local lengths is still d. This is exactly the tropical multiplicity in dimension zero. □

Corollary 5.8.2 (Transverse hypersurfaces)**.** *The stable product of the r tropical binomial hypersurfaces is $d[w_0]$, equal to the tropical cycle of Z.*

Proof. Write $m_i = b_i \bar{m}_i$ with $\bar{m}_i$ primitive. The ith hypersurface has weight b_i. The primitive normal covectors $\bar{m}_i$ are real-linearly independent. The transverse lattice-index rule gives multiplicity $|\det(\bar{m}_1, \dots, \bar{m}_r)|$ for the weight-one hyperplanes; multiplying their weights gives $|\det(m_1, \dots, m_r)| = d$. The support is their unique intersection point w_0. This uses the transverse part of the tropical intersection construction, not a general lifting theorem for intersections. □

Example 5.8.3. Consider in a three-dimensional torus the equations

$$x^2 = a, \qquad xy^3 = b, \qquad yz^5 = c.$$

The exponent matrix has determinant 30, so the intersection has length 30. Its tropical point satisfies

$$w_x = \frac{v(a)}{2}, \qquad w_y = \frac{v(b) - w_x}{3}, \qquad w_z = \frac{v(c) - w_y}{5}.$$

The first hypersurface has weight two. The other two exponent vectors are primitive and give weight-one hypersurfaces. Their three primitive normal vectors have determinant 15; multiplying by the weight two recovers 30. Thus neither the individual weights nor the primitive intersection index alone gives the answer. The computation remains valid in characteristics two, three and five.

For non-binomial equations, a transverse intersection of tropical supports need not by itself identify an algebraic scheme intersection without the relevant lifting and proper-intersection hypotheses. The binomial proof succeeds because one finite free algebra controls both the generic fibre and the initial fibre. In the general theory, flatness, initial degenerations and multiplicity comparison replace that explicit algebra. This is the point at which the geometric results used by Gubler and collaborators enter the calculus.

References. [Gub13, Sections 5 and 13]; [GK17, Section 1]. The finite-free and initial-fibre calculations are proved here; the general tropical multiplicity and intersection conventions are those of the cited sources.

5.9 Real weights and valued extensions

A tropical point is a real vector; it is not required to be the valuation vector of a point over the original ground field. There is a direct construction of a complete valued extension realizing any prescribed vector. It explains why the definition by initial degenerations remains meaningful for arbitrary real weights.

Proposition 5.9.1. *For every $w = (w_1, \dots, w_r) \in \mathbb{R}^r$, there is a complete valued extension L/K and elements $t_i \in L^\times$ with $v(t_i) = w_i$. The absolute value on L restricts to the given one on K.*

Proof. On the Laurent polynomial ring $K[T_1^{\pm 1}, \dots, T_r^{\pm 1}]$ take the weighted Gauss norm

$$\left| \sum_m a_m T^m \right|_w = \max_m |a_m| e^{-\langle m, w \rangle}.$$

It is multiplicative. One proof uses the associated graded ring: the initial homogeneous part of a nonzero polynomial is nonzero, and the graded Laurent polynomial ring over the graded residue field is a domain, so initial parts of a product cannot multiply to zero. Equivalently, refine the real weight order by a monomial order to select a leading exponent; the leading coefficients multiply without cancellation. Extend the norm to the fraction field by $|f/g|_w = |f|_w / |g|_w$, then complete. The images t_i of T_i have the prescribed norms, while constants keep their original norms. □

This is a valued extension construction, not a finite-extension assertion. A finite extension cannot introduce an arbitrary real value over a discretely

valued field: its value group remains a finite-index enlargement. Thus strictification by a general analytic extension and descent through a finite field extension are different operations.

5.10 The closure algebra and all initial relations

Fix L, t as above and put $R = L^\circ$, $\mathfrak{m} = L^{\circ\circ}$ and $k = R/\mathfrak{m}$. Translate the given subscheme $X \subset T$ by t^{-1}, so its ideal becomes $I_t \subset L[z_1^{\pm 1}, \dots, z_r^{\pm 1}]$. The closure algebra in the torus over R is

$$A = R[z_1^{\pm 1}, \dots, z_r^{\pm 1}]/J, \qquad J = I_t \cap R[z_1^{\pm 1}, \dots, z_r^{\pm 1}].$$

Proposition 5.10.1. *The algebra A is torsion-free over R and hence flat over this valuation ring. Its special-fibre ideal is generated by the reductions of all normalized polynomials in I_t: divide each nonzero polynomial by one of its coefficients of maximal absolute value and then reduce.*

Proof. The map $A \to L[z^{\pm 1}]/I_t$ is injective by the definition of J. Its target has no R-torsion, because every nonzero element of R is invertible in L. Thus A is torsion-free. We use the standard valuation-ring criterion that every torsion-free module is flat.

A normalized polynomial has coefficients in R, with at least one unit coefficient, so it belongs to J and reduces to a nonzero initial polynomial. Conversely, take $g \in J$. If all its coefficients belong to $\mathfrak{m}$, its reduction is zero. Otherwise a coefficient of maximal absolute value is a unit of R; dividing by that coefficient changes its reduction by a nonzero scalar in k and gives one of the stated normalized relations. These two observations identify the ideals in $k[z^{\pm 1}]$. □

The proof shows why dividing by a small scalar is necessary even if it is not a unit in R. The relation belongs to the generic-fibre ideal, where that scalar is invertible; taking the saturated closure then retains its normalized reduction. Reducing only an arbitrary list of original equations computes a potentially larger scheme. The explicit example with $f_2 - f_1 = \pi(y - 1)$ earlier in this chapter is exactly this saturation phenomenon.

If t' is another choice with the same valuation vector, the coordinate ratios t'_i/t_i have absolute value one after passage to a common valued extension.

Their reductions are units. They induce a torus translation of the special fibre, so geometric component lengths and the resulting tropical multiplicities are unchanged. This is the elementary part of independence from the translating point. General valued-base-change invariance of tropical multiplicities remains the comparison theorem used in their definition.

5.10.1 Where the lifting theorem enters

Suppose an analytic point of X has tropicalization w. Evaluation in its completed residue field sends each normalized relation to zero, so its reductions cannot generate the unit ideal in the appropriate residue torus. This gives the elementary implication from an analytic point to nonempty initial degeneration. The reverse implication requires lifting from a nonempty reduction to an analytic point of the prescribed tropical fibre. It is the geometric content of the fundamental theorem, not a consequence of the termwise minimum rule alone.

Likewise, flatness of a closure does not by itself identify the initial scheme obtained from any unsaturated presentation. The closure, its relevant analytic domain, and the reduction map must refer to the same normalized ideal. Keeping these objects aligned prevents a common circular argument in informal explanations of tropicalization.

5.11 Hypersurface weights from Newton edges

At a generic point of a maximal face of a tropical hypersurface, the active exponents form a segment in the Newton subdivision. Let its endpoints be m_0 and $m_0 + dq$, where $q \in M$ is primitive and $d > 0$. The initial polynomial has the form

$$\chi^{m_0} P(\chi^q), \qquad P(z) = c_0 + c_1 z + \cdots + c_d z^d, \quad c_0 c_d \neq 0.$$

Proposition 5.11.1. *The tropical multiplicity of this maximal face is d, the lattice length of the dual Newton edge. This remains valid in inseparable residue characteristic.*

Proof. Extend q to a basis of M. Since χ^{m_0} is a unit, the initial scheme becomes

$$\operatorname{Spec} k[z^{\pm 1}]/(P(z)) \ \times \ (\mathbb{G}_m)^{r-1}.$$

The first algebra has dimension d over k. Indeed P has degree d and nonzero constant term, so z is already invertible in $k[z]/(P)$ and localization does not change that quotient. After algebraic closure it decomposes into local Artinian algebras at its roots, whose lengths sum to d. The product with the torus preserves these generic component lengths. No separability assumption is used. □

In a two-dimensional transverse slice at a codimension-two face, the dual Newton cell is a polygon. List its oriented edge vectors as $d_i q_i$ around the boundary. Their sum is zero. After choosing dual oriented lattice bases, rotating each primitive q_i to the corresponding primitive outgoing normal u_i gives

$$\sum_i d_i u_i = 0.$$

By Proposition 5.11.1, the d_i are exactly the hypersurface weights. This proves balancing for the hypersurface in that local slice, and hence modulo the tangent lattice of the original face. The Newton-subdivision correspondence supplies the duality of the slice and polygon; the actual balancing equation is the closed-polygon identity. Higher-codimension ideals need the more general multiplicity theorem.

5.11.1 The sign of the tropical divisor

The tropical polynomial $F(w) = \min_m(v(a_m) + \langle m, w\rangle)$ is concave. With the positive Hessian convention of this book, its effective hypersurface current is therefore

$$\delta_{\mathrm{Trop}(V(f))} = \mathrm{d}'\mathrm{d}''[-F].$$

The corner coefficient across a maximal face is the jump between the two active slopes, and the primitive normal calculation gives the lattice length d. This proves the formula from the corner identity. It is a formula on the real tropical space. It must not be replaced by an equality $\log|f| = -F \circ \mathrm{trop}$ on every analytic point: cancellation makes that equality fail away from the Gauss section, as seen at the beginning of the chapter.

5.12 Why the weight is locally constant on the regular locus

Proposition 5.12.1. *Let C be a balanced pure d-dimensional tropical cycle with constant weights on a polyhedral complex. If an open part of its support is an open subset of a single d-dimensional affine space, its weight is locally constant there.*

Proof. At a facet introduced by the subdivision inside this affine space, precisely two local maximal sides occur. Their tangent lattices agree, and their outgoing primitive classes in the rank-one quotient are opposite. Balancing says $m_+n - m_-n = 0$, so $m_+ = m_-$. In a sufficiently small affine ball, two generic interior points can be joined by a polygonal path avoiding the codimension-two skeleton and crossing only facets. Repeated application of this equality identifies the weights along the path. Points on lower-dimensional subdivision faces inherit this common local weight. □

This is an alternative explanation of local constancy once the balancing theorem has been established. It does not say that the initial schemes are all isomorphic, nor that one weight can be extended unchanged across a singular tropical vertex. In computations, choose a point inside each maximal cell to obtain its multiplicity, and then use the separate weights in the balancing equation at their common face.

References. Initial degenerations and the lifting theorem are treated in [Gub13, Sections 5–6 and 13]; the closure and hypersurface calculations above make the normalization and multiplicities explicit. For the valuation-ring flatness criterion, see `https://stacks.math.columbia.edu/tag/0539`.

5.13 Initial ideals and analytic tropical fibres

Let K be a complete non-Archimedean field, put $v = -\log|\cdot|$, and write $M = \mathbf{Z}^n$. For $w \in \mathbf{R}^n$ and $f = \sum_m a_m \chi^m \in K[M]$, put

$$\|f\|_w = \max_m |a_m| e^{-\langle m,w\rangle}, \qquad \mu_w(f) = \min_{a_m \neq 0} \big(v(a_m) + \langle m, w\rangle\big).$$

A term is active if it attains this maximum, or equivalently this minimum. Denote by $\mathcal{T}_w$ the completion of $K[M]$ for $\|\cdot\|_w$. The multilaurent presentation of Theorem 1.17.2 identifies it, as a normed algebra, with

$$K\{r_1^{-1}T_1, r_1S_1, \dots, r_n^{-1}T_n, r_nS_n\}/(T_iS_i - 1)_i, \qquad r_i = e^{-w_i}.$$

Lemma 5.13.1 (Coordinate bases for weighted subspaces)**.** *Let $E = K^q$ have the norm $\|(a_i)\| = \max_i |a_i| r_i$, where every $r_i > 0$. For every s-dimensional subspace $W \subset E$ there are distinct indices $b_1, \dots, b_s$ and a basis $g_1, \dots, g_s$ of W such that*

$$(g_i)_{b_j} = \delta_{ij}, \qquad \|g_i\| = r_{b_i}, \qquad \Big\|\sum_i c_i g_i\Big\| = \max_i |c_i| r_{b_i}.$$

The same assertions hold after every isometric field extension, with the same basis and weights. If all $r_i = 1$, reduction of the unit ball of W has dimension s over $\widetilde{K}$, and its reduction commutes with residue-field extension.

Proof. Choose a nonzero vector in W, choose a coordinate at which its weighted norm is attained, and divide by that coordinate. This gives g_1 with pivot b_1, pivot coefficient one, and norm r_{b_1}. Apply induction to the kernel in W of the b_1-coordinate. Its basis vectors have zero b_1-coordinate. Subtract their pivot multiples from g_1 to make all other pivot coordinates of g_1 zero. Each subtraction has norm at most r_{b_1}: its scalar coefficient c occurs in a pivot coordinate of weight r_b and satisfies $|c| r_b \le r_{b_1}$. The b_1-coordinate stays one, so the norm stays r_{b_1}. This completes the construction.

The ultrametric inequality gives the upper bound for a linear combination. Its pivot coordinates are exactly the c_i, giving the reverse bound. These calculations remain valid over an isometric extension. For unit weights the unit ball is therefore $\bigoplus_i K^\circ g_i$, its reduction has basis $\overline{g}_i$, and the identical description over the extension proves the last assertion. □

Proposition 5.13.2 (The tropical fibre algebra)**.** *Let $I \subset K[M]$ be an ideal and $X = \operatorname{Spec}(K[M]/I)$. For every real weight w, the fibre $X_w = \{x \in X^{\mathrm{an}} : -\log|\chi^{e_i}(x)| = w_i\}$ is the affinoid space*

$$X_w = \mathcal{M}(\mathcal{T}_w / I\mathcal{T}_w).$$

The ideal $I\mathcal{T}_w$ is closed and is the norm closure of I in $\mathcal{T}_w$.

Proof. Theorem 1.18.3 makes every ideal of $\mathcal{T}_w$ closed. Write an element of $I\mathcal{T}_w$ as a finite sum $\sum_j h_j f_j$ with $f_j \in I$ and $h_j \in \mathcal{T}_w$. Approximating the finitely many h_j by Laurent polynomials approximates this sum by elements of I. Thus its closure is precisely $I\mathcal{T}_w$.

An analytic point in the indicated fibre bounds every Laurent polynomial by $\|\cdot\|_w$, so its seminorm extends to $\mathcal{T}_w$ and kills $I\mathcal{T}_w$. Conversely, the bounds

on T_i, S_i and the identities $T_i S_i = 1$ force $|T_i| = r_i$ at every point of the displayed quotient. Restriction to $K[M]/I$ gives the corresponding point of X^{an}, by Proposition 2.10.1. The two constructions are inverse, continuous in the evaluation topologies, and identify the analytic structures by the rational-domain universal property of Theorem 2.31.2, applied to the finitely many coordinate and inverse-coordinate inequalities. □

Theorem 5.13.3 (The tropical fundamental theorem over the original field)**.** *With the preceding notation, the following conditions on $w \in \mathbf{R}^n$ are equivalent:*

1. $w \in \mathrm{Trop}(X) := \mathrm{trop}(X^{\mathrm{an}})$*;*
2. $\mathcal{T}_w / I\mathcal{T}_w \neq 0$*;*
3. *every nonzero $f \in I$ has at least two active terms at w.*

This holds with trivial valuation, arbitrary real radii, and nonreduced X.

Proof. The equivalence of the first two conditions follows from Proposition 5.13.2 and Theorem 1.20.1. If f has a unique active term $a_m \chi^m$, then

$$f = a_m \chi^m (1 + h), \qquad \|h\|_w < 1.$$

Lemma 1.1.4 makes f invertible in $\mathcal{T}_w$, so the quotient is zero. Conversely, if that quotient is zero, then $1 \in I\mathcal{T}_w$. By the closure assertion in Proposition 5.13.2, there is $g \in I$ with $\|1 - g\|_w < 1$. Its constant coefficient has absolute value one and every other term has weighted size less than one. Thus g has a unique active term, contradicting the third condition. □

Proposition 5.13.4 (Initial ideals and scalar extension)**.** *Put $R = K^\circ$ and $k = \widetilde{K}$. Let $J = I \cap R[M]$ and let $\overline{J}$ be its image in $k[M]$. For any complete isometric extension L/K, with residue field l, one has*

$$\overline{IL[M] \cap L^\circ[M]} = \overline{J}\, l[M].$$

Moreover, $R[M]/J$ is flat over R, its special fibre is $k[M]/\overline{J}$, and this special fibre is nonempty exactly when $0 \in \mathrm{Trop}(X)$.

Proof. For a finite set $F \subset M$, let V_F be its coordinate span and $W_F = I \cap V_F$. Every vector space over K is torsion-free and therefore flat by [StacksAlg, Tag

0539], applied to the valuation ring K itself. Tensoring the kernel sequence $W_F \to V_F \to K[M]/I$ thus gives

$$(IL[M]) \cap (V_F \otimes_K L) = W_F \otimes_K L.$$

Apply Lemma 5.13.1 with unit weights. The reductions of the unit balls on the two sides therefore differ exactly by scalar extension from k to l. Taking the union over finite supports proves the ideal identity.

The closure algebra injects into $K[M]/I$, hence has no R-torsion. It is flat by the valuation-ring criterion, [StacksAlg, Tag 0539]. Its special fibre has the stated presentation. If $\overline{J}$ contains a monomial, lifting the equality expressing this monomial as a finite sum of reduced relations gives an element of I whose reduction is that monomial. It has a unique active term. Conversely, normalize an element of I having a unique active term by its largest coefficient; its reduction is a unit monomial in $\overline{J}$. Thus $\overline{J}$ is proper exactly when every relation has at least two active terms. Theorem 5.13.3 finishes the proof. □

Proposition 5.13.5 (Completion retains the initial relations)**.** *Let $\widehat{I} = I\mathcal{T}_0$, let $\mathcal{T}_0^{\leq 1}$ denote the Gauss unit ball, and reduce this unit ball coefficientwise to $k[M]$. Then*

$$\operatorname{red}(\widehat{I} \cap \mathcal{T}_0^{\leq 1}) = \overline{J}.$$

Proof. One inclusion follows from $I \subset \widehat{I}$. Given $h \in \widehat{I}$ with $\|h\|_0 \leq 1$, approximate it by $g \in I$ with $\|g - h\|_0 < 1$, using Proposition 5.13.2. Then $\|g\|_0 \leq 1$ and g, h have identical reductions. This proves the reverse inclusion. Coefficientwise reduction is a Laurent polynomial because only finitely many coefficients of a convergent Laurent series can have absolute value one. □

Theorem 5.13.6 (Initial degenerations at arbitrary real weights)**.** *Choose a complete isometric extension L/K and $t \in (L^\times)^n$ with $v(t) = w$. Such a choice exists for every real w. Write I_t for the image of $IL[M]$ under $\chi^m \mapsto t^m \chi^m$, and set*

$$\operatorname{in}_{w,t,L}(X) = \operatorname{Spec}\left(\widetilde{L}[M]/\overline{I_t \cap L^\circ[M]}\right).$$

After residue-field extension and translation by a point of the residue torus, this scheme is independent of L, t. It is nonempty exactly when $w \in \operatorname{Trop}(X)$. For every complete isometric extension E/K,

$$\operatorname{Trop}(X_E) = \operatorname{Trop}(X).$$

Proof. Extend the multiplicative weighted Gauss norm on $K[M]$ to its fraction field and complete. The coordinate monomials give the required t, with the original absolute value on K. Multiplicativity is the Gauss calculation of Proposition 1.2.2. Theorem 2.35.5 places any two choices in a common complete valued extension. Proposition 5.13.4 compares the resulting initial ideals under that extension. For two translating points in the common field, the ratios t'_i/t_i have absolute value one; their residues induce a torus translation between the two special fibres.

Theorem 2.33.13, applied to the fibre algebra of Proposition 5.13.2, shows that every tropical fibre is nonempty before a complete extension exactly when it is nonempty afterwards. This proves tropical base-change invariance. Translation by t identifies the fibre at w with the unit-radius fibre. Proposition 5.13.4 and Theorem 5.13.3 now give the assertion about nonemptiness. □

Proposition 5.13.7 (Properness of toric tropicalization)**.** *The map* trop : $X^{\mathrm{an}} \to \mathbf{R}^n$ *is continuous and topologically proper. Its image is closed.*

Proof. Continuity follows from evaluation on the invertible characters. The inverse image of a closed box is the closed analytic subspace defined by I in the corresponding affinoid polyannulus; Proposition 2.24.1 gives its compactness. For a compact subset of $\mathbf{R}^n$, choose a box containing it; its inverse image is closed in that compact polyannulus, hence compact. Closedness of the image also follows directly from Theorem 5.13.3: its complement is the union, over $f \in I$, of the open loci where one term of f is uniquely active. □

References. These arguments reconstruct the initial-degeneration and lifting mechanisms of [Gub13, Sections 3–5]. The analytic lifting step here is the closed-ideal approximation in the actual tropical-fibre algebra; the proof uses the previously established affinoid spectrum and base-change results.

5.14 A combinatorial calculus of corners

Let N be a lattice and put $V = N \otimes_{\mathbf{Z}} \mathbf{R}$. A rational polyhedron may have arbitrary real translation constants; its direction space is rational. All complexes in this section are finite. Write $N_\sigma = N \cap \operatorname{span}_{\mathbf{R}}(\sigma - \sigma)$. A real

weighted d-cycle is a pure d-dimensional rational polyhedral complex with constant real weights w_σ on its facets, satisfying

$$\sum_{\sigma \supset \tau} w_\sigma u_{\sigma/\tau} = 0 \quad \text{in } V/N_{\tau,\mathbf{R}}.$$

Here $u_{\sigma/\tau}$ is the primitive outward generator of the rank-one lattice N_σ/N_τ. Subdivision preserves a weight on each subdivided facet; cells of weight zero may be discarded. We identify cycles after a common weighted subdivision. A piecewise affine function on a cycle is continuous and affine on the cells of some finite rational subdivision. Its slopes may be real.

Lemma 5.14.1 (Primitive vectors and saturated lattices)**.** *A primitive vector of $\mathbf{Z}^n$ extends to a lattice basis. A saturated subgroup $A \subset \mathbf{Z}^n$ is generated by part of a basis and has a free quotient. The group $\mathrm{GL}_n(\mathbf{Z})$ is generated by row interchanges, sign changes, and adding an integer multiple of one row to another.*

Proof. The Euclidean algorithm, applied successively to coordinate pairs, uses these invertible row operations to send a vector to $(d, 0, \dots, 0)$, where d is the greatest common divisor of its coordinates. For a primitive vector $d = 1$; applying the inverse row-operation matrix to the standard basis proves the first assertion. If $A \neq 0$ is saturated, divide a nonzero element by the greatest common divisor of its coordinates. Saturation puts the resulting primitive vector a in A. The image of A in $\mathbf{Z}^n/\mathbf{Z}a \simeq \mathbf{Z}^{n-1}$ is saturated: if a nonzero integer multiple of a class lies in that image, a representative has that multiple in A, since $\mathbf{Z}a \subset A$, and hence belongs to A. Induction and lifting a quotient basis now give a basis whose first members generate A.

For an invertible integral matrix, the greatest common divisor of its first column is one, since any common divisor would divide its determinant ± 1. Row operations reduce that column to the first standard vector. The lower-right block is again invertible over $\mathbf{Z}$; reduce it by induction, and then clear the remaining entries of the first row using the lower identity rows. This reduces the entire matrix to the identity and proves the generation assertion. □

Lemma 5.14.2 (Refinement and products of balanced complexes)**.** *Finite rational polyhedral complexes admit common rational refinements. Balancing is invariant under refinement, and the Cartesian product of two balanced cycles, with product weights, is balanced. The star at a face and the quotient of that star by the direction space of the face are balanced.*

Proof. Take all affine supporting hyperplanes of the finitely many faces, and cut every polyhedron by them. The resulting sign cells and their faces form a common refinement. At a new codimension-one face lying inside an old facet, there are two incident pieces with the same weight and opposite primitive normals. At a subdivision of an old codimension-one face, the old normal equation is unchanged. All other new faces have smaller dimension. These observations prove refinement invariance in both directions.

A codimension-one face of a product is either $\tau \times \rho$ or $\sigma \times \nu$, where exactly one factor has codimension one. Its normal equation is the normal equation in that factor multiplied by the weight of the other factor. For a star, the normals and weights are the original ones at the faces containing the chosen face. Its direction lattice is saturated, so passage to the quotient preserves the primitive normal quotient lattices and the balancing equations. □

Definition 5.14.3 (The corner operator)**.** Let C be a balanced d-cycle and φ a piecewise affine function on it. On a subdivision on which φ is affine, let φ_σ denote its linear part. At a codimension-one face τ choose any real linear extension $\ell_\tau : V \to \mathbf{R}$ of φ_τ, and choose a lattice lift $n_{\sigma/\tau}$ of each primitive normal. The corner $D_\varphi C$ has weight

$$w_{D_\varphi C}(\tau) = \sum_{\sigma \supset \tau} w_\sigma (\varphi_\sigma - \ell_\tau)(n_{\sigma/\tau}). \tag{C}$$

For $d = 0$ the corner is zero.

Proposition 5.14.4 (Well-defined corner weights)**.** *Formula* (C) *is independent of all normal lifts, linear extensions, and adapted subdivisions. It is linear in C and in φ, and $D_\ell C = 0$ for every globally affine function ℓ. It commutes with integral affine lattice isomorphisms.*

Proof. Changing a lift adds a vector in $N_{\tau,\mathbf{R}}$, on which $\varphi_\sigma - \ell_\tau$ vanishes by continuity. Changing ℓ_τ by a linear form h vanishing on that space changes the sum by $-h(\sum_\sigma w_\sigma n_{\sigma/\tau}) = 0$, by balancing. At a new face inside an old

affine piece, the two contributions cancel; in fact the same linear extension can be used on both sides. At a subdivided old face, the formula has the old value. Lemma 5.14.2 now compares any two adapted subdivisions. The formula gives both linearities. For a globally affine function choose its common linear part as every ℓ_τ; all summands vanish. A lattice isomorphism transports the saturated lattices, their primitive generators, and this same formula, proving the last assertion. □

Theorem 5.14.5 (Balancing and commutation of corners)**.** *For every balanced real cycle C and continuous piecewise affine functions φ, ψ on it, $D_\varphi C$ is balanced and*

$$D_\psi D_\varphi C = D_\varphi D_\psi C.$$

Proof. Both assertions are local at an original face of codimension two. Take its star and quotient by its direction space. Subtract linear extensions of the two functions along that face. By Proposition 5.14.4, this changes neither corner. We have reduced to a two-dimensional balanced rational fan, with both functions homogeneous and linear on its cones. Cut cones with lineality by rational hyperplanes if necessary, so that the cones are pointed; their two-dimensional cones then have two extreme rays. Let the distinct primitive ray vectors be $v_1, \dots, v_s$. For a cone with extreme rays v_i, v_j put

$$d_{ij} = [N_{\langle v_i, v_j \rangle} : \mathbf{Z}v_i + \mathbf{Z}v_j], \qquad a_{ij} = a_{ji} = w_{ij}/d_{ij};$$

put $a_{ij} = 0$ if there is no such cone. Modulo $\mathbf{R}v_i$, its primitive normal is the class of v_j/d_{ij}. Indeed v_i is primitive in the rank-two saturated lattice; Lemma 5.14.1 completes it to a basis, in which $v_j = bv_i + d_{ij}e$ with the appropriate sign of e. Balancing at v_i therefore says

$$\sum_j a_{ij} v_j = \lambda_i v_i \quad \text{for some } \lambda_i \in \mathbf{R}.$$

The weight of the first corner on the ray i is consequently

$$b_i(\varphi) = \sum_j a_{ij}\varphi(v_j) - \lambda_i \varphi(v_i). \tag{R}$$

One may use v_j/d_{ij} instead of a lattice lift in (C), since their difference is tangential. Symmetry of a_{ij} and the displayed balancing equations give

$$\sum_i b_i(\varphi) v_i = \sum_j \varphi(v_j) \sum_i a_{ij} v_i - \sum_i \lambda_i \varphi(v_i) v_i = 0.$$

This is balancing for the corner at the origin. The coefficient of the second corner at that origin is

$$\sum_i b_i(\varphi)\psi(v_i) = \sum_{i,j} a_{ij}\varphi(v_j)\psi(v_i) - \sum_i \lambda_i \varphi(v_i)\psi(v_i),$$

which is symmetric in φ, ψ. Hence the two orders agree. For a cycle of dimension one, its corner is a zero-cycle, whose further corner is zero; in dimension zero both assertions are immediate. The local calculation and refinement invariance prove the theorem in every dimension. □

Lemma 5.14.6 (Restriction, products, and quotient directions)**.** *The corner depends only on the restricted function on the support of its cycle. If p is projection from a product, then*

$$D_{\varphi \circ p}(C \times E) = (D_\varphi C) \times E.$$

Taking a star commutes with the corner after replacing the function by its local piecewise linear part. If a fan has a rational lineality space L, and the function is pulled back from its quotient by L, the corner is the pullback of the corner on that quotient, with the same weights.

Proof. All terms of (C) are determined by the restricted affine functions. A codimension-one face of $C \times E$ belonging to the E factor has zero weight for this corner, since the function has a common affine extension there. At a face belonging to the C factor, the formula is the corner weight in C multiplied by the weight in E. Stars leave the normal quotient lattices unchanged. For the last assertion, $N \cap L$ is saturated; Lemma 5.14.1 gives a lattice splitting of N over $N/(N \cap L)$. The fan is then a product of L with its quotient, so the product calculation applies. □

References. The corner construction follows Allermann–Rau, *First steps in tropical intersection theory*, Section 3. The proof above gives the full rank-two calculation, including its tangential correction, before using any differential forms or toric Chow theory.

5.15 The standard-chart homotopy

Let K be complete and nontrivially valued, with its original absolute value. Fix $0 < |a| < 1$ and integers $r, q \geq 0$. Put

$$A = K\langle X_0, \dots, X_r, T_1, \dots, T_q \rangle / (X_0 \cdots X_r - a), \qquad V = \mathcal{M}(A).$$

Proposition 3.26.1 gives the continuous Gauss section $s : \Delta_a \to V$, where

$$\Delta_a = \{(w_0, \dots, w_r) \in \mathbb{R}^{r+1}_{\geq 0} : \sum_i w_i = v(a)\}.$$

After eliminating $X_0 = a/(X_1 \cdots X_r)$, $s(w)$ is the Gauss point of radii e^{-w_i} in X_i, $1 \leq i \leq r$, and radius 1 in every T_j. Let $\Sigma_a = s(\Delta_a)$ and write $\mathrm{val} : V \to \Delta_a$ for the values of $X_0, \dots, X_r$.

Lemma 5.15.1 (Mixed Gauss displacement)**.** *For $x \in V$, write $E = \mathcal{H}(x)$, $\xi_i = X_i(x) \in E^\times$ and $b_j = T_j(x) \in E$. For $0 < t \leq 1$, take the Gauss absolute value of radius t in all variables $S_1, \dots, S_r, Z_1, \dots, Z_q$ over E, and extend it to the completed rational function field E_t. The substitutions*

$$\begin{aligned} X_i &\longmapsto \xi_i(1 + S_i) && (1 \leq i \leq r), \\ X_0 &\longmapsto \xi_0 \prod_{i=1}^{r} (1 + S_i)^{-1}, \\ T_j &\longmapsto b_j + Z_j && (1 \leq j \leq q). \end{aligned}$$

define a bounded homomorphism $A \to E_t$ and hence a point $H(x,t) \in V$. Set $H(x,0) = x$. For every $t \in [0,1]$,

$$|X_i(H(x,t))| = |X_i(x)| \quad (0 \leq i \leq r), \qquad |T_j(H(x,t))| = \max\{|T_j(x)|, t\}.$$

Proof. The Gauss absolute value is multiplicative by Proposition 1.2.2. For $t > 0$ the polynomial $1 + S_i$ has norm 1, so it is nonzero and invertible in the rational function field, also when $t = 1$. All displayed coordinate images have norm at most one, and their product in the first $r + 1$ coordinates is a. Convergent restricted series can therefore be evaluated in the complete field E_t. This yields a homomorphism from the Tate algebra, contractive for its Gauss norm, which kills the defining relation. It descends contractively for the quotient norm on A. Multiplicativity and the exact Gauss norms of $1 + S_i$ and $b_j + Z_j$ give the stated coordinate values. At $t = 0$ they are the values at x, as required. □

Theorem 5.15.2 (A jointly continuous strong deformation retraction)**.** *The map $H : V \times [0,1] \to V$ of Lemma 5.15.1 is jointly continuous and is a strong deformation retraction onto Σ_a. More precisely,*

$$\begin{aligned} H(x,0) &= x, & H(x,1) &= s(\mathrm{val}(x)), \\ H(s(w),t) &= s(w), & \mathrm{val}(H(x,t)) &= \mathrm{val}(x). \end{aligned}$$

It commutes with complete isometric ground-field extension.

Proof. We first give a formula proving joint continuity, including at $t = 0, 1$. Let $P \in K[X_1^{\pm 1}, \dots, X_r^{\pm 1}, T]$ and choose $D \in \mathbb{N}^r$ so that $Q = X^D P$ is a polynomial. Expand the finite polynomial

$$Q\big(X_1(1+S_1), \dots, X_r(1+S_r), T+Z\big) = \sum_{\beta,\gamma} Q_{\beta\gamma}(X,T) S^\beta Z^\gamma.$$

Since the factors $1 + S_i$ have norm one, the exact formula is

$$|P(H(x,t))| = \frac{\max_{\beta,\gamma} |Q_{\beta\gamma}(x)|\, t^{|\beta|+|\gamma|}}{\prod_{i=1}^{r} |X_i(x)|^{D_i}}, \tag{H}$$

where the term of degree zero uses $0^0 = 1$. Its right side is continuous in (x, t): it is a finite maximum divided by a continuous positive function. Every X_i is invertible on V. The Laurent polynomial algebra is dense in A by Proposition 3.26.1. All evaluations $H(x, t)$ are contractive for one fixed quotient norm, so approximating an arbitrary $f \in A$ by Laurent polynomials gives uniform approximation of the real functions $|f(H(x, t))|$ on $V \times [0, 1]$. Thus all such evaluation functions are continuous, which is precisely continuity into the Berkovich spectrum.

At $t = 1$, translation of any variable of the unit Gauss norm by a scalar of norm at most one is isometric. Indeed, the ultrametric inequality makes substitution contractive on polynomials, and the inverse translation gives the reverse bound; the assertion then extends to fractions. Apply this to $S_i \mapsto 1 + S_i$ and $Z_j \mapsto b_j + Z_j$. For every Laurent polynomial $P = \sum c_{\alpha\mu} X^\alpha T^\mu$,

$$|P(H(x,1))| = \max_{\alpha,\mu} |c_{\alpha\mu}| \prod_{i=1}^{r} |X_i(x)|^{\alpha_i}.$$

This is the defining Gauss norm of $s(\mathrm{val}(x))$, including radius one for every free coordinate T_j. Density gives the same equality on A.

Let $x = s(w)$. The norm of every monomial after displacement is at most its original Gauss norm, so $|P(H(x, t))| \le |P(x)|$. In (H), the constant coefficient is $Q(x)$ and has norm equal to the Gauss norm of Q at $s(w)$. Consequently the same formula gives the reverse inequality. Thus H fixes every skeletal point. The identity at time zero is its definition, and preservation of val is the coordinate formula in the lemma. These are all strong deformation retraction identities.

For a complete isometric extension L/K and a point $x' \in V_L$ over x, the completed residue-field embedding $\mathcal{H}(x) \hookrightarrow \mathcal{H}(x')$ is isometric. Formula (H) therefore gives the same value on every original Laurent polynomial before and after projection. Density proves commutation on A. At the endpoint the standard Gauss section over L restricts to that over K, with unchanged real radii. □

Proposition 5.15.3 (The maximum-parameter composition law)**.** *For all $x \in V$ and $s, t \in [0, 1]$,*

$$H(H(x, s), t) = H(x, \max(s, t)).$$

Proof. When one parameter is zero the assertion is immediate. For positive parameters, realize the completed residue field of $H(x, s)$ inside the completed Gauss rational field used in its definition. The injection comes from its actual evaluation homomorphism, after quotienting its kernel, taking fractions, and completing. Gauss extension preserves the norm of every coefficient. Iterating the construction therefore uses independent variables (S, Z) of radius s and (S', Z') of radius t, and substitutes

$$X_i \mapsto \xi_i(1 + S_i)(1 + S'_i), \qquad T_j \mapsto b_j + Z_j + Z'_j.$$

The two sets of variables are symmetric, so assume $s \leq t$. Set

$$W_i = S_i + S'_i + S_i S'_i, \qquad V_j = Z_j + Z'_j.$$

The rational coordinate change from (S, Z, S', Z') to (S, Z, W, V) is an isometry of Gauss fields with radii (s, s, t, t). To see this directly, its inverse formulas are

$$S'_i = (W_i - S_i)/(1 + S_i), \qquad Z'_j = V_j - Z_j.$$

The displayed images have their assigned norm bounds, because $s \leq t$ and $|1 + S_i| = 1$, including when $s = t = 1$. Hence substitution of an old polynomial has norm at most its original Gauss norm. Its image is a new polynomial divided by a product of powers of $1 + S_i$, of norm one. Applying the reverse substitution to that numerator gives the reverse bound. This proves equality for polynomials and then for their fractions. The same argument for $V = Z + Z'$ is linear and has no denominator.

After this isometry the iterated substitution is simply $X_i \mapsto \xi_i(1 + W_i)$ and $T_j \mapsto b_j + V_j$. It does not involve (S, Z); its Gauss norm is the single displacement norm of radius t. This proves the identity on Laurent polynomials, and uniform approximation proves it on all of A. □

Proposition 5.15.4 (Monomial and bounded affine covariance)**.** *On a torus times a unit polydisc, the mixed displacement construction is unchanged by a torus coordinate change $Y_i = c_i X^{m_i}$ whose exponent matrix lies in $\mathrm{GL}_r(\mathbb{Z})$, together with a free-coordinate change $U = BT + b$ with B, B^{-1} having entries in K° and $b \in (K^\circ)^q$. Whenever such a change identifies two standard chart domains, it intertwines their entire homotopies. In particular the construction is independent of which monomial coordinate is eliminated from $X_0 \cdots X_r = a$.*

Proof. For $0 < t < 1$, the normalized relative displacement of Y_i is

$$W_i = \prod_j (1 + S_j)^{m_{ij}} - 1.$$

Integer binomial coefficients have norm at most one, including those for negative exponents. The binomial series therefore converge for radius t, and $|W_i| \le t$. The inverse integral matrix gives the inverse substitutions with the same bounds. The two maps of radius-t Tate algebras are contractive inverses, hence isometric. At $t = 1$ use instead the variables $R_i = 1 + S_i$. Their radius-one Gauss norm is unchanged by translation. A unimodular monomial change permutes the Laurent monomial basis, so preserves the Gauss norm and its extension to fractions. At time zero both constructions are evaluation at the original point.

For the free variables, the relative displacement of $BT + b$ is BZ. Both B and its inverse are contractive for the radius-t coordinate maximum norm, so their induced Gauss substitutions are isometric. The constant b merely changes the center. These isometries identify all polynomial evaluations defining the homotopies; density identifies the analytic evaluations. Finally, changing the eliminated coordinate expresses the new independent monomial variables in the old ones by an integral invertible exponent matrix and powers of a, precisely the torus changes just treated. □

References. The construction uses explicit Gauss displacement on the standard semistable chart. At the endpoint, evaluation in a rational Gauss field retains the norm-one denominators and their multiplicative inverses.

5.16 Finite polyhedral initial data

Write $S = K[x_0, \dots, x_n]$ with its usual grading, and let $H \subset S$ be homogeneous. For $\omega \in \mathbf{R}^{n+1}$, pass to a complete isometric extension realizing the coordinate weights, translate the variables, and reduce normalized homogeneous relations. This gives a homogeneous ideal $\mathrm{in}_\omega(H)$ in a polynomial ring over the residue field. The arguments of Proposition 5.13.4 apply degree by degree to polynomial rings as well. We compare ideals after residue-field extension and diagonal change of coordinates.

Lemma 5.16.1 (The Hilbert function of an initial ideal)**.** *For every degree d, the degree-d component of $\mathrm{in}_\omega(H)$ is the reduction of the unit ball of the translated subspace H_d. Consequently*

$$\dim_{\widetilde{L}}\big(\widetilde{L}[x]/\,\mathrm{in}_\omega(H)\big)_d = \dim_K (S/H)_d.$$

Proof. The reduction of the unit ball of H_d is a vector subspace. Multiplying a normalized homogeneous relation of degree $e \le d$ by a monomial of degree $d-e$ gives a normalized relation in H_d. Conversely, every reduced vector in H_d is one of the homogeneous initial relations defining the ideal. Thus its degree-d part is exactly this reduced subspace. After translating, Lemma 5.13.1 with unit weights gives its dimension $\dim_K H_d$. Subtract this from the number of degree-d monomials. □

Lemma 5.16.2 (Finitely many monomial ideals with a fixed Hilbert function)**.** *In a polynomial ring in finitely many variables, there are only finitely many monomial ideals with a prescribed Hilbert function. This finite set depends only on that function and the number of variables, not on the coefficient field.*

Proof. Suppose there were infinitely many distinct such ideals. Enumerate the monomials by nondecreasing degree. Successively retain an infinite subfamily on which membership of the next monomial is constant. The limiting membership choices define a monomial ideal Q: divisibility preserves membership because it does so in every ideal of every sufficiently late subfamily. In each degree its standard monomials have the prescribed cardinality. The polynomial ring is Noetherian by [StacksAlg, Tag 00FN], so Q has finitely many monomial generators, all of degree at most some D.

An infinite subfamily agrees with Q on all monomials of degree at most D. Each member therefore contains Q. Since its standard-monomial count equals that of Q in every degree, the containment is equality. This contradicts the distinctness of the infinite subfamily. All choices involved only sets of exponent vectors. □

Proposition 5.16.3 (A uniform degree bound for homogeneous initial ideals)**.** *There is an integer D depending only on the Hilbert function of S/H such that every $\mathrm{in}_\omega(H)$, over every residue extension, is generated in degrees at most D. The same D works for all further ordinary initial ideals of these ideals.*

Proof. Fix a degree-compatible monomial order. For a homogeneous ideal J over a field, let $\mathrm{LM}(J)$ be the monomial ideal generated by its leading monomials. Its standard monomials form a basis of the quotient in every degree. Indeed, finitely many leading monomials generate $\mathrm{LM}(J)$ by Noetherianity. Choose homogeneous elements of J having these leading monomials. Polynomial division by them terminates because the monomial order is well ordered and reduces every homogeneous polynomial to a linear combination of standard monomials. Such a combination cannot belong to J unless it is zero, since otherwise its leading monomial would belong to $\mathrm{LM}(J)$. The same division proves that the chosen elements generate J.

By Lemma 5.16.1, all ideals under consideration have the Hilbert function of H. Their monomial leading ideals therefore belong to the finite list of Lemma 5.16.2. Take D to be the largest degree among the minimal monomial generators in this list. Lifting these generators as in the preceding paragraph proves the bound. An ordinary weight initial ideal has the same Hilbert function, by the same degreewise argument over a trivially valued field, so the bound applies to it as well. □

Lemma 5.16.4 (Plücker coordinates and weighted row reduction)**.** *Let $W \subset K^F$ have dimension s, where F is a finite set of exponent vectors, and let C be a full-rank row matrix for W. For an s-element column set B with $\det C_B \neq 0$, put*

$$p_B(\omega) = v(\det C_B) + \sum_{b \in B} \langle b, \omega \rangle.$$

If B minimizes $p_B(\omega)$, the rows g_b of $C_B^{-1}C$, indexed by $b \in B$, satisfy

$$(g_b)_{b'} = \delta_{bb'}, \qquad v((g_b)_m) + \langle m, \omega \rangle \geq \langle b, \omega \rangle.$$

Their initial forms form a basis of the initial subspace. Which coefficients survive is determined by equalities among the finitely many functions p_B.

Proof. The pivot identity follows from the matrix inverse. Cramer's rule expresses $(g_b)_m$, up to sign, as the ratio of the minor obtained by replacing b by m to $\det C_B$. Minimality gives the displayed inequality. The resulting rows have distinct pivots and weighted norms $e^{-\langle b,\omega\rangle}$; the pivot-coordinate argument of Lemma 5.13.1 applies. After translating and dividing the b-th row by the b-th translating monomial, its pivot is one. Reduction therefore gives independent spanning rows. A nonpivot coefficient survives exactly when the corresponding minor has the same value of p as B. □

Theorem 5.16.5 (Finite polyhedral strata for initial degenerations)**.** *There is a finite polyhedral complex covering $\mathbf{R}^{n+1}$, defined by integer linear forms and constants in $v(K^\times)$, such that on the relative interior of each cell the schemes defined by $\mathrm{in}_\omega(H)$ are isomorphic after residue extension and diagonal torus translation. For an ideal $I \subset K[M]$, there is consequently a finite integral $v(K^\times)$-affine polyhedral complex on $\mathbf{R}^n$ for which $\mathrm{Trop}(\operatorname{Spec} K[M]/I)$ is the support of a subcomplex.*

Proof. Take D from Proposition 5.16.3. For each $d \le D$ apply Lemma 5.16.4 to H_d. Cut weight space by all equalities between the finitely many affine functions p_B occurring in these degrees. Their normals are integral and their constant differences belong to $v(K^\times)$. The resulting finite hyperplane arrangement is the required polyhedral complex.

Fix weights ω, ω' in the same relative interior. In each degree choose one common minimizing pivot set B; the same coefficients in its row basis survive at both weights. Realize both weights in a common valued extension, with translating points t, t'. For every surviving exponent difference $m-b$, the unit ratio $(t'/t)^{m-b}$ has norm one. On the subgroup $A \subset \mathbf{Z}^{n+1}$ generated by all these differences, their residues define a character $\rho : A \to \widetilde{L}^\times$. This character extends to the full exponent lattice after a finite algebraic residue extension. To see this directly, adjoin the standard basis vectors successively. When adjoining e_i, if no positive multiple of e_i belongs to the current subgroup, give e_i any nonzero value. Otherwise let $q > 0$ be least with qe_i in that subgroup, adjoin a root b of $T^q - \rho(qe_i)$, and set $\rho(a+me_i) = \rho(a)b^m$. This is well defined and extends ρ. A root is obtained by quotienting by an irreducible factor of that polynomial. After finitely many steps, translation by the resulting torus

point sends every surviving coefficient in every row at ω to its coefficient at ω'. The ideals are generated by these rows in degrees at most D, so the translation identifies the ideals.

For the Laurent ideal, take the homogeneous kernel

$$H = \ker\bigl(K[x_0,\dots,x_n] \longrightarrow (K[M]/I)[z,z^{-1}]\bigr), \qquad x_0 \mapsto z, \quad x_i \mapsto z\chi^{e_i}.$$

Localizing all the variables and putting $x_0 = 1$ recovers I. Initial relations commute with this operation: multiplication by a monomial shifts every term valuation by the same amount, and any Laurent relation can first be multiplied by a monomial and homogenized. Restrict the constructed complex to $\omega_0 = 0$. Nonemptiness of the resulting initial scheme is constant on every relative interior. By Theorem 5.13.6 this is exactly membership in $\mathrm{Trop}(X)$. That set is closed by Proposition 5.13.7; hence it contains the closure of every selected relative interior. It is therefore a union of closed cells, together with all their faces. □

Theorem 5.16.6 (Initial degenerations describe local stars)**.** *Let $X \subset \operatorname{Spec} K[M]$ be a closed subscheme and $w \in \mathrm{Trop}(X)$. Give the residue field of an initial degeneration $Y = \mathrm{in}_w(X)$ the trivial absolute value. Then*

$$\mathrm{Trop}(Y) = \mathrm{Star}_w \mathrm{Trop}(X).$$

Here the right side consists of those u for which $w + \varepsilon u \in \mathrm{Trop}(X)$ for all sufficiently small positive ε. More precisely, for every fixed u, the schemes $\mathrm{in}_{w+\varepsilon u}(X)$ and $\mathrm{in}_u(Y)$ are equivalent after residue extension and translation for all sufficiently small positive ε.

Proof. Homogenize as in Theorem 5.16.5, and write $\omega = (0,w)$ and $v = (0,u)$. Choose the common degree bound D of Proposition 5.16.3, also valid for the further initial ideals. In every degree at most D, minimize the pair

$$\left(p_B(\omega),\ \sum_{b\in B}\langle b, v\rangle\right).$$

There are finitely many pairs. Thus a lexicographically minimizing basis is minimizing for $p_B(\omega+\varepsilon v)$ whenever $\varepsilon > 0$ is sufficiently small. Cramer's rule then says that its surviving row entries are exactly those which first survive at ω and then have minimal v-weight. After the first reduction these rows

are a basis of the degree piece of $\mathrm{in}_\omega(H)$; their second reduction is therefore its degree piece at v, by Lemma 5.16.4.

We spell out the coefficient comparison. Fix a complete field L and t of value ω. Adjoin Gauss variables q_i of values εv_i to L and complete their fraction field, obtaining E. Use tq for the one-step translation. For a row entry c_{bm} put $a = m - b$. An entry survives both operations precisely when

$$v(c_{bm}) + \langle a, \omega \rangle = 0, \qquad \langle a, v \rangle = 0.$$

Its coefficient after the first reduction is $\overline{c_{bm} t^a} \in \widetilde{L}$. The ordinary weight initial form over the trivially valued field retains this coefficient. The one-step coefficient in $\widetilde{E}$ is $\overline{c_{bm} t^a}\, \overline{q^a}$. Thus the discrepancy is exactly the character $a \mapsto \overline{q^a}$ on the subgroup generated by the surviving exponent differences. Extend this character to the whole exponent lattice by the finite root-adjunction argument in the preceding theorem. A single torus translation then compares all the row generators.

For completeness, the ordinary initial ideal obtained by retaining the terms of least v-weight agrees, up to this same kind of translation, with the valued-extension definition of in_v over a trivially valued field. Indeed, introduce Gauss variables s_i of values v_i and divide each selected row by its pivot monomial. For a retained exponent difference a one has $\langle a, v \rangle = 0$, so the only new coefficient factor is the character $a \mapsto \overline{s^a}$. This also extends after algebraic residue extension. The common degree bound identifies the full homogeneous ideals from the compared rows. Localizing and dehomogenizing gives the claimed scheme equivalence. The nonemptiness criterion of Theorem 5.13.6, over both fields, gives the equality of supports. □

References. The classical finite Gröbner-complex and local-star theorems are discussed in [Gub13, Sections 10 and 12–13]. The proof here uses finite-dimensional weighted elimination, a Hilbert-function degree bound, and the analytic fundamental theorem already proved above.

5.17 Pushforwards and the projection formula

For an affine map, the same letter denotes its linear part when applied to direction vectors or direction lattices.

Lemma 5.17.1 (The index along a facet)**.** *Let $f : N \to N'$ be a lattice homomorphism, and let $\tau \prec \sigma$ be rational polyhedra of dimensions $d-1, d$. Suppose f is injective on $N_{\sigma,\mathbf{R}}$ and that $\tau' = f(\tau)$ is a facet of $\sigma' = f(\sigma)$. Put*

$$a_\sigma = [N'_{\sigma'} : f(N_\sigma)], \qquad a_\tau = [N'_{\tau'} : f(N_\tau)].$$

There is a positive integer $m_{\sigma/\tau}$ such that

$$f(u_{\sigma/\tau}) = m_{\sigma/\tau} u_{\sigma'/\tau'} \mod N'_{\tau'}, \qquad a_\sigma = a_\tau m_{\sigma/\tau}.$$

Proof. Injectivity and saturation give $f(N_\sigma) \cap N'_{\tau'} = f(N_\tau)$: the inverse image of $N'_{\tau',\mathbf{R}}$ inside $N_{\sigma,\mathbf{R}}$ is $N_{\tau,\mathbf{R}}$. The image of the primitive outward generator in the rank-one quotient is a positive integer $m_{\sigma/\tau}$ times the target generator. The finite groups fit into the exact sequence

$$0 \longrightarrow N'_{\tau'}/f(N_\tau) \longrightarrow N'_{\sigma'}/f(N_\sigma) \longrightarrow N'_{\sigma'}/(N'_{\tau'} + f(N_\sigma)) \longrightarrow 0.$$

The final group has order $m_{\sigma/\tau}$, so taking orders proves the equality. □

Theorem 5.17.2 (Balanced pushforward)**.** *Let C be a finite balanced real d-cycle in $N_{\mathbf{R}}$ and let $f : N_{\mathbf{R}} \to N'_{\mathbf{R}}$ be an integral affine map. There is a well-defined balanced d-cycle f_*C. After subdivision so that images are cells, its weight on a d-dimensional target cell σ' is*

$$w_{f_*C}(\sigma') = \sum_{\dim \sigma = d,\ f(\sigma) = \sigma'} w_C(\sigma)[N'_{\sigma'} : f(N_\sigma)]. \tag{P}$$

Facets on which the dimension drops contribute zero. Pushforward is linear and functorial under composition of integral affine maps.

Proof. Images of rational polyhedra are rational polyhedra: eliminate one coordinate at a time from their finite systems of affine inequalities, pairing a lower and an upper bound for that coordinate. Equalities are eliminated by substitution. This finite elimination also permits real constants. Choose a common target subdivision of all images and pull its defining hyperplanes back to the source. On every facet where f is injective, source cells correspond bijectively to the target cells lying in its image. Formula (P) therefore defines a finite weighted complex. Adding more hyperplanes merely subdivides it with the same generic weights; two choices are compared by their combined hyperplanes. This also proves invariance under a source subdivision.

Fix a target face τ' of dimension $d-1$ and sum over all source faces τ of dimension $d-1$ with $f(\tau)=\tau'$. Multiply the balancing equation at τ, after applying the linear part of f, by $a_\tau=[N'_{\tau'}:f(N_\tau)]$. For an incident full-rank facet σ, Lemma 5.17.1 replaces its term by $w_C(\sigma)a_\sigma u_{f(\sigma)/\tau'}$. If f has rank less than d on σ, its normal maps into $N'_{\tau',\mathbf{R}}$ and gives zero in the normal quotient. Each full-rank σ with $f(\sigma)\supset\tau'$ occurs exactly once: the inverse image of the target facet is its unique facet mapping to τ'. The sum is consequently the balancing equation for the weights (P). This proves balancedness, including faces bordering a zero-weight region.

Linearity follows from the sum. For a second map g, a facet contributing to $(g\circ f)_*C$ has full rank at both stages, while a dimension drop cannot be reversed. On a contributing facet the two finite indices multiply, since g is injective on the relevant direction space. After a common subdivision, summing these products proves $(g\circ f)_*=g_*f_*$. □

Theorem 5.17.3 (The corner projection formula)**.** *Let f and C be as in Theorem 5.17.2, and let φ be a continuous piecewise affine function on the target. Then*

$$f_*(D_{\varphi\circ f}C)=D_\varphi(f_*C).$$

Proof. Refine the complexes so that all images are cells and φ is affine on each image. For a target $(d-1)$-cell τ', extend the linear part of $\varphi|_{\tau'}$ to a linear form ℓ on the target. On any source face τ mapping onto τ', use $\ell\circ f$ in the corner formula. For an incident facet σ of full image dimension, the difference $\varphi_{f(\sigma)}-\ell$ vanishes tangentially to τ'. Lemma 5.17.1 therefore gives

$$a_\tau\big(\varphi_{f(\sigma)}-\ell\big)\big(f(n_{\sigma/\tau})\big)=a_\sigma\big(\varphi_{f(\sigma)}-\ell\big)(n_{f(\sigma)/\tau'}).$$

A facet collapsed to dimension $d-1$ has $f(N_{\sigma,\mathbf{R}})=N'_{\tau',\mathbf{R}}$, so its term vanishes. A lower-rank facet cannot contain such a τ. Multiply by $w_C(\sigma)$ and sum over τ and its incident facets. As in the pushforward proof, every full-rank facet over a target facet containing τ' occurs once. The resulting sum is exactly the corner weight of f_*C at τ'. If no full-rank facet occurs, both sides have weight zero there. This proves equality of the cycles. □

Corollary 5.17.4 (A constant weighted degree)**.** *Let C be a finite balanced d-cycle and $f : N_{\mathbf{R}} \to \mathbf{R}^d$ an integral affine map. For a point b outside the images of all cells of dimension at most $d-1$ and all rank-deficient facets, the number*

$$\deg_f(C;b) = \sum_{x \in |C|,\ f(x)=b} w_C(\sigma_x)[\mathbf{Z}^d : f(N_{\sigma_x})]$$

is finite and independent of b. It is the constant weight of f_*C on $\mathbf{R}^d$, allowing zero.

Proof. Each permitted preimage is in a full-rank facet, and each such facet contains at most one preimage. Formula (P) gives the displayed sum. Extend a target subdivision to a subdivision of all of $\mathbf{R}^d$, assigning weight zero to the omitted regions. Across every codimension-one face, the two full-dimensional normal vectors are opposite primitive vectors, so balancing equates their weights. Any two generic points can be joined by a polygonal path avoiding the codimension-two faces and crossing only finitely many codimension-one faces: choose its intermediate vertices outside the finitely many affine subspaces causing a nontransverse crossing. For $d = 1$ an interval suffices; for $d = 0$ there is only one target point. The weight is therefore constant throughout the target. □

References. The lattice-index method is the one used by Gathmann–Kerber–Markwig, *Tropical fans and the moduli spaces of tropical curves*, Proposition 2.25, and by Allermann–Rau, *First steps in tropical intersection theory*, Proposition 4.8. The index and cancellation arguments have been given here for finite affine complexes and real weights.

5.18 Displacement through a standard Jacobian chart

Retain the standard algebra A and its integral coordinates from Lemma 5.15.1. Let $F = (F_1, \dots, F_m)$ be represented by restricted series in $X_0, \dots, X_r, T, Y_1, \dots, Y_m$ with coefficients in K°. Write $J = \partial F/\partial Y$ and consider the nonzero affinoid algebra

$$B = A\langle Y_1, \dots, Y_m, Q\rangle/(F_1, \dots, F_m, Q \det J - 1). \tag{J}$$

We use its displayed quotient norm. In particular the coordinate images and $J^{-1} = Q\,\mathrm{adj}(J)$ have norm at most one. All the statements below keep this presentation over the original valued field.

Lemma 5.18.1 (Hensel lifting at a Banach-algebra center)**.** *Let C be a complete non-Archimedean Banach algebra, let $G \in C\langle V_1, \dots, V_m\rangle^m$ have norm at most one, and let $v \in C^m$ have norm at most one. Suppose $D = G'(v)$ is invertible with $\|D^{-1}\| \le 1$ and $\|G(v)\| \le \delta < 1$. There is a unique root $v + h$ with $\|h\| < 1$, and it satisfies $\|h\| \le \delta$. Its Jacobian inverse has norm at most one. Moreover, if two such systems differ in coefficient norm by at most ϵ, and their roots both lie at distance less than one from the same center with an inverse Jacobian of norm at most one for the first system at that center, their roots differ by at most ϵ.*

Proof. Translation by the bounded tuple v is a contractive substitution. Expanding gives

$$G(v + H) = e + DH + N(H), \qquad e = G(v),$$

where N has terms of degree at least two and coefficient norms at most one. This assertion follows first for polynomials by the binomial formula, then for restricted series by uniform truncation. For $\|h\|, \|k\| \le \rho < 1$, telescoping each monomial gives

$$\|N(h)\| \le \|h\|^2, \qquad \|N(h) - N(k)\| \le \rho\|h - k\|.$$

Consequently $h \mapsto -D^{-1}e - D^{-1}N(h)$ maps the closed ball of radius δ to itself and is a contraction of ratio at most δ. Starting at zero, successive differences tend to zero geometrically; completeness gives a fixed point. Any other root at distance less than one lies with it in a ball of some radius $\rho < 1$, where the same estimate proves equality. The Jacobian changes by a matrix of norm at most δ, so a matrix geometric series gives the asserted inverse bound.

For the perturbation assertion, let the two roots be z, w, and write the first system as G. Telescoping its translated monomials gives

$$G(z) - G(w) = (D + E)(z - w), \qquad \|E\| \le \max(\|z - v\|, \|w - v\|) < 1.$$

The matrix $D + E$ has inverse norm at most one. One term on the left is zero; the other has norm at most ϵ because w is a root of the second system. This proves the bound. All constructions use Banach-algebra operations, so nilpotents are retained. □

Proposition 5.18.2 (The bounded displacement expansion)**.** *Put $U = (S_1, \dots, S_r, Z_1, \dots, Z_q)$. There are bounded K-linear maps $D_\nu : B \to B$, $\nu \in \mathbb{N}^{r+q}$, with $D_0 = \mathrm{id}$ and $\|D_\nu b\| \le \|b\|$, such that for every $0 < \delta < 1$ the series*

$$\sigma_\delta(b) = \sum_\nu D_\nu(b) U^\nu \in B\{\delta^{-1}U\}$$

defines a contractive homomorphism, extending the base displacement

$$X_i \mapsto X_i(1 + S_i)\ (i \ge 1), \quad X_0 \mapsto X_0 \prod_i (1 + S_i)^{-1}, \quad T_j \mapsto T_j + Z_j.$$

It is characterized by this base formula and $\|\sigma_\delta(Y) - Y\| < 1$. In fact

$$\|\sigma_\delta(b) - b\| \le \delta\|b\|, \qquad D_\nu(bc) = \sum_{\alpha+\beta=\nu} D_\alpha(b) D_\beta(c).$$

Proof. Work in $C_\delta = B\{\delta^{-1}U\}$. The base formulas define a contractive homomorphism by geometric inversion and bounded substitution. On each integral coordinate the difference from its original image has norm at most δ. Telescoping monomials, passing to restricted series, and taking quotient infima therefore give $\|\sigma_\delta(a) - a\| \le \delta\|a\|$ for $a \in A$.

Apply the base displacement to the coefficients of F. At the center $Y \in B^m$ the new error has norm at most δ, since $F(Y) = 0$. Its Jacobian differs from $J(Y)$ by a matrix of norm at most δ. As $J(Y)^{-1} = Q\,\mathrm{adj}(J(Y))$ has norm at most one, the new Jacobian also has inverse norm at most one. Lemma 5.18.1 gives the unique root $Y + h_\delta$, with $\|h_\delta\| \le \delta$. The determinant at that root has an inverse Q_δ of norm at most one and $\|Q_\delta - Q\| \le \delta$, again by the geometric inverse formula. These images kill all relations in (J) and give a contractive homomorphism $B \to C_\delta$. The same telescoping and quotient-infimum argument gives its displayed difference bound on every $b \in B$.

If $\delta' \le \delta < 1$, restriction from radius δ to radius δ' carries this root to a root at distance less than one from Y. Uniqueness identifies it with the root constructed at radius δ'. Thus all coefficient expansions agree. Evaluation at $U = 0$ is the identity, by the same uniqueness. Write their coefficients as $D_\nu(b)$. Contractivity at every radius gives

$$\|D_\nu(b)\|\delta^{|\nu|} \le \|b\|.$$

Letting δ increase to one proves $\|D_\nu(b)\| \le \|b\|$. The homomorphism identity at any positive radius, together with uniqueness of coefficients, gives the convolution identity and K-linearity. □

Theorem 5.18.3 (The strict-time analytic lift)**.** *For $x \in \mathcal{M}(B)$ and $0 \le t < 1$, the formula*

$$|b(H_B(x,t))| = \max_{\nu} |D_\nu(b)(x)| t^{|\nu|}, \qquad 0^0 = 1, \tag{E}$$

defines a jointly continuous map $H_B : \mathcal{M}(B) \times [0,1) \to \mathcal{M}(B)$ with $H_B(x,0) = x$. For the structure map $f : \mathcal{M}(B) \to V$,

$$f(H_B(x,t)) = H(f(x),t).$$

If b has quotient norm at most one, then $|b(H_B(x,t))| < 1$ if and only if $|b(x)| < 1$. Thus every strict-time trajectory preserves its reduction prime in the displayed integral model.

Proof. For $t > 0$, evaluate the coefficients of $\sigma_\delta(b)$ in $\mathcal{H}(x)$ with $t < \delta < 1$ and then use the multiplicative radius-t Gauss norm. This gives a bounded homomorphism into a complete Gauss field and hence an actual analytic point. Its norm is (E). At $t = 0$ the constant coefficient gives x. The base-coordinate formulas identify the composite with f.

On $\mathcal{M}(B) \times [0,\delta]$, choose $\delta < \delta' < 1$. The restricted series $\sigma_{\delta'}(b)$ has uniformly small tails on this product. Each finite truncation of (E) is a finite maximum of continuous functions, so its uniform limit is continuous. These intervals cover $[0,1)$ and prove joint continuity. For a norm-at-most-one element b, its constant coefficient is $b(x)$ and its nonconstant terms have norms at most t. Thus a constant coefficient of norm one stays of norm one; if it is smaller than one, every term is bounded by $\max(|b(x)|, t) < 1$. This proves the reduction assertion. □

Proposition 5.18.4 (Functoriality and the composition law)**.** *A contractive homomorphism between two presentations (J) over the same standard base commutes with their coefficient operators D_ν. The lifts therefore commute with such morphisms, with integral Laurent restrictions, and with complete isometric ground-field extension. They satisfy*

$$H_B(H_B(x,s),t) = H_B(x, \max(s,t)) \qquad (0 \le s,t < 1).$$

Proof. Let $\psi : B \to C$ be the homomorphism in question. Applying ψ to the lifted Y-tuple gives a root of the base-displaced B-equations in $C\{\delta^{-1}U\}$. Applying the C-displacement to $\psi(Y)$ gives another such root. Both differ

from $\psi(Y)$ by norm at most δ, and the original Jacobian inverse has norm at most one. Lemma 5.18.1 identifies them. The inverse determinant and then every element of B agree by bounded substitution. Comparing coefficients proves the assertion. The same proof treats a complete isometric scalar extension using the identical equations.

For an integral Laurent restriction adjoining g^{-1}, both g and g^{-1} have norm at most one. The displacement of g differs from g by norm at most δ; factoring out g and using geometric inversion extends the map uniquely to the inverse, with the same bound. This proves the restriction assertion. It also shows that two bounded presentations identified over the fixed base give the same displacement.

For the composition law, take independent variables (S, Z) and (S', Z') of radii $s, t < 1$. The two successive lifts solve the base-displaced equations whose relative coordinates are

$$W_i = S_i + S'_i + S_i S'_i, \qquad V_j = Z_j + Z'_j.$$

The resulting Y-tuple differs from the original one by norm at most $\max(s, t) < 1$. The single lift evaluated at (W, V) has the same property and solves the same equations. Hensel uniqueness identifies the two tuples. The isometric Gauss-coordinate change in Proposition 5.15.3 then identifies their norms with the single displacement of radius $\max(s, t)$. This gives the identity for all $b \in B$, and the zero-parameter cases are immediate. □

Lemma 5.18.5 (The pointwise endpoint)**.** *For every $x \in \mathcal{M}(B)$, the formula*

$$|b(H_B(x, 1))| = \sup_{\nu} |D_\nu(b)(x)| = \lim_{t \uparrow 1} |b(H_B(x, t))|$$

defines a point of $\mathcal{M}(B)$ and extends its trajectory continuously to $[0, 1]$. Each endpoint evaluation is lower semicontinuous as a function of x. The endpoint commutes with all strict-time maps and is idempotent:

$$H_B(H_B(x, 1), t) = H_B(x, 1) = H_B(H_B(x, t), 1), \qquad 0 \le t \le 1.$$

Proof. For each b, the strict-time value is nondecreasing and bounded by $\|b\|$. Its limit is the displayed coefficient supremum: each fixed coefficient is attained in the limit, and their supremum bounds every time value. Passing to limits in the multiplicative, ultrametric, and scalar identities proves that this is a bounded multiplicative seminorm extending the given absolute value.

Evaluation topology gives convergence to its point, hence continuity of the trajectory at time one. A supremum of continuous coefficient evaluations is lower semicontinuous.

Fix $t < 1$. Continuity of the strict-time map in x and Proposition 5.18.4 allow $s \uparrow 1$ in $H_B(H_B(x,s),t) = H_B(x,\max(s,t))$. The resulting identity is $H_B(H_B(x,1),t) = H_B(x,1)$. In the opposite order, pointwise convergence applied to the fixed point $H_B(x,t)$ gives $H_B(H_B(x,t),1) = H_B(x,1)$. Finally let $t \uparrow 1$ in the first equality, again using the continuous trajectory of the fixed point $H_B(x,1)$, to get idempotence. □

References. The construction is an explicit Banach-algebra Hensel lift of mixed Gauss displacement. The norm estimates are uniform in the real parameter below one.

5.19 Pure dimension and geometric weights

The initial scheme is taken in the sense of Theorem 5.13.6. A scheme is pure of dimension d if every irreducible component of its underlying space has dimension d; embedded components are allowed.

Lemma 5.19.1 (Finite filtrations and zero-dimensional algebras)**.** *Every finite module over a Noetherian ring has a finite filtration whose nonzero successive quotients are $A/\mathfrak{p}$ for prime ideals $\mathfrak{p}$. A zero-dimensional algebra of finite type over a field k is finite-dimensional over k.*

Proof. Finite free modules over a Noetherian ring are Noetherian by induction on their rank, using the first-coordinate ideal and the kernel of that coordinate; hence their quotients are Noetherian. In a nonzero finite module choose a nonzero element whose annihilator is maximal among annihilators of nonzero elements. Such a maximum exists by the ascending chain condition. Its annihilator $\mathfrak{p}$ is prime: if ab annihilates the element and b does not, the annihilator of its nonzero b-multiple is the same maximal annihilator, so $a \in \mathfrak{p}$. This yields a submodule $A/\mathfrak{p}$. Repeat in the quotient. The resulting ascending chain of submodules terminates.

For the second assertion, finite-type k-algebras are Noetherian by [StacksAlg, Tag 00FN]. Apply the filtration to the algebra itself. Each prime quotient is a zero-dimensional finite-type domain. Its fraction field has transcendence degree zero by [StacksAlg, Tag 00P0]. Its finitely

many algebra generators are therefore algebraic over k, so the quotient is a finite-dimensional field. A finite extension of these vector spaces is finite-dimensional. □

Proposition 5.19.2 (Pure dimension of initial schemes)**.** *Let $X \subset \operatorname{Spec} K[M]$ be pure of dimension d. For every $w \in \operatorname{Trop}(X)$, every initial scheme $\operatorname{in}_{w,t,L}(X)$ is pure of dimension d.*

Proof. By [StacksAlg, Tag 02NK], extension from K to L preserves pure dimension. Translate by t and work at weight zero over L. Write $R = L^\circ$, $k = \widetilde{L}$ and $I \subset L[M]$ for the translated ideal. Its finitely many minimal primes are $P_1, \dots, P_s$. Let

$$B = R[M]/(I \cap R[M]), \qquad B_i = R[M]/(P_i \cap R[M]).$$

Each B_i is a domain of finite type containing R, and its generic fibre is the dimension-d domain $L[M]/P_i$. The exact valuation-ring dimension theorem [StacksAlg, Tag 00QK] says that, whenever $B_i \otimes_R k$ is nonzero, its spectrum is equidimensional of dimension d. Its hypotheses require finite type, not finite presentation or a discrete valuation.

The identity

$$\sqrt{I \cap R[M]} = \sqrt{I} \cap R[M] = \bigcap_i (P_i \cap R[M])$$

shows that $\operatorname{Spec} B$ is the union of the closed subsets $\operatorname{Spec} B_i$. Taking the inverse image of this finite union in the closed fibre shows that $\operatorname{Spec}(B \otimes_R k)$ is the union of the nonempty pure dimension-d special fibres just described. A finite union of pure dimension-d closed subsets is pure of dimension d: every irreducible component is a maximal irreducible component among them, and a proper containment of irreducible closed subsets strictly lowers their finite dimensions. The closed fibre is nonempty by Theorem 5.13.6. □

Proposition 5.19.3 (The tangent torus of an initial stratum)**.** *Use the finite complex constructed in Theorem 5.16.5. Let w lie in the relative interior of a cell $\tau \subset \operatorname{Trop}(X)$, put $L_\tau = \operatorname{span}_{\mathbf{R}}(\tau - \tau)$ and $M_0 = M \cap L_\tau^\perp$. The ideal of $Y = \operatorname{in}_w(X)$ is generated by elements of $k[M_0]$, after multiplication of its generators by unit characters. Consequently it is extended from its intersection with $k[M_0]$, and*

$$Y \simeq T_{N_\tau} \times Z_\tau, \qquad Z_\tau = \operatorname{Spec}\big(k[M_0]/(I_Y \cap k[M_0])\big), \quad N_\tau = N \cap L_\tau.$$

The product uses a lattice splitting, while the quotient scheme Z_τ is intrinsic to Y and N_τ.

Proof. Use the finitely many homogeneous pivot rows in the proof of Theorem 5.16.5, and divide each row by its pivot character. A surviving exponent difference a satisfies one of the affine Plücker equalities defining the cell. That equality holds throughout the relative interior of τ, hence its linear part vanishes on L_τ. Thus $a \in M_0$ after dehomogenization. These rows generate the initial Laurent ideal, so it is extended from an ideal $J_0 \subset k[M_0]$.

The lattice N_τ is saturated. Lemma 5.14.1 splits $N = N_\tau \oplus Q$, and dually $k[M]$ is a Laurent polynomial ring over $k[M_0]$. Comparing coefficients in these additional Laurent variables shows $J_0 k[M] \cap k[M_0] = J_0$. The quotient ring is therefore $(k[M_0]/J_0)[N_\tau^\vee]$, proving the assertion including its scheme structure. □

Lemma 5.19.4 (A zero tropical support forces dimension zero)**.** *Let Z be a nonempty closed subscheme of a torus over a complete field k. If $\mathrm{Trop}(Z) \subseteq \{0\}$, then Z is zero-dimensional and its coordinate algebra is finite over k.*

Proof. Suppose an integral component has positive dimension, with function field F. By [StacksAlg, Tag 00P0], F/k has positive transcendence degree. At least one of its torus-coordinate generators, say z_1, is transcendental over k. Extend z_1 to a maximal algebraically independent subset $z_1, \dots, z_e$ of a finite field-generating set. Then F is finite over $k(z_1, \dots, z_e)$. Give the latter field its Gauss absolute value with $|z_1| \neq 1$ and arbitrary positive radii on the other variables, and complete it to E. The Gauss norm is multiplicative by Proposition 1.2.2.

The finite nonzero E-algebra $D = F \otimes_{k(z_1,\dots,z_e)} E$ has a maximal proper ideal: choose one of maximal E-dimension. Its quotient is a finite field extension E'/E, which has the extending complete absolute value by Lemma 1.12.1. The induced map $F \to E'$ is injective, since it is a unital map from a field. Restricting its norm to the coordinate algebra gives an analytic point of Z by Proposition 2.10.1, with $|z_1| \neq 1$. This contradicts the tropical-support hypothesis. All components therefore have dimension zero. Apply Lemma 5.19.1 to obtain finiteness over k. □

Theorem 5.19.5 (Pure dimension of algebraic tropicalization)**.** *If $X \subset \operatorname{Spec} K[M]$ is nonempty and pure of dimension d, then $\operatorname{Trop}(X)$ is a nonempty finite pure d-dimensional integral $v(K^\times)$-affine polyhedral complex. At the relative interior of each maximal cell σ, the quotient Z_σ of Proposition 5.19.3 is a nonempty finite scheme over the residue field.*

Proof. The Gauss-field construction in Lemma 5.19.4, with arbitrary radii and allowing an empty transcendence basis, gives an analytic point on each nonempty integral component. Thus the tropicalization is nonempty. The finite complex is Theorem 5.16.5. Fix a maximal cell σ, a relative-interior point w, and put $s = \dim \sigma$. The local star at w is the vector space L_σ. By Theorem 5.16.6, $Y = \operatorname{in}_w(X)$ has tropical support L_σ. By Proposition 5.19.3, $Y = T_{N_\sigma} \times Z_\sigma$. Every analytic point of Z_σ gives a point of this product by assigning value one to the first torus coordinates. It follows that $\operatorname{Trop}(Z_\sigma) \subseteq \{0\}$. The quotient is nonempty, since Y is nonempty. Lemma 5.19.4 makes Z_σ finite.

Every component of $T_{N_\sigma} \times Z_\sigma$ has dimension s: a prime quotient of its finite coefficient algebra is a finite field extension, and its Laurent polynomial ring has transcendence degree s, so [StacksAlg, Tag 00P0] applies. Proposition 5.19.2 gives dimension d for the same components. Thus $s = d$. This holds for every maximal cell. □

Definition 5.19.6 (The geometric tropical weight)**.** At $w \in \operatorname{relint}(\sigma)$ as in Theorem 5.19.5, define

$$m_X(\sigma) = \dim_k \Gamma(Z_\sigma, \mathcal{O}_{Z_\sigma}).$$

Proposition 5.19.7 (Geometric multiplicity and its invariance)**.** *The weight is a positive integer. It is independent of the chosen weight-realizing field, translating point, and lattice splitting. It equals the sum of the generic lengths of the irreducible components of the initial scheme after a residue extension splitting its finite transverse algebra. In particular it equals the usual sum of geometric generic lengths over an algebraic closure.*

Proof. Let $A = \Gamma(Z_\sigma, \mathcal{O})$, a nonzero finite-dimensional k-algebra. Finite-dimensional scalar extension preserves its dimension. A finite field extension k'/k splits the characteristic polynomials of a finite set of algebra generators of A. Every residue field of $A \otimes_k k'$ is then k', because the images of those

generators are roots in k'. Choose a composition series by successively taking a proper submodule of maximal vector-space dimension. Its factors are the one-dimensional residue fields. Localizing the series at each maximal ideal counts each factor exactly once, and gives

$$\dim_k A = \dim_{k'}(A \otimes_k k') = \sum_z \operatorname{length}\big((A \otimes_k k')_z\big).$$

Adjoining Laurent variables and localizing at a generic point tensors the local Artinian factor with the rational function field of the corresponding torus. Its composition series tensors to a composition series with the same number of simple factors. Thus the displayed summands are exactly the generic lengths of $Y_{k'}$. The intrinsic quotient description in Proposition 5.19.3 makes the lattice splitting irrelevant. Theorem 5.13.6 compares all other choices by residue extension and a torus translation; translation preserves the tangent subtorus and induces a translation on its quotient. The same finite dimension is therefore obtained from every choice. □

Proposition 5.19.8 (Constancy, refinement, and complete base change)**.** *The integer $m_X(\sigma)$ is constant on the relative interior of each maximal initial stratum. The resulting weighted polyhedral complex is independent of its choice of initial-stratum refinement, and every complete isometric extension of K preserves its support and weights.*

Proof. Theorem 5.16.5 compares the initial schemes at two relative-interior points by a single torus translation after residue extension. The cell has the same tangent lattice at both points, so the comparison descends to their transverse finite quotients. Proposition 5.19.7 equates the dimensions. A subdivision retains the tangent lattice on its dimension-d pieces and therefore retains the weights. Any two initial-stratum complexes have a common refinement; at a point belonging to both relative interiors their weights are computed from the identical initial scheme and tangent lattice. Finally, the initial scalar-extension identity in Proposition 5.13.4, together with the common-extension comparison in Theorem 5.13.6, gives the same finite quotient after residue extension. Its dimension is unchanged. □

References. The classical pure-dimension and multiplicity statements are reconstructed here from the exact valuation-ring dimension theorem of the Stacks Project, the finite Plücker construction above, and the finite transverse algebra of an initial stratum.

5.20 Stable intersection from the diagonal

Retain the finite real cycle convention of Section 5.14. Put $n = \operatorname{rank} N$. Choose a lattice basis of $N^\vee$, and let x_i, y_i be the corresponding coordinates on $N_{\mathbf{R}} \times N_{\mathbf{R}}$. Set

$$h_i(x,y) = \max\{0, x_i - y_i\}, \qquad \mathcal{D}_{xy} = D_{h_1} \cdots D_{h_n},$$
$$\Delta = \{(x,x) : x \in N_{\mathbf{R}}\}.$$

Lemma 5.20.1 (A basis-independent diagonal operator)**.** *For every balanced cycle Z in $N_{\mathbf{R}} \times N_{\mathbf{R}}$, the cycle $\mathcal{D}_{xy}Z$ is balanced, is supported in $|Z| \cap \Delta$, and is independent of the chosen basis of $N^\vee$. Interchanging x and y preserves this operator.*

Proof. By Theorem 5.14.5, successive corners are balanced and their order does not matter. A corner by $\max(0, \ell)$ is supported where $\ell = 0$, since off that hyperplane the function is affine. It follows inductively that the final support lies in Δ.

Every change of integral basis is a composition of row interchanges, sign changes, and addition of an integer multiple of one row to another. Row interchanges use commutation of corners. A sign change replaces $\max(0, \ell)$ by $\max(0, -\ell) = \max(0, \ell) - \ell$, which has the same corner operator. For a row operation $\ell_i \mapsto \ell_i + a\ell_j$, perform the j-th corner first. Its support is contained in $\ell_j = 0$, where the old and new i-th functions coincide. Restriction locality, Lemma 5.14.6, proves equality. The elementary row operations generate $\mathrm{GL}_n(\mathbf{Z})$ by Lemma 5.14.1, so this proves basis independence. Interchanging x, y changes each linear form to its negative and is already covered. □

Definition 5.20.2 (The stable product)**.** For balanced cycles C, E of dimensions p, q in $N_{\mathbf{R}}$, define

$$C \cdot E = \pi_*\big(\mathcal{D}_{xy}(C \times E)\big), \qquad \pi(x,y) = x.$$

The projection on the support in Δ is a lattice isomorphism. The product has dimension $p + q - n$, and is zero when that number is negative.

Theorem 5.20.3 (The ring of balanced cycles)**.** *The stable product is bilinear, commutative, associative, and compatible with weighted subdivision and integral affine lattice isomorphisms. Its unit is* $[N_{\mathbf{R}}]$ *with weight one, and* $|C \cdot E| \subseteq |C| \cap |E|$*.*

Proof. Bilinearity and refinement invariance follow from those of products, corners, and pushforwards. The preceding lemma gives the support assertion, commutativity, and invariance under changes of lattice basis. Simultaneous translation of x, y leaves their differences unchanged, proving affine invariance.

For the unit, begin with $N_{\mathbf{R}} \times E$. Impose $x_i = y_i$ one coordinate at a time by its corner. Before the i-th step, x_i is still a free lattice coordinate, so the two sides of $x_i - y_i = 0$ have opposite primitive normals and the slope jump of $\max(0, x_i - y_i)$ is exactly one. Each facet thus keeps the weight of its E factor. At a face inherited from E, the function has a common affine extension on either side away from that equality and contributes zero. The final cycle is the diagonal copy of E with its original weights. Projection therefore gives $[N_{\mathbf{R}}] \cdot E = E$.

For associativity, work on $C \times E \times F$ with coordinates x, y, z. By product locality, imposing the $x = y$ diagonal first gives the diagonal copy of $(C \cdot E) \times F$. On this support the functions $\max(0, x_i - z_i)$ and $\max(0, y_i - z_i)$ agree. Hence the lift of $(C \cdot E) \cdot F$ to the small diagonal is

$$\mathcal{D}_{yz}\mathcal{D}_{xy}(C \times E \times F).$$

Starting instead with $E \cdot F$ gives $\mathcal{D}_{xy}\mathcal{D}_{yz}(C \times E \times F)$. The corner operators commute, so the two cycles agree before projection to x. If an intermediate dimension is negative, both expressions vanish by the same commuting-operator calculation. This proves associativity in all degrees. □

Lemma 5.20.4 (Complementary fans and displacement)**.** *Let* A, B *be finite balanced fans in* $N_{\mathbf{R}}$*, of dimensions* p, q *with* $p+q = n$*. For a vector* b *outside the finitely many lower-dimensional images of faces under* $(x, y) \mapsto x - y$*, the coefficient of their stable product at the origin is*

$$\sum_{\substack{\sigma \in A^{(p)},\ \tau \in B^{(q)} \\ b \in \operatorname{relint}(\sigma - \tau) \\ N_{\sigma,\mathbf{R}} + N_{\tau,\mathbf{R}} = N_{\mathbf{R}}}} w_A(\sigma) w_B(\tau) [N : N_\sigma + N_\tau]. \tag{F}$$

In particular this sum is independent of the generic vector b, including for signed real weights.

Proof. Put $Z = A \times B$ and $f(x, y) = x - y$. Its dimension is n. By Corollary 5.17.4, $f_*Z = c[N_{\mathbf{R}}]$, where c is the generic weighted degree. On a full-rank product facet, f is injective because the two direction spaces have dimensions adding to n. Its lattice image is $N_\sigma + N_\tau$, and a generic inverse image exists precisely when $b \in \operatorname{relint}(\sigma - \tau)$. Thus c is the displayed sum.

On $N_{\mathbf{R}}$ let $g_i(z) = \max(0, z_i)$. Repeated use of Theorem 5.17.3 gives

$$f_*\big(D_{g_1 \circ f} \cdots D_{g_n \circ f} Z\big) = D_{g_1} \cdots D_{g_n}\big(c[N_{\mathbf{R}}]\big) = c[0].$$

The final equality is the successive unit slope jumps of the coordinate hyperplanes. The source is a zero-dimensional fan supported on the diagonal, so it is a multiple of the origin. Pushforward does not change a zero-dimensional point weight. By the definition of the diagonal product this multiple is also the coefficient of $A \cdot B$. It is therefore c, as required. □

Theorem 5.20.5 (The local fan displacement formula)**.** *Let C, E be finite balanced real cycles in $N_{\mathbf{R}}$, of dimensions p, q, and put $r = p + q - n \geq 0$. Choose a common polyhedral refinement of their intersection and let γ be an r-dimensional face. At $a \in \operatorname{relint}(\gamma)$, take the two local stars and quotient by $L = N_{\gamma,\mathbf{R}}$. Their dimensions are complementary in $N_{\mathbf{R}}/L$. The weight of $C \cdot E$ on γ is the sum (F) for these quotient fans. Equivalently it is the weighted small-displacement limit of the transverse intersections of C and $E + \varepsilon b$, with lattice factors $[N : N_\sigma + N_\tau]$.*

Proof. Stars commute with corners and products. Both stars at a have L as a lineality space: the minimal face of either original complex containing a contains the local directions of γ. Since N_γ is saturated, Lemma 5.14.1 allows us to choose $N = N_\gamma \oplus Q$. Write the two stars as $L \times A$ and $L \times B$. The diagonal cuts in the L coordinates give the diagonal copy of L with weight one, by the unit calculation in Theorem 5.20.3. The remaining cuts give $A \cdot B$. Their dimensions satisfy

$$\dim A + \dim B = (p - r) + (q - r) = n - r = \operatorname{rank} Q.$$

The preceding lemma computes the coefficient at their origin, which is the desired weight on γ. For every contributing pair of original facets, N_γ is contained in both direction lattices, so

$$[N/N_\gamma : (N_\sigma + N_\tau)/N_\gamma] = [N : N_\sigma + N_\tau].$$

Finally choose a small neighborhood of a meeting only the faces that contain its minimal faces. In this neighborhood the original cells are translates of their local tangent cones. For a generic b and sufficiently small $\varepsilon > 0$, a transverse displaced pair approaching a is therefore exactly a pair counted by the quotient-fan formula; the signs agree because $x \in C \cap (E + \varepsilon b)$ means $x - y = \varepsilon b$. Only finitely many cells occur, so the incidence conditions are constant for all sufficiently small positive ε. Summing their weights gives the claimed weighted limit. There is no additional support in the relative interior of an intersection cell of dimension $s > r$: quotienting its common s-dimensional lineality gives a product of dimension $r - s < 0$, which is zero. Cells of dimension less than r cannot carry an r-cycle. Thus refining the r-faces gives all coefficients of the cycle. □

Corollary 5.20.6 (Pointwise smooth weights)**.** *Suppose the weights of C, E are smooth functions on their cells, extending smoothly to their faces, and that their values satisfy balancing at every point of every face. Freezing the weights at each point and applying the local fan formula defines a smooth-weighted balanced product. It is independent of subdivision and generic displacement, and is bilinear, commutative, and associative.*

Proof. At a fixed point, the frozen stars are constant-weight balanced fans. Theorem 5.20.5 makes their product coefficient well-defined. On any fixed prospective intersection face, this coefficient is a finite sum of products of restrictions of the original weights multiplied by constant lattice indices. It is therefore smooth. At a lower face, freezing again identifies the normal equation with balancing of the product of the frozen fans. Refinement and the two bracketings of a triple product are compared by the same frozen fan calculation, using Theorem 5.20.3. The equalities hold at every point, hence as equalities of smooth coefficient functions. □

References. The diagonal construction is due to Allermann–Rau, *First steps in tropical intersection theory*, Section 9. For the stable fan description see Jensen–Yu, *Stable intersections of tropical varieties*, Section 2. Here basis independence is proved by elementary row operations, and the displacement formula by the balanced pushforward of the difference map.

5.21 Continuity at the endpoint of a Jacobian chart

Lemma 5.21.1 (Finite control of bounded coefficients)**.** *Let C be a nonzero affinoid algebra, and let $R \subseteq C$ be a Noetherian subring all of whose elements have norm at most one. For a family $c_\nu \in R$ indexed by $\mathbb{N}^d$, put*

$$q(x,t) = \max_\nu |c_\nu(x)| t^{|\nu|} \quad (0 \le t < 1), \qquad q(x,1) = \sup_\nu |c_\nu(x)|.$$

There is a finite set of indices E for which

$$q(x,1) = \max_{\nu \in E} |c_\nu(x)| \qquad (x \in \mathcal{M}(C)).$$

If the family is nonzero and $N = \max_{\nu \in E} |\nu|$, then $t^N q(x,1) \le q(x,t) \le q(x,1)$ for $0 < t \le 1$. Consequently q is jointly continuous on $\mathcal{M}(C) \times [0,1]$.

Proof. The ideal of R generated by all c_ν is finitely generated. Express a finite generating list in terms of the original family and take the union of its finite index sets; thus finitely many actual c_ν, $\nu \in E$, generate the same ideal. Every other coefficient is a finite sum of these generators multiplied by elements of R. Since all those multipliers have norm at most one, evaluation gives the claimed finite maximum at every x. If the ideal is zero, the conclusion is immediate. Otherwise retaining those same terms in $q(x,t)$ gives the lower bound $t^N q(x,1)$; the upper bound is its definition. In particular

$$0 \le q(x,1) - q(x,t) \le 1 - t^N.$$

At time one this is uniform convergence to a continuous finite maximum. For times bounded above by $\delta < 1$, the coefficient bound makes the tails at most δ^{n+1} after total degree n, so finite maxima converge uniformly and give joint continuity there as well. □

Lemma 5.21.2 (Polynomial implicit equations have finite endpoint control)**.** *Let C be an affinoid algebra receiving the standard algebra A contractively. Suppose the system $G \in K^\circ[X_0, \dots, X_r, T, Y]^m$ has a root $y \in C^m$ of norm at most one and its determinant has an inverse $q \in C$ of norm at most one. Lift this root under the base displacement by Lemma 5.18.1. For every polynomial $P \in K[X_0, \dots, X_r, T, Y, Q]$, the norm of P evaluated at the displaced base coordinates, the lifted root, and the inverse determinant extends jointly continuously to $\mathcal{M}(C) \times [0,1]$ by taking its coefficient supremum at time one.*

Proof. First assume the coefficients of P belong to K°. Let $R \subseteq C$ be the ring generated over the image of $\mathbb{Z}$ by the finitely many coefficients of G, P,

the base-coordinate images, the coordinates of y, and q. Every generator has norm at most one, hence so does every element of R. The ring $\mathbb{Z}$ is Noetherian, since each ideal is generated by its least positive element, or is zero. Its image is a quotient, and [StacksAlg, Tag 00FN] therefore makes R Noetherian.

We check that every formal displacement coefficient lies in this single ring. The base displacement has coefficients in R: the inverse factors $(1+S_i)^{-1}$ have integral coefficients. Write the lifted root as $y + h(U)$, with $h(0) = 0$. In each positive total degree, the coefficient of an as-yet-unknown term of h in $G_\sigma(y+h)$ is the original matrix $G'(y)$ times that coefficient; all remaining terms use only already determined lower degrees. The inverse matrix $q\,\mathrm{adj}(G'(y))$ has its entries in R. Induction on total degree therefore puts every coefficient of h in R. This formal recursion agrees with the analytic root at each radius less than one by uniqueness. Recursively inverting its determinant puts every coefficient of the new inverse determinant in R as well. Polynomial substitution then does the same for P.

Lemma 5.21.1 now applies to this coefficient family and proves the assertion. For arbitrary P, multiply it by a nonzero scalar making its finitely many coefficients integral, apply the result, and divide the real-valued norm function by the absolute value of that scalar. Such a scalar exists because the ground-field valuation is nontrivial. □

Theorem 5.21.3 (Joint continuity for restricted-series Jacobian charts)**.** *For every presentation* (J)*, the pointwise endpoint of Lemma 5.18.5 makes*

$$H_B : \mathcal{M}(B) \times [0,1] \longrightarrow \mathcal{M}(B)$$

jointly continuous. Its full homotopy commutes with the contractive morphisms, integral Laurent restrictions and complete isometric extensions of Proposition 5.18.4.

Proof. We first treat a polynomial representative $b = P(X,T,Y,Q) \in B$. Choose polynomial systems $F^{(n)} \in K^\circ[X,T,Y]^m$ with $\|F^{(n)} - F\| \le \epsilon_n < 1$ and $\epsilon_n \to 0$, using the fixed restricted-series representatives. Differentiation is contractive for the unit-radius Gauss norms, so the Jacobians are within the same bound. At the original tuple $Y \in B^m$, the system $F^{(n)}$ has error at

most ϵ_n and an inverse Jacobian of norm at most one. Thus Lemma 5.18.1, now in B, gives an exact root Y_n with

$$\|Y_n - Y\| \le \epsilon_n, \qquad Q_n = \det((F^{(n)})'(Y_n))^{-1},$$
$$\|Q_n\| \le 1, \qquad \|Q_n - Q\| \le \epsilon_n.$$

Displace the base and lift the two roots, obtaining Y_δ and $Y_{n,\delta}$ in $B\{\delta^{-1}U\}$ for $\delta < 1$. Both roots lie at distance at most $\max(\delta, \epsilon_n) < 1$ from the original tuple Y. Their displaced systems differ by at most ϵ_n, since base substitution is contractive. The perturbation part of Lemma 5.18.1 therefore gives, uniformly for every $\delta < 1$,

$$\|Y_{n,\delta} - Y_\delta\| \le \epsilon_n.$$

The determinant difference has the same bound, by telescoping and the derivative bound. Both inverses have norm at most one, so the identity $u^{-1} - v^{-1} = u^{-1}(v - u)v^{-1}$ gives $\|Q_{n,\delta} - Q_\delta\| \le \epsilon_n$. Let $P_n(U)$ denote P evaluated on the displaced base, $Y_{n,\delta}$ and $Q_{n,\delta}$. This definition is compatible with all radii. If C_P is the maximum absolute value of the coefficients of P, telescoping its finitely many monomials yields

$$\|P_n(U) - \sigma_\delta(b)\|_\delta \le C_P \epsilon_n \qquad (0 < \delta < 1). \tag{A}$$

By Lemma 5.21.2, the norm functions defined by $P_n(U)$, with coefficient suprema at time one, are jointly continuous on the full product. Inequality (A) bounds the differences of their real-valued norms from $|b(H_B(x,t))|$ by $C_P\epsilon_n$ for all x and $t < 1$. Passing to the already established pointwise limits gives the same bound at $t = 1$. Thus $|b(H_B(x,t))|$ is a uniform limit of continuous functions on the full product and is continuous there.

Finally polynomial representatives are dense in the quotient Banach algebra B. Contractivity of every strict-time evaluation and of the endpoint seminorm gives

$$\big|\,|b(H_B(x,t))| - |b'(H_B(x,t))|\,\big| \le \|b - b'\| \quad (0 \le t \le 1).$$

Uniform polynomial approximation proves continuity for every $b \in B$, hence joint continuity into its spectrum. The previously proved strict-time functorial identities pass to the endpoint by pointwise convergence. □

Theorem 5.21.4 (Retraction to the inverse-image skeleton)**.** *For the structure map $f : \mathcal{M}(B) \to V$, put $\Sigma_B = f^{-1}(\Sigma_a)$. Then H_B is a strong deformation retraction onto Σ_B. It satisfies the maximum-parameter composition law for all $s, t \in [0,1]$. Every fibre of $\Sigma_B \to \Sigma_a$ is finite.*

Proof. The fibre over any $z \in V$ is the affinoid spectrum of $B\widehat{\otimes}_A\mathcal{H}(z)$ by Theorem 2.33.11. The equations $F = 0$, $Q \det J - 1 = 0$ in the $m+1$ variables (Y, Q) have the block Jacobian

$$\begin{pmatrix} J & 0 \\ Q\,\partial(\det J)/\partial Y & \det J \end{pmatrix}, \qquad \det = (\det J)^2 \in B^\times.$$

The matrix criterion Theorem 2.41.7 therefore gives a rank-zero quasi-smooth presentation at every fibre point. Lemma 2.50.1 makes every such point isolated. The fibre is compact, so this discrete space is finite.

If $x \in \Sigma_B$, the projected trajectory is constant because the standard homotopy fixes Σ_a. Thus the continuous trajectory $H_B(x,t)$ lies in a finite discrete fibre; connectedness of $[0,1]$ makes it constant. Conversely every endpoint projects to $s(\mathrm{val}(f(x))) \in \Sigma_a$, so the endpoint image is precisely Σ_B. The time-zero identity, joint continuity and skeletal fixedness prove the strong deformation retraction assertion. The composition law was proved for strict times and, when a parameter is one, in Lemma 5.18.5. The preceding fibre calculation gives the final statement. □

References. The finite coefficient argument uses a finitely generated ring over the image of $\mathbb{Z}$; it imposes no Noetherian hypothesis on K°. Uniform Hensel perturbation then passes from polynomial to restricted-series Jacobian equations.

5.22 Initial multiplicities and balancing

Lemma 5.22.1 (Lengths on a curve and on its normalization)**.** *Let B be a pure one-dimensional finite-type algebra over a field k. Let $s \in B$ be a nonzerodivisor such that B/sB is finite-dimensional over k. For each minimal prime P_i, put*

$$\nu_i = \mathrm{length}_{B_{P_i}}(B_{P_i}), \qquad B_i = B/P_i,$$

and let $\overline{B}_i$ be the normalization of B_i in its fraction field. Then

$$\dim_k(B/sB) = \sum_i \nu_i \sum_{\substack{q \in \operatorname{Spec} \overline{B}_i \text{ closed} \\ s(q)=0}} [\kappa(q) : k] \operatorname{ord}_q(s). \tag{L}$$

Proof. For every finite B-module M, the kernel and cokernel of multiplication by s are finite over the finite-dimensional algebra B/sB. Define

$$\chi_s(M) = \dim_k(M/sM) - \dim_k \ker(s : M \to M).$$

The six-term kernel-cokernel sequence for a short exact sequence shows that χ_s is additive. Apply the prime filtration of Lemma 5.19.1 to B. A factor B/P with nonminimal P has dimension zero, hence is finite over k, and its kernel and cokernel have equal dimensions. It contributes zero. Localizing the filtration at a minimal prime P_i counts exactly ν_i copies of its residue field. The nonzerodivisor s avoids each minimal prime: in the finite-length local ring B_{P_i} every element of its maximal ideal is nilpotent, whereas multiplication by s remains injective. Thus

$$\dim_k(B/sB) = \sum_i \nu_i \dim_k(B_i/sB_i).$$

The normalization $\overline{B}_i$ is a finite B_i-module by [StacksAlg, Tag 0BXR]. Its quotient $Q_i = \overline{B}_i/B_i$ is annihilated by a nonzero element of B_i, obtained by clearing denominators of finitely many module generators. Hence Q_i has zero-dimensional support and is finite over k, by the same prime-filtration argument. Therefore $\chi_s(Q_i) = 0$. Multiplication by s is injective on B_i and on $\overline{B}_i$, so additivity gives

$$\dim_k(B_i/sB_i) = \dim_k(\overline{B}_i/s\overline{B}_i).$$

Every closed local ring of the normal curve $\operatorname{Spec} \overline{B}_i$ is a discrete valuation ring by [StacksAlg, Tag 0B8Y]. Its quotient by s has length $\operatorname{ord}_q(s)$, as is seen from the powers of a uniformizer. A composition series of the finite-dimensional quotient, localized at its finitely many maximal ideals, gives the residue-degree-weighted sum in (L). □

Proposition 5.22.2 (A ray multiplicity is a boundary-order sum)**.** *Let k have the trivial absolute value and let $C \subset \operatorname{Spec} k[M]$ be a pure one-dimensional closed subscheme. Write C_i for its integral components, with generic lengths ν_i, and let $\overline{C}_i$ be normal proper curves with function fields $k(C_i)$. At a closed point $p \in \overline{C}_i$, define $a_{i,p} \in N$ by*

$$\langle m, a_{i,p} \rangle = \operatorname{ord}_p(\chi^m).$$

If $u \in N$ is primitive and $\mathbf{R}_{\geq 0}u$ is a ray of $\operatorname{Trop}(C)$, its geometric initial multiplicity is

$$m_C(u) = \sum_i \nu_i \sum_{\substack{p \in \overline{C}_i \text{ closed} \\ a_{i,p} = e_{i,p}u,\ e_{i,p} > 0}} [\kappa(p) : k] e_{i,p}. \tag{B}$$

Proof. By Lemma 5.14.1, choose torus coordinates $x_1, \dots, x_n$ in which $u = e_1$. An integral lattice change transports weighted Laurent norms and their initial ideals, so it preserves the multiplicity definition. Let $I \subset k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ define C, and put

$$J = I \cap k[x_1, x_2^{\pm 1}, \dots, x_n^{\pm 1}], \qquad B = k[x_1, x_2^{\pm 1}, \dots, x_n^{\pm 1}]/J.$$

This is the coordinate algebra of the schematic closure in $\mathbf{A}^1 \times \mathbf{G}_m^{n-1}$. Multiplication by x_1 is injective, and $B[x_1^{-1}]$ is the coordinate algebra of C. Every Laurent relation, multiplied by a suitable power of x_1, has lowest x_1-exponent zero and belongs to J. Its ordinary initial form in direction e_1 is its reduction modulo x_1. Conversely, every nonzero reduction of an element of J is such an initial form. Thus there is an equality of ideals, and hence of schemes,

$$\operatorname{in}_{e_1}(C) \simeq \mathbf{G}_m \times \operatorname{Spec}(B/x_1 B).$$

The ordinary initial ideal over a trivially valued field agrees with the initial-degeneration definition after extension and translation, by the explicit coefficient comparison in Theorem 5.16.6. The initial scheme is pure of dimension one by Proposition 5.19.2; consequently $B/x_1 B$ is zero-dimensional, hence finite by Lemma 5.19.1. Its dimension is exactly $m_C(u)$. The closure $\operatorname{Spec} B$ is pure one-dimensional: each of its minimal components meets $x_1 \neq 0$, and its function field is that of a component of C, of transcendence degree one. Its generic lengths are the same ν_i, since localization at the corresponding minimal primes is unchanged by inverting x_1.

For precision about the completion, take the closure of each C_i in $(\mathbf{P}^1)^n$ and normalize it. Normalization is finite by [StacksAlg, Tag 0BXR]. The resulting curve is proper: projective space is proper by [StacksAlg, Tag 01WC]; finite products follow from proper base change and composition, and closed subspaces from proper closed immersions [StacksAlg, Tags 01W3–01W5]. A finite morphism is proper because it is of finite type, separated, and universally closed by [StacksAlg, Tag 01WM]. Here finiteness implies integrality by the determinant trick: multiplication by an algebra element on finitely many module generators satisfies the monic determinant relation. The normal proper model of a one-variable function field is unique up to its unique function-field-preserving isomorphism by [StacksAlg, Tag 0BY1]; hence this model may be used to compute the stated orders. Normalization commutes with restriction to an open set, since integral closure commutes with localization. Indeed, clearing the finitely many denominators in a monic relation shows that an element integral after localization becomes integral before localization after multiplying by one denominator.

The normalization of each component of $\operatorname{Spec} B$ is therefore the inverse image of $\mathbf{A}^1 \times \mathbf{G}_m^{n-1}$ in $\overline{C}_i$. Its points above $x_1 = 0$ are exactly those with

$$\operatorname{ord}_p(x_1) > 0, \qquad \operatorname{ord}_p(x_j) = 0 \quad (2 \le j \le n),$$

equivalently $a_{i,p} = e_{i,p}u$ with $e_{i,p} > 0$. Applying Lemma 5.22.1 with $s = x_1$ gives (B), including the residue degrees and generic component lengths. □

Theorem 5.22.3 (Balancing for a trivially valued curve)**.** *For every pure one-dimensional closed torus subscheme C over a trivially valued field, its tropicalization with the geometric initial multiplicities is a balanced one-dimensional fan. In a pointed subdivision,*

$$\sum_{\rho} m_C(\rho) u_\rho = 0 \quad \text{in } N,$$

where u_ρ is the primitive generator of the ray ρ.

Proof. Since $v(k^\times) = 0$, all hyperplanes in the finite construction of Theorem 5.16.5 pass through zero. Theorem 5.19.5 therefore makes the tropicalization a finite pure one-dimensional fan. For every component C_i, only finitely many points of $\overline{C}_i$ have a nonzero vector $a_{i,p}$, because each of the finitely many coordinate rational functions has only finitely many zeros and poles. Every such vector lies in $\operatorname{Trop}(C_i)$: the absolute value $f \mapsto \exp(-\operatorname{ord}_p(f))$

on its function field is trivial on k and, after completion, gives an analytic point of C_i with that tropical vector. Its positive rescalings give the same ray.

Conversely, Proposition 5.22.2 expresses the positive weight on each tropical ray as a sum over these points, so none is missing. For every character $m \in M$, the degree of its principal divisor on each proper integral curve is zero by [StacksAlg, Tag 02RU]:

$$\sum_p [\kappa(p) : k]\langle m, a_{i,p}\rangle = 0.$$

Multiply by ν_i, sum over i, and group the nonzero vectors by their primitive rays. Formula (B) turns the result into $\langle m, \sum_\rho m_C(\rho)u_\rho\rangle = 0$. Since this holds for every m, it is the required lattice-vector equality. □

Proposition 5.22.4 (Weighted initial stars and tangent quotients)**.** *Let X be pure of dimension d, let τ be a cell of its initial-stratum complex of dimension s, and take $w \in \operatorname{relint}(\tau)$. Write $\operatorname{in}_w(X) = T_{N_\tau} \times Z_\tau$ as in Proposition 5.19.3. Give its residue field the trivial valuation. Then Z_τ is pure of dimension $d-s$, and*

$$\operatorname{Trop}(Z_\tau) = \big(\operatorname{Star}_w \operatorname{Trop}(X)\big)/L_\tau.$$

The equality preserves the weights on maximal cells, using the quotient lattice N/N_τ.

Proof. A prime quotient of the coordinate algebra of Z_τ, with s independent Laurent variables adjoined, has dimension increased by s, by [StacksAlg, Tag 00P0]. Proposition 5.19.2 therefore proves the pure-dimension assertion. For every analytic point of Z_τ, its completed residue field admits Gauss extensions with arbitrary values of the additional torus coordinates. Hence the tropical support of the product is exactly $L_\tau \times \operatorname{Trop}(Z_\tau)$. Apply Theorem 5.16.6 to obtain the displayed support equality.

Take a direction u whose class in $N_{\mathbf{R}}/L_\tau$ lies in the relative interior of a maximal cell of a common refinement of the quotient star and the initial complex of Z_τ. For small $\varepsilon > 0$, Theorem 5.16.6 identifies $\operatorname{in}_{w+\varepsilon u}(X)$ with $\operatorname{in}_u(\operatorname{in}_w(X))$ after residue extension and torus translation. In the split coordinate ring, all equations for Z_τ use quotient characters only. Taking their initial forms therefore gives

$$\operatorname{in}_u(\operatorname{in}_w(X)) \simeq T_{N_\tau} \times \operatorname{in}_{\bar{u}}(Z_\tau).$$

The maximal tangent lattice on the left is the inverse image in N of the maximal tangent lattice in N/N_τ. These are saturated lattices. Quotienting the two schemes by that maximal tangent torus gives the same finite transverse algebra after residue extension. Its dimension is unchanged. Proposition 5.19.7 gives equality of the weights; Proposition 5.19.8 removes the chosen refinements. □

Theorem 5.22.5 (Balancing with actual initial multiplicities)**.** *Let K be any complete real-valued non-Archimedean field, and let $X \subset \operatorname{Spec} K[M]$ be a pure d-dimensional closed subscheme. Its tropicalization, endowed with the geometric initial multiplicities of Definition 5.19.6, is a balanced integral weighted polyhedral complex. Thus at every codimension-one cell τ,*

$$\sum_{\sigma \supset \tau} m_X(\sigma) u_{\sigma/\tau} = 0 \quad \text{in } N/N_\tau,$$

where $u_{\sigma/\tau}$ denotes the primitive generator towards σ. The assertion is stable under subdivision and complete isometric base extension.

Proof. The case $d = 0$ has no balancing condition. If $d > 0$, choose $w \in \operatorname{relint}(\tau)$. Proposition 5.22.4 expresses the initial scheme as $T_{N_\tau} \times C$ with C a pure one-dimensional closed torus subscheme over a trivially valued residue field. Its tropical rays are precisely the images of the cells $\sigma \supset \tau$ in the quotient lattice. Their primitive vectors are $u_{\sigma/\tau}$, and their actual geometric weights are $m_X(\sigma)$ by that proposition. Theorem 5.22.3 gives exactly the displayed equation. Refinement and base-change compatibility are those of Proposition 5.19.8. □

References. This is the classical reduction of tropical balancing to the divisor of a character on a proper curve. The length calculation (L) and the exact identity $\operatorname{in}_{e_1}(C) = \mathbf{G}_m \times \operatorname{Spec}(B/x_1 B)$ identify the boundary-order coefficients with the scheme-theoretic initial weights.

5.23 The homotopy of a strictly semistable formal model

Throughout this section K is complete and nontrivially valued, $O = K^\circ$, and $\pi \in O$ satisfies $0 < |\pi| < 1$. Formal algebras carry their π-adic topology. A formally étalemap below has the unique lifting property for square-zero extensions of discrete algebras annihilated by a power of π.

Lemma 5.23.1 (A bounded presentation for a formal étalechart)**.** *Let $A_0 \to B_0$ be an adic, topologically finitely presented, formally étalehomomorphism of complete admissible O-algebras. There are polynomials $F_1, \dots, F_m \in A_0[Y_1, \dots, Y_m]$ and an isomorphism*

$$B_0 \simeq A_0[Y_1, \dots, Y_m, Q]/(\widehat{F_1, \dots, F_m}, Q \det(\partial F/\partial Y) - 1). \tag{FJ}$$

Here completion is π-adic. If A_0 is a standard semistable algebra, the generic fibre has a presentation (J)*, with all coefficients integral and all coordinates bounded by one.*

Proof. Put $\bar{A} = A_0/\pi A_0$ and $\bar{B} = B_0/\pi B_0$. Topological finite presentation makes $\bar{B}$ finitely presented over $\bar{A}$. Its unique square-zero lifting property follows from that of B_0: a target $\bar{A}$-algebra is annihilated by π, and every lifted A_0-map therefore factors through $\bar{B}$. By [StacksAlg, Tag 00UR], $\bar{A} \to \bar{B}$ is étale. The global presentation [StacksAlg, Tag 00U9] gives

$$\bar{B} = \bar{A}[Y_1, \dots, Y_m]/(\bar{F}_1, \dots, \bar{F}_m), \qquad \det(\partial \bar{F}/\partial Y) \in \bar{B}^\times.$$

Lift the finitely many polynomial coefficients to A_0, and write C for the uncompleted algebra on the right of (FJ). Then $C/\pi C \simeq \bar{B}$. We check its algebraic unique lifting property directly. Across a square-zero ideal I in a target ring, choose any lifts y of a prescribed solution modulo I. Its determinant is a unit: a lift of a unit modulo a nilpotent ideal is a unit, by a finite geometric inverse. The error $F(y)$ lies in I^m. The correction

$$h = -F'(y)^{-1} F(y) \in I^m$$

gives $F(y + h) = 0$, since all terms quadratic in h vanish. The same linear identity proves uniqueness; Q must be the inverse determinant at the corrected root. Thus C has the required unique lifting property.

The identity modulo π now lifts successively to mutually inverse maps

$$C/\pi^n C \;\simeq\; B_0/\pi^n B_0 \qquad (n \geq 1).$$

Indeed the kernels of the maps from level $n+1$ to level n are square-zero. The lifting properties of C and B_0 respectively lift the two directions, and uniqueness makes both composites the identity and makes the isomorphisms compatible in n. Taking inverse limits, and using the assumed completeness of B_0, proves (FJ). Finally Proposition 3.14.7 and Lemma 3.14.8 identify the generic algebra with the stated restricted-series quotient. Coefficients in a standard A_0 have integral restricted-series lifts in its defining coordinates, so this is exactly a bounded presentation of type (J). □

Lemma 5.23.2 (Intrinsic bounds on displacement coefficients)**.** *For a bounded presentation (J), every analytic unit $v \in B^\times$ satisfies*

$$|v(H_B(x,t))| = |v(x)|, \qquad |D_\nu(v)(x)| \le |v(x)| \quad (x \in \mathcal{M}(B),\ 0 \le t \le 1).$$

If $b \in B$ satisfies $|b(y)| \le 1$ at every $y \in \mathcal{M}(B)$, then $|D_\nu(b)(x)| \le 1$ for all ν, x. Consequently, for $0 < \delta < 1$ and $E = \mathcal{H}(x)$, the evaluated displacement has

$$\left\| \frac{\sigma_{\delta,x}(v)}{v(x)} - 1 \right\|_\delta \le \delta, \qquad \|\sigma_{\delta,x}(b) - b(x)\|_\delta \le \delta.$$

Proof. The constant coefficient in the displacement expansion gives $|c(H_B(x,t))| \ge |c(x)|$ for every $c \in B$. Apply this to v and v^{-1}. Multiplicativity makes the product of the two left sides and the product of the two right sides both equal to one, so both inequalities are equalities. For each coefficient,

$$|D_\nu(v)(x)|t^{|\nu|} \le |v(x)| \qquad (0 < t < 1).$$

Letting t increase to one proves the coefficient bound for a unit. For the second assertion, $H_B(x,t)$ is an actual point of $\mathcal{M}(B)$, so $|b(H_B(x,t))| \le 1$; the same limiting argument gives the coefficient bound. Since D_0 is the identity, the remaining terms have positive total degree, and their Gauss norms at radius δ are at most δ. These arguments use only the values of the functions on the analytic space, not a comparison of quotient norms. □

Theorem 5.23.3 (Independence of the semistable chart)**.** *Suppose the same affinoid algebra B has two bounded presentations of type (J), possibly over different standard semistable algebras. Their homotopies on $\mathcal{M}(B)$ are equal at every time in $[0,1]$.*

Proof. Use unprimed and primed notation for the two presentations. Fix x, put $E = \mathcal{H}(x)$, and fix $0 < \delta < 1$. In $E\{\delta^{-1}U\}$ define

$$\Phi_i = \frac{\sigma_{\delta,x}(X_i')}{X_i'(x)} - 1 \quad (1 \le i \le r'), \qquad \Psi_j = \sigma_{\delta,x}(T_j') - T_j'(x).$$

The functions X_i' are global analytic units because their product is the nonzero scalar a'. The T_j' and the primed implicit coordinates have values at most one everywhere. Lemma 5.23.2 therefore gives $\|\Phi_i\|, \|\Psi_j\| \le \delta$. Substitution $U' = (S', Z') \mapsto (\Phi, \Psi)$ defines a contractive E-homomorphism

$$\theta : E\{\delta^{-1}U'\} \longrightarrow E\{\delta^{-1}U\}.$$

The two maps $\sigma_{\delta,x}$ and $\theta\sigma'_{\delta,x}$ have identical images on the primed base coordinates: this is the definition for X'_i with $i \geq 1$ and for T'_j, and the product relation gives it for X'_0. Their images of the primed implicit tuple solve the same displaced equations and both lie within δ of $Y'(x)$. At that center the new Jacobian differs by at most δ from the original invertible Jacobian of inverse norm at most one. The geometric inverse formula and Lemma 5.18.1 therefore identify the two roots. Their inverse determinants coincide as well.

The primed presentation topologically generates B. Both maps are bounded for its topology, so equality on its coordinates implies equality on every $b \in B$. Contractivity of θ now gives

$$\|\sigma_{\delta,x}(b)\|_\delta \leq \|\sigma'_{\delta,x}(b)\|_\delta.$$

Reversing the two presentations gives the opposite inequality. These Gauss norms are precisely evaluation at the two time-δ points, so the points agree. They agree at zero by definition and at one by the pointwise endpoint formula. □

Proposition 5.23.4 (Restriction to formal open subsets)**.** *Let $\mathfrak{U} = \mathrm{Spf}(B_0)$ admit a topologically finitely presented formal étalemap to a standard semistable formal chart. Its homotopy preserves the generic fibre of every formal open subset. On an admissible affine formal open $\mathfrak{V} \subseteq \mathfrak{U}$, its restriction equals the homotopy obtained from the restricted chart, independently of its bounded presentation.*

Proof. Choose a bounded presentation by Lemma 5.23.1. A principal formal open $D(g)$, $g \in B_0$, has generic fibre $|g| = 1$ by Proposition 3.17.1. The integral function g has values at most one. The constant coefficient of its displacement gives $|g(H_B(x,t))| \geq |g(x)|$, so a point where $|g| = 1$ stays in that domain at every time. General formal opens are unions of principal formal opens, and their generic fibres are the corresponding unions by Proposition 3.17.2; hence they are preserved as well.

Write $\mathfrak{V} = \mathrm{Spf}(C_0)$. An open immersion is formally étale: a square-zero thickening does not change underlying points, so a map whose reduction factors through that open factors through it uniquely. Composing unique lifting properties shows that the restricted map to the standard base is formally étale. We also verify relative finite presentation. By Theorem 3.14.4, choose finite presentations $B_0 = O\langle W\rangle/I$ and $C_0 = O\langle Z\rangle/J$. Lift the images of the

finitely many W_i to integral restricted series $h_i(Z)$. The two substitution maps give the presentation

$$C_0 \simeq B_0\langle Z\rangle/(J(Z),\, W_i - h_i(Z) : i).$$

The relations are finite; the same construction applies directly to the map from the standard base to C_0. Thus Lemma 5.23.1 applies over that original base. The integral map $B_0 \to C_0$ gives a contractive map on the displayed generic algebras: each integral presentation coordinate of B maps into C_0, whose elements have norm at most one by Proposition 3.14.7; bounded substitution and quotient infima give contractivity. Thus Proposition 5.18.4 and Theorem 5.21.3 identify the restricted homotopy with that on $\mathfrak{V}_\eta$. Theorem 5.23.3 removes the chosen presentation. □

Theorem 5.23.5 (Global semistable deformation)**.** *Let $\mathfrak{X}$ be a quasi-compact admissible formal O-scheme with Hausdorff generic fibre X. Suppose it has a finite affine formal open cover $\mathfrak{U}_i$ with topologically finitely presented formal étalemaps*

$$f_i : \mathfrak{U}_i \longrightarrow \operatorname{Spf} O\langle X_{i0}, \dots, X_{ir_i}, T_{i1}, \dots, T_{iq_i}\rangle/(X_{i0}\cdots X_{ir_i} - a_i),$$
$$0 < |a_i| < 1.$$

There is a canonical jointly continuous homotopy $H_{\mathfrak{X}} : X \times [0,1] \to X$ whose restriction to each $U_i = (\mathfrak{U}_i)_\eta$ is the constructed chart homotopy. It is a strong deformation retraction onto the closed compact subset

$$\Sigma(\mathfrak{X}) = H_{\mathfrak{X}}(X, 1), \qquad \Sigma(\mathfrak{X}) \cap U_i = (f_i)_\eta^{-1}(\Sigma_{a_i}).$$

It satisfies the maximum composition law, is independent of the affine cover and the semistable coordinate maps, and restricts to the corresponding homotopy on every quasi-compact formal open. Under a complete isometric extension L/K, the projection $X_L \to X$ commutes with these homotopies.

Proof. The preceding presentation lemma and Theorem 5.21.4 give a jointly continuous strong deformation H_i on each U_i. On an overlap of two formal charts, choose a principal formal open of the first chart contained in the second. It is admissible by Proposition 3.17.1. Its generic fibre is preserved by both homotopies, and Proposition 5.23.4 identifies their restrictions with the two homotopies of that affine algebra. Those homotopies coincide by Theorem 5.23.3. Such formal affine opens cover the overlap, and their generic

fibres cover the analytic overlap by Proposition 3.17.2. Thus all H_i agree on overlaps.

Each U_i is compact and hence closed in the Hausdorff space X, and the finite family covers X. The product sets $U_i \times [0,1]$ consequently form a finite closed cover. The maps H_i glue to a continuous map: the inverse image of any closed subset of X is the finite union of its closed inverse images in these closed product sets. All homotopy and composition identities hold on each preserved U_i, hence globally.

A point in U_i is fixed by the global endpoint exactly when it is fixed by the local endpoint, namely when it lies over Σ_{a_i}. The endpoint is idempotent, so its image is precisely its fixed set. Since X is a finite union of compact sets, it is compact; its endpoint image is compact and hence closed. This proves the displayed local description and strong deformation assertion. A second formal atlas has common affine formal refinements with the first one; the same overlap argument proves independence. It also proves restriction to a quasi-compact formal open. Finally, base change of a chart retains its integral Jacobian presentation. The local scalar-extension compatibility of Theorem 5.21.3 therefore gives the asserted projection identity on the cover, and hence globally. □

5.24 The faces of an étalesemistable chart

Let $A_0 = O\langle X_0, \dots, X_r, T_1, \dots, T_q\rangle/(X_0 \cdots X_r - a)$, where $0 < |a| < 1$, and let $A_0 \to B_0$ be a topologically finitely presented formally étalemap of admissible algebras. Write $f : X = \mathcal{M}(B) \to \mathcal{M}(A)$ for the generic map, $\bar{f}$ for the special-fibre map, and $k = \widetilde{K}$. For a nonempty subset $I \subseteq \{0, \dots, r\}$ put

$$\begin{aligned} \Delta_I &= \{w \in \Delta_a : w_j = 0 \text{ for } j \notin I\}, \\ D_I &= \{X_i = 0 \ (i \in I), \quad X_j \neq 0 \ (j \notin I)\} \subseteq \operatorname{Spec}(A_0 \otimes_O k), \\ S_I &= \bar{f}^{-1}(D_I). \end{aligned}$$

The stratum D_I is the integral smooth variety with coordinates $X_j^{\pm 1}$ for $j \notin I$ and $T_1, \dots, T_q$. Denote its generic point by ξ_I and the finite set of generic points of S_I by Q_I. The empty inverse image has $Q_I = \varnothing$.

Lemma 5.24.1 (The residue field over a relative open face)**.** *Let $w \in \mathrm{relint}(\Delta_I)$ and let $z_w \in \Sigma_a$ be its standard Gauss point. Specialization sends z_w to ξ_I, and the induced extension*

$$F_I := \kappa(\xi_I) = k(X_j : j \notin I, T_1, \dots, T_q) \longrightarrow h_w := \widetilde{\mathcal{H}(z_w)}$$

is purely transcendental. In particular, for every finite extension l/F_I, the tensor product $l \otimes_{F_I} h_w$ is a field, of degree $[l : F_I]$ over h_w.

Proof. Choose $i_0 \in I$ and eliminate X_{i_0} using the product relation. The independent Gauss variables are the remaining X_i and the T_j. The variables X_j with $j \notin I$ and all the T_j have weight zero; all other original X_i have strictly positive weight. Their residues therefore give exactly the generic point ξ_I: every nonzero polynomial in the zero-weight variables has Gauss norm one after normalizing an integral coefficient.

In the residue-lattice proof of Proposition 3.8.1, let L_w be the lattice of exponent vectors whose weight belongs to $v(K^\times)$. The subgroup generated by the zero-weight coordinate vectors is a direct summand of the full exponent lattice, hence is saturated in L_w. Lemma 5.14.1 extends its displayed coordinate basis to a basis of L_w. The Gauss residue formula with this basis expresses h_w as the rational function field over k in the residues of the zero-weight variables and finitely many additional normalized Laurent monomials. It is therefore purely transcendental over the indicated subfield F_I.

Finally, write $h_w = F_I(U_1, \dots, U_s)$. The ring $l \otimes_{F_I} h_w$ is the localization of $l[U_1, \dots, U_s]$ by the nonzero polynomials over F_I. It is an integral domain and is finite-dimensional over h_w, hence is a field: multiplication by a nonzero element is an injective endomorphism of a finite-dimensional vector space and is therefore surjective. Tensoring a basis of l/F_I gives its stated degree. □

Proposition 5.24.2 (Actual skeletal fibres and residual strata)**.** *For $w \in \mathrm{relint}(\Delta_I)$, specialization induces a bijection*

$$f^{-1}(z_w) \xrightarrow{\sim} Q_I.$$

For the point $x_{\eta,w}$ labelled by $\eta \in Q_I$, put $E_w = \mathcal{H}(z_w)$ and $l_\eta = \kappa(\eta)$. Then $\mathcal{H}(x_{\eta,w})/E_w$ is the unramified finite extension lifting the finite separable residue extension

$$l_\eta \otimes_{F_I} h_w \,/\, h_w.$$

Its degree is $[l_\eta : F_I]$, and its value group is that of E_w.

Proof. Use Lemma 5.23.1 to write $B_0 = \widehat{C}$ with $C = A_0[Y,Q]/(F, Q\det J - 1)$. The square-zero correction in that proof and [StacksAlg, Tag 00UR] show that C is an étaleA_0-algebra. Base change along the bounded evaluation $A_0 \to E_w^\circ$ gives an étalealgebra C_E over E_w° by [StacksAlg, Tag 00U2, part (3)]. It is flat by part (6) of the same lemma, so multiplication by π is injective. Its π-adic completion D_0 is an admissible E_w°-algebra by Lemma 3.14.8 and Theorem 3.14.4. The explicit restricted presentation gives

$$D = D_0[1/\pi] = B\widehat{\otimes}_A E_w,$$
$$D_0/E_w^{\circ\circ} D_0 = (B_0 \otimes_O k) \otimes_{A_0 \otimes_O k} h_w.$$

For the second identity one simply reduces the finite list of equations in the restricted presentation: reduction of an integral restricted series is a polynomial, and reduction of the ideal is generated by the reductions of its displayed generators. The first identity is the completed-tensor fibre presentation of Theorem 2.33.11.

The special-fibre map is étale, and its fibre over ξ_I is a finite disjoint union $\coprod_{\eta \in Q_I} \operatorname{Spec}(l_\eta)$ with each l_η/F_I finite separable, by [StacksAlg, Tag 02GL]. Here the fibre points are exactly the generic points of S_I. Indeed S_I is smooth over k and hence normal by [StacksAlg, Tags 036U, 056T]; flat going down, [StacksAlg, Tag 00HS], makes every component dominate the integral D_I, and the zero-dimensional generic fibre has no proper chains of primes. The preceding lemma now identifies the special fibre of D_0 with

$$\prod_{\eta \in Q_I} (l_\eta \otimes_{F_I} h_w),$$

a finite product of fields. This is in particular reduced, so Lemma 4.12.1 identifies it with the canonical reduction of D. Every prime of this reduction is minimal. Theorem 3.16.7 therefore gives one and only one actual analytic point over each factor, and these exhaust the fibre. If the product is empty, the integral quotient norm would have 1 of norm less than one unless $D = 0$; thus the analytic fibre is empty in exactly that case. This proves the bijection and its specialization labels.

For the valued-field assertion, fix a factor $\lambda = l_\eta \otimes_{F_I} h_w$ and apply Lemma 4.15.2 over the complete field E_w. It supplies a finite complete field L/E_w of degree $[\lambda : h_w]$, residue field λ, and unchanged value group. Lift the coordinates of the corresponding special-fibre point arbitrarily into L°. The finitely many equation errors have norms strictly below one and the Jacobian determinant has norm one. Lemma 5.18.1 corrects them to an exact

bounded solution in L. This gives a point of $\mathcal{M}(D)$ with that residue label, hence the point already identified, and an isometric embedding

$$\mathcal{H}(x_{\eta,w}) \hookrightarrow L.$$

The embedding first holds for the fraction field of the evaluated affinoid algebra and then for its completion. Its residue field contains λ. Lifts of a basis of λ/h_w are E_w-linearly independent: normalize any putative relation by a coefficient of largest absolute value and reduce. Hence the left field has degree at least $[\lambda : h_w]$, while the embedding into L gives the opposite inequality. It equals L, proving the assertion. □

Lemma 5.24.3 (The unique stratum along a face)**.** *Let $\eta \in Q_I$ and let $\varnothing \neq J \subseteq I$. There is exactly one $\eta_J \in Q_J$ whose closure contains η. For $J' \subseteq J$, the generization of η_J labelled by J' is $\eta_{J'}$.*

Proof. The closed subscheme Z_J of the standard special fibre cut out by $X_j = 0$ for $j \in J$ is an affine space: at least one factor of the product relation is zero, so the remaining coordinates are free. Its pullback Y_J is étaleover Z_J, hence smooth and normal by [StacksAlg, Tags 036U, 056T]. Its local ring at η is therefore a domain. Flat going down for $\bar{f}$ gives a prime below η lying over the generic point ξ_J of Z_J. It lies in S_J. Every such prime is a component generic point of Y_J: going down forces minimal primes below it to lie over the generic point of Z_J, and the étalegeneric fibre is a finite disjoint union of fields. The domain $\mathcal{O}_{Y_J,\eta}$ has only one minimal prime, so there is precisely one such component through η. This proves existence and uniqueness. If $J' \subseteq J$, the successive generization lies over $\xi_{J'}$ and its closure contains η; uniqueness identifies it with $\eta_{J'}$. □

Nonempty face generizations. In the formally étalesemistable chart above, for each $\eta \in Q_I$ and $\varnothing \neq J \subseteq I$ there is exactly one $\eta_J \in Q_J$ whose closure contains η. For $\varnothing \neq J' \subseteq J$, the J'-generization of η_J is $\eta_{J'}$.

Here is the proof on this range. The formal chart is topologically finitely presented. Reduction modulo π has finite presentation and inherits the unique square-zero lifting property from discrete tests killed by π. It is étaleby [StacksAlg, Tag 00UR]. Base change to k and to the actual coordinate face Z_J preserves étaleness, and the resulting map $Y_J \to Z_J$ is flat [StacksAlg, Tag 00U2(3), (6)]. As J is nonempty, Z_J is an affine space over k. In the

local étalepresentation of Y_J over that polynomial base, the square Jacobian is invertible [StacksAlg, Tag 00U9]. The same minor in the presentation over k proves smoothness [StacksAlg, Tag 00T7(1)]. Thus the relevant local ring of Y_J at η is a normal domain [StacksAlg, Tag 056T].

Flat going down gives a prime below η over the generic point of Z_J [StacksAlg, Tag 00HS]. Every such prime is minimal in Y_J. Indeed any prime below it still lies over that generic point, whose étalefibre is a finite disjoint union of spectra of fields [StacksAlg, Tag 02GL(2)]. The local domain has one minimal prime, giving the required unique label. For nonempty J', the successive generization over the generic point of $Z_{J'}$ specializes to η and hence is the unique $\eta_{J'}$.

Theorem 5.24.4 (A finite simplex realization of the chart skeleton)**.** *For each $\eta \in Q_I$, the points $x_{\eta,w}$ on* $\mathrm{relint}(\Delta_I)$ *extend uniquely to a continuous map*

$$s_\eta : \Delta_I \longrightarrow X, \qquad s_\eta(w) = x_{\eta_J,w} \quad \text{if } w \in \mathrm{relint}(\Delta_J).$$

It is a homeomorphism onto its image. The finite collection of these closed simplices covers $\Sigma_B = f^{-1}(\Sigma_a)$, and the restriction to a face Δ_J is s_{η_J}. More precisely, Σ_B is homeomorphic to the quotient of the finite disjoint union of the Δ_I, one copy for each $\eta \in Q_I$, by exactly these face identifications. In these charts all face maps are the coordinate face inclusions, integral affine for the lattices

$$N_I = \{(n_i)_{i\in I} \in \mathbb{Z}^I : \textstyle\sum_i n_i = 0\}.$$

The intersection with the generic fibre of a formal open subset is a union of closed faces in this finite realization.

Proof. First work over the relative open face. The primes in Q_I are distinct and incomparable. Choose an integral function $g \in B_0$ whose reduction is outside η and inside every other member of Q_I, by choosing one element for each other prime and multiplying. If there is no other prime, take $g = 1$. By the actual fibre description, the locus $|g| = 1$ selects exactly $x_{\eta,w}$ over each w in this open face. For any compact subset $W \subset \mathrm{relint}(\Delta_I)$, the set

$$\{x \in f^{-1}(\{z_w : w \in W\}) : |g(x)| = 1\}$$

is closed in the compact affinoid space X, and maps continuously and bijectively onto W. It is therefore homeomorphic to W. Every point of the relative open face has a compact relative neighborhood contained in that face, proving continuity of $w \mapsto x_{\eta,w}$ throughout the relative open face.

Consider the closure of its graph in the compact space $\Delta_I \times X$. Let (w, x) be a point of that closure and let $J = \{j : w_j > 0\}$. Continuity of f gives $f(x) = z_w$, so the actual fibre theorem labels x by some $\theta \in Q_J$. If an integral function h has reduction outside η, then $|h(x_{\eta,u})| = 1$ at every interior u. Continuity makes $|h(x)| = 1$. Thus the reduction prime θ is contained in η. Since $\sum_i w_i = v(a) > 0$, J is nonempty. By nonempty face generization, $\theta = \eta_J$. Every $w \in \Delta_I$ is a limit of interior points; compactness supplies a graph limit above it. We have proved that this limit is unique. The closed graph therefore projects continuously and bijectively onto the compact Hausdorff simplex Δ_I, hence homeomorphically. Its second projection is the desired continuous extension. Its composition with the base coordinate map is the identity, so it is injective and is a homeomorphism onto its compact image.

The boundary formula and transitivity for nonempty faces prove the face restriction assertion. There are finitely many I and finitely many $\eta \in Q_I$, and the fibre theorem shows that their relative interiors partition Σ_B. Two closed cells meet at a point of support J exactly when their J-generizations agree; hence their intersection is a union of the indicated common closed faces. The induced map from the finite face quotient to Σ_B is therefore a continuous bijection. Its source is compact and its target Hausdorff, so it is a homeomorphism. Coordinate inclusions preserve the displayed integer lattices, giving the final affine assertion.

On a relative open cell the specialization label is constant. If this label belongs to a given formal open, all its generizations belong to that open as well. Consequently inclusion of an open cell includes its whole closed simplex. The specialization description of formal-open generic fibres then gives the asserted union of closed faces. □

5.25 Integral-affine coordinates on the skeleton

Lemma 5.25.1 (Units on an open polyannulus)**.** *Let L be a complete non-Archimedean field and let $\Omega \subset \mathbf{R}^d$ be a nonempty convex open set. Let $P_\Omega \subset (\mathbf{G}_{m,L}^d)^{\mathrm{an}}$ be the analytic domain defined by $v(Z) \in \Omega$, and denote its Gauss section by $w \mapsto z_w$. For every analytic unit $g \in \mathcal{O}(P_\Omega)^\times$ there are $c \in v(L^\times)$ and $m \in \mathbf{Z}^d$ such that*

$$v(g(z_w)) = c + \langle m, w \rangle \qquad (w \in \Omega).$$

The assertion includes $d = 0$, when $P_\Omega = \mathcal{M}(L)$.

Proof. Suppose first that $d > 0$. On the inverse image of a closed rectangular box $W = \prod_j [\alpha_j, \beta_j]$ contained in Ω, the arbitrary-radius domain presentations of Theorem 2.31.2 give the product of closed annulus algebras; their actual analytic sections are these algebras by Theorem 2.39.4. Iterate Theorem 2.13.2, with the other coordinates as Banach coefficients. This gives a unique Laurent expansion $\sum_{m \in \mathbf{Z}^d} c_m Z^m$, and the norm is

$$\max_m |c_m| \max_{w \in W} \exp(-\langle m, w \rangle).$$

The weighted coefficients tend to zero outside finite subsets. To check the multivariable assertion directly, apply the one-coordinate normal-form operators successively to polynomial families. Their bounds and commuting actions on different exponent pairs extend them to the complete weighted coefficient space. Their composite kernel is the sum of the coordinate relations, and their common section is the Laurent monomial representative. The finite-tail condition is exactly the weighted c_0 condition of Lemma 2.13.1. Thus this iteration also proves uniqueness and all restriction formulas.

Use boxes with nonempty interior. On overlapping interiors, uniqueness makes their coefficients equal. Every two such boxes can be connected by a finite chain of overlapping smaller boxes inside the convex set Ω. Hence a function on P_Ω has a single Laurent coefficient family, converging on every compact box, and

$$\log |g(z_w)| = \max_m \{\log |c_m| - \langle m, w \rangle\}.$$

This is convex in w. Apply the same argument to g^{-1}. Multiplicativity of the Gauss seminorm says that its logarithm is the negative of the displayed function. A function both convex and concave on an open convex set is affine: equality in the two convexity inequalities gives its affine interpolation on every line segment. At an interior point w_0, a nonzero coefficient c_m attains the maximum. The difference between the affine function and this one affine term is nonnegative on Ω and zero at w_0; an affine function with these properties is zero, by varying a small distance in both directions along every vector. Thus $v(g(z_w)) = v(c_m) + \langle m, w \rangle$. If $d = 0$, the unit is an element of $L^\times$, which proves the same claim. □

Lemma 5.25.2 (A rational-origin tube over a semistable chart)**.** *Let L be complete and nontrivially valued, and let $0 < |b| < 1$. Put*

$$D_0 = L^\circ\langle Z_0, \dots, Z_d\rangle/(Z_0 \cdots Z_d - b).$$

Let $D_0 \to E_0$ be a topologically finitely presented formally étalemap of admissible algebras, and let q be a rational point of its special fibre lying over $Z_0 = \cdots = Z_d = 0$. The generic map induces an analytic isomorphism

$$]q[\xrightarrow{\sim} P_b := \{z \in \mathcal{M}(D) : |Z_i(z)| < 1 \text{ for every } i\}.$$

After eliminating Z_0, this is the open polyannulus with logarithmic domain $w_j > 0$ and $\sum_{j=1}^{d} w_j < v(b)$.

Proof. Choose the bounded presentation of Lemma 5.23.1, with coordinates Y, Q, equations $F(Y) = 0$ and $Q \det F'(Y) = 1$. Lift the residue coordinates of q to constants y_0, q_0 in L°. Choose integral restricted-series lifts of all the coefficients in the variables $Z_0, \dots, Z_d$. Their evaluations at $Z = 0, Y = y_0$ have F-error of norm less than one, and the Jacobian determinant has norm one. These assertions follow by reduction at the specified rational point q.

Let W be a compact logarithmic box inside the displayed open simplex, with affinoid inverse image P_W. All $|Z_i|$ are at most some $\rho < 1$ there. Every nonconstant monomial in their integral series has norm at most ρ. Thus, in the Banach algebra $\mathcal{O}(P_W)$, the error $F(y_0)$ has norm at most some $\delta < 1$, and $F'(y_0)$ has inverse of norm at most one. Indeed its determinant differs by a term of norm less than one from its constant unit, so the geometric inverse applies. Lemma 5.18.1 produces the unique root y_W with $|y_W - y_0| \le \delta$; set $q_W = (\det F'(y_W))^{-1}$. This gives a bounded analytic section $P_W \to \mathcal{M}(E)$, specializing to q. Uniqueness makes the sections agree on overlaps, so they glue on all of P_b. When $d = 0$ use the same argument in L, where $Z_0 = b$.

We check the inverse as an analytic map. Over P_W, cover the tube by affinoid subdomains on which $|Y - y_0| \le \lambda < 1$, increasing λ if necessary so that $\lambda \ge \delta$; one may impose the analogous bound on $Q - q_0$. The original coordinate tuple and the pullback of y_W are roots of the same system in the Banach algebra of such a domain. Both lie within a radius less than one of y_0, where the same Jacobian inverse estimate holds. Hensel uniqueness makes them equal as algebra elements, and then the inverse-determinant equation identifies Q. Consequently the section and the generic map are inverse on these subdomains. Every point of the tube lies in one of them, since its

residue coordinates are those of q. This proves the analytic isomorphism, including its compatibility for every compact box in the open simplex. □

Theorem 5.25.3 (Units along an étaleskeletal simplex)**.** *Use the notation and hypotheses of Theorem 5.24.4. Put* $\Gamma = v(K^\times)$*. For every* $\eta \in Q_I$ *and every* $g \in B^\times$*, the function*

$$w \longmapsto v(g(x_{\eta,w})) \qquad (w \in \Delta_I)$$

is Γ*-integral-affine: after eliminating any one coordinate in* I*, it has an integral linear part and a constant term in* Γ*.*

Proof. Let H_0 be the completion of $K(X_j : j \notin I, T_1, \dots, T_q)$ in the unit Gauss norm. Its residue field is F_I, by Proposition 3.8.1, and its value group is $|K^\times|$: a nonzero polynomial attains its Gauss norm at a coefficient, fractions have the same property for their values, and completion does not change the value group since a nonzero convergent limit has the same norm as every sufficiently close term.

Send the zero-weight variables to these fixed elements of H_0°. The remaining formal base is

$$H_0^\circ\langle X_i : i \in I\rangle / \left(\prod_{i\in I} X_i - b\right), \qquad b = a\left(\prod_{j\notin I} X_j\right)^{-1}, \qquad |b| = |a|.$$

Base-change the algebraic square-Jacobian presentation of Lemma 5.23.1 and then complete. The result is an admissible formally étalealgebra over this base. The justification is the same explicit completion argument in Proposition 5.24.2: the algebraic presentation is flat, so it has no scalar torsion; Lemma 3.14.8 preserves this under completion, and Theorem 3.14.4 gives the admissible finite presentation.

Above the all-zero special point this model has the residual point with field $l_\eta = \kappa(\eta)$, a finite separable extension of F_I. Lemma 4.15.2 supplies one finite extension L_0/H_0 with residue field l_η and unchanged value group. Base-change the presentation once more to L_0°. The multiplication map $l_\eta \otimes_{F_I} l_\eta \to l_\eta$ chooses a rational residual point q above the origin. Thus all fields, models, and the point q have been fixed independently of w.

Lemma 5.25.2 now gives an analytic map from the entire open simplex polyannulus over L_0 to X, by its inverse section followed by the two base-change projections. Its Gauss point with logarithms w maps to z_w on the

original standard base. Indeed, after eliminating one X_i with $i \in I$, the coefficient Gauss norm in the zero-weight variables followed by the Gauss norm in the remaining positive-weight variables is exactly the original Gauss norm. This identity first holds for Laurent polynomials and then for all functions by bounded continuity. The specialization of the image is η, by the chosen multiplication map of residue fields. Proposition 5.24.2 therefore identifies this image with the actual point $x_{\eta,w}$.

The pullback of g is an analytic unit on this single open polyannulus. Lemma 5.25.1 gives the required integral-affine formula on $\operatorname{relint}(\Delta_I)$, with constant term in $v(L_0^\times) = \Gamma$. The closed-cell section is continuous by Theorem 5.24.4, and g is nowhere zero. Hence both sides extend continuously and agree on all of Δ_I. The proof also covers a zero-dimensional face. □

Theorem 5.25.4 (The integral-affine skeleton of a formal model)**.** *Let $\mathfrak{X}$ satisfy Theorem 5.23.5. Its canonical fixed set $\Sigma(\mathfrak{X})$ is a finite Γ-integral-affine face-glued cell complex of closed simplices. Distinct cells may have the same vertex set. Each affine formal semistable chart supplies exactly the cells of Theorem 5.24.4. The cell structure and its integral-affine transition maps are independent of the chosen formal chart atlas. The homotopy of Theorem 5.23.5 is a jointly continuous strong deformation retraction onto this finite complex.*

Proof. Write sp for specialization to $\mathfrak{X}_s$. In one chart, the relative interior of the cell labelled by η is precisely

$$E_\eta = \{x \in \Sigma(\mathfrak{X}) : \operatorname{sp}(x) = \eta\}.$$

This is the actual-fibre classification in Proposition 5.24.2. There is no restriction to the chart on the right: the inverse image under specialization of its formal open set is its generic domain, by Proposition 3.17.2. Thus whenever a second chart contains η, the same set E_η is one of its relative open cells. For clarity, choose $x \in E_\eta$. The canonical fixed-set identity makes its image in the second standard base a Gauss point. Its specialization is the generic point of the unique coordinate stratum whose vanishing coordinates are those belonging to η. The actual-fibre classification therefore puts η in that chart's corresponding Q_I. The same classification identifies its whole relative open cell with E_η. This proves intrinsically which residual generic points label cells, without choosing a global ordering of the special-fibre branches. Its closure in the global Hausdorff generic fibre is the closed simplex supplied by either

chart, since each generic affinoid chart is compact and closed. Consequently the cells themselves are independent of charts.

Compare two coordinate systems on such a cell. Choose a common principal formal neighbourhood of η in their overlap. Its generic domain contains the entire closed cell: it contains E_η and is closed, or equivalently one may use the formal-open subcomplex assertion of Theorem 5.24.4. It is admissible by Proposition 3.17.1, and its map to either standard base has a bounded presentation by Proposition 5.23.4. Every monomial coordinate of the second chart is an analytic unit on this common domain, since the product of those coordinates is a nonzero scalar. Theorem 5.25.3 therefore makes each of its logarithmic coordinates a Γ-integral-affine function of the first ones.

Reverse the two charts to obtain the same assertion for the inverse coordinate change. The linear parts are mutually inverse maps of the integer tangent lattices $N_I = \{(n_i) \in \mathbf{Z}^I : \sum n_i = 0\}$. They are therefore unimodular isomorphisms. Indeed, the coordinate compositions are identities on a relative open simplex, so their derivatives are inverse real linear maps; both preserve the displayed lattices. The affine constants belong to Γ, and all face maps are the already proved coordinate face inclusions.

There are finitely many cells because the given formal atlas and the cell lists in its charts are finite. Their intersections are unions of common faces. To see this for cells in different charts, first intersect the second cell with the first formal open set. The formal-open subcomplex assertion expresses this intersection as a finite union of its closed faces. Each such face now lies in the first chart, where the common-face property has already been proved. These faces are the same global cells by the intrinsic description E_η above. Finally the map from the finite face-gluing quotient of the compact simplices to $\Sigma(\mathfrak{X})$ is a continuous bijection from a compact space to a Hausdorff space, hence is a homeomorphism. This proves the asserted finite complex and its intrinsic integral-affine structure. The previously proved global homotopy has exactly this fixed set and remains the claimed strong deformation retraction. □

5.26 Faithful rational coordinates on a curve skeleton

Throughout this section, K is algebraically closed, complete and nontrivially valued. Let X be a smooth projective integral K-curve and let $\mathcal{X}/K^\circ$ be a

projective model satisfying Theorem 4.23.3. Write $\Sigma = \Sigma(\mathcal{X})$. It is a compact finite metric graph. Its finite edges have the logarithmic length defined by the nodal annuli.

Lemma 5.26.1 (Rational approximation of a nonzero germ)**.** *Every point of Σ has algebraic support the generic point of X. If $x \in \Sigma$ and g is an analytic germ with $g(x) \neq 0$, there is $f \in K(X)^\times$ such that $|f| = |g|$ on an ordinary neighborhood of x on which both are defined. If $|g(x)| = 1$, $f(x)$ and $g(x)$ can be required to have the same residue.*

Proof. At a component vertex, Proposition 4.24.2 gives residual transcendence degree one. At an interior annular Gauss point, Proposition 3.8.1 gives either residual transcendence degree one or a nonzero relative value-group rank. A point supported at an algebraic closed point would have completed residue field K, by algebraic closedness and Proposition 2.10.1. These alternatives exclude closed algebraic support; an integral algebraic curve has only generic and closed points.

By the same description of analytification, $\mathcal{H}(x)$ is the completion of $K(X)$ for the norm of x. Choose $f \in K(X)$ with $|f(x) - g(x)| < |g(x)|$. This makes $f \neq 0$. The rational function and its inverse are regular on a Zariski open containing the generic point, so their analytifications are defined near x. Continuity preserves the displayed strict inequality on a smaller ordinary neighborhood. The ultrametric inequality gives $|f| = |g|$ there. When $|g(x)| = 1$, the same inequality says that the residues agree. □

Lemma 5.26.2 (Prescribing finitely many residual orders)**.** *Let C be an irreducible component of the special fibre of $\mathcal{X}$, and let $p_1, \dots, p_m$ be distinct closed points of its normalization C^ν. For each i there is $u_i \in k(C)^\times$ with*

$$\operatorname{ord}_{p_j}(u_i) = \delta_{ij}.$$

Proof. The field $k = \widetilde{K}$ is algebraically closed: lift a monic polynomial to K° and factor it over K; dominance of the leading term makes every root integral, and reduction gives the factorization over k. Choose a projective embedding of C. A hyperplane avoiding the finitely many images of the p_j exists: the forbidden coefficients form a finite union of proper linear subspaces over the infinite field k. Its complement on C is affine. The normalization is finite by [StacksAlg, Tag 0BXR], so its inverse image is an affine normal curve $\operatorname{Spec} B$

containing all p_j. Its local rings at these points are discrete valuation rings by [StacksAlg, Tag 0B8Y].

Write $\mathfrak{m}_j$ for their maximal ideals. A local uniformizer at p_i can be written a_i/b_i, with $a_i, b_i \in B$ and $b_i \notin \mathfrak{m}_i$; hence a_i has order one at p_i. The ideals $\mathfrak{m}_i^2$ and $\mathfrak{m}_j$, $j \neq i$, are pairwise comaximal. Indeed, powers of comaximal ideals remain comaximal by expanding a sufficiently high power of $1 = a + b$. For two comaximal ideals I, J, write $a + b = 1$ with $a \in I, b \in J$; then $br + as$ has prescribed residues r modulo I and s modulo J. Induction gives the same assertion for finitely many such ideals. Choose $u_i \in B$ congruent to a_i modulo $\mathfrak{m}_i^2$ and to 1 modulo each $\mathfrak{m}_j$, $j \neq i$. Its orders are exactly those claimed, so it is a nonzero rational function. □

Proposition 5.26.3 (Rational functions are piecewise integral affine)**.** *For every $f \in K(X)^\times$, the function $-\log|f|$ is finite and continuous on Σ, and is integral affine on every edge of a finite subdivision.*

Proof. All skeletal points have generic algebraic support, so f and f^{-1} are analytic near Σ. At a vertex, the finitely many incident graph branches are among the residual directions of Theorem 4.25.3. Theorem 4.24.5 makes $-\log|f|$ integral affine on a sufficiently short initial segment of every one of them.

For an open nodal edge choose an algebraic affine neighborhood of its residual node. Its formal generic fibre contains the entire nodal tube. Express $f = P/Q$ in the fraction field of the generic coordinate algebra. The functions P, Q are analytic on that tube. In its pointed annulus coordinate, their norms on every compact subinterval of radii are finite maxima of Laurent monomial norms, by Proposition 4.4.1 and its first, nonunit, assertion. The two nonzero functions have positive Gauss norm at every radius. Subtracting their logarithmic profiles proves finite piecewise integral affineness on each compact interior subinterval. The already affine endpoint segments leave one such compact subinterval. There are finitely many edges, so one finite common subdivision works. □

Proposition 5.26.4 (Local rational coordinates of lattice slope one)**.** *Every point of Σ has a graph neighborhood W and finitely many rational functions whose negative logarithmic norms give an injective map $W \to \mathbf{R}^m$. On each sufficiently short edge of W, one coordinate has slope 1 or -1. At a model*

vertex the incident branches can be sent to distinct positive coordinate axes, after subtracting the value at the vertex.

Proof. For a point interior to an edge, its nodal annulus coordinate z is an analytic unit, and $-\log|z|$ is its length coordinate. Lemma 5.26.1 gives a rational function with the same norm on a neighborhood of the point. Restricting to a small interval proves the assertion there, for every real radius.

At a vertex v_C, label its incident branches by the distinct normalization points $p_1, \dots, p_m$, counting the two ends of a loop separately. Choose the residual functions u_i of Lemma 5.26.2. By Proposition 4.24.2, each is the residue of a norm-one element of $\mathcal{H}(v_C)$. The density of $K(X)$ in this field gives $f_i \in K(X)^\times$ of norm one with residue u_i. The slope formula Theorem 4.24.5 then gives, on the j-th short branch with length parameter τ,

$$\big(-\log|f_1|, \dots, -\log|f_m|\big)(\gamma_j(\tau)) = \tau e_j.$$

There are only finitely many pairs (i, j), so one shortening works for all of them. The resulting star is mapped injectively, with primitive edge direction vectors. An isolated vertex needs no local coordinates. □

Theorem 5.26.5 (Faithful tropicalization of a compact model skeleton)**.** *There is a nonempty Zariski open $U \subset X$ containing Σ in U^{an} and a closed immersion $\varphi : U \hookrightarrow \mathbf{G}_m^n$ such that $\varphi_{\mathrm{trop}}|_\Sigma$ is a homeomorphism onto its image. After a finite subdivision of Σ, its restriction to each edge is integral affine with primitive derivative vector. Thus each edge has its original logarithmic length measured in the primitive lattice length of its tropical image.*

Proof. Compactness gives finitely many neighborhoods from Proposition 5.26.4. Collect their coordinate functions into a map $F : \Sigma \to \mathbf{R}^a$. It is injective on each of these neighborhoods. Let V be the union of their Cartesian squares in $\Sigma \times \Sigma$. This is an open neighborhood of the diagonal. The compact set $E = (\Sigma \times \Sigma) \setminus V$ contains no diagonal point.

Any two distinct points $x, y \in \Sigma$ are separated by the absolute value of a nonzero rational function. Indeed, choose an algebraic affine open of X. It contains the algebraic support of every skeletal point, so its analytification contains x, y. Their distinct seminorms on its coordinate ring are separated by some element of that ring. By generic support, its two values are nonzero. The inequality of its two norms persists on a neighborhood of (x, y). Finitely many such neighborhoods cover E; append their rational functions to F. The

resulting map G is injective on all of Σ. It is a homeomorphism onto its image because its source is compact and its target is Hausdorff.

Proposition 5.26.3 gives a finite common subdivision on which all coordinates of G are integral affine. On the interior of each resulting edge, one of the original local coordinates has slope 1 or -1. Its derivative vector is therefore primitive. Refining further retains this property.

It remains to realize these coordinates in a closed torus immersion. Choose a nonempty affine open $U_0 = \operatorname{Spec} A$ of X. Write the finitely many rational functions of G as fractions of nonzero elements of A, and invert all their numerators and denominators. This gives a finite-type algebra B and an affine open on which these functions are units. Choose finitely many nonzero algebra generators of B, and invert them as well, obtaining B'. The chosen generators and their inverses generate B', so they give a surjection from a Laurent polynomial algebra onto B'. Append the functions defining G to this generating list. The resulting map $U = \operatorname{Spec} B' \to \mathbf{G}_m^n$ is a closed immersion. Every inverted element is nonzero in $K(X)$, so its zero locus contains no skeletal point; hence $\Sigma \subset U^{\mathrm{an}}$. Adding coordinates retains injectivity and a derivative coordinate equal to 1 or -1. Apply Proposition 5.26.3 once more to the appended functions and take a finite common refinement. All coordinates of the closed immersion are then affine on every subdivided edge, while the primitive derivative coordinate is retained. The primitive derivative formula identifies the two edge lengths, as asserted. □

References. This is the rational-coordinate form of faithful tropicalization for compact curve skeleta. The proof constructs local lattice-slope-one coordinates first, then uses compact pair separation and an affine algebra presentation to obtain a closed torus immersion.

5.27 Faithful curve coordinates over the original field

Lemma 5.27.1 (An affine neighborhood of finitely many projective points)**.** *Let Z be a projective scheme over any field k and let $q_1, \dots, q_m$ be distinct closed points. They lie in a common affine open of Z. If C is a projective integral curve and $p_1, \dots, p_m$ are distinct closed points of its normalization, there are functions $u_i \in k(C)^\times$ satisfying*

$$\operatorname{ord}_{p_j}(u_i) = \delta_{ij}.$$

Proof. Embed Z in $\mathbf{P}^n_k$ and let P_i be the homogeneous prime defining q_i. The P_i are pairwise incomparable. For each $j \neq i$, choose a homogeneous element of $P_j \setminus P_i$, and let g_i be their product; if $m = 1$, put $g_1 = 1$. Then g_i vanishes at the other selected points and not at q_i. Choose a coordinate $x_{a_i} \notin P_i$ and an integer $D \geq \max(1, \max_i \deg g_i)$. The homogeneous polynomial

$$F = \sum_i g_i x_{a_i}^{D - \deg g_i}$$

is outside every P_i: modulo P_i its only nonzero summand is the i-th. The standard affine open $D_+(F)$ in projective space therefore gives the required affine open on Z. This construction works over finite fields as well.

Apply this to the finitely many distinct images of the p_j on C. Its inverse image under normalization is affine, because normalization is finite by [StacksAlg, Tag 0BXR]. Write it as $\operatorname{Spec} B$, and let $\mathfrak{m}_j$ correspond to p_j. By [StacksAlg, Tag 0B8Y], $B_{\mathfrak{m}_i}$ is a discrete valuation ring. A local uniformizer is a_i/b_i with $a_i, b_i \in B$ and $b_i \notin \mathfrak{m}_i$, so a_i has order one at p_i. The ideals $\mathfrak{m}_i^2$ and $\mathfrak{m}_j$, $j \neq i$, are pairwise comaximal. The elementary Chinese remainder construction in Lemma 5.26.2 uses only these comaximalities, not any assumption on k: for $I + J = B$ and $a + b = 1$ with $a \in I, b \in J$, the element $br + as$ has prescribed residues r modulo I and s modulo J, and induction treats finitely many ideals. Choose u_i congruent to a_i modulo $\mathfrak{m}_i^2$ and to 1 modulo every other $\mathfrak{m}_j$. This gives precisely the displayed orders, regardless of the residue degrees of the p_j. □

In the rest of this section, K is complete and nontrivially valued, with its original absolute value. Let X be a smooth projective integral K-curve with a projective model $\mathcal{X}/K^\circ$ satisfying Theorem 4.23.3, or its characteristic-zero form Theorem 4.28.5. Thus all nodes used for the finite graph are split rational nodes with the stated pointed charts. Put $\Sigma = \Sigma(\mathcal{X})$.

Lemma 5.27.2 (Generic support and rational approximation over K)**.** *Every $x \in \Sigma$ has generic algebraic support. If g is an analytic germ at x with $g(x) \neq 0$, some $f \in K(X)^\times$ satisfies $|f| = |g|$ on an ordinary neighborhood of x. If $|g(x)| = 1$, the residues of $f(x)$ and $g(x)$ can also be made equal.*

Proof. For a finite extension E/K, Lemma 1.12.1 makes E complete, and its proof shows that every value in $v(E^\times)$ has a positive integer multiple in $v(K^\times)$. Moreover $\widetilde{E}/\widetilde{K}$ is algebraic: for $b \in E$ of norm one, take a polynomial relation over K, divide its coefficients by a coefficient of maximal norm, and

reduce. The resulting polynomial over $\widetilde{K}$ is nonzero and annihilates the residue of b. Thus the residual transcendence degree and relative rational value-group rank of a finite extension are both zero.

At a component vertex, Proposition 4.24.2 gives residual transcendence degree one. At an annular Gauss point, Proposition 3.8.1 gives the sum of these two invariants equal to one. If a skeletal point had closed algebraic support, its residue field would be a finite extension of K by [StacksAlg, Tag 00P0], and Proposition 2.10.1 would identify its completed residue field with that same finite field. This contradicts the preceding invariants. An integral algebraic curve has only generic and closed points, so its support is generic.

Consequently $\mathcal{H}(x)$ is the completion of $K(X)$ for the norm of x. Choose $f \in K(X)$ with $|f(x) - g(x)| < |g(x)|$. The rational function and its inverse are regular on a Zariski open containing the generic support; their analytifications are defined near x. Continuity preserves the strict inequality on an ordinary neighborhood, and the ultrametric inequality gives equality of the norms there. For norm-one values the error has norm less than one, which also identifies the residues. □

Proposition 5.27.3 (Integral affine coordinates on the original skeleton)**.** *For every $f \in K(X)^\times$, the function $-\log|f|$ is finite and continuous on Σ and integral affine on the edges of a finite subdivision. Every point of Σ has a graph neighborhood on which finitely many such functions give an injective map with a derivative coordinate equal to 1 or -1 on each short edge.*

Proof. By Lemma 5.27.2, f and its inverse are analytic near every skeletal point. At a vertex, each incident graph branch is a rational normalization point above a split node, by Lemma 4.24.1. Theorem 4.24.5, over the original field, makes the logarithmic norm affine with integral slope on a short initial segment. On the interior of a nodal edge, express $f = P/Q$ on an algebraic affine neighborhood of the residual node. Both functions are analytic on the nodal tube. On each compact radius subinterval their nonzero Gauss norms are finite maxima of Laurent monomial norms by Proposition 4.4.1. Hence their logarithmic difference has finitely many integral affine pieces. The endpoint segments leave a compact interior interval; doing this for the finitely many edges proves the first assertion.

At an interior point, approximate its annular coordinate z by Lemma 5.27.2. The rational norm agrees near that point with $|z|$, so its negative logarithm is an oriented length coordinate of slope one. At a vertex v_C, list

the finitely many distinct normalization points p_j corresponding to incident edges, counting the two branches of a loop separately. Lemma 5.27.1 gives $u_i \in k(C)^\times$ with orders δ_{ij}. Proposition 4.24.2 identifies these with norm-one residues in $\mathcal{H}(v_C)$. Approximate their lifts by $f_i \in K(X)$ with error of norm less than one. Theorem 4.24.5 gives, after a simultaneous shortening of finitely many branches,

$$(-\log|f_1|,\dots,-\log|f_m|)(\gamma_j(t)) = te_j.$$

These coordinates embed the small graph star in distinct positive coordinate axes and have primitive derivative vectors. An isolated vertex requires no coordinates. □

Theorem 5.27.4 (Original-field faithful tropicalization of a split model skeleton)**.** *There is a nonempty Zariski open $U \subset X$ with $\Sigma \subset U^{\mathrm{an}}$ and a closed immersion over the original field K,*

$$\varphi : U \hookrightarrow \mathbf{G}_{m,K}^n$$

whose tropicalization restricts to a homeomorphism on Σ. On a finite subdivision of its edges, it is integral affine with primitive derivative vector. Consequently it preserves every edge length with the original absolute value, when the image is measured by primitive lattice length.

Proof. Take finitely many graph neighborhoods from Proposition 5.27.3 covering the compact graph. Gather their rational coordinate functions into F. The union V of the squares of these neighborhoods is open in $\Sigma \times \Sigma$ and contains the diagonal. On every pair in V the coordinates of F separate distinct points. The complement E is compact and disjoint from the diagonal.

Any two distinct $x, y \in \Sigma$ are distinguished by a nonzero rational function over K. Indeed, any algebraic affine open of X contains their generic algebraic supports, and the two distinct seminorms on its coordinate ring differ on some element. Both values are nonzero by generic support. The strict inequality of absolute values persists on an ordinary neighborhood of (x, y). Finitely many such pair neighborhoods cover E. Append their rational functions to F, obtaining a globally injective continuous map G. A continuous injection of this compact graph into a Hausdorff real vector space is a homeomorphism onto its image.

Choose a nonempty affine open $\operatorname{Spec} A \subset X$. Invert the finitely many nonzero numerators and denominators of the coordinates of G, obtaining B,

and then invert finitely many nonzero algebra generators of B, obtaining B'. Those generators and their inverses generate B'. Together with the coordinates of G they therefore give a surjection from a Laurent polynomial K-algebra onto B', hence the required closed immersion for $U = \operatorname{Spec} B'$. Every function inverted is nonzero at every generic-support point, so $\Sigma \subset U^{\mathrm{an}}$. All choices and functions are defined over K.

Apply Proposition 5.27.3 to all final coordinates and take a finite common subdivision. On each resulting edge, one of the local coordinates has derivative 1 or -1 on a nonempty subinterval, hence on the entire edge because it is affine there. The full integer derivative vector is therefore primitive. Appending coordinates does not remove that coordinate or destroy injectivity. The primitive lattice parameter on the image is exactly the original annular length parameter, which proves the length assertion. □

References. The construction extends the compact-model-skeleton argument to the original field. Homogeneous finite-point separation replaces infinite-field hyperplane avoidance, and the residue/value invariants of finite extensions replace algebraic closedness in the support argument.

5.28 Geometric degrees and tropical lattice indices

Write $R = K^\circ$, $\mathfrak{m} = K^{\circ\circ}$ and $k = \widetilde{K}$. In the first two lemmas, K is complete and nontrivially valued, and $\pi \in R$ satisfies $0 < |\pi| < 1$. The formal algebras carry their quotient π-adic topologies. For a lattice M, the notation $R\langle M\rangle$ denotes the Gauss unit ball in $K\langle M\rangle$.

Lemma 5.28.1 (The quotient model of an initial degeneration)**.** *Let $I \subset K[M]$ be an ideal and set*

$$P = K\langle M\rangle, \qquad P_0 = R\langle M\rangle, \qquad J = IP.$$

Then

$$B_0 = P_0/(J \cap P_0)$$

is an admissible topologically finitely presented R-algebra, with generic algebra $B = P/J$. Its special algebra is

$$B_0/\mathfrak{m}B_0 \simeq k[M]/\overline{I \cap R[M]},$$

where the bar denotes coefficientwise reduction in $k[M]$.

Proof. The ideal J is closed by Theorem 1.18.3. The algebra B_0 embeds in P/J, so it has no R-torsion. It is a quotient of a restricted power-series algebra in finitely many variables: use generators for M and their inverses. Consequently Theorem 3.14.4 gives a finite presentation, and Proposition 3.14.5 gives completeness and separatedness. Every restricted Laurent series becomes integral after multiplication by a power of π. The same is true of every element of J, whence

$$B_0[1/\pi] = P/J.$$

The elements of P_0 of Gauss norm less than one form $\mathfrak{m}P_0$. To see this, choose a coefficient of maximal absolute value in a nonzero restricted series. Dividing by that coefficient gives an element of P_0; the coefficient belongs to $\mathfrak{m}$ precisely when the original norm is less than one. The reverse inclusion follows from the ultrametric inequality applied to a finite sum. Thus $P_0/\mathfrak{m}P_0 = k[M]$. Proposition 5.13.5 identifies the image of $J \cap P_0$ in this ring with $\overline{I \cap R[M]}$, proving the special-fibre formula. □

Lemma 5.28.2 (Finite formal models at the Gauss point)**.** *Let*

$$A_0 = R\langle U_1^{\pm 1}, \dots, U_d^{\pm 1}\rangle, \qquad A = A_0[1/\pi],$$

and let $A_0 \to B_0$ be a formal homomorphism to an admissible topologically finitely presented R-algebra. Suppose that $\overline{B}_0 = B_0/\mathfrak{m}B_0$ is finite over $k[U_1^{\pm 1}, \dots, U_d^{\pm 1}]$. Then B_0 is finite over A_0, and the generic algebra $B = B_0[1/\pi]$ has its canonical finite-A-module topology. If H is the completed residue field of the Gauss point of $\mathcal{M}(A)$, then

$$\dim_H (B \widehat{\otimes}_A H) = \dim_{k(U)} \left(\overline{B}_0 \otimes_{k[U^{\pm 1}]} k(U) \right). \tag{5.28.1}$$

Proof. Choose bounded formal generators $y_1, \dots, y_s$ of B_0, giving a surjection $A_0\langle Y_1, \dots, Y_s\rangle \to B_0$. Each $\overline{y}_i$ satisfies a monic polynomial over the residual Laurent ring. Lift its coefficients to obtain $P_i \in A_0[Y_i]$, monic of degree $r_i > 0$, with all lower coefficients of Gauss norm at most one. Since $P_i(y_i) \in \mathfrak{m}B_0$, this element is a finite sum of scalars in $\mathfrak{m}$ times elements of B_0. Lift those elements through the formal presentation. This produces $G_i \in A_0\langle Y\rangle$ with

$$G_i(y) = P_i(y_i), \qquad \|G_i\| < 1.$$

There are finitely many equations, so $\theta = \max_i \|G_i\| < 1$ is a common contraction bound.

Apply Lemma 3.16.4 over A. Its convergent division formula expresses every bounded restricted series, modulo the $P_i - G_i$, as a linear combination of the monomials Y^ν with $0 \leq \nu_i < r_i$. The coefficient maps have norm at most one, so a series in $A_0\langle Y\rangle$ has all its remainder coefficients in A_0. The images of these finitely many monomials therefore generate B_0 over A_0.

The same calculation compares the generic norms. Give B the quotient norm from $A\langle Y\rangle$ and let $\|\cdot\|_{\mathrm{fin}}$ be the quotient norm from the finite free A-module on the remainder monomials. A finite remainder expression gives $\|b\|_B \leq \|b\|_{\mathrm{fin}}$. Conversely, apply the bounded division formula to each representative of b and take the infimum of its norms. It gives $\|b\|_{\mathrm{fin}} \leq \|b\|_B$. Hence these quotient norms agree; the equivalent admissible norms give the same Banach topology.

Let $S \subset A_0$ consist of the elements of Gauss norm one, and put $V = S^{-1}A_0$. The Gauss norm is multiplicative by Proposition 1.2.2. Every nonzero restricted Laurent series a has a coefficient $c \in K^\times$ of maximal absolute value, so that $a = cs$ with $s \in S$. Every element of $\mathrm{Frac}(A)^\times$ is consequently of the form cs/t, where $c \in K^\times$ and $s, t \in S$. It belongs to V exactly when $|c| \leq 1$. Thus V is the Gauss valuation ring in $\mathrm{Frac}(A)$, with residue field $k(U)$: the reductions of S are all the nonzero Laurent polynomials. Its completion is H°. Indeed, an element of norm at most one in the completion of $\mathrm{Frac}(A)$ can be approximated, with error less than one, by elements of norm at most one, which belong to V.

The finite V-module $M = S^{-1}B_0$ is torsion-free. Every nonzero element of V is a nonzero scalar from R times a unit; multiplication by that scalar is injective on B_0 and remains injective after localization. A finite torsion-free module over a valuation ring is free. One can see this directly by choosing a generating set of minimal size: a nonzero relation has a coefficient dividing all the others, and torsion-freeness then expresses the corresponding generator in terms of the rest. This contradicts minimality.

The rank of M is therefore its residue dimension over $k(U)$, which is the right-hand side of (5.28.1). Tensoring its free basis with H computes $\dim_H(B \otimes_A H)$ with the same value. The norm comparison above and Proposition 2.33.19 identify this algebraic tensor product with $B\widehat{\otimes}_A H$. □

Proposition 5.28.3 (Initial multiplicity and an actual fibre degree)**.** *Let $X \subset T_N$ be a pure d-dimensional closed subscheme over a complete real-valued non-Archimedean field K. Let w belong to the relative interior of a maximal cell σ of an initial-stratum complex, and suppose that $q : N \to \mathbf{Z}^d$ restricts to an injective map on N_σ with finite index*

$$e = [\mathbf{Z}^d : q(N_\sigma)].$$

After a complete isometric extension realizing w, made nontrivially valued, translate w to zero. The induced monomial map from the actual tropical fibre X_w to the d-dimensional unit polycircle is finite and norm-admissible. Its fibre algebra over the completed Gauss residue field H has dimension $e\, m_X(\sigma)$. Multiplying the target coordinates by constants of absolute value one preserves these conclusions.

Proof. Theorem 5.13.6 supplies the required complete extension and translation, and Proposition 5.19.8 preserves the weight. The quotient model of Lemma 5.28.1 has the full initial algebra as its special algebra. By Proposition 5.19.3 and Theorem 5.19.5, a lattice splitting identifies it with

$$D[N_\sigma^\vee], \qquad \dim_k D = m_X(\sigma),$$

where D is a finite k-algebra.

In this presentation the target characters reduce to $d_j \chi^{a_j}$, with $d_j \in D^\times$. The a_j form a basis of a sublattice of index e in $N_\sigma^\vee$. Choose representatives for its cosets. Every character has a unique expression as one of these representatives times a product of the χ^{a_j}. Substitution of $\chi^{a_j} = d_j^{-1} U_j$ consequently makes $D[N_\sigma^\vee]$ free of rank e over $D[U^{\pm 1}]$. The asserted independence follows by comparing character exponents, since the d_j are units. A k-basis of D then gives a basis of size $e \dim_k D$ over $k[U^{\pm 1}]$.

The normalized monomials and their inverses belong to the quotient formal model, so this is the reduction of a formal homomorphism from the unit-polycircle model. Apply Lemma 5.28.2. It proves the finiteness, norm comparison and actual Gauss fibre dimension. A unit constant changes only one of the coefficients d_j in the preceding calculation. The calculation uses a basis of the entire finite algebra D, including its nilpotents. □

Lemma 5.28.4 (A finite free locus)**.** *Let $f : X \to Y$ be a dominant morphism of affine finite-type K-schemes. Suppose that Y is integral and that the generic fibre is nonempty and zero-dimensional. Put $F = K(Y)$ and*

$$\delta = \dim_F \Gamma(X_F, \mathcal{O}_{X_F}).$$

There is a nonzero $h \in \Gamma(Y, \mathcal{O}_Y)$ such that $f^{-1}(Y_h) \to Y_h$ is finite free of rank δ.

Proof. Write $Y = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$. Lemma 5.19.1 makes the generic algebra B_F finite-dimensional. Each of a finite set of A-algebra generators of B satisfies a monic polynomial over F. Clear the finitely many coefficient denominators. Each resulting polynomial equality holds in B_F, so its error in the localized B is killed by a further nonzero element of A. After inverting the product of these finitely many elements, all the monic equalities hold. The bounded monomials in the generators then give a finite set of module generators for this localization of B.

Choose $c_1, \ldots, c_\delta$ among those monomials whose images form an F-basis of B_F. Express the other module generators in that basis. Clear their finitely many coefficient denominators and then kill their finitely many errors by another localization in $A \setminus \{0\}$. The c_i now generate the localized module. They are independent: a relation maps to an F-linear relation between their images, and its coefficients vanish because the localized A injects into F. The product of all elements inverted is the required h, and the displayed basis has size δ. □

Theorem 5.28.5 (The algebraic tropical degree formula)**.** *Let $\varphi : T_N \to T_{N'}$ be a homomorphism of split tori over a complete real-valued non-Archimedean field K, and write $a : N \to N'$ for its lattice map. Let $X \subset T_N$ be a pure d-dimensional closed subscheme and $Y \subset T_{N'}$ an integral d-dimensional closed subscheme. Suppose that $\varphi|_X : X \to Y$ is dominant with nonempty zero-dimensional generic fibre of scheme degree δ.*

There is a finite union E_{trop} of polyhedra of dimension less than d in $\mathrm{Trop}(Y)$ such that, for $w' \notin E_{\mathrm{trop}}$, its tropical preimages form a finite set $\{w_i\}_i$. Each w_i lies in the relative interior of a maximal initial cell σ_i, while w' lies in the relative interior of a maximal initial cell σ', and

$$\delta\, m_Y(\sigma') = \sum_i m_X(\sigma_i)[N'_{\sigma'} : a(N_{\sigma_i})]. \tag{5.28.2}$$

For the polyhedral pushforward of Theorem 5.17.2, this is the weighted-cycle identity

$$a_* \operatorname{Trop}(X) = \delta \operatorname{Trop}(Y).$$

Proof. Step 1: choose the finite free locus and the regular weights. Choose h by Lemma 5.28.4, and let $E = Y \setminus Y_h$ have its reduced closed structure. Every irreducible component of E has dimension less than d: a chain of primes above a nonzero prime of the domain $\Gamma(Y, \mathcal{O}_Y)$ becomes one step longer after adjoining its zero prime. Apply Theorem 5.19.5 to the finitely many components of E. Their tropicalizations form a finite union of lower-dimensional polyhedra.

Also exclude the nonmaximal cells of $\operatorname{Trop}(Y)$, the images of nonmaximal cells of $\operatorname{Trop}(X)$, and the images of maximal source cells on which a has rank less than d. Intersect these images with the target complex. The finite polyhedral elimination and refinement in Theorem 5.17.2 give a finite union E_{trop} of polyhedra of dimension less than d. It cannot contain the relative interior of a maximal target cell.

Fix $w' \notin E_{\mathrm{trop}}$. Each of its tropical preimages is in the relative interior of a maximal source cell on whose direction space a is injective. Each of finitely many such cells contributes at most one point. Thus the preimage set is finite. Near each w_i, the image of its full-dimensional source cell is contained in the local tropical support at w', which is the direction space of σ'. The dimensions are equal, so

$$a(N_{\sigma_i})_{\mathbf{R}} = N'_{\sigma',\mathbf{R}}.$$

The indices in (5.28.2) are therefore finite.

Step 2: decompose the actual inverse-image algebra. Use Theorem 5.13.6 to make one complete isometric extension L/K realizing w' and all the w_i, with nontrivial valuation. The finite free algebra on Y_h remains free of rank δ after extension, and Proposition 5.19.8 preserves all the weights. In particular, the subsequent argument uses the same finite free module even if Y_L is reducible or nonreduced.

Write $Y_{w'} = \mathcal{M}(C)$ for the actual target tropical fibre over L. Since $w' \notin \operatorname{Trop}(E)$, h has no zero on this fibre. It is therefore a unit in C by Theorem 1.20.1, and $Y_{w'} \subset (Y_h)^{\mathrm{an}}$. The explicit tensor description in Theorem 2.33.26 shows that the inverse image is $\mathcal{M}(D)$, where D is a finite C-algebra free of rank δ.

Every point of this inverse image has tropical weight one of the w_i. Its subset of weight w_i is the actual affinoid domain $X_{w_i} = \mathcal{M}(B_i)$ of Proposition 5.13.2. These subsets are open and closed: they are fibres of a continuous map with values in a finite Hausdorff set. Applying the structure sheaf property of Theorem 2.39.4 to this finite disjoint cover gives

$$D \simeq \prod_i B_i. \tag{5.28.3}$$

The inverse image is nonempty. Indeed, $C \neq 0$ and $\delta > 0$, so $D \neq 0$; the affinoid nonemptiness theorem 1.20.1 gives a point of $\mathcal{M}(D)$. Consequently the set of w_i is nonempty.

Step 3: compare all degrees over one Gauss field. The saturated lattice $N'_{\sigma'}$ is a direct summand of N' by Lemma 5.14.1. Choose $q : N' \to \mathbf{Z}^d$ whose restriction to it is an isomorphism. Divide the corresponding target monomials by their values at a translating point of weight w'. This gives one unit-polycircle algebra $A = L\langle U_1^{\pm 1}, \dots, U_d^{\pm 1}\rangle$ mapping to C and to every B_i. After the respective source translations, the induced monomials differ from those for $q \circ a$ by constants of absolute value one, because $a(w_i) = w'$.

Let H be the completed Gauss residue field of A. Proposition 5.28.3 gives

$$\begin{aligned} \dim_H(C \widehat{\otimes}_A H) &= m_Y(\sigma'), \\ \dim_H(B_i \widehat{\otimes}_A H) &= m_X(\sigma_i)[N'_{\sigma'} : a(N_{\sigma_i})]. \end{aligned}$$

All the maps are finite with the canonical finite-module norms, so Proposition 2.33.19 identifies these completed tensors with the algebraic ones. Since D is free of rank δ over C, tensoring its free basis over A with H gives

$$\dim_H(D \otimes_A H) = \delta \dim_H(C \otimes_A H).$$

Taking dimensions in (5.28.3) proves (5.28.2).

Theorem 5.22.5 makes both weighted tropicalizations balanced cycles. Choose compatible source and target subdivisions as in the polyhedral pushforward theorem. The equality holds on a dense relative open subset of every maximal target cell. Both weights are constant on each such cell by Proposition 5.19.8 and the definition of polyhedral pushforward. This proves the weighted-cycle identity. □

Remark 5.28.6 (Finite algebras and nonreduced targets). Every degree in the proof is the dimension of a finite algebra. For the map $t \mapsto t^p$ in characteristic p, both this degree and the lattice index are p, although each geometric fibre has one underlying point.

The same proof applies to a pure d-dimensional target Y if there is an open subset meeting all its generic points on which the source algebra is finite free of one fixed rank δ. The complement again has dimension less than d, and Steps 2 and 3 use precisely this free module. For comparison, the closed immersion

$$T^d \hookrightarrow T^d \times \operatorname{Spec} K[\epsilon]/(\epsilon^2), \qquad \epsilon \longmapsto 0,$$

is finite and dominant on underlying spaces, but its generic module is not free. The source weight is one and the target weight is two. Thus its residue-field degree one does not give the displayed degree formula. Realize the target as a torus subscheme by writing $\epsilon = V - 1$ and imposing $(V-1)^2 = 0$.

References. The weighted-cycle identity is the classical Sturmfels–Tevelev multiplicity formula; its trivial- and nontrivial-valuation forms are reviewed in [GRW16, §8]. The proof above derives it from the initial quotient model and the actual finite analytic fibres. The short valuation-module and generic-basis arguments agree with the algebraic facts in the Stacks Project, Tags 0539, 00NZ, 02NW and 051S.

5.29 Analytic coordinates on a skeletal chart

Let K be complete and nontrivially valued, with its specified real-valued absolute value. Put $O = K^\circ$, $k = \widetilde{K}$, $v = -\log|\cdot|$, and $\Gamma = v(K^\times)$. Consider a topologically finitely presented formally étalemap of admissible algebras

$$A_0 = O\langle X_0, \dots, X_r, T_1, \dots, T_q\rangle/(X_0 \cdots X_r - a) \longrightarrow B_0, \qquad 0 < |a| < 1.$$

Write $B = B_0 \otimes_O K$, $Y = \mathcal{M}(B)$, and $\Sigma \subset Y$ for the inverse image of the standard skeleton. The notation Δ_I, Q_I, and $x_{\eta,w}$ is that of Proposition 5.24.2 and Theorem 5.24.4. In particular, Σ is a compact finite complex, and a point over the relative interior of Δ_I is specified by w and $\eta \in Q_I$. A finite piecewise Γ-integral-affine function has integral linear parts and constants in Γ on a finite subdivision defined by integral linear inequalities with constants in Γ.

Proposition 5.29.1 (Finite piecewise-affine values)**.** *Let $g \in B$ be nonzero at every point of Σ. Then $v(g)|_\Sigma$ is finite piecewise Γ-integral-affine. One finite hyperplane subdivision of the standard simplex induces a suitable subdivision on every skeletal cell.*

Proof. Assume that Σ is nonempty. Compactness gives $\alpha = \min_\Sigma |g| > 0$. Use the bounded square-Jacobian presentation of Lemma 5.23.1, with root coordinates $Y_1, \dots, Y_m$, inverse determinant Q, and integral polynomial equations $F(Y) = 0$ whose coefficients belong to A_0. Give its generic algebra the quotient Banach norm in which all these coordinates have norm at most one. Choose a polynomial $P(X, T, Y, Q)$ such that

$$\|g - P(X, T, Y, Q)\| < \varepsilon < \alpha.$$

Let $M \geq 1$ bound the absolute values of its coefficients. The difference of two monomials, expanded one factor at a time, shows that P is M-Lipschitz on tuples of norm at most one.

Approximate the finitely many restricted coefficients of F by integral polynomials in X, T, with error $\delta < 1$ and $M\delta < \varepsilon$, obtaining G. At the original root, both $G(Y)$ and $G'(Y) - F'(Y)$ have norm at most δ. The inverse of $F'(Y)$ has norm at most one, since its adjugate is integral and Q has norm at most one. The geometric inverse gives the same bound for $G'(Y)^{-1}$. Lemma 5.18.1 produces $Y_G \in B^m$ with

$$G(Y_G) = 0, \qquad \|Y_G - Y\| \leq \delta, \qquad \|Y_G\| \leq 1.$$

The determinant and reciprocal identities imply that $Q_G = (\det G'(Y_G))^{-1}$ has norm at most one and $\|Q_G - Q\| \leq \delta$. Hence

$$g_G = P(X, T, Y_G, Q_G), \qquad \|g_G - g\| < \alpha, \qquad |g_G(x)| = |g(x)| \quad (x \in \Sigma).$$

Put

$$R = K[X_0, \dots, X_r, T]/(X_0 \cdots X_r - a),$$
$$C = R[Y, Q]/(G, Q \det G' - 1), \qquad F_0 = \mathrm{Frac}(R).$$

Eliminating X_0 identifies R with a Laurent polynomial ring in $X_1, \dots, X_r$ and a polynomial ring in T, so it is a domain. The square-zero correction by the inverse Jacobian proves that C/R is formally étale. It is finitely presented, and therefore étaleby [StacksAlg, Tag 00UR]. Its affine generic fibre is a finite product of finite separable extensions of F_0 by [StacksAlg,

Tag 02GL]; finiteness here follows from quasi-compactness of this discrete fibre. Thus the image of P satisfies a polynomial over F_0.

At a skeletal point, evaluation on R is its standard Gauss norm and has zero kernel. The evaluation $C \to \mathcal{H}(x)$ therefore extends to $C \otimes_R F_0$. In particular this generic fibre is nonzero. Clear the denominators of an annihilating polynomial and remove its largest common power of the variable. Because $g_G(x) \neq 0$, the resulting relation holds at every skeletal point and has nonzero constant and leading coefficients:

$$\sum_{i=0}^{s} A_i g_G^i = 0, \qquad A_i \in R.$$

There are at least two nonzero terms. Write $\lambda_i(w) = v(A_i(z_w))$ for each nonzero A_i. A Laurent polynomial Gauss norm makes λ_i the minimum of finitely many Γ-integral-affine functions. At least two terms of the displayed relation have the smallest valuation, whence

$$v(g(x_{\eta,w})) \in \left\{ \frac{\lambda_i(w) - \lambda_j(w)}{j - i} : i < j,\ A_i A_j \neq 0 \right\}.$$

Subdivide the standard simplex by comparisons of coefficient monomials and then by comparisons of the resulting affine candidates. Multiplying the equations by positive integers leaves integral hyperplanes with constants in Γ. On a connected relative open piece the distinct candidates have a fixed strict order. A continuous selection from them must select one throughout that piece. This gives finitely many rational-affine formulas, with a common subdivision on all the skeletal cells.

To establish integrality, fix a relative open cell. The proof of Theorem 5.25.3, before its use of the unit hypothesis, constructs a single coefficient field L_0 with $v(L_0^\times) = \Gamma$ and an open polyannulus parametrizing this cell's Gauss points. Pull back g to that polyannulus. The Laurent expansion constructed in Lemma 5.25.1 gives

$$\log |g(z_w)| = \max_m \{\log |c_m| - \langle m, w \rangle\}.$$

On a full-dimensional open piece where the left side is affine, choose a Laurent term attaining the maximum at an interior point. The difference between the left side and this term is affine, nonnegative, and zero at an interior point, so it is identically zero there. That piece consequently has an integral slope and a constant in Γ. Continuity gives the same assertion on its faces. Each

lower-dimensional piece is a face of a full-dimensional piece in its original cell. An original zero-dimensional cell is evaluated in L_0 itself. This proves the assertion on every cell. □

Lemma 5.29.2 (Small scalar shifts)**.** *For $h \in B$, at most finitely many $c \in K$ make $h + c$ vanish at a point of Σ. Consequently a nonvanishing shift can be chosen with arbitrarily small positive $|c|$.*

Proof. On each relative open cell use the fixed coefficient field and polyannulus from the preceding proof. If $h + c$ vanishes at one Gauss point, every Laurent coefficient vanishes. Its pullback is then the zero function on that polyannulus. Since the inclusion of K is isometric, at most one scalar has this property on a given cell. The same argument for a zero-dimensional cell takes place in its coefficient field. There are finitely many cells. Powers of any $\pi \in K$ with $0 < |\pi| < 1$ give infinitely many distinct nonzero scalars converging to zero, so they avoid the finite exceptional set. □

Theorem 5.29.3 (Faithful analytic coordinates)**.** *There are an ordinary open analytic neighbourhood V of Σ in Y and an analytic map $\varphi : V \to (\mathbf{G}^N_{m,K})^{\mathrm{an}}$ whose tropicalization is a homeomorphism on Σ onto its image. On every cell of a finite subdivision it is integral affine and induces an isomorphism onto the saturated lattice of its image.*

Proof. Include the units $X_0, \dots, X_r$, whose product is a; their valuations distinguish the base weights. For each I and $\eta \in Q_I$, choose $\bar{h}_\eta$ in the special-fibre algebra that belongs to every other prime in Q_I and does not belong to η. Indeed these primes are incomparable, since they lie in the same zero-dimensional étalefibre; multiplying elements of $\eta' \setminus \eta$ for $\eta' \neq \eta$ gives the required element. Omit the construction when Q_I has fewer than two members. Lift $\bar{h}_\eta$ to $h_\eta \in B_0$ and choose $0 < |c_\eta| < 1$ by Lemma 5.29.2 so that $g_\eta = h_\eta + c_\eta$ is nonzero on Σ. Specialization gives, for $w \in \operatorname{relint} \Delta_I$,

$$|g_\eta(x_{\eta,w})| = 1, \qquad 0 < |g_\eta(x_{\eta',w})| < 1 \quad (\eta' \neq \eta).$$

Thus this finite list separates points over the same weight.

Let V be the complement of the zero loci of the g_η. It is an ordinary open neighbourhood of Σ. Each g_η is an analytic unit on V: near any point choose a positive lower bound for its norm and use the inverse on the corresponding rational domain, as in Theorem 2.31.2. These units and the X_i define φ. Its

tropicalization is continuous and injective on Σ, hence is a homeomorphism onto its image by compactness.

Apply Proposition 5.29.1 to the finite list. The union of its hyperplanes gives a common finite subdivision, compatible with all coordinate faces and their identifications. On a subcell the linear map is, after eliminating one standard coordinate, the graph $n \mapsto (n, Ln)$ of an integral map. Projection to the first coordinates is an integral left inverse. If a point of its real image belongs to the ambient integer lattice, its first coordinates belong to the subcell's saturated tangent lattice; the entire point is then the corresponding integral graph point. This proves the claimed lattice equality. □

5.30 Rational coordinates on a projective model

The passage to rational functions uses an algebraic model. The projective assumption also provides a common affine open on which the functions separating residual strata are bounded. The following elementary construction works over a finite residue field as well.

Lemma 5.30.1 (Common affine neighbourhoods of residual points)**.** *Let O be a local ring with residue field k, and let $P \hookrightarrow \mathbf{P}^n_O$ be a closed immersion. Every finite subset $Z \subset |P_k|$ is contained in an affine open of P. If P_k is of finite type over k, there is a finite family of affine opens $U_j \subset P$ such that each U_j contains Z and every two points of P_k lie together in some U_j.*

Proof. Represent the points of Z by homogeneous primes of $k[x_0, \dots, x_n]$. Discard any prime contained in another one in this finite list. Avoiding the remaining maximal primes also avoids all the discarded ones. Denote these pairwise incomparable primes by $\mathfrak{p}_1, \dots, \mathfrak{p}_s$. For $i \neq j$ choose a homogeneous element $a_{ij} \in \mathfrak{p}_j \setminus \mathfrak{p}_i$; one exists because both ideals are homogeneous. Put $h_i = \prod_{j \neq i} a_{ij}$. Choose a coordinate x_{ℓ_i} outside $\mathfrak{p}_i$ and an integer $D \geq 1$ at least all the degrees of the h_i. Then

$$\bar{H} = \sum_{i=1}^{s} h_i x_{\ell_i}^{D - \deg h_i}$$

is homogeneous of positive degree and is outside every $\mathfrak{p}_i$: modulo $\mathfrak{p}_i$ only its ith summand survives, and that summand is nonzero. For $s = 1$ the empty product is one, so the same construction applies. For $Z = \varnothing$ any coordinate suffices.

Lift the coefficients to a homogeneous polynomial H over O. The standard open $D_+(H)$ in projective space is affine by [StacksAlg, Tag 01MB]. Its intersection with P is therefore an affine open containing Z.

Apply the first assertion to $Z \cup \{p, q\}$ for every $p, q \in P_k$. The open products $U_k \times_k U_k$ obtained in this way cover $P_k \times_k P_k$: the two projections of a point of the product form such a pair. This product has a finite affine cover, obtained from the pairwise products of a finite affine cover of P_k. Each affine scheme is quasi-compact: refine an open cover by standard opens $D(f_\alpha)$. Their elements generate the unit ideal, since otherwise [StacksAlg, Tag 00E0(2)] would give a maximal ideal outside the cover. An expression for 1 uses finitely many f_α and gives a finite subcover. Choose a finite subcover. Finally, every underlying pair (p, q) is the image of a point of the scheme product. Indeed $\kappa(p) \otimes_k \kappa(q)$ is a nonzero ring: if $1 \otimes 1 = 0$, this equality is witnessed by finitely many defining bilinear relations. Let $E \subset \kappa(p)$ and $F \subset \kappa(q)$ be the finite-dimensional k-subspaces spanned by 1 and every entry in these relations. Choose linear functionals on E and F that take 1 to 1. Their product annihilates all the chosen relations but takes $(1, 1)$ to 1, a contradiction. By [StacksAlg, Tag 00E0(2)], the nonzero tensor ring has a maximal ideal, which is prime and provides the desired point. Thus the finite subcover has the stated pair property. □

Fix a projective, flat, finitely presented O-scheme $\mathcal{X}$ with smooth integral generic fibre X. Assume that $\mathcal{X}_s$ has a finite affine open cover $\mathcal{U}_\lambda = \operatorname{Spec} C_\lambda$, where each $\mathcal{U}_\lambda$ admits an étale$O$-morphism to

$$\operatorname{Spec} O[X_0, \dots, X_{r_\lambda}, T_1, \dots, T_{q_\lambda}]/(X_0 \cdots X_{r_\lambda} - a_\lambda), \qquad 0 < |a_\lambda| < 1.$$

Here an affine open cover of $\mathcal{X}_s$ means affine opens of $\mathcal{X}$ whose special fibres cover $\mathcal{X}_s$. Complete along any $\pi \in O$ with $0 < |\pi| < 1$. Lemma 4.13.1 and Theorem 4.13.2 identify the completion with an admissible model of X^{an}. The completed charts are formally étaleand form a finite formal-open cover: a square-zero lifting problem modulo π^m is the algebraic étalelifting problem, and its compatible solutions give the completed lifting property. Consequently Theorem 5.25.4 gives its compact finite integral-affine skeleton S.

Proposition 5.30.2 (Skeletal valuations of rational functions)**.** *Every point of S is a norm on $K(X)$. For every $f \in K(X)^\times$, the function $v(f)|_S$ is continuous and finite piecewise Γ-integral-affine. Moreover the set of specializations of points of S is finite and is the set of generic residual-stratum points occurring in the completed charts.*

Proof. Consider a chart meeting S, and write $C_K = C_\lambda \otimes_O K$. It is a nonzero domain because $\mathcal{U}_{\lambda,K}$ is a nonempty affine open of the integral scheme X. The algebraic generic chart base

$$R = K[X_0, \dots, X_r, T]/(X_0 \cdots X_r - a)$$

is a domain. The map $R \to C_K$ is étale, hence flat by [StacksAlg, Tag 00U2(3),(6)]. A nonzero flat algebra over a domain is torsion free, so this map is injective. For $F_0 = \mathrm{Frac}(R)$, the algebra $C_K \otimes_R F_0$ is a nonzero domain and an affine étalealgebra over F_0. By [StacksAlg, Tag 02GL] it is a finite separable field extension. Since it contains C_K and is contained in $\mathrm{Frac}(C_K)$, it equals $\mathrm{Frac}(C_K) = K(X)$.

For a point x of the chart skeleton, evaluation on R is the standard Gauss norm. It is nonzero on every nonzero element of R. Thus the map $C_K \to \mathcal{H}(x)$ extends to $C_K \otimes_R F_0 = K(X)$. This is a unital map from a field, hence injective. The absolute value on $\mathcal{H}(x)$ gives the asserted norm on $K(X)$. All points of S belong to one of these charts.

On each chart write $f = g/h$ with nonzero $g, h \in C_K$. Their images are analytic on its completed generic fibre and are nonzero at every skeletal point by the preceding argument. Proposition 5.29.1 applies to both. Their valuation difference is continuous and has finitely many integral-affine formulas there. The finite cover gives the assertion globally, with the formulas agreeing on common faces by evaluation of the same rational function.

By Proposition 5.24.2, a point of a chart skeleton over $w \in \mathrm{relint}\,\Delta_I$ specializes to its label $\eta \in Q_I$. Conversely every such label has its nonempty relative open simplex. Each Q_I is finite, there are finitely many I and charts, and the formal specializations agree on overlaps. This proves the last assertion. □

Theorem 5.30.3 (Faithful rational tropicalization of a projective skeleton)**.** *In the stated projective algebraic model setting, there are a dense open $U \subset X$ with $S \subset U^{\mathrm{an}}$ and a closed immersion $\varphi : U \hookrightarrow \mathbf{G}_{m,K}^{N}$ such that $\operatorname{trop} \circ \varphi^{\mathrm{an}}|_{S}$ is a homeomorphism onto its image. On every cell of a finite subdivision of S, this map is Γ-integral-affine and identifies the tangent lattice with the saturated lattice of its image. All coordinates belong to the original field $K(X)$ and use the specified absolute value of K.*

Proof. Algebraic chart coordinates. For each chart include its multiplicative coordinates $X_0, \dots, X_r$ as nonzero rational functions on X. Proposition 5.30.2 makes all their norms finite and positive on all of S. On each relative open cell of their own chart their valuations are the standard simplex coordinates. Thus they distinguish two points with the same residual label and provide an integral left inverse on each cell's tangent lattice.

Separation of residual labels. Let $Z \subset \mathcal{X}_s$ be the finite set of skeletal specializations from the preceding proposition. Use Lemma 5.30.1 to choose one affine open $\mathcal{W} = \operatorname{Spec} D$ of $\mathcal{X}$ containing all of Z. Every point of S belongs to the generic fibre of $\widehat{\mathcal{W}}$, because this generic fibre is the inverse image of $\mathcal{W}_s$ under specialization; see Lemma 4.13.1 and the formal-open restriction statement in Proposition 5.23.4. Elements of D are therefore bounded by one at every point of S.

View the members of Z as distinct prime ideals of $D \otimes_O k$. For each unordered distinct pair choose an orientation $(\mathfrak{p}, \mathfrak{q})$ with $\mathfrak{p} \not\subseteq \mathfrak{q}$, choose $\bar{h} \in \mathfrak{p} \setminus \mathfrak{q}$, and lift it to $h \in D$. Flatness embeds D in the coordinate ring of its generic open, hence in $K(X)$. Since $\bar{h} \neq 0$, this is a nonzero rational function. At a skeletal point specializing to $\mathfrak{p}$ its norm is less than one; at one specializing to $\mathfrak{q}$ its norm is one. These values follow directly from the definition of specialization for a bounded formal function. Add all these finitely many lifts to the coordinate list.

The resulting tropical map is injective on S. Indeed the new functions distinguish different specializations. If x, y have the same specialization η, choose an algebraic chart containing η. Both points belong to its completed generic fibre and its skeleton, by formal-open restriction. In that chart they have the same residual label, and equality of the multiplicative coordinate values gives the same w. The bijection in Proposition 5.24.2 then gives $x = y$.

A closed immersion of a dense open. Let $f_1, \dots, f_M$ denote the list just constructed. Choose a nonempty affine open $V = \operatorname{Spec} A$ of X and nonzero

algebra generators $t_1, \dots, t_b$ for the finitely generated integral K-algebra A; zero generators can be omitted. Write $f_i = p_i/q_i$ with nonzero $p_i, q_i \in A$, and put

$$s = \prod_j t_j \prod_i p_i q_i, \qquad U = \operatorname{Spec} A[1/s].$$

Append the functions t_j, p_i, q_i to the list. They are units on U, and the induced homomorphism from the Laurent polynomial coordinate ring of the torus onto $A[1/s]$ is surjective: it contains the generators t_j of A and the inverse of their displayed product s. This gives a closed immersion. The norm statement in Proposition 5.30.2 says that every point of S has generic scheme support. It therefore belongs to the analytification of every nonempty open of X, in particular to U^{an}.

Lattices and topology. All coordinates of the final list are finite piecewise Γ-integral-affine on S. Subdivide each original cell so that these finitely many coordinates are affine. Its original chart coordinates remain among them, and projection to those coordinates gives an integral left inverse. The graph argument in Theorem 5.29.3 shows that the resulting map on every such subcell is unimodular. Their images are finitely many Γ-integral polyhedra. To see this explicitly, the coordinate projection identifies each image with its source polyhedron, and its other coordinates are the graphs of integral-affine formulas with constants in Γ. Cut the finite family of image polyhedra by their supporting hyperplanes. As in Lemma 5.14.2, this gives a finite common polyhedral subdivision. Pulling it back under the injective map gives a subdivision compatible with every face identification in S, and preserves the integral left inverses on tangent lattices. Finally S is compact, and its continuous injective map to $\mathbf{R}^N$ is a homeomorphism onto its image. □

References. The faithful tropicalization theorem is a reconstruction, for the stated projective unmarked algebraic models, of the method in Gubler–Rabinoff–Werner, *Skeletons and tropicalizations*, arXiv:1404.7044v3, Section 9. The affine construction above uses homogeneous prime avoidance; its projective standard-open input is [StacksAlg, Tag 01MB].

5.31 Finite projections of analytic tropical fibres

Let K be complete for its given real-valued non-Archimedean absolute value. The first two results allow the trivial valuation and generalized affinoid algebras. For an affinoid space X, write

$$\partial_K X = X \setminus \mathrm{Int}(X/\mathcal{M}(K)).$$

For a tuple $r = (r_1, \dots, r_d)$ of positive real numbers put

$$A_r = K\{r^{-1}T, rS\}/(T_iS_i - 1)_{i=1}^d, \qquad \mathbb{T}_r = \mathcal{M}(A_r).$$

Thus $\mathbb{T}_r$ is the exact-radius polytorus $|T_i| = r_i$. All quotient ideals below are actual algebraic ideals, and all tensor products with a hat are separated completed tensor products.

Theorem 5.31.1 (Boundary avoidance gives an actual finite algebra)**.** *Let $X = \mathcal{M}(B)$ be a generalized K-affinoid space and let $g_1, \dots, g_d \in B^\times$. For $r_i > 0$ set*

$$F_r = \{x \in X : |g_i(x)| = r_i \text{ for every } i\},$$
$$B_r = B\{r^{-1}U, rV\}/(U_i - g_i, V_i - g_i^{-1})_{i=1}^d.$$

Suppose $F_r \cap \partial_K X = \varnothing$. Then B_r is finite over A_r under $T_i \mapsto U_i$, $S_i \mapsto V_i$, and its given affinoid norm is equivalent to its canonical finite-module norm. The spectrum of this algebra is the actual affinoid domain F_r, with its induced analytic structure. The assertion includes the empty domain and uses the original field and absolute value.

Proof. The equations defining B_r are a joint Weierstrass presentation: they impose $|g_i| \le r_i$ and $|g_i^{-1}| \le r_i^{-1}$. Theorems 2.31.1 and 2.31.2 identify its spectrum, algebra, and domain mapping property with F_r. They also cover the zero algebra. We may therefore assume $B \neq 0$.

Choose positive real numbers a_i, b_i such that

$$a_i < r_i < b_i, \qquad \|g_i\| < b_i, \qquad \|g_i^{-1}\| < a_i^{-1}.$$

The larger closed polyannulus has algebra

$$C = K\{b^{-1}T, aS\}/(T_iS_i - 1)_{i=1}^d.$$

Substitution gives a bounded map $C \to B$ by Proposition 1.9.3. The exact-radius polytorus is an affinoid domain of $\mathcal{M}(C)$. Its algebra is A_r: applying

the rational universal property in both directions identifies the joint inequalities $|T_i| \le r_i$ and $|S_i| \le r_i^{-1}$ with its displayed presentation.

For $x \in F_r$, the hypothesis says that $x \in \mathrm{Int}(\mathcal{M}(B)/\mathcal{M}(K))$. Apply the affinoid composition formula 2.44.1 to

$$\mathcal{M}(B) \longrightarrow \mathcal{M}(C) \longrightarrow \mathcal{M}(K).$$

It gives $x \in \mathrm{Int}(\mathcal{M}(B)/\mathcal{M}(C))$. By the actual rational base-change presentation and Proposition 2.33.8,

$$B\widehat{\otimes}_C A_r \simeq B_r$$

boundedly, with bounded inverse. The bounded maps from B and A_r to B_r agree on C, so the pushout gives one map. In the pushout the images of g_i and g_i^{-1} are the images of the two coordinates of A_r; their powers therefore have bounds by constant multiples of r_i^m and r_i^{-m}. Bounded substitution gives the reverse map from B_r. Both composites fix all coefficient and coordinate images, hence are identities by polynomial density. The base-change assertion of Proposition 2.33.21 now puts every point of $\mathcal{M}(B_r)$ in its relative interior over $\mathbb{T}_r$. Theorem 2.43.8 proves both finite generation over A_r and equivalence with the canonical finite module norm. □

Proposition 5.31.2 (The factors of a finite projected level)**.** *Let $f_1, \dots, f_n \in B^\times$ define*

$$t : X \longrightarrow \mathbb{R}^n, \qquad t(x) = (-\log|f_1(x)|, \dots, -\log|f_n(x)|).$$

Let $Q = (q_{ji}) \in M_{d\times n}(\mathbb{Z})$ and $w' \in \mathbb{R}^d$. Suppose

$$w' \notin Q(t(\partial_K X)),$$
$$Q^{-1}(w') \cap t(X) = \{w_1, \dots, w_s\}$$

is finite. Put $g_j = \prod_i f_i^{q_{ji}}$ and $r_j = \exp(-w'_j)$. Then the actual algebras of the fibres $t^{-1}(w_i)$ give a bounded product decomposition

$$B_r \simeq \prod_{i=1}^{s} B_{w_i}.$$

Each B_{w_i} is finite over A_r with its canonical module norm. For every $\xi \in \mathbb{T}_r$, with $H = \mathcal{H}(\xi)$, these are actual finite analytic fibre algebras and

$$\dim_H(B_r\widehat{\otimes}_{A_r}H) = \sum_{i=1}^{s} \dim_H(B_{w_i}\widehat{\otimes}_{A_r}H).$$

The empty finite set has zero product algebra and degree zero.

Proof. The monomials g_j are analytic units and satisfy $(-\log|g_j|)_j = Q \circ t$. The boundary condition therefore gives the hypothesis of Theorem 5.31.1. Its finite algebra B_r represents $(Q \circ t)^{-1}(w')$. The continuous map t on this space has the finite image $\{w_1, \dots, w_s\}$. Each inverse image is consequently open and closed. It is also the actual rational domain defined by

$$|f_j| = \exp(-(w_i)_j) \quad (1 \le j \le n).$$

The rational universal property identifies this domain with the one obtained by imposing the same equalities directly on X. These domains are pairwise disjoint and form a finite affinoid cover of $\mathcal{M}(B_r)$. The norm-strict equalizer of Theorem 2.38.1 is thus precisely the asserted bounded product isomorphism, since all intersections have zero algebra. Each factor is a direct summand of the finite A_r-module B_r. Theorem 1.18.3 identifies its induced and canonical module norms.

By Proposition 2.33.19, tensoring these finite modules with H is already complete. Algebraic tensor product preserves the finite direct sum underlying the product, so dimensions add as displayed. Theorem 2.33.11 identifies these tensor algebras with the actual analytic fibres over ξ. □

5.32 Finite quotient models retaining moment coordinates

Now assume that K is nontrivially valued, and put $O = K^\circ$, $\mathfrak{m} = K^{\circ\circ}$, $k = O/\mathfrak{m}$. Choose $\pi \in O$ with $0 < |\pi| < 1$ and set

$$A_0 = O\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle, \qquad A = A_0[1/\pi].$$

The integral algebra A_0 is the unit ball for the Laurent Gauss norm, and $\widetilde{A} = k[T_1^{\pm 1}, \dots, T_d^{\pm 1}]$. These identifications follow from the actual Laurent normal form 1.17.2 and the multiplicative Gauss norm 1.2.2.

Theorem 5.32.1 (A finite formal model containing prescribed bounded elements)**.** *Let $A \to D$ be a bounded map of generalized K-affinoid algebras. Assume that D is finite over A and its given norm is equivalent to its canonical finite-A-module norm. Let $h_1, \dots, h_m \in D$ be power-bounded elements. There is an admissible topologically finitely presented O-algebra $D_0 \subset D$, finite over A_0, such that*

$$D_0[1/\pi] = D, \qquad h_1, \dots, h_m \in D_0.$$

It can be chosen as the actual image of a finite restricted-series presentation $A_0\langle Y\rangle \to D$. Its generic-fibre Banach structure is the given one on D.

Let ξ be the Gauss point of $\mathcal{M}(A)$ and $H = \mathcal{H}(\xi)$. For this model put

$$C_0 = (D_0/\mathfrak{m}D_0) \otimes_{k[T^{\pm 1}]} k(T_1, \dots, T_d).$$

Then C_0 is a finite-dimensional algebra, and

$$\begin{aligned} \dim_H(D\widehat{\otimes}_A H) &= \dim_{k(T)} C_0 \\ &= \sum_{\mathfrak{n}\in \operatorname{Max} C_0} \operatorname{length}_{(C_0)_\mathfrak{n}}((C_0)_\mathfrak{n})[\kappa(\mathfrak{n}) : k(T)]. \end{aligned} \tag{5.32.1}$$

For fixed $A \to D$, this total degree is independent of the chosen presentation and of the chosen admissible finite A_0-model with generic algebra D.

Proof. For $D = 0$ use $D_0 = 0$. Otherwise choose finite A-module generators $b_1, \dots, b_\ell$. Multiplying each by a sufficiently small nonzero power of π, obtain generators $e_1, \dots, e_\ell$ with $\|e_j\| \le 1$. Append the prescribed h_i to this list, and denote the resulting power-bounded elements by $z_1, \dots, z_N$. By Proposition 1.9.3, evaluation defines a bounded surjection

$$q : P = A\langle Y_1, \dots, Y_N\rangle \longrightarrow D, \qquad Y_i \longmapsto z_i.$$

It is norm-admissible. Indeed the first ℓ images still generate D as an A-module, and Theorem 1.18.3 gives uniformly controlled coefficients for a linear lift in these generators. That linear polynomial is a lift through q. Consequently the quotient norm from P is equivalent to the given norm on D. This presentation also proves that D is strict over K.

Put $P_0 = A_0\langle Y_1, \dots, Y_N\rangle$, $J = \ker q$, and

$$D_0 = P_0/(J \cap P_0) \subset D.$$

The map on the right is injective by construction. In particular D_0 has no O-torsion. Since P_0 is topologically finite type over O, Proposition 3.14.5, applied to the finite P_0-module $P_0/(J \cap P_0)$, proves that its quotient π-adic topology is complete and separated. Theorem 3.14.4 makes it topologically finitely presented. Also $P_0[1/\pi] = P$, by bounding the coefficients of a restricted series, whence $D_0[1/\pi] = D$. All the h_i lie in D_0.

It remains to prove finiteness over A_0. The finite map $A \to D$ is a map of strict affinoids. By Theorem 3.16.5, $\widetilde{D}$ is finite over $\widetilde{A}$. For each z_i choose a

monic relation for its residue over $k[T^{\pm 1}]$, and lift its coefficients to a monic $p_i(Z) \in A_0[Z]$. Then

$$w_i = p_i(z_i) \in D^{\circ\circ}.$$

In the quotient norm from q, $\|w_i^n\| \to 0$; topological nilpotence is unchanged by the equivalent norm just established. For each of the finitely many i, choose $a_i \geq 1$ and an actual lift $G_i \in P$ of $w_i^{a_i}$ with $\|G_i\| < 1$. Such a lift exists by the definition of the quotient infimum once $\|w_i^{a_i}\| < 1$. Since there are finitely many lifts,

$$\theta = \max_i \|G_i\| < 1.$$

In particular $G_i \in P_0$ and $p_i(Y_i)^{a_i} - G_i \in J \cap P_0$. The strict Laurent coefficient norm of a nonzero G_i attains its maximum at a coefficient in K. Choosing such a coefficient c_i gives $G_i = c_i G_i'$ with $G_i' \in P_0$ and $c_i \in \mathfrak{m}$. The zero lift has the same conclusion. Thus every G_i vanishes modulo $\mathfrak{m} D_0$.

Reduction of P_0 is the polynomial algebra $k[T^{\pm 1}, Y_1, \ldots, Y_N]$, since restricted coefficients are eventually smaller than one. The images of the Y_i in $D_0/\mathfrak{m}D_0$ satisfy the monic equations $\overline{p}_i(Y_i)^{a_i} = 0$. The bounded powers below their degrees span this quotient over $k[T^{\pm 1}]$. Lemma 5.28.2 now applies to the actual admissible quotient $A_0 \to D_0$. It proves that D_0 is finite over A_0, compares its generic norm with the canonical finite module norm, and gives the first equality in (5.32.1).

For the second equality, decompose the finite-dimensional Artinian algebra C_0 into its local factors by [StacksAlg, Tag 00JB]. A composition series of the factor $(C_0)_{\mathfrak{n}}$ has $\mathrm{length}_{(C_0)_{\mathfrak{n}}}((C_0)_{\mathfrak{n}})$ successive quotients equal to $\kappa(\mathfrak{n})$. Taking vector-space dimensions proves the formula. In particular the local lengths are those of the actual quotient algebra. Finally any admissible finite A_0-model of the same D has finite reduction, so Lemma 5.28.2 identifies its total degree with the same left side. This proves independence. □

For an application to the preceding projected fibres, suppose $r_j = |c_j|$ with $c_j \in K^{\times}$; the change of coordinate $T_j \mapsto c_j T_j$ identifies the target with the unit polytorus over K. If the full moment weight w_i also has $\exp(-(w_i)_a) = |b_a|$ with $b_a \in K^{\times}$, the functions f_a/b_a and b_a/f_a have absolute value one on $t^{-1}(w_i)$. They are power bounded by Theorem 1.20.2. Including both in the prescribed list places the normalized moment characters and their inverses in D_0. The resulting finite model and the degree in (5.32.1) use these original-field coordinates with the original absolute value.

5.33 Common models for finite moment projections

Fix a complete nontrivially valued field K with its given real-valued absolute value, and write

$$O = K^{\circ}, \qquad \mathfrak{m} = K^{\circ\circ}, \qquad k = O/\mathfrak{m}.$$

Choose $\pi \in O$ with $0 < |\pi| < 1$. An *actual admissible model* of a K-affinoid algebra D is an admissible topologically finitely presented O-subalgebra $D_0 \subset D$ for which $D_0[1/\pi] = D$ and the generic-fibre Banach topology is the given topology of D. The special algebra is always the actual quotient $\overline{D}_0 = D_0/\mathfrak{m}D_0$.

Proposition 5.33.1 (Simultaneous finite models)**.** *For $1 \le i \le r$, let*

$$A_{i,0} = O\langle U_{i,1}^{\pm 1}, \dots, U_{i,d}^{\pm 1} \rangle, \qquad A_i = A_{i,0}[1/\pi],$$

and let $A_i \to D$ be bounded homomorphisms. Assume that D is finite over every A_i and that its given norm is equivalent to each canonical finite-module norm.

Every actual admissible model $D_0 \subset D$ containing the images of all the $U_{i,j}$ and $U_{i,j}^{-1}$ is finite over every $A_{i,0}$. Such models exist and may contain any prescribed finite list of power-bounded elements of D.

More generally, let $E_{1,0}, \dots, E_{s,0}$ be actual admissible models of D, each containing the target coordinates and their inverses for at least one of the maps $A_i \to D$. There is an actual admissible model $C_0 \subset D$ containing all the $E_{j,0}$ and all target coordinates and inverses. It is finite over every $A_{i,0}$ and over every $E_{j,0}$.

Proof. An actual admissible model D_0 is a bounded open O-submodule of D. Indeed, take a formal presentation $O\langle Z \rangle \twoheadrightarrow D_0$. Its generic presentation has the quotient Banach topology. The image of the Gauss unit ball is bounded and contains every element whose quotient norm is strictly less than one, by the definition of the quotient infimum. Equivalence with the given norm proves the assertion. In particular, every element of D_0 is power bounded: all its powers lie in the bounded set D_0. The same argument shows that πD_0 is open in D.

The images of $U_{i,j}$ and their inverses define a continuous formal map $A_{i,0} \to D_0$. Restricted-series substitution is valid in the complete π-adic

ring D_0, and agrees in D with the given bounded homomorphism. Fix i, and choose formal generators $z_1, \dots, z_N$ of D_0 over O. The generic algebra D is strict, since it is the generic fibre of this model. Theorem 3.16.5 makes its canonical reduction $\widetilde{D}$ finite over $k[U_i^{\pm 1}]$. For each z_a, lift a monic equation of its canonical residue to a monic polynomial $p_a(Z) \in A_{i,0}[Z]$. Its value $p_a(z_a)$ is topologically nilpotent in D. Since πD_0 is open, some positive power of this value belongs to πD_0. Consequently the image $\overline{z}_a \in \overline{D}_0$ satisfies a monic equation $\overline{p}_a(Z)^{n_a} = 0$ over $k[U_i^{\pm 1}]$.

The algebra $\overline{D}_0$ is generated by the finitely many $\overline{z}_a$ over this Laurent ring. The powers below the degrees of these monic equations span it as a module. Thus $\overline{D}_0$ is finite over $k[U_i^{\pm 1}]$. Lemma 5.28.2 gives finiteness of D_0 over $A_{i,0}$. This argument applies separately to every i and does not require flatness over $A_{i,0}$.

For existence, use Theorem 5.32.1 with $A_1 \to D$. Append to its prescribed list all target coordinates and inverses, and all formal generators of all $E_{j,0}$. The target elements are power bounded because the maps $A_i \to D$ are bounded. The model generators are power bounded by the first paragraph. The resulting actual model C_0 contains every $E_{j,0}$: restricted series in its formal generators converge in C_0, and their values in D are the original values. It is finite over all $A_{i,0}$ by the part already proved. If $A_{i,0}$ maps into $E_{j,0}$, any finite list of $A_{i,0}$-module generators of C_0 also generates it over $E_{j,0}$. This proves the last assertion. For $D = 0$ all the models may be taken to be zero. □

Definition 5.33.2 (Top-dimensional special cycles)**.** For a finite-type k-algebra R of dimension at most d, put

$$[\operatorname{Spec} R]_d = \sum_{\substack{\mathfrak{p} \in \operatorname{Min} R \\ \dim(R/\mathfrak{p}) = d}} \operatorname{length}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}})[\operatorname{Spec}(R/\mathfrak{p})].$$

For a finite morphism of finite-type k-schemes, the pushforward of an integral component is its image multiplied by the full degree of the extension of function fields. Pushforward of cycles is extended additively. These are cycles before passage to rational equivalence.

Proposition 5.33.3 (Component lengths under a finite model comparison)**.** *Let $D_0 \subset C_0 \subset D$ be actual admissible models of the same affinoid algebra, both finite over*

$$A_0 = O\langle U_1^{\pm 1}, \dots, U_d^{\pm 1}\rangle.$$

The inclusion induces a finite morphism $\nu : \operatorname{Spec} \overline{C}_0 \to \operatorname{Spec} \overline{D}_0$, and

$$\nu_*[\operatorname{Spec} \overline{C}_0]_d = [\operatorname{Spec} \overline{D}_0]_d.$$

More precisely, for every d-dimensional minimal prime $\mathfrak{p}$ of $\overline{D}_0$,

$$\operatorname{length}((\overline{D}_0)_{\mathfrak{p}}) = \sum_{\mathfrak{q} \mapsto \mathfrak{p}} \operatorname{length}((\overline{C}_0)_{\mathfrak{q}})[\kappa(\mathfrak{q}) : \kappa(\mathfrak{p})], \tag{5.33.1}$$

where the sum is over the d-dimensional minimal primes of $\overline{C}_0$ above $\mathfrak{p}$.

Proof. Finite A_0-module generators of C_0 also generate it over D_0, so ν is finite. Both special algebras are finite over $k[U^{\pm 1}]$ and are Noetherian by [StacksAlg, Tag 00FN]. Their components have dimension at most d. By [StacksAlg, Tag 00P0], a component has dimension d exactly when it dominates $\operatorname{Spec} k[U^{\pm 1}]$: its function field is finite over the fraction field of the corresponding prime quotient of $k[U^{\pm 1}]$, and a nonzero prime imposes a nonzero polynomial relation on its d Laurent generators.

Let $S \subset A_0$ be the elements of Laurent Gauss norm one. The Gauss-localization argument of Lemma 5.28.2 identifies

$$V = S^{-1}A_0, \qquad F = \operatorname{Frac}(A_0), \qquad \kappa(V) = k(U).$$

Here V is a valuation ring, and every nonzero element of V is a nonzero scalar from O times a unit. The finite modules $M = S^{-1}D_0$ and $N = S^{-1}C_0$ are therefore torsion free over V and free by the finite-generator argument in that lemma. They are subalgebras of the same generic algebra

$$E = D \otimes_{A_0[1/\pi]} F.$$

Both span E over F. They consequently have the same rank n. Their residue algebras are

$$B = \overline{D}_0 \otimes_{k[U^{\pm 1}]} k(U), \qquad C = \overline{C}_0 \otimes_{k[U^{\pm 1}]} k(U).$$

They retain the full generic local rings of the corresponding special fibres. If $n = 0$, there are no components to compare.

By [StacksAlg, Tag 00JB], B and C are products of their Artinian local factors. Fix a maximal ideal $\mathfrak{p}$ of B, and let $\bar{b} \in B$ be the idempotent which is one on $B_{\mathfrak{p}}$ and zero on the other factors. Choose any lift $b \in M$. Multiplication by b on the two free V-modules M and N has the same characteristic polynomial: after scalar extension to F, both operators are multiplication by the same element on E. Equality over V follows from $V \subset F$.

Reducing this equality to $k(U)$ gives

$$t^{n-a}(t-1)^{a} = t^{n-a'}(t-1)^{a'}, \qquad a = \dim_{k(U)} B_{\mathfrak{p}}, \quad a' = \dim_{k(U)}(\bar{b}C).$$

The factors t and $t-1$ are relatively prime in every characteristic, so $a = a'$. The idempotent $\bar{b}$ selects exactly the local factors $C_{\mathfrak{q}}$ whose maximal ideals contract to $\mathfrak{p}$ in B. Taking a composition series of each Artinian local factor therefore gives

$$\operatorname{length}(B_{\mathfrak{p}})[\kappa(\mathfrak{p}) : k(U)] = \sum_{\mathfrak{q} \mapsto \mathfrak{p}} \operatorname{length}(C_{\mathfrak{q}})[\kappa(\mathfrak{q}) : k(U)].$$

Divide by the positive integer $[\kappa(\mathfrak{p}) : k(U)]$ and use the tower formula for field degrees. Localization identifies these local rings with those in (5.33.1), proving that formula and the cycle identity. No local factor has been replaced by its reduced quotient. □

5.34 The residual cycle of a finite analytic moment fibre

Let N be a lattice, $M = N^{\vee}$, and $\mathbb{T}_N = \operatorname{Spec} k[M]$. A normalized moment tuple on D is a homomorphism

$$h : M \longrightarrow D^{\times}$$

such that every character value and its inverse are power bounded. A basis and its inverses suffice for this condition. For $Q : N \to \mathbf{Z}^d$, put

$$A_Q = K\langle U_1^{\pm 1}, \dots, U_d^{\pm 1} \rangle, \qquad U_j \longmapsto h(Q^{*} e_j).$$

We consider those Q for which this map is finite and the given norm on D is equivalent to the canonical finite-A_Q-module norm. Write H_Q for the completed residue field of the Gauss point of $\mathcal{M}(A_Q)$, and put

$$\delta_Q = \dim_{H_Q}(D \mathbin{\widehat{\otimes}}_{A_Q} H_Q).$$

An actual model is *compatible with h* if it contains the values of a basis of M and their inverses.

Theorem 5.34.1 (A residual moment cycle independent of the projection)**.** *Suppose at least one such finite projection Q exists. For any actual model D_0 compatible with h, the residual moment morphism*

$$\phi_0 : \operatorname{Spec} \overline{D}_0 \longrightarrow \mathbb{T}_N$$

is finite, and the effective cycle

$$\mathcal{C}_d(D, h) = (\phi_0)_*[\operatorname{Spec} \overline{D}_0]_d$$

is independent of the model and of the finite projection used to construct it. If $\mathcal{C}_d(D, h) = \sum_Z c_Z[Z]$, then every finite projection Q restricts to a finite dominant map $q_Z : Z \to \mathbb{T}_{\mathbf{Z}^d}$, and

$$\delta_Q = \sum_Z c_Z[k(Z) : k(U_1, \dots, U_d)]. \tag{5.34.1}$$

The cycles also respect monomial changes of moment coordinates. Precisely, let $a : N \to N'$ and $h' = h \circ a^*$, and suppose that h' admits a finite projection to a d-dimensional unit polytorus for the same actual algebra D. The residual monomial map is finite on every component of $\mathcal{C}_d(D, h)$ and

$$\mathcal{C}_d(D, h') = a_* \mathcal{C}_d(D, h).$$

Proof. Proposition 5.33.1 supplies compatible models and makes every such model finite over the formal target of every finite projection. Fix one projection Q. The map $k[U^{\pm 1}] \to \overline{D}_0$ factors through the residual moment homomorphism $k[M] \to \overline{D}_0$. Its finite module generators therefore also generate $\overline{D}_0$ over $k[M]$. This proves that ϕ_0 is finite.

For two compatible models, the same proposition gives a common model C_0 containing both and finite over both. All three models are finite over the target of the fixed Q. Proposition 5.33.3 identifies the pushforwards of the top-dimensional cycle of $\operatorname{Spec} \overline{C}_0$ with those of both models. Their residual moment maps agree. Pushforwards of these finite cycles compose because degrees of finite extensions of function fields multiply. Pushing to $\mathbb{T}_N$ proves the asserted independence, including comparison of models initially constructed using different projections.

For any other finite projection, use the same compatible model. Each d-dimensional component of its special fibre dominates $\mathbb{T}_{\mathbf{Z}^d}$ by the dimension argument in Proposition 5.33.3. Its residual moment image Z is also d-dimensional. The coordinate ring of Z lies between the image of $k[U^{\pm 1}]$ and the coordinate domain of the source component. The latter is finite over $k[U^{\pm 1}]$. Every generator of the intermediate ring is consequently integral over $k[U^{\pm 1}]$, and finitely many generators satisfying monic equations make it finite. Thus q_Z is finite and dominant.

By Lemma 5.28.2 and the Artinian length formula in Theorem 5.32.1,

$$\delta_Q = \sum_{\dim(\overline{D}_0/\mathfrak{p})=d} \operatorname{length}((\overline{D}_0)_{\mathfrak{p}})[\kappa(\mathfrak{p}) : k(U)].$$

Group the terms having the same residual moment image Z. The definition of c_Z and the field-degree tower formula give exactly (5.34.1).

For the last assertion, a model compatible with h is also compatible with h'. It is finite over a target of a finite projection for h'. On every component, the coordinate ring of the residual h-image is intermediate between the ring of the residual h'-image and a finite algebra over the latter. The same monic-equation argument proves finiteness of the induced map of image components. The two residual moment maps now compose through a, and multiplication of the full field degrees proves the cycle identity. □

Theorem 5.34.2 (The degree of a subtorus-invariant residual cycle)**.** *In the preceding theorem let $L \subset N$ be a saturated sublattice of rank d. Suppose every reduced integral support Z of $\mathcal{C}_d(D,h)$ is stable, as a closed subscheme, under translation by the split subtorus $\mathbb{T}_L$. Let $\rho : \mathbb{T}_N \to \mathbb{T}_{N/L}$ be the quotient map. There are distinct closed points $s_1, \dots, s_t$ of $\mathbb{T}_{N/L}$ and positive integers $c_1, \dots, c_t$ with*

$$\mathcal{C}_d(D,h) = \sum_{j=1}^{t} c_j[\rho^{-1}(s_j)].$$

The empty sum is allowed. For every finite projection Q for which $Q|_L$ is injective, put

$$e_Q = [\mathbf{Z}^d : Q(L)].$$

Then

$$\delta_Q = e_Q \mu, \qquad \mu = \sum_{j=1}^{t} c_j[\kappa(s_j) : k] \in \mathbf{Z}_{\geq 0}. \tag{5.34.2}$$

In particular, δ_Q/e_Q is independent of all these projections and is positive exactly when the residual cycle is nonzero.

Proof. Lemma 5.14.1 gives a splitting $N = L \oplus N_1$. Put $M_0 = (N/L)^\vee$ and $M_L = L^\vee$. Then $k[M] = k[M_0][M_L]$, and translation by $\mathbb{T}_L$ is the character grading by M_L.

Let $I \subset k[M]$ be the prime ideal of an integral support Z. The scheme-theoretic invariance hypothesis says that the coaction sends I to $I \otimes_k k[M_L]$. If $f = \sum_\alpha f_\alpha \chi^\alpha \in I$, with $f_\alpha \in k[M_0]$, independence of the characters in the second tensor factor shows that each $f_\alpha \chi^\alpha$ belongs to I. Multiplication by the inverse character then puts f_α in $I \cap k[M_0]$. Thus

$$I = (I \cap k[M_0])k[M], \qquad k[Z] = (k[M_0]/(I \cap k[M_0]))[M_L].$$

The coefficient algebra is a finite-type domain. Since $\dim Z = d$, the dimension–transcendence-degree equality [StacksAlg, Tag 00P0] makes its fraction field algebraic over k. Its finitely many algebra generators are therefore algebraic over k. They span a finite-dimensional domain by their monic equations, and this domain is a field. It is $\kappa(s)$ for a closed point s of $\mathbb{T}_{N/L}$. This proves that $Z = \rho^{-1}(s)$. Distinct integral supports give distinct points, establishing the first formula.

Fix one of these points, and write $k' = \kappa(s)$. Under the splitting its fibre has coordinate ring $k'[M_L]$. The target characters for Q have the form

$$U_i \longmapsto d_i \chi^{a_i}, \qquad d_i \in (k')^\times.$$

The a_i generate a sublattice of M_L of index e_Q; equivalently, this is the dual index of the injective map $Q|_L$. Choose one representative for each coset. Every character has a unique expression as a representative times a product of the χ^{a_i}, and the units d_i can be absorbed into its coefficient. These representatives are therefore a basis of $k'[M_L]$ over $k'[U^{\pm 1}]$. A k-basis of k' gives a basis of size $e_Q[k' : k]$ over $k[U^{\pm 1}]$. This is the same character-coset calculation as in Proposition 5.28.3. The generic field degree of $\rho^{-1}(s) \to \mathbb{T}_{\mathbf{Z}^d}$ is consequently $e_Q[k' : k]$. Substitution in (5.34.1) of Theorem 5.34.1 gives (5.34.2). Full residue-field degrees, including inseparable degrees, enter this computation. The positive coefficients c_j already contain the unreduced generic lengths from the actual models. □

Theorem 5.34.3 (Projection comparison at a flat analytic tropical germ)**.** *Let $X = \mathcal{M}(B)$ be a generalized K-affinoid space and let $f_1, \dots, f_n \in B^\times$ define the moment map*

$$t : X \longrightarrow \mathbf{R}^n, \qquad t(x) = (-\log|f_1(x)|, \dots, -\log|f_n(x)|).$$

Let $w \in \mathbf{R}^n$, and choose $b_j \in K^\times$ with $|b_j| = \exp(-w_j)$. Put $h_j = f_j/b_j$, let $D = B_w$ be the actual algebra of $F_w = t^{-1}(w)$, and set $N = \mathbf{Z}^n$. Suppose there are a saturated rank-d sublattice $L \subset N$ and a neighborhood W of w such that

$$t(X) \cap W \subset w + L_{\mathbf{R}}, \qquad F_w \cap \partial_K X = \varnothing.$$

Assume that the normalized moment tuple on D admits a finite projection to a d-dimensional unit polytorus with its canonical finite-module norm.

Every integral support of $\mathcal{C}_d(D, h)$ is a fibre of $\mathbb{T}_N \to \mathbb{T}_{N/L}$ over a closed point. Consequently, for any finite projections Q_0, Q_1 of this same actual algebra with $Q_i|_L$ injective,

$$\frac{\dim_{H_{Q_0}}(D \widehat{\otimes}_{A_{Q_0}} H_{Q_0})}{[\mathbf{Z}^d : Q_0(L)]} = \frac{\dim_{H_{Q_1}}(D \widehat{\otimes}_{A_{Q_1}} H_{Q_1})}{[\mathbf{Z}^d : Q_1(L)]} = \sum_s c_s [\kappa(s) : k] \in \mathbf{Z}_{\geq 0},$$

where $\mathcal{C}_d(D, h) = \sum_s c_s [\rho^{-1}(s)]$.

Proof. The functions h_j and h_j^{-1} have absolute value one on F_w, so they are power bounded in D by Theorem 1.20.2. Theorem 5.34.1 therefore applies. If its cycle is zero, its support assertion is empty and the degree formula follows from Theorem 5.34.2.

Choose $\varepsilon > 0$ so small that the closed box $\{v : |v_j - w_j| \leq \varepsilon \text{ for all } j\}$ is contained in W. Let $Y \subset X$ be its inverse image, the actual rational affinoid domain

$$e^{-\varepsilon} \leq |h_j| \leq e^{\varepsilon} \quad (1 \leq j \leq n).$$

Its presentation by the h_j and h_j^{-1} is the actual joint Laurent presentation of Theorem 2.31.2. At every $x \in F_w$, all the relative coordinate inequalities are strict, since $1 < e^{\varepsilon}$. Thus $x \in \operatorname{Int}(Y/X)$. The boundary hypothesis and the composition inclusion in Proposition 2.33.21 give

$$F_w \subset \operatorname{Int}(Y/K).$$

Imposing the equalities $|h_j| = 1$ on Y gives the same actual algebra D as imposing them on X, again by the rational universal property.

Put $M_0 = M \cap L_{\mathbf{R}}^{\perp}$. For $u \in M_0$, the unit $h_u = \prod_j h_j^{u_j}$ has absolute value one at every point of Y, because $t(Y) - w \subset L_{\mathbf{R}}$. Theorem 1.20.2 and bounded substitution, Proposition 1.9.3, give a morphism

$$h_u : Y \longrightarrow \mathcal{M}(K\langle T^{\pm 1}\rangle).$$

For $x \in F_w$, apply the forward implication for absolute evaluation to the actual collar Y, using $x \in \operatorname{Int}(Y/K)$. The image S_x of $\widetilde{\mathcal{O}(Y)}^{\mathrm{gr}}$ in $\widetilde{\mathcal{H}(x)}^{\mathrm{gr}}$ is finite over $\widetilde{K}^{\mathrm{gr}}$. Choose finitely many homogeneous module generators for S_x. The component of $\widetilde{K}^{\mathrm{gr}}$ that takes any given generator to degree one has dimension at most one over k. Thus $(S_x)_1$ is a finite-dimensional k-algebra. The identity $|h_u| = 1$ on Y gives spectral radius one, and $|h_u(x)| = 1$ makes its degree-one initial image the nonzero residue of $h_u(x)$ in $\widetilde{\mathcal{H}(x)}$. This residue lies in $(S_x)_1$ and is therefore algebraic over k.

Choose an actual model D_0 compatible with h and a d-dimensional minimal prime $\mathfrak{p}$ of $\overline{D}_0$. The formal specialization theorem 3.16.15 supplies $x \in \mathcal{M}(D) = F_w$ specializing to $\mathfrak{p}$. By the definition of specialization, evaluation induces an injection

$$\overline{D}_0/\mathfrak{p} \hookrightarrow \widetilde{\mathcal{H}(x)},$$

and hence an injection of its fraction field. The preceding paragraph shows that the residue of every h_u, $u \in M_0$, is algebraic over k in this fraction field.

Let Z be the residual moment image of this component. Its coordinate ring is the image of $k[M]$ in that field, and $\dim Z = d$ by finiteness of the residual moment map. The image of $k[M_0]$ is a finitely generated algebraic domain over k, hence a finite field k' by monic equations and finite-dimensional linear algebra. It corresponds to a closed point s of $\mathbb{T}_{N/L}$. Split $N = L \oplus N_1$ using Lemma 5.14.1. The coordinate ring of Z is then an integral quotient of $k'[L^{\vee}]$. A nonzero kernel would give a nonzero Laurent polynomial relation on its d character generators and lower the transcendence degree below d. This contradicts $\dim Z = d$ and [StacksAlg, Tag 00P0]. The quotient map is therefore an isomorphism. Thus $Z = \rho^{-1}(s)$ as a closed subscheme. This treats every support of the residual cycle, and Theorem 5.34.2 now gives the displayed degree comparison. □

5.35 Purity and degrees near a projected analytic level

Throughout this section K is complete for its specified real-valued non-Archimedean absolute value. Generalized affinoid algebras and the trivial valuation are allowed. An affinoid algebra B is pure of analytic dimension d if $\dim_K \mathcal{M}(B/\mathfrak{p}) = d$ for every minimal prime $\mathfrak{p}$ of B. We use analytic dimension before strictification; it need not equal the Krull dimension of a generalized affinoid algebra.

Lemma 5.35.1 (Components under a free-radius extension)**.** *Let B be a generalized K-affinoid algebra, let r be a finite K-free polyradius, and put $L = K_r$ and $B_L = B\widehat{\otimes}_K L$. If B is integral, then B_L is integral. For arbitrary nonzero B, extension of ideals gives a bijection*

$$\operatorname{Min}(B) \longrightarrow \operatorname{Min}(B_L), \qquad \mathfrak{p} \longmapsto \mathfrak{p}B_L.$$

Moreover

$$B_L/\mathfrak{p}B_L \simeq (B/\mathfrak{p})_L, \qquad \dim_L \mathcal{M}(B_L/\mathfrak{p}B_L) = \dim_K \mathcal{M}(B/\mathfrak{p}).$$

Consequently B is pure of analytic dimension d if and only if B_L is pure of analytic dimension d.

Proof. Suppose first that B is integral. By Lemma 2.52.1, there is $x \in \mathcal{M}(B)$ with kernel zero. In the coefficient description of Theorem 1.16.2, its Gauss lift is

$$\left|\sum_{\nu} b_\nu T^\nu\right|_{\sigma_r(x)} = \max_{\nu} |b_\nu(x)| r^\nu$$

by Theorem 1.19.1. A nonzero coefficient family has a nonzero coefficient, whose value at x is positive. Thus this multiplicative seminorm has kernel zero, and B_L is a domain. No freeness over $\mathcal{H}(x)$ is used.

The actual quotient identity follows from Theorem 1.18.8. Applied to $B/\mathfrak{p}$, the preceding argument shows that $\mathfrak{p}B_L$ is prime. Its contraction is $\mathfrak{p}$, by the constant embedding in $(B/\mathfrak{p})_L$. Every prime of B_L contracts to a prime of B containing a minimal prime, by [StacksAlg, Tag 00E0(4)]. It therefore contains one of the primes $\mathfrak{p}B_L$. These extended primes cannot contain one another, since their contractions are distinct minimal primes. They are exactly the minimal primes of B_L. Finally apply Theorem 3.21.8 to each quotient. This proves the dimension formula and both purity implications. □

Lemma 5.35.2 (Dimension at maximal ideals of a strict affinoid)**.** *Let B be a nonzero strict K-affinoid algebra and let $\mathfrak{m}$ be a maximal ideal. Then*

$$\dim B_{\mathfrak{m}} = \max_{\substack{\mathfrak{p}\in \operatorname{Min}(B)\\ \mathfrak{p}\subseteq \mathfrak{m}}} \dim(B/\mathfrak{p}).$$

In particular, if B is pure of dimension d, then every maximal localization has dimension d.

Proof. First let $R = K\langle T_1, \dots, T_e\rangle$ and let $\mathfrak{n}$ be maximal. Theorem 1.12.3 makes $E = R/\mathfrak{n}$ a finite extension of K. Write a_i for the image of T_i; boundedness of the residue point gives $|a_i| \leq 1$. A finite K-basis of E, together with the norm comparison of Lemma 1.12.1, identifies $E\langle T\rangle$ with the finite free R-module $R\otimes_K E$: expanding coefficients in that basis preserves convergence in both directions. This ring extension is integral by [StacksAlg, Tag 00GK].

In $E\langle T\rangle$, the coordinate ideals

$$(0) \subset (T_1 - a_1) \subset \cdots \subset (T_1 - a_1, \dots, T_e - a_e)$$

form a strict chain of prime ideals. Translation by the a_i is a bounded automorphism, and each quotient is a Tate algebra in the remaining variables. The last ideal contracts to $\mathfrak{n}$. Incomparability for an integral extension, [StacksAlg, Tag 00GT], makes all contractions in this chain distinct. Hence $\dim R_{\mathfrak{n}} \geq e$. The reverse inequality follows from $\dim R = e$, proved in Proposition 3.21.1.

Now suppose that B is integral, of dimension e. Choose a finite injection $R = K\langle T_1, \dots, T_e\rangle \hookrightarrow B$ by Theorem 1.12.2 and Proposition 3.21.1. The contraction $\mathfrak{n} = \mathfrak{m} \cap R$ is maximal: its quotient is a K-subalgebra of the finite extension $B/\mathfrak{m}$, hence is a finite-dimensional domain and a field by [StacksAlg, Tag 00GS(1)]. The ring R is normal by Theorem 1.13.1. Going down for an integral extension over a normal domain, [StacksAlg, Tag 00H8], lifts a chain of length e ending at $\mathfrak{n}$ to a chain ending at the specified $\mathfrak{m}$. Thus $\dim B_{\mathfrak{m}} = e$.

For general B, apply this result to $B/\mathfrak{p}$ for each minimal prime contained in $\mathfrak{m}$. These are finitely many, by [StacksAlg, Tag 00FR] and Theorem 1.11.5. Their localized spectra cover $\operatorname{Spec}(B_{\mathfrak{m}})$: any prime in a chain contains a minimal prime of B, by [StacksAlg, Tag 00E0(4)]. A chain is therefore a chain in one such quotient. Taking the maximal possible length gives the displayed equality. □

Theorem 5.35.3 (Purity of affinoid domains)**.** *Let $X = \mathcal{M}(B)$ be pure of analytic dimension d. Every nonempty affinoid domain $V = \mathcal{M}(C) \subseteq X$ is pure of analytic dimension d.*

Proof. First assume that K is nontrivially valued and that B and C are strict. The restriction $B \to C$ is flat by Proposition 2.38.2. If $\mathfrak{n}$ is maximal in C, then $C/\mathfrak{n}$ is finite over K by Theorem 1.12.3. The image of B in this field is a finite-dimensional domain, hence a field by [StacksAlg, Tag 00GS(1)]. Thus $\mathfrak{m} = \mathfrak{n} \cap B$ is maximal. Flat going down, [StacksAlg, Tag 00HS], and Lemma 5.35.2 give

$$\dim C_{\mathfrak{n}} \geq d.$$

On the other hand, Proposition 3.22.8 gives $\dim C \leq d$. Analytic and Krull dimensions agree here because both algebras are strict.

Fix a minimal prime $\mathfrak{q}$ of C. The minimal primes are finite by [StacksAlg, Tag 00FR]. Choose an element h in all the other minimal primes and outside $\mathfrak{q}$: for each other prime choose an element outside $\mathfrak{q}$ and take their product; take $h = 1$ if there is no other prime. By Lemma 2.52.1, the integral affinoid $C/\mathfrak{q}$ has a point where h does not vanish. Choose $c \in K^{\times}$ with $|c| \leq |h|$ at that point. The strict Laurent domain

$$\mathcal{M}\big((C/\mathfrak{q})\langle U\rangle/(hU - c)\big)$$

is nonempty by Theorem 2.31.1. A maximal residue field of its algebra is finite over K. Its map from C has maximal kernel $\mathfrak{n}$, contains $\mathfrak{q}$, and does not contain h. Hence $\mathfrak{q}C_{\mathfrak{n}}$ is the only minimal prime of $C_{\mathfrak{n}}$. The spectra of this local ring and its quotient by that prime have the same prime chains. It follows that

$$d \leq \dim C_{\mathfrak{n}} = \dim(C/\mathfrak{q})_{\mathfrak{n}} \leq \dim(C/\mathfrak{q}) \leq d.$$

Every minimal-prime quotient of C consequently has dimension d.

For arbitrary K, B, and C, choose one free-radius extension $L = K_r$ making both algebras strict, by Theorem 1.17.6. If K is trivially valued, include a radius different from 1 among the additional radii, so that L is nontrivially valued. Theorem 2.36.9 identifies $\mathcal{M}(C_L)$ with the corresponding affinoid domain in $\mathcal{M}(B_L)$. Lemma 5.35.1 makes B_L pure of Krull dimension d and ensures $C_L \neq 0$. The strict case makes C_L pure of dimension d. Apply the same lemma to C to descend this purity, with analytic dimension, to the original field. □

For a nonempty closed rectangular box $P = \prod_{j=1}^{d}[\alpha_j, \beta_j] \subset \mathbb{R}^d$, allow $\alpha_j = \beta_j$ and put

$$\ell_j = e^{-\beta_j}, \qquad u_j = e^{-\alpha_j}, \qquad A_P = K\{u^{-1}T, \ell S\}/(T_j S_j - 1)_j.$$

Its spectrum $\mathbb{T}_P$ is the closed polyannulus $\ell_j \le |T_j| \le u_j$. For $s \in P$, let η_s be its Gauss point, with $|\sum_m a_m T^m|_{\eta_s} = \max_m |a_m| e^{-\langle m,s\rangle}$, and put $H_s = \mathcal{H}(\eta_s)$.

Proposition 5.35.4 (Gauss degrees over a closed polyannulus)**.** *The algebra A_P is an integral domain of analytic dimension d, and every η_s has kernel zero. Let D be a finite A_P-algebra with its canonical affinoid norm. Then*

$$\dim_{H_s}(D \widehat{\otimes}_{A_P} H_s) = \dim_{\mathrm{Frac}(A_P)}(D \otimes_{A_P} \mathrm{Frac}(A_P))$$

for every $s \in P$. In particular this integer is independent of s.

If $D \neq 0$ is pure of analytic dimension d, then every minimal prime of D contracts to zero in A_P, the displayed integer is positive, and $\mathcal{M}(D) \to \mathbb{T}_P$ is surjective. Every point above η_s satisfies $d_K(x) = d$.

Proof. Iterating Theorem 2.13.2 gives unique Laurent expansions in A_P, with the finite-tail condition for the norm

$$\max_m |a_m| \max_{s \in P} e^{-\langle m,s\rangle}.$$

The operators for different coordinate pairs commute on polynomials, and their bounds extend this assertion to the completed coefficient spaces. Thus their common section and actual relation kernel give the indicated expansion also when some intervals have length zero. The weighted Gauss argument of Proposition 1.2.2 is multiplicative at each s. Every nonzero expansion has a nonzero coefficient, so its value at every η_s is positive. This proves both assertions about integrality and kernels.

At any point of $\mathbb{T}_P$, Laurent polynomials are dense in the evaluated affinoid algebra. Approximating a nonzero denominator closely enough preserves its value and approximates its inverse. The completed residue field is therefore the completion of a field generated by at most d elements over K. Lemma 3.22.3 bounds its residual dimension by d. At η_s this field is the weighted Gauss completion of $K(T_1, \dots, T_d)$, whose residual dimension is d by Lemma 3.22.4. Theorem 3.22.7 now gives $\dim_K \mathbb{T}_P = d$.

The kernel-zero map $A_P \to H_s$ extends to its fraction field. Since D is finite, Proposition 1.18.7 identifies the completed tensor in the statement with the ordinary tensor. Extension of a finite-dimensional vector space from $\mathrm{Frac}(A_P)$ to H_s preserves its dimension. This proves the displayed equality, without a flatness assumption on D.

Suppose now that $D \neq 0$ is pure of analytic dimension d, and fix a minimal prime $\mathfrak{q}$ of D. Set $E = D/\mathfrak{q}$. Choose a free-radius extension L/K making A_P and E strict. Then E_L is integral by Lemma 5.35.1, and has Krull dimension d by Theorem 3.21.8. The same statements hold for $(A_P)_L$. The finite-module comparison in Theorem 1.18.8 preserves the finite module presentation, so E_L is finite over $(A_P)_L$. If the kernel J of this map were nonzero, any prime chain in $(A_P)_L/J$ could be extended by the zero prime of the domain $(A_P)_L$. Hence $\dim((A_P)_L/J) \leq d-1$. The finite injection $(A_P)_L/J \hookrightarrow E_L$ is integral by [StacksAlg, Tag 00GK], so [StacksAlg, Tag 00OK] gives a contradiction. Thus $J = 0$. The isometric scalar embedding then shows that $A_P \to E$ is injective as well. This holds for every minimal prime, and in particular $A_P \to D$ is injective.

The generic algebra is consequently nonzero, proving positivity. For an arbitrary $y \in \mathbb{T}_P$ with kernel $\mathfrak{p}$, lying over for the integral injection $A_P \hookrightarrow D$, [StacksAlg, Tag 00GQ], makes $D \otimes_{A_P} \kappa(\mathfrak{p})$ nonzero. Extending this vector space to $\mathcal{H}(y)$ preserves nonzeroness. The finite tensor comparison and Theorem 1.20.1 therefore give a point of the actual analytic fibre, identified by Theorem 2.33.11. This proves surjectivity. Finally Lemma 3.22.6 and the Gauss residual-dimension calculation give $d_K(x) = d$ at every point above η_s. □

Theorem 5.35.5 (A finite polyannulus neighborhood of a level)**.** *Let $X = \mathcal{M}(B)$, let $g_1, \dots, g_d \in B^\times$, and set*

$$q : X \longrightarrow \mathbb{R}^d, \qquad q(x) = (-\log|g_1(x)|, \dots, -\log|g_d(x)|).$$

Suppose $s_0 \in q(X) \setminus q(\partial_K X)$. There is a closed rectangular neighborhood P of s_0, with s_0 in its interior, such that $Y = q^{-1}(P)$ is an actual affinoid domain of X, $Y \subset \mathrm{Int}(X/K)$, and

$$g : Y \longrightarrow \mathbb{T}_P$$

is finite with the canonical finite-module norm on its algebra.

If X is pure of analytic dimension d, this map is surjective. Its Gauss-fibre dimension is one fixed positive integer on P. For every $s \in P$, its

restriction to the exact-radius polytorus at s is the actual algebra of the whole level $q^{-1}(s)$.

Proof. The set $\partial_K X$ is closed, by the openness of affinoid interior in Proposition 2.33.21. It is compact by Proposition 2.24.1; its image under q is compact and hence closed. Choose a closed box P around s_0 disjoint from that image. The simultaneous inequalities on g_j and g_j^{-1} give the domain Y with its actual affinoid algebra, by Theorems 2.31.1 and 2.31.2.

Choose a larger closed polyannulus $\mathbb{T}_{P^+}$ whose coordinate bounds strictly exceed the norms of all g_j and g_j^{-1} and which contains $\mathbb{T}_P$. Bounded substitution, as in Theorem 5.31.1, gives $X \to \mathbb{T}_{P^+}$. At every point of Y, the affinoid composition formula 2.44.1 makes this morphism inner, since the point is in $\mathrm{Int}(X/K)$. The actual relative pushout of Proposition 2.33.8 identifies the algebra of Y with

$$B\widehat{\otimes}_{A_{P^+}} A_P.$$

Indeed both presentations impose exactly the stated bounds on the images of g_j and g_j^{-1}; the rational universal maps in both directions fix these images and the coefficients. Interior is preserved by base change, by Proposition 2.33.21. Hence every point of Y is inner over $\mathbb{T}_P$. Theorem 2.43.8 proves finiteness and the norm assertion.

If X is pure of analytic dimension d, then Y is nonempty and pure of the same dimension by Theorem 5.35.3. Proposition 5.35.4 gives surjectivity and a constant positive Gauss degree. Base change from A_P to the exact-radius polytorus algebra imposes the equalities $|g_j| = e^{-s_j}$, so the same actual rational universal property identifies it with the whole level algebra. This includes every real s, without changing the original absolute value. □

Proposition 5.35.6 (Constant degrees for separated groups of moment fibres)**.** *Let $X = \mathcal{M}(B)$ be pure of analytic dimension d, let $f_1, \dots, f_n \in B^\times$ define $t : X \to \mathbb{R}^n$, and let $Q \in M_{d\times n}(\mathbb{Z})$. Suppose*

$$s_0 \notin Q(t(\partial_K X)), \qquad Q^{-1}(s_0) \cap t(X) = \{w_1, \dots, w_m\} \neq \varnothing$$

is finite. Choose pairwise disjoint neighborhoods N_i of the w_i. There are closed rectangular boxes $W_i \subset N_i$ with $w_i \in \mathrm{Int}(W_i)$ and a closed rectangular neighborhood P of s_0 such that

$$Y = (Q \circ t)^{-1}(P) = \coprod_{i=1}^{m} Y_i, \qquad Y_i = Y \cap t^{-1}(W_i), \qquad t(Y_i) \subset \mathrm{Int}(W_i).$$

This is a finite disjoint affinoid cover and gives a bounded product decomposition $D = \prod_i D_i$ over A_P. Each D_i is finite over A_P, and there are positive integers δ_i such that

$$\dim_{H_s}(D_i \widehat{\otimes}_{A_P} H_s) = \delta_i \qquad (s \in P).$$

At $s = s_0$, the algebra being measured is the actual finite analytic fibre of B_{w_i} over the projected Gauss point. Thus each original moment fibre has positive projected Gauss degree, extending to the constant Gauss degree of its separated group. Whenever $Q^{-1}(s) \cap t(X)$ is finite, δ_i is the sum of the actual projected Gauss degrees of its moment fibres whose values lie in W_i.

Proof. Take closed boxes W_i as stated, still pairwise disjoint. Put $q = Q \circ t$; its coordinates are the logarithms of the units $g_j = \prod_k f_k^{q_{jk}}$. The compact set

$$X \setminus t^{-1}\left(\bigcup_i \operatorname{Int}(W_i)\right)$$

has q-image disjoint from s_0. Shrink the box supplied by Theorem 5.35.5 to avoid this compact image. Then every point of Y belongs to exactly one Y_i, and its t-image lies in the interior of the corresponding box. Each Y_i is the actual Laurent domain obtained by imposing the coordinate bounds defining W_i, by Theorem 2.31.2. The sets are both open and closed in Y.

The norm-strict equalizer of Theorem 2.38.1 for this disjoint cover is the asserted product isomorphism: all pairwise intersections have zero algebra. Its factors are direct summands of the finite A_P-module D, so they are finite; their induced norms are canonical by Theorem 1.18.3. Each contains $t^{-1}(w_i)$ and is nonempty; Theorem 5.35.3 makes it pure of analytic dimension d. Proposition 5.35.4 gives the constants $\delta_i > 0$.

At s_0, the imposed equalities $Q \circ t = s_0$ leave only $w_1, \dots, w_m$ as possible moment values. Within W_i the remaining domain is exactly $t^{-1}(w_i)$. The rational universal property identifies its algebra with B_{w_i}, and identifies the base change of D_i to the exact-radius polytorus with that algebra. Laurent polynomial density identifies the completed residue fields of its Gauss point and η_{s_0}. The finite tensor comparison of Proposition 1.18.7 and Theorem 2.33.11 now identify the tensor in the statement with the actual fibre. If the joint moment image over another $s \in P$ is finite, apply Proposition 5.31.2 at s and retain the factors belonging to W_i. Finite products commute with these tensors, so their dimensions add, proving the last assertion. □

5.36 Gauss degrees at arbitrary real radii

Let K be complete for its specified real-valued non-Archimedean absolute value, which may be trivial. For $r \in \mathbb{R}^d_{>0}$ put

$$A_{K,r} = K\{r^{-1}T, rU\}/(T_iU_i - 1)_{i=1}^d.$$

The actual Laurent presentation of Theorem 1.17.2 identifies this algebra isometrically with the convergent Laurent series

$$\sum_{m\in\mathbb{Z}^d} a_m T^m, \qquad |a_m|r^m \longrightarrow 0 \quad \text{outside finite subsets},$$

$$\left\|\sum_m a_m T^m\right\|_r = \max_m |a_m| r^m.$$

By Proposition 1.2.2 and density of finite Laurent sums, this norm is multiplicative. Write $\xi_{K,r}$ for its spectrum point and $H_{K,r}$ for the completion of $\mathrm{Frac}(A_{K,r})$ for this norm. Thus $H_{K,r} = \mathcal{H}(\xi_{K,r})$. Empty tuples give $A_{K,r} = H_{K,r} = K$.

Proposition 5.36.1 (Monomial covers of Gauss fields)**.** *Let $C = (c_{ji}) \in M_d(\mathbb{Z})$ have nonzero determinant, put $s_j = \prod_i r_i^{c_{ji}}$, and set $e = |\det C|$. The substitution $S_j \mapsto \prod_i T_i^{c_{ji}}$ gives an isometric inclusion $A_{K,s} \hookrightarrow A_{K,r}$. If R is a set of representatives for $\mathbb{Z}^d/C^{\mathrm{t}}\mathbb{Z}^d$, then*

$$A_{K,r} = \bigoplus_{u\in R} A_{K,s}T^u, \qquad \left\|\sum_{u\in R} a_u T^u\right\|_r = \max_{u\in R} r^u \|a_u\|_s. \tag{5.36.1}$$

In particular the map is finite free of rank e, with the canonical finite-module Banach structure. Its actual fibre over $\xi_{K,s}$ is the field

$$A_{K,r}\widehat{\otimes}_{A_{K,s}} H_{K,s} = H_{K,r}, \qquad [H_{K,r} : H_{K,s}] = e.$$

The Gauss point is the only point of this fibre.

Proof. The substitution preserves each Laurent coefficient and its weighted size, since $s^v = r^{C^{\mathrm{t}}v}$. The exponent map is injective, so it preserves the coefficient norm and the finite-tail condition.

We first check the integer index. Integer row and column operations diagonalize a nonsingular integer matrix for this purpose. Move a nonzero entry to

the upper left. If an entry in its row or column is not divisible by it, division with remainder followed by a swap strictly decreases the positive absolute value of the pivot. After finitely many such reductions the pivot divides its entire row and column. Clear both by integer elementary operations and repeat on the remaining nonsingular square block. These operations are unimodular. A diagonal matrix has quotient index the product of the absolute values of its diagonal entries, which is its absolute determinant. Applying the same operations in transposed order also proves

$$[\mathbb{Z}^d : C\mathbb{Z}^d] = [\mathbb{Z}^d : C^{\mathsf{t}}\mathbb{Z}^d] = |\det C|.$$

Every exponent has a unique expression $u + C^{\mathsf{t}}v$ with $u \in R$ and $v \in \mathbb{Z}^d$. Grouping a convergent Laurent series by these finitely many cosets gives (5.36.1). Each grouped series converges: its weighted coefficients are those of the original series, divided by the fixed positive number r^u. Conversely a finite sum of such grouped series has a vanishing weighted tail. Disjoint exponents prove uniqueness and the norm equality. Since R is finite, its weighted maximum norm is equivalent to the usual maximum norm of a free $A_{K,s}$-module.

Put $F_s = \operatorname{Frac}(A_{K,s})$. The localization $A_{K,r} \otimes_{A_{K,s}} F_s$ is a finite-dimensional domain over F_s, hence a field: multiplication by a nonzero element is an injective linear endomorphism and is therefore surjective. It is thus $\operatorname{Frac}(A_{K,r})$, with the same coset basis. Clearing a common denominator in $A_{K,s}$ proves the weighted norm identity for coefficients in F_s as well.

Its completion is consequently $\bigoplus_{u \in R} H_{K,s} T^u$ with that weighted norm. Multiplication extends continuously, and its norm remains multiplicative by approximating two elements with elements of the dense valued field. It is a finite-dimensional domain over $H_{K,s}$, hence a field. By the definition of $H_{K,r}$ this completed field is $H_{K,r}$, of degree $|R| = e$. Proposition 2.33.19 identifies the algebraic tensor with the stated completed tensor; Theorem 2.33.11 identifies its spectrum with the actual analytic fibre.

Finally, a bounded multiplicative seminorm on the field $H_{K,r}$ is at most its given multiplicative norm, by applying its bound to successive powers. Applying the same inequality to the inverse of a nonzero element gives the reverse inequality. Its spectrum over $H_{K,s}$ is therefore a single point, with completed residue field $H_{K,r}$. □

For a finite $A_{K,r}$-algebra D whose given norm is equivalent to its canonical finite-module norm, define its Gauss fibre and degree by

$$E_{K,r}(D) = D\widehat{\otimes}_{A_{K,r}} H_{K,r}, \qquad \delta_{K,r}(D) = \dim_{H_{K,r}} E_{K,r}(D). \tag{5.36.2}$$

These are an actual finite analytic fibre algebra and a nonnegative integer by Proposition 2.33.19 and Theorem 2.33.11.

Proposition 5.36.2 (Scalar extension of a finite Gauss fibre)**.** *Let L/K be any complete isometric extension, and put $D_L = D\widehat{\otimes}_K L$. There are natural bounded algebra isomorphisms*

$$(A_{K,r})_L \simeq A_{L,r}, \qquad E_{L,r}(D_L) \simeq E_{K,r}(D) \otimes_{H_{K,r}} H_{L,r},$$

and the coefficient inclusion induces an isometric embedding $H_{K,r} \hookrightarrow H_{L,r}$. In particular

$$\delta_{L,r}(D_L) = \delta_{K,r}(D).$$

For either of these finite fibre algebras E over its indicated field H, its degree is

$$\dim_H E = \sum_{\mathfrak{p} \in \operatorname{Max} E} \operatorname{length}_{E_{\mathfrak{p}}}(E_{\mathfrak{p}})[\kappa(\mathfrak{p}) : H].$$

Proof. The actual relative pushout 2.33.8 extends the Laurent relation presentation coefficientwise. Its norm is the Laurent coefficient norm over L, by Theorem 1.17.2. The coefficient inclusion preserves the Gauss norm, so it extends first to the fraction fields and then to an isometric embedding $H_{K,r} \hookrightarrow H_{L,r}$.

The two bounded pushouts of D and L identify

$$D_L \simeq D\widehat{\otimes}_{A_{K,r}} A_{L,r}.$$

By Proposition 2.33.19, this is finite over $A_{L,r}$ with its canonical module norm. Applying Corollary 2.33.9 to the Gauss point, and then the same finite-base-change proposition, gives

$$\begin{aligned} D_L\widehat{\otimes}_{A_{L,r}} H_{L,r} &\simeq D\widehat{\otimes}_{A_{K,r}} H_{L,r} \\ &\simeq \left(D\widehat{\otimes}_{A_{K,r}} H_{K,r}\right) \otimes_{H_{K,r}} H_{L,r}. \end{aligned}$$

The tensors on the last line are already complete because the first factor is finite-dimensional over $H_{K,r}$. A basis over $H_{K,r}$ becomes a basis after this field extension, proving the degree equality, including degree zero.

A finite-dimensional algebra over a field is Artinian. Its local product decomposition is [StacksAlg, Tag 00JB]. In a composition series of $E_{\mathfrak{p}}$ every simple quotient is $\kappa(\mathfrak{p})$: the annihilator of a generator of a simple module is maximal, and this local ring has only one maximal ideal. Taking dimensions gives the displayed sum. Apply this argument to each of the two actual fibre algebras. □

Theorem 5.36.3 (Integral models computing an arbitrary-radius degree)**.** *Let D be as above. Let L/K be a complete nontrivially valued isometric extension, choose $b_j \in L^\times$ with $|b_j| = r_j$, and write $S_j = T_j/b_j$. Put*

$$O_L = L^\circ, \quad k_L = \widetilde{L}, \quad A_{L,0} = O_L\langle S_1^{\pm 1}, \dots, S_d^{\pm 1}\rangle.$$

For any finite list of power-bounded elements of D_L there is an admissible topologically finitely presented algebra $D_{L,0} \subset D_L$, finite over $A_{L,0}$, containing that list and having generic algebra D_L. For every such finite model put

$$C_{L,0} = (D_{L,0}/L^{\circ\circ}D_{L,0}) \otimes_{k_L[S^{\pm 1}]} k_L(S_1, \dots, S_d).$$

Then

$$\delta_{K,r}(D) = \sum_{\mathfrak{p} \in \operatorname{Max} C_{L,0}} \operatorname{length}_{(C_{L,0})_{\mathfrak{p}}}((C_{L,0})_{\mathfrak{p}})[\kappa(\mathfrak{p}) : k_L(S)]. \tag{5.36.3}$$

In particular this integer is independent of L, of the scalars b_j, and of the finite model. Such an extension L always exists.

If D is the actual full fibre at a weight w of analytic units $f_1, \dots, f_n$, and is finite with its canonical module norm over the polytorus of their monomials f^{q_j}, one may choose L realizing all $\exp(-w_i)$. For $a_i \in L^\times$ of these absolute values, the model may be chosen to contain all f_i/a_i and a_i/f_i. The target normalizers can be taken as $b_j = \prod_i a_i^{q_{ji}}$.

Proof. The coordinate change $S_j = T_j/b_j$ is an isometric isomorphism between $A_{L,r}$ and the unit-polytorus algebra over L; it sends their Gauss points to each other. Proposition 5.36.2 gives D_L its canonical finite-module Banach structure. Apply Theorem 5.32.1 over L. It constructs the required actual integral quotient model, with the prescribed elements, and computes its special-fibre length sum as $\delta_{L,r}(D_L)$. The same theorem gives this equality for every admissible finite model with that generic algebra. Proposition 5.36.2 identifies this number with $\delta_{K,r}(D)$, proving (5.36.3) and all the independence assertions.

To obtain an extension realizing any finite list of radii, append one radius $0 < \rho < 1$ and take the completion of the fraction field of its Laurent algebra for the multiplicative Gauss norm. Its coordinate elements realize the prescribed radii and ρ, so this is a complete nontrivially valued extension of K with the original absolute value on K.

In the last assertion the full fibre is the actual simultaneous Laurent domain $|f_i| = \exp(-w_i)$. The relative pushout 2.33.8 shows that its base change is the same actual Laurent domain over L. On every point of this domain both f_i/a_i and a_i/f_i have absolute value one. Theorem 1.20.2 makes these elements power bounded, also with the canonical equivalent norm. They may therefore be included in the finite prescribed list. The displayed choice of b_j makes the normalized target coordinates exactly the indicated monomials in these elements. □

Proposition 5.36.4 (Calibration under an isogeny of the target)**.** *Let $A_{K,r} \to D$ be finite with its canonical Banach structure, and let C and s be as in Proposition 5.36.1. Regard D also as an $A_{K,s}$-algebra by composition. It has the canonical finite-$A_{K,s}$-module Banach structure, and*

$$\delta_{K,s}(D) = |\det C|\, \delta_{K,r}(D).$$

More generally let D be an actual full moment-fibre algebra at $w \in \mathbb{R}^n$, and let $Q_0, Q_1 : \mathbb{Z}^n \to \mathbb{Z}^d$ have rank d and the same rational row space. Suppose the two monomial maps make D finite with its canonical module norms over their polytori at $Q_i w$. Let $N \subset \mathbb{Z}^n$ have rank d and suppose $Q_i|_N$ is injective. Writing δ_{Q_i} for the corresponding Gauss degree, one has

$$\frac{\delta_{Q_0}}{[\mathbb{Z}^d : Q_0(N)]} = \frac{\delta_{Q_1}}{[\mathbb{Z}^d : Q_1(N)]}.$$

Proof. Choose a finite free $A_{K,r}$-module surjecting onto D and give D its quotient norm. The finite weighted basis in (5.36.1) identifies that free module with a finite free $A_{K,s}$-module with an equivalent norm. The same quotient therefore gives the canonical finite-$A_{K,s}$-module topology on D.

Proposition 5.36.1 and associativity of the finite tensors identify

$$D \otimes_{A_{K,s}} H_{K,s} \simeq D \otimes_{A_{K,r}} \left(A_{K,r} \otimes_{A_{K,s}} H_{K,s}\right) \simeq D \otimes_{A_{K,r}} H_{K,r}.$$

Their completed versions are the same by Proposition 2.33.19. The algebra on the right has dimension $\delta_{K,r}(D)$ over $H_{K,r}$, and $[H_{K,r} : H_{K,s}] = |\det C|$, giving the first formula.

For the second assertion, equality of the rational row spaces gives an invertible rational matrix B with $Q_1 = BQ_0$. Choose a positive integer m clearing its denominators and set $C_0 = mB$, $C_1 = mI_d$. Then both C_i are nonsingular integer matrices and $C_0Q_0 = C_1Q_1$. These equal matrices define the same monomial map on the same actual algebra D, with the same target radii. Applying the first assertion twice gives

$$|\det C_0|\delta_{Q_0} = |\det C_1|\delta_{Q_1}.$$

For each i, the finite chain $C_iQ_i(N) \subset C_i\mathbb{Z}^d \subset \mathbb{Z}^d$ has index

$$[\mathbb{Z}^d : C_iQ_i(N)] = |\det C_i|[\mathbb{Z}^d : Q_i(N)].$$

Indeed representatives for the two successive quotients give representatives for the composite quotient, and C_i identifies the first quotient with $\mathbb{Z}^d/Q_i(N)$. The left sides for $i = 0, 1$ agree because $C_0Q_0 = C_1Q_1$. Dividing the two displayed identities proves the calibrated equality. □

5.37 Weights on interior flat tropical germs

Theorem 5.37.1 (Positive intrinsic weights at arbitrary real flat germs)**.** *Let $X = \mathcal{M}(B)$ be nonempty and pure of analytic dimension d over the original complete field K. Let $f_1, \dots, f_n \in B^\times$ and put $t(x) = (-\log|f_i(x)|)_i$. Fix $w \in t(X)$ and a saturated rank-d sublattice $N_0 \subset \mathbb{Z}^n$. Suppose that, for some open neighborhood W of w,*

$$t(X) \cap W \subset w + (N_0)_{\mathbb{R}}, \qquad F_w = t^{-1}(w) \subset \mathrm{Int}(X/K).$$

Let D_w be the actual affinoid algebra of the whole fibre F_w. For every integral map $Q : \mathbb{Z}^n \to \mathbb{Z}^d$ whose restriction to N_0 is injective, the monomials f^{q_j} make D_w finite, with its canonical module norm, over the exact-radius polytorus algebra at Qw. Its Gauss degree $\delta_Q(w)$ is positive, and

$$m_{X,f}(w) = \frac{\delta_Q(w)}{[\mathbb{Z}^d : Q(N_0)]} \in \mathbb{Z}_{>0}$$

is independent of Q. This assertion allows arbitrary real w and the trivial valuation. It uses the actual fibre algebra, including its nilpotents.

More precisely, after any complete nontrivial isometric extension L/K realizing the numbers e^{-w_i}, choose $a_i \in L^\times$ of these absolute values. Form

the residual moment cycle of $D_w \widehat{\otimes}_K L$ *with characters* f_i/a_i. *If* $\rho : \mathbb{T}_{\mathbb{Z}^n,\widetilde{L}} \to \mathbb{T}_{\mathbb{Z}^n/N_0,\widetilde{L}}$ *is the quotient, that cycle has the form* $\sum_s c_s[\rho^{-1}(s)]$, *and*

$$m_{X,f}(w) = \sum_s c_s[\kappa(s) : \widetilde{L}].$$

This integer is independent of the extension, normalizing elements, and compatible admissible model.

Proof. Transverse integral maps Q exist by the saturated-lattice splitting in Lemma 5.14.1. Choose a closed rectangular box P containing w in its interior and contained in W. Its inverse image $Y = t^{-1}(P)$ is the actual simultaneous Laurent domain, by Theorem 2.31.2. At every point of F_w, the finitely many coordinate inequalities defining Y are strict. Its relative presentation therefore puts $F_w \subset \mathrm{Int}(Y/X)$. Proposition 2.33.21 and the hypothesis give $F_w \subset \mathrm{Int}(Y/K)$.

The linear map Q is injective on $(N_0)_{\mathbb{R}}$. Since $t(Y) \subset w + (N_0)_{\mathbb{R}}$, the entire level of $Q \circ t|_Y$ at Qw is F_w. Theorem 5.31.1, applied to Y and its units f^{q_j}, makes its actual level algebra finite with the canonical module norm over the target polytorus. The level is the same affinoid domain as $F_w \subset X$: the rational universal property gives bounded inverse identifications of their algebras. Thus the finite algebra is precisely D_w.

Theorem 5.35.3 makes the nonempty domain F_w pure of analytic dimension d. Apply Proposition 5.35.4 to its finite map to the exact-radius polytorus, regarded as a polyannulus with degenerate intervals. Its Gauss degree is positive. This construction works for every indicated Q over K itself.

Choose a complete extension L as in Theorem 5.36.3, realizing all e^{-w_i} and one radius strictly between zero and one. The actual full fibre of X_L at w is $(F_w)_L$, by Theorem 2.36.9. Every point of X_L restricts to a point of X with the same character values, so $t(X_L) \cap W \subset w + (N_0)_{\mathbb{R}}$. Every point of $(F_w)_L$ is interior over L by the base-change assertion of Proposition 2.33.21. Proposition 5.36.2 preserves each finite map, its canonical module norm, and its Gauss degree.

Normalize by the a_i and apply Theorem 5.34.3 over L. Its hypotheses have just been verified for the same actual fibre algebra and for every Q. It identifies the residual supports with the indicated cosets and gives

$$\delta_{Q,L}(w) = [\mathbb{Z}^d : Q(N_0)] \sum_s c_s[\kappa(s) : \widetilde{L}].$$

Scalar extension preserves $\delta_Q(w)$, and its positivity has already been proved over K. This proves integrality, positivity, and independence of the projection. The left side divided by its fixed lattice index is defined over the original field and absolute value; it proves independence of L and the a_i. Independence of the compatible model also follows from Theorem 5.34.1. □

Theorem 5.37.2 (Local constancy on an interior flat germ)**.** *Under the hypotheses of Theorem 5.37.1, there is an open neighborhood V of w such that*

$$t(X) \cap V = (w + (N_0)_{\mathbb{R}}) \cap V.$$

Every v in this set has its whole fibre in $\mathrm{Int}(X/K)$, satisfies the same flat-germ hypotheses with lattice N_0, and has

$$m_{X,f}(v) = m_{X,f}(w).$$

The value at a weight is unchanged by restriction to an affinoid domain containing its whole moment fibre, whenever the two weighted germs satisfy these hypotheses.

Proof. The boundary $\partial_K X$ is compact, by openness of interior and compactness of X in Propositions 2.33.21 and 2.24.1. Its tropical image is compact and does not contain w. Choose the box P in the preceding proof also disjoint from that image. Then $Y = t^{-1}(P)$ lies in $\mathrm{Int}(X/K)$ and is pure of analytic dimension d, by Theorem 5.35.3.

Fix a transverse integral Q; one exists by choosing a basis of N_0 extending to a basis of $\mathbb{Z}^n$, using Lemma 5.14.1, and projecting onto the first d coordinates. The entire level of $Q \circ t|_Y$ at Qw is $F_w \subset \mathrm{Int}(Y/K)$, as above. Theorem 5.35.5 therefore supplies a closed rectangular neighborhood P' of Qw for which

$$Y' = (Q \circ t|_Y)^{-1}(P') \longrightarrow \mathbb{T}_{P'}$$

is finite and surjective, with a fixed positive Gauss degree δ at every $s \in P'$.

The affine restriction $Q : w + (N_0)_{\mathbb{R}} \to \mathbb{R}^d$ is an isomorphism. Its inverse sends P' exactly to $t(Y')$: inclusion in one direction follows from $t(Y) \subset w + (N_0)_{\mathbb{R}}$, and the other follows from surjectivity of $Y' \to \mathbb{T}_{P'}$. Take an open neighborhood

$$V \subset \mathrm{Int}(P) \cap Q^{-1}(\mathrm{Int}(P'))$$

of w. The preceding equality proves $t(X) \cap V = (w + (N_0)_{\mathbb{R}}) \cap V$. For v in this set, its whole fibre in X belongs to Y', since $v \in \mathrm{Int}(P)$ and $Qv \in P'$; in

particular it is interior over K. The local flat-germ condition at v follows by restricting to a sufficiently small neighborhood inside W, since $v + (N_0)_{\mathbb{R}} = w + (N_0)_{\mathbb{R}}$.

Restriction of the finite polyannulus map to its exact level at $s = Qv$ has algebra D_v, by Theorem 5.35.5 and injectivity of Q on the affine plane. Laurent polynomials are dense both in this target's exact-radius algebra and in the polyannulus algebra evaluated at η_s. Their completed Gauss residue fields are therefore the same completion of $K(T_1, \dots, T_d)$ with its weighted Gauss norm. Finite base change consequently identifies the Gauss fibre of D_v with that of Y' over η_s. Thus $\delta_Q(v) = \delta$ for every such v, including w. Theorem 5.37.1 gives

$$m_{X,f}(v) = \frac{\delta}{[\mathbb{Z}^d : Q(N_0)]} = m_{X,f}(w).$$

Finally an affinoid domain containing the whole fibre has the same actual fibre domain, with the same analytic structure. This follows from domain restriction and the two universal factorizations in Theorem 2.36.9. The local image equality just proved, applied to both domains, and inclusion of their moment images identify their tangent planes: both have dimension d. Their saturated tangent lattices are consequently equal. The fibre algebra and its monomial maps to each target polytorus are unchanged, so their Gauss degrees, and hence their calibrated weights, agree. □

Theorem 5.37.3 (Geometric pushforward under a finite locally free map)**.** *Let $p : X \to Y$ be a finite morphism of generalized K-affinoid spaces, both pure of analytic dimension d. Suppose its algebra is finite locally free of constant rank $\Delta > 0$ over the target algebra, with the canonical module Banach structure. Let unit moment tuples f_X and f_Y define $t_X : X \to \mathbb{R}^n$ and $t_Y : Y \to \mathbb{R}^{n'}$, and suppose that*

$$t_Y \circ p = a \circ t_X$$

comes from the corresponding identity of characters for an integral map $a : \mathbb{Z}^n \to \mathbb{Z}^{n'}$. Let $w' \in t_Y(Y)$ have its whole fibre in $\mathrm{Int}(Y/K)$, and suppose that the local tropical image at w' is contained in $w' + (N_Y)_{\mathbb{R}}$ for a saturated rank-d lattice $N_Y \subset \mathbb{Z}^{n'}$. Assume

$$a^{-1}(w') \cap t_X(X) = \{w_1, \dots, w_s\}$$

is finite and nonempty, and that the local tropical image at each w_i is contained in $w_i + (N_i)_{\mathbb{R}}$ for a saturated rank-d lattice $N_i \subset \mathbb{Z}^n$.

Then $a(N_i) \subset N_Y$ has finite index for every i, all the weights in Theorem 5.37.1 are defined, and

$$\Delta\, m_{Y,f_Y}(w') = \sum_{i=1}^{s} m_{X,f_X}(w_i)[N_Y : a(N_i)].$$

The locally free rank, the fibre algebras, and their lengths are taken before reduction. No restriction on the real weights or on the characteristic or value group of K is imposed.

Proof. Theorem 2.33.18 makes p everywhere interior relative to Y. The composition inclusion in Proposition 2.33.21 therefore gives

$$p^{-1}(t_Y^{-1}(w')) \subset \mathrm{Int}(X/K).$$

In particular the whole fibre at every w_i is interior. Theorem 5.37.1 defines its positive intrinsic weight and that at w'. Theorem 5.37.2 shows that the local images fill neighborhoods in their stated planes.

For v sufficiently near w_i in $w_i+(N_i)_{\mathbb{R}}$, one has $v \in t_X(X)$ and $a(v) \in t_Y(Y)$ sufficiently near w'. It follows that $a((N_i)_{\mathbb{R}}) \subset (N_Y)_{\mathbb{R}}$. Saturation of N_Y implies $a(N_i) \subset N_Y$. If the restriction of a to $(N_i)_{\mathbb{R}}$ had nonzero kernel, a small interval through w_i in that kernel would give infinitely many elements of $a^{-1}(w') \cap t_X(X)$. This contradicts the finite-set hypothesis. Hence this restriction is injective, and $a(N_i)$ has finite index in N_Y.

Choose an integral $Q : \mathbb{Z}^{n'} \to \mathbb{Z}^d$ transverse to N_Y. Take a small closed box in the target moment space, with w' in its interior and with its inverse image $Y_0 \subset Y$ contained in the specified flat-germ neighborhood. As in the proof of Theorem 5.37.1, the entire level of $Q \circ t_Y|_{Y_0}$ at Qw' is $t_Y^{-1}(w')$ and is interior over K in Y_0. Put $X_0 = p^{-1}(Y_0)$. Theorem 2.36.9 gives its actual affinoid algebra, and Proposition 2.33.19 preserves the finite canonical module structure. Ordinary base change of a locally free module preserves its rank: its local bases base change to local bases.

The entire level of $Qa \circ t_X|_{X_0}$ at Qw' is $p^{-1}(t_Y^{-1}(w'))$. Indeed, the target moment image in Y_0 lies in the plane on which Q is injective. This level is interior in X_0 by finite relative interior and composition. Its moment

values are exactly $w_1, \dots, w_s$. Proposition 5.31.2 therefore identifies its actual algebra with

$$\prod_{i=1}^{s} D_{w_i}.$$

Each factor is the original whole moment-fibre algebra on X, since that whole fibre already belongs to X_0.

Let $D'_{w'}$ be the actual whole target-fibre algebra, A the exact-radius target polytorus at Qw', and H its Gauss field. The algebra of the displayed source level is also the actual finite base change of the algebra of X along the algebra map from Y to $D'_{w'}$. It is consequently finite locally free of rank Δ over $D'_{w'}$. After tensoring over A with H, it is finite locally free of that rank over the Artinian algebra

$$E' = D'_{w'} \widehat{\otimes}_A H.$$

All these finite tensors are ordinary tensors with their complete canonical norms by Proposition 2.33.19.

By [StacksAlg, Tag 00JB], E' is a finite product of Artinian local algebras. On each factor, a local basis of the finite locally free module at its unique maximal ideal is a global basis: a principal open neighborhood of that ideal is the entire spectrum, and its defining element is a unit. Its H-dimension is therefore Δ times the H-dimension of that factor. Summing the dimensions and using the actual product of source fibres gives

$$\Delta\, \delta_Q(w') = \sum_{i=1}^{s} \delta_{Qa}(w_i).$$

Apply Theorem 5.37.1 to every term. Since Q is injective on N_Y, counting representatives in the subgroup chain $Qa(N_i) \subset Q(N_Y) \subset \mathbb{Z}^d$ gives

$$[\mathbb{Z}^d : Qa(N_i)] = [\mathbb{Z}^d : Q(N_Y)]\,[N_Y : a(N_i)].$$

Cancel the common positive first factor. This proves the stated geometric pushforward formula with all lengths and full field degrees already incorporated in the intrinsic weights. □

5.38 Local tropical geometry at interior fibres

Local polyhedrality of analytic moment images belongs to the theory developed by Berkovich and Ducros. For the compact-space and point-germ theorems, see A. Ducros, *Espaces de Berkovich, polytopes, squelettes et théorie*

des modèles, Confluentes Math. **4** (2012), 1250007, Theorems 3.2 and 3.4, DOI: 10.1142/S1793744212500077. The proof here reconstructs the local star at a whole interior fibre from the graded germ reductions and finite affinoid projections established earlier in the book.

For a tuple of invertible analytic functions $h = (h_1, \dots, h_n)$, write

$$t_h(y) = (-\log|h_1(y)|, \dots, -\log|h_n(y)|), \qquad h^m = \prod_{i=1}^{n} h_i^{m_i} \quad (m \in \mathbf{Z}^n).$$

The tropicalization of an algebraic torus subscheme over a trivially valued field is its actual analytic moment image, with the convention of Theorem 5.13.6.

Lemma 5.38.1 (Lifting strict sectors from a residue field)**.** *Let K be a complete non-Archimedean field with its specified absolute value, let (X, x) be a good K-analytic germ, and suppose that $x \in \mathrm{Int}(X/K)$. Let $h_1, \dots, h_n$ be invertible analytic functions near x such that $|h_i(x)| = 1$. Put $k = \widetilde{K}$, and let*

$$R = k[\widetilde{h}_1^{\pm 1}, \dots, \widetilde{h}_n^{\pm 1}] \subset \widetilde{\mathcal{H}(x)}, \qquad Z = \operatorname{Spec} R \subset \mathbf{G}^n_{m,k}.$$

If $u \in \mathrm{Trop}(Z)$ and $m_1, \dots, m_s \in \mathbf{Z}^n$ satisfy $\langle m_j, u\rangle > 0$ for every j, then every neighbourhood of x contains a point y for which

$$|h^{m_j}(y)| < 1 \qquad (1 \le j \le s).$$

In particular the corresponding strict rational sector contains moment values arbitrarily close to $0 = t_h(x)$.

Proof. Choose an analytic point of Z whose moment value is u. Its kernel is a prime $\mathfrak{p} \subset R$, and its absolute value induces a valuation ring $V_{\mathfrak{p}}$ in $\kappa(\mathfrak{p})$ containing the trivially valued field k. Let $B \subset R_{\mathfrak{p}}$ be the inverse image of $V_{\mathfrak{p}}$ under the residue map. This is a local subring of $\mathrm{Frac}(R) \subset \widetilde{\mathcal{H}(x)}$. Indeed, its maximal ideal is the inverse image of the maximal ideal of $V_{\mathfrak{p}}$, and an element outside that inverse image is a unit of $R_{\mathfrak{p}}$ whose inverse again lies in B. The character $\widetilde{h}^{m_j}$ is a unit of $R_{\mathfrak{p}}$, and its image in $\kappa(\mathfrak{p})$ has absolute value $e^{-\langle m_j, u\rangle} < 1$. It therefore belongs to the maximal ideal of B.

By [StacksAlg, Tag 00IA], there is a valuation ring $V \subset \widetilde{\mathcal{H}(x)}$ dominating B. It contains k, and each $\widetilde{h}^{m_j}$ belongs to its maximal ideal. Consequently $\widetilde{h}^{-m_j} \notin V$. Lemma 3.19.20 extends V to a graded valuation ring

$$W \subset F_x := \widetilde{\mathcal{H}(x)}^{\mathrm{gr}} \quad \text{containing } \widetilde{K}^{\mathrm{gr}}, \qquad W_1 = V.$$

No assertion about the rank of this valuation is needed.

Choose an ordinary affinoid neighbourhood of x on which the h_i are units. The ordinary local test preserves interiority over $\mathcal{M}(K)$ on this neighbourhood. The absolute evaluation clause makes the image of its graded algebra in F_x finite over $\widetilde{K}^{\mathrm{gr}}$. Every graded valuation ring containing $\widetilde{K}^{\mathrm{gr}}$ contains this image: apply the integral-closedness assertion of Lemma 3.19.1 to its homogeneous elements. Thus

$$\mathscr{R}(X,x) = \mathscr{P}_{\widetilde{K}^{\mathrm{gr}}}(F_x),$$

using the actual germ reduction of Proposition 3.19.4. In particular $W \in \mathscr{R}(X,x)$.

Suppose that a neighbourhood of x contains no point satisfying all the strict inequalities in the statement. Choose a smaller ordinary affinoid neighbourhood of x inside it on which all h_i are units. The finitely many Weierstrass domains

$$|h^{-m_j}| \leq 1 \qquad (1 \leq j \leq s)$$

cover this representative. They are a finite closed cover, hence admissible by Lemma 2.39.2, and all contain x. Each inequality is active at x. Proposition 3.19.6 says that their germ reductions cover $\mathscr{R}(X,x)$. By Lemma 3.19.5, the j-th reduction consists of the valuation rings containing $\widetilde{h}^{-m_j}$. None contains W, a contradiction. The case $s = 0$ is immediate. Finally continuity of t_h allows the neighbourhood to be chosen inside the inverse image of any prescribed neighbourhood of 0. □

Lemma 5.38.2 (A finite tropical basis over a trivially valued field)**.** *Let k be a field with the trivial absolute value, and let $J \subset k[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ be an ideal. There are finitely many nonzero $p_1, \dots, p_a \in J$ such that*

$$\mathrm{Trop}(\operatorname{Spec} k[T^{\pm 1}]/J) = \bigcap_{j=1}^{a} \mathrm{Trop}(V(p_j)).$$

For $J = 0$ the empty list is allowed. The set on either side is the support of a finite rational polyhedral fan. At every weight outside this set, some p_j has a unique term of least weight.

Proof. Form the homogeneous ideal $H \subset k[x_0, \dots, x_n]$ used in Theorem 5.16.5; localization in all variables followed by $x_0 = 1$ recovers J. Use

ordinary weight initials, taking terms of least weight. In the present trivial valuation, the hyperplane arrangement in that theorem is central. On each relatively open cone C its proof gives one fixed ordinary homogeneous ideal $H_C = \mathrm{in}_\omega(H)$ over k: the minimizing pivot sets and the surviving entries in their row bases are fixed on C, and these rows generate the ideal in the uniformly bounded degrees of Proposition 5.16.3. Restrict this finite arrangement to $\omega_0 = 0$.

For any cone C outside the tropicalization, localization of H_C in all variables is the unit ideal, by Theorem 5.13.6 and the localization argument in Theorem 5.16.5. Hence H_C contains a monomial x^{α_C}, of some degree e_C. In the finite-dimensional space H_{e_C}, choose a fixed row matrix and consider every column set B with a nonzero maximal minor. There are finitely many such sets. For every one include all members of its pivot row basis in a list $\mathcal{G}_C$.

Fix $\omega \in C$ and choose a minimizing pivot set B for this degree. Lemma 5.16.4 and Lemma 5.16.1 show that the initial rows form a basis of $(H_C)_{e_C}$ and have the identity matrix in the pivot columns. Since the single monomial vector x^{α_C} belongs to this space, its column must be a pivot: otherwise its pivot coordinates would all be zero, forcing the vector to be zero. Its pivot coordinates then show that the corresponding initial row is exactly x^{α_C}. Thus a member of the finite list $\mathcal{G}_C$ has a monomial initial at every $\omega \in C$.

Dehomogenize the lists for the finitely many excluded cones. These give nonzero Laurent polynomials in J. A homogeneous polynomial of fixed degree remains nonzero after $x_0 = 1$, since its remaining exponent vector determines the exponent of x_0. Every weight outside the tropicalization has a unique least term in one of the resulting Laurent polynomials. Such a weight is outside that polynomial's hypersurface tropicalization by the strong triangle inequality. Conversely, an actual analytic point of the subscheme is an analytic point of every one of these hypersurfaces, with the same moment value. This proves the equality. The fan assertion is the central arrangement already constructed. If J is the unit ideal, the polynomial 1 gives the assertion directly. □

Lemma 5.38.3 (The actual image of a finite affinoid map)**.** *Let $A \to D$ be a bounded homomorphism of generalized K-affinoid algebras, finite with*

its canonical module norm, and put $I = \ker(A \to D)$. *Its analytic image is exactly* $\mathcal{M}(A/I)$, *with the quotient affinoid structure, and*

$$\dim_K \mathcal{M}(D) = \dim_K \mathcal{M}(A/I) \leq \dim_K \mathcal{M}(A)$$

when $D \neq 0$. *If* D *is pure of analytic dimension* d, *so is* A/I.

Proof. The quotient is an actual affinoid algebra by Corollary 1.18.4. The injection $A/I \hookrightarrow D$ is integral by [StacksAlg, Tag 00GK]. For $z \in \mathcal{M}(A/I)$ with kernel $\mathfrak{p}$, lying over, [StacksAlg, Tag 00GQ], makes $D \otimes_{A/I} \kappa(\mathfrak{p})$ nonzero. It is a finite-dimensional vector space. Extending it to $\mathcal{H}(z)$ keeps it nonzero. Proposition 1.18.7 identifies this algebra with the actual completed fibre algebra, which has an analytic point by Theorem 1.20.1. The fibre identification of Theorem 2.33.11 therefore gives a point above z. This proves the image assertion.

At every point of $\mathcal{M}(D)$, the completed residue field is finite over that of its image, by Lemma 3.22.6. Taking the maxima in Theorem 3.22.7 on the two spaces gives equality of their analytic dimensions. The dimension inequality follows from the same maximum formula, since the quotient space has the same completed residue fields as its points in $\mathcal{M}(A)$.

Suppose D is pure of dimension d and let $\mathfrak{p}$ be a minimal prime of A/I. A prime of D above $\mathfrak{p}$ contains a minimal prime $\mathfrak{q}$ of D, by [StacksAlg, Tag 00E0(4)]. Its contraction equals $\mathfrak{p}$ by minimality. Thus $(A/I)/\mathfrak{p} \hookrightarrow D/\mathfrak{q}$ is a finite injection. These quotient algebras have the canonical finite module norms by Theorem 1.18.3. Apply the dimension equality just proved to this injection. The right-hand algebra has analytic dimension d, and hence so does the left-hand algebra. This holds for every minimal prime. □

Proposition 5.38.4 (A finite analytic image around a whole moment fibre)**.** *Let* $X = \mathcal{M}(B)$ *be a generalized affinoid space over a complete field* K *with its specified absolute value, let* $f_1, \dots, f_n \in B^\times$, *and put* $t = t_f$. *If* $w \in t(X)$ *and* $F_w = t^{-1}(w) \subset \mathrm{Int}(X/K)$, *there is a closed box* P *about* w, *with* w *in its interior, such that* $Y = t^{-1}(P)$ *lies in* $\mathrm{Int}(X/K)$ *and its moment map to the closed polyannulus* $\mathbb{T}_P$ *is finite. For* $A_P = \mathcal{O}(\mathbb{T}_P)$, $D_P = \mathcal{O}(Y)$, *and* $I_P = \ker(A_P \to D_P)$, *one has*

$$t(X) \cap P = t\big(\mathcal{M}(A_P/I_P)\big).$$

Every point of the right-hand analytic space has an actual lift to Y. *If* X *is pure of analytic dimension* d, *the closed analytic image* $\mathcal{M}(A_P/I_P)$ *is pure*

of dimension d. For every $v \in P$, restriction to the exact-radius polytorus at v gives the actual algebra of the whole fibre F_v.

Proof. Apply the first assertion of Theorem 5.35.5 to the complete tuple of n moment functions. That assertion places no equality between n and the dimension of X. It gives the stated domain, finite map, canonical module norm, and the exact-level identifications. Lemma 5.38.3 gives the actual image and the pointwise lifting assertion. In the pure case, Theorem 5.35.3 makes Y pure of dimension d, and the last assertion of the same lemma gives the purity of its finite image. □

Proposition 5.38.5 (A residual fan bounds and meets the actual germ)**.** *Let K be complete and nontrivially valued, put $O = K^\circ$ and $k = \widetilde{K}$, and let $X = \mathcal{M}(B)$ be a generalized affinoid space. Let $h_1, \dots, h_n \in B^\times$, and suppose the nonempty whole fibre $F_0 = t_h^{-1}(0)$ belongs to $\mathrm{Int}(X/K)$. Let D be its actual affinoid algebra, and let $D_0 \subset D$ be an actual admissible model, finite over*

$$O\langle T_1^{\pm 1}, \dots, T_n^{\pm 1}\rangle, \qquad T_i \longmapsto h_i.$$

Set

$$\overline{D}_0 = D_0/K^{\circ\circ}D_0, \quad J = \ker(k[T^{\pm 1}] \longrightarrow \overline{D}_0),$$
$$R = k[T^{\pm 1}]/J, \qquad Z = \operatorname{Spec} R.$$

Then $C = \mathrm{Trop}(Z)$ is the nonempty support of a finite rational polyhedral fan of dimension at most $\dim_K X$. There is a neighbourhood V of 0 such that

$$t_h(X) \cap V \subset C.$$

For every $u \in C$ and every finite list $m_j \in \mathbf{Z}^n$ with $\langle m_j, u\rangle > 0$, the strict sector $\{v : \langle m_j, v\rangle > 0 \text{ for all } j\}$ contains points of $t_h(X)$ arbitrarily close to 0.

Proof. The algebra $\overline{D}_0$ is finite over its subalgebra R. Thus $R \hookrightarrow \overline{D}_0$ is integral by [StacksAlg, Tag 00GK], and lying over, [StacksAlg, Tag 00GQ], applies to every prime of R. Formal specialization 3.16.15 is surjective on $\operatorname{Spec}\overline{D}_0$. If $\mathfrak{q}$ lies above $\mathfrak{p} \subset R$ and $x \in F_0$ specializes to $\mathfrak{q}$, evaluation gives injections

$$R/\mathfrak{p} \hookrightarrow \overline{D}_0/\mathfrak{q} \hookrightarrow \widetilde{\mathcal{H}(x)}.$$

Indeed a model element has zero residue at x exactly when its class belongs to $\mathfrak{q}$; every nonzero residue consequently has absolute value one before reduction. These injections extend to the fraction fields.

The finite-type algebra R is Noetherian, by [StacksAlg, Tag 00FO], and has finitely many minimal primes, by [StacksAlg, Tag 00FR]. For any such prime $\mathfrak{p}$, the preceding injection and Theorem 3.22.7 give

$$\dim(R/\mathfrak{p}) = \operatorname{trdeg}_k \operatorname{Frac}(R/\mathfrak{p}) \leq \operatorname{trdeg}_k \widetilde{\mathcal{H}(x)} \leq d_K(x) \leq \dim_K X,$$

where the equality is [StacksAlg, Tag 00P0]. Apply Theorem 5.19.5 to each reduced irreducible component of Z and take a common refinement of their finite fans. The tropicalization of their finite union is their union, since every analytic seminorm has a prime kernel containing one minimal prime by [StacksAlg, Tag 00E0(4)]. This proves the fan and dimension assertions. Nonemptiness also follows from specialization of any point of the nonempty fibre.

Take a finite tropical basis $p_1, \ldots, p_a$ for J from Lemma 5.38.2. Lift each nonzero coefficient to an element of O of absolute value one, and let $P_j(T) \in O[T^{\pm 1}]$ be the resulting polynomial. Its value $P_j(h)$ belongs to $K^{\circ\circ} D_0$. By the definition of this ideal it is a finite sum $\sum_\ell c_\ell d_\ell$, with $|c_\ell| < 1$ and $d_\ell \in D_0$. Model elements have absolute value at most one at every point of F_0, as in Theorem 3.16.15. It follows that

$$\max_{x \in F_0} |P_j(h)(x)| < 1.$$

Choose a real number $c_j < 1$ strictly larger than this maximum. The closed subset of X on which $|P_j(h)| \geq c_j$ is compact by Proposition 2.24.1 and has moment image avoiding 0. After shrinking a common neighbourhood V of 0, one therefore has $|P_j(h)(x)| < c_j$ whenever $t_h(x) \in V$. Shrink V again so that $e^{-\langle m, v\rangle} > c_j$ for every exponent m occurring in p_j and every $v \in V$. If $v \notin C$, the last assertion of Lemma 5.38.2 supplies an index j for which the least value of $\langle m, v\rangle$ among these exponents occurs exactly once. At a point x with $t_h(x) = v$, that term is the unique largest term of $P_j(h)(x)$, so $|P_j(h)(x)| = e^{-\langle m, v\rangle} > c_j$. This contradiction proves the asserted inclusion in C.

Finally let $u \in C$ and choose an actual analytic point of Z with moment value u, with kernel $\mathfrak{p}$. Choose $\mathfrak{q}$ above $\mathfrak{p}$ and $x \in F_0$ specializing to $\mathfrak{q}$, as in the first paragraph. The subalgebra generated by the residues of the $h_i^{\pm 1}$ in $\widetilde{\mathcal{H}(x)}$ is exactly $R/\mathfrak{p}$. The chosen analytic point factors through this algebra. If the list of inequalities is empty, the assertion is immediate. Otherwise the chosen point induces a valuation ring in $\operatorname{Frac}(R/\mathfrak{p})$. The residue injection already constructed identifies this ring with a local subring of $\widetilde{\mathcal{H}(x)}$. By

[StacksAlg, Tag 00IA], a valuation ring of that field dominates this local subring. Each residue of h^{m_j} lies in its maximal ideal, because its value at the chosen residual point is $e^{-\langle m_j,u\rangle} < 1$. Lemma 3.19.20 extends the dominating valuation ring to a graded valuation ring W containing $\widetilde{K}^{\mathrm{gr}}$ and with that ring as its degree-one part. Put $g_j = h^{-m_j}$, and let γ_j be its degree-one initial value at x. Thus no γ_j belongs to W.

For an arbitrary prescribed neighbourhood of x, choose an actual smaller rational neighbourhood N by Lemma 2.41.1, with all its defining inequalities strict at x. The strict inequalities give relative interior and Proposition 2.33.21 composes it with the original analytic interior. Thus $x \in \mathrm{Int}(N/K)$. Take the bounded coordinate maps from the actual rational presentations. The residue field at x is unchanged by Proposition 2.33.10. For an inner presentation of N/K, the finite-generation part of Theorem for the actual absolute admissible weighted Tate quotient makes the evaluated graded ring finite over $\widetilde{K}^{\mathrm{gr}}$: all new variables vanish at their assigned degrees. Every homogeneous element of that image therefore belongs to W. Indeed it satisfies a homogeneous monic equation, and dividing by its leading power would otherwise put 1 in the maximal homogeneous ideal of W.

Suppose the desired strict sector has no point in N. Then for each $y \in N$ one has $|g_j(y)| \le 1$ for at least one j. For each i form the actual domain

$$V_i = \{y \in N : |g_i/g_j(y)| \le 1 \text{ for every } j\}.$$

All ratios are units, so these are actual denominator-one radius-one domains containing x. Total comparability in the graded valuation ring orders the finitely many γ_i by divisibility. Choose i with $\gamma_i/\gamma_j \in W$ for all j. The graded reduction image of the actual V_i at x is finite over the image generated by the reduction of N and these degree-one ratios. For this use present N by an absolute weighted Tate algebra, append the ratios by the actual coefficientwise quotient of Proposition 1.17.3, and invoke only the admissible Tate-presentation clause of Theorem . Any surviving class of an original coordinate comes from its class in N by spectral contraction, and any surviving appended class is the stated ratio. The residue identification along $V_i \to N$ is the rational Proposition 2.33.10. Graded integral closedness (Lemma 3.19.1) puts the image of V_i in W. On V_i, however, $|g_i(y)| \le |g_j(y)| \le 1$ for some j at each point. Thus g_i has spectral seminorm at most one on V_i, by the spectral maximum formula 1.20.3. Its class at x is $\gamma_i \in W$, a contradiction. This proves realization of the strict signs in every neighbourhood of x. Continuity of t_h proves the assertion about values approaching 0. The choice of

x used the whole fibre; it did not restrict to the fibre above one target Gauss point. □

Strict sectors at a normalized interior point. Let x be an interior point of an affinoid over the complete ground field, and let analytic units h_i have absolute value one at x. Let $R \subset \widetilde{\mathcal{H}(x)}$ be the Laurent algebra generated over the residue field by their degree-one initial units. Suppose a residual analytic point of R, over the trivially valued residue field, satisfies the prescribed finitely many signs $|\widetilde{h}^{m_j}| < 1$. Every ordinary neighborhood of x then contains a point with $|h^{m_j}| < 1$ for every j.

To apply the preceding minimum-ratio argument at this fixed point, let $\mathfrak{q}$ be the kernel of the residual point. Its valuation ring lies in $\mathrm{Frac}(R/\mathfrak{q})$. Take the inverse image of that ring under the residue map $R_{\mathfrak{q}} \to \mathrm{Frac}(R/\mathfrak{q})$. The inverse image is local. An element whose residue is a unit of the valuation ring is a unit of $R_{\mathfrak{q}}$ and its inverse also belongs to the inverse image. This gives an actual local subring of $\widetilde{\mathcal{H}(x)}$ with all the prescribed monomial residues in its maximal ideal. A dominating valuation ring exists by [StacksAlg, Tag 00IA]. By Lemma 3.19.20 it extends to a graded valuation ring W containing the graded ground field. For $g_j = h^{-m_j}$, each initial value γ_j is outside W.

Take a strict rational neighborhood N inside the prescribed neighborhood. Strict inequalities and forward interior composition make x inner in N over the ground field. In an inner weighted presentation all added coordinate classes evaluate to zero at the assigned degrees. The fixed absolute presentation result therefore puts the evaluated graded ring of N inside W by graded integral closedness. If the desired signs fail everywhere on N, some $|g_j| \leq 1$ at each point there. Choose i with all γ_i/γ_j in W using total homogeneous divisibility. On the actual radius-one quotient domain where all $|g_i/g_j| \leq 1$, append those ratios to the same absolute weighted presentation. The surviving old coordinate classes come from N by spectral contraction. The new ones evaluate to the stated ratios. The same fixed absolute result and graded integral closedness put the evaluated graded ring of the quotient inside W. But on that domain $|g_i| \leq 1$ everywhere, so the spectral maximum formula puts γ_i in W, a contradiction.

Theorem 5.38.6 (The local tropical star of a whole interior fibre)**.** *Let $X = \mathcal{M}(B)$ be nonempty and pure of analytic dimension d over a complete non-Archimedean field K with its specified absolute value. Let $f_1, \dots, f_n \in B^\times$, put $t = t_f$, and suppose*

$$w \in t(X), \qquad F_w = t^{-1}(w) \subset \operatorname{Int}(X/K).$$

Choose a complete nontrivially valued isometric extension L/K and elements $a_i \in L^\times$ with $|a_i| = e^{-w_i}$. Let D_w be the actual algebra of F_w, put $D = (D_w)_L$, and set $h_i = f_i/a_i$. There is an actual admissible model $D_0 \subset D$, finite over $L^\circ\langle T_1^{\pm 1}, \dots, T_n^{\pm 1}\rangle$ by $T_i \mapsto h_i$. For any such model set

$$Z = \operatorname{Spec} \operatorname{im}\left(\widetilde{L}[T^{\pm 1}] \longrightarrow D_0/L^{\circ\circ}D_0\right).$$

Then $C = \operatorname{Trop}(Z)$ is the support of a finite pure d-dimensional rational polyhedral fan, and there is an open neighbourhood V of w such that

$$t(X) \cap V = (w + C) \cap V.$$

Its support C is independent of the extension, normalizing elements, and compatible model. The theorem allows the trivial absolute value on K, arbitrary real w, and nonreduced B.

Proof. Proposition 5.38.4, restricted to the exact level at w, makes D_w finite with its canonical norm over the n-dimensional exact-radius polytorus. Theorem 5.36.3 constructs an extension realizing all the radii, and gives the indicated actual finite model over every such L. The actual domain base-change theorem 2.36.9 identifies $\mathcal{M}(D)$ with the whole normalized zero fibre in X_L. It is contained in $\operatorname{Int}(X_L/L)$ by the base-change assertion of Proposition 2.33.21. Moreover

$$t_h(X_L) = t(X) - w, \qquad \dim_L X_L = \dim_K X = d,$$

by Theorems 2.33.13 and 3.21.8. No purity assertion after this scalar extension is used.

Apply Proposition 5.38.5 to X_L. It gives a nonempty finite rational fan C of dimension at most d, an inclusion of $t(X) - w$ in C near 0, and actual moment values approaching w in every strict rational sector meeting C. The last assertion means sectors containing a specified point of C with all their

defining inequalities strict. The boundary $\partial_K X$ is compact, by Propositions 2.33.21 and 2.24.1. Its moment image is compact and avoids w. Choose $\varepsilon > 0$ such that in the open Euclidean ball $B_\varepsilon(w)$ one has

$$t(X) \cap B_\varepsilon(w) \subset w + C, \qquad t(\partial_K X) \cap B_\varepsilon(w) = \varnothing.$$

Fix a maximal cone σ of this fan and put $r = \dim \sigma$. The set

$$E_\sigma = t(X) \cap \big(w + \operatorname{relint} \sigma\big) \cap B_\varepsilon(w)$$

is nonempty. If $\sigma = \{0\}$ this follows from $w \in t(X)$. Otherwise choose a nonzero rational vector $u \in \operatorname{relint} \sigma$. There is a strict rational sector containing u which meets C only in $\operatorname{relint} \sigma$. To see this, choose an integral linear form ℓ with $\ell(u) > 0$ and use the slice $\ell(v) = 1$. In this slice a sufficiently small rational open box around $u/\ell(u)$ avoids the finitely many other cones and the proper faces of σ. Its defining inequalities, after multiplication by the positive $\ell(v)$ and clearing denominators, together with $\ell(v) > 0$, define the required strict sector. Sector lifting from the preceding proposition, followed by $t_h(X_L) = t(X) - w$, gives points of E_σ arbitrarily close to w.

Take $v \in E_\sigma$ and put $N_\sigma = \mathbf{Z}^n \cap \operatorname{span}_{\mathbf{R}} \sigma$. This is saturated of rank r: a rational basis of the span becomes integral after clearing denominators. Since the fan is finite and $v - w$ belongs to the relative interior of a maximal cone, the moment image in a sufficiently small neighbourhood of v lies in $v + (N_\sigma)_{\mathbf{R}}$. Choose a closed rectangular box P_v in that neighbourhood, with v in its interior, and put $Y_v = t^{-1}(P_v)$. Choose an integral projection $Q : \mathbf{Z}^n \to \mathbf{Z}^r$ that is injective on N_σ, using Lemma 5.14.1. Its monomials are analytic units on Y_v. The entire level of $Q \circ t|_{Y_v}$ at Qv is exactly $F_v = t^{-1}(v)$, since Q is injective on this affine plane.

Every point of F_v is interior over K, because the chosen ball avoids $t(\partial_K X)$. The defining inequalities of Y_v are strict on F_v, so its actual rational presentation puts $F_v \subset \operatorname{Int}(Y_v/X)$. Composition in Proposition 2.33.21 therefore gives $F_v \subset \operatorname{Int}(Y_v/K)$. Theorem 5.31.1 now makes the actual algebra of F_v finite over the r-dimensional exact-radius polytorus at Qv. The rational universal property 2.31.2 identifies this level algebra with the original full fibre algebra in X. By Theorem 5.35.3 its analytic dimension is d. Lemma 5.38.3 and Proposition 5.35.4 give

$$d \le r.$$

The residual fan bound gives $r \le d$. Thus every maximal cone has dimension d.

At every point of E_σ, the hypotheses of Theorem 5.37.2 are now satisfied over the original field: X is pure of analytic dimension d, the whole fibre is interior, and the local image lies in the rank-d saturated tangent plane. That theorem makes E_σ relatively open in

$$U_\sigma = \big(w + \operatorname{relint}\sigma\big) \cap B_\varepsilon(w).$$

It is relatively closed there because $t(X)$ is compact. The set U_σ is convex and nonempty. Starting at a point of E_σ, join it to any point of U_σ by its line segment. The inverse image of E_σ on $[0,1]$ is open and closed and contains 0. If it were not all of $[0,1]$, the supremum of the initial interval it contains would contradict either its closedness or its openness. Thus $E_\sigma = U_\sigma$. This applies also to the zero-dimensional case, where $U_\sigma = \{w\}$. The relative interiors of the maximal cones are dense in their union. Closedness of $t(X)$ therefore fills all their faces inside the ball. Together with the opposite inclusion already proved, this yields

$$t(X) \cap B_\varepsilon(w) = (w + C) \cap B_\varepsilon(w).$$

Finally, two choices of normalizers, extension, or model give conical supports with the same germ at 0 by this equality. Multiplying any vector by a sufficiently small positive real number puts it in that common neighbourhood. Homogeneity of the two supports then makes them equal everywhere. □

Corollary 5.38.7 (Local polyhedrality without a purity assumption)**.** *Let $X = \mathcal{M}(B)$ be a generalized affinoid space over any complete non-Archimedean field with its specified absolute value, and let $t = t_f$ be the moment map of finitely many analytic units. If $w \in t(X)$ and the whole fibre F_w is interior over K, there is a finite rational polyhedral fan C and a neighbourhood V of w such that*

$$t(X) \cap V = (w + |C|) \cap V.$$

More precisely, $|C|$ is the union of the pure fans supplied by Theorem 5.38.6 for the irreducible components of X meeting F_w, using their analytic dimensions.

Proof. The generalized affinoid algebra B is Noetherian by Theorem 1.18.3, applied to the ideals of the finite module B over itself, and has finitely many minimal primes $\mathfrak{p}_i$, by [StacksAlg, Tag 00FR]. The closed analytic subspaces $X_i = \mathcal{M}(B/\mathfrak{p}_i)$ cover X: the prime kernel of every analytic point contains a

minimal prime by [StacksAlg, Tag 00E0(4)], and the point factors boundedly through that quotient. The quotient affinoid structure is again given by Corollary 1.18.4. Each nonempty X_i is pure of its analytic dimension, since its algebra is a domain.

The closed immersion $X_i \to X$ is finite, hence everywhere inner by Theorem 2.33.18. Composition in Proposition 2.33.21 therefore makes every point of $X_i \cap F_w$ interior over K. For those i for which this intersection is nonempty, apply Theorem 5.38.6 to X_i. For every other i, Proposition 2.24.1 makes $t(X_i)$ compact. It avoids w and can be removed by shrinking the neighbourhood of w. Intersect the finitely many resulting neighbourhoods. The equality for X follows by taking their union. Subdividing by the finitely many rational hyperplanes used to define all the cones gives a common rational polyhedral refinement, and hence C with the stated support. □

5.39 Finite polyhedrality of actual analytic moment images

Throughout this section K is complete for its specified real-valued non-Archimedean absolute value. It may be trivially valued, and affinoid presentation radii may be arbitrary positive real numbers. The dimension is analytic dimension. The polyhedral inequality descriptions have integral linear parts and arbitrary real constants.

Theorem 5.39.1 (A finite Gauss-fibre cover of the analytic boundary)**.** *Let $X = \mathcal{M}(A)$ be a nonempty generalized affinoid space of analytic dimension d. There are finitely many elements $a_j \in A$, with positive spectral radii $r_j = \rho_A(a_j)$, and bounded maps*

$$K\{r_j^{-1}T\} \longrightarrow A, \qquad T \longmapsto a_j,$$

with the following property. Let η_j be the radius-r_j Gauss point, let $H_j = \mathcal{H}(\eta_j)$ with its specified absolute value, and put

$$X_j = \mathcal{M}_{H_j}\big(A \widehat{\otimes}_{K\{r_j^{-1}T\}} H_j\big).$$

The natural maps identify these actual analytic fibres with compact subsets of X, preserve completed residue fields, and satisfy

$$\partial_K X = \bigcup_j X_j, \qquad \dim_{H_j} X_j \le d - 1$$

for every nonempty X_j. The empty list is allowed. In particular an affinoid of analytic dimension zero has empty absolute boundary.

Proof. Choose homogeneous algebra generators $\bar{a}_j$ of $\widetilde{A}^{\mathrm{gr}}$ over $\widetilde{K}^{\mathrm{gr}}$ by Theorem 3.18.7, and discard zero generators. If $\bar{a}_j$ has degree r_j, choose a lift a_j in the spectral filtration. Nonvanishing of its initial form means exactly that $\rho_A(a_j) = r_j > 0$. Theorem 1.20.2, together with the spectral formula 1.20.3, gives a constant C_j with $\|a_j^m\| \le C_j r_j^m$ for every $m \ge 0$. Proposition 1.9.3 therefore gives the displayed bounded substitution map. This uses the original admissible norm on A; it does not require $\|a_j\| \le r_j$.

Fix $x \in X$ and let S_x be the image of $\widetilde{A}^{\mathrm{gr}}$ in $\widetilde{\mathcal{H}(x)}^{\mathrm{gr}}$. Write u_j for the image of $\bar{a}_j$. The graded innerness criterion 3.18.10, applied to evaluation $A \to \mathcal{H}(x)$ over K, says that x is interior precisely when S_x is finite over $\widetilde{K}^{\mathrm{gr}}$.

A homogeneous element u of degree r satisfying any nonzero polynomial relation over the graded base satisfies a monic homogeneous relation. Indeed, take a nonzero homogeneous component of that polynomial relation, for the grading with $\deg T = r$. Its highest nonzero coefficient is a nonzero homogeneous element of the graded field $\widetilde{K}^{\mathrm{gr}}$, and is therefore invertible. Divide by that coefficient. Thus every such u is integral. If all the u_j are integral, their finitely many monic relations express S_x as a finite module, generated by monomials with bounded exponents. Conversely, in a finite algebra every element is integral: expressing multiplication by it on a finite generating family and applying the adjugate to the resulting square matrix gives a monic annihilating polynomial, which annihilates 1 as well. Consequently

$$x \in \partial_K X \iff \text{some } u_j \text{ satisfies no nonzero homogeneous polynomial relation over } \widetilde{K}^{\mathrm{gr}}.$$

For a fixed j, this last condition is equivalent to the image of x in the radius-r_j disc being η_j. To prove one direction, let $P(T) = \sum_i c_i T^i$ be a nonzero polynomial over K, and put $R = \max_i |c_i| r_j^i$. The terms attaining this maximum give a nonzero homogeneous polynomial in the initial variable. If u_j has no relation, its value is nonzero. Hence

$$|P(a_j(x))| = R.$$

Weighted polynomials are dense in $K\{r_j^{-1}T\}$, so these equalities identify the bounded seminorm on the whole disc algebra with the Gauss seminorm. Conversely, at that Gauss seminorm one has $|a_j(x)| = r_j$. Lift the finitely many

coefficients of a proposed nonzero homogeneous relation for u_j to elements of K with their exact prescribed sizes. All its terms have one common weighted size. A relation of initial forms would make the evaluated polynomial have strictly smaller size, contrary to the Gauss equality. This proves the equivalence and the asserted equality of boundary sets.

Theorem 2.33.11 constructs X_j as the actual generalized H_j-affinoid fibre. It gives its homeomorphism onto the full inverse image of η_j and the isometric identification of its completed residue field at a point z with $\mathcal{H}_A(x)$ at the corresponding point $x \in X$. The field H_j is the completion of $K(T)$ for the radius-r_j Gauss absolute value: weighted polynomial density also gives density of their fractions. Lemma 3.22.4 gives $d(H_j/K) = 1$. The tower formula of Lemma 3.22.2 therefore gives

$$d_{H_j}(z) = d(\mathcal{H}_A(x)/H_j) = d_K(x) - 1 \leq d - 1.$$

Taking the maximum on the nonempty affinoid X_j and using Theorem 3.22.7 proves the dimension bound. If $d = 0$, a nonempty fibre would have a point with negative relative invariant, which is impossible. Thus the boundary is empty in that case. Compactness of the fibres is Proposition 2.24.1. □

Theorem 5.39.2 (Finite polyhedral upper bounds without an interior hypothesis)**.** *Let $X = \mathcal{M}(A)$ be a nonempty generalized affinoid space of analytic dimension d, and let analytic units $f_1, \dots, f_n$ define $t : X \to \mathbb{R}^n$. For every $w \in t(X)$ there are a finite rational polyhedral fan C of dimension at most d and a neighbourhood V of w such that*

$$t(X) \cap V \subset (w + |C|) \cap V.$$

Consequently $t(X)$ is contained in the support of a finite integral affine polyhedral complex of dimension at most d.

Proof. Choose a complete isometric extension L/K containing elements with sizes equal to every radius in a fixed presentation of A, and elements c_i with $|c_i| = e^{-w_i}$; require also that L be nontrivially valued. Such a field is explicit: take the completion of a rational function field with weighted Gauss norms assigning these finitely many positive radii to its variables, and add a variable of size strictly between zero and one. The Gauss construction is Proposition 1.2.2; its valued completion is the one in Lemma 3.22.4. The actual scalar presentation 2.33.5, followed by rescaling the finitely many presentation variables, makes X_L strict. Its dimension is d by Theorem 3.21.8.

Put $h_i = f_i/c_i$. Scalar spectral surjectivity 2.33.13 gives

$$t_h(X_L) = t(X) - w.$$

It suffices to prove the upper bound at zero for this actual strict space. Its whole fibre $F = t_h^{-1}(0)$ is a nonempty strict affinoid domain by Theorem 2.31.2. Let D be its algebra. It has an admissible integral model D_0 by Proposition 3.14.7. The elements $h_i^{\pm 1}$ are power-bounded in D, since their values on F are one and Theorem 1.20.2 applies. Adjoin them to D_0 by Theorem 3.16.8. The resulting algebra is still an actual admissible model of the same D, and contains all these elements. Use this enlarged model in what follows.

Write $k = \widetilde{L}$, $\mathfrak{m} = L^{\circ\circ}$, and

$$J = \ker\bigl(k[T_1^{\pm 1}, \dots, T_n^{\pm 1}] \longrightarrow D_0/\mathfrak{m}D_0\bigr), \qquad R = k[T^{\pm 1}]/J.$$

Thus R injects into the actual special-fibre algebra. For every minimal prime $\mathfrak{p}$ of R, localization preserves this injection, by [StacksAlg, Tag 00HT]. The algebra $(D_0/\mathfrak{m}D_0) \otimes_R R_{\mathfrak{p}}$ is nonzero, so it has a prime by [StacksAlg, Tag 00E0(1)]. Its contraction is the unique prime of $R_{\mathfrak{p}}$. It therefore gives a prime $\mathfrak{q}$ of the special fibre lying above $\mathfrak{p}$; no finiteness of the injection $R \to D_0/\mathfrak{m}D_0$ is required here. Actual specialization surjectivity, Theorem 3.16.15, supplies $x \in F$ specializing to $\mathfrak{q}$. The induced injections give

$$\operatorname{Frac}(R/\mathfrak{p}) \hookrightarrow \widetilde{\mathcal{H}(x)}.$$

By [StacksAlg, Tag 00P0] and Theorem 3.22.7,

$$\dim(R/\mathfrak{p}) = \operatorname{trdeg}_k \operatorname{Frac}(R/\mathfrak{p}) \leq d_L(x) \leq d.$$

The residue field here is the same as for $x \in X_L$, by Corollary 2.33.15.

The finite-type k-algebra R is Noetherian by [StacksAlg, Tag 00FO], and has finitely many minimal primes, by [StacksAlg, Tag 00FR]. Its tropical support is the union of the supports of the corresponding integral quotients: every analytic prime kernel contains a minimal prime by [StacksAlg, Tag 00E0(4)]. Theorem 5.19.5 bounds the dimension of each of these algebraic tropicalizations by d. Lemma 5.38.2 gives a finite rational fan C with support $\operatorname{Trop}(\operatorname{Spec} R)$ and finitely many nonzero $p_j \in J$ cutting out that support tropically. Its dimension is therefore at most d.

Lift every nonzero coefficient of each p_j to an element of L° of absolute value one, obtaining a Laurent polynomial P_j. Since $p_j \in J$, $P_j(h) \in \mathfrak{m}D_0$.

Express this membership as a finite sum $\sum_\ell b_{j\ell} d_{j\ell}$ with $|b_{j\ell}| < 1$ and $d_{j\ell} \in D_0$. Every $d_{j\ell}$ has absolute value at most one on F, by the actual model presentation. Hence there is a real $c_j < 1$ with $|P_j(h)| < c_j$ throughout F. The compact subset of X_L where $|P_j(h)| \geq c_j$ has compact moment image missing zero. Avoiding these finitely many images gives a neighbourhood V_0 of zero where all these strict inequalities hold on the entire inverse image. This uses the compact spectrum of Proposition 2.24.1.

Shrink V_0 further so that, for every monomial of every P_j, its norm at a weight $v \in V_0$ is greater than c_j. At zero all these finitely many norms equal one, so continuity permits the simultaneous choice. If an actual moment value $v \in V_0$ lay outside $|C|$, some p_j would have a unique least-weight term there. The same term of $P_j(h)$ would have strictly largest absolute value at any point over v, and that value would exceed c_j. The non-Archimedean triangle equality would give $|P_j(h)| > c_j$, a contradiction. Thus $t_h(X_L) \cap V_0 \subset |C|$. Translating back proves the local assertion over the original field and norm.

Finally, at every $w \in t(X)$ choose a closed rectangular box centred at w inside its upper-bound neighbourhood. Finitely many box interiors cover the compact set $t(X)$. The union of the intersections of these boxes with the corresponding translated finite fans contains $t(X)$. It is a finite union of integral affine polytopes, all of dimension at most d. Cutting by their finitely many defining hyperplanes gives a finite common complex, as in Lemma 5.14.2. This proves the final assertion. □

Theorem 5.39.3 (Global polyhedrality and the tropicalized boundary)**.** *Let X be a nonempty generalized affinoid space of analytic dimension d over K, and let t be the moment map of any finite tuple of analytic units. The actual image $t(X)$ is the support of a finite integral affine polyhedral complex of dimension at most d. The actual boundary image $t(\partial_K X)$ is likewise finite polyhedral, of dimension at most $d-1$; it is empty when $d = 0$.*

If X is pure of analytic dimension d, every point of $t(X) \setminus t(\partial_K X)$ has the pure d-dimensional local fan of Theorem 5.38.6.

Proof. We induct on the analytic dimension, simultaneously for all complete specified ground fields, all positive real radii, and all unit tuples. Theorem 5.39.2 gives a finite polyhedral upper bound of dimension zero when $d = 0$, hence a finite set containing $t(X)$. Any subset of a finite set is a finite zero-dimensional complex. Theorem 5.39.1 makes the boundary empty. This establishes the induction base.

Assume the assertion for every analytic dimension less than d. First suppose X is pure of dimension d. Use Theorem 5.39.1 to cover its actual boundary by the finitely many actual affinoid fibres X_j over their specified fields H_j, each of dimension at most $d-1$. The restrictions of all original moment functions remain analytic units there. The homeomorphisms and residue-field identifications in that theorem preserve their absolute values. By induction,

$$B = t(\partial_K X) = \bigcup_j t(X_j)$$

is a finite union of integral affine polytopes of dimension at most $d-1$. In particular it admits a finite complex structure of that dimension.

Put $S = t(X)$. By Theorem 5.39.2, choose a finite complex $\mathcal{P}$ of dimension at most d whose support contains S. The upper-bound construction uses bounded boxes, so all its cells may be taken to be polytopes. Since $B \subset S$, cutting by the defining hyperplanes of both complexes gives a common refinement of $\mathcal{P}$ in which B is a subcomplex.

For a d-cell σ of this refinement, its relative interior misses B. The set $S \cap \operatorname{relint}(\sigma)$ is relatively closed because S is compact. It is also relatively open. Indeed, near one of its points no other cell of the finite complex meets that point, so the actual moment image is contained in the affine span of σ. The whole moment fibre is interior, since its value does not belong to B. Theorem 5.37.2, with the saturated rank-d tangent lattice of σ, fills a neighbourhood in that affine plane. This proves relative openness. The relative interior of σ is convex and therefore connected, by the line-segment open-and-closed argument used in Theorem 5.38.6. Thus the intersection is either empty or all of $\operatorname{relint}(\sigma)$.

Select the cells for which it is nonempty and let T be their union with all their faces. Closedness of S shows $T \subset S$. We claim $S = B \cup T$. If $v \in S \setminus (B \cup T)$, choose a neighbourhood missing the closed sets B and T. The whole fibre at v is interior, so Theorem 5.38.6 presents S there by a pure d-dimensional translated fan. A relatively open part of a d-cone in this fan cannot lie in the finitely many affine spans of the lower-dimensional cells of $\mathcal{P}$. To check this, choose a line direction in its span outside the finitely many proper direction subspaces; each remaining proper affine span meets a line in that direction at at most one point. Such a direction exists: in a basis of the d-plane, the vectors $(1, b, \dots, b^{d-1})$, for integers b, avoid each fixed proper linear subspace except at finitely many b, by applying a nonzero

linear equation for that subspace. Consequently arbitrarily close to v there is a point of S in the relative interior of a d-cell of $\mathcal{P}$. That cell is selected, contradicting the chosen neighbourhood's disjointness from T. This proves the claim. The finite union $B \cup T$ is polyhedral of dimension at most d.

For a general $X = \mathcal{M}(A)$ of dimension d, the Noetherian algebra A has finitely many minimal primes, by Theorem 1.18.3 and [StacksAlg, Tag 00FR]. The actual quotient affinoids of Corollary 1.18.4 $\mathcal{M}(A/\mathfrak{p})$ cover X, because every analytic prime kernel contains a minimal prime, [StacksAlg, Tag 00E0(4)]. Each quotient is a domain, and is pure of its analytic dimension. This dimension is at most d: its points have the same completed residue fields as their images in X, so apply Theorem 3.22.7. The pure case just proved handles the components of dimension d, and induction handles the others. Their finite union of moment images proves the assertion for $t(X)$. The boundary assertion for this general X follows directly from its own finite Gauss-fibre cover and the induction hypothesis, exactly as in the pure case. No purity after a coefficient extension is used.

Finally, the stated local purity off the boundary image is precisely Theorem 5.38.6, whose full-fibre hypothesis holds at every such value. □

5.40 Generic cycles in actual analytic Gauss fibres

Throughout, K is complete for its specified real-valued non-Archimedean absolute value. The valuation may be trivial. All affinoid algebras may be generalized, and finite algebra maps carry their canonical finite-module Banach structures. For a positive real polyradius r, write $R = A_{K,r}$ and $H = H_{K,r}$ for the actual Laurent algebra and its Gauss field in Section 5.36.

Proposition 5.40.1 (A Gauss fibre sees the generic component lengths)**.** *Let A be a generalized K-affinoid algebra, and let $A \to D$ represent an affinoid domain of $\mathcal{M}(A)$. Suppose that a bounded map $R \to D$ is finite, and put*

$$E = D\widehat{\otimes}_R H.$$

Then $E = D \otimes_R H$ is a finite-dimensional H-algebra and is flat as an ordinary A-module. Every prime of E contracts to a minimal prime of A.

For a finite A-module M and $\mathfrak{p} \in \operatorname{Min}(A)$, put

$$\beta_{\mathfrak{p}}(M) = \operatorname{length}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}), \qquad \lambda_{\mathfrak{p}}(E) = \dim_H(E/\mathfrak{p}E).$$

These are finite nonnegative integers, and

$$\dim_H(M \otimes_A E) = \sum_{\mathfrak{p} \in \operatorname{Min}(A)} \beta_{\mathfrak{p}}(M)\lambda_{\mathfrak{p}}(E). \tag{5.40.1}$$

The term $\lambda_{\mathfrak{p}}(E)$ is positive exactly when a prime of E contracts to $\mathfrak{p}$.

If an ideal $I \subset A$ is contained in no minimal prime, then $IE = E$. If $M_{\mathfrak{p}} = 0$ at every minimal prime, then $M \otimes_A E = 0$.

Proof. The Laurent Gauss norm is multiplicative and has zero kernel, by the actual presentation and norm calculation in Theorem 1.17.2 and Proposition 1.2.2. Hence R is a domain and $R \to H$ factors through $\operatorname{Frac}(R)$. Localization is flat by [StacksAlg, Tag 00HT(1)]. A module over a field is flat by the ideal criterion [StacksAlg, Tag 00HD], since that field has only the zero and unit ideals. It follows that H is flat over R.

Proposition 2.33.19 identifies E with the displayed ordinary tensor, with its complete canonical finite-dimensional norm. The homomorphism $D \to E$ is flat: tensoring a D-module with E is tensoring its underlying R-module with H. The domain map $A \to D$ is flat by Proposition 2.38.2. Composing these two exact tensor functors proves that $A \to E$ is flat.

The algebra E is Artinian and is a finite product of its local factors E_j, by [StacksAlg, Tag 00JB]. This includes the empty product when $E = 0$. Let $\mathfrak{q}_j$ be the maximal ideal of a nonzero factor, and let $\mathfrak{p}_j$ be its contraction in A. A strictly smaller prime of A would, by flat going down [StacksAlg, Tag 00HS], lift to a strictly smaller prime under the corresponding prime of E. Every prime of an Artinian ring is minimal. Thus $\mathfrak{p}_j$ is minimal in A.

The ring A is Noetherian by Theorem 1.18.3, and has finitely many minimal primes by [StacksAlg, Tag 00FR]. For any such prime $\mathfrak{p}$, the local ring $A_{\mathfrak{p}}$ is Noetherian of dimension zero, by [StacksAlg, Tag 00FN] and the minimality of $\mathfrak{p}$. It is Artinian by [StacksAlg, Tag 00KH]. Therefore every finite module over it has finite length.

Fix j and put $\mathfrak{p} = \mathfrak{p}_j$. Every element of $A \setminus \mathfrak{p}$ maps to a unit of E_j, so $A \to E_j$ factors through $A_{\mathfrak{p}}$. The factor E_j is flat over A, being a direct summand of E, and hence is flat over $A_{\mathfrak{p}}$ by [StacksAlg, Tag 00HT(2)]. Take a composition series of $M_{\mathfrak{p}}$. Every simple quotient is $\kappa(\mathfrak{p})$: a simple module over a commutative local ring is its residue field. Tensoring this series with E_j remains exact and gives

$$\dim_H(M \otimes_A E_j) = \beta_{\mathfrak{p}}(M)\, \dim_H(E_j/\mathfrak{p}E_j).$$

The quotient on the right is nonzero, since $\mathfrak{p}E_j \subset \mathfrak{q}_j$.

For a different minimal prime $\mathfrak{p}' \neq \mathfrak{p}_j$, choose an element of $\mathfrak{p}'$ outside $\mathfrak{p}_j$. Its image is a unit of E_j, so $E_j/\mathfrak{p}'E_j = 0$. Consequently $E/\mathfrak{p}E$ is exactly the product of $E_j/\mathfrak{p}E_j$ over the factors with contraction $\mathfrak{p}$. Summing the preceding dimension identity proves (5.40.1) and the positivity assertion.

Finally, if I is contained in no minimal prime, then for each j it contains an element outside $\mathfrak{p}_j$. Thus $IE_j = E_j$, and hence $IE = E$. The last assertion follows from the displayed length formula. □

Theorem 5.40.2 (Weights of finite modules on an interior flat germ)**.** *Let $Y = \mathcal{M}(A)$ be nonempty and pure of analytic dimension d. Let a unit moment tuple f define $t : Y \to \mathbb{R}^n$, and let $w \in t(Y)$. Suppose that the entire fibre $F_w = t^{-1}(w)$ lies in $\mathrm{Int}(Y/K)$ and that, for some neighborhood W of w and a saturated rank-d lattice $N \subset \mathbb{Z}^n$,*

$$t(Y) \cap W \subset w + N_{\mathbb{R}}.$$

For $\mathfrak{p} \in \mathrm{Min}(A)$, put $Y_{\mathfrak{p}} = \mathcal{M}(A/\mathfrak{p})$. Set $m_{\mathfrak{p}}(w) = 0$ if its moment fibre at w is empty; otherwise this fibre satisfies the same flat-germ hypotheses, and let $m_{\mathfrak{p}}(w)$ be its positive intrinsic weight.

Let M be a finite A-module. For any integral $Q : \mathbb{Z}^n \to \mathbb{Z}^d$ injective on N, let D_w be the actual algebra of F_w, let R_Q be the polytorus algebra at Qw, and let H_Q be its Gauss field. Put

$$E_Q = D_w \widehat{\otimes}_{R_Q} H_Q, \qquad e_Q = [\mathbb{Z}^d : Q(N)], \qquad \delta_Q(M, w) = \dim_{H_Q}(M \otimes_A E_Q).$$

Then

$$\frac{\delta_Q(M, w)}{e_Q} = \sum_{\mathfrak{p} \in \mathrm{Min}(A)} \beta_{\mathfrak{p}}(M) m_{\mathfrak{p}}(w) \in \mathbb{Z}_{\geq 0}. \tag{5.40.2}$$

In particular this integer, denoted $m_{M,f}(w)$, is independent of Q and is additive in short exact sequences of finite A-modules. For $M = A$ it is the intrinsic weight of Y:

$$m_{Y,f}(w) = \sum_{\mathfrak{p} \in \mathrm{Min}(A)} \mathrm{length}(A_{\mathfrak{p}}) m_{\mathfrak{p}}(w).$$

These module weights are unchanged by restriction to an affinoid domain containing the whole fibre, whenever the two underlying pure-dimensional germs satisfy the stated flat-germ hypotheses.

Proof. Theorem 5.37.1 gives every finite map $R_Q \to D_w$, with its canonical norm and actual Gauss fibre. Apply Proposition 5.40.1 to the domain $F_w \subset Y$.

The quotient $A \to A/\mathfrak{p}$ is finite with its canonical norm. Its spectral map is everywhere interior relative to Y by Theorem 2.33.18. The composition inclusion in Proposition 2.33.21 therefore makes every point over F_w interior over K. The affinoid $Y_{\mathfrak{p}}$ is pure of analytic dimension d by the assumed purity of A, and its moment image is contained in that of Y. If its fibre at w is nonempty, Theorem 5.37.1 applies with the same lattice N.

The actual domain base-change theorem 2.36.9 and finite tensor comparison 2.33.19 identify its full fibre algebra with $D_w/\mathfrak{p}D_w$. This is finite over R_Q. Its actual Gauss algebra is

$$(D_w/\mathfrak{p}D_w)\widehat{\otimes}_{R_Q} H_Q = E_Q/\mathfrak{p}E_Q.$$

For a nonempty fibre its dimension is $e_Q m_{\mathfrak{p}}(w)$ by the flat-germ weight theorem. For an empty fibre the actual algebra is zero, by Theorem 1.20.1, and the same identity holds with $m_{\mathfrak{p}}(w) = 0$. Substitution in (5.40.1) proves (5.40.2).

All tensors in this argument also have their canonical analytic meaning for M. The finite module $M \otimes_A D_w$ has its complete canonical norm by Proposition 1.18.7. A quotient of a finite sum of copies of the finite canonical R_Q-module D_w gives its canonical finite-R_Q-module topology: compose the two finite free quotient presentations. Applying the same tensor comparison over R_Q shows that $M \otimes_A E_Q$ is already complete. Thus no separated completion discards a submodule in the length calculation.

Flatness of $A \to E_Q$ makes its tensor functor exact, proving additivity. For $M = A$ the numerator is the ordinary Gauss degree of D_w, so the formula gives exactly $m_{Y,f}(w)$.

Finally let $A \to A_V$ represent a domain containing F_w, and replace M by $M \otimes_A A_V$. The actual full fibre algebra is still D_w, by Theorem 2.36.9. The saturated tangent lattice is unchanged by Theorem 5.37.2. For a common Q, associativity gives

$$(M \otimes_A A_V) \otimes_{A_V} E_Q = M \otimes_A E_Q.$$

Both the Gauss dimension and its lattice divisor are the same, which proves the restriction assertion. □

5.41 Finite analytic pushforward with generic cycle degrees

Theorem 5.41.1 (Finite pushforward with full generic lengths)**.** *Let $p : X = \mathcal{M}(B) \to Y = \mathcal{M}(A)$ be finite, with the canonical finite-A-module norm on B. Suppose that X and Y are pure of analytic dimension d. Let unit moment tuples define $t_X : X \to \mathbb{R}^n$ and $t_Y : Y \to \mathbb{R}^{n'}$, and let $a : \mathbb{Z}^n \to \mathbb{Z}^{n'}$ be integral. Assume the corresponding identity of characters, so that $t_Y \circ p = a \circ t_X$.*

Let $w' \in t_Y(Y)$ have its whole fibre in $\mathrm{Int}(Y/K)$, and suppose that the local target image is contained in $w' + (N_Y)_{\mathbb{R}}$ for a saturated rank-d lattice N_Y. Assume that

$$a^{-1}(w') \cap t_X(X) = \{w_1, \dots, w_s\}$$

is finite and nonempty and that the local image at each w_i is contained in $w_i + (N_i)_{\mathbb{R}}$ for a saturated rank-d lattice N_i.

For every i, the lattice $a(N_i)$ is contained in N_Y with finite index. Write $m_{\mathfrak{p}}(w')$ for the component weights of Y in Theorem 5.40.2. Then

$$\sum_{i=1}^{s} m_{X,f_X}(w_i)[N_Y : a(N_i)] = m_{B,f_Y}(w') = \sum_{\mathfrak{p} \in \mathrm{Min}(A)} \beta_{\mathfrak{p}}(B) m_{\mathfrak{p}}(w'). \quad (5.41.1)$$

The coefficients are the full generic pushforward lengths:

$$\beta_{\mathfrak{p}}(B) = \sum_{\substack{\mathfrak{q} \in \mathrm{Min}(B) \\ \mathfrak{q} \cap A = \mathfrak{p}}} \mathrm{length}_{B_{\mathfrak{q}}}(B_{\mathfrak{q}})[\kappa(\mathfrak{q}) : \kappa(\mathfrak{p})].$$

Consequently, if a positive rational number Δ satisfies

$$\beta_{\mathfrak{p}}(B) = \Delta \ \mathrm{length}(A_{\mathfrak{p}})$$

for every component with nonempty moment fibre at w', then

$$\Delta \, m_{Y,f_Y}(w') = \sum_{i=1}^{s} m_{X,f_X}(w_i)[N_Y : a(N_i)].$$

For a reduced integral target, Δ is the positive integer $\dim_{\mathrm{Frac}(A)}(B \otimes_A \mathrm{Frac}(A))$. If B is also integral, it is $[\mathrm{Frac}(B) : \mathrm{Frac}(A)]$.

Proof. Finite relative interior, Theorem 2.33.18, and the composition inclusion 2.33.21 put every whole source fibre over w' in $\mathrm{Int}(X/K)$. Theorem 5.37.1 therefore defines all source and target weights. Their local images fill relative neighborhoods in the specified planes by Theorem 5.37.2. The equality of moment maps then gives $a((N_i)_{\mathbb{R}}) \subset (N_Y)_{\mathbb{R}}$. Saturation gives $a(N_i) \subset N_Y$. A nonzero vector in the kernel of $a|_{(N_i)_{\mathbb{R}}}$ would produce a small interval of source moment weights above w', contrary to the stated finite set. Thus $a|_{N_i}$ is injective, and its image has finite index in N_Y.

Choose an integral $Q : \mathbf{Z}^{n'} \to \mathbf{Z}^d$ injective on N_Y by Lemma 5.14.1. Inside the target flat-germ neighbourhood choose a closed box $P = \prod_j [\alpha_j, \beta_j]$ with w' in its interior. Write $g_j = f_{Y,j}$ and $Y_0 = t_Y^{-1}(P)$. The inequalities

$$|g_j| \le e^{-\alpha_j}, \qquad |g_j^{-1}| \le e^{\beta_j}$$

give a single rational presentation over A with denominator 1. Theorems 2.31.1 and 2.31.2 identify its quotient, with its quotient norm, as the affinoid domain algebra of Y_0. At every y in the full fibre $F' = t_Y^{-1}(w')$, the two coordinates for each j are strictly below their displayed presentation radii. Thus $y \in \mathrm{Int}(Y_0/Y)$ by Definition 2.7.1. Since $F' \subset \mathrm{Int}(Y/K)$ by hypothesis, Proposition 2.33.21 gives $F' \subset \mathrm{Int}(Y_0/K)$.

The image of Y_0 lies in $w' + (N_Y)_{\mathbf{R}}$, on which Q is injective. Its entire projected level at Qw' is therefore F'. For every norm-admissible affinoid test over Y, mapping into F' is equivalent to mapping into Y_0 and then into this projected level. The rational domain universal property gives inverse bounded maps between the two representing algebras. Denote this algebra by $D'_{w'}$ with its original characters.

By Theorem 2.36.9, $X_0 = p^{-1}(Y_0)$ and $p^{-1}(F')$ are the actual pullbacks of the two target domains. Proposition 2.33.19 preserves the finite canonical module structure of $X_0 \to Y_0$. Theorem 2.33.18 and the composition inclusion put every point above F' in $\mathrm{Int}(X_0/K)$. By the character identities, the whole Qa-level on X_0 at Qw' is $p^{-1}(F')$. Its source moment values are precisely the w_i: every original source point with $t_X = w_i$ already maps into F'. The same universal property identifies the projected source level with the pulled-back domain over X, and each exact w_i-domain in X_0 with the original whole domain in X. Proposition 5.31.2 now splits the projected level algebra into the product of their actual algebras. Consequently there are bounded algebra isomorphisms

$$\prod_i D_{w_i} \;\simeq\; B \widehat{\otimes}_A D'_{w'} \;\simeq\; B \otimes_A D'_{w'}.$$

The last isomorphism uses the given finite canonical norm and Proposition 2.33.19. The maps respect the projected target characters, since they come from restriction and pullback of the specified character identity.

Let R_Q be the exact-radius target at Qw' and H_Q its Gauss field. All finite tensors are ordinary complete canonical tensors by Proposition 2.33.19. Tensoring the preceding product with H_Q and taking dimensions gives

$$\dim_{H_Q}(B \otimes_A E_Q) = \sum_i \delta_{Qa}(w_i).$$

Theorem 5.40.2 evaluates the left side as $[\mathbb{Z}^d : Q(N_Y)]m_{B,f_Y}(w')$. Theorem 5.37.1 evaluates each right-hand term as $[\mathbb{Z}^d : Qa(N_i)]m_{X,f_X}(w_i)$. Counting representatives in $Qa(N_i) \subset Q(N_Y) \subset \mathbb{Z}^d$ gives

$$[\mathbb{Z}^d : Qa(N_i)] = [\mathbb{Z}^d : Q(N_Y)]\,[N_Y : a(N_i)].$$

Cancellation proves (5.41.1).

For the coefficient formula, fix a minimal prime $\mathfrak{p}$ of A. The ring $A_{\mathfrak{p}}$ is Artinian by the first proposition. Its finite algebra $B_{\mathfrak{p}} = B \otimes_A A_{\mathfrak{p}}$ has finite length as an $A_{\mathfrak{p}}$-module, and hence is Artinian as a ring. Decompose it into its local factors by [StacksAlg, Tag 00JB]. They are $B_{\mathfrak{q}}$ for primes $\mathfrak{q}$ of B over $\mathfrak{p}$. Such primes are minimal in B: a smaller prime would have the same contraction, contradicting incomparability [StacksAlg, Tag 00GT] for the finite, hence integral, map [StacksAlg, Tag 00GK]. Every simple quotient in a composition series of $B_{\mathfrak{q}}$ is $\kappa(\mathfrak{q})$, whose length over $A_{\mathfrak{p}}$ is its full finite dimension over $\kappa(\mathfrak{p})$. Adding these lengths gives the asserted coefficient formula.

The target formula in Theorem 5.40.2 now proves the assertion with Δ. If A is an integral domain, its unique generic local ring is $\mathrm{Frac}(A)$ and has length one. Thus $\beta_{(0)}(B)$ is the displayed vector-space dimension. It is positive because (5.41.1) has a positive left side. If B is integral, this positivity first makes $A \to B$ injective: a nonzero element of its kernel would become invertible and annihilate the generic algebra. Localizing the domain B at the nonzero elements of A is therefore injective. The resulting finite-dimensional domain is a field and hence is precisely $\mathrm{Frac}(B)$. This proves the last assertion. □

Proposition 5.41.2 (The free locus of a finite analytic module)**.** *Let A be a generalized K-affinoid algebra and let M be a finite A-module. The set*

$$U_M = \{\mathfrak{p} \in \operatorname{Spec}(A) : M_{\mathfrak{p}} \text{ is free over } A_{\mathfrak{p}}\}$$

is open. It is a finite union of principal opens $D(g_j)$ on each of which M_{g_j} is finite free. If M is free at all minimal primes, then the ideal $I = (g_1, \dots, g_t)$ is contained in no minimal prime. This condition holds for every finite M when A is reduced.

Let

$$U_M^{\mathrm{an}} = \{y \in \mathcal{M}(A) : \ker(y) \in U_M\}.$$

For every affinoid domain $V \subset U_M^{\mathrm{an}}$, its restriction $M \otimes_A A_V$ is finite locally free as an ordinary A_V-module. Its local bases give bounded isomorphisms with the canonical free-module norms on affinoid domains where the corresponding g_j is a unit.

Suppose M is free at all minimal primes, and let E be any actual Gauss algebra in Proposition 5.40.1. Then $IE = E$. On a local factor E_j whose residue prime contracts to $\mathfrak{p}$, the module $M \otimes_A E_j$ is free of rank $\operatorname{rank}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$.

Proof. The ring A is Noetherian and every finite module is finitely presented, by Theorem 1.18.3. Suppose $M_{\mathfrak{p}}$ is free of rank r. A basis can be represented by elements of M: multiply its finitely many members by their respective denominators, which are units in $A_{\mathfrak{p}}$. They give a map $A^r \to M$ whose localization at $\mathfrak{p}$ is an isomorphism. Its kernel and cokernel are finite and become zero at $\mathfrak{p}$ by exactness of localization, [StacksAlg, Tag 00HT]. For each member of finite generating lists, vanishing after localization supplies an annihilator outside $\mathfrak{p}$. The product of these finitely many annihilators is an element $g \notin \mathfrak{p}$ for which $M_g \simeq A_g^r$. This proves openness and gives a principal free neighborhood of every point of U_M.

Let I be the ideal generated by all these elements g. It is generated by finitely many of them, since A is Noetherian: express a finite generating set of I using finitely many of the original g's. Thus $U_M = D(I) = \bigcup_j D(g_j)$ with the asserted free bases. If every minimal prime belongs to U_M, none contains I. When A is reduced, each $A_{\mathfrak{p}}$ at a minimal prime is a reduced Artinian local ring, by Proposition 5.40.1. Its maximal ideal is nilpotent by [StacksAlg, Tag 00JB] and hence zero; it is a field. The finite module $M_{\mathfrak{p}}$ is therefore free.

Empty domains are immediate. Let $V = \mathcal{M}(A_V) \subset U_M^{\mathrm{an}}$ be nonempty. There is no point of V where all g_j vanish. The quotient A_V/IA_V is an actual affinoid algebra with its canonical norm, by Theorem 2.33.18, and has empty spectrum. Theorem 1.20.1 makes it zero. Hence $IA_V = A_V$ and the principal opens $D(g_j)$ cover $\mathrm{Spec}(A_V)$. The given algebraic bases base change on these opens and prove that $M \otimes_A A_V$ is finite locally free. If $W \subset V$ is an affinoid domain on which g_j is a unit, its corresponding basis gives an algebraic isomorphism

$$M \otimes_A A_W \simeq A_W^{r_j}.$$

Both maps are bounded for the canonical norms by Theorem 1.18.3. Such domains cover V: compactness, Proposition 2.24.1, gives a positive lower bound for $\max_j |g_j(y)|$, and the resulting finitely many Laurent domains $|g_j| \geq \varepsilon$ cover. Their actual algebras are given by Theorem 2.31.2.

Finally, $IE = E$ follows from Proposition 5.40.1. For a local factor over $\mathfrak{p}$, the map $A \to E_j$ factors through $A_{\mathfrak{p}}$. Tensor its prescribed free basis with E_j to obtain the stated rank. □

Theorem 5.41.3 (Finite neighborhoods of a relative interior level)**.** *Let $p : X = \mathcal{M}(B) \to Y = \mathcal{M}(A)$ be any affinoid morphism, and let a unit moment tuple on Y define $t_Y : Y \to \mathbb{R}^{n'}$. Fix $w' \in t_Y(Y)$ and suppose*

$$p^{-1}(t_Y^{-1}(w')) \subset \mathrm{Int}(X/Y).$$

There is a closed rectangular box P about w', with w' in its interior, for which

$$Y_0 = t_Y^{-1}(P), \qquad X_0 = p^{-1}(Y_0), \qquad p_0 : X_0 \longrightarrow Y_0$$

are actual affinoid domains and a finite morphism with the canonical module Banach structure. The box may be chosen inside any prescribed neighborhood of w'.

Suppose in addition that X and Y are pure of analytic dimension d, that w' is a whole interior flat germ of t_Y with saturated lattice N_Y, and that unit moment coordinates on X and an integral lattice map a satisfy the character identity $t_Y \circ p = a \circ t_X$. Suppose

$$a^{-1}(w') \cap t_X(X) = \{w_1, \ldots, w_s\}$$

is finite and nonempty, and that the local source image at each w_i lies in $w_i + (N_i)_{\mathbb{R}}$ for a saturated rank-d lattice N_i. Write A_0 and B_0 for the actual

algebras of Y_0 and X_0. For a sufficiently small box as above, all the original source weights are defined and

$$\sum_i m_{X,f_X}(w_i)[N_Y : a(N_i)] = m_{B_0,f_Y}(w') = \sum_{\mathfrak{p}\in \mathrm{Min}(A_0)} \mathrm{length}_{(A_0)_{\mathfrak{p}}}((B_0)_{\mathfrak{p}}) m_{(Y_0)_{\mathfrak{p}},f_Y}(w').$$

The component weight is zero for an empty fibre. The resulting integer is independent of the chosen box. If the displayed generic module lengths are Δ times those of A_0 on all components meeting the fibre, then the middle term is $\Delta m_{Y,f_Y}(w')$.

Proof. By Proposition 2.33.21, the relative boundary $Z = X \setminus \mathrm{Int}(X/Y)$ is closed. It is compact by Proposition 2.24.1. The compact set $(t_Y \circ p)(Z)$ does not contain w', so choose a closed box P about w' avoiding it and inside the prescribed neighborhood. The simultaneous coordinate bounds give the actual domain Y_0 by Theorem 2.31.2. Theorem 2.36.9 gives the actual domain X_0, with algebra $B\widehat{\otimes}_A A_0$.

Every point of X_0 comes from $\mathrm{Int}(X/Y)$. Base change of relative interior in Proposition 2.33.21 makes p_0 everywhere interior. Theorem 2.43.8 proves that B_0 is finite over A_0 and has its canonical module norm. This proves the first assertion over the original field and absolute value.

For the remaining assertions choose P also inside the target flat-germ neighborhood. Purity of X_0 and Y_0 follows from Theorem 5.35.3. The whole target fibre is contained in Y_0; all its defining inequalities are strict at that fibre. The relative domain presentation and the composition inclusion 2.33.21 therefore make it interior over K in Y_0. Its weight is the original one by Theorem 5.37.2.

Each entire original source fibre at w_i is contained in X_0, because $a(w_i) = w'$. It is interior over K in X by the assumed relative interior and the target absolute interior. It is also interior in X_0 by the finite relative interior of p_0 and composition. The local flat containment persists under restriction. Theorem 5.37.1 defines these weights; Theorem 5.37.2 identifies their values before and after restriction. The complete set of source moment values over w' is unchanged. Theorem 5.41.1, applied to the actual finite map p_0, now gives the displayed formula and the assertion involving Δ.

For independence of the box, intersect any two sufficiently small boxes. Their intersection is again a closed box with w' in its interior. The asso-

ciated finite maps restrict to the same actual map on this intersection, by Theorem 2.36.9 and Proposition 2.33.19. The finite module over either target restricts to this same module, and the intersection contains the whole target moment fibre. The restriction assertion in Theorem 5.40.2 proves equality of the two middle terms. This proves the claimed independence. □

5.42 Cycles of finite analytic modules

Theorem 5.42.1 (Module cycles and finite geometric pushforward)**.** *Let $Y = \mathcal{M}(A)$ be pure of analytic dimension d, with a unit moment tuple f_Y and moment map t_Y. Let $\Omega \subset \mathbb{R}^{n'}$ be open and convex, with $\Omega \cap t_Y(\partial_K Y) = \varnothing$. Suppose a finite pure d-dimensional integral affine polyhedral complex $\mathcal{C}_Y$ has*

$$t_Y(Y) \cap \Omega = |\mathcal{C}_Y| \cap \Omega.$$

For every finite A-module M, the values $m_{M,f_Y}(v)$ are constant on $\operatorname{relint}(\sigma) \cap \Omega$ for each d-cell σ. They define a nonnegative integral balanced d-cycle, denoted $\operatorname{Trop}_{f_Y,\Omega}(M)$. These cycles are additive in short exact sequences and independent of the chosen polyhedral subdivision. For $M = A$ this is the analytic cycle of Theorem 5.43.2.

Let $p : X = \mathcal{M}(B) \to Y$ be finite, with X pure of analytic dimension d. Suppose unit moment coordinates on X and an integral map $a : \mathbb{Z}^n \to \mathbb{Z}^{n'}$ satisfy the corresponding character identity $t_Y \circ p = a \circ t_X$. Suppose also that a finite pure d-dimensional integral affine complex $\mathcal{C}_X$ satisfies

$$t_X(X) \cap a^{-1}(\Omega) = |\mathcal{C}_X| \cap a^{-1}(\Omega).$$

Then lattice-index pushforward on these open sets gives

$$a_* \operatorname{Trop}_{f_X, a^{-1}(\Omega)}(B) = \operatorname{Trop}_{f_Y,\Omega}(B). \tag{5.42.1}$$

On the left B is the structure module of X; on the right it is its finite A-module. If $\Delta \in \mathbb{Q}_{\geq 0}$ satisfies

$$\beta_{\mathfrak{p}}(B) = \Delta \operatorname{length}(A_{\mathfrak{p}}) \qquad (\mathfrak{p} \in \operatorname{Min}(A)),$$

then the right side is $\Delta \operatorname{Trop}_{f_Y,\Omega}(A)$. The same statement allows a positive rational Δ for a nonreduced target.

Proof. Form the finite A-algebra

$$A \oplus M, \qquad (a,m)(a',m') = (aa', am' + a'm),$$

with its canonical Banach structure from Theorem 2.33.18. The ideal $0 \oplus M$ has square zero. Every multiplicative seminorm annihilates it, so projection and the coefficient inclusion identify its spectrum with that of A. Its minimal primes are $\mathfrak{p} \oplus M$, and the corresponding integral quotients are $A/\mathfrak{p}$. Consequently $Y_M = \mathcal{M}(A \oplus M)$ is pure of analytic dimension d and has the same moment image under the units $(f_{Y,j}, 0)$.

Every point of Y_M above a fibre over Ω is interior over K: the map $Y_M \to Y$ is finite and hence relatively interior, and all points of Y with moment value in Ω are interior by hypothesis. Apply the composition inclusion 2.33.21. Theorem 5.43.2 therefore gives positive balanced analytic cycles for both Y and Y_M on the same complex over Ω.

At an interior point v of a d-cell, let E_Q be the actual target Gauss algebra for a transverse projection.

For the domain comparison at v, use $f_i = f_{Y,i}$ and $r_i = \exp(-v_i)$ in the following exact-level calculation.

Let A be a generalized affinoid over the fixed complete non-Archimedean field K, let M be finite over A, and give the idealization $A_M = A \oplus M$ its canonical finite-algebra Banach structure from Theorem 2.33.18. For units $f_i \in A^\times$ and positive radii r_i, put

$$D = A\{r^{-1}T, rS\}/(T_i - f_i, S_i - f_i^{-1})_i, \qquad D_M = A_M\{r^{-1}T, rS\}/(T_i - (f_i,0), S_i - (f_i^{-1},0))_i$$

These use actual ideals and the displayed quotient norms. The rational point and geometric factorization clauses of Theorems 2.31.1 and 2.31.2, with denominator 1 and the simultaneous upper bounds r_i, r_i^{-1}, identify the algebras with the whole exact-radius domains below and above, respectively. The lifted unit has inverse $(f_i^{-1}, 0)$. Thus the inequalities upstairs describe exactly the inverse image of the whole domain. The same descriptions include the empty domain.

There is a canonical bounded isomorphism, with bounded inverse,

$$D_M \simeq D \widehat{\otimes}_A A_M.$$

Indeed, for a commutative Banach K-algebra C with a bounded unital map $\phi : A_M \to C$, the bounded factorization clause of Theorem 2.31.2 says that extending ϕ across D_M requires exactly the estimates

$$\|\phi((f_i, 0))^m\| \le c_i^+ r_i^m, \qquad \|\phi((f_i^{-1}, 0))^m\| \le c_i^- r_i^{-m} \quad (m \ge 0)$$

for some constants $c_i^{\pm}$. The factorization is unique. These are precisely the estimates for extending $\phi|_A$ across D. Thus D_M represents the pairs of bounded maps from A_M and D agreeing on A. Their balanced product on finite sums $\sum_j d_j \otimes b_j$ has norm bounded by the product of the two operator bounds times $\max_j \|d_j\|\|b_j\|$. Taking the projective infimum and passing through separation and completion gives the unique bounded map from $D\widehat{\otimes}_A A_M$. The two universal factorizations are inverse because their composites fix both structural maps. This comparison preserves the actual algebras over A_M, hence also the maps from the exact-radius target torus defined by the common projecting monomials in the units.

Proposition 2.33.19 gives the Gauss algebra of Y_M as

$$(A \oplus M) \otimes_A E_Q = E_Q \oplus (M \otimes_A E_Q).$$

Thus its dimension is the sum of the two dimensions. Theorem 5.40.2 and the common lattice divisor give

$$m_{Y_M}(v) = m_Y(v) + m_{M,f_Y}(v).$$

Subtracting the two balanced cycles proves constancy and balancing for the module weights. Their nonnegativity and integrality are (5.40.2). Additivity follows from the exact-sequence assertion of that theorem, and common-refinement invariance follows from the same property of the two analytic cycles. This proves the first part.

For the finite map p, every point of X whose moment value lies in $a^{-1}(\Omega)$ is interior over K, again by finite relative interior and composition. Thus the stated source analytic cycle is defined. Exclude from the target the lower-dimensional cells of $\mathcal{C}_Y$, the images of all cells of $\mathcal{C}_X$ of dimension less than d, and the images of those d-cells on whose tangent space a has rank less than d. Linear images of the finitely many polyhedra are polyhedra, by the finite elimination and refinement construction in Theorem 5.17.2. Each excluded set has dimension at most $d-1$. For $d = 0$ the nonempty cells are points and every restriction has rank zero. The exclusions therefore remove no point, so every target point in the support is already in the complement. For $d > 0$ the

complement is dense in the relative interior of every target d-cell: a line in its affine span with direction outside the finitely many excluded proper direction subspaces meets those affine subspaces in at most finitely many points. Such a line direction exists by the finite-polynomial argument in Theorem 5.43.1.

Fix a target point v in this complement and in the target support. Every source moment value above it lies in the relative interior of a d-cell on which a is injective. There is at most one such point per cell, so the full inverse image in $t_X(X)$ is finite. If it is nonempty, all the hypotheses of Theorem 5.41.1 hold: the target and source germs are flat at cell interiors, and the target whole fibre is interior. The theorem identifies the lattice-index sum of source weights with $m_{B,f_Y}(v)$.

If this finite inverse image is empty, the actual inverse image of the entire target moment fibre is empty. Its algebra is $B \otimes_A D_v$, by finite base change and Theorem 2.36.9. Theorem 1.20.1 makes this algebra zero. Hence $B \otimes_A E_Q = 0$ and $m_{B,f_Y}(v) = 0$, which is also the empty lattice-index sum.

The map a on the actual source support over Ω is proper. For a compact subset $C \subset \Omega$, its inverse image is the closed subset $t_X(X) \cap a^{-1}(C)$ of the compact set $t_X(X)$. Refine the target along the finitely many cell images and pull the defining hyperplanes back to the source, as in Theorem 5.17.2. The lattice-index sum then defines the local pushforward; facets with smaller-dimensional image contribute zero. On the generic part of every resulting target d-cell, this sum equals the module weight just computed. Both are constant there, so the cycles agree. This proves (5.42.1). The last assertion follows on every such cell from Theorem 5.40.2 and its formula for the structure module A. □

Theorem 5.42.2 (Finite-module pushforward and composition)**.** *Keep the finite morphism $p : X = \mathcal{M}(B) \to Y = \mathcal{M}(A)$, the moment tuples, the lattice map a, and the given complexes over Ω in Theorem 5.42.1. For every finite B-module M, regarded also as a finite A-module, one has*

$$a_* \operatorname{Trop}_{f_X, a^{-1}(\Omega)}(M) = \operatorname{Trop}_{f_Y,\Omega}(M).$$

Its target generic coefficients are

$$\operatorname{length}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \sum_{\substack{\mathfrak{q} \in \operatorname{Min}(B) \\ \mathfrak{q} \cap A = \mathfrak{p}}} \operatorname{length}_{B_{\mathfrak{q}}}(M_{\mathfrak{q}})[\kappa(\mathfrak{q}) : \kappa(\mathfrak{p})] \qquad (\mathfrak{p} \in \operatorname{Min}(A)).$$

These pushforwards respect finite composition. More precisely, let $Z = \mathcal{M}(C) \to X$ also be finite with Z pure of analytic dimension d, and let a unit tuple f_Z and an integral map b satisfy the corresponding character identity. Assume a finite pure integral complex presents the actual f_Z-image on $(ab)^{-1}(\Omega)$. For every finite C-module L,

$$(ab)_* \operatorname{Trop}_{f_Z,(ab)^{-1}(\Omega)}(L) = a_* b_* \operatorname{Trop}_{f_Z,(ab)^{-1}(\Omega)}(L) = \operatorname{Trop}_{f_Y,\Omega}(L).$$

Here ab means $a \circ b$, and the middle pushforwards use the indicated inverse open sets.

If P is a finite locally free A-module of constant rank r, then the tensor projection formula is

$$a_* \operatorname{Trop}_{f_X,a^{-1}(\Omega)}(M \otimes_A P) = \operatorname{Trop}_{f_Y,\Omega}(M \otimes_A P) = r \operatorname{Trop}_{f_Y,\Omega}(M).$$

Proof. The square-zero algebra $B \oplus M$ is finite over B and over A, with the canonical finite-module norm, by Theorem 2.33.18. It has the same analytic points, moment image, and pure analytic dimension as B, as proved in Theorem 5.42.1. Apply that theorem to its finite map to Y. Its source cycle is the sum of the structure cycle of X and the module cycle of M. Its target module cycle is the sum of the cycles of B and M as A-modules. The theorem for $B \to A$ and subtraction give the asserted equality. Subtraction is valid on a common finite refinement because each lattice-index coefficient is a finite linear sum of source coefficients. The same compact actual support gives properness for the square-zero source, including modules with zero generic weight.

For a minimal prime $\mathfrak{p}$ of A, decompose the finite Artinian algebra $B_{\mathfrak{p}}$ into its local factors $B_{\mathfrak{q}}$, as in the proof of Theorem 5.41.1. The finite module $M_{\mathfrak{p}}$ decomposes into the corresponding $M_{\mathfrak{q}}$. A composition series of each factor as a $B_{\mathfrak{q}}$-module has residue field $\kappa(\mathfrak{q})$ as every simple quotient. Its length over $A_{\mathfrak{p}}$ is the full field degree $[\kappa(\mathfrak{q}) : \kappa(\mathfrak{p})]$. Adding these lengths proves the displayed coefficient formula.

The composite of two finite canonical affinoid maps is finite and canonical: finite module generators compose, and composing their finite free quotient presentations gives the same Banach topology. Apply the first assertion to $C \to B$, then to $B \to A$ with L as a finite B-module, and finally to $C \to A$. All source support above the stated open sets is interior by finite relative interior and Proposition 2.33.21. The given complexes therefore meet the

same hypotheses at each step. The three equalities give the composition formula.

Finally, tensor P with an actual Artinian Gauss algebra E_Q used to compute any of the target module weights. On a local factor E_j, the map from A factors through the local ring at the contracted prime. A local free basis of P of rank r therefore gives $P \otimes_A E_j \simeq E_j^r$. Consequently

$$\dim_{H_Q}\big((M \otimes_A P) \otimes_A E_Q\big) = r \, \dim_{H_Q}(M \otimes_A E_Q).$$

Divide by the common lattice index and apply Theorem 5.40.2. This proves the target tensor equality on every generic cell. The first assertion, applied to the finite B-module $M \otimes_A P$, gives its pushforward equality. All these tensors have their canonical complete finite-module norms by Proposition 1.18.7. □

Theorem 5.42.3 (Pushforward on a relative interior weight domain)**.** *Let $p : X = \mathcal{M}(B) \to Y = \mathcal{M}(A)$ be an affinoid morphism, with X and Y pure of analytic dimension d. Let a unit tuple f_Y define $t_Y : Y \to \mathbb{R}^{n'}$. Write $\partial(X/Y) = X \setminus \mathrm{Int}(X/Y)$ and put*

$$U_p = \mathbb{R}^{n'} \setminus \big(t_Y(\partial_K Y) \cup (t_Y \circ p)(\partial(X/Y))\big).$$

This is open. There is a canonical local nonnegative integral balanced d-cycle $\mathcal{C}_{p,f_Y}$ on U_p, with support $(t_Y \circ p)(X) \cap U_p$. It is the local analytic cycle

$$\mathcal{C}_{p,f_Y} = \mathrm{Trop}_{p^* f_Y}(X)|_{U_p}.$$

Every $w \in U_p \cap t_Y(Y)$ has a closed target moment box P about w, with w in its interior, such that

$$Y_0 = t_Y^{-1}(P), \qquad X_0 = p^{-1}(Y_0), \qquad p_0 : X_0 \longrightarrow Y_0$$

is an actual finite canonical affinoid map. Writing $A_0 = \mathcal{O}(Y_0)$ and $B_0 = \mathcal{O}(X_0)$, there is an open neighbourhood $W \subset \mathrm{int}(P)$ of w on which

$$\mathcal{C}_{p,f_Y}|_W = \mathrm{Trop}_{f_Y|Y_0}(B_0)|_W.$$

These finite-module descriptions agree on overlaps and under further target-box restriction. At a flat target germ $v \in W \cap t_Y(Y)$ their weight is

$$m_{\mathcal{C}_{p,f_Y}}(v) = \sum_{\mathfrak{p} \in \mathrm{Min}(A_0)} \mathrm{length}_{(A_0)_{\mathfrak{p}}}((B_0)_{\mathfrak{p}}) m_{(Y_0)_{\mathfrak{p}},f_Y}(v),$$

where an empty component fibre has weight zero.

Suppose a unit tuple f_X and an integral map $a : \mathbb{Z}^n \to \mathbb{Z}^{n'}$ satisfy the character identity $p^* f_Y = f_X^{a^*}$. All source weights in $a^{-1}(U_p) \cap t_X(X)$ have whole interior fibres, and

$$a_*\big(\mathrm{Trop}_{f_X}(X)|_{a^{-1}(U_p)}\big) = \mathcal{C}_{p,f_Y}.$$

The pushforward uses the entire actual source support and is proper over U_p.

Proof. The absolute boundary of Y and the relative boundary of X are closed, by Proposition 2.33.21, and compact, by Proposition 2.24.1. Their moment images are compact, so U_p is open. If $(t_Y \circ p)(x) \in U_p$, then x is interior relative to Y and $p(x)$ is interior over K. The composition inclusion in Proposition 2.33.21 makes x interior over K. Consequently

$$U_p \cap (t_Y \circ p)(\partial_K X) = \varnothing.$$

Theorem 5.45.3, applied on X to the pullback unit tuple $p^* f_Y$, defines the asserted local cycle and its support.

Fix $w \in U_p \cap t_Y(Y)$. The entire inverse image of its target moment fibre is relatively interior by definition of U_p. Theorem 5.41.3 therefore supplies the stated actual finite map, and permits P to be chosen inside U_p. Choose an open box $W \subset \mathrm{int}(P)$ about w. The whole target fibre at every $v \in W$ satisfies the box inequalities strictly. It is interior in Y_0 relative to Y, by the actual rational presentation, and interior in Y over K, by $P \subset U_p$. Composition makes it interior in Y_0 over K. Thus W belongs to the whole interior weight domain of Y_0. Theorem 5.35.3 gives purity of Y_0 and of every nonempty X_0.

The finite-map assertion of Theorem 5.45.3, with the same moment tuple on source and target and the identity lattice map, now gives

$$\mathrm{Trop}_{p_0^*(f_Y|Y_0)}(X_0)|_W = \mathrm{Trop}_{f_Y|Y_0}(B_0)|_W.$$

The whole pullback moment fibre at every $v \in W$ is the same in X_0 and X, and it is absolute interior in both. For X_0 this follows from finite relative interior and the target interior just checked. The flat-germ restriction assertion of Theorem 5.37.2 identifies their weights on every local generic cell. Hence the source cycle in this display is $\mathcal{C}_{p,f_Y}|_W$. If X_0 is empty, its algebra and module cycle are zero by Theorem 1.20.1, and the same identity holds.

For completeness, the finite-module descriptions also compare directly. Intersect two boxes at a weight in the overlap of their open interiors. The intersection has that weight in its interior and retains the whole target fibre and its entire inverse image. The actual domain and finite tensor comparisons, Theorem 2.36.9 and Proposition 2.33.19, identify both restricted finite algebras with the algebra of this same inverse-image domain. The whole-fibre restriction assertion of Theorem 5.40.2 gives equal module weights on all generic cells of a common local refinement. This proves the overlap and restriction statements. The generic-length expression is the formula of that theorem for the actual finite A_0-module B_0.

Finally, the equality $t_Y \circ p = a \circ t_X$ and the first paragraph show that every whole f_X-fibre above U_p is absolute interior. Apply the finite-map theorem 5.45.3 to the identity map of X, with source tuple f_X, target tuple $p^* f_Y$, and lattice map a, and restrict its equality to U_p. Its target module is the structure module of X, so the result is exactly the asserted pushforward. For a compact subset $C \subset U_p$, the complete inverse image $t_X(X) \cap a^{-1}(C)$ is closed in the compact set $t_X(X)$. This proves the stated properness on the actual support. □

Theorem 5.42.4 (The exceptional image of a generically free module)**.** *Let $Y = \mathcal{M}(A)$ be nonempty and pure of analytic dimension d, and let $I \subset A$ be an ideal contained in no minimal prime. Put $Z = \mathcal{M}(A/I)$, with its actual closed-subspace structure. If $d = 0$, then Z is empty. If Z is nonempty, then*

$$\dim_K Z \leq d - 1.$$

For any unit tuple f defining $t : Y \to \mathbb{R}^n$, the entire actual image $t(Z)$ is the support of a finite integral affine polyhedral complex of dimension at most $d-1$; for $d = 0$ it is empty. Put $U_Y = \mathbb{R}^n \setminus t(\partial_K Y)$. The intersection $t(Z) \cap U_Y$ is relatively closed in $t(Y) \cap U_Y$, and its complement is relatively open and dense there.

In particular, let M be a finite A-module which is free at every minimal prime, and let I_M define its nonfree locus as in Proposition 5.41.2. For every

$$w \in \big(t(Y) \cap U_Y\big) \setminus t(\mathcal{M}(A/I_M)),$$

there is a closed moment box P about w such that the whole domain $Y_P = t^{-1}(P)$ is in the analytic free locus and $M \otimes_A \mathcal{O}(Y_P)$ is finite locally free. If A is integral and the generic rank of M is r, this local rank is constantly r

on $\operatorname{Spec}\mathcal{O}(Y_P)$. *These conclusions apply to every finite module over reduced* A. *For a finite algebra* $M = B$, *the corresponding actual finite map over* Y_P *is finite locally free, with the same constant rank in the integral-target case.*

Proof. The finite quotient A/I has its actual generalized affinoid structure by Theorem 2.33.18. Choose a free-radius extension $L = K_s$ making A and A/I strict, by Theorem 1.17.6. The quotient comparison of Theorem 1.18.8 gives

$$(A/I)_L = A_L/IA_L.$$

Lemma 5.35.1 identifies the minimal primes of A_L with $\mathfrak{p}A_L$ for $\mathfrak{p} \in \operatorname{Min}(A)$, and their contractions are $\mathfrak{p}$. Thus IA_L is contained in no minimal prime of A_L. The analytic-dimension theorem 3.21.8 gives $\dim A_L = d$, since A_L is strict.

Every chain of primes of A_L/IA_L lifts to a chain $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_\ell$ of primes of A_L containing IA_L. By [StacksAlg, Tag 00E0(4)] applied to the zero ideal in A_L and the prime $\mathfrak{q}_0$, there is a minimal prime $\mathfrak{p}_L \subseteq \mathfrak{q}_0$ of A_L. The inclusion $\mathfrak{p}_L \subset \mathfrak{q}_0$ is strict, because IA_L is not contained in $\mathfrak{p}_L$. Prepending $\mathfrak{p}_L$ therefore gives a chain of length $\ell + 1$ in A_L. Hence $\ell + 1 \leq d$. Put $R = A_L/IA_L$. If $R \neq 0$, then [StacksAlg, Tag 00E0(2)] gives a maximal ideal $\mathfrak{m}$ of R. The quotient $R/\mathfrak{m}$ is a field, so $\mathfrak{m}$ is prime. There is thus a chain of length zero in this case, and the bound on all chains gives $\dim R \leq d - 1$. The quotient is zero when $d = 0$. The faithful quotient comparison detects zero before extension, and Theorem 3.21.8 identifies the dimension of every nonzero quotient with that of $\mathcal{M}(A/I)$ over the original K.

If Z is empty, its image has the empty complex. Otherwise apply Theorem 5.39.3 to Z and the restrictions of the original units. The dimension inequality just proved gives the asserted finite complex of dimension at most $d - 1$ for the entire actual image. Compactness of Z and its image, Proposition 2.24.1, proves relative closedness on U_Y.

At every point of $t(Y) \cap U_Y$, Theorem 5.38.6 presents the ambient actual image by a finite pure d-dimensional fan. The exceptional complex is contained in a finite union of affine subspaces of dimension at most $d - 1$. No such finite union contains a relatively open part of a d-cell. For example, choose a line direction in its tangent plane outside the finitely many proper direction subspaces; the finite-polynomial construction in Theorem 5.43.1 supplies one. Every short line segment in that direction meets each excluded affine subspace at most once. Since a pure finite complex is the closure of

the union of its maximal-cell interiors, this proves density of the complement at every point. Openness follows from the relative closedness already proved. For $d = 0$ the exceptional set is empty and the assertion is immediate.

For the module M, Proposition 5.41.2 makes I_M an ideal avoiding every minimal prime, and identifies its analytic zero set with the nonfree locus. Its image under t is compact. Choose a closed box P about the given w, with w in its interior, contained in U_Y and disjoint from that compact image. The actual domain Y_P is given by Theorem 2.31.2. It lies entirely in the analytic free locus, so the analytic restriction assertion of Proposition 5.41.2 makes the restricted module finite locally free.

If A is integral, a free basis of $M_{\mathfrak{p}}$ at any prime of the free locus becomes a basis over $\operatorname{Frac}(A)$ after further localization. Its rank is therefore the same integer r at every such prime. The principal free bases in the proof of that proposition consequently all have rank r. They base change to the indicated cover of $\operatorname{Spec}\mathcal{O}(Y_P)$, proving the constant-rank assertion. Reduced A has the required generic freeness by the same proposition. For a finite algebra B, finite base change and its canonical complete norm, Proposition 2.33.19, identify the restricted algebra with $B \otimes_A \mathcal{O}(Y_P)$. The module assertion is exactly finite local freeness of this actual morphism. □

5.43 Balancing from finite analytic degrees

Fix a complete real-valued non-Archimedean field K, with its specified absolute value. The valuation may be trivial. Let $X = \mathcal{M}(B)$ be a generalized K-affinoid space, pure of analytic dimension d, and let $f_1, \dots, f_n \in B^\times$. Write

$$t : X \longrightarrow N_{\mathbb{R}}, \qquad t(x) = (-\log|f_1(x)|, \dots, -\log|f_n(x)|), \qquad N = \mathbb{Z}^n.$$

At a whole interior flat d-dimensional germ, use the positive integer $m_{X,f}$ of Theorem 5.37.1. All tangent lattices are saturated in the ambient lattice.

Theorem 5.43.1 (The normal equation at a polyhedral wall)**.** *Suppose $d \geq 1$, $w \in t(X)$, and $t^{-1}(w) \subset \operatorname{Int}(X/K)$. Let $N_\tau \subset N$ be saturated of rank $d-1$. For $1 \leq i \leq r$, let $N_i \subset N$ be saturated of rank d and contain N_τ, and let $u_i \in N_i$ lift a primitive generator of N_i/N_τ. Suppose the half-planes*

$$C_i = (N_\tau)_{\mathbb{R}} + \mathbb{R}_{\geq 0} u_i$$

are distinct, and that on an open neighborhood W of w the actual moment image is

$$t(X) \cap W = \left(w + \bigcup_{i=1}^{r} C_i \right) \cap W.$$

For each i, the weight $m_{X,f}$ is a fixed positive integer m_i on the relative interior of $w + C_i$ sufficiently near w. These integers satisfy

$$\sum_{i=1}^{r} m_i u_i = 0 \quad \text{in } N/N_\tau.$$

Proof. The boundary of X is compact, by Propositions 2.33.21 and 2.24.1. Its moment image is compact and misses w. Shrink W to avoid that image. Away from the common wall, distinct half-planes have disjoint relative interiors. Near such a point the image is an open part of its affine d-plane, and its whole fibre is interior. Theorems 5.37.1 and 5.37.2 give a positive locally constant integer there. Intersecting the relative interior with a sufficiently small open box about w gives a convex set. A locally constant function is constant on each segment in that set: otherwise the supremum of the initial interval of constancy contradicts local constancy. This gives the integers m_i.

By Lemma 5.14.1, choose a homomorphism $q : N \to \mathbb{Z}^{d-1}$ whose restriction to N_τ is an isomorphism. Subtracting from each u_i its unique element of N_τ with the same q-image gives representatives satisfying $q(u_i) = 0$, without changing the half-planes or the asserted normal equation. Let $\lambda : N \to \mathbb{Z}$ vanish on N_τ and satisfy $\lambda(u_i) \neq 0$ for every i. Put

$$Q = (q, \lambda) : N \longrightarrow \mathbb{Z}^d.$$

Since N_i/N_τ is generated by the class of u_i, the chosen basis of N_τ, followed by u_i, is a basis of N_i. In these bases $Q|_{N_i}$ has matrix $\operatorname{diag}(I_{d-1}, \lambda(u_i))$. Consequently

$$Q|_{(N_i)_{\mathbb{R}}} \text{ is injective}, \qquad [\mathbb{Z}^d : Q(N_i)] = |\lambda(u_i)|. \tag{5.43.1}$$

Take a small closed rectangular box $P \subset W$ with w in its interior, and let $Y = t^{-1}(P)$ be the actual affinoid domain given by Theorem 2.31.2. It is pure of analytic dimension d by Theorem 5.35.3. At each point of $t^{-1}(w)$ all inequalities defining Y are strict. Thus that fibre is interior in Y over X, and composition in Proposition 2.33.21 makes it interior in Y over K.

The whole level of $Q \circ t|_Y$ at $s_0 = Qw$ is $t^{-1}(w)$. Indeed, for $v - w = b + au_i \in C_i$, with $b \in (N_\tau)_{\mathbb{R}}$ and $a \geq 0$, the equation $Q(v-w) = 0$ first gives $a\lambda(u_i) = 0$ and then $q(b) = 0$. Thus $a = b = 0$. Theorem 5.35.5, applied to the monomial units with exponent matrix Q on Y, gives a closed box P' about s_0 and an actual finite map

$$Z = (Q \circ t|_Y)^{-1}(P') \longrightarrow \mathbb{T}_{P'}.$$

Its Gauss-fibre dimension is a fixed positive integer δ on P'. Every point of Z is interior in Y over K.

For a sufficiently small $\varepsilon > 0$, both $s_\pm = s_0 \pm \varepsilon e_d$ lie in the interior of P'. The moment values in Y over s_+ are exactly

$$v_i^+ = w + \frac{\varepsilon}{\lambda(u_i)} u_i, \qquad \lambda(u_i) > 0,$$

and those over s_- are exactly

$$v_i^- = w - \frac{\varepsilon}{\lambda(u_i)} u_i, \qquad \lambda(u_i) < 0.$$

To see this, the first $d-1$ coordinate equations give $b = 0$, and the last gives the displayed coefficient of u_i. Choose ε so that all these finitely many points belong to the interior of P; the local image hypothesis then also proves their existence. Distinct half-planes give distinct points on each side.

Every whole moment fibre at these values in X belongs to Y. The projected levels avoid the absolute boundary of Y, since they belong to $Z \subset \mathrm{Int}(Y/K)$. Proposition 5.31.2 therefore decomposes each actual level algebra into the product of its whole moment fibre algebras, before reduction. Theorem 5.35.5 identifies these level algebras with the exact-radius restrictions of the same finite algebra over $\mathbb{T}_{P'}$. Their Gauss fields are the same weighted completions of the target Laurent rational function field. Consequently their Gauss dimensions are both δ, and the product decompositions give

$$\delta = \sum_{\lambda(u_i)>0} \delta_Q(v_i^+), \qquad \delta = \sum_{\lambda(u_i)<0} \delta_Q(v_i^-).$$

By Theorem 5.37.1 and (5.43.1), these identities read

$$\sum_{\lambda(u_i)>0} m_i \lambda(u_i) = \sum_{\lambda(u_i)<0} m_i(-\lambda(u_i)).$$

Hence $\lambda(\sum_i m_i u_i) = 0$ for every such λ.

These integral covectors separate points of the free quotient N/N_τ. Explicitly, in a basis of that quotient use covectors with coordinates $(1, a, a^2, \dots, a^{h-1})$, where $h = \operatorname{rank}(N/N_\tau)$ and $a \in \mathbb{Z}$. Their values on any fixed nonzero vector are a nonzero polynomial in a, with finitely many roots. If $\sum_i m_i u_i$ had nonzero class, choose an integer a avoiding the roots for that class and for all the nonzero classes of the u_i. The resulting covector would contradict the preceding equality. Thus the normal equation holds in N/N_τ. □

Theorem 5.43.2 (The analytic weights on a given polyhedral image)**.** *Let $\Omega \subset N_{\mathbb{R}}$ be open and convex, with $\Omega \cap t(\partial_K X) = \varnothing$. Suppose a finite pure d-dimensional integral affine polyhedral complex $\mathcal{C}$ satisfies*

$$t(X) \cap \Omega = |\mathcal{C}| \cap \Omega.$$

Discard cells not meeting Ω. Each d-cell σ has a well-defined positive integer m_σ: it is $m_{X,f}(v)$ for any $v \in \operatorname{relint}(\sigma) \cap \Omega$. For every $(d-1)$-cell τ meeting Ω these integers obey

$$\sum_{\sigma \supset \tau} m_\sigma u_{\sigma/\tau} = 0 \quad \text{in } N/N_\tau.$$

Thus they define a positive integral balanced d-cycle on Ω. It is unchanged by a common polyhedral refinement or by an affinoid restriction containing all moment fibres over Ω and satisfying the same hypotheses.

Proof. At an interior point of a d-cell, the finiteness and face intersection property of the complex give a neighborhood in which its support is an open part of that cell's affine span. The whole fibre is interior by the condition on Ω. Theorems 5.37.1 and 5.37.2 therefore give a positive locally constant integer on $\operatorname{relint}(\sigma) \cap \Omega$. This intersection is convex, so the segment argument in the preceding proof makes the integer constant.

Choose $w \in \operatorname{relint}(\tau) \cap \Omega$. Such a point exists whenever τ meets the open set Ω: points of its relative interior approach every point of the polyhedron. Only cells containing τ occur sufficiently near w. In a finite inequality presentation of an incident d-cell, inequalities strict at w remain strict in a small neighborhood; the remaining inequality in its affine span cuts out the half-space pointing into that cell from τ. Its direction is $(N_\tau)_{\mathbb{R}} + \mathbb{R}_{\geq 0} u_{\sigma/\tau}$. Distinct incident cells give distinct half-spaces, since otherwise their relative

interiors would intersect. Theorem 5.43.1 now gives the displayed equation. For $d = 0$ there is no normal equation to check.

On a subdivision, the generic interior of each new d-cell has the same whole fibres and tangent lattice as its old cell, so its weight is the old weight. Lemma 5.14.2 identifies the resulting cycles under common refinement. For an affinoid restriction as in the statement, Theorem 5.37.2 identifies the actual fibre algebras, tangent lattices and weights. The same argument on a common refinement proves the asserted equality of cycles. □

5.44 Adding moment coordinates

Theorem 5.44.1 (Pushforward after adding analytic unit coordinates)**.** *In the setting above let $g_1, \dots, g_m \in B^\times$ and put*

$$\widehat{t} = (t, (-\log|g_j|)_j) : X \longrightarrow \mathbb{R}^{n+m}, \qquad \pi : \mathbb{Z}^{n+m} \longrightarrow \mathbb{Z}^n.$$

Let $\Omega \subset \mathbb{R}^n$ be open and convex and avoid $t(\partial_K X)$. Suppose finite pure d-dimensional integral affine complexes $\mathcal{C}$ and $\widehat{\mathcal{C}}$ satisfy

$$t(X) \cap \Omega = |\mathcal{C}| \cap \Omega,$$
$$\widehat{t}(X) \cap \pi^{-1}(\Omega) = |\widehat{\mathcal{C}}| \cap \pi^{-1}(\Omega).$$

Equip both with their analytic weights from Theorem 5.43.2. Then on Ω

$$\pi_*[\widehat{\mathcal{C}}] = [\mathcal{C}].$$

More explicitly, outside a finite union of polyhedra of dimension at most $d-1$, each $v \in |\mathcal{C}| \cap \Omega$ lies in the relative interior of a d-cell σ, its full inverse image in $\widehat{t}(X)$ is finite, and

$$m_\sigma = \sum_{\substack{u \in \widehat{t}(X) \\ \pi(u) = v}} m_{\widehat{\sigma}(u)} [N_\sigma : \pi(N_{\widehat{\sigma}(u)})]. \tag{5.44.1}$$

Here u lies in the relative interior of a unique d-cell $\widehat{\sigma}(u)$, and π is injective on its tangent space. The equality uses all source moment fibres over v.

Proof. The set $\pi^{-1}(\Omega)$ is open and convex and misses $\widehat{t}(\partial_K X)$. Thus the indicated weights exist on both complexes and are balanced by Theorem 5.43.2.

Remove from Ω the cells of $\mathcal{C}$ of dimension less than d, the images of all cells of $\widehat{\mathcal{C}}$ of dimension less than d, and the images of those d-cells on whose

tangent space π has rank less than d. Each removed set is contained in a finite union of affine subspaces of dimension at most $d-1$. The images are polyhedra: eliminating one real coordinate from a finite list of linear inequalities replaces the upper and lower bounds for that coordinate by their pairwise comparisons, and preserves the inequalities not involving it; iterating gives a finite inequality presentation of a linear image. Rational coefficients remain rational. This also supplies a finite polyhedral refinement adapted to all these images, by cutting along the finitely many resulting supporting hyperplanes.

The complement is dense in the relative interior of every d-cell meeting Ω. To verify this directly, in its affine span choose a line direction outside the finitely many proper linear subspaces parallel to the excluded affine subspaces. A line in that direction meets each of them at at most one point. Arbitrarily short segments therefore contain points of the complement. Such a direction exists, for example by the finite-polynomial argument in the proof of Theorem 5.43.1.

Fix v in this complement and in the target support. Every $u \in \widehat{t}(X)$ above v is in $\pi^{-1}(\Omega)$, so it lies in the relative interior of a unique d-cell of $\widehat{\mathcal{C}}$, and π has full rank on that cell. The affine restriction is injective, so there is at most one such point for each of the finitely many cells. There is at least one, since $t = \pi \circ \widehat{t}$. This proves finiteness and nonemptiness of the full inverse image.

Near v the target image is contained in its affine d-plane, and near every u the joint image is contained in its affine d-plane. The whole target fibre is interior because $v \notin t(\partial_K X)$; the whole source fibres are contained in it. Apply Theorem 5.37.3 to the identity morphism $X \to X$, with source moment tuple (f, g), target moment tuple f, and lattice map π. Its algebra is finite free of rank one over itself, with its original Banach structure. All its full-fibre hypotheses have just been checked. The resulting formula is exactly (5.44.1).

The restriction of π to the actual joint support over Ω is proper: above a compact subset of Ω its inverse image is a closed subset of the compact set $\widehat{t}(X)$. On the finite refinement constructed above, the weight defining π_* at a generic point of a target d-cell is the finite sum in (5.44.1); cells with smaller-dimensional image contribute no d-weight. The formula agrees with the analytic target weight on a dense subset of every such cell. Both are constant there, so the weighted cycles are equal. □

5.45 Local analytic tropical cycles

Lemma 5.45.1 (A finite presentation near a compact projected fibre)**.** *Let $S \subset \mathbb{R}^n$ be compact, let $a : \mathbb{R}^n \to \mathbb{R}^m$ be an integral linear map, and let $w \in \mathbb{R}^m$. Suppose that, at each $u \in S \cap a^{-1}(w)$, there are a neighbourhood V_u and a finite pure d-dimensional rational polyhedral fan C_u such that*

$$S \cap V_u = (u + |C_u|) \cap V_u.$$

There are an open convex neighbourhood W of w and a finite pure d-dimensional integral affine polyhedral complex $\mathcal{D}$ such that

$$|\mathcal{D}| \subset S, \qquad S \cap a^{-1}(W) = |\mathcal{D}| \cap a^{-1}(W).$$

If the fibre is empty, $\mathcal{D}$ may be empty. The neighbourhood W may be required to lie in any prescribed neighbourhood of w.

Proof. For each u in the compact fibre, take a closed rectangular box P_u centred at u, contained in V_u, and with u in its interior. The interiors of finitely many such boxes $P_{u_1}, \dots, P_{u_s}$ cover the fibre. Put

$$T = \bigcup_{j=1}^{s} \big((u_j + |C_{u_j}|) \cap P_{u_j}\big) = \bigcup_{j=1}^{s} (S \cap P_{u_j}).$$

For every d-cone σ of C_{u_j}, the polytope $(u_j + \sigma) \cap P_{u_j}$ has dimension d: the box contains a neighbourhood of its cone vertex. Every smaller-dimensional cone is a face of a d-cone. Thus T is a finite union of d-dimensional integral affine polytopes and their faces. Cut these polytopes by all the finitely many hyperplanes in their inequality and affine-hull presentations. Their cells and faces form a finite common refinement $\mathcal{D}$ with support T. Each cell is contained in the closure of a d-cell. Indeed, in any original d-polytope its relative interior is dense, and the finitely many proper cutting hyperplanes have empty relative interior. Hence this complex is pure of dimension d. This is also the explicit refinement construction of Lemma 5.14.2.

The set

$$E = S \setminus \bigcup_{j=1}^{s} \operatorname{int}(P_{u_j})$$

is compact and has no point over w. Its image $a(E)$ is compact and misses w. Choose a sufficiently small open convex neighbourhood W of w disjoint from

that image. Every point of $S \cap a^{-1}(W)$ belongs to one of the box interiors, hence to T. Conversely $T \subset S$, proving both asserted inclusions. If the fibre is empty, apply the same compactness argument to $E = S$ and take the empty complex. Shrinking W proves the last assertion. □

Definition 5.45.2 (A local weighted polyhedral cycle)**.** A local integral d-cycle on an open subset U of a real lattice space is a collection of finite weighted integral affine polyhedral complexes on an open cover of U, with integer weights on their d-cells, which agree after common refinement on every overlap. It is balanced if each local complex satisfies the primitive-normal balancing equations. Complexes with all weights zero represent the zero cycle. Lattice-index pushforward is computed on such local presentations when the map on their support is proper and the required source presentations are finite near each target point.

Theorem 5.45.3 (Analytic cycles on whole interior weight domains)**.** *Let $Y = \mathcal{M}(A)$ be a generalized affinoid space, pure of analytic dimension d, over any complete non-Archimedean field with its specified absolute value. Let f_Y be a finite tuple of analytic units, let $t_Y : Y \to \mathbb{R}^{n'}$ be its moment map, and put*

$$U_Y = \mathbb{R}^{n'} \setminus t_Y(\partial_K Y).$$

There is a unique local positive integral balanced d-cycle

$$\mathrm{Trop}_{f_Y}(Y)|_{U_Y}$$

whose support is $t_Y(Y) \cap U_Y$ and whose weights at flat d-dimensional germs are the actual Gauss-degree weights. For every finite A-module M there is a local nonnegative integral balanced cycle $\mathrm{Trop}_{f_Y}(M)|_{U_Y}$ with the module weights of Theorem 5.40.2. It is additive in short exact sequences and is unchanged by common subdivision. The structure module gives the preceding positive cycle.

Let $p : X = \mathcal{M}(B) \to Y$ be finite, with X pure of analytic dimension d. Let f_X be a unit tuple and let $a : \mathbb{Z}^n \to \mathbb{Z}^{n'}$ satisfy the actual character identities $p^* f_{Y,j} = \prod_i f_{X,i}^{a_{ji}}$, so that $t_Y \circ p = a \circ t_X$. The entire source support over U_Y lies in its whole interior weight domain, and

$$a_*\big(\mathrm{Trop}_{f_X}(X)|_{a^{-1}(U_Y)}\big) = \mathrm{Trop}_{f_Y}(B)|_{U_Y}. \tag{5.45.1}$$

In particular, adjoining any finite list of analytic unit coordinates to a moment tuple and projecting back to that tuple preserves its local analytic cycle.

Proof. The set U_Y is open: the absolute boundary is closed in the compact affinoid spectrum by Propositions 2.33.21 and 2.24.1. At a point of $t_Y(Y) \cap U_Y$, Theorem 5.38.6 gives a translated finite pure rational fan presenting the actual image on a sufficiently small open convex neighbourhood $W \subset U_Y$. Theorem 5.43.2 gives its positive integral balanced weights. At a point of $U_Y \setminus t_Y(Y)$ take a neighbourhood disjoint from the compact image and use the zero cycle.

These local presentations agree on overlaps. Cut along the finitely many defining hyperplanes of two presentations. On the relative interior of every resulting d-cell meeting the overlap, the actual image has the same flat germ and saturated tangent lattice. Both weights are therefore m_{Y,f_Y} of that same whole fibre. The common refinement has equal weights, which is the required agreement. This also proves uniqueness. Theorem 5.42.1 on each of these neighbourhoods gives the module cycles. Their weights agree on overlaps by Theorem 5.40.2; its exact-sequence formula proves the stated additivity.

For the finite morphism, any point x with $a(t_X(x)) \in U_Y$ maps to an absolute interior point of Y. Finite morphisms are everywhere relatively inner by Theorem 2.33.18. Composition in Proposition 2.33.21 makes x an absolute interior point of X. Thus

$$a^{-1}(U_Y) \cap t_X(\partial_K X) = \varnothing.$$

Every whole source moment fibre above this set is interior. Theorem 5.38.6 therefore applies at each of its weights.

Fix $w \in U_Y$. Apply Lemma 5.45.1 to the compact set $S = t_X(X)$ and its complete inverse image over w. It supplies a finite pure source complex on $a^{-1}(W)$ for some open convex neighbourhood $W \subset U_Y$ of w. If $w \in t_Y(Y)$, shrink W inside one target fan neighbourhood already constructed. The two finite complexes then satisfy exactly the hypotheses of Theorem 5.42.1. Its equality proves (5.45.1) over W. If the source inverse image over w is empty, the compactness part of the lemma allows W to miss $a(t_X(X))$; the empty-fibre argument in that same theorem makes the module cycle zero there. If $w \notin t_Y(Y)$, both sides are zero on a smaller neighbourhood disjoint from $t_Y(Y)$.

The map on the actual source support is proper over U_Y: the inverse image of a compact subset of U_Y is closed in the compact set $t_X(X)$. These finite local presentations therefore define the lattice-index pushforward, and their equalities glue to the displayed identity. Finally take $p = \mathrm{id}_Y$, $B = A$, and

the coordinate projection from the enlarged moment tuple. Its finite module is the structure module, giving the last assertion. □

5.46 Integration on analytic moment cycles

Definition 5.46.1 (Lattice normalization for real superforms)**.** On an open subset of $N_{\mathbb{R}}$, a real superform of bidegree (p, q) is a smooth function with values in $\bigwedge^p N'^*_{\mathbb{R}} \otimes \bigwedge^q N''^*_{\mathbb{R}}$. All one-form generators anticommute. The operators d' and d'' differentiate coefficients and wedge the corresponding primed or double-primed differential on the left. The involution J exchanges the two copies and fixes functions. In particular $d'' = Jd'J$, as is checked on a coefficient and on the constant one-form generators.

For an affine d-plane with tangent lattice M, choose an integral basis and its affine coordinates $x_1, \dots, x_d$. Use the positive top generator

$$\omega_M = d'x_1 \wedge d''x_1 \wedge \cdots \wedge d'x_d \wedge d''x_d.$$

The lattice-normalized integral of $c\omega_M$ on a polyhedron is the ordinary real integral of c in these coordinates. A form is restricted to a polyhedron by restricting both cotangent factors to its affine span. Its support on a weighted complex is the closure, in that complex, of the points in the relative interiors of its maximal cells where this restriction is nonzero, after discarding cells of weight zero. Compact support below refers to this support.

Lemma 5.46.2 (The real polyhedral integration calculation)**.** *Lattice-normalized integration of smooth top superforms with compact support is well-defined and additive under finite polyhedral subdivision. Let $F : \sigma \to \tau$ be an affine isomorphism of d-dimensional polyhedra whose linear part maps the tangent lattice N_σ into N_τ. If that lattice map is injective, then*

$$\int_\sigma F^*\alpha = [N_\tau : F(N_\sigma)] \int_\tau \alpha.$$

A linear map of rank less than d has zero pullback in bidegree (d, d).

Suppose $d \geq 1$, and let β have bidegree $(d-1, d)$ near a d-polyhedron σ, with compact tangential support on σ. For each facet τ choose a basis $v_1, \dots, v_{d-1}$ of N_τ and a primitive lift $u_{\sigma/\tau} \in N_\sigma$ pointing into σ. Write $d\mu_\tau$ for the real measure giving this basis covolume one. Then

$$\int_\sigma d'\beta = (-1)^{d(d+1)/2} \sum_{\tau \prec \sigma} \int_\tau \beta(v'_1, \dots, v'_{d-1}; v''_1, \dots, v''_{d-1}, u''_{\sigma/\tau})\, d\mu_\tau. \quad (5.46.1)$$

The facet expression is independent of the chosen basis and of the lift modulo N_τ.

Proof. We first give the elementary real integration facts used in the calculation. A closed bounded box in $\mathbb{R}^r$ is sequentially compact: repeatedly bisect its sides, retain a closed subbox containing infinitely many terms of a sequence, and choose terms with increasing indices in the nested boxes. Their diameters tend to zero, so completeness of the real coordinates gives a convergent subsequence in the original box. A continuous function on a compact subset is bounded and uniformly continuous. Otherwise a sequence witnessing failure of either assertion has a convergent subsequence; in the second case take pairs at distance tending to zero. Continuity at the common limit is a contradiction.

Let P be a bounded polytope in $\mathbb{R}^r$, and let c be continuous on a neighbourhood of P. In a fixed bounding box use a cubical grid of side length h. The grid boxes meeting a given affine hyperplane have total volume at most Ch. To see this, choose a nonzero coordinate coefficient in an equation for the hyperplane. For each box in the other $r-1$ coordinates, the possible remaining coordinate ranges over an interval of length at most a constant times h, so only boundedly many of the remaining grid intervals are met. There are at most $C'h^{-(r-1)}$ such boxes in the fixed bounding region. The boundary of P is contained in finitely many such hyperplanes. Its grid boxes therefore have total volume tending to zero.

The upper and lower grid sums for c on the boxes wholly inside P differ by at most the bounding volume times the modulus of continuity of c at distance $\sqrt{r}h$. The boundary boxes add an error bounded by $Ch\sup|c|$. Both errors tend to zero. Refining two grids gives the same limit, which defines the integral. The same bounds allow signed functions and prove additivity under a finite polyhedral dissection: its additional faces lie in finitely many hyperplanes and contribute zero. They also prove that lower-dimensional polyhedra have zero r-dimensional integral. For unbounded polyhedra, intersect with a box containing the compact support in its interior; a larger box gives the same value.

We will use iterated integration in these coordinates. The same construction proves it for the functions just considered. On each line parallel to a coordinate axis, the characteristic function of a polytope times a continuous function is piecewise continuous on finitely many intervals. The upper and lower step functions of a rectangular grid bound its one-dimensional integrals.

Integrating these step functions in the remaining variables gives exactly their rectangular grid sums. Their difference tends to zero by the preceding estimates. Thus both orders of integration have the same limit. This argument also proves existence of the required outer integrals; it does not assume an interchange theorem.

One-dimensional translation and scaling follow directly by translating or scaling interval subdivisions. Interchanging coordinates follows from the equality of iterated integrals. A shear $x_i \mapsto x_i + cx_j$ preserves the integral, because after fixing all other coordinates it is a translation in x_i. Every invertible real matrix is a product of shears, coordinate interchanges and nonzero diagonal scalings, by Gaussian elimination. Hence ordinary real change of variables for an invertible affine map with linear matrix L has factor $|\det L|$. All the intermediate regions are polyhedra, so the preceding integral constructions apply at every step.

In integral bases, pullback of a top superform contributes $(\det L)^2$, one determinant from each exterior factor. The scalar change of variables contributes $|\det L|^{-1}$. Their product is $|\det L|$. For a full-rank integral lattice map this is $[N_\tau : F(N_\sigma)]$, by the integer basis calculation in Lemma 5.14.1. A change of integral basis has determinant ± 1, proving independence of the normalization. If the rank is less than d, the pullback on $\bigwedge^d$ is zero. This proves the first assertions.

For completeness, the one-dimensional derivative formula also follows from the same integral. If g is continuous, the difference quotient of $G(t) = \int_a^t g(s)\,ds$ lies between the minimum and maximum of g on the intervening interval; hence $G' = g$. A function with zero derivative is constant by the mean value argument: a continuous function on a closed interval attains its extrema by the compactness just proved, an interior extremum of a differentiable function has derivative zero, and subtraction of the line through the endpoint values gives Rolle's argument. Applying this to a C^1 function minus the primitive of its derivative gives

$$\int_a^b h'(s)\,ds = h(b) - h(a).$$

Apply this identity on the coordinate lines through a polyhedron, followed by the established iterated integration. For a coordinate x_j, almost every such line meets a bounded convex polytope in an interval. Its endpoints lie on facets; the projections of intersections of distinct facets have dimension

at most $r-2$ and have zero $(r-1)$-dimensional integral by the grid estimate. On a facet written $x_j = \ell(\widehat{x}_j)$ the endpoint sign is the sign of the j-th outward normal component. Give the facet its Euclidean measure in orthonormal affine coordinates; the proved linear change of variables computes this measure in any other affine parametrization. The graph's Euclidean area factor is $\sqrt{1+|\nabla\ell|^2}$, obtained by taking the determinant of the Gram matrix $I+\nabla\ell\,\nabla\ell^{\mathsf{t}}$. The absolute value of the corresponding unit-normal component is its reciprocal. Thus the endpoint integral is exactly the facet integral of that normal component times the coefficient. Facets parallel to these lines have zero normal component and give no term. Summing over j proves the divergence formula on a polytope for a C^1 vector field. The argument applies to an unbounded polyhedron with compact support by cutting with a box outside the support; all the new facet terms vanish.

We now apply this coefficient formula to β. Put $s_d = (-1)^{d(d-1)/2}$. In coordinates for any integral basis of N_σ, write its tangential restriction uniquely as

$$\beta = s_d \sum_{j=1}^{d} (-1)^{j-1} b_j \, d'x_1 \wedge \cdots \widehat{d'x_j} \cdots \wedge d'x_d \wedge d''x_1 \wedge \cdots \wedge d''x_d.$$

Direct differentiation gives $d'\beta = (\sum_j \partial_j b_j)\omega_{N_\sigma}$. At a facet, the chosen basis of N_τ followed by $u_{\sigma/\tau}$ is an integral basis of N_σ: the quotient is free of rank one and its indicated element is primitive. This is again Lemma 5.14.1. Use its affine coordinates $(t_1,\dots,t_{d-1},s)$, with the facet in $s=0$ and the polyhedron locally in $s\ge 0$. The divergence contribution of this facet is $-b_d(t,0)\,dt_1\cdots dt_{d-1}$. On the other hand,

$$\beta(v'_1,\dots,v'_{d-1};v''_1,\dots,v''_{d-1},u''_{\sigma/\tau}) = s_d(-1)^{d-1}b_d(t,0).$$

The ratio of these two expressions is $(-1)^d s_d = (-1)^{d(d+1)/2}$. A unimodular change of cell coordinates has absolute determinant one; the two exterior factors and the scalar change of variables above therefore preserve this facet calculation. This proves (5.46.1) for all facets with one common sign. A basis change in N_τ contributes its determinant twice, and a change of the lift by a tangent vector gives a repeated vector in the double-primed alternating factor. These observations prove the last assertion. □

Theorem 5.46.3 (Integration and closedness of a local balanced cycle)**.** *Let C be a local weighted integral d-cycle on an open subset $U \subset N_{\mathbb{R}}$ in the sense of Definition 5.45.2. For smooth superforms with compact support on C, the cell formula*

$$I_C(\alpha) = \sum_{\sigma} m_\sigma \int_\sigma \alpha$$

defines a linear functional in bidegree (d,d), independent of finite local presentations and subdivisions. It is nonnegative on forms whose tangential top coefficients are nonnegative whenever all weights are nonnegative. If C is balanced, then

$$I_C(d'\beta) = 0, \qquad I_C(d''\gamma) = 0$$

for compactly supported forms of bidegrees $(d-1,d)$ and $(d,d-1)$, respectively. For $d = 0$ the functional is the weighted sum of values and both differential assertions are empty.

If an integral affine map a is proper on the support of C, has finite source presentations near the target support, and defines the lattice-index pushforward a_*C, then

$$I_C(a^*\alpha) = I_{a_*C}(\alpha)$$

for every smooth α with compact ambient support in the target open set.

Proof. Let K be the support of the form on C. Around each point of this compact set choose a finite presentation and a smaller closed rectangular box contained in its open domain. Finitely many box interiors cover K. Intersect the finitely many presented cells with these boxes and take a common hyperplane refinement. On overlaps their weights agree by the definition of a local cycle. Count each resulting cell once with that weight. This gives a finite polyhedral description on an open neighbourhood of K in the support. Cells or artificial faces outside that neighbourhood meet no nonzero tangential coefficient of the form. The integral may therefore be computed on this finite description.

Two such descriptions have a common refinement near K. The first assertion of Lemma 5.46.2 proves that the two sums coincide. Linearity and nonnegativity follow from the corresponding inequalities for real grid sums.

For β, use such a finite description around its support. Any artificial exterior facet lies outside that support; all the adjacent tangential coefficients of β vanish in a relative neighbourhood, so their boundary contractions vanish

there as well. On a genuine $(d-1)$-face τ, formula (5.46.1) adds the normal lifts with their weights. Its contribution is

$$(-1)^{d(d+1)/2}\int_{\tau}\beta(v'_1,\dots,v'_{d-1};v''_1,\dots,v''_{d-1},(\sum_{\sigma\supset\tau}m_\sigma u_{\sigma/\tau})'')\,d\mu_\tau.$$

Balancing puts the vector in the last slot in $(N_\tau)_{\mathbb{R}}$. The double-primed factor alternates, so this expression is zero. Summing the finitely many faces gives $I_C(d'\beta)=0$. The identities $d''=Jd'J$ and $J\omega_{N_\sigma}=(-1)^d\omega_{N_\sigma}$ give the assertion for γ by exchanging the two exterior factors. This verifies both closedness statements with the stated normalization.

For the projection formula, properness makes the support of the pullback on C compact. Choose finite source and target presentations near these compact sets and refine along all cell images. On a source d-cell of full rank, the affine map is injective on its affine span. The change-of-variables formula of Lemma 5.46.2 multiplies its contribution by $[N_\tau : a(N_\sigma)]$. On a source cell of rank less than d the pullback vanishes. Grouping the remaining finitely many contributions over each target cell gives exactly its lattice-index pushforward weight. This is the asserted identity. □

Theorem 5.46.4 (Analytic moment integrals and finite change of charts)**.** *Let $X=\mathcal{M}(B)$ be a generalized affinoid space, pure of analytic dimension d, over a complete non-Archimedean field with its specified absolute value. For a unit tuple f write $t_f : X\to\mathbb{R}^n$ and $U_f=\mathbb{R}^n\setminus t_f(\partial_K X)$. For every finite B-module M, the formula*

$$I_{M,f}(\alpha)=I_{\mathrm{Trop}_f(M)|_{U_f}}(\alpha)$$

defines a linear functional on smooth top forms whose tangential support is compact in U_f. It is nonnegative when their tangential top coefficients are nonnegative. It is additive in short exact sequences of modules and satisfies

$$I_{M,f}(d'\beta)=0,\qquad I_{M,f}(d''\gamma)=0$$

for smooth compactly supported tests of the indicated complementary bidegrees in U_f.

Let $p : X=\mathcal{M}(B)\to Y=\mathcal{M}(A)$ be finite, with Y also pure of analytic dimension d. Suppose unit tuples f_X, f_Y and an integral map a satisfy $t_{f_Y}\circ p = a\circ t_{f_X}$. For every finite B-module M, regarded also as a finite A-module, and every smooth top form α with compact ambient support in U_{f_Y},

$$I_{M,f_X}(a^*\alpha)=I_{M,f_Y}(\alpha). \tag{5.46.2}$$

The support of the left-hand test on its actual source cycle is compact and lies in its whole interior weight domain; there α is understood to be extended by zero outside U_{f_Y}.

In particular, let f, g be two unit tuples on the same X, let π_f, π_g be the projections from the joint tuple (f, g), and let α, β be top forms with compact ambient support in U_f, U_g, respectively. Extend them by zero outside these open sets. If

$$\pi_f^* \alpha = \pi_g^* \beta$$

on every tangential d-plane of the actual joint local moment image over $\pi_f^{-1}(U_f) \cup \pi_g^{-1}(U_g)$, then $I_{B,f}(\alpha) = I_{B,g}(\beta)$.

Adjoining unit coordinates is faithful on tangential superforms in every bidegree: on $t_f(X) \cap U_f$, a smooth superform is zero on all tangential faces if and only if its pullback is zero on all tangential faces of the actual joint image over U_f.

Proof. Theorem 5.45.3 constructs the local balanced module cycles with non-negative integral weights. Their definition uses the actual Gauss algebras and generic local lengths of Theorem 5.40.2. The functional, its positivity, and both differential identities are therefore Theorem 5.46.3. The exact-sequence equality of module weights gives the corresponding equality of these finite cell integrals.

First assume that the target units pull back to the actual monomials in the source units prescribed by a. For this finite map, every point whose target weight belongs to U_{f_Y} is interior in X, by finite relative interior and composition, as proved in Theorem 5.45.3. The complete source moment support over a compact subset of U_{f_Y} is compact. Near each target weight, Lemma 5.45.1 supplies a finite source presentation retaining the entire inverse support; the target has the local fan presentation of Theorem 5.38.6. Theorem 5.42.2 on these finite presentations gives

$$a_*\big(\mathrm{Trop}_{f_X}(M)|_{a^{-1}(U_{f_Y})}\big) = \mathrm{Trop}_{f_Y}(M)|_{U_{f_Y}}.$$

The local identities agree after refinement because the module weights refer to the same actual fibres. The projection formula of Theorem 5.46.3 proves (5.46.2). Notice that its real factor is the lattice index, whereas the target coefficients are the full finite-module generic lengths. Both appear in the proved cycle formula.

For a general equality of moment maps as in the statement, use the joint source tuple $(f_X, p^* f_Y)$. Its actual image is the graph of a. The case just proved applies both to its projection onto f_X with the identity morphism and to its projection onto f_Y with the morphism p. The first graph projection is an integral isomorphism, with inverse $u \mapsto (u, a(u))$. It identifies saturated tangent lattices and has index one on every cell. The second projection is a after this identification. The corresponding finite lattice-index sums therefore give the same cycle formula for the stated moment-map equality, and its integral follows from the same real projection formula. All these comparisons are over U_{f_Y}; finite relative interior and composition give the required absolute interior on the whole source fibres.

Apply this result to $p = \mathrm{id}_X$ with source tuple (f, g) and target tuple f, and then with target tuple g. The source structure module is B in both cases. A compactly supported smooth form on U_f extends smoothly by zero to $\mathbb{R}^n$: near the complement of U_f its support is absent. The support of its pullback on the joint actual image is a closed subset of the compact set $t_{(f,g)}(X) \cap \pi_f^{-1}(\operatorname{supp} \alpha)$. Every point of that set lies over a whole interior f-fibre, so the joint cycle is defined there. The same holds for β. Their tangential equality gives equal cell integrals on a common finite presentation. The two identity projection formulas now give the asserted equality of original integrals.

Finally consider a form η of any bidegree on the target moment image over U_f. If its tangential restrictions vanish, refine the joint and target local presentations so that every source cell maps into one target cell. Pullback on tangent spaces then gives zero on the source. Conversely suppose the joint pullback vanishes. Near any target point take the finite presentations supplied above, and remove the lower-dimensional target cells and the images of lower-dimensional or rank-dropping source cells, exactly as in Theorem 5.44.1. On the dense remaining part of every target d-cell, the positive target weight is the sum of the positive source weights times their lattice indices. Hence at least one full-rank source cell lies above every such point. Its tangent map is a real linear isomorphism onto the target tangent plane, so pullback is injective on both exterior factors. The tangential value of η is zero there. Continuity of its smooth coefficients extends this vanishing over the entire target d-cell. Every smaller face is contained in such a cell, and restriction to its tangent space preserves zero. This proves the converse, including the bidegrees above d, where all these restrictions are already zero. □

5.47 Weights on partial affinoid domains

Throughout this section K is complete for its specified real-valued non-Archimedean absolute value. Trivial valuation and arbitrary positive affinoid radii are allowed. Purity means analytic purity as in Section 5.35. All fibre algebras are the actual affinoid algebras, with their nilpotents.

Lemma 5.47.1 (Finite affinoid domains and idempotents)**.** *Let $A \to B$ represent a generalized affinoid domain $U \subseteq \mathcal{M}(A)$. Suppose that B is finite over A and that its given norm is equivalent to its canonical finite-module norm. There is a unique idempotent $e \in A$ such that the restriction map identifies B with $eA = A/(1-e)A$, boundedly in both directions. Moreover,*

$$U = \{x : |e(x)| = 1\}, \qquad A \simeq eA \times (1-e)A.$$

The set U is open and closed, and the displayed product is a bounded isomorphism with bounded inverse. These assertions include the empty domain, for which $e = 0$.

Proof. The self-intersection of the domain has actual algebra $B\widehat{\otimes}_A B \simeq B$, and the isomorphism is multiplication, by Theorem 2.36.9. Proposition 1.18.7 identifies this tensor with $B \otimes_A B$. Thus the two tensor arrows from B agree, and $A \to B$ is an epimorphism of ordinary rings. The finite epimorphism theorem, [StacksAlg, Tag 04VT], makes it surjective.

Put $J = \ker(A \to B)$. It is finitely generated by Theorem 1.18.3. The domain restriction is flat by Proposition 2.38.2. Tensoring $J \hookrightarrow A$ with A/J is therefore injective. Its source is J/J^2 and its map to A/J is zero, so $J = J^2$. By [StacksAlg, Tag 00EH], $J = (1-e)A$ for an idempotent e. Multiplication by e identifies A/J with eA. If another idempotent e' has the same kernel, then $e(1-e') = e'(1-e) = 0$, so $e = ee' = e'$. This proves uniqueness.

Every point takes an idempotent to 0 or 1. The points killing $(1-e)A$ are exactly those with $|e| = 1$. The quotient norm is canonical, so they are precisely the points of B. The two level sets of $|e|$ are open and closed. The algebraic maps in the product decomposition and their inverses are maps between finite canonical A-modules, hence bounded by Theorem 1.18.3. The factors are actual affinoid quotients by Corollary 1.18.4. Finally the empty-domain algebra is zero by Theorem 1.20.1, giving $e = 0$. □

Theorem 5.47.2 (Inclusion–exclusion on interior fibre covers)**.** *Let $X = \mathcal{M}(A)$ be pure of analytic dimension d, let $f_1, \dots, f_n \in A^\times$, and put $t = (-\log|f_1|, \dots, -\log|f_n|)$. Fix $w \in t(X)$ and a saturated rank-d lattice $N \subset \mathbb{Z}^n$. Suppose that*

$$F_w = t^{-1}(w) \subset \mathrm{Int}(X/K), \qquad t(X) \cap W \subset w + N_{\mathbb{R}}$$

for an open neighborhood W of w. Let $V_1, \dots, V_q \subseteq X$ be affinoid domains such that

$$F_w \subseteq \bigcup_{i=1}^{q} V_i, \qquad F_w \cap V_i \subseteq \mathrm{Int}(V_i/K) \quad (1 \le i \le q).$$

For nonempty $I \subseteq \{1, \dots, q\}$, write $V_I = \bigcap_{i \in I} V_i$, $A_I = \mathcal{O}(V_I)$ and $F_I = F_w \cap V_I$. Let D and D_I be the actual algebras of F_w and F_I.

Each F_I is an open and closed affinoid domain of F_w. There are unique idempotents $e_i \in D$ such that, with $e_I = \prod_{i \in I} e_i$, restriction gives bounded isomorphisms

$$D_I \simeq e_I D, \qquad \prod_{i=1}^{q} (1 - e_i) = 0. \tag{5.47.1}$$

For every integral $Q : \mathbb{Z}^n \to \mathbb{Z}^d$ injective on N, let R_Q be the exact-radius polytorus algebra at Qw, let H_Q be its Gauss field, and put

$$E_Q = D \widehat{\otimes}_{R_Q} H_Q, \qquad E_{Q,I} = D_I \widehat{\otimes}_{R_Q} H_Q, \qquad c_Q = [\mathbb{Z}^d : Q(N)].$$

Then $E_{Q,I} = e_I E_Q$. For every finite D-module P,

$$\dim_{H_Q}(P \otimes_D E_Q) = \sum_{\varnothing \neq I \subseteq \{1,\dots,q\}} (-1)^{|I|+1} \dim_{H_Q}(P \otimes_D E_{Q,I}). \tag{5.47.2}$$

All these tensors also agree with their completed tensors for the canonical finite-module norms.

In particular, let M be a finite A-module and set $M_I = M \otimes_A A_I$. Write $m_{M_I,f}(w) = 0$ if F_I is empty. For a nonempty F_I, its whole fibre is interior in V_I and its module weight is defined by Theorem 5.40.2. Then

$$m_{M,f}(w) = \sum_{\varnothing \neq I \subseteq \{1,\dots,q\}} (-1)^{|I|+1} m_{M_I,f}(w). \tag{5.47.3}$$

For $M = A$ this is the same identity for the intrinsic weights $m_{X,f}$ and $m_{V_I,f}$. If the selected parts $F_w \cap V_i$ are pairwise disjoint, it reduces to the sum of their weights.

There is an open neighborhood W' of w on which the same domains cover every whole fibre. For every $v \in t(X) \cap W' = (w + N_{\mathbb{R}}) \cap W'$, all the preceding assertions apply at v. The empty intersections remain empty, and each module weight in (5.47.3) is constant on this neighborhood in the indicated plane.

Proof. Finite intersections are actual affinoid domains by Corollary 2.36.10. Every nonempty one is pure of analytic dimension d by Theorem 5.35.3. If $x \in F_w \cap V_i$, the composition equality 2.44.1 makes x interior in V_i/X. The domain interior theorem 2.44.3 says that it is an ordinary interior point of V_i in X. Consequently, if $x \in F_I$, the intersection of these finitely many neighborhoods is a neighborhood of x contained in V_I. The same domain theorem, followed by the composition inclusion in Proposition 2.33.21, gives

$$F_I \subset \operatorname{Int}(V_I/K).$$

Thus no extra boundary condition on the intersections is required.

The whole flat-germ theorem 5.37.1 makes D and every nonzero D_I finite canonical modules over the same R_Q. It applies to V_I because its moment image is contained in that of X, and hence in the same affine plane near w. An empty F_I has zero algebra by Theorem 1.20.1. The domain base-change theorem 2.36.9 identifies $D \to D_I$ with the actual domain restriction to $F_I \subset F_w$.

This restriction is also finite with its canonical D-module norm. Indeed, any finite R_Q-generating family of D_I also generates it over D. Controlled lifts through its canonical R_Q-presentation, followed by the bounded map $R_Q \to D$, bound the quotient norm from the corresponding D-presentation by a constant times the given norm of D_I. Conversely the bounded D-action bounds its given norm by that quotient norm. These are exactly the two finite-generator norm comparisons of Theorem 1.18.3.

Lemma 5.47.1, first for each singleton I, supplies the unique e_i. Domain intersections and their completed tensor presentations identify

$$D_I \simeq \widehat{\bigotimes}_{i \in I, D} e_i D \simeq e_I D.$$

Here completion changes nothing by Proposition 1.18.7; multiplication gives the last algebraic identification. If $h = \prod_i (1 - e_i)$ were nonzero, the nonzero affinoid quotient hD would have a point by Theorem 1.20.1. Its image in F_w

would belong to none of the V_i, a contradiction. This proves (5.47.1) with the actual algebras, including their nilpotents.

Finite base change, Proposition 2.33.19, gives $E_{Q,I} = e_I E_Q$. Put $L = P \otimes_D E_Q$, a finite-dimensional H_Q-vector space. For each $J \subseteq \{1, \dots, q\}$ set

$$a_J = \prod_{j \in J} e_j \prod_{j \notin J} (1 - e_j).$$

These are pairwise orthogonal idempotents with sum 1, and $a_\varnothing = 0$. Therefore $L = \bigoplus_{J \neq \varnothing} a_J L$. On the summand $a_J L$, e_I acts as 1 precisely when $I \subseteq J$, and otherwise as 0. Its coefficient in the right side of (5.47.2) is

$$\sum_{\varnothing \neq I \subseteq J} (-1)^{|I|+1} = 1,$$

as follows by expanding $\prod_{j \in J}(1 - 1) = 0$. This proves that formula.

The finite D-module P is finite over R_Q. Composing finite free quotient presentations gives equivalent canonical norms for this restriction of scalars. Proposition 1.18.7 consequently identifies every tensor just used with its complete canonical finite-dimensional tensor. In particular, if $P = M \otimes_A D$, associativity identifies its factor $P \otimes_D E_{Q,I}$ with $M_I \otimes_{A_I} E_{Q,I}$. Dividing (5.47.2) by c_Q and applying Theorem 5.40.2 to X and the nonempty V_I proves (5.47.3), independently of Q. The structure-module case agrees with Theorem 5.37.1. Pairwise disjoint parts make every intersection of size at least two empty.

We finally choose the neighborhood uniformly. The ordinary interiors of the V_i cover F_w, as proved above. The complement of their union in compact X is compact, and its moment image misses w. Also the compact sets $\partial_K X$ and $\partial_K V_i$ have moment images missing w. Shrink W to avoid all these images. Compactness and openness of interior are Propositions 2.24.1 and 2.33.21. The resulting neighborhood has whole fibres covered by the same V_i, and each selected fibre is interior in its own domain. The intersection argument above applies unchanged. Every V_I whose fibre at w is empty has compact moment image missing w; shrinking again keeps it empty throughout the neighborhood.

Theorem 5.37.2 makes the image of X equal to the indicated plane nearby. For each fixed V_I, apply the finite generic-component formula in Theorem 5.40.2 to M_I. A component with empty fibre at w remains absent nearby by compactness. Every other component has a whole interior flat fibre, as verified in that theorem's proof, and its intrinsic weight is locally constant

by Theorem 5.37.2. There are only finitely many components and intersections. One final shrinking gives their simultaneous local constancy and all the stated assertions. □

References. The algebraic inputs concerning finite epimorphisms and idempotent ideals are the cited Stacks lemmas. The contribution here is their application to the actual generalized analytic fibre algebras and the resulting boundary-sensitive localization of Gauss and tropical module weights. This is a foundational reconstruction; no claim of novelty in the literature is made.

5.48 Global additivity over affinoid covers

Use the global top-dimensional chains $\mathcal{T}_f(M)$ and their lattice-normalized integrals $\mathcal{I}_{M,f}$ from the finite-module chain and weights and the finite-cell functional. The original complete field and absolute value are fixed; trivial valuation and arbitrary positive presentation radii are allowed.

Theorem 5.48.1 (Global additivity for a finite affinoid cover)**.** *Let $X = \mathcal{M}(A)$ be nonempty and pure of analytic dimension d, let M be a finite A-module, and let a finite tuple of analytic units f define $t : X \to \mathbb{R}^n$. Suppose that $V_1, \dots, V_q$ is a finite affinoid-domain cover of X. For nonempty $I \subseteq \{1, \dots, q\}$ put*

$$V_I = \bigcap_{i \in I} V_i, \qquad A_I = \mathcal{O}(V_I), \qquad M_I = M \otimes_A A_I, \qquad f_I = f|_{V_I}.$$

An empty intersection contributes the zero chain and zero integral. Then, as integral weighted d-chains in the same $\mathbb{R}^n$,

$$\mathcal{T}_f(M) = \sum_{\varnothing \neq I \subseteq \{1,\dots,q\}} (-1)^{|I|+1} \mathcal{T}_{f_I}(M_I). \tag{5.48.1}$$

In particular, for every smooth real superform α of bidegree (d,d) defined on a neighborhood of the actual compact image $t(X)$,

$$\mathcal{I}_{M,f}(\alpha) = \sum_{\varnothing \neq I \subseteq \{1,\dots,q\}} (-1)^{|I|+1} \mathcal{I}_{M_I,f_I}(\alpha). \tag{5.48.2}$$

The identities are unchanged by finite refinements of the cover or of the polyhedral complexes used to represent the chains.

Proof. Corollary 2.36.10 makes every V_I an actual affinoid domain of X, with its compatible restriction algebra. Every nonempty V_I is pure of analytic dimension d by Theorem 5.35.3. The finite module M_I has its canonical complete norm by Theorem 1.18.3. Thus all the nonzero terms in the claimed identity have the stated analytic meaning.

Theorem 5.39.3 makes the actual images $t(X)$ and $t(V_I)$ finite integral affine polyhedral sets. It also makes

$$E = t(\partial_K X) \cup \bigcup_{\varnothing \neq I \subseteq \{1,\dots,q\}} t(\partial_K V_I)$$

a finite polyhedral subset of $t(X)$ of dimension at most $d-1$; for $d=0$ it is empty. By Lemma 5.14.2, choose one finite common subdivision of $t(X)$ in which every $t(V_I)$ and E is a subcomplex, and which also refines the representatives of all the finitely many chains under consideration.

Fix a d-cell σ of this subdivision and a point $w \in \operatorname{relint}(\sigma)$. Its relative interior misses E. Since the whole image has dimension at most d, near w no cell other than σ is present. Hence the local moment image is contained in the affine d-plane $w + (N_\sigma)_{\mathbb{R}}$, where N_σ is its saturated tangent lattice. The whole fibre F_w is interior in X, and each $F_w \cap V_i$ is interior in V_i, by the definition of E. The domains cover F_w because they cover X.

Apply Theorem 5.47.2 to this fibre and the finite module M. It gives

$$m_{M,f}(w) = \sum_{\varnothing \neq I \subseteq \{1,\dots,q\}} (-1)^{|I|+1} m_{M_I, f_I}(w),$$

with zero for the empty selected fibres. The common subdivision ensures that $\operatorname{relint}(\sigma)$ is either contained in $t(V_I)$ or disjoint from it. By the finite-module chain and weights, the terms in this identity are exactly the coefficients of the corresponding global chains on σ. Thus these coefficients agree on every d-cell. Lower-dimensional cells carry no coefficient in a d-chain. This proves (5.48.1); it also covers the case $\dim t(X) < d$, when there are no d-cells.

All these chains have compact polyhedral support contained in $t(X)$. Apply the finite-cell functional to the same smooth form on each term. Their common representatives have finitely many cells, and the integral is the sum of the cell integrals times their integer coefficients. The coefficient identity proves (5.48.2). Subdivision invariance is part of the finite-cell functional. Repeating the argument for any finite refinement of the affinoid cover gives the same global chain $\mathcal{T}_f(M)$ and the same integral, which proves the final assertion. □

5.49 Global moment chains and finite analytic chart data

Theorem 5.49.1 (Global finite-module tropical chains)**.** *Let $X = \mathcal{M}(A)$ be pure of analytic dimension d over a complete non-Archimedean field with its specified absolute value, and let a unit tuple f define $t_f : X \to \mathbb{R}^n$. For every finite A-module M there is a canonical finite nonnegative integral d-dimensional polyhedral chain*

$$\mathcal{T}_f(M).$$

Its support is contained in $t_f(X)$. On any common finite subdivision of the actual image and $t_f(\partial_K X)$, the weight of a d-cell is the module weight $m_{M,f}(v)$ at any point of its relative interior. Cells of dimension less than d give no d-chain term. The chain is additive in short exact sequences, independent of subdivision, and balanced off $t_f(\partial_K X)$. For $M = A$ every occurring d-cell has positive weight.

For a smooth (d,d)-superform α on an open neighbourhood of the actual compact image, define

$$\mathcal{I}_{M,f}(\alpha) = \sum_{\sigma} m_\sigma \int_\sigma \alpha.$$

This is independent of subdivision and is nonnegative when the tangential top coefficients of α are nonnegative. It agrees with $I_{M,f}$ of Theorem 5.46.4 for tests supported in the whole interior weight domain.

Let $p : \mathcal{M}(B) \to \mathcal{M}(A)$ be finite, with both spaces pure of analytic dimension d, and suppose moment tuples satisfy $t_{f_A} \circ p = a \circ t_{f_B}$ for an integral map a. For every finite B-module M,

$$a_* \mathcal{T}_{f_B}(M) = \mathcal{T}_{f_A}(M), \qquad \mathcal{I}_{M,f_B}(a^*\alpha) = \mathcal{I}_{M,f_A}(\alpha).$$

On the right M is its finite A-module. In particular, adjoining analytic unit coordinates preserves these chains under projection and preserves the corresponding integrals.

Proof. Theorem 5.39.3 gives a finite polyhedral image of dimension at most d and a finite boundary image of dimension at most $d-1$. Choose one finite common subdivision with the boundary image as a subcomplex. At a point in the relative interior of a d-cell, the whole moment fibre is therefore interior, and the local image is contained in that cell's affine d-plane. Theorem 5.40.2 defines the nonnegative integer weight there. It is locally constant,

by Theorem 5.42.1 on small convex neighbourhoods with the actual local fan presentations of Theorem 5.38.6. The relative interior of the cell is convex, so the weight is constant there. For the structure module it is positive by Theorem 5.37.1.

On a common subdivision of two choices, the generic weights compute the same whole-fibre module invariant. Thus their chains agree. Additivity follows from the exact-sequence formula of Theorem 5.40.2. On the complement of the boundary image the chain is the local balanced module cycle of Theorem 5.45.3. All cells with nonzero weights are compact polytopes: the actual moment image is compact, and the complex is finite. Consequently the integral is a finite sum of the integrals constructed in Lemma 5.46.2. That lemma gives subdivision independence and positivity. Restriction to the whole interior weight domain gives exactly the previously defined local integral.

First suppose that the compatibility is an identity of analytic characters: each target unit pulls back to the monomial in the source units prescribed by a. For this finite morphism, take finite global subdivisions of the two actual images and their boundary images. Refine along the images of all source cells, using the finite linear elimination and hyperplane construction in Theorem 5.17.2. Outside a finite union of polyhedra of dimension at most $d-1$ in each target d-cell, the target whole fibre is interior, and every source weight above it lies in a full-rank d-cell. The entire inverse weight set is finite, with at most one point per such source cell. Finite relative interior and composition imply absolute interior for every point of those source fibres.

Near such a target weight, Lemma 5.45.1 retains the complete source inverse support in a finite pure presentation. Theorem 5.42.2 then identifies the finite lattice-index sum with the target weight of M as an A-module. Its empty-inverse- image case gives zero on both sides. Thus the generic weight of every target d-cell is exactly its pushforward weight. The resulting equality of finite chains is $a_*\mathcal{T}_{f_B}(M) = \mathcal{T}_{f_A}(M)$. This proves the chain formula under the character identity. In particular it applies to the identity morphism and adjoining any analytic unit coordinates.

Now suppose only the equality of moment maps in the statement. On the source use the joint tuple $(f_B, p^* f_A)$. Its actual image is the graph of a over $t_{f_B}(\mathcal{M}(B))$. Projection of that graph onto its first factor is an integral isomorphism: its inverse is $u \mapsto (u, a(u))$. On the saturated tangent lattices of every corresponding cell these two maps are inverse, so the index of this

projection is one. The augmentation case just proved identifies the graph chain with $\mathcal{T}_{f_B}(M)$ under this isomorphism. The second projection satisfies an actual character identity with the target tuple. Its pushforward is $\mathcal{T}_{f_A}(M)$ by the already proved case, and on the graph it is a after the first projection. The equality of the finite lattice-index sums therefore proves the chain formula under the stated moment-map equality as well.

The full actual source support is compact, so this pushforward is proper. On each full-rank source cell, Lemma 5.46.2 gives the factor $[N_\tau : a(N_\sigma)]$ in the integral. On a rank-dropping cell the top-degree pullback is zero. Grouping these finite cell integrals by their target cells proves the integral identity. No ambient support cutoff is needed here, because all the integration polytopes are compact. Taking p to be the identity and a a coordinate projection proves the last assertion. □

Theorem 5.49.2 (Faithfulness after adjoining analytic units)**.** *Let X be any generalized affinoid space, and let f, g be finite unit tuples. Write*

$$S = t_f(X), \qquad \widehat{S} = t_{(f,g)}(X), \qquad \pi : \widehat{S} \longrightarrow S.$$

Restrict a smooth ambient superform to every tangential face of these actual finite polyhedral sets. These restrictions are independent of subdivision. Such a form on S has all tangential restrictions zero if and only if its pullback has all tangential restrictions zero on $\widehat{S}$.

Tangential equality is preserved by restriction from an affinoid space to any affinoid domain, using the restricted unit coordinates.

Proof. Both actual images have finite polyhedral structures by Theorem 5.39.3, and π is surjective on their actual point sets. A finite subdivision does not change tangential vanishing: the relative interiors of the full-dimensional pieces in each original cell are dense there, and continuity of the coefficients extends vanishing from them to the cell and its faces. The converse follows by restricting an alternating covector to a smaller tangent space.

Refine the target by the images of all source cells and refine the source by the inverse images of the resulting faces. Finite inequality elimination and hyperplane subdivision, as in Theorem 5.17.2, make this a finite polyhedral refinement. Every source cell then maps into a target cell. If the form vanishes tangentially on the target, pullback through these tangent maps vanishes on the source.

For the converse, fix a target cell τ of dimension r. Its relative interior is covered by the images of the finitely many source cells mapping into it. Remove their image faces of dimension less than r. The remaining subset of relint(τ) is dense: in its affine span, a line direction outside finitely many proper linear subspaces meets the corresponding proper affine spans in at most finitely many points. Such a direction is obtained by the polynomial vectors $(1, b, \dots, b^{r-1})$ avoiding finitely many roots; for $r = 0$ no exclusion is needed. At a point v of this dense set some source cell σ has an image of dimension r containing v in its relative interior.

There is a preimage u of v in relint(σ). To verify this elementary convex assertion, choose u_0 in that relative interior. A small ball in its affine span lies in σ; its linear image contains a small ball in the image span, since a surjective map of finite-dimensional real vector spaces has a linear right inverse. Thus $\pi(u_0)$ is in the relative interior of the image. Since v is also there, for some $\varepsilon > 0$ the point $v' = v + \varepsilon(v - \pi(u_0))$ still belongs to that image. Choose $u' \in \sigma$ over v'. Then

$$u = \frac{u' + \varepsilon u_0}{1 + \varepsilon}$$

maps to v and is interior in σ, because the positive coefficient of u_0 brings a scaled open ball with it.

The tangent map from σ onto τ is surjective. Its pullback on each exterior cotangent factor is injective, by a linear right inverse. Vanishing of the joint pullback at u therefore forces the tangential value of the original form at v to be zero. Density and continuity prove vanishing throughout τ. This applies to every cell, including lower-dimensional maximal cells of the actual image, and proves the converse.

Finally the moment image of an affinoid domain is a finite polyhedral subset of the old image, again by Theorem 5.39.3. Refine both complexes by all their defining hyperplanes. Each new cell of the subset lies in a cell of the old complex, so its tangent space is a subspace of that cell's tangent space. Restriction preserves tangential vanishing and hence tangential equality. □

Definition 5.49.3 (Finite affinoid moment-chart data)**.** Let $X = \mathcal{M}(A)$ be pure of analytic dimension d. Finite moment-chart data of bidegree (p, q) consist of a finite affinoid-domain cover $X = \bigcup_{i=1}^{s} V_i$, a finite analytic unit tuple f_i on each V_i, and a smooth (p, q)-superform α_i on a neighbourhood of $t_{f_i}(V_i)$. On every intersection $V_i \cap V_j$, the two forms must have equal tangential pullbacks to the actual joint moment image for (f_i, f_j). A refinement

may restrict the domains, adjoin unit coordinates, and replace a superform by one with the same tangential pullback. Data are identified if they agree on a common finite affinoid refinement.

Theorem 5.49.4 (An integral independent of finite affinoid chart data)**.** *Fix a finite A-module M. For finite moment-chart data of bidegree (d,d) on X, put $V_I = \bigcap_{i\in I} V_i$, $A_I = \mathcal{O}(V_I)$ and $M_I = M \otimes_A A_I$. Choose any $i \in I$ to represent the restricted form on V_I, and set*

$$\mathcal{I}_M(\alpha) = \sum_{\varnothing\neq I\subseteq\{1,\dots,s\}} (-1)^{|I|+1}\mathcal{I}_{M_I,f_i|_{V_I}}(\alpha_i). \tag{5.49.1}$$

An empty intersection contributes zero. The expression is independent of each chosen i, of the finite cover, of the unit tuples and of the representatives within the equivalence of Definition 5.49.3. It is linear in the form and additive in short exact sequences of finite A-modules. It is nonnegative if all chart representatives have nonnegative tangential top coefficients. For a single global moment presentation it is exactly the global integral given by the finite-cell functional.

Proof. Finite intersections are actual affinoid domains by Corollary 2.36.10; their algebras and restriction maps are the actual domain ones. Every nonempty intersection is pure of analytic dimension d by Theorem 5.35.3. Thus all the displayed integrals exist by the finite-cell functional. Compatibility of the chart data restricts to further intersections by Theorem 5.49.2. On V_I, the same-affinoid coordinate augmentation identifies the integrals of any two chosen representatives with the integral of their equal joint pullbacks. This proves independence of the choice of i.

First suppose the data come from a single tuple on X. The domain-cover identity of Theorem 5.48.1 gives exactly (5.49.1). The same identity holds after restricting a single chart presentation to any of its affinoid subdomains.

Now compare compatible data on two finite covers $(V_i)_{i=1}^s$ and $(W_j)_{j=1}^t$. The module on every overlap is the restriction of the same M, by the actual domain tensor identifications of Proposition 1.18.7 and Theorem 2.36.9. For each nonempty I, cover V_I by the $V_I\cap W_j$ and apply the preceding single-chart identity to its chosen representative. Substitution in (5.49.1) gives

$$\sum_{I\neq\varnothing}\sum_{J\neq\varnothing}(-1)^{|I|+|J|}\mathcal{I}_{M|_{V_I\cap W_J}}(\alpha|_{V_I\cap W_J}).$$

The two representatives on each of these intersections have equal integrals by their joint-coordinate comparison. Interchange the two finite sums and apply the domain-cover identity to each W_J. This recovers precisely the expression for the second cover. Thus the value is unchanged under compatible replacement or refinement. For data identified after a common refinement, apply this comparison to each of the two original covers and that common refinement. This proves all claimed independences.

Linearity is termwise. For a short exact sequence of finite A-modules, domain flatness on a single actual domain keeps it exact on every V_I. The global module-chain additivity of the finite-module chain and weights gives additivity of each integral, and hence of the finite alternating sum. The assertion about a single global presentation was proved in the second paragraph.

To verify positivity, order the cover and group the alternating sum by the largest index j in I. The j-th group uses the single representative α_j and is the integral of the chain

$$C_j = \sum_{J \subseteq \{1,\dots,j-1\}} (-1)^{|J|} \mathcal{T}_{f_j|_{V_j \cap V_J}}(M|_{V_j \cap V_J}),$$

with the empty intersection in this formula understood to be V_j. Each C_j is nonnegative. Indeed, choose a common finite subdivision of these actual moment images and their boundary images by the common finite subdivision lemma. The latter have dimension at most $d-1$ by Theorem 5.39.3. At a generic point of a d-cell, apply Theorem 5.47.2 on V_j to the family consisting of V_j itself and the domains $V_j \cap V_i$ for $i < j$. It supplies idempotents e_i in the actual whole-fibre algebra; the member V_j ensures the cover hypothesis. For a transverse Gauss algebra E_Q and its common lattice divisor c_Q, the weight of C_j is

$$\frac{1}{c_Q} \dim_{H_Q} \left(\prod_{i<j} (1 - e_i) \right) \left(M|_{V_j} \otimes_{A_{V_j}} E_Q \right) \geq 0.$$

This is the elementary expansion of the product of the complementary idempotents, with all module structures retained. It is already an integer because it is the displayed alternating sum of integer module weights. Thus every generic coefficient of C_j is nonnegative. Its integral against α_j is nonnegative by the nonnegative cell integral on each cell. Summing these finitely many groups proves positivity. □

5.50 Integration on compact good analytic spaces

The complete field K and its real-valued non-Archimedean absolute value are fixed. Trivial valuation and arbitrary positive affinoid radii are allowed. Compact means quasi-compact and Hausdorff.

Lemma 5.50.1 (Affinoid intersections in a separated good space)**.** *Let X be a good Hausdorff K-analytic space. Any finite intersection of affinoid domains of X is an affinoid domain, with the empty space allowed. Two finite affinoid-domain covers therefore have a common finite affinoid refinement. If X is compact, it has a finite affinoid cover whose ordinary interiors cover X.*

Proof. Lemma 2.53.3, applied to $X \to \mathcal{M}(K)$, makes the analytic diagonal $\Delta : X \to X \times_K X$ a closed immersion. Let $U = \mathcal{M}(A)$ and $V = \mathcal{M}(B)$ be affinoid domains of X. Theorem 2.40.11 makes

$$U \times_K V = \mathcal{M}(A \widehat{\otimes}_K B)$$

an actual affinoid domain of $X \times_K X$. The pullback of Δ to this domain is an affinoid closed subspace. Indeed, the coherent defining ideal restricts to a finite ideal by Theorem 2.39.6, and its quotient is the actual affinoid quotient of Corollary 1.18.4. The arbitrary-domain restriction argument in Lemma 2.53.2 identifies this quotient with the entire diagonal pullback, including its analytic structure.

This pullback is $U \times_X V$. By the domain universal property of Theorem 2.40.8, a map to it is exactly a map to X whose image lies in both U and V. Thus it is the actual analytic domain $U \cap V$. There is no identification of an analytic product with an ordinary product of point sets in this argument. Iteration proves the finite-intersection assertion, using domain transitivity in Proposition 2.40.5. The pairwise intersections of two finite covers form the claimed common refinement. Affinoid domains are compact by Proposition 2.24.1, and hence closed in the Hausdorff space X. Lemma 2.39.2 makes every finite cover by such domains admissible. Finally, goodness supplies an affinoid neighbourhood of each point. Compactness gives a finite subcover of their ordinary interiors. □

Definition 5.50.2 (Pure dimension on a good space)**.** A good analytic space is pure of analytic dimension d if every point has an ordinary affinoid neighbourhood pure of analytic dimension d in the sense of Section 5.35.

Lemma 5.50.3 (Purity is local on affinoid covers)**.** *Let $V_1, \dots, V_s$ be a finite affinoid-domain cover of a nonempty affinoid X. Then X is pure of analytic dimension d if and only if every nonempty V_i is so. Consequently every nonempty affinoid domain of a good Hausdorff space pure of analytic dimension d is pure of analytic dimension d.*

Proof. The forward implication is Theorem 5.35.3. For the converse write $X = \mathcal{M}(A)$ and $V_i = \mathcal{M}(C_i)$, omitting empty members. Fix a minimal prime $\mathfrak{p}$ of A. The actual quotient $A/\mathfrak{p}$ is affinoid by Corollary 1.18.4, and its spectrum is nonempty by Theorem 1.20.1. Theorem 2.36.9 gives an affinoid-domain cover of this spectrum with algebras

$$D_i = C_i \widehat{\otimes}_A (A/\mathfrak{p}) = C_i/\mathfrak{p}C_i.$$

The equality with the ordinary quotient follows from Proposition 1.18.7, applied to the finite canonical A-module $A/\mathfrak{p}$.

Every nonzero D_i is pure of analytic dimension d. To see this, let $\mathfrak{q}$ be a prime of C_i minimal over $\mathfrak{p}C_i$. The domain map $A/\mathfrak{p} \to D_i$ is flat by Proposition 2.38.2. Flat going down, [StacksAlg, Tag 00HS], shows that the minimal prime $\mathfrak{q}/\mathfrak{p}C_i$ contracts to the zero prime of the domain $A/\mathfrak{p}$: otherwise going down to zero would give a strictly smaller prime. Hence $\mathfrak{q} \cap A = \mathfrak{p}$. Choose a minimal prime $\mathfrak{q}_0 \subseteq \mathfrak{q}$ of C_i by [StacksAlg, Tag 00E0(4)]. Its contraction is a prime contained in the minimal prime $\mathfrak{p}$, and so equals $\mathfrak{p}$. Thus $\mathfrak{p}C_i \subseteq \mathfrak{q}_0$. Minimality over $\mathfrak{p}C_i$ gives $\mathfrak{q}_0 = \mathfrak{q}$. The minimal-prime quotient of D_i in question is therefore the minimal-prime quotient $C_i/\mathfrak{q}$, whose analytic dimension is d by the hypothesis on C_i.

A nonempty affinoid whose minimal-prime quotients all have dimension d itself has dimension d. Indeed, each point kernel contains a minimal prime by [StacksAlg, Tag 00E0(4)]. Passage to that quotient does not change its evaluated fraction field or its completed residue field. The maximum formula of Theorem 3.22.7 consequently identifies the dimension with the maximum of those component dimensions. Applying the domain-cover formula 3.22.8 to the displayed cover now gives

$$\dim_K \mathcal{M}(A/\mathfrak{p}) = d.$$

This holds for every minimal prime of A, proving purity. All dimensions here are analytic; no generalized Krull dimension comparison is used.

For the last assertion let V be a nonempty affinoid domain of the good Hausdorff space. Its compact underlying space is covered by finitely many pure affinoid neighbourhoods W_j of points of X. By Lemma 5.50.1, the sets $V \cap W_j$ are affinoid domains of W_j. Their nonempty members are pure of dimension d by Theorem 5.35.3. They cover V, so the implication just proved applies to V. □

Lemma 5.50.4 (Finite module data on an analytic cover)**.** *Let (V_i) be a finite affinoid-domain cover of a good Hausdorff analytic space X. Put $A_i = \mathcal{O}(V_i)$ and $A_{ij} = \mathcal{O}(V_i \cap V_j)$. Finite A_i-modules M_i, with isomorphisms*

$$A_{ij} \otimes_{A_i} M_i \simeq A_{ij} \otimes_{A_j} M_j$$

satisfying the identity and cocycle conditions, determine a coherent $\mathcal{O}_{X_G}$-module $\mathcal{F}$, uniquely up to the isomorphism inducing the given identifications. Conversely every coherent module gives such data. The construction is compatible with morphisms, finite affinoid refinements and short exact sequences. For every affinoid domain $W \subseteq V_i$,

$$\Gamma(W, \mathcal{F}) = \mathcal{O}(W) \otimes_{A_i} M_i$$

with its canonical finite-module norm.

Proof. All finite intersections are affinoid by Lemma 5.50.1. Theorem 2.39.6 associates to M_i a coherent sheaf $\widetilde{M_i}$ on $(V_i)_G$ and identifies its restrictions with the displayed finite tensors. It turns the given module isomorphisms into sheaf isomorphisms on the intersections, with the same cocycle condition.

For any analytic domain $T \subseteq X$, define $\mathcal{F}(T)$ to be the families of sections of $\widetilde{M_i}$ on $T \cap V_i$ whose restrictions agree under these isomorphisms. The domain and covering calculus of Proposition 2.40.5 shows that an admissible cover of T restricts to one of each $T \cap V_i$. Existence and uniqueness of glued sections for the sheaves $\widetilde{M_i}$ therefore prove the sheaf axiom for these families. On a domain contained in V_i, its i-th section determines all the others by restriction and the transition isomorphisms. The cocycle condition makes them compatible. Thus $\mathcal{F}|_{(V_i)_G}$ is $\widetilde{M_i}$; in particular it is coherent. Indeed, finite generation and finite generation of the kernels of maps from finite free sheaves are both checked on this admissible cover, where $\widetilde{M_i}$ is coherent. The same description proves uniqueness and glues compatible module morphisms.

Conversely, sections of a coherent sheaf on each V_i are finite modules by Theorem 2.39.6. That theorem gives both their restriction isomorphisms and the displayed formula on W. Theorem 1.18.3 gives the canonical norms and boundedness of all these module maps, while Proposition 1.18.7 identifies the finite ordinary and completed tensors. The two constructions are inverse on every chart and hence inverse globally by the sheaf axiom. Refinements simply restrict these same sheaves. Finally exactness is local on the cover, and the affinoid equivalence in Theorem 2.39.6 is exact. This proves the assertion for short exact sequences. □

Definition 5.50.5 (Moment-chart data on a compact good space)**.** Let X be compact, good and pure of analytic dimension d. Finite moment-chart data of bidegree (p,q) consist of a finite affinoid-domain cover (V_i), a finite tuple of analytic units f_i on each V_i, and a smooth real (p,q)-superform α_i on a neighbourhood of $t_{f_i}(V_i)$. On each intersection the two forms have equal tangential pullbacks to the actual joint moment image. Data are identified when they agree on a common finite affinoid refinement, allowing adjoining unit coordinates and equality of tangential pullbacks. These are the conventions of Definition 5.49.3, with X no longer required to be affinoid. The same definition is used on a compact analytic domain Q that is a finite union of affinoid domains in an ambient good Hausdorff space pure of dimension d. Its presenting domains are affinoid domains of Q; Q itself may fail to be good.

Theorem 5.50.6 (The integral on a compact good analytic space)**.** *Let X be a compact good K-analytic space pure of analytic dimension d, let $\mathcal{F}$ be a coherent $\mathcal{O}_{X_G}$-module, and let α be finite moment-chart data of bidegree (d,d). For a presenting cover $(V_i)_{i=1}^{s}$ put*

$$V_I = \bigcap_{i\in I} V_i, \qquad M_I = \Gamma(V_I, \mathcal{F}).$$

For nonempty I, choose any $i \in I$. Then

$$\mathcal{I}_{X,\mathcal{F}}(\alpha) = \sum_{\varnothing \neq I \subseteq \{1,\dots,s\}} (-1)^{|I|+1} \mathcal{I}_{M_I, f_i|_{V_I}}(\alpha_i) \tag{5.50.1}$$

is independent of these choices and of the representative of the finite chart data. Empty intersections contribute zero. The integral is linear in α, additive in short exact sequences of coherent modules, and nonnegative if the chart

representatives have nonnegative tangential top coefficients. It may equivalently be constructed from compatible finite module data as in Lemma 5.50.4. For affinoid X it agrees with Theorem 5.49.4.

More generally, let Y be good, Hausdorff and pure of analytic dimension d, and let $Q \subseteq Y$ be the union of finitely many affinoid domains of Y. The same construction and all these conclusions hold for coherent modules and finite moment-chart data on the compact analytic domain Q, including when Q is not good.

Proof. The intersections V_I are actual affinoid domains by Lemma 5.50.1, and their nonempty members are pure of dimension d by Lemma 5.50.3. Lemma 5.50.4 makes M_I a finite module, identified with the restriction of any M_i for $i \in I$. Thus every nonzero term is given by the finite-cell functional on the actual affinoid. Compatibility of the chart forms restricts to every further domain by Theorem 5.49.2. On V_I, adjoining two chosen tuples identifies their integrals with the integrals of their joint pullbacks, by the same-affinoid coordinate augmentation. The pullbacks have the same tangential top coefficients. Their cell integrals agree, proving independence of i.

Consider two compatible presentations on finite covers (V_i) and (W_j). On each V_I the domains $V_I \cap W_j$ form a finite affinoid cover. Apply Theorem 5.48.1 to the chosen single-chart representative on V_I and the module $\Gamma(V_I, \mathcal{F})$. Inserting its formula in (5.50.1) gives the finite double sum

$$\sum_{I \neq \varnothing} \sum_{J \neq \varnothing} (-1)^{|I|+|J|} \mathcal{I}_{\mathcal{F}|_{V_I \cap W_J}}(\alpha|_{V_I \cap W_J}).$$

The modules are the same restrictions of $\mathcal{F}$, and the two local form representatives have equal integrals by the joint-coordinate argument just given. Interchange the finite sums and apply the cover formula on each W_J. This recovers the expression for the second presentation. If two data sets agree only after a common refinement, compare each with that refinement. Lemma 5.50.1 provides the needed finite intersections. Restriction preserves tangential equality by Theorem 5.49.2, and the same-affinoid coordinate augmentation preserves the integral. This proves all the asserted independences.

Linearity follows after passing to a common cover and adjoining the two tuples on each member, where addition is ordinary addition of their pulled-back superforms. For a short exact sequence of coherent modules, the affinoid

exact equivalence 2.39.6 makes its section sequence on every V_I exact. Additivity of the affinoid global chains in the finite-module chain and weights therefore proves additivity of the displayed finite sum.

For positivity, order the cover and use the largest index j of I as the representative in each term. The j-th group is the integral against α_j of the finite d-chain

$$C_j = \sum_{J \subseteq \{1,\dots,j-1\}} (-1)^{|J|} \mathcal{T}_{f_j|_{V_j \cap V_J}} \big(\Gamma(V_j \cap V_J, \mathcal{F})\big),$$

where $V_j \cap V_\varnothing$ means V_j. All moment tuples in this formula are restrictions of the single tuple f_j on the affinoid V_j. By Theorem 5.39.3 and the common finite subdivision lemma, choose a finite common subdivision of these actual images and their boundary images. The boundary images have dimension at most $d-1$, and are empty for $d = 0$.

At the relative interior of any d-cell, its whole fibre in V_j and all its parts in $V_j \cap V_i$ are interior. Near that point the actual image lies in the cell's saturated d-plane. Apply Theorem 5.47.2 on V_j to the family consisting of V_j itself and the $V_j \cap V_i$, $i < j$. The member V_j ensures the cover hypothesis. Write D for its whole-fibre algebra, $e_i \in D$ for the overlap idempotents, E_Q for a transverse Gauss algebra, and c_Q for the common positive lattice index. The coefficient of C_j on this cell is

$$\frac{1}{c_Q} \dim_{H_Q} \left(\prod_{i<j} (1 - e_i) \right) \big(M_j \otimes_{\mathcal{O}(V_j)} E_Q \big) \geq 0.$$

Indeed, expansion of the product gives exactly the alternating sum of the idempotent-restricted module dimensions in C_j. This retains all nilpotent lengths. The coefficient is already an integer, being an alternating sum of the integer module weights. Thus C_j is nonnegative on every d-cell. If its images have dimension less than d, the chain is zero and the assertion is immediate. The nonnegative cell integral on each cell makes its integral against α_j nonnegative. Summing over j proves positivity.

The coherent-module formulation and compatible-module formulation coincide by Lemma 5.50.4. When X itself is affinoid, its defining expression is the one in Theorem 5.49.4, so the affinoid comparison follows.

Finally consider Q in the second assertion. Each of its finitely many affinoid pieces is compact and closed in the Hausdorff space Y. Their finite cover

is a quasi-net by Lemma 2.39.2, so the analytic-domain definition 2.40.4 gives Q its actual analytic-domain structure. It is compact. By the transitivity in Proposition 2.40.5, every affinoid domain of Q is also an affinoid domain of Y. Its finite intersections are therefore affinoid by Lemma 5.50.1 applied to Y, and every nonempty one is pure of dimension d by Lemma 5.50.3 applied to Y.

A coherent sheaf on Q_G restricts to finite modules on these affinoids by Theorem 2.39.6. Conversely, compatible finite modules glue on Q_G by the same section-family construction as in Lemma 5.50.4: for each analytic $T \subseteq Q$, take compatible sections on $T \cap V_i$. The finite cover is admissible, so the sheaf axiom and coherence of its chart restrictions give the coherent glued sheaf, with the stated affinoid restriction tensors. Thus all modules, intersections and finite cover comparisons in the preceding proof are available on Q. Each invocation of the affinoid cover formula occurs on a single affinoid intersection, and the complementary-idempotent positivity calculation occurs on a single affinoid V_j. Neither uses goodness of Q. Those arguments prove precisely the same independence, exact-sequence additivity and positivity on Q, as claimed. □

5.51 Finite pushforward for compact chart integrals

Theorem 5.51.1 (Finite pushforward on compact analytic domains)**.** *Let $p : Y \to X$ be a norm-admissible finite morphism of good Hausdorff analytic spaces pure of the same analytic dimension d, over a complete field with its specified real-valued absolute value. Let $Q \subseteq X$ be a finite union of affinoid domains, and put $R = p^{-1}(Q)$. Let $\mathcal{F}$ be coherent on Y_G. Then R is a finite union of affinoid domains of Y, and both Q and R are compact analytic domains. Neither is required to be good.*

Pulling back the unit tuples defines a pullback of finite moment-chart data on Q to such data on R. For every top-degree datum α,

$$\mathcal{I}_{R,\mathcal{F}|_R}(p^*\alpha) = \mathcal{I}_{Q,(p_*\mathcal{F})|_Q}(\alpha). \tag{5.51.1}$$

Successive applications agree for composable finite morphisms between spaces of the same pure analytic dimension. If $(p_*\mathcal{F})|_Q$ is locally free of constant rank ρ, then

$$\mathcal{I}_{R,\mathcal{F}|_R}(p^*\alpha) = \rho\,\mathcal{I}_{Q,\mathcal{O}_Q}(\alpha).$$

All conclusions allow trivial valuation, arbitrary positive affinoid radii, and nonreduced spaces and modules.

Proof. Choose a finite affinoid-domain cover (V_i) of Q. Each V_i is an affinoid domain of X by the domain transitivity of Proposition 2.40.5. Finiteness of p makes $W_i = p^{-1}(V_i)$ an actual affinoid domain of Y, whose algebra is a finite canonical module over $\mathcal{O}(V_i)$. These domains cover R. The finite-union assertion of Theorem 5.50.6 therefore gives the indicated compact analytic-domain structures. By Lemma 5.50.1, every finite intersection V_I is affinoid, and

$$W_I = \bigcap_{i \in I} W_i = p^{-1}(V_I)$$

is its full finite affinoid inverse image. Every nonempty such domain is pure of dimension d by Lemma 5.50.3.

Suppose that (f_i, α_i) presents the given chart data on the V_i. Its proposed pullback has tuple $p^* f_i$ and the same ambient superform α_i on W_i. Units remain units. On $W_i \cap W_j$ the actual joint moment image is contained in the joint moment image on $V_i \cap V_j$. Both are finite polyhedral sets by Theorem 5.39.3. Refine their complexes by their finitely many defining hyperplanes, using Lemma 5.14.2. Each cell of the source image lies in a cell of the target image, and its tangent space is a subspace of that cell's tangent space. Tangential equality on the target consequently restricts to tangential equality on the source. This proves compatibility of the pulled-back data in every bidegree. The same argument applies to a common refinement of two representatives. Inverse images of its domains are actual finite affinoid domains, and pullback commutes with adjoining units. Thus the pullback depends only on the equivalence class of the data.

Proposition 2.35.15 makes $p_* \mathcal{F}$ coherent and identifies, for each nonempty I,

$$\Gamma(V_I, p_* \mathcal{F}) = \Gamma(W_I, \mathcal{F}).$$

Write $A_I = \mathcal{O}(V_I)$, $B_I = \mathcal{O}(W_I)$ and $M_I = \Gamma(W_I, \mathcal{F})$. The module M_I is finite over B_I by Theorem 2.39.6, and finite over A_I because B_I is finite over A_I. The cited direct-image theorem gives precisely this restriction of scalars, with its compatible canonical norms and domain restrictions.

Assume first that W_I is nonempty and put $f = f_i|_{V_I}$. The two tuples are f and its literal pullback $p^* f$ on the full inverse domain. The character identity therefore holds with the identity map on the moment lattice.

By Theorem 5.39.3, both actual moment images are finite polyhedral sets. The absolute boundary image of V_I has dimension at most $d-1$ and is empty if $d=0$. Use Lemma 5.14.2 to subdivide the two images and that boundary image together. At a relative-interior point w of a target d-cell, choose a small convex open neighbourhood in moment space disjoint from the boundary image and the other cells. The target image is presented there by the given cell. If the source image contains w, it contains that cell and has the same local presentation. The whole target fibre is absolutely interior there. Finiteness gives relative interior everywhere on the actual source affinoid by Theorem 2.33.18, so Proposition 2.33.21 gives absolute interior of the whole source fibre there. With these pure local presentations, Theorem 5.42.2 compares the same finite module M_I over B_I and A_I. The index on the common saturated tangent lattice is one.

If the source image misses w, its compactness lets us avoid it on a neighbourhood of w. For the entire target simultaneous level algebra D_w, the inverse level algebra $B_I \otimes_{A_I} D_w$ is the actual finite pullback by Theorems 2.33.19 and 2.36.9. It is zero by Theorem 1.20.1, since its spectrum is empty. Thus $M_I \otimes_{A_I} D_w = 0$, and further tensoring with the transverse Gauss field gives zero. The target module weight is zero by Theorem 5.40.2, in agreement with the empty source coefficient.

These generic coefficients determine the closed finite-module chains on the common subdivision. Pairing the same ambient form with these equal chains by the finite-cell integral gives

$$\mathcal{I}_{M_I,p^*f_i|_{W_I}}(\alpha_i) = \mathcal{I}_{M_I,f_i|_{V_I}}(\alpha_i),$$

where the right side regards M_I as an A_I-module. An empty inverse image has zero module on both sides. Multiply by $(-1)^{|I|+1}$ and sum over the nonempty index sets. The two finite sums are exactly the compact chart integrals in Theorem 5.50.6, including its non-good-domain variant. This proves (5.51.1).

For a further finite map $q : Z \to Y$, the inverse image of every W_i is again the full finite affinoid inverse image. Apply the same equality first to q and then to p. Pullback of unit functions is associative, and

$$p_*(q_*\mathcal{G}) = (p \circ q)_*\mathcal{G}$$

on sections of every analytic domain, since both sides take sections on its full inverse image. These identities prove the stated compatibility for successive applications.

Finally suppose the indicated direct image is locally free of constant rank ρ on Q_G. Refine its local trivializing cover by affinoid domains and intersect them with the presenting chart domains. The quasi-net and compactness assertions of Lemma 2.39.2 give a finite such refinement. The compact integral is unchanged by this refinement. On every new V_I its finite module N_I is free of rank ρ. At each minimal prime $\mathfrak{p}$ of A_I, $\operatorname{length}((N_I)_{\mathfrak{p}}) = \rho \operatorname{length}((A_I)_{\mathfrak{p}})$. The exact module-weight formula of Theorem 5.40.2 therefore multiplies every top-dimensional global-chain coefficient by ρ. Lower-dimensional cells give no top-dimensional term. Applying the finite-cell integral to the scaled finite-module chain gives

$$\mathcal{I}_{N_I,f_i}(\alpha_i) = \rho\, \mathcal{I}_{A_I,f_i}(\alpha_i).$$

The same finite cover sum proves the last formula. In particular the argument preserves all generic Artinian lengths before multiplying by ρ. □

5.52 Smooth moment germs and compactly supported integration

The complete field and its specified absolute value remain fixed. We use the real superform conventions of Definition 5.46.1.

Lemma 5.52.1 (Tangential equality on a finite affinoid cover)**.** *Let V be a generalized affinoid space, let f be a finite tuple of analytic units, and let $V = \bigcup_{i=1}^{s} V_i$ be a finite affinoid-domain cover. A smooth ambient superform η near $t_f(V)$ has zero tangential restriction on every face of $t_f(V)$ if and only if the same holds on each $t_f(V_i)$.*

Tangential equality on a finite polyhedral set is preserved by d', d'', J, and multiplication by another superform. These assertions hold in every bidegree and in every dimension.

Proof. All the actual images are finite polyhedral by Theorem 5.39.3, and $t_f(V) = \bigcup_i t_f(V_i)$ as point sets. Cut their complexes by the finitely many supporting hyperplanes of all their cells, as in Lemma 5.14.2. This makes every subimage a subcomplex of the resulting complex on the union. Each cell of that union is contained in one subimage: a point of its relative interior belongs to a cell of one subcomplex, and the common face property then puts the entire cell in that subcomplex. Thus vanishing on the subimages

implies vanishing on every cell of the common refinement. Independence of subdivision, proved in Theorem 5.49.2, gives vanishing on the original image. The converse is restriction to a polyhedral subset, as in that theorem.

Fix a cell and choose affine coordinates on its span. Restriction of both cotangent factors commutes with differentiation along that span. If a tangential coefficient vanishes on the relative interior, all its derivatives in tangent directions vanish there. Their continuity extends the assertion to its faces. This proves the claims for d' and d''. The operations J and wedge product commute directly with restriction of alternating covectors. Applying these observations to the difference of two forms proves preservation of tangential equality. □

Definition 5.52.2 (Ordinary smooth moment germs)**.** Let X be a good Hausdorff analytic space and let $x \in X$. A smooth moment representative of bidegree (p, q) at x is a triple (V, f, α), where V is an ordinary affinoid neighbourhood of x, f is a finite tuple of analytic units on V, and α is a smooth real (p, q)-superform on an open neighbourhood of $t_f(V)$. Two representatives are equivalent if, on some ordinary affinoid neighbourhood $W \subseteq V \cap V'$ of x, their pullbacks to the actual joint image for (f, f') have equal tangential restrictions on every face.

For an ordinary open set $U \subseteq X$, write $\mathcal{A}_X^{p,q}(U)$ for the families of such germs at the points of U which are locally represented by one smooth moment representative. Here locally means on the ordinary interior of an affinoid neighbourhood contained in U. The support of a section is the complement of the open set on which all its germs are zero. A top-degree section on a pure-dimensional space is positive if locally it has representatives with nonnegative tangential top coefficients.

On a finite union of affinoid domains, finite moment-chart data mean a finite affinoid cover with unit tuples and smooth ambient superforms whose joint tangential pullbacks agree on every overlap, identified under common finite refinement. This uses the same presentation convention as Definition 5.49.3 in arbitrary dimension; purity is imposed when integrating top-degree forms.

Theorem 5.52.3 (The sheaf of ordinary smooth moment forms)**.** *The relation in Definition 5.52.2 is an equivalence relation. The spaces $\mathcal{A}_X^{p,q}$ form sheaves on the ordinary topology, with well-defined wedge product, J, d' and d''. These operations satisfy the usual exterior-algebra rules,*

$$(d')^2 = (d'')^2 = 0, \qquad d'd'' + d''d' = 0.$$

Every analytic morphism between good Hausdorff spaces over the specified field induces a functorial pullback on these sheaves, commuting with all these operations.

For any affinoid domain $V \subseteq U$, a section on U determines finite moment-chart data on V, canonically up to common refinement and tangential equality. This assignment commutes with further affinoid restriction and all the displayed operations. If the section has zero germ at every point of V, the induced finite chart data are zero. The same assertions hold on a finite union of affinoid domains contained in U.

Proof. We first record the neighbourhood choices used below. In an affinoid spectrum its evaluation topology has a basis of finitely many strict bounds on absolute values. At a point satisfying such bounds, replace each upper bound by a slightly smaller positive real bound still strict at the point, and each positive lower bound by a slightly larger one still strict there. Their weak inequalities define an actual affinoid domain by Theorem 2.31.2 and Corollary 2.36.10. It contains an ordinary neighbourhood of the point and is contained in the prescribed open set. Goodness and domain transitivity 2.40.5 therefore give a basis of ordinary affinoid neighbourhoods in X. Intersections of two such neighbourhoods are affinoid by Lemma 5.50.1 and remain ordinary neighbourhoods of the common point. No choice of a ground-field element with a prescribed real absolute value is needed for these inequalities.

Reflexivity and symmetry of the relation are immediate. For transitivity, restrict three representatives to a common smaller ordinary affinoid neighbourhood and adjoin all three tuples. The two given equalities pull back to that joint image and imply equality of the first and third forms there. Faithfulness under adding coordinates, Theorem 5.49.2, descends this equality to their two-tuple joint image. This proves transitivity. The same argument shows that one may shrink a representative or add unit coordinates without changing its germ.

Sums and wedge products are computed on a common smaller neighbourhood with the joint tuple. The preceding faithfulness and transitivity make

the result independent of the representatives. Apply J, d' and d'' to the ambient forms. Lemma 5.52.1 shows that these operations preserve all the germ relations, so they descend. Their algebraic identities can be checked on the coefficients and constant one-form generators of a representative. Equality of mixed partial derivatives follows from the real calculus proved in Lemma 5.46.2: the two iterated fundamental-theorem expressions for a rectangular second difference are equal; dividing by the two side lengths and letting them tend to zero identifies the continuous mixed derivatives. With the anticommuting one-form generators, the stated differential identities and the graded product rules follow.

The sheaf axiom is pointwise. Families agreeing on overlaps give one family of germs, and its local representatives are the given local representatives. This proves both existence and uniqueness of gluing. All the operations commute with restriction, as their representative constructions do.

Let $h : Y \to X$ be such a morphism and let $y \in Y$. Choose a representative (V, f, α) at $h(y)$ and an ordinary affinoid neighbourhood W of y mapping into V. Continuity of h and the affinoid basis give W. The analytic-domain universal property 2.40.8 makes the restriction an actual morphism $W \to V$. Use (W, h^*f, α) as the pulled-back representative. Its actual moment image is contained in $t_f(V)$. Both sets are finite polyhedral by Theorem 5.39.3. Tangential vanishing restricts to any finite polyhedral subset: a common subdivision puts each of its cells inside a cell of the larger set, where restriction of alternating covectors preserves zero. Thus an equality on a smaller target germ gives an equality on a smaller source germ. This proves well-defined pullback. Composition is checked on the pulled-back unit functions, and the ambient form is unchanged in these coordinates. Pullback consequently commutes with the displayed operations and is functorial.

Now let $V \subseteq U$ be affinoid and let α be a section on U. At every point of V choose a representing neighbourhood W and then an affinoid neighbourhood W' contained in the ordinary interior of W. The preceding basis construction permits this choice. Finitely many of the ordinary interiors of these W' cover the compact set V, by Proposition 2.24.1. The intersections $V \cap W'$ are affinoid domains by Lemma 5.50.1. Restrict the chosen tuples and ambient forms to them.

These representatives agree on every entire overlap. Indeed, each point of an overlap lies in the ordinary interiors of the two original representing neighbourhoods W, so equality of its germs supplies a smaller ordinary affi-

noid neighbourhood on which the joint forms agree. Finitely many such neighbourhoods cover the compact overlap by their interiors. Intersect them with the overlap and apply Lemma 5.52.1 to the fixed joint tuple there. It upgrades the local equalities to tangential equality on the whole joint image of the overlap. Thus the restricted representatives are finite moment-chart data on V.

The same argument applied to two choices shows that they agree on the finite common refinement by all pairwise intersections. It proves independence and compatibility with further restriction. The operations are computed on these same refinements, so commute with the assignment. If every germ on V is zero, apply the overlap argument to each presenting chart and the zero representative. A finite cover suffices by compactness, and Lemma 5.52.1 makes each entire chart representative tangentially zero.

For a finite union of affinoid domains, perform the construction on its finitely many members. The same compact-overlap argument proves compatibility and independence on their union. This also proves the zero-germ assertion there. Its analytic-domain structure is supplied by the finite closed-cover quasi-net in Lemma 2.39.2 and the analytic-domain definition 2.40.4. □

Theorem 5.52.4 (Integration of compactly supported smooth moment forms)**.** *Let X be good, Hausdorff and pure of analytic dimension d, and let $\mathcal{F}$ be a coherent $\mathcal{O}_{X_G}$-module. There is a canonical linear functional*

$$\int_{X,\mathcal{F}} : \mathcal{A}^{d,d}_{X,c}(X) \longrightarrow \mathbb{R}$$

on sections of compact support. It is nonnegative on positive sections and additive in short exact sequences of coherent modules.

Choose finitely many affinoid domains whose ordinary interiors cover the support, and let Q be their union. The functional is the finite-chart integral of the induced data on Q and $\mathcal{F}|_{Q_G}$ from Theorem 5.50.6. It is independent of this compact neighbourhood and of all the presenting charts. It agrees with the earlier affinoid and compact-space integrals whenever those are applied to the induced data.

Proof. Put $S = \operatorname{supp}(\alpha)$. The zero section has integral zero, so first suppose S is nonempty. The affinoid-neighbourhood basis in the proof of Theorem 5.52.3 and compactness of S give the required finite family. Its union Q is a compact analytic domain by the finite-union assertion in Theorem 5.50.6.

Its ordinary interior in X contains S. Theorem 5.52.3 supplies canonical finite chart data on Q. The restriction of a coherent sheaf is coherent, as is seen from its finite-module presentations on affinoid domains and Theorem 2.39.6. The compact-domain assertion of Theorem 5.50.6 therefore defines a real number, even when Q is not good.

We prove independence of Q. Let Q_1, Q_2 be two choices and put

$$O = \mathrm{Int}_{\mathrm{top}}(Q_1, X) \cap \mathrm{Int}_{\mathrm{top}}(Q_2, X), \qquad Z = X \setminus S.$$

Both sets are open, $S \subseteq O$, and $O \cup Z = X$. Cover the compact set $Q_1 \cup Q_2$ by the ordinary interiors of finitely many affinoid domains W_l, each contained either in O or in Z. Goodness and the affinoid basis justify this choice. Mark as central the members contained in O, and mark the remaining members as zero members. Every central member belongs to both Q_1 and Q_2.

Fix finite affinoid presentations $(V_i^{(a)})$ of Q_a, for $a = 1, 2$. A finite affinoid cover of Q_a is given by all the central W_l, together with the domains $V_i^{(a)} \cap W_l$ for the zero members. These are actual affinoid intersections by Lemma 5.50.1 applied in X, and they cover Q_a since the W_l cover $Q_1 \cup Q_2$. The section has zero germ at every point of a zero member. The zero-germ assertion of Theorem 5.52.3 makes its induced finite chart data zero on each zero piece and on every further intersection involving such a piece.

Use these covers to compute the two integrals by the finite inclusion-exclusion formula of Theorem 5.50.6. If a covering member needs several charts, first replace it by its finite presenting chart cover; the independence statement of that theorem allows this refinement. Every term involving a zero piece is zero. The surviving terms involve only the same central members, with the same restricted coherent module and the same section. Choose their finite presenting charts once for both covers. The two finite sums are then identical. This proves independence of the compact neighbourhood. The same argument with no central members proves zero when the support is empty.

Independence of the representatives on a fixed Q is Theorem 5.50.6. For two compactly supported sections, choose one finite affinoid neighbourhood of the union of their supports. Its finite-chart integral is linear, which proves linearity of the functional. If α is positive, compactness permits the finite presenting charts on Q to be chosen from positive local representatives and zero representatives outside its support. Positivity of that same theorem gives the claimed inequality. On every affinoid intersection an exact sequence of

coherent modules has an exact sequence of finite section modules by Theorem 2.39.6. Additivity of the compact-domain integral therefore proves the asserted coherent-module additivity.

The comparisons with the earlier integrals follow by using the same finite affinoid cover and the same inclusion-exclusion expression. In the compact case one may compare any support neighbourhood with X itself by the same central-and-zero cover argument. □

5.53 Smooth cutoffs on good analytic spaces

Lemma 5.53.1 (Compact smooth cutoffs)**.** *Let X be a good Hausdorff analytic space over the specified complete field. The sheaf $\mathcal{A}_X^{0,0}$ identifies with a subsheaf of continuous real-valued functions. For every point x and ordinary open neighbourhood $O \subseteq X$, there is a section $\chi \in \mathcal{A}_X^{0,0}(X)$ such that*

$$0 \le \chi \le 1, \qquad \chi = 1 \text{ on an ordinary neighbourhood of } x,$$
$$\operatorname{supp}(\chi) \text{ is compact in } O.$$

The assertion allows a trivially valued ground field.

Proof. A degree-zero representative evaluates to the continuous function $\alpha \circ t_f$. Two representatives have equal functions near a point exactly when their joint degree-zero form vanishes at every point of the actual joint image on a smaller ordinary neighbourhood. This is precisely the germ relation of Definition 5.52.2. Thus evaluation is injective on germs and identifies the indicated sheaf with a subsheaf of continuous functions. It is closed under composition with a smooth real function: in a representative, compose that function with the smooth ambient coefficient.

We use a smooth real profile ρ with values in $[0,1]$, equal to zero on $(-\infty, 0]$ and to one on $[1,\infty)$. Here is an elementary construction. The power series $e^u = \sum_{n\ge 0} u^n/n!$ and all its termwise derivatives converge uniformly on each bounded interval: after any fixed number of derivatives, their tails are bounded by a constant times a geometric series, once the index exceeds twice the interval bound. Applying the fundamental theorem from Lemma 5.46.2 to the partial sums proves termwise differentiation. The derivative of $e^u e^{-u}$ is zero, so this product equals its value 1 at zero. Hence $e^{-u} = 1/e^u$; positivity follows for $u \ge 0$ from the series and for $u < 0$ from this reciprocal identity. For $t > 0$ put $E(t) = e^{-1/t}$, and put $E(t) = 0$ for $t \le 0$. Each right derivative

is a polynomial in $1/t$ times $e^{-1/t}$. Every such expression tends to zero at zero, since $e^u \geq u^{m+1}/(m+1)!$ for $u > 0$. Induction with difference quotients therefore proves that E is smooth with every derivative zero at zero. The function

$$\rho(t) = \frac{E(t)}{E(t) + E(1-t)}$$

has the required properties; its denominator is positive.

Choose an ordinary affinoid neighbourhood $W = \mathcal{M}(A)$ of x. In its evaluation topology, choose finitely many strict upper and positive lower bounds on absolute values of functions $a_j \in A$ whose intersection contains x and is contained in $O \cap \mathrm{Int}_{\mathrm{top}}(W, X)$. The neighbourhood basis used in Theorem 5.52.3 permits this choice. For each upper condition $|a_j| < b_j$, choose

$$|a_j(x)| < b_{j,0} < b_{j,1} < b_j.$$

Use the factor

$$1 - \rho\left(\frac{\log|a_j| - \log b_{j,0}}{\log b_{j,1} - \log b_{j,0}}\right),$$

with value one where $a_j = 0$. For each lower condition $|a_j| > c_j > 0$, choose

$$c_j < c_{j,0} < c_{j,1} < |a_j(x)|$$

and use the factor

$$\rho\left(\frac{\log|a_j| - \log c_{j,0}}{\log c_{j,1} - \log c_{j,0}}\right),$$

with value zero where $a_j = 0$.

Each factor is a smooth moment section on W. At a zero of a_j it is constant on an ordinary neighbourhood, hence has a constant representative. At a point where a_j is nonzero, choose a smaller ordinary affinoid neighbourhood where $|a_j|$ is bounded below by a positive real constant. The unit criterion 1.20.1 makes its restriction an analytic unit. The displayed expression is then a smooth function of the moment coordinate $-\log|a_j|$. These representatives agree as continuous functions, and hence as germs by the first paragraph. The sheaf theorem 5.52.3 glues them.

Multiply the finitely many factors. The resulting χ_W lies between zero and one and equals one on an ordinary neighbourhood of x. Its support is contained in the subset $C \subseteq W$ defined by the weak upper bounds $|a_j| \leq b_{j,1}$ and weak lower bounds $|a_j| \geq c_{j,0}$. This subset is compact by Proposition 2.24.1,

is closed in the Hausdorff space X, and is contained in $O \cap \mathrm{Int}_{\mathrm{top}}(W, X)$ by the strictly nested choices of bounds. On the open cover

$$X = \mathrm{Int}_{\mathrm{top}}(W, X) \cup (X \setminus C)$$

glue χ_W on the first member and zero on the second. They agree on the overlap. The resulting section has all the stated properties. All thresholds were positive real numbers; the construction never required them to be absolute values of ground-field constants. □

Theorem 5.53.2 (Finite smooth partitions near a compact set)**.** *Let X be good and Hausdorff, let $S \subseteq X$ be compact, and let $O_1, \dots, O_m$ be ordinary open sets covering S. There are compactly supported sections $\chi_i \in \mathcal{A}_X^{0,0}(X)$ such that*

$$0 \le \chi_i \le 1, \qquad \mathrm{supp}(\chi_i) \subseteq O_i,$$
$$\sum_i \chi_i = 1 \quad \text{on an ordinary neighbourhood of } S.$$

If X is pure of analytic dimension d, $\mathcal{F}$ is coherent, and $\alpha \in \mathcal{A}_{X,c}^{d,d}(X)$ has support in S, then

$$\int_{X,\mathcal{F}} \alpha = \sum_i \int_{O_i, \mathcal{F}|_{(O_i)_G}} (\chi_i \alpha)|_{O_i}.$$

Each term is a compactly supported integral on the indicated open analytic subspace.

Proof. The case $S = \varnothing$ uses zero functions. Otherwise, apply Lemma 5.53.1 at each point of S, with one O_i containing it. Compactness gives finitely many resulting functions b_j whose loci where they equal one cover S by ordinary neighbourhoods. Group them by the chosen index and put

$$b_i' = \sum_{j \text{ assigned to } i} b_j, \qquad s = \sum_i b_i'.$$

Thus $s \ge 1$ on S and each b_i' has compact support contained in O_i. Choose a smooth real function η with $0 \le \eta \le 1$, equal to zero for $t \le 1/4$ and to one for $t \ge 1/2$, by the profile construction in that lemma. The function

$$h(t) = \begin{cases} \eta(t)/t, & t > 0, \\ 0, & t \le 0 \end{cases}$$

is smooth, since it is identically zero for $t \le 1/4$. Set $\chi_i = h(s)b_i'$. Smooth composition and the sheaf algebra of Theorem 5.52.3 make these global smooth moment sections. Since $0 \le b_i' \le s$, they take values in $[0,1]$, and their supports are contained in those of b_i'. Their sum is $\eta(s)$, which equals one on the open neighbourhood $\{s > 1/2\}$ of S. This proves the partition assertion.

The equality $\alpha = \sum_i \chi_i \alpha$ holds as sections: it holds on a neighbourhood of its support, and every germ of α off that support is zero. The support of $\chi_i\alpha$ is compact and lies in O_i. By the affinoid-neighbourhood basis, one may choose its finite compact affinoid neighbourhood entirely inside O_i. The same finite-chart formula then computes its integral in X and in O_i, by Theorem 5.52.4. Open subspaces are good by that same neighbourhood basis and Hausdorff by restriction; their nonempty affinoid charts have dimension d by Lemma 5.50.3. Thus all the indicated integrals exist, and linearity gives the displayed formula. □

5.54 Finite projection for compactly supported smooth forms

Theorem 5.54.1 (Finite projection for smooth moment forms)**.** *Let $p : Y \to X$ be a norm-admissible finite morphism of good Hausdorff analytic spaces pure of the same analytic dimension d, over the specified complete field. Let $\mathcal{F}$ be coherent on Y_G. For every $\alpha \in \mathcal{A}^{d,d}_{X,c}(X)$, the form $p^*\alpha$ has compact support and*

$$\int_{Y,\mathcal{F}} p^*\alpha = \int_{X,p_*\mathcal{F}} \alpha.$$

These identities are compatible with composition of finite morphisms. If $p_*\mathcal{F}$ is locally free of constant rank ρ on the analytic G-site over an ordinary open neighbourhood of $\operatorname{supp}(\alpha)$, then

$$\int_{Y,\mathcal{F}} p^*\alpha = \rho \int_{X,\mathcal{O}_X} \alpha.$$

The absolute value is unchanged, and the assertions allow trivial valuation and nonreduced spaces.

Proof. Put $S = \operatorname{supp}(\alpha)$. The ordinary smooth-germ pullback is defined by Theorem 5.52.3. If $y \notin p^{-1}(S)$, a representative of α vanishes on an ordinary

neighbourhood of $p(y)$. Its pullback vanishes near y, by continuity of p and the definition of pullback on representatives. Thus

$$\operatorname{supp}(p^*\alpha) \subseteq p^{-1}(S).$$

Choose finitely many affinoid domains $V_i \subseteq X$ whose ordinary interiors cover S, using the affinoid neighbourhood basis in Theorem 5.52.3. Put $Q = \bigcup_i V_i$. Finiteness of p makes $W_i = p^{-1}(V_i)$ actual affinoid domains, and $R = \bigcup_i W_i = p^{-1}(Q)$ is compact by Theorem 5.51.1. The set $p^{-1}(S)$ is closed in R: S is compact and therefore closed in the Hausdorff space X. It is consequently compact. The support of the pullback is closed and contained in this compact set, which proves the first assertion.

For every i, the open set $p^{-1}(\operatorname{Int}_{\mathrm{top}}(V_i, X))$ is contained in W_i, and therefore in $\operatorname{Int}_{\mathrm{top}}(W_i, Y)$. These interiors cover $p^{-1}(S)$ and hence the support of the pullback. Thus Q and R are valid compact neighbourhoods for the two integrals in Theorem 5.52.4; neither finite union needs to be good.

The coherent direct image $p_*\mathcal{F}$ exists by Proposition 2.35.15. The finite chart data induced on R by the ordinary section $p^*\alpha$ are the pullback of the data induced by α on Q. To check this, take the finite presenting affinoid cover supplied by Theorem 5.52.3. On its full inverse images both constructions use the same pulled-back analytic units and the same ambient superform. Their relations agree on every overlap by that theorem's restriction and pullback compatibility. If further affinoid restriction is needed, both sets of data are compared on that same finite common refinement.

Theorem 5.51.1 now gives

$$\mathcal{I}_{R,\mathcal{F}|_R}(p^*\alpha) = \mathcal{I}_{Q,(p_*\mathcal{F})|_Q}(\alpha).$$

By Theorem 5.52.4, these are precisely the two ordinary compact-support integrals in the statement.

For composable finite maps, functoriality of ordinary pullback is Theorem 5.52.3. Direct images compose on sections over full inverse images, as in Theorem 5.51.1. Applying the proved identity twice therefore gives the identity for the composite.

Finally, under the local rank hypothesis choose all V_i inside the indicated ordinary open neighbourhood of S. The scalar-rank assertion of Theorem 5.51.1 applies on Q. It multiplies the structure-module integral by ρ, with the generic Artinian lengths retained. Independence of compact neigh-

bourhoods in Theorem 5.52.4 identifies that structure-module integral with the one on X. □

5.55 Graded independence and pointed moment images

Let K be a complete non-Archimedean field with its specified real-valued absolute value, possibly trivial. For an analytic point x, write $F_x = \widetilde{\mathcal{H}(x)}^{\mathrm{gr}}$ and $k = \widetilde{K}^{\mathrm{gr}}$. The initial form of a nonzero $a \in \mathcal{H}(x)$ is denoted by $\widetilde{a}$; it is a homogeneous unit of degree $|a|$. Algebraic independence of homogeneous elements means ordinary polynomial independence over the underlying graded ring k. The kernel of polynomial evaluation is graded, so this is equivalent to the absence of nonzero homogeneous polynomial relations. For a graded field generated by finitely many such elements, trdeg_k means the transcendence degree of the fraction fields of the underlying graded domains, or equivalently the size of a maximal algebraically independent subtuple of the homogeneous generators.

Lemma 5.55.1 (Norm rigidity from a graded algebraic initial form)**.** *Let X be a good K-analytic space, let $x \in X$, and let f be an analytic unit on an ordinary neighbourhood of x. If $\widetilde{f(x)} \in F_x$ is algebraic over k, there is an ordinary neighbourhood U of x on which*

$$|f(y)| = |f(x)| \qquad (y \in U).$$

For a finite tuple of units the same assertion holds simultaneously when all its initial forms are algebraic over k. No interior or purity hypothesis is required.

Proof. Put $r = |f(x)| > 0$ and $u = \widetilde{f(x)}$. Take a nonzero homogeneous relation

$$\sum_{j \in J} \bar{c}_j u^j = 0$$

with $J \subset \mathbf{Z}_{\geq 0}$ finite, all $\bar{c}_j$ nonzero and homogeneous, and all the terms of the same degree $R > 0$. Such a relation exists because the kernel of evaluation in the graded polynomial ring is graded. It has at least two distinct exponents, since each $\bar{c}_j u^j$ is nonzero. Lift $\bar{c}_j$ to $c_j \in K$ with $|c_j| r^j = R$, and put $P(T) = \sum_{j \in J} c_j T^j$. Cancellation of the initial forms gives $|P(f(x))| < R$.

Choose c with $|P(f(x))| < c < R$. By continuity, after shrinking to an ordinary neighbourhood U of x,

$$|P(f(y))| < c.$$

Write $a = \min J$ and $b = \max J$; then $a < b$. For a positive real q near r, the sizes of the terms are

$$|c_j| q^j = R(q/r)^j.$$

If $q > r$, the unique largest size is the $j = b$ term; if $q < r$, it is the $j = a$ term. Choose $\varepsilon > 0$ so that the maximum of these finitely many sizes is strictly larger than c whenever $0 < |\log(q/r)| < \varepsilon$. Continuity of $|f|$ allows us to shrink U further to $|\log(|f(y)|/r)| < \varepsilon$. If $|f(y)| \neq r$, the non-Archimedean strong triangle inequality gives

$$|P(f(y))| = \max_{j \in J} |c_j| |f(y)|^j > c,$$

a contradiction. The finite-tuple assertion follows by intersecting the finitely many resulting neighbourhoods. □

Lemma 5.55.2 (Independent initial forms survive Gauss normalization)**.** *Let $V = \mathcal{M}(A)$ be a generalized K-affinoid space, $x \in V$, and let $f_1, \dots, f_s \in A^\times$ have algebraically independent initial forms in F_x over k. Put $r_i = |f_i(x)|$. Give $K(S_1, \dots, S_s)$ the weighted Gauss absolute value with $|S_i| = r_i$ and write L for its completion. There exists an analytic point $x_L \in V_L$ above x with*

$$\widetilde{f_1/S_1}(x_L), \dots, \widetilde{f_s/S_s}(x_L)$$

algebraically independent over $\widetilde{L}$. Moreover $x \in \mathrm{Int}(V/K)$ implies $x_L \in \mathrm{Int}(V_L/L)$.

Proof. Put $H = \mathcal{H}(x)$ and let E be the completion of $H(S_1, \dots, S_s)$ for the same weighted Gauss norm. The given norm on $K(S_1, \dots, S_s)$ is its restriction, so L embeds isometrically in E. Evaluation at x gives a bounded map $A \to H \subset E$. Together with $L \to E$ it induces a bounded map $A \widehat{\otimes}_K L \to E$ and hence a point x_L above x.

Write $u_i = \widetilde{f_i(x)}$ and $s_i = \widetilde{S}_i$. The defining Gauss norm says that the graded Laurent polynomial ring $F_x[s_1^{\pm 1}, \dots, s_s^{\pm 1}]$ embeds in $\widetilde{E}^{\mathrm{gr}}$: a nonzero finite sum has a maximal Gauss term and its initial form does not vanish. Here polynomial independence holds even when two weighted monomials have the

same numerical degree, because the Gauss norm does not cancel distinct monomials. The u_i are independent over k by assumption, while the s_i are independent over the entire F_x. Consequently the u_i/s_i are algebraically independent over the ordinary fraction field of $k[s_1^{\pm 1}, \dots, s_s^{\pm 1}]$: any polynomial relation, after clearing denominators and powers of the s_i, would be a polynomial relation in the u_i with coefficients in $k[s_1^{\pm 1}, \dots, s_s^{\pm 1}]$; independence of the u_i, followed by independence of the s_i, makes each coefficient zero. The degree-one residue field $\widetilde{L}$ is a subfield of this fraction field, as follows directly by reducing fractions for the weighted Gauss norm. Thus the displayed normalized initial forms are independent over $\widetilde{L}$. The inclusion of the graded residue field of $\mathcal{H}(x_L)$ in that of E preserves such independence.

Finally, base-change preservation of relative interior in Proposition 2.33.21, applied over the point $\mathcal{M}(K)$, gives the last assertion. The argument did not change the originally specified absolute value on K. □

Theorem 5.55.3 (Local openness for graded-independent moments)**.** *Let $V = \mathcal{M}(A)$ be a generalized K-affinoid space, let $x \in \mathrm{Int}(V/K)$, and let $f_1, \dots, f_s \in A^\times$ have algebraically independent homogeneous initial forms over k in F_x. Put*

$$t_f(y) = (-\log|f_1(y)|, \dots, -\log|f_s(y)|), \qquad w = t_f(x).$$

For every ordinary neighbourhood U of x in V, the set $t_f(U)$ contains an ordinary Euclidean neighbourhood of w in $\mathbf{R}^s$. In particular, for every ordinary affinoid neighbourhood W of x contained in V, its actual finite polyhedral moment image has dimension s. The assertion holds for the trivial valuation, arbitrary real radii and every analytic dimension of V.

Proof. For $s = 0$ the image is the one-point space $\mathbf{R}^0$, so there is nothing to prove. Assume $s > 0$. Use Lemma 5.55.2 and write $h_i = f_i/S_i$ on V_L. The point x_L is interior, every $|h_i(x_L)| = 1$, and the reductions of the h_i are algebraically independent over $\widetilde{L}$. Thus the actual residue algebra

$$\widetilde{L}[\widetilde{h}_1^{\pm 1}, \dots, \widetilde{h}_s^{\pm 1}] \simeq \widetilde{L}[T_1^{\pm 1}, \dots, T_s^{\pm 1}]$$

is a Laurent polynomial algebra. Its tropicalization over the trivially valued residue field is all of $\mathbf{R}^s$: for any real vector v, the weighted Gauss seminorm on its Laurent ring has moment v.

Let C be a nonempty open rational homogeneous polyhedral cone in $\mathbf{R}^s$. Choose a vector $v \in C$ and integer vectors $m_1, \dots, m_a$ such that the strict

inequalities $\langle m_j, z\rangle > 0$ define a nonempty smaller cone contained in C and contain v. The normalized sector argument applied to (V_L, x_L) and (h_i) says that every ordinary neighbourhood of x_L contains a point where $\langle m_j, t_h\rangle > 0$ for all j. For any $\delta > 0$ apply it to the neighbourhood

$$p^{-1}(U) \cap t_h^{-1}(B_\delta(0))$$

of x_L, where $p : V_L \to V$ is projection. The projected point belongs to U and its moment displacement $t_f(y) - w = t_h(y_L)$ belongs to $C \cap B_\delta(0)$.

Choose an ordinary affinoid neighbourhood $W \subseteq U$ of x by the affinoid-neighbourhood basis established in the proof of Theorem 5.52.3. Apply the previous paragraph with U replaced by the ordinary interior of W. Theorem 5.39.3 gives a finite integral affine polyhedral complex whose support is exactly $t_f(W)$. Because $w \in t_f(W)$, the cells containing w determine a finite union C_W of closed rational polyhedral tangent cones, and for some $\epsilon > 0$ one has

$$t_f(W) \cap B_\epsilon(w) = (w + C_W) \cap B_\epsilon(w).$$

If $C_W \neq \mathbf{R}^s$, its closed conical support has a complementary nonempty open rational homogeneous cone. The preceding sector lifting then produces a moment displacement in this cone arbitrarily close to zero, contradicting the displayed equality. Hence $C_W = \mathbf{R}^s$ and the displayed equality puts $B_\epsilon(w)$ inside $t_f(W) \subseteq t_f(U)$. The dimension assertion follows because the moment image is a subset of $\mathbf{R}^s$ containing a ball. □

Corollary 5.55.4 (The two extreme ranks of pointed tropical germs)**.** *Let V, x be as in Theorem 5.55.3, and let $f = (f_1, \dots, f_n)$ be any finite unit tuple. Put*

$$s = \operatorname{trdeg}_{\widetilde{K}^{\mathrm{gr}}} \widetilde{K}^{\mathrm{gr}}(\widetilde{f_1(x)}, \dots, \widetilde{f_n(x)}).$$

If $s = 0$, there is an ordinary neighbourhood U_0 of x on which t_f is constant. If $s = n$, for every ordinary neighbourhood U of x, the germ $(t_f(U), t_f(x))$ is the full Euclidean germ $(\mathbf{R}^n, t_f(x))$. More generally, if s of the initial forms are algebraically independent, the projection of $t_f(U)$ to those s coordinates contains an ordinary Euclidean neighbourhood of its base point.

Proof. When $s = 0$, each initial form is algebraic over $\widetilde{K}^{\mathrm{gr}}$, so apply Lemma 5.55.1 to the finite tuple. When $s = n$ the tuple itself satisfies Theorem 5.55.3. The projection assertion follows by applying the same theorem to the indicated subtuple. □

Lemma 5.55.5 (Finite tropical witnesses for homogeneous graded relations)**.** *Let k be a graded field, let $r_1, \dots, r_n > 0$, and let $u_1, \dots, u_n$ be nonzero homogeneous elements of degrees r_i in a graded field extension F/k. Put*

$$I = \ker\bigl(k[Z_1^{\pm 1}, \dots, Z_n^{\pm 1}] \longrightarrow F,\ Z_i \longmapsto u_i\bigr), \qquad s = \operatorname{trdeg}_k k(u_1, \dots, u_n).$$

The Laurent polynomial ring is graded by $\deg Z_i = r_i$, and I is homogeneous. Define $C \subset \mathbf{R}^n$ by requiring that for every nonzero homogeneous relation $P = \sum_m c_m Z^m \in I$ the minimum of $\langle m, q\rangle$ over its nonzero terms is attained at least twice. Then C is the support of a finite pure s-dimensional rational polyhedral fan. There are finitely many nonzero homogeneous relations $P_1, \dots, P_N \in I$ whose tropical hypersurfaces already cut out C. If $I = 0$, the list is empty and $s = n$.

Proof. Adjoin algebraically independent homogeneous variables S_i of degrees r_i and let ℓ be the graded fraction field of $k[S_1^{\pm 1}, \dots, S_n^{\pm 1}]$. The degree-one part ℓ_1 is an ordinary field. As the degrees r_i lie in the support of ℓ, replacing Z_i by $S_i H_i$ identifies the degree-one part of

$$(k[Z^{\pm 1}]/I) \otimes_k \ell$$

with a finitely generated ℓ_1-algebra

$$B = \ell_1[H_1^{\pm 1}, \dots, H_n^{\pm 1}]/J.$$

The tensor algebra injects into the graded fraction field of $F[S_1^{\pm 1}, \dots, S_n^{\pm 1}]$. Indeed polynomial independence of the S_i over F preserves the original kernel I after scalar extension, and localization at nonzero homogeneous elements of $k[S_i^{\pm 1}]$ remains injective. Thus B is a domain.

Choose a maximal algebraically independent subtuple of the u_i over k, with s members. The ratios of those members to their corresponding S_i are algebraically independent over ℓ_1: a relation would, after clearing denominators and powers of the S_i, contradict joint independence of the chosen u_i and all the S_i. Every remaining u_j is algebraic over the graded field generated by the chosen subtuple. Its nonzero homogeneous polynomial equation, after the substitutions $u_i = S_i H_i$ and division by a homogeneous unit of ℓ, is an ordinary polynomial equation for H_j over ℓ_1 and the chosen H_i. Consequently

$$\dim B = \operatorname{trdeg}_{\ell_1} \operatorname{Frac} B = s,$$

using the finite-type dimension theorem [StacksAlg, Tag 00P0], already an authorized leaf in the frozen foundation.

The ordinary field ℓ_1 is given the trivial absolute value. Apply the finite tropical basis Lemma 5.38.2 to J. Write $T = \mathrm{Trop}(\operatorname{Spec} B)$. It is a finite rational polyhedral fan; the algebraic tropical pure-dimension theorem 5.19.5, applied to the integral Laurent variety $\operatorname{Spec} B$, gives pure dimension s. A nonzero homogeneous original relation becomes an ordinary Laurent relation after normalization. Therefore $T \subseteq C$. The difficult reverse inclusion requires descending finite witnesses from J to I; we establish that inclusion next.

We now descend a *finite* tropical basis. Choose a finite basis $Q_1, \dots, Q_a \in J$ from the preceding lemma, and let $S \subset \mathbf{Z}^n$ be the finite union of their exponent supports. Put

$$M_S = \bigoplus_{m \in S} kZ^m, \qquad N_S = I \cap M_S.$$

These are graded k-modules, but no assertion about flatness of an infinite coefficient extension is needed. Write $s_i = S_i$ for these auxiliary graded variables.

Fix $q \notin T$. Some normalized Q_j has a unique least q-weight monomial, say with exponent $m_0 \in S$. Rewrite it in the variables $Z_i = S_i H_i$, obtaining

$$Q = \sum_{m \in S} a_m(s) Z^m \in \ell[Z^{\pm 1}],$$

where every nonzero $a_m(s)$ is homogeneous and Q has one common total graded degree. The coefficient at m_0 is nonzero, and all coefficients at other exponents of q-weight at most $\langle m_0, q \rangle$ are zero. Choose a common nonzero *homogeneous* denominator $d(s) \in k[s_1^{\pm 1}, \dots, s_n^{\pm 1}]$ of its finitely many coefficients. Then

$$d(s)Q = \sum_{a \in A} s^a P_a(Z), \qquad P_a \in M_S,$$

for one finite set $A \subset \mathbf{Z}^n$. Every nonzero P_a is homogeneous: in a fixed monomial $s^a Z^m$, the total degree of dQ determines the degree of its k-coefficient, and therefore the total degree of the corresponding terms of P_a.

The normalized relation is zero after substituting $Z_i = u_i$ in the graded polynomial ring $F[s_i^{\pm 1}]$. The s_i are algebraically independent over F, so comparison of their Laurent coefficients gives $P_a(u) = 0$ for every a; hence $P_a \in N_S$. The nonzero Laurent polynomial $d(s)a_{m_0}(s)$ has some nonzero coefficient, so one P_a has a nonzero m_0-coordinate. Since all other coordinates of q-weight at most $\langle m_0, q \rangle$ vanish before multiplication by d, they vanish

in every P_a. This chosen homogeneous P_a has the single initial monomial $c_{m_0} Z^{m_0}$. Thus $q \notin C$, proving $C \subseteq T$ and hence $C = T$.

It remains to replace the resulting weight-dependent choice by a finite list. A *circuit* of N_S is a nonzero homogeneous vector whose exponent support is minimal among supports of nonzero vectors of N_S. For each possible support there is at most one circuit up to multiplication by a nonzero homogeneous scalar: if two had that support, eliminate one coordinate by such a scalar, producing a nonzero vector of smaller support unless they were proportional. There are at most $2^{|S|} - 1$ supports, so there are finitely many circuits up to proportionality.

Given a homogeneous $P \in N_S$ with unique least term m_0, choose a homogeneous vector with nonzero m_0-coordinate, support contained in $\operatorname{supp} P$, and support minimal under those two conditions. If some nonzero vector had support properly contained in that support, either it has a nonzero m_0-coordinate, contradicting minimality, or it can be scaled to cancel a different coordinate without changing the m_0-coordinate, again contradicting minimality. The chosen vector is thus a circuit. Its support lies in $\operatorname{supp} P$ and contains m_0, so its initial at q is still a single monomial. Taking one representative of each circuit gives a finite family of homogeneous original relations, and a vector is outside C precisely when one of these has a single least term. This proves the assertion. □

Lemma 5.55.6 (The actual residual algebra after free Gauss normalization)**.** *Let $V = \mathcal{M}(A)$, $x \in V$, and $f_1, \dots, f_n \in A^\times$ be arbitrary analytic units. Put $r_i = |f_i(x)|$, $F_x = \widetilde{\mathcal{H}(x)}^{\mathrm{gr}}$, $k = \widetilde{K}^{\mathrm{gr}}$, and $u_i = \widetilde{f_i(x)}$. Let L be the completion of $K(S_1, \dots, S_n)$ for the weighted Gauss norm $|S_i| = r_i$, without assuming these radii are free over the value group of K. There is a point $x_L \in V_L$ above x for which the residue algebra*

$$\widetilde{L}[\widetilde{f_1/S_1}(x_L)^{\pm 1}, \dots, \widetilde{f_n/S_n}(x_L)^{\pm 1}]$$

is precisely the ordinary degree-one algebra B constructed in Lemma 5.55.5 from the graded relation ideal of the u_i. If $x \in \operatorname{Int}(V/K)$, then $x_L \in \operatorname{Int}(V_L/L)$.

Proof. Put $H = \mathcal{H}(x)$ and let E be the completion of $H(S_1, \dots, S_n)$ for the same weighted Gauss norm. The formula on finite sums gives an isometric embedding $L \hookrightarrow E$. It also proves the graded identities

$$\widetilde{L}^{\mathrm{gr}} = \operatorname{Frac}^{\mathrm{gr}}(k[s_1^{\pm 1}, \dots, s_n^{\pm 1}]), \quad \widetilde{E}^{\mathrm{gr}} = \operatorname{Frac}^{\mathrm{gr}}(F_x[s_1^{\pm 1}, \dots, s_n^{\pm 1}]),$$

where $s_i = \widetilde{S}_i$ and the fraction fields invert nonzero homogeneous elements. Indeed the maximum Gauss term of a nonzero polynomial has nonzero initial polynomial, distinct Laurent monomials do not cancel, and the initial form of a rational fraction is the quotient of its numerator and denominator initial forms. Passing to the completion adds no homogeneous initial values: approximate every nonzero element by a rational fraction with smaller error.

The bounded evaluation $A \to H$ and the isometric map $L \to E$ produce a bounded map $A\widehat{\otimes}_K L \to E$ and hence x_L above x. The completed residue field $\mathcal{H}(x_L)$ embeds isometrically into E. The graded injection

$$F_x \otimes_k \widetilde{L}^{\mathrm{gr}} \longrightarrow \widetilde{E}^{\mathrm{gr}}$$

comes from the algebraic independence of the s_i over F_x, followed by homogeneous localization. Consequently the kernel of evaluation of the normalized initials u_i/s_i over $\widetilde{L} = (\widetilde{L}^{\mathrm{gr}})_1$ is exactly the degree-one normalization of the scalar extension of the original homogeneous relation ideal I. By the definition of B in Lemma 5.55.5, this is the asserted ordinary residue algebra. The base-change interior inclusion in Proposition 2.33.21 gives the last assertion. □

Lemma 5.55.7 (A graded residual fan bounds a pointed moment germ)**.** *Let $V = \mathcal{M}(A)$ be a generalized K-affinoid space, $x \in V$, and let $f = (f_1, \dots, f_n)$ be a tuple of analytic units. Put $w = t_f(x)$, set $u_i = \widetilde{f_i(x)} \in F_x$, and let C_x be the graded tropical fan of Lemma 5.55.5. There are an ordinary affinoid neighbourhood W_0 of x and $\varepsilon > 0$ such that*

$$t_f(W_0) \cap B_\varepsilon(w) \subseteq (w + C_x) \cap B_\varepsilon(w).$$

This upper bound has pure fan dimension $s = \operatorname{trdeg}_k k(u_1, \dots, u_n)$ and requires no interior or purity hypothesis on V.

Proof. Choose the finite homogeneous relations $P_j = \sum_m \bar{c}_{jm} Z^m$ from Lemma 5.55.5, and lift each nonzero $\bar{c}_{jm}$ to $c_{jm} \in K$ with its exact absolute value. Because P_j is homogeneous, there is an $R_j > 0$ such that

$$|c_{jm}| \prod_i |f_i(x)|^{m_i} = R_j$$

for every monomial of P_j. Put $F_j = \sum_m c_{jm} f^m \in A$. The graded relation gives $|F_j(x)| < R_j$. Choose real b_j with $|F_j(x)| < b_j < R_j$. By continuity, all

the inequalities $|F_j(y)| < b_j$ hold on one ordinary affinoid neighbourhood W_0 of x. Shrink $\varepsilon > 0$ so that for every $v \in B_\varepsilon(w)$, every monomial of the finite list has size

$$|c_{jm}| \exp(-\langle m, v\rangle) > b_j.$$

If $v = t_f(y)$ does not belong to $w + C_x$, then $q = v - w \notin C_x$ and some P_j has one unique least q-weight exponent. All terms of $F_j(y)$ have sizes $R_j \exp(-\langle m, q\rangle)$; the unique least exponent is therefore the unique largest term, and the strong triangle inequality gives $|F_j(y)| > b_j$. This contradicts the choice of W_0. The fan and dimension assertions are precisely those of Lemma 5.55.5. □

Theorem 5.55.8 (Stabilization and purity of pointed tropical germs)**.** *Let $V = \mathcal{M}(A)$ be a generalized K-affinoid space, $x \in \mathrm{Int}(V/K)$, and $f = (f_1, \dots, f_n)$ a tuple of analytic units. Put $w = t_f(x)$ and*

$$s = \mathrm{trdeg}_{\widetilde{K}^{\mathrm{gr}}} \widetilde{K}^{\mathrm{gr}}(\widetilde{f_1(x)}, \dots, \widetilde{f_n(x)}).$$

There is a finite pure s-dimensional rational polyhedral fan $C_x \subseteq \mathbf{R}^n$ and an ordinary affinoid neighbourhood W_0 of x such that for every *ordinary neighbourhood U of x contained in W_0, there is $\varepsilon_U > 0$ with*

$$t_f(U) \cap B_{\varepsilon_U}(w) = (w + C_x) \cap B_{\varepsilon_U}(w).$$

Thus the pointed moment-image germ is independent of the choice of sufficiently small ordinary neighbourhood. Its dimension is exactly s, for every complete real-valued ground field, including the trivial valuation, with the original absolute value fixed. The statement allows nonreduced or nonpure A.

Proof. Let C_x and W_0 be supplied by Lemmas 5.55.5 and 5.55.7. The finite upper relation applies to every smaller neighbourhood $U \subseteq W_0$.

For the opposite inclusion, use Lemma 5.55.6 to make the simultaneous weighted Gauss extension with all n radii and its specified point x_L above x. That lemma identifies the ordinary residual Laurent algebra generated by the normalized initial forms $\widetilde{f_i/S_i}(x_L)$ with the algebra B of Lemma 5.55.5. Its tropicalization is therefore exactly C_x, and x_L is interior over L.

Fix a nonzero vector $q \in C_x$ and a strict rational open homogeneous sector S containing q. The normalized sector argument, applied to the normalized

tuple $h_i = f_i/S_i$ at x_L, supplies points with t_h in S in every ordinary neighbourhood of x_L. To require arbitrarily small displacements, intersect this neighbourhood with $t_h^{-1}(B_\delta(0))$ for any $\delta > 0$. For any prescribed ordinary neighbourhood U of x the inverse image of U under $V_L \to V$ is an ordinary neighbourhood of x_L. Projection then gives points $y \in U$ with $t_f(y) - w \in S$ and $\|t_f(y) - w\| < \delta$.

Choose an ordinary affinoid neighbourhood W of x contained in $U \cap W_0$. The actual image $t_f(W)$ is a finite integral affine polyhedral set by Theorem 5.39.3. In a sufficiently small Euclidean ball about w, it equals $w + D_W$ for the finite closed rational polyhedral tangent fan D_W formed by the cells containing w. By the upper inclusion, $D_W \subseteq C_x$. If some nonzero $q \in C_x$ were absent from D_W, the finite closed conical set D_W would have a strict rational open sector about q disjoint from it. The preceding residue-sector lifting into W would contradict the local equality with $w + D_W$. Since $0 \in D_W$, this proves $D_W = C_x$. Hence $t_f(W)$ agrees near w with $w + C_x$. It is contained in $t_f(U)$, while the upper inclusion for $U \subseteq W_0$ gives the reverse containment near w. Shrinking the Euclidean ball once more proves the displayed equality. The finite pure-dimensional fan calculation in Lemma 5.55.5 gives the dimension assertion. Every field extension was isometric and every projected moment used the original norm on K. □

5.56 Proper morphisms and top-dimensional moment pushforward

Throughout this section, K is a complete non-Archimedean field with its specified real-valued absolute value. The valuation may be trivial. All analytic spaces and completed residue fields use this absolute value. A finite morphism is norm-admissible finite. Compactness and properness have the meanings of Definition 2.53.1. For a unit tuple $f = (f_1, \dots, f_n)$ write $t_f = (-\log|f_1|, \dots, -\log|f_n|)$. The exact polytorus at $w \in \mathbb{R}^n$ is the affinoid domain of the analytic torus on which $|T_j| = e^{-w_j}$ for every j. Empty spaces have zero moment chain and zero integral.

Lemma 5.56.1 (A finite neighbourhood of the entire analytic fibre)**.** *Let $p : Z \to Y$ be a compact morphism of good Hausdorff analytic spaces. Suppose that $y \in Y$, that the underlying set of the analytic fibre Z_y is finite, and*

that every point above y belongs to $\mathrm{Int}(Z/Y)$*. There is an ordinary open neighbourhood N of y such that the full restriction*

$$p^{-1}(N) \longrightarrow N$$

is finite. One may require N to lie in any prescribed ordinary neighbourhood of y. Empty fibres are allowed.

Consequently, if p is proper, the locus of target points with finite analytic fibres is ordinary open, and the full restriction of p to that locus is finite.

Proof. Choose an affinoid neighbourhood V of y inside the prescribed target neighbourhood. It is compact by Proposition 2.24.1. Its inverse image Z_V is compact Hausdorff by compactness of p. The topological identification of Theorem 2.33.11, applied on charts, identifies the points of Z_y with the points of $p^{-1}(y)$.

Suppose first that this fibre consists of $z_1, \dots, z_r$. A finite subset of a Hausdorff space has disjoint ordinary open neighbourhoods of its points: choose separating neighbourhoods for every pair and take the finitely many intersections. Theorem 2.49.1 therefore supplies pairwise disjoint ordinary open neighbourhoods W_i of z_i, ordinary open neighbourhoods N_i of y, and finite maps $W_i \to N_i$. The same theorem permits the W_i to be contained in the chosen disjoint neighbourhoods and in $p^{-1}(\mathrm{Int}_{\mathrm{top}}(V, Y))$.

The set

$$C = Z_V \setminus \bigcup_{i=1}^{r} W_i$$

is compact. Its image is closed in the Hausdorff affinoid space V and does not contain y. Choose N inside $\mathrm{Int}_{\mathrm{top}}(V, Y) \cap \bigcap_i N_i$ and disjoint from $p(C)$. Then

$$p^{-1}(N) = \coprod_{i=1}^{r} \big(W_i \cap p^{-1}(N)\big).$$

Every map in this finite disjoint union is the base change of $W_i \to N_i$, hence is finite by Proposition 2.33.19 and the gluing of Theorem 2.40.8. A finite disjoint union of finite morphisms is finite: over an affinoid target its algebra is the finite product of the finite canonical algebras of the summands. The product norm is the canonical finite-module norm, and its coordinate idempotents identify its spectrum and sheaf with the indicated disjoint union. This proves finiteness of the full restriction.

If the fibre is empty, use $C = Z_V$. The same compact-image argument gives $p^{-1}(N) = \varnothing$, which is the finite space represented by the zero algebra.

For a proper morphism the interior condition holds everywhere. The neighbourhood just constructed consists of finite-fibre points, since finite analytic fibres are finite discrete spaces by Lemma 2.35.1. Thus the finite-fibre locus is open. On this open locus the restrictions are finite on an ordinary open cover of the target; the target-local definition of a finite analytic morphism glues them to the full finite restriction. □

Theorem 5.56.2 (The finite locus at maximal-dimensional analytic points)**.** *Let $p : X \to Y$ be a proper morphism of good Hausdorff analytic spaces, with $\dim_K X \leq d$. If $y \in Y$ satisfies $d_K(y) = d$, then its entire analytic fibre is finite, and p is finite over an ordinary neighbourhood of y. If Y is pure of analytic dimension d, the ordinary finite-fibre locus is open and dense in Y.*

Proof. We first verify the zero-dimensional fact needed for the actual fibre. A nonempty generalized affinoid space of analytic dimension zero has a finite underlying set. Indeed choose a strictifying complete extension by Theorem 1.17.6. Theorem 3.21.8 makes the strict algebra after this extension zero-dimensional. Finite normalization, Theorem 1.12.2, and Proposition 3.21.1 make it finite over the new ground field itself. Its actual strict norm is the canonical finite-module norm by Proposition 1.14.4. Its spectrum is finite by Lemma 2.35.1. The scalar projection onto the original spectrum is surjective by Theorem 2.33.13, so the original spectrum is finite too. This argument concerns analytic dimension, with Krull dimension used only after strictification.

Put $H = \mathcal{H}(y)$. To see that the actual analytic fibre X_y is good, choose at each z above y ordinary affinoid neighborhoods of y and z, the latter mapping into the former. The completed-residue fibre of the resulting affinoid map is affinoid by Theorem 2.33.11. It contains the intersection with the ordinary neighborhood of z and thus is an ordinary affinoid neighborhood in the whole fibre. Its underlying topology is that of $p^{-1}(y)$, and the completed residue field at a point z is the original $\mathcal{H}_X(z)$ with its structural isometric inclusion $H \hookrightarrow \mathcal{H}_X(z)$. These assertions follow on each affinoid chart from Theorem 2.33.11, and agree under the domain restrictions of Theorem 2.40.11. The fibre is compact Hausdorff: it is closed in the compact inverse image of any affinoid neighbourhood of y.

For every point z in this fibre, the valued-field tower formula, Lemma 3.22.2, gives

$$d_H(z) = d_K(z) - d_K(y) \leq \dim_K X - d \leq 0.$$

Here the point-dimension bound is Theorem 3.22.9, and $d_H(z)$ is nonnegative. It is consequently zero. The maximum formula of Theorem 3.22.7 makes every nonempty affinoid chart of X_y zero-dimensional over H. Compactness supplies finitely many such charts, each with a finite underlying set by the first paragraph. Thus the entire fibre is finite. Properness supplies relative interiority of all its points, so Lemma 5.56.1 gives the asserted ordinary finite neighbourhood, including the empty-fibre case.

If Y is pure of dimension d, every nonempty ordinary open subset contains a nonempty affinoid neighbourhood V of dimension d, by goodness and Lemma 5.50.3. The Abhyankar maximum formula supplies a point of V with $d_K(y) = d$. Corollary 2.33.15 leaves its completed residue field unchanged under domain restriction, so it is a point of the finite locus just proved. That locus therefore meets every nonempty ordinary open subset. Its openness is Lemma 5.56.1, proving density. □

Theorem 5.56.3 (Finite moment collars on compact analytic spaces)**.** *Let Z be a compact good Hausdorff analytic space of analytic dimension at most d, and let f be a finite tuple of global analytic units on Z. Choose a finite affinoid-domain cover $(U_i)_{i=1}^{r}$ whose ordinary interiors cover Z, and put*

$$E = \bigcup_{i=1}^{r} t_f(\partial_K U_i).$$

Then E is a compact finite integral-affine polyhedral set of dimension at most $d-1$; it is empty if $d = 0$. For every $w \notin E$ there is a closed rectangular box P about w, with w in its interior, for which

$$Z_P = t_f^{-1}(P) \longrightarrow \mathbb{T}_P$$

is finite. In particular Z_P is an actual affinoid domain of Z. The box can be chosen in any prescribed neighbourhood of w. The inverse image here is the entire inverse image, and restriction to any exact polytorus in P gives the entire original moment fibre with its analytic structure.

Proof. The cover exists by Lemma 5.50.1. Each U_i has dimension at most d by Theorem 3.22.9. The boundary-image assertion of Theorem 5.39.3 proves the claims about E. Each boundary is closed by Proposition 2.33.21 and compact by Proposition 2.24.1, so its image and their finite union are compact.

Let $q : Z \to \mathbb{G}_m^{n,\mathrm{an}}$ be the actual analytic map defined by f, and let $\mathbb{T}_w$ be the exact polytorus at w. The map q is compact: the inverse image of a compact analytic domain in the Hausdorff torus is a closed subset of the compact Hausdorff space Z.

For each i with $w \in t_f(U_i)$, the entire fibre in U_i avoids $\partial_K U_i$. Proposition 5.38.4 supplies a finite map from its full moment neighbourhood to a closed target polyannulus about w. If $w \notin t_f(U_i)$, compactness of this image supplies a box about w with empty inverse image in U_i. Intersect these finitely many boxes and shrink slightly. We obtain a box Q about w such that every restriction

$$U_i \cap t_f^{-1}(Q) \longrightarrow \mathbb{T}_Q$$

is finite, including the empty restrictions.

Fix a point $a \in \mathbb{T}_w$. Its entire fibre in Z is a finite union of fibres of these finite maps, and is therefore finite. At a point z of this fibre, choose i for which $z \in \mathrm{Int}_{\mathrm{top}}(U_i, Z)$. Since w satisfies all inequalities of Q strictly, the affinoid $U_i \cap t_f^{-1}(Q)$ is an ordinary neighbourhood of z, and $\mathbb{T}_Q$ is an ordinary neighbourhood of a. Finite morphisms are everywhere interior by Theorem 2.33.18. Ordinary local testing, Proposition 2.44.6, gives $z \in \mathrm{Int}(Z/\mathbb{G}_m^{n,\mathrm{an}})$. Lemma 5.56.1 now gives an ordinary neighbourhood N_a of a over which the full map q is finite. The union $N = \bigcup_{a \in \mathbb{T}_w} N_a$ is an ordinary open neighbourhood of the whole exact polytorus, and $q^{-1}(N) \to N$ is finite by target locality.

Choose a closed box Q' about w inside the prescribed real neighbourhood. The complement $\mathbb{T}_{Q'} \setminus N$ is compact and its tropical image does not contain w. A smaller closed box P about w can therefore be chosen so that $\mathbb{T}_P \subset N$. The full restriction $q^{-1}(\mathbb{T}_P) \to \mathbb{T}_P$ is finite, proving the assertion. The analytic-domain inverse-image construction 2.40.11 identifies this source with $t_f^{-1}(P)$. Finite base change gives the actual exact-level algebras. No point or factor of the original inverse image has been removed. □

Corollary 5.56.4 (Proper morphisms are finite near generic moment levels)**.** *Let $p : X \to Y$ be a proper morphism of good Hausdorff analytic spaces, with $\dim_K X \le d$. Let $V \subseteq Y$ be an affinoid domain and let f be a unit tuple on V. There is a compact finite integral-affine polyhedral set $E \subseteq \mathbb{R}^n$ of dimension at most $d-1$ such that, for every $w \notin E$, one can choose a closed moment box P about w for which*

$$V_P = t_f^{-1}(P), \qquad X_P = p^{-1}(V_P), \qquad p_P : X_P \longrightarrow V_P$$

are actual affinoid domains and a finite morphism. Empty domains are allowed. The box can be made arbitrarily small.

In particular, if $\mathrm{Fin}(p_V)$ is the ordinary finite-fibre locus of $p_V : p^{-1}(V) \to V$, then

$$t_f\big(V \setminus \mathrm{Fin}(p_V)\big) \subseteq E.$$

The same inclusion holds for the image of every positive-dimensional analytic fibre. When V is pure of dimension d, the complement of E is dense on every d-dimensional cell of its actual moment image.

Proof. Properness makes $Z = p^{-1}(V)$ compact Hausdorff. It is good by the actual product charts. At a point of Z, choose an ordinary affinoid neighborhood U of its image in Y and ordinary affinoid neighborhoods X_0 and V_0 of its projection points in X and V, both mapping into U. The affinoid product $X_0 \times_U V_0$ identifies with the intersection of the two inverse images in $X \times_Y V$ by Theorem 2.40.11. It contains the intersection of the inverse images of their ordinary interiors, an ordinary neighborhood of the chosen point. Its dimension is at most d, since it is an analytic domain of X. Apply Theorem 5.56.3 to Z and the unit tuple p^*f. It gives the set E and a finite map $X_P \to \mathbb{T}_P$. The intermediate domain V_P is affinoid by the actual simultaneous Laurent inequalities of Theorem 2.31.2. Write their algebras as

$$R_P = \mathcal{O}(\mathbb{T}_P) \longrightarrow A_P = \mathcal{O}(V_P) \longrightarrow B_P = \mathcal{O}(X_P).$$

The algebra B_P is finite over R_P, and the same finite generating family makes it finite over A_P.

Here its given norm is the canonical finite-A_P-module norm. Indeed, using Theorem 1.18.3, choose R_P-module generators $b_1, \dots, b_m$ and their controlled lifts for the canonical R_P-module norm. They are also A_P-module generators. Boundedness of $R_P \to A_P$ makes the quotient A_P-module norm bounded above by a constant times the given norm of B_P. Conversely, the bounded

analytic map $A_P \to B_P$ and the finitely many b_i bound the given norm by a constant times that quotient norm. The two norms are therefore equivalent. This proves norm-admissible finiteness over V_P.

Every point of V of weight $w \notin E$ lies in the ordinary interior of V_P, because the defining moment inequalities are strict there. Its full fibre is consequently finite. Lemma 5.56.1 identifies the resulting ordinary finite locus. A positive-dimensional analytic fibre cannot be the finite fibre of a finite morphism: its affinoid pieces would be finite algebras over the completed target residue field and hence have analytic dimension zero, by finite residue extensions and the Abhyankar maximum formula. Finally E has dimension at most $d-1$, so it cannot contain a relatively open subset of a d-cell. This proves the density assertion exactly on those cells. □

Theorem 5.56.5 (Moment chains on compact analytic domains)**.** *Let Z be a compact good Hausdorff analytic space pure of dimension d, let $\mathcal{F}$ be coherent on Z_G, and let f be a global unit tuple. There is a canonical finite nonnegative integral d-chain $\mathcal{T}_{Z,f}(\mathcal{F})$, with support contained in the actual compact image $t_f(Z)$. For any finite affinoid-domain cover $(U_i)_{i=1}^r$, it is given by*

$$\mathcal{T}_{Z,f}(\mathcal{F}) = \sum_{\varnothing \neq I \subseteq \{1,\dots,r\}} (-1)^{|I|+1} \mathcal{T}_{f|_{U_I}}\big(\Gamma(U_I,\mathcal{F})\big), \qquad U_I = \bigcap_{i\in I} U_i.$$

Empty intersections contribute zero. This chain is additive in short exact sequences of coherent modules. For every smooth (d,d)-superform α defined near $t_f(Z)$,

$$\int_{\mathcal{T}_{Z,f}(\mathcal{F})} \alpha = \mathcal{I}_{Z,\mathcal{F}}(t_f^*\alpha).$$

Here the right side is the compact finite-chart integral.

The same construction applies to a finite union of affinoid domains in a good Hausdorff pure d-dimensional ambient space, even if the union is not good. It is additive by finite inclusion–exclusion for covers by such compact analytic domains. The analogous inclusion–exclusion identity holds for compatible finite moment-chart integrals, without requiring one global tuple.

If $Z_P = t_f^{-1}(P)$ is an affinoid moment domain, then

$$\mathcal{T}_{Z,f}(\mathcal{F})|_{\mathrm{int}(P)} = \mathcal{T}_{f|_{Z_P}}\big(\Gamma(Z_P,\mathcal{F})\big)|_{\mathrm{int}(P)}.$$

If global unit tuples g, f on Z satisfy $t_f = a \circ t_g$ for an integral lattice map a, then

$$a_*\mathcal{T}_{Z,g}(\mathcal{F}) = \mathcal{T}_{Z,f}(\mathcal{F}).$$

This pushforward is proper on the actual support. Cells on which a drops rank contribute zero.

Proof. All finite intersections are actual affinoid domains by Lemma 5.50.1, and every nonempty one is pure of dimension d by Lemma 5.50.3. Its sections of $\mathcal{F}$ form a finite module by Theorem 2.39.6. Thus the displayed finite sum is a well-defined integral d-chain. By Theorem 5.39.3, the actual image of Z is the finite union of the polyhedral images of the U_i, so it is compact and finite polyhedral of dimension at most d.

To compare covers (U_i) and (V_j), apply Theorem 5.48.1 on each U_I to its affinoid cover $(U_I \cap V_j)_j$. The first display becomes

$$\sum_{I\neq\varnothing}\sum_{J\neq\varnothing}(-1)^{|I|+|J|}\mathcal{T}_{f|_{U_I\cap V_J}}\big(\Gamma(U_I\cap V_J,\mathcal{F})\big).$$

Interchanging the finite sums and applying the same theorem on each V_J gives the expression for the second cover. This also proves independence under further refinement. Exact sequences remain exact on every affinoid intersection by the coherent equivalence, and the affinoid chains are additive by the additivity of finite-module chains; hence so is this chain.

For nonnegativity, order the cover and group each term by its largest index j. Use the supporting-hyperplane construction of Lemma 5.14.2 to subdivide all images and boundary images together. Consider the relative interior of a d-cell. All occurring full fibres and their overlaps are interior. The complementary-idempotent calculation in Theorem 5.50.6, using Theorem 5.47.2, identifies the coefficient of the j-th group with

$$\frac{1}{e_Q}\dim_{H_Q}\left(\prod_{i<j}(1-e_i)\right)\big(M_j\otimes_{\mathcal{O}(U_j)}E_Q\big).$$

Here $M_j=\Gamma(U_j,\mathcal{F})$, E_Q is the actual transverse Artinian Gauss algebra, e_Q is its lattice index, and the e_i are the idempotents of the overlap domains. Expanding the product gives exactly that group's alternating sum of module dimensions. The displayed coefficient is nonnegative, and the chain coefficient is an integer because it was defined as an alternating sum of integral weights. This proves positivity, including all module nilpotents. The integral identity is precisely the same finite sum as the defining formula of Theorem 5.50.6.

For a finite affinoid union in a good Hausdorff ambient space, the domain, intersection, purity and coherence statements are the ones verified in the last

paragraph of that theorem. The preceding double-cover argument therefore applies verbatim to the chains and to compatible finite chart data. We give the finite-union bookkeeping explicitly. In the commutative ring generated by symbols $[U]$ for the finitely many affinoid domains involved, impose $[U][V] = [U \cap V]$; all these symbols are idempotent. A finite covering family is represented by

$$1 - \prod_i (1 - [U_i]).$$

Expanding gives its inclusion–exclusion expression. Products represent intersection covers: the Boolean identity

$$(1 - \prod_i (1 - x_i))(1 - \prod_j (1 - y_j)) = 1 - \prod_{i,j} (1 - x_i y_j)$$

holds for commuting idempotents. To verify it, expand the identity decomposition into the products which choose either each variable or its complement. On every such product the two sides are the same zero or one. A union of compact domains represented by such covering expressions is consequently represented by $1 - \prod_l (1 - P_l)$. Expansion first by the P_l, or first by their affinoid constituents, gives the same finite sum of intersection symbols. Cover independence just proved permits either presentation of any intersection. Applying its chain or compatible-integral value proves the asserted finite inclusion–exclusion formula.

For the restriction assertion, intersect the chosen cover with Z_P. On the interior of P, the whole moment fibre in every U_I is unchanged. Refine by the finitely many boundary images of both U_I and $U_I \cap Z_P$. At every remaining generic d-cell, the whole-fibre module restriction assertion of Theorem 5.40.2 identifies their weights. Equality on these generic cells is equality of d-chains. The restricted inclusion–exclusion sum is the affinoid cover expression on Z_P, proving the claim.

Finally apply the same-affinoid chain projection to the identity map of each U_I, with its two restricted tuples. It gives the claimed lattice pushforward on each term. Finite linearity of the lattice-index formula gives the equality for their sum. The actual source support is compact, so the pushforward is proper. The rank-dropping convention is the one in that affinoid theorem and is unchanged by finite summation. □

Theorem 5.56.6 (Proper moment pushforward with generic lengths)**.** *Let $p : X \to Y$ be proper, where X and Y are good Hausdorff and pure of the same analytic dimension d. Let $\mathcal{F}$ be coherent on X_G. For an affinoid domain $V \subseteq Y$ and a unit tuple f on V, put $Z = p^{-1}(V)$ and*

$$\mathcal{P}_{p,\mathcal{F},f} = \mathcal{T}_{Z,p^*f}(\mathcal{F}|_Z).$$

This is a finite nonnegative integral d-chain. There is a compact finite polyhedral set E of dimension at most $d-1$ such that at every generic d-cell point $w \notin E$ it has the following description. There is an actual finite restriction

$$p_P : X_P = p^{-1}(V_P) \longrightarrow V_P = t_f^{-1}(P), \qquad w \in \operatorname{int}(P),$$

with affinoid algebras A_P, B_P. For the finite B_P-module $M_P = \Gamma(X_P, \mathcal{F})$ its coefficient is

$$m_{\mathcal{P}}(w) = \sum_{\mathfrak{p} \in \operatorname{Min}(A_P)} \operatorname{length}_{(A_P)_{\mathfrak{p}}}((M_P)_{\mathfrak{p}}) m_{\mathcal{M}(A_P/\mathfrak{p}),f}(w),$$

$$\operatorname{length}_{(A_P)_{\mathfrak{p}}}((M_P)_{\mathfrak{p}}) = \sum_{\substack{\mathfrak{q} \in \operatorname{Min}(B_P) \\ \mathfrak{q} \cap A_P = \mathfrak{p}}} \operatorname{length}_{(B_P)_{\mathfrak{q}}}((M_P)_{\mathfrak{q}}) [\kappa(\mathfrak{q}) : \kappa(\mathfrak{p})].$$

An empty component fibre contributes zero. Residue-field degrees in this expression are full degrees, including their inseparable factors. The coefficient and the chain do not depend on the finite collar or the affinoid cover of Z.

If a global unit tuple g on Z and an integral map a satisfy $t_f \circ p = a \circ t_g$, then

$$a_* \mathcal{T}_{Z,g}(\mathcal{F}|_Z) = \mathcal{P}_{p,\mathcal{F},f}.$$

In particular its generic coefficient is the full lattice-index sum over all full-rank source cells above that target cell; contracted cells contribute zero. For every smooth (d,d)-superform α defined near $t_f(V)$,

$$\mathcal{I}_{Z,\mathcal{F}}(p^* t_f^* \alpha) = \int_{\mathcal{P}_{p,\mathcal{F},f}} \alpha.$$

Proof. The space Z is compact Hausdorff and good as in Corollary 5.56.4. Its affinoid domains are domains of X and are pure of dimension d by Lemma 5.50.3. Thus Z is pure of dimension d, and Theorem 5.56.5 defines the displayed chain with its stated positivity and finiteness.

Choose the exceptional set in Corollary 5.56.4, and enlarge it by $t_f(\partial_K V)$. On a finite common subdivision of the actual image, the source-chain image

and this exceptional set, obtained by Lemma 5.14.2, a generic point w of a d-cell avoids the exceptional set. Near it, the target image lies in the affine span of that cell, with its saturated rank-d tangent lattice. The proper finite-collar theorem gives p_P; choose the box small enough that w satisfies its inequalities strictly. Every point of the entire target fibre is absolute interior in V and interior in the domain V_P. Domain interior and composition, Propositions 2.44.5 and 2.33.21, make it absolute interior in V_P. Every point of its full inverse image is absolute interior in X_P, because p_P is finite. The nonempty domains V_P and X_P are pure of dimension d, and M_P is finite over both B_P and A_P by Proposition 2.35.15.

The restriction statement in Theorem 5.56.5 identifies the source chain near w with that of X_P. Apply the one-affinoid chain calculation in finite pushforward to $p_P : X_P \to V_P$ and the single domain V_P. The finite collar gives norm-admissibility, and the coefficient module is M_P on the full inverse image and the same module regarded over A_P on the target. The source tuple is the literal pullback of $f|_{V_P}$; the map of moment lattices is the identity. The calculation identifies the two finite chains, with index one on each common saturated top tangent lattice and zero target module weight where the source fibre is empty. The generic-length formula of Theorem 5.40.2 is the first equality in the statement. The Artinian local factor calculation of Theorem 5.42.2 is the second. It uses the actual local module lengths, and its simple factors have dimension the full field degree over the target residue field. Thus neither nilpotents nor inseparability have been removed.

Two collars compare on a smaller moment box retaining the whole target fibre and its entire inverse image. Finite base change identifies their restricted algebras and modules with the algebra and module of that same actual inverse image. The whole-fibre module restriction assertion of Theorem 5.40.2 gives the same coefficient. Cover independence also follows from the already canonical compact chain.

The lattice pushforward is the last assertion of Theorem 5.56.5 on Z. It requires only the displayed equality of moment maps; the actual joint-character graph argument is included in its affinoid prerequisite. A common finite refinement of the compact source image gives the finite full-rank lattice-index sum on each generic target cell. Its support is proper, and rank-dropping source cells have zero top-degree contribution. Finally the compact-chain integral identity gives the last display directly. Every construction used the

complete inverse image Z, and every finite collar retained its entire inverse image over the target box. □

Theorem 5.56.7 (Proper projection from generic coefficient agreement)**.** *Let $p : X \to Y$ be proper between good Hausdorff analytic spaces pure of the same dimension d. Let $\mathcal{F}$ and $\mathcal{G}$ be coherent on X_G and Y_G, respectively, and fix $\Delta \in \mathbb{Q}_{\geq 0}$. Let $D \subseteq Y$ be a closed analytic subspace whose defining ideal, on every affinoid domain of Y, is contained in no minimal prime. The empty subspace is permitted.*

Suppose that whenever $W = \mathcal{M}(A)$ is an affinoid domain of $Y \setminus D$ for which $p^{-1}(W) = \mathcal{M}(B) \to W$ is finite, the modules

$$M = \Gamma(p^{-1}(W), \mathcal{F}), \qquad L = \Gamma(W, \mathcal{G})$$

satisfy, for every $\mathfrak{p} \in \operatorname{Min}(A)$,

$$\operatorname{length}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \Delta \operatorname{length}_{A_{\mathfrak{p}}}(L_{\mathfrak{p}}). \tag{5.56.1}$$

Then for every affinoid domain $V \subseteq Y$ and unit tuple f on V,

$$\mathcal{P}_{p,\mathcal{F},f} = \Delta \mathcal{T}_f\big(\Gamma(V, \mathcal{G})\big).$$

For every ordinary smooth moment form $\alpha \in \mathcal{A}^{d,d}_{Y,c}(Y)$, the pullback has compact support and

$$\int_{X,\mathcal{F}} p^*\alpha = \Delta \int_{Y,\mathcal{G}} \alpha.$$

These scalar formulas multiply under composition when the corresponding coefficient conditions hold for the two maps.

For $\mathcal{F} = \mathcal{O}_X$ and $\mathcal{G} = \mathcal{O}_Y$, condition (5.56.1) uses the ratio of the full generic local lengths. If A is reduced, its denominator is one, so a nonzero such ratio is a positive integer. If an actual finite restriction has integral rings A, B and an injective structural map, its numerator is $[\operatorname{Frac}(B) : \operatorname{Frac}(A)]$, with the full inseparable degree retained.

Proof. Fix V and f. The actual closed subspace $D \cap V$ has moment image of dimension at most $d-1$ by Theorem 5.42.4. This image is compact and finite polyhedral. Add it to the exceptional polyhedral set of Theorem 5.56.6 and take a common finite subdivision by Lemma 5.14.2. At a generic point w of every remaining d-cell, choose the proper finite collar small enough that its

entire target domain V_P misses $D \cap V$. This is possible because the moment image of $D \cap V$ is compact and does not contain w.

Apply (5.56.1) to $W = V_P$. The generic-length formula in Theorem 5.56.6 and the module-weight formula in Theorem 5.40.2 give

$$m_{\mathcal{P}}(w) = \Delta\, m_{\Gamma(V_P,\mathcal{G}),f}(w) = \Delta\, m_{\Gamma(V,\mathcal{G}),f}(w).$$

The second equality is the whole-fibre restriction assertion of that theorem: the full fibre lies in V_P, and the restriction of the coherent module is its actual domain tensor. Equality on the generic part of every d-cell is equality of the finite d-chains. The excluded sets have dimension at most $d-1$ and contain no d-cell. This proves the first assertion, including components of the source whose moment image has dimension smaller than d.

The chain equality and Theorem 5.56.5 give the corresponding integral equality for a single smooth moment representative on any affinoid V. We pass carefully to ordinary forms. Let $S = \operatorname{supp}(\alpha)$. Pullback of representatives in Theorem 5.52.3 gives

$$\operatorname{supp}(p^*\alpha) \subseteq p^{-1}(S).$$

Choose a finite union Q of affinoid domains of Y whose ordinary interiors cover S. Every full inverse image of one of these domains is compact by properness and good by Corollary 2.46.4. Choose a finite affinoid cover of each such inverse image. Thus $R = p^{-1}(Q)$ is a finite union of affinoid domains in X. It is compact Hausdorff, and $p^{-1}(S)$ is closed in it. Consequently the pullback support is compact. The ordinary interior of R contains it, since it contains $p^{-1}(\operatorname{Int}_{\mathrm{top}}(Q, Y))$.

Represent the induced finite chart data of α on Q by a finite affinoid cover (V_i), with one moment tuple and one smooth representative on each member, as in Theorem 5.52.3. Put $Z_i = p^{-1}(V_i)$. Each Z_i is compact good and admits a finite affinoid cover; the same holds for $Z_I = p^{-1}(V_I)$, where $V_I = \bigcap_{i\in I} V_i$. Their pulled-back finite chart representatives agree with the data induced by $p^*\alpha$, because both use the same analytic unit pullbacks on every affinoid refinement. The compact-domain inclusion–exclusion assertion of Theorem 5.56.5 computes the integral on R as the alternating sum of the integrals on the Z_I. On each Z_I, use a representative from any $i \in I$. The single-chart equality just proved identifies this term with Δ times the corresponding integral on V_I. The alternating sum is therefore Δ times the finite-chart integral on Q. Theorem 5.52.4 identifies these two compact-neighbourhood integrals with the stated ordinary compact-support integrals.

For two composable maps satisfying the respective conditions, the pullback of a compactly supported form under the second map has compact support by the assertion just proved. Apply the two integral formulas successively and use the functoriality of ordinary smooth pullback in Theorem 5.52.3. The factors multiply.

Finally, for reduced A, every $A_{\mathfrak{p}}$ at a minimal prime is a field, as checked in Proposition 5.41.2, so its own length is one and the length of a finite module is an integer. For a finite injection of integral rings $A \to B$, localizing B at the nonzero elements of A gives a finite-dimensional domain over $\mathrm{Frac}(A)$, hence a field. It contains B and is contained in $\mathrm{Frac}(B)$; conversely every nonzero element of B is invertible in that field, so it is $\mathrm{Frac}(B)$. Its vector-space dimension is the claimed full degree for this actual finite restriction. □

Corollary 5.56.8 (Proper modifications and generic finite free degree)**.** *Keep $p : X \to Y$ and $D \subseteq Y$ as in Theorem 5.56.7. If p is an isomorphism over $Y \setminus D$ and coherent modules $\mathcal{F}, \mathcal{G}$ are identified there by $\mathcal{F} \simeq p^{*}\mathcal{G}$, then*

$$\int_{X,\mathcal{F}} p^{*}\alpha = \int_{Y,\mathcal{G}} \alpha$$

for every compactly supported ordinary smooth top form. In particular a proper modification which is an isomorphism away from such a closed subspace preserves the structure-module integral. Additional source components mapped into D are allowed, provided X remains pure of dimension d.

More generally, suppose that over $Y \setminus D$ the map is finite and its coherent direct image of $\mathcal{F}$ is locally free of constant rank r on the analytic G-site. Then

$$\int_{X,\mathcal{F}} p^{*}\alpha = r \int_{Y,\mathcal{O}_Y} \alpha.$$

The rank may be the full degree of a finite inseparable map; separability is not required.

Proof. In the isomorphism case the two finite modules on every affinoid restriction disjoint from D are identified. Their generic local lengths are equal, so Theorem 5.56.7 applies with $\Delta = 1$.

For the rank assertion, fix a target affinoid moment chart and exclude the actual moment image of its intersection with D, as in the proof of that theorem. On a sufficiently small proper finite collar about any remaining generic d-cell point, the whole target domain is disjoint from D. The direct

image is therefore analytically locally free of rank r on that domain. The rank formula in Theorem 5.51.1, proved there by an actual finite affinoid trivializing refinement, identifies its finite-module moment chain with r times the structure chain. Equivalently, on that refinement each actual Artinian Gauss factor carries r copies of the original module; inclusion–exclusion gives the same equality on the whole domain. Thus the compact source chain equals r times the target structure chain on every generic d-cell. The ordinary-form argument in Theorem 5.56.7 now gives the displayed equality without replacing analytic local freeness by an unproved algebraic-local assertion.

If a source component maps into D, its image under every pulled-back target tuple lies in the finite polyhedral image of D of dimension at most $d-1$. Its pulled-back top chain and top-form integral are zero. The preceding proof retains it in every full inverse image; its zero contribution is a dimension calculation. All remaining factors are the full finite-module factors, so the field degrees, including inseparable degrees, retain their original normalization. □

5.57 Rational coordinates and initial models along a skeleton

Let K be complete for its specified nontrivial real-valued absolute value. Write $O = K^\circ$, $k = \widetilde{K}$, and $v = -\log|\cdot|$, and fix $\pi \in O$ with $0 < |\pi| < 1$. We use the algebraic model setting of Theorem 5.30.3: $\mathcal{X}$ is a given projective, flat, finitely presented O-scheme, its generic fibre X is smooth and integral, and a finite collection of affine opens covering $\mathcal{X}_s$ has étalemaps to

$$\operatorname{Spec} R_\lambda, \qquad R_\lambda = O[X_0, \dots, X_{r_\lambda}, T_1, \dots, T_{q_\lambda}]/(X_0 \cdots X_{r_\lambda} - a_\lambda), \quad 0 < |a_\lambda| < 1.$$

Let $S \subset X^{\mathrm{an}}$ be the compact skeleton of the completed model. All rational functions below belong to the original field $K(X)$.

For such a chart $\mathcal{V} = \operatorname{Spec} C$ and a nonempty $I \subset \{0, \dots, r\}$, write S_I for its residual stratum over

$$D_I = \{X_i = 0 \ (i \in I),\ X_j \neq 0 \ (j \notin I)\}$$

and Q_I for the finite set of generic points of S_I. These are the strata and labels of Proposition 5.24.2. If $s = x_{\eta,w}$, then $I = \{i : w_i > 0\}$ and $\eta \in Q_I$. An irreducible residual component in this section is a component of this locally closed stratum.

Lemma 5.57.1 (Algebraic coordinates on a common affine open)**.** *There is an affine open $\mathcal{W} = \operatorname{Spec} D$ of $\mathcal{X}$ containing every specialization of S. There is a finite collection of affine étalesemistable charts $\mathcal{V}_\lambda = \operatorname{Spec} C_\lambda$ contained in $\mathcal{W}$ whose completed generic fibres cover S. For each pair (I, η) in a chart one may choose $h_{\lambda,I,\eta} \in C_\lambda$ such that*

$$E_{\lambda,I,\eta} := (S_I)_{\overline{h}_{\lambda,I,\eta}}$$

is a nonempty smooth integral open of the component labelled by η and meets no other component of S_I.

Fix a faithful rational tuple F_0 as in Theorem 5.30.3. There are a finite list $P \subset D \setminus \{0\}$, a dense open

$$U = \operatorname{Spec} D_K \left[1/\prod_{p\in P} p\right] \subset X,$$

and a finite tuple $F \subset K(X)^\times$ with the following properties. The open U is contained in every $\mathcal{V}_{\lambda,K}$ and contains S analytically. The tuple F contains P, F_0, finite nonzero O-algebra generating sets of D and every C_λ, the chart coordinates, and every $h_{\lambda,I,\eta}$. Each member of F is a ratio of members of P, or itself a member of P. All members of F are units on U, and they define a closed immersion $U \hookrightarrow \mathbf{G}_{m,K}^{|F|}$. Its restriction to S is faithful and unimodular on a finite subdivision.

Proof. Proposition 5.30.2 makes the set Z of skeletal specializations finite. Apply Lemma 5.30.1 to obtain $\mathcal{W}$. For each point of Z in an original chart, choose a principal open neighbourhood in that chart contained in $\mathcal{W}$. There are only finitely many choices. A principal open is étaleover its chart, and the restricted map to the standard model is étaleby [StacksAlg, Tag 00U2]. At every skeletal point specializing to the chosen point, the defining function of this principal open has norm one. Lemma 4.13.1 and Proposition 5.23.4 therefore show that these completed opens cover S and have the restricted skeletons claimed.

The algebras D and C_λ are finitely presented over O: apply [StacksAlg, Tag 01TQ] to the restrictions of the finitely presented structural morphism to these affine opens. They are O-flat. Thus they inject into their generic coordinate rings, which in turn inject into $K(X)$.

For fixed (λ, I), the distinct primes in Q_I are incomparable. For each $\eta' \neq \eta$, choose an element of the prime of η' outside the prime of η, and

multiply these finitely many elements. They may be chosen in $C_\lambda \otimes_O k$, by taking the inverse images of the stratum primes in this ring. Lift the product to $h_{\lambda,I,\eta} \in C_\lambda$. If Q_I is a singleton, take $h_{\lambda,I,\eta} = 1$. The stratum is smooth over k, by étalebase change and smooth composition [StacksAlg, Tags 00U2, 036U]; it is reduced by [StacksAlg, Tag 056T]. The principal open thus has exactly the chosen minimal prime and is integral. It is nonempty, smooth, and contains η. In particular each chosen h is a nonzero element of $K(X)$.

Let A be the finite list consisting of F_0, nonzero algebra generators for the charts, their standard coordinates, and the chosen h's. For every $f \in A$ write $f = p_f/q_f$ with nonzero $p_f, q_f \in D$. This is possible because $\mathrm{Frac}(D) = K(X)$; denominators from K can be cleared by a power of π. For each λ, choose $b_\lambda \in D\backslash\{0\}$ whose principal generic open is contained in the nonempty open $\mathcal{V}_{\lambda,K}$ of $\mathcal{W}_K$. Such a principal open exists by the affine open basis; again clear a scalar denominator to put b_λ in D. Let P contain these b_λ, all p_f, q_f, and a finite nonzero generating set for D over O, and put $F = A \cup P$. Zero generators can be omitted.

Every point of S is a norm on $K(X)$ by Proposition 5.30.2. It consequently has generic scheme support and belongs to U^{an}. The elements b_λ give $U \subset \mathcal{V}_{\lambda,K}$. The Laurent coordinate map associated with F surjects onto $D_K[1/\prod P]$: its image contains the generators of D_K and the inverses of every member of P. This proves the closed immersion assertion. Finally all added rational coordinates have finite piecewise $v(K^\times)$-integral-affine values on S by the same proposition. Retaining F_0 retains injectivity and an integral left inverse on every tangent lattice. A common finite subdivision, followed by the integral graph argument in Theorem 5.30.3, gives the last assertion. □

Proposition 5.57.2 (A weighted chart with one residual component)**.** *Fix a chart* $\operatorname{Spec} C$*, a label* $\eta \in Q_I$*, and its separator* h*, and put* $E = (S_I)_{\bar{h}}$*. Let* $w \in \operatorname{relint} \Delta_I$ *and* $s = x_{\eta,w}$*. Let* L/K *be a complete isometric extension in whose value group all* w_i *occur. Choose* $c_i \in L^\times$ *with* $v(c_i) = w_i$ *and* $\prod_i c_i = a$*, taking* $c_j = 1$ *for* $j \notin I$*. Write* $l = \widetilde{L}$*.*

There is an admissible L°*-model* B_0 *of the actual domain in the completed chart over* L *given by*

$$|X_i| = |c_i| \quad (0 \le i \le r), \qquad |h| = 1.$$

Its special fibre is

$$\operatorname{Spec}(B_0/\mathfrak{m}_L B_0) \simeq E_l \times \mathbf{G}_{m,l}^{|I|-1}.$$

This fibre is smooth and geometrically reduced, and B_0 is the spectral unit ball of $B = B_0[1/\pi]$. Every Shilov point of $\mathcal{M}_L(B)$ restricts to s. Consequently, for every $g \in C$ nonzero as a rational function on X,

$$\rho_B(g) = |g(s)| > 0, \qquad |g(\xi)| = |g(s)| \quad (\xi \in \Gamma_B).$$

Proof. Use the separator supplied by Lemma 5.57.1. Choose $i_0 \in I$. Choose the c_i for $i \neq i_0$ first and then put $c_{i_0} = a/\prod_{i \neq i_0} c_i$. The relation $\sum_i w_i = v(a)$ gives its required norm. Consider the algebraic base change

$$A_w = L^\circ[U_0^{\pm 1}, \dots, U_r^{\pm 1}, T_1, \dots, T_q]/(\prod_i U_i - 1),$$
$$C_w = (C \otimes_R A_w)[1/h],$$
$$B_0 = \widehat{C_w},$$

where $X_i \mapsto c_i U_i$ and completion is π-adic. Eliminating U_{i_0} makes A_w a Laurent polynomial algebra with polynomial variables over L°; it is flat by the coefficientwise ideal criterion [StacksAlg, Tag 00HD]. The map $A_w \to C_w$ is étaleby [StacksAlg, Tag 00U2]. Thus C_w is finitely presented and flat over L°, and Lemma 4.13.1 applies.

Its explicit bounded-coordinate presentation imposes the original chart bounds, $|U_i| = 1$, and $|h^{-1}| \leq 1$. As h was already integral, these are exactly the displayed domain conditions. This proves the assertion about the actual analytic domain, with its analytic structure. On reduction, c_i vanishes for $i \in I$ and equals one for $i \notin I$. The coordinates U_j for $j \notin I$ are therefore the invertible stratum coordinates X_j. After eliminating U_{i_0}, the remaining U_i, $i \in I \setminus \{i_0\}$, are independent invertible variables. Reducing the displayed finite presentation gives precisely $E_l \times \mathbf{G}_{m,l}^{|I|-1}$, also for nondiscrete valuations: integral restricted series reduce to polynomials. Smoothness and geometric reducedness follow from [StacksAlg, Tags 00U2, 036U, 056T]. Lemma 4.12.1 identifies B_0 with B° and its special fibre with $\widetilde{B}$.

Let $\xi \in \Gamma_B$. Its reduction is a component generic point by Theorem 3.16.7. The reduced algebra is étale, hence flat, over $l[U_i^{\pm 1} : i \neq i_0, T]$ after the above identification. Flat going down, [StacksAlg, Tag 00HS], makes its minimal prime contract to the zero prime of this Laurent polynomial algebra. The image of ξ in its analytic unit polytorus times polydisc therefore reduces to the generic point. It is the Gauss point: divide any nonzero restricted series by a coefficient of maximum norm; its nonzero polynomial reduction cannot

vanish at that generic point. Polynomial approximation, or Proposition 1.2.2, gives the Gauss value for every series.

Restriction to K gives the standard point z_w. Moreover the reduction of ξ contracts to η in the original stratum. Indeed the field extension $k \to l$ is flat by [StacksAlg, Tag 00HD], and going down applied to $E_l \to E$ contracts a minimal prime to the unique minimal prime of E; adjoining the independent Laurent variables has the same property. Compatibility of algebraic completion, scalar extension and specialization is the explicit presentation comparison of Lemma 4.28.1. Proposition 5.24.2 now says that the restricted point is exactly $x_{\eta,w} = s$. Taking maxima on the finite Shilov boundary gives the last formula. Its value is positive by Proposition 5.30.2. □

Lemma 5.57.3 (Localization at a common Shilov value)**.** *Let L be complete and nontrivially valued, and let B_0 be an admissible L°-algebra with reduced special fibre, and put $B = B_0[1/\pi]$. Suppose $g_1, \dots, g_t \in B$ and $b_1, \dots, b_t \in L^\times$ satisfy*

$$|g_j(\xi)| = |b_j| \quad (\xi \in \Gamma_B, 1 \le j \le t).$$

Put $u_j = g_j/b_j$. Then $u_j \in B_0$, and

$$B_0' = B_0\langle V_1, \dots, V_t\rangle/(u_jV_j - 1 : 1 \le j \le t)$$

is an admissible model of the domain $|g_j| = |b_j|$ for all j. Its reduction is $\widetilde{B}[1/\bar{u}_1 \cdots \bar{u}_t]$, and this open meets every irreducible component, including every geometric component. Its Shilov points are the original Shilov points, regarded in the restricted domain. In particular a smooth special fibre remains smooth after this localization.

Proof. If $B = 0$, then $B_0 = 0$ by π-torsion-freeness, and every assertion is immediate. Assume $B \neq 0$. The Shilov maximum formula gives $\rho_B(u_j) = 1$, and Lemma 4.12.1 gives $u_j \in B_0$. At each component generic point its residue is nonzero by Theorem 3.16.7 and the equality $|u_j(\xi)| = 1$. Apply Proposition 3.17.1 successively. It gives the displayed admissible algebra, the exact domain $|u_j| = 1$, and the principal localization on reduction. Its special fibre is still reduced, so the reduced-model lemma and the generic-point theorem apply again. All original generic points survive. Their original analytic lifts belong to the restricted domain and are the unique lifts of those generic points. This identifies the Shilov boundaries.

After any field extension of the residue field, each minimal prime contracts to an original minimal prime by [StacksAlg, Tags 00HD, 00HS]. None can therefore contain any $\bar{u}_j$. Thus the localization retains every geometric component as well. Finally, restriction to an open preserves smoothness, as in [StacksAlg, Tag 036U]. □

Theorem 5.57.4 (Actual initial models along a projective skeleton)**.** *In the stated model setting, the tuple F of Lemma 5.57.1 can be chosen so that the resulting closed immersion $\varphi : U \hookrightarrow \mathbf{G}_{m,K}^{N}$ has the following additional property. Let $s \in S$, put $\omega = \mathrm{trop}_{\varphi}(s)$, and choose a refined chart containing its specialization, with label (I, η). For every complete isometric L/K realizing the coordinates of ω in its value group, the actual initial scheme of U_L at ω is isomorphic to a principal open of*

$$E_{\lambda,I,\eta,\widetilde{L}} \times \mathbf{G}_{m,\widetilde{L}}^{|I|-1}$$

which meets every geometric irreducible component. In particular this initial scheme is smooth and geometrically reduced. The quotient norm of the normalized tropical-fibre presentation is its spectral norm, and its integral quotient model is the spectral unit ball.

Proof. Use precisely the finite lists constructed in Lemma 5.57.1. Fix s and its chart, and write $h = h_{\lambda,I,\eta}$. The normalization field L exists by Theorem 5.13.6; the argument below works for every such L. Choose the standard-coordinate normalizers with product a_λ as in Proposition 5.57.2. Let B_0 be that proposition's selected weighted model.

We first locate the entire tropical fibre. Every point of the actual fibre

$$Y_\omega = \mathrm{trop}_{\varphi}^{-1}(\omega) \subset U^{\mathrm{an}}$$

has, on every algebra generator of the selected chart, the same absolute value as s, which is at most one. Algebraically U is contained in this chart by construction. Hence the actual bounded-coordinate description of Lemma 4.13.1 puts the entire fibre inside the completed chart. The coordinates X_i impose its weights w_i, and the coordinate h imposes $|h| = |h(s)| = 1$. The same argument over L places the entire base-changed fibre in $\mathcal{M}_L(B_0[1/\pi])$. This also shows explicitly how points outside the selected formal chart are excluded: their chart-generator norms cannot satisfy all these equalities.

For each $p \in P$, choose $b_p \in L^\times$ with $|b_p| = |p(s)|$. The function p is regular on this chart because $\mathcal{V}_\lambda \subset \mathcal{W}$. Proposition 5.57.2 gives this same norm at every Shilov point of $B_0[1/\pi]$. Lemma 5.57.3 therefore gives

$$B_0' = B_0\langle V_p : p \in P\rangle/((p/b_p)V_p - 1 : p \in P).$$

Its reduction is a principal open of the stated smooth product and meets every geometric component.

Its generic fibre is exactly $Y_{\omega,L}$. In one direction this follows from the preceding containment and the equalities $|p| = |p(s)|$. In the other direction those equalities make all members of P invertible and thus put a point in U_L^{an}. Every remaining coordinate of F is p_f/q_f with $p_f, q_f \in P$, so it has the required absolute value as well. The presentations of rational domains identify the analytic structures, not only the sets of points; equivalently, use Proposition 2.33.12 and the universal property of these displayed equations.

It remains to identify the integral model with the actual initial quotient. Choose normalizers b_f for all coordinates of F. Changing them by norm-one constants only changes torus coordinates. Put

$$T_0 = L^\circ\langle Z_f^{\pm 1} : f \in F\rangle, \qquad T = T_0[1/\pi], \qquad Z_f \longmapsto f/b_f.$$

All these images and their inverses belong to B_0'. For $f = p_f/q_f$ this follows by using the normalized p_f, q_f and a norm-one scalar. The map $T_0 \to B_0'$ is surjective, as can be checked on its actual restricted presentation. Every chart algebra generator u is the image of $b_u Z_u$, with $b_u \in L^\circ$ because $|u(s)| \leq 1$. Every $U_i^{\pm 1}$ is the image of $Z_{X_i}^{\pm 1}$ multiplied by a norm-one scalar. The generator h^{-1} is the image of a norm-one scalar times Z_h^{-1}. Finally the generators $(p/b_p)^{-1}$ are the images of Z_p^{-1}. These are all the generators in the displayed construction of B_0'. Their lifts are integral Laurent polynomials of Gauss norm at most one. Substitution lifts every integral restricted series in them to T_0, by Proposition 1.2.3. Thus the map is surjective, rather than merely having dense image.

Let $\mathfrak{a}$ be the algebraic ideal of the closed immersion $U_L \hookrightarrow \mathbf{G}_{m,L}^N$, in these normalized coordinates. Proposition 5.13.2 identifies the generic map with

$$T/\mathfrak{a}T \;\simeq\; B_0'[1/\pi].$$

Since B_0' is L°-flat and is embedded in its generic algebra, the kernel of $T_0 \to B_0'$ is precisely $\mathfrak{a}T \cap T_0$. Consequently

$$T_0/(\mathfrak{a}T \cap T_0) \;\simeq\; B_0'.$$

Proposition 5.13.5 identifies the reduction of this quotient with the actual initial scheme of the algebraic immersion. Its special fibre was computed above. Lemma 4.12.1 now identifies the quotient norm with the spectral norm and the quotient model with the spectral unit ball. The arbitrary-real-weight interpretation and independence of normalization are those of Theorem 5.13.6; the calculation has retained every component throughout. □

Corollary 5.57.5 (Shilov fibres and geometric generic lengths)**.** *For the immersion of Theorem 5.57.4, the original-K tropical fibre at $\operatorname{trop}_{\varphi}(s)$ has Shilov boundary $\{s\}$ for every $s \in S$. The inverse of the faithful map on S is therefore a continuous section over $\operatorname{trop}_{\varphi}(S)$ selecting the unique Shilov point.*

For $s = x_{\eta,w}$, let g_{η} be the number of irreducible components of $E_{\lambda,I,\eta,\bar{k}}$, where $\bar{k}$ is an algebraic closure of k. After any normalization and algebraically closed residue extension, the sum of the generic component lengths of the actual initial scheme at $\operatorname{trop}_{\varphi}(s)$ equals g_{η}. In particular this sum is one if the corresponding residual component is geometrically integral. If every residual component in these charts is geometrically integral, this conclusion holds at every point of S. At weights lying in relative interiors of maximal tropical cells, the sum is the tropical multiplicity of Proposition 5.19.7.

Proof. Normalize the weights over a complete isometric extension L/K. Theorem 5.57.4 and Lemma 5.57.3 identify the Shilov points of the base-changed fibre with those of the selected weighted chart. Every one restricts to s by Proposition 5.57.2. For any analytic function g on the original tropical fibre, its base change consequently satisfies

$$\max_{Y_{\omega,L}} |g| = |g(s)|.$$

The map $Y_{\omega,L} \to Y_{\omega}$ is surjective by Theorem 2.33.13, with scalar compatibility supplied by Proposition 5.13.2. The same maximum formula therefore holds on Y_{ω}. The singleton $\{s\}$ is a boundary, and the existence and minimality of the Shilov boundary in Corollary 3.12.5 give the assertion. The faithful map on the compact S is a homeomorphism onto its image, so its inverse is continuous and has the stated values.

Over an algebraically closed residue field the initial scheme is smooth and hence reduced. Each of its generic local rings is consequently a field and

has length one. The principal localization in Theorem 5.57.4 removes no geometric component. Taking the product with a split torus changes no component count: the Laurent polynomial ring of an integral component is a domain, and the finite minimal-prime decomposition gives all components this way.

For completeness, this count is unchanged by extending the algebraically closed residue field. Every irreducible component over an algebraically closed field is geometrically irreducible by [StacksAlg, Tag 020J]. Thus its base change remains irreducible. The components stay distinct and incomparable: contraction of a prime under a field extension preserves their original defining prime, by faithful flatness of a nonzero extension of fields and [StacksAlg, Tags 00HD, 00HP, 00HQ]. Their finite union still covers the base change. To compare k with the residue field l of a chosen normalization, take an algebraic closure of l and inside it the algebraic closure of k consisting of elements algebraic over k. Applying the preceding argument gives exactly g_η. The asserted length sum follows. For a maximal-cell weight, Proposition 5.19.7 identifies this sum with its registered tropical weight. □

References. This construction reconstructs the rational-coordinate and initial-fibre methods of Gubler–Rabinoff–Werner, *Skeletons and tropicalizations*, arXiv:1404.7044v3, Sections 9–10, in the stated projective algebraic model scope. The non-Archimedean inputs are the internally proved results cited above. The scheme-theoretic initial quotient is identified by its integral presentation, and the arithmetic component count is retained; no independent-priority claim is made.

Chapter 6

Charts, skeleta and faithful tropicalization

Chapter 4 constructed the annular and semistable curve skeleta. We use them here to compare tropical maps, integral lattice indices and canonical analytic lifts. Section 3.8 supplies the residue-dimension calculation for the higher-dimensional local model.

6.1 Skeleta and tropical images

6.1.1 The Gauss section of a torus

For $w \in N_{\mathbb{R}}$, define a weighted Gauss norm on the Laurent algebra by

$$\left| \sum_{m} a_m \chi^m \right|_{\sigma(w)} = \max_{m} |a_m| e^{-\langle m, w \rangle}. \tag{6.1.1}$$

It is a multiplicative norm and therefore a point of T^{an}. The minus sign in the exponential is forced by $v = -\log|\cdot|$. In particular,

$$-\log|\chi^m|_{\sigma(w)} = \langle m, w \rangle, \qquad \mathrm{trop} \circ \sigma = \mathrm{id}_{N_{\mathbb{R}}}.$$

The set $S(T) = \sigma(N_{\mathbb{R}})$ is the torus skeleton. The map σ is continuous and identifies $N_{\mathbb{R}}$ with this subset of T^{an}; a separate deformation-retraction theorem makes $\sigma \circ \mathrm{trop}$ a strong deformation retraction. Having a section alone would not prove that theorem.

Equation (6.1.1) also gives the exact identity

$$-\log|f|_{\sigma(w)} = F_f(w).$$

The Gauss norm retains the largest size among the terms, even when several sizes agree. It does not perform the cancellation that can occur at other points in the same tropical fibre.

References. [GRW16, Sections 3–4]; Proposition 5.5.1.

Example 6.1.1 (The ambient section misses the hypersurface)**.** Let $X = V(1 + z_1 + z_2) \subset \mathbb{G}_m^2$. For every $w \in \mathbb{R}^2$,

$$|1 + z_1 + z_2|_{\sigma(w)} = \max(1, e^{-w_1}, e^{-w_2}) > 0.$$

Thus $\sigma(w) \notin X^{\mathrm{an}}$, including when $w \in \mathrm{Trop}(X) = L$. A point of X^{an} must annihilate its defining equation. The ambient Gauss point does not.

Consequently, restricting σ to L does not produce a section $L \to X^{\mathrm{an}}$. A section for a subvariety, when a theorem supplies one, must choose different analytic points. Likewise, the ambient torus deformation retraction need not preserve X^{an}.

This elementary example is a useful safeguard. The tropicalization is a subset of a real vector space; a skeleton is a subset of an analytic space. A map can identify suitable pieces of them, but the identification is additional structure or an additional theorem.

6.1.2 Annuli, lengths, and integral slopes

The annular skeleton and its retraction were constructed in Section 4.3 and illustrated in Figure 4.1. Its coordinate $t = -\log|T|$ has integral slope one. We now compare the logarithmic metric with lattice indices and model multiplicities.

For the local semistable equation $UV = a$, with $0 < |a| < 1$, the annular skeleton has coordinate $t = v(U)$ and interval

$$0 \le t \le v(a), \qquad v(V) = v(a) - t.$$

If $a = \pi^b$ over a discretely valued field, its length is $b\lambda$, where $\lambda = -\log|\pi|$. It becomes b only after the explicit normalization $\lambda = 1$.

For a curve with a chosen semistable model, annular pieces assemble into a metric graph. Over a non-algebraically-closed residue field, residual degrees can contribute weights to its slope formulas. The model, the graph, its metric, and these weights must all be specified. A smallest skeleton need not contain enough cells to represent every differential form: a curve with good reduction can have a one-point minimal skeleton, although its local differential calculus is not zero.

There are also several kinds of boundary. The endpoints of a closed annulus belong to its analytic boundary, whereas an open annulus is boundaryless. A facet in a chosen polyhedral subdivision is a combinatorial notion. A component removed from an algebraic compactification is yet another notion. Balancing away from an analytic boundary cannot be silently applied at the endpoint of a restricted annulus.

References. [GRW16, Sections 3–4 and 9–10].

For a discretely valued field with uniformizer π, write $\lambda = -\log|\pi|$. The annulus $|\pi|^m \le |T| \le 1$ has length $m\lambda$. Its integral slope for the function $-\log|T|$ is one, while the change in function value is $m\lambda$. The integer m, the real length, and the slope are different quantities. If an algebraic intersection length is m, the associated logarithmic height has the factor λ in this normalization. By contrast, the total mass of the Chern measure of a degree-one model line is one.

Example 6.1.2 (Power maps on annuli)**.** Let $0 < r < R$ and $d \ge 1$. The map $T \mapsto T^d$ takes $A[r, R]$ to $A[r^d, R^d]$. If $t = -\log|T|$, the target skeletal coordinate is dt. Thus the target interval has d times the source length, and the image of the source integral tangent lattice has index d in the target lattice. The map is finite of degree d: the equation $T^d = S$ is monic and the source algebra is free of rank d over the target algebra. This includes the inseparable case. The integral slope and algebraic degree retain the multiplicity that a count of distinct fibres can miss.

6.1.3 A semistable local model and its simplex

Over a discretely valued field, a basic strictly semistable model has an étale local chart

$$\operatorname{Spec}\big(K^\circ[x_0, \dots, x_d]/(x_0 \cdots x_r - \pi)\big), \qquad 0 \le r \le d.$$

Write $\lambda = v(\pi) > 0$. Its skeleton is locally described by

$$t_i = v(x_i) \geq 0, \qquad \sum_{i=0}^{r} t_i = \lambda.$$

This is an r-simplex. Its dimension is r, while the generic fibre has dimension d. The remaining coordinates describe directions not recorded by this simplex.

On the special fibre, the equation becomes $x_0 \cdots x_r = 0$: the components are the coordinate hyperplanes. Their intersections are normal-crossing strata. For $r = 1$ the transversal picture is a node; for larger r, several components meet normally. These semistable equations control the singularities of the special fibre. Over a general valuation ring we do not additionally assert that the underlying formal model is a noetherian regular scheme. A stratum where more components meet corresponds to a simplex of higher dimension. Thus a skeleton records incidence and valuation data, not every residual algebraic direction.

6.1.4 Faithful tropicalization and multiplicity-one sections

In this subsection, let K be complete, algebraically closed, and nontrivially valued. Let $(\mathfrak{X}, H)$ be a strictly semistable pair in the proper algebraic setting of Gubler–Rabinoff–Werner. Its skeleton $S(\mathfrak{X}, H)$ lies in the analytic generic fibre, with unbounded directions when the horizontal divisor H is present. A tropicalization is *faithful* on this skeleton if it is injective there and unimodular on its polyhedral pieces: it preserves the integral tangent lattices, not just the topology.

Theorem 6.1.3 (GRW, Theorem 9.5)**.** *In the strictly semistable-pair setting of [GRW16], there exists a finite collection of nonzero rational functions on the generic fibre whose tropicalization is faithful on $S(\mathfrak{X}, H)$.*

The rational functions are regular units on a suitable dense open containing the relevant skeletal points. The assertion does not say that every preassigned moment map is faithful. Nor does it say that the entire analytic space is homeomorphic to the skeleton. The added rational functions can have divisors outside H; the resulting tropicalization need not factor through the skeletal retraction [GRW16, Remark 9.6].

A different theorem starts with an embedding of a very affine integral variety into a torus. On a subset Z of its tropicalization where the tropical

multiplicity is one, each tropical fibre has a unique Shilov boundary point. These points give a continuous partial section $Z \to X^{\mathrm{an}}$. If Z is contained in the closure of its interior in the whole tropicalization, this is the unique continuous section on Z [GRW16, Theorem 10.6]. This statement concerns multiplicity-one fibres, and is not obtained by restricting the ambient torus Gauss section.

On a curve with good reduction, a natural compact skeleton may consist of one point. If all differential forms factored through that point, every positive-degree form would vanish. Local annular and punctured-disc domains still admit nonconstant tropical coordinates and nonzero forms. Therefore global factorization through a chosen small skeleton is a property of special data, not a general consequence of the existence of a skeleton.

6.1.5 Payne's inverse limit: what is recovered

For a quasiprojective variety over a complete algebraically closed nontrivially valued field, Payne identifies its analytification with the inverse limit of extended tropicalizations over the appropriate system of toric embeddings:

$$X^{\mathrm{an}} \simeq \varprojlim_{\iota} \mathrm{Trop}_{\iota}(X).$$

This is a homeomorphism of topological spaces [Pay09, Theorem 4.2]. The transition maps remember how a larger collection of characters projects to a smaller one.

The reason a whole system is needed can already be seen in one tropical fibre. A fixed tuple of values $|f_i(x)|$ may be shared by many analytic points. Adding more functions can distinguish them. The inverse limit requires compatible values for all such choices and recovers the seminorm, whereas one finite tropicalization records only part of it.

There are three different questions here:

Question	Additional structure needed
Recover the underlying topology	The inverse system of extended tropicalizations.
Recover analytic functions and their norms	Analytic or formal models and compatible sheaves; a bare topological homeomorphism is insufficient.
Recover forms, currents, and their operations	Test spaces, quotient relations, refinements, and continuity of the operations.

A smooth form has locally finite tropical presentations by definition. This fact is not a consequence that every current factors through one finite tropicalization. Likewise, a Hodge theorem on selected tropical spaces does not automatically pass through this inverse limit: the relevant complexes, transition maps, cohomology, and exactness must first be compared.

6.2 Fans and extended tropicalization: keeping the boundary

6.2.1 Cones specify which limits are admitted

A torus excludes the zero and pole loci of its characters. This is convenient locally, but a projective variety cannot in general be embedded in one torus. To describe a compactification, tropical geometry must allow some character valuations to tend to $+\infty$.

A rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$ is generated by finitely many lattice vectors with nonnegative real coefficients. It is strongly convex if it contains no line. Its dual cone and character monoid are

$$\sigma^{\vee} = \{m \in M_{\mathbb{R}} : \langle m, w\rangle \geq 0 \text{ for every } w \in \sigma\}, \qquad S_{\sigma} = \sigma^{\vee} \cap M.$$

The associated affine toric variety is

$$U_{\sigma} = \operatorname{Spec} K[S_{\sigma}].$$

The monoid consists precisely of the characters permitted to extend as regular functions on this affine chart. For $\sigma = \mathbb{R}_{\geq 0} \subset \mathbb{R}$, it is $\mathbb{N}$, and $U_{\sigma} = \mathbb{A}^1$; for the zero cone it is all of M, and the chart is the torus.

A fan Σ is a finite collection of strongly convex rational polyhedral cones, closed under faces, with pairwise intersections common faces. The affine

varieties U_σ glue to the normal toric variety X_Σ. A cone of dimension k corresponds to a torus orbit of dimension $r-k$. The fan is complete if its support is all of $N_\mathbb{R}$; this is equivalent to properness of X_Σ.

Two adjectives must be separated. A cone is *simplicial* when its ray generators are linearly independent over $\mathbb{R}$. It is *smooth* or unimodular when its primitive ray generators are part of a $\mathbb{Z}$-basis of N. The cone spanned by $(1,0),(1,2)$ is simplicial but not smooth. This distinction matters when a theorem assumes a simplicial fan: it has not thereby assumed a smooth toric variety.

References. [Gub13, Section 6.2–6.3 and Appendix A]; [GK17, Appendix A.7].

6.2.2 The topology is part of the construction

Set $\mathbb{R}_\infty = \mathbb{R}\cup\{+\infty\}$, with addition extended by $a+\infty=\infty$. Neighbourhoods of $+\infty$ contain a tail $(c,+\infty]$. The map $t\mapsto e^{-t}$, with $e^{-\infty}=0$, identifies this space with $\mathbb{R}_{\geq 0}$.

Definition 6.2.1 (Extended tropicalization of a toric chart)**.** Define

$$U_\sigma^{\mathrm{trop}} = \mathrm{Hom}_{\mathrm{mon}}(S_\sigma, \mathbb{R}_\infty),$$

where a monoid homomorphism is additive and sends 0 to 0. Give this set the topology of pointwise convergence. A finite set of monoid generators realizes it as a subspace of a finite product of copies of $\mathbb{R}_\infty$.

For a global fan these extended charts assemble into the space

$$N_\Sigma = \coprod_{\sigma\in\Sigma} N(\sigma), \qquad N(\sigma) = N_\mathbb{R}/\operatorname{span}_\mathbb{R}(\sigma). \tag{6.2.1}$$

The extended charts glue along the opens associated to common faces, giving the topology on this disjoint union. It is not the disjoint-union topology. The stratum $N(0)=N_\mathbb{R}$ is dense, and a sequence can converge from it to a lower-dimensional stratum.

For $x\in U_\sigma^{\mathrm{an}}$, the rule

$$m \longmapsto -\log|\chi^m(x)|, \qquad -\log 0 = +\infty,$$

is a monoid homomorphism and gives the extended tropicalization map. These maps glue to $X_\Sigma^{\mathrm{an}}\to N_\Sigma$. The orbit corresponding to σ maps to $N(\sigma)$. For a closed subvariety of X_Σ, its extended tropicalization is the image under this map.

References. [Pay09, Section 3, especially Remark 3.1]; [BGK25, Section 1.13].

Example 6.2.2 (The line, affine line and projective line)**.** For $\mathbb{G}_m$, the target is $\mathbb{R}$. For $\mathbb{A}^1$ with coordinate T, it is $\mathbb{R} \cup \{+\infty\}$, and $T = 0$ maps to $+\infty$. For $\mathbb{P}^1$, the two rays of its complete fan add two endpoints:

$$N_\Sigma = \mathbb{R} \cup \{-\infty, +\infty\}.$$

The point $T = 0$ corresponds to $+\infty$; the point $T = \infty$, with local coordinate T^{-1}, corresponds to $-\infty$. These are toric boundary points. The proper analytic space $\mathbb{P}^{1,\mathrm{an}}$ itself has empty analytic boundary.

For $\mathbb{P}^2$, use the dense-torus coordinates $z_1 = X_1/X_0$, $z_2 = X_2/X_0$, with $x = v(z_1)$, $y = v(z_2)$. Its fan has rays $(1,0), (0,1), (-1,-1)$, and its maximal cones are the sectors between consecutive rays. The extended tropicalization has one dense $\mathbb{R}^2$, three one-dimensional strata $\mathbb{R}$, and three zero-dimensional strata. A useful topological model is the closed triangle

$$(x, y) \longmapsto \frac{(1, e^{-x}, e^{-y})}{1 + e^{-x} + e^{-y}}.$$

The interior is $\mathbb{R}^2$, open sides are the one-dimensional strata, and vertices are the zero-dimensional strata. This model is a homeomorphism, not an identification of integral affine structures with the usual affine structure of a bounded triangle.

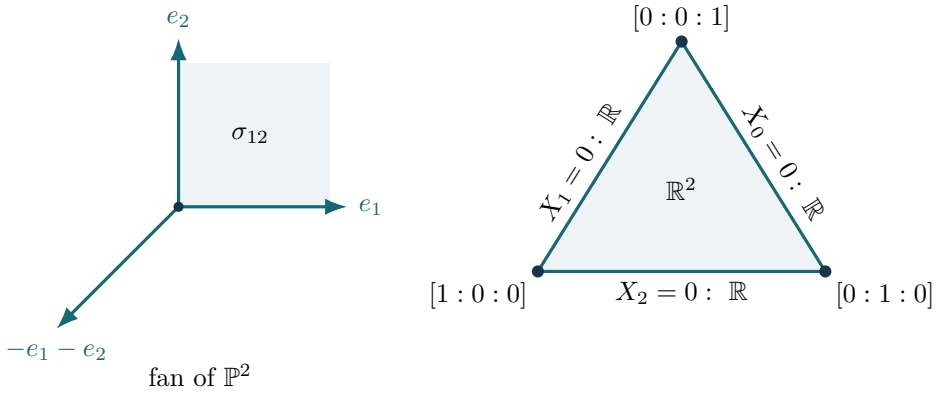


Figure 6.1: A cone of dimension k produces a boundary stratum of dimension $2 - k$. Thus the two-dimensional cone σ_{12} gives the vertex $[1:0:0]$, while the ray e_1 gives the open side $X_1 = 0$. The triangle records topology and strata.

6.2.3 Boundary compatibility and Green functions

A function or form defined on the dense $N_{\mathbb{R}}$ need not extend across these strata. The smooth tropical calculus imposes compatibility with projection to the lower-dimensional stratum: near the boundary, the form must come from that stratum in the directions that disappear. On $\mathbb{R}_{\infty}$, an ordinary smooth function in this sense is constant for all sufficiently large t, locally near $+\infty$; a positive-degree form vanishes there. A function such as t instead has the affine growth expected of a Green potential for a boundary divisor.

This is why extended tropicalization contributes more than a compact picture. It records where a divisor lies and what growth a Green function must have. BGK25 develops the compatible delta-calculus, Green functions and star-products needed to compute local heights on a suitable tropical toric compactification. Its subdivision theorems and boundary conditions are additional results; the affine stable-intersection construction alone does not supply them.

There is a complementary global lesson. Over a complete, algebraically closed, nontrivially valued field, Payne's theorem identifies the analytification of a quasiprojective variety with the inverse limit of its extended tropicalizations over all quasiprojective toric embeddings. One tropicalization forgets information; the compatible system recovers it. This topological reconstruction does not say that every current, metric or potential is represented on a single finite tropicalization.

References. [Pay09, Theorem 4.2]; [BGK25, Sections 1.7–1.11 and 3]. For a systematic further introduction, see [MS15].

Example 6.2.3 (Limits in the extended tropical plane)**.** In the triangle coordinates above, the sequences (n, n), $(n, 0)$, and $(-n, -n)$ have respective limits

$$(1, 0, 0), \qquad (1/2, 0, 1/2), \qquad (0, 1/2, 1/2).$$

The first sequence escapes through the interior of the cone generated by e_1, e_2 and reaches its zero-dimensional stratum. The second escapes along the ray e_1 and reaches the one-dimensional stratum $X_1 = 0$. The third escapes along $-e_1 - e_2$ and reaches the one-dimensional stratum $X_0 = 0$. A divergent sequence need not approach a toric fixed point.

6.3 Skeleta, slope formulas, and tropical degree

6.3.1 Skeleta of semistable pairs

For a strictly semistable pair $(\mathfrak{X}, H)$, the skeleton is assembled from the standard simplices of local semistable equations, with unbounded directions contributed by the horizontal divisor H. Its faces correspond to strata. The resulting integral-affine structure remembers the orders of vanishing of functions along model components and horizontal divisors.

In dimension one this is a metrized graph. In higher dimension, the normal direction along a codimension-one face carries data involving the neighbouring strata and intersection numbers. The higher-dimensional slope formula is therefore not obtained by summing arbitrary Euclidean normal derivatives. The model geometry specifies the correct weights and normalizations.

6.3.2 Slope formulas and principal divisors

If a rational function has divisor supported in the allowed boundary, its logarithmic absolute value restricts to an integral-affine function on each canonical cell of the skeleton. The slope formula expresses the balancing of its divisor data. On a curve it reduces to the familiar sum of outgoing slopes. In higher dimension it uses the incidence and intersection data of the semistable model.

This result is a bridge between algebraic principal divisors and tropical principal divisors. It tells us that the polyhedral data obtained from an analytic function obey the conservation law needed for integration. Conversely, an arbitrary PL function on a chosen skeleton need not be the logarithm of one analytic unit on a neighbourhood; lifting prescribed slopes is a separate problem.

6.3.3 The Sturmfels–Tevelev mechanism

For a generically finite map between suitable varieties in tori, the pushforward of the tropical cycle carries lattice-index weights. In a region where the map is combinatorially well behaved, the target weight is obtained by summing

$$m_\sigma[N_\tau : F(N_\sigma)]$$

over source cells mapping onto the target cell, with the generic degree of the algebraic map accounted for in the cycle pushforward formula. This is the tropical version of counting sheets of a finite map, including ramification and multiplicity.

GRW develops related degree formulas for skeleta and tropicalizations. The lesson for CLD is direct: if a form is presented by a larger moment map and then projected, the calibrated integral must transform by precisely these degrees. The lesson for GK is equally direct: a tropical intersection product has to descend through the same refinement maps.

6.3.4 Existence and scope of faithful tropicalization

The faithful tropicalization theorem says that suitable rational functions can be chosen so that a specified skeleton is mapped injectively and unimodularly on its cells. It does not say that every tropicalization is faithful. Nor does it say that an arbitrary analytic datum factors through the retraction to that skeleton.

The distinction can already be seen on an annulus: the coordinate T^m has slope m and fails unimodularity when $|m| > 1$ as an integer lattice index, while T supplies the primitive parameter. Additional functions can separate cells that a first tropicalization identifies. In higher-dimensional proofs, constructing enough such functions is a geometric theorem involving models and rational functions.

The rational functions supplied for faithfulness may have divisors outside the original boundary divisor. Therefore the theorem should not be strengthened to a global factorization through the original skeleton without checking the actual functions and their domains.

6.3.5 Multiplicity one and the tropical skeleton

For a tropical fibre of multiplicity one, the unique Shilov boundary point gives a canonical candidate for a section of tropicalization. The continuity theorem in GRW applies on the multiplicity-one locus in its stated setting. The stronger uniqueness assertion for a continuous section includes an additional closure-of-interior hypothesis. These are different assertions.

The later paper *Tropical Skeletons* studies the collection of Shilov boundary points of tropical fibres and its topology and functorial behaviour. This is

a more intrinsic construction relative to a tropicalization than simply choosing a semistable skeleton in advance. It helps explain when combinatorial points have canonical analytic lifts.

For the differential calculus, however, even a canonical section of one tropicalization does not eliminate all other charts. Equality of currents still requires every actual test. Chapter 24 explains why a topological inverse-limit theorem or a faithful section does not by itself transfer a Hodge theory.

References. [GRW16, Theorems 6.12, 8.4, 9.5 and 10.6]; [GRW17]; [Gub13].

Tropical contractions and SYZ fibrations. Yamamoto's 2026 theorem gives a further geometric use of tropicalization. For the Batyrev–Borisov degenerations over $k((t))$ specified in [Yam26, Section 3], with k algebraically closed of characteristic zero, a good unimodular subdivision and irreducible generic fibre, tropicalization identifies the essential skeleton piecewise integrally affinely with the dual intersection complex. Composing with a tropical contraction gives an affinoid torus fibration off its discriminant, with the prescribed integral affine structure [Yam26, Theorem 1.1]. This is a structured degeneration theorem; it supplies no general descent theorem for CLD currents or cohomology.

6.4 Recovering an affine analytification from finite coordinate data

Payne's theorem has a useful elementary core: every seminorm identity involves only finitely many functions. We prove a precise affine version using actual tropical images. This also identifies the point at which a topological reconstruction stops short of a differential or cohomological reconstruction.

Let A be a finitely generated reduced K-algebra and $X = \operatorname{Spec} A$. Fix a finite set F_0 of algebra generators. Let $\mathscr{F}$ be the directed set of finite subsets $F \subset A$ containing F_0. The functions in F give a closed embedding $\iota_F : X \hookrightarrow \mathbb{A}^F$. Use absolute-value coordinates and set

$$T_F = \{(|f(x)|)_{f\in F} : x \in X^{\mathrm{an}}\} \subset \mathbb{R}^F_{\geq 0}.$$

For $F \subset G$, coordinate projection maps T_G onto T_F. With $v = -\log|\cdot|$, the same construction takes values in $(\mathbb{R} \cup \{+\infty\})^F$. The value $+\infty$ records a zero and is essential for embeddings into affine space.

Theorem 6.4.1 (Affine coordinate reconstruction)**.** *The natural map*

$$\Phi : X^{\mathrm{an}} \longrightarrow \varprojlim_{F \in \mathscr{F}} T_F$$

is a homeomorphism. Here the analytic space on the left is described by multiplicative seminorms on A extending the absolute value of K, and every T_F is the actual image defined above.

Proof. A point in the inverse limit gives compatible numbers $s_f \geq 0$ for every $f \in A$: choose a finite F containing f and read its coordinate. Directedness and compatibility show that the answer does not depend on the choice. For $a, b \in A$, choose one finite set containing $a, b, a+b, ab, 0, 1$. Its coordinate vector belongs to T_F, so it is realized by an analytic point. Consequently

$$s_0 = 0, \quad s_1 = 1, \quad s_{ab} = s_a s_b, \quad s_{a+b} \leq \max(s_a, s_b).$$

Adding any $c \in K$ to the finite set gives $s_c = |c|$. Hence $f \mapsto s_f$ is a multiplicative seminorm extending K, and determines a point of X^{an}. Its image is the original compatible family. This proves surjectivity. If two analytic points have the same family, they have the same value on every $f \in A$, so they are the same seminorm. This proves injectivity.

Every coordinate of Φ is an evaluation map, so Φ is continuous. Conversely, the topology of X^{an} is generated by all evaluations $x \mapsto |f(x)|$, and each of these is a coordinate projection from some T_F. The inverse map is therefore continuous. No compactness assumption on X^{an} is needed for this final step. □

The proof uses actual images, not an unproved identification with a combinatorial tropical variety. The theorem equating analytic images with algebraically defined tropicalizations is a separate input. Likewise, gluing affine descriptions into the toric-embedding formulation for a quasiprojective variety is the content of the global theorem cited in Section 6.1.5. Keeping these ingredients separate makes the reconstruction argument reusable.

Proposition 6.4.2 (Finite-coordinate neighbourhoods)**.** *Every ordinary neighbourhood of a point in X^{an} contains a neighbourhood defined by finitely many conditions on values $|f_i|$. On the open subset where all f_i are nonzero, these conditions can be written using a finite tropicalization by analytic units.*

Proof. The evaluation topology has a subbasis consisting of inverse images of open subsets of $\mathbb{R}_{\geq 0}$ under single evaluations. Finite intersections form a basis, proving the first assertion. On the common nonvanishing locus, each regular f_i is an analytic unit and the change of coordinate $s \mapsto -\log s$ is a homeomorphism from $\mathbb{R}_{>0}$ to $\mathbb{R}$. This gives the second assertion. If some f_i vanishes at the point, extended coordinates or a different local construction are required; deleting that coordinate would discard information. □

6.5 What a refinement must preserve

The preceding theorem recovers points and topology because their defining relations are finite. Differential forms involve an additional quotient: two presentations may agree after restriction to tangent spaces of the tropical support. Integration also involves weights, and current convergence involves a test topology. These three structures cannot be read from the underlying inverse-limit topology alone.

Example 6.5.1 (A homeomorphism that loses the lattice)**.** The map $F : \mathbb{R} \to \mathbb{R}$, $t \mapsto 2t$, is a homeomorphism and is integral affine, but its tangent-lattice map $\mathbb{Z} \to \mathbb{Z}$ has index two. For a top superform $\alpha = a(y)d'y \wedge d''y$,

$$\int_{\mathbb{R}} F^*\alpha = 2\int_{\mathbb{R}} \alpha.$$

Thus topological injectivity does not give unimodularity, nor does it preserve calibrated integration with weight one. This is why faithful tropicalization includes an integral-affine requirement.

Example 6.5.2 (One tropical coordinate does not determine all tests)**.** On the analytic torus with coordinate z, the map $x \mapsto -\log|z(x)|$ forgets the values of $z - a$ for $a \in K^\times$. On the open locus $z \neq a$, the latter function is a unit and produces the additional smooth coordinate $-\log|z-a|$. Points with the same $|z|$ can have different $|z - a|$. A test depending on this additional coordinate therefore need not be a pullback from the first tropical line. An identity checked only against functions of $-\log|z|$ has not yet been checked against all original smooth tests.

To pass a calculation through chart refinements, one consequently needs: pullback preserving the quotient of forms; compatibility of d' and d'' with that

pullback; the weighted pushforward formula for integration; and continuity on the chosen spaces of compactly supported tests. Chapters 7–9 establish or explicitly import these ingredients in their stated categories. An inverse limit of spaces supplies none of them merely by being a homeomorphism.

The same distinction applies to cohomology. A direct limit of complexes requires actual transition maps of complexes. An identification with a complex on X^{an} then requires exactness or a comparison argument, and a statement about cohomology requires a further passage through kernels and images. This is why a tropical Hodge theorem cannot be transferred to Berkovich cohomology solely by citing Payne's topological theorem. It does not make such a transfer impossible; it specifies the additional mathematical work.

References. [Pay09, Theorem 4.2]; [GRW16, Sections 9–10]. The affine reconstruction above is proved directly for the explicitly defined system of actual images.

6.6 Tropical fibres and their Shilov points

Section 3.4 established the criterion for a singleton Shilov boundary. We now apply it to tropical fibres, using the formal-model comparison rather than identifying two different reductions. Now let U be an integral closed subvariety of a split torus over a complete, algebraically closed, nontrivially valued field. For a tropical point w with coordinates in the value group, the fibre U_w is strictly affinoid. Its algebra A_w is a quotient of a Laurent affinoid algebra. Distinguish its quotient Gauss norm from its spectral seminorm:

$$\|f\|_{\mathrm{quot}} = \inf_{g \mapsto f} \|g\|_w, \qquad \|f\|_{\mathrm{sp}} = \sup_{x \in U_w} |f(x)|.$$

A quotient norm is submultiplicative and need not itself be an analytic point. The polyhedral formal model has the initial degeneration as special fibre; the canonical formal model has the canonical reduction as special fibre. The latter is reduced, and these two special fibres need not agree.

GRW Sections 10.2–10.5 supply a finite comparison of the two formal models and the associated special-fibre projection formula. Under equidimensionality, tropical multiplicity one forces a single component of the canonical reduction, hence a singleton Shilov boundary. The unique point is the spectral seminorm. Identifying it with the quotient norm requires the stronger

reducedness condition on the initial degeneration. For arbitrary real w, the valued-extension and boundary-descent results are additional inputs. This discussion separates the geometric theorem from the elementary norm argument above.

6.7 Adding coordinates without changing the lattice

Lemma 6.7.1. *Let N be a lattice. Suppose $A : N \to \mathbb{Z}^r$ is injective with saturated image. For every integral map $B : N \to \mathbb{Z}^s$, the map (A, B) has saturated image.*

Proof. The cokernel of A is finitely generated and torsion-free, hence free. Therefore A admits an integral left inverse R. Composing R with projection onto the first factor gives an integral left inverse of (A, B):

$$(R \circ \mathrm{pr}_1) \circ (A, B) = \mathrm{id}_N .$$

The image of (A, B) is thus a direct summand, in particular saturated.

□

For tropicalizations, this lemma applies to each tangent lattice after taking a common integral-affine subdivision. It proves that adding coordinates preserves unimodularity wherever it was already established. The existence of a finite subdivision on a semistable skeleton is a separate assertion about the restrictions of rational functions.

Example 6.7.2. On an annular skeleton, the map $t \mapsto mt$ is injective for every nonzero integer m, but its lattice index is $|m|$. The map $t \mapsto (mt, t)$ is unimodular because projection to the second coordinate is an integral left inverse. More generally, $t \mapsto (at, bt)$ is unimodular exactly when $\gcd(a, b) = 1$. Thus injectivity and preservation of the integral lattice are distinct requirements.

6.8 Finite separation on a compact skeleton

Proposition 6.8.1. *Let S be a compact polyhedral space, and let $\mathcal{F}$ be a family of continuous real functions separating its points. Suppose a finite tuple F_0 drawn from $\mathcal{F}$ is locally injective on S and unimodular on a finite subdivision. Assume every further function is integral affine on a finite subdivision. Then a finite tuple extending F_0 gives a faithful embedding of S into a real vector space.*

Proof. Choose finitely many open sets V_i covering S such that F_0 is injective on each V_i. Set

$$O = \bigcup_i (V_i \times V_i), \qquad C = (S \times S) \setminus O.$$

The set O is an open neighborhood of the diagonal. Consequently C is compact and contains no diagonal point. For every pair (x, y) in C, some function f in $\mathcal{F}$ separates x and y. The inequality $f(x) \neq f(y)$ persists on an open neighborhood of the pair. Compactness gives finitely many such functions which separate every pair in C.

Together with F_0 these functions separate all points of S: a pair in O is separated by F_0, and a pair outside O by one of the added functions. Take a common finite subdivision. Lemma 6.7.1 preserves unimodularity on its cells. Finally, a continuous injection from a compact space into a Hausdorff real vector space is a homeomorphism onto its image.

□

Remark 6.8.2. Local injectivity on S is stronger than injectivity on each closed cell separately. Different cells meeting at a vertex may still be folded onto one another. Proposition 6.8.1 does not remove this hypothesis. Nor does its compactness argument apply directly to a noncompact skeleton with horizontal boundary.

In the GRW construction, semistable coordinates first supply integral-affine embeddings of individual canonical polyhedra. The global separation of different polyhedra uses additional rational functions and model geometry. Proposition 6.8.1 explains one possible finite assembly step when its stronger local hypothesis holds; it is not a replacement for the complete global theorem.

6.9 From faithful coordinates to differential calculus

Faithfulness concerns a specified subset of the analytic space. Its conclusions are injectivity and unimodularity on that subset. It does not say that every analytic function, smooth form, or current factors through a retraction onto the subset. A current is still tested on the full admitted test space, and a representative still requires compatibility under refinements.

The distinction matters already for a curve with good reduction: its smallest proper skeleton may be one point, while annular domains carry nonzero differential forms. Enlarging the system of charts recovers these directions. A finite faithful tropicalization of one skeleton therefore cannot by itself transfer a global Hodge or Green theory.

References. [GRW16, Section 9 and Sections 10.1–10.6]; [GRW17, Sections 2 and 8].

6.10 Pure dimension away from the tropicalized boundary

A moment map on a d-dimensional analytic space can have a lower-dimensional tropical image. Affinoidness and avoidance of the analytic boundary restore the expected dimension. The proof is useful beyond the dimension theorem: it turns a tropical projection into a finite morphism of affinoid spaces and then reads dimension at a Shilov boundary point.

Lemma 6.10.1 (A simultaneous integral projection)**.** *Let $P \subset \mathbb{R}^n$ be a finite polyhedral complex of dimension at most d, and let $B \subset P$ be contained in finitely many polyhedra of dimension at most $d-1$. For $\omega \in P \setminus B$, there is an integral linear map $Q : \mathbb{R}^n \to \mathbb{R}^d$ which is injective on every face of P and satisfies $Q\omega \notin Q(B)$. Assume that the stated finite polyhedral cover of B consists of closed polyhedra, none containing ω.*

Proof. For each face σ of P, require Q to be injective on its tangent space L_σ. For each polyhedron τ in the chosen cover of B, require injectivity on the linear span of $\tau - \omega$. The first spaces have dimension at most d, and the second have dimension at most $\dim \tau + 1 \leq d$. For each of these finitely many subspaces, injectivity is a nonempty open condition on the space of real

$d \times n$ matrices; its complement is given by vanishing of all relevant maximal minors and has empty interior. The intersection of the conditions is therefore nonempty and open. It contains a rational matrix. Clearing denominators gives an integral matrix without changing injectivity.

If $b \in B$ had $Qb = Q\omega$, choose a covering polyhedron τ containing b. Then $b - \omega \neq 0$ would lie in the kernel of Q on $\operatorname{span}(\tau - \omega)$, a contradiction. □

In the analytic application, B is compact and polyhedral, so a finite subdivision provides a closed polyhedral cover avoiding ω. The last condition in the lemma is material. Merely knowing that ω is not in a subset of a polyhedron would not imply that projection separates it from the whole covering polyhedron.

Theorem 6.10.2 (Affinoid tropical purity)**.** *Let K be complete and nontrivially valued, let X be a strict affinoid space of pure dimension d, and let $\varphi : X \to (\mathbb{G}_m^n)^{\mathrm{an}}$ be a moment map. Then $\varphi_{\mathrm{trop}}(X)$ has pure dimension d at every point of*

$$\varphi_{\mathrm{trop}}(X) \setminus \varphi_{\mathrm{trop}}(\partial X).$$

Open proof obligation (RC2). Prove the local germ and finite-projection step in the stated dimension, support and base-change scope; GRPL Theorem 6.4 and the proof of 6.6.

Remark 6.10.3. A constant moment map from a positive-dimensional proper space has a point as tropical image, although the source is boundaryless. This does not contradict the theorem, because the source is not affinoid. Conversely, a constant moment map from a positive-dimensional affinoid space sends its boundary to the same point, so the asserted open set is empty. Both hypotheses have visible roles.

This result concerns smooth tropicalization by analytic units. It does not prove purity of the image of a subvariety under the canonical tropicalization of a general abelian variety. That map comes from Raynaud uniformization and may forget an abelian part with good reduction. The distinction becomes important in Chapter 21, where canonical measures can live on cells of several dimensions.

References. [GRPL, Sections 6.1–6.5 and Theorem 6.6]. The proof above expands the finite-projection and Shilov-boundary argument; analytic dimension, properness, reduction and base-change results are explicit geometric inputs.

6.11 Matroids and the local models of tropical manifolds

Balancing is enough for Stokes, but not for all cohomological dualities. The matroidal local models used in tropical Poincaré duality satisfy a stronger incidence structure. This section constructs their fans from finite combinatorial data and proves balancing directly.

Definition 6.11.1. A matroid on a finite set E is specified by a rank function $r : 2^E \to \mathbb{Z}_{\geq 0}$ satisfying $0 \leq r(A) \leq |A|$, monotonicity, and submodularity

$$r(A \cup B) + r(A \cap B) \leq r(A) + r(B).$$

Its closure is $\mathrm{cl}(A) = \{e : r(A \cup \{e\}) = r(A)\}$, with A included. A flat is a set equal to its closure. A circuit is a minimal dependent set, where dependence means $r(A) < |A|$. We assume the matroid is loopless, meaning $r(\{e\}) = 1$ for every e. Its rank is $r(E)$.

For vectors $(v_e)_{e \in E}$ over a field, $r(A) = \dim \mathrm{span}\{v_e : e \in A\}$ gives the basic example. The axioms do not require such a representation. Thus a matroidal tropical model need not arise from a linear subspace over the particular ground field under consideration.

Lemma 6.11.2. *For $e \notin A$, one has $e \in \mathrm{cl}(A)$ if and only if some circuit C satisfies $e \in C \subseteq A \cup \{e\}$.*

Proof. Choose a maximal independent subset B of A. Its cardinality is $r(A)$, and adding e is dependent exactly when $r(A \cup \{e\}) = r(A)$. In that case a minimal dependent subset of $B \cup \{e\}$ is a circuit and must contain e, since B is independent. Conversely, in a circuit C, deleting any element leaves an independent set of rank $|C| - 1 = r(C)$. Thus $e \in \mathrm{cl}(C \setminus \{e\})$, and closure is monotone, giving the assertion for A. The facts about maximal independent sets and monotone closure follow from the rank axioms by adding one element at a time and applying submodularity. □

6.11.1 The circuit and flag descriptions

Let $N = \mathbb{Z}^E/\mathbb{Z}\mathbf{1}$, where $\mathbf{1} = \sum_{e \in E} e_e$. Define the Bergman support by the minimum convention

$$|B(M)| = \big\{ w \in \mathbb{R}^E/\mathbb{R}\mathbf{1} : \#\operatorname{argmin}_{e \in C} w_e \geq 2 \ \mathit{extforeverycircuit} C \big\}.$$

Adding a constant to all coordinates does not change this condition. For a nonempty proper flat F, let e_F be the class of its indicator vector. A flag of proper flats $\mathscr{F} = (F_1 \subsetneq \cdots \subsetneq F_k)$ gives the cone

$$\sigma_{\mathscr{F}} = \mathbb{R}_{\geq 0} e_{F_1} + \cdots + \mathbb{R}_{\geq 0} e_{F_k}.$$

Proposition 6.11.3. *The cones $\sigma_{\mathscr{F}}$ form a fan with support $|B(M)|$. If $r = r(E)$, this fan is pure of dimension $r-1$. Each cone is unimodular in the lattice of its linear span.*

Proof. A weight w satisfies the circuit condition exactly when each nonempty proper upper level set $F_c = \{e : w_e > c\}$ is a flat. If such a level set is not flat, choose $e \notin F_c$ in its closure. Lemma 6.11.2 gives a circuit contained in $F_c \cup \{e\}$, with e its unique minimum. Conversely, a unique minimum e in a circuit is separated from the other circuit weights by a level c, making e lie in the closure of F_c but not in F_c.

The distinct upper level sets form a nested flag. Subtracting the minimum weight expresses w uniquely as a positive linear combination of their indicator vectors. Allowing some coefficients to vanish gives the faces of the cones. Thus intersections of cones are their common flag faces, proving the fan assertion. Strict inclusions of flats increase rank. Every flag can be filled to one containing a flat of each rank $1, \ldots, r-1$, by adjoining an element outside the current flat and taking closure. Hence maximal cones have $r-1$ generators.

To check the lattice, put $F_0 = \varnothing$, $F_{k+1} = E$, and choose an element $b_i \in F_i \setminus F_{i-1}$ for $1 \leq i \leq k+1$. For $1 \leq i \leq k$, the integral coordinate difference $w \mapsto w_{b_i} - w_{b_{i+1}}$ descends to N and takes the value δ_{ij} on e_{F_j}. These differences give an integral left inverse to the generator map $\mathbb{Z}^k \to N$. Its image is saturated, proving both linear independence and unimodularity. □

Theorem 6.11.4 (Weight-one balancing)**.** *Assign weight one to each maximal cone of the flag fan. The resulting weighted fan is balanced.*

Proof. A codimension-one flag omits exactly one rank between two successive retained flats $A \subset B$, with $r(B) = r(A) + 2$; allow $A = \varnothing$ and $B = E$. The maximal cones containing it correspond to intermediate flats G of rank $r(A)+1$. In the rank-two minor $(M|B)/A$, the sets $G \setminus A$ are its rank-one flats. They partition $B \setminus A$: each element has a unique rank-one closure, and

two such closures either agree or are disjoint. If there are s of them, their indicator vectors satisfy

$$\sum_{G} e_G = e_B + (s-1)e_A.$$

Modulo the span of the codimension-one cone, the right side vanishes; when $B = E$, its vector already vanishes in N. By the preceding unimodularity argument, the image of each e_G is the primitive outgoing generator for its adjacent cone. Thus the displayed identity is exactly the balancing equation with all weights one. Rank one has no balancing faces to check. □

The proof uses only the rank-two intervals in the lattice of flats. This explains why matroidal fans have particularly controlled local stars. It also separates two uses of the word smooth: a unimodular cone is a lattice property, whereas being a tropical manifold requires every local model to have the specified matroidal form. A fan all of whose individual cones are unimodular need not have that local form.

References. [AK]; the use of these local models in tropical Poincaré duality is [JSS, Definition 4.15 and Theorem 4.33].

6.12 Accessible skeleta and the limits of functoriality

A semistable skeleton is constructed from a model and comes with its associated retraction. The tropical skeleton of a fixed torus embedding is defined through Shilov boundary points of tropical fibres. Neither description by itself supplies a class stable under all the operations needed in analytic geometry. Ducros–Thuillier introduce accessible skeleta to address this functorial problem. We retain the coefficient data c and the Γ-strict category of their paper. Locally in the G-topology, an accessible skeleton is covered by compact pieces that are images of elementary classical skeleta under analytic morphisms into compact analytic domains. The image is not required to consist only of points with maximal possible Abhyankar rank in the target. This is one reason the accessible category is larger than the classical category.

Theorem 6.12.1 (Ducros–Thuillier, fixed version)**.** *In the category of [DT26], accessible skeleta form the smallest admissible class containing the standard torus skeleta. This class contains classical skeleta, is stable under finite intersections and the specified piecewise linear subspaces, and has the following functorial properties.*

1. *If $f : Y \to X$ and $T \subset Y$ is accessible, with $f|_T$ topologically proper, then $f(T)$ is accessible.*
2. *If $\Sigma \subset X$ is accessible and every fibre over Σ is empty or zero-dimensional, then $f^{-1}(\Sigma)$ is accessible.*
3. *The function $x \mapsto d_K(x)$ is locally constant on an accessible skeleton. Its locus where $d_K(x) = \dim_x X$ is open and closed and is classical.*

Open proof obligation (RC2). Prove the accessible-skeleton geometric image assertion; this block explicitly offers DT26 and APW26 as imported proof.

An accessible skeleton is not, by its definition, a deformation retract of the ambient space. Nor does the image theorem assign the tropical multiplicities needed to push differential forms. To use it in an integration formula one still needs weights, lattice normalizations and compatibility with calibration. To use it in a Green-form construction one additionally needs descent in the chosen sheaf of actual forms. These are separate inputs. There is a further distinction from faithfulness. Faithful tropicalization concerns the behaviour of a chosen map on a chosen skeleton. In the strong faithfulness used in [APW26], injectivity on the relevant dual complex is supplemented by weight one on its top-dimensional tropical faces. The additional weight condition allows multiplicity formulas to control other branches above generic tropical points. A homeomorphism on the chosen skeleton alone supplies no such control.

References. [DT26, Theorems 9.1.12 and 9.2.3]; [APW26, Section 12.4].

Part II

Smooth forms and currents

Chapter 7

Superforms and positivity cones

The real differential algebra fixes the signs of integration, curvature and the divisor formula. We develop its positivity cones through their matrix representations, then prove the domination estimates used for currents. The distinction between ordinary and strong positivity is essential in middle degree.

7.1 The differential algebra

Definition 7.1.1 (Real superforms)**.** Let $V = \mathbb{R}^n$ with coordinates $x_1, \dots, x_n$, and let $\Omega \subset V$ be open. Write V'^* and V''^* for two copies of V^*. The algebra of superforms is

$$A^{p,q}(\Omega) = C^\infty(\Omega) \otimes \bigwedge^p V'^* \otimes \bigwedge^q V''^*.$$

All one-form generators anticommute, including generators from different copies. Thus a form is uniquely written as

$$\alpha = \sum_{|I|=p,\ |J|=q} \alpha_{IJ}\, d'x_I \wedge d''x_J, \qquad d'x_I = d'x_{i_1} \wedge \cdots \wedge d'x_{i_p},$$

where indices in a multiindex are increasing. The total degree is $p+q$. In particular,

$$\alpha \wedge \beta = (-1)^{(p+q)(r+s)} \beta \wedge \alpha \quad (\beta \in A^{r,s}).$$

Definition 7.1.2. The involution J is the algebra automorphism exchanging $d'x_i$ and $d''x_i$ and fixing functions. A (p,p)-form is symmetric if $J\alpha = (-1)^p\alpha$.

Moving the primed generators to the left gives

$$J(d'x_I \wedge d''x_J) = (-1)^{pq} d'x_J \wedge d''x_I. \tag{7.1.1}$$

Since p^2 and p have the same parity, symmetry in bidegree (p,p) means precisely $\alpha_{IJ} = \alpha_{JI}$. This is the symmetry that will become ordinary matrix symmetry.

Definition 7.1.3. The differentials are the total-degree derivations determined by

$$d'f = \sum_i \partial_i f \, d'x_i, \qquad d''f = \sum_i \partial_i f \, d''x_i,$$
$$d'd'x_i = d'd''x_i = d''d'x_i = d''d''x_i = 0.$$

For the displayed expansion of α, these definitions give

$$d'\alpha = \sum_{I,J,i} \partial_i \alpha_{IJ} \, d'x_i \wedge d'x_I \wedge d''x_J,$$
$$d''\alpha = (-1)^p \sum_{I,J,i} \partial_i \alpha_{IJ} \, d'x_I \wedge d''x_i \wedge d''x_J.$$

Proposition 7.1.4. *The operators satisfy*

$$(d')^2 = (d'')^2 = 0, \quad d'd'' + d''d' = 0, \quad d'' = Jd'J.$$

For a smooth function f, one has

$$\mathrm{d'd''}f = \sum_{i,j} \partial_i\partial_j f \, d'x_i \wedge d''x_j.$$

Proof. The terms in $(d')^2\alpha$ cancel in pairs: the second derivatives are symmetric in the two new indices, whereas the primed generators are antisymmetric. The same argument applies to d''. In $d'd''\alpha + d''d'\alpha$, the derivatives again agree, and exchanging the two new generators reverses the sign. The identity with J can be checked on functions and the one-form generators, after which the derivation rule proves it on the algebra. Applying d' to $d''f$ gives the last formula. □

Remark 7.1.5 (A sign that must be corrected in the 2012 text). The displayed commutativity exponent in [CLD12, §1.2.6, p. 14] is $pp' + qq'$. Read literally, it would make $d'x_i$ commute with $d''x_j$, contrary to (7.1.1) and to the total-degree Leibniz rule used later in that text. The displayed exponent in [CLD12, §3.1.4, p. 36] is also incompatible with those formulas. Throughout these notes the exponent is $(p+q)(p'+q')$. This is an explicit correction of the displayed formulas, not a second convention being used implicitly.

Proposition 7.1.6 (Conversion of the two operators)**.** *Let J be the involution in Definition 7.1.1. The algebra automorphism in [CLD25, §1.1.2] satisfies*

$$J_{25}(d'x_i) = d''x_i, \qquad J_{25}(d''x_i) = -d'x_i.$$

On forms of bidegree (p,q) one has

$$J_{25} = (-1)^q J, \qquad J_{25}^2 = (-1)^{p+q}\,\mathrm{id}\,. \tag{7.1.2}$$

For a form of type (p,p), the old symmetry condition $J\alpha = (-1)^p\alpha$ is equivalent to $J_{25}\alpha = \alpha$. The strong generators and the signed squares $(-1)^{p(p-1)/2}\beta\wedge J\beta$, with β of type $(p,0)$, are unchanged when J is replaced by J_{25}. Hence the three cones in Definition 7.2.2 agree under this conversion.

Proof. Every double-primed generator contributes one extra minus sign under J_{25}, giving the first equality. Applying it twice gives $(-1)^{p+q}J^2$. The symmetry assertion follows by setting $q = p$. A form β of type $(p,0)$ contains no double-primed factor, so $J_{25}\beta = J\beta$. This proves the statement about the generators and, by the complementary wedge definitions, all three cones. □

Put $s_p = (-1)^{p(p-1)/2}$ and fix the positive top form

$$\mathrm{vol}_n = d'x_1 \wedge d''x_1 \wedge \cdots \wedge d'x_n \wedge d''x_n = s_n\, d'x_{1\cdots n} \wedge d''x_{1\cdots n}.$$

Integration means $\int_\Omega f\,\mathrm{vol}_n = \int_\Omega f\,dx_1\cdots dx_n$. The choice is compatible with the lattice normalization when the coordinates form a lattice basis. Under an arbitrary linear change of coordinates the coefficient of this top form changes by the square of the determinant; the normalization of the integral must therefore be transformed as well.

7.2 Symmetric forms as matrices

Remark 7.2.1 (Matrix normalization). The factor s_p removes the sign caused by collecting all primed generators. For a real $m \times m$ matrix H, where $m = \binom{n}{p}$, define

$$\alpha_H = s_p \sum_{I,J} H_{IJ}\, d'x_I \wedge d''x_J. \tag{7.2.1}$$

Thus H is symmetric exactly when α_H is symmetric. If $a = \sum_I a_I d'x_I$, then

$$s_p\, a \wedge Ja = \alpha_{aa^t}.$$

Definition 7.2.2. A symmetric form α_H is positive if H is positive semidefinite. It is strongly positive if H is a finite sum of matrices aa^t with $a \in \bigwedge^p V^*$ decomposable. It is weakly positive if $z^t H z \geq 0$ for every decomposable p-vector $z \in \bigwedge^p V$.

The zero element is permitted in all three cones. For a smooth form these conditions are imposed at every point. In the definition of strong positivity, the pointwise decomposition need not have smoothly varying summands. This chapter uses the pointwise cone condition; no smooth choice of eigenvectors is needed.

Proposition 7.2.3. *A symmetric form is positive if and only if it is a finite sum $s_p \sum_\nu a_\nu \wedge Ja_\nu$, without a decomposability restriction. There are inclusions*

$$C^p_s \subset C^p_+ \subset C^p_w.$$

All three cones coincide for $p = 0, 1, n-1, n$.

Proof. The spectral theorem gives $H = \sum_\nu \lambda_\nu v_\nu v_\nu^t$ with $\lambda_\nu \geq 0$ precisely when H is positive semidefinite. Take $a_\nu = \sqrt{\lambda_\nu} v_\nu$. The converse follows from $z^t a a^t z = (a^t z)^2$. The two cone inclusions follow from the same formula.

In the four indicated degrees every exterior vector and covector is decomposable. For degree $n-1$, a nonzero exterior vector is contraction of a volume vector by a nonzero covector. A basis in which that covector is the last dual basis vector expresses the contraction as a single wedge of $n-1$ basis vectors. Thus the decomposability restrictions disappear. □

Proposition 7.2.4 (Hessian criterion)**.** *Let $u \in C^\infty(\Omega)$. Then*

$$d'd''u = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} d'x_i \wedge d''x_j.$$

This form is positive if and only if the real Hessian of u is positive semidefinite at every point. If Ω is convex, this is equivalent to convexity of u.

Proof. The displayed formula follows from the definitions. A symmetric $(1,1)$-form is represented by a real symmetric matrix. Evaluating along a real direction v gives the quadratic expression $v^{\mathsf{T}} \operatorname{Hess}(u) v$. The three positivity cones in this degree agree with the cone of positive semidefinite matrices. Finally, the restriction of u to the segment $x + tv$ has second derivative $v^{\mathsf{T}} \operatorname{Hess}(u)(x + tv)v$; the one-dimensional criterion for convexity proves the last assertion. □

Proposition 7.2.5 (Wedge duality)**.** *The wedge pairing between symmetric forms in complementary bidegrees is nondegenerate. Under this pairing the positive cone is self-dual across complementary degrees, and the strongly positive and weakly positive cones are dual.*

Proof. Let $q = n - p$. For increasing multiindices I of size p and A of size q, let $C_{I,A}$ be the sign of $dx_I \wedge dx_A$ relative to $dx_1 \wedge \cdots \wedge dx_n$ when $A = I^c$, and zero otherwise. The matrix C is a signed permutation matrix. Expanding the wedge gives

$$\alpha_H \wedge \alpha_K = \operatorname{tr}(HCKC^t) \operatorname{vol}_n. \tag{7.2.2}$$

Indeed the scalar sign is $s_p s_q (-1)^{pq} / s_n = 1$. Since C is invertible, nondegeneracy follows. Testing against all positive semidefinite K is equivalent to H being positive semidefinite: rank-one tests give all inequalities $z^t H z \geq 0$.

The complement map C takes decomposable q-covectors to decomposable p-vectors. To see this, choose an orthonormal basis adapted to the span of the covectors in their wedge; its complementary basis gives the image under C, up to a scalar and sign. Thus testing against strongly positive K gives exactly weak positivity of H.

For the reverse duality, the strong cone is closed. Normalize its nonzero generators by $\operatorname{tr}(aa^t) = 1$. The resulting set is compact, being the image of the closed set of unit decomposable covectors. Its convex hull is compact in finite dimension. Every nonzero element of the cone is its trace times a point

of that convex hull. A convergent sequence in the cone has bounded traces, so its limit remains in the cone. The finite-dimensional separation theorem now identifies the double dual with the cone itself. □

7.3 A positive control form

The current estimates will require a fixed form dominating every symmetric coefficient matrix. Define

$$\omega = \sum_i d'x_i \wedge d''x_i, \qquad \beta_p = \frac{\omega^p}{p!}.$$

Lemma 7.3.1. *One has $\beta_p = \alpha_{\mathrm{Id}_m}$, where $m = \binom{n}{p}$. In particular β_p is strongly positive and lies in the interior of the ordinary positive cone.*

Proof. The degree-two factors in ω commute, and a repeated factor has zero wedge. Every set of p distinct indices therefore occurs $p!$ times. Moving the double-primed factors to the right contributes s_p, giving (7.2.1) with the identity matrix. The result is a sum of decomposable squares. If a symmetric matrix E has operator norm less than one, $\mathrm{Id}_m + E$ is positive definite; this proves the interior assertion. □

Proposition 7.3.2 (Coefficient domination)**.** *Let $\eta = \alpha_H$ be a smooth symmetric form supported in a compact set $K \Subset \Omega$. Choose $0 \le \chi \in C_c^\infty(\Omega)$ with $\chi = 1$ near K. If $M = \sup_K \|H(x)\|_{\mathrm{op}}$, then*

$$M\chi\beta_p + \eta \ge 0, \qquad M\chi\beta_p - \eta \ge 0. \tag{7.3.1}$$

Both inequalities mean ordinary positivity.

Proof. On K, every eigenvalue of $H(x)$ lies in $[-M, M]$, so $M\mathrm{Id}_m \pm H(x)$ is positive semidefinite. Off K, $H = 0$ and $M\chi\mathrm{Id}_m$ is positive semidefinite. This proves the assertions at every point, without choosing eigenvectors smoothly. □

Consequently every compactly supported symmetric form is a difference of two compactly supported positive forms. This is the decomposition needed for positive currents. Requiring both summands to be strongly positive changes the statement, and the next section shows that the stronger requirement is impossible in general.

Remark 7.3.3. One may replace the operator norm by a coefficient norm. If $|H_{IJ}| \le A$, then $\|H\|_{\mathrm{op}} \le mA$: for $|v| = 1$, $|(Hv)_I| \le A \sum_J |v_J| \le A\sqrt{m}$, and summing over I gives the bound. Constants in chart estimates may therefore depend on the number of coefficient directions, but not on derivatives of the test form.

7.4 The Plücker obstruction

Take $n = 4$ and $p = 2$. Order the exterior coordinates by

$$12,\ 13,\ 14,\ 23,\ 24,\ 34.$$

Define the symmetric matrix

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Its quadratic form is

$$z^t B z = 2(z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{23}). \tag{7.4.1}$$

Lemma 7.4.1. *The right side vanishes for every decomposable $z \in \bigwedge^2 \mathbb{R}^4$. Moreover $B^2 = \mathrm{Id}_6$ and $\operatorname{tr} B = 0$.*

Proof. If $z = a \wedge b$, then $z \wedge z = a \wedge b \wedge a \wedge b = 0$. Expanding in the volume basis gives

$$z \wedge z = 2(z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{23})\, e_1 \wedge e_2 \wedge e_3 \wedge e_4.$$

This proves the relation. The matrix B exchanges the first and sixth coordinates, exchanges the second and fifth with a minus sign, and exchanges the third and fourth. Applying it twice is the identity, and all its diagonal entries vanish. □

Theorem 7.4.2 (Failure of strong-cone spanning)**.** *In bidegree* $(2,2)$ *on* $\mathbb{R}^4$*, the strongly positive cone does not span the space of symmetric forms. The weakly positive cone contains the nonzero line* $\mathbb{R}\alpha_B$*. In particular it is not pointed.*

Proof. The nonzero linear functional $\ell(H) = \operatorname{tr}(BH)$ vanishes on every strong generator aa^t, because $\operatorname{tr}(Baa^t) = a^tBa = 0$ for decomposable a. Hence it vanishes on the linear span of the strong cone. But $\ell(B) = \operatorname{tr}(B^2) = 6$, so that span is proper. The same Plücker relation on decomposable vectors gives $z^t(tB)z = 0$ for every $t \in \mathbb{R}$. Thus every $t\alpha_B$ is weakly positive. □

Example 7.4.3 (A positive form which is not strongly positive)**.** Let $a = e_{12}+e_{34}$ in exterior coordinate notation and put $H = aa^t$. Then H is positive semidefinite, while $\operatorname{tr}(BH) = 2$. Since this functional vanishes on the entire strong span, α_H is not even a difference of strongly positive forms.

This example pinpoints the interior error. The form β_2 is strongly positive, but it cannot lie in the interior of the strong cone in the full symmetric space. Every matrix $M\mathrm{Id}_6 + H$ in Example 7.4.3 has $\ell(M\mathrm{Id}_6 + H) = 2$, regardless of M, and therefore is not strongly positive. Enlarging the coefficient of a fixed strong form cannot repair this obstruction.

Remark 7.4.4 (Relation to the sources). This is the obstruction in [BGJK, Example 2.3.6]. It contradicts the real statement of [CLD12, Lemma 5.2.3], even after symmetry is imposed. That lemma's proof uses the interior assertion just disproved. The analytic Lemma 5.3.3 cites it through tropical charts; its printed proof therefore loses this input. An analytic counterexample would additionally require a faithful realization of the relevant forms and positivity through the analytic charts, and is not supplied by the real calculation alone.

Similarly, [GK19, Remark 1.12(i)] asserts that weak positivity of both α and $-\alpha$ implies $\alpha = 0$. The form α_B disproves this assertion in its stated real setting. The analytic uses of this remark, including Lemma 2.16, require a separate analysis of their realization and quotient relations. We do not label those analytic statements disproved solely by this example.

7.5 Pullbacks and products

The corrected coefficient estimates use ordinary positivity. Before passing to currents, we record the operations that preserve this cone. These proofs also explain which cone conditions are available when multiplying a current by a smooth form.

Proposition 7.5.1. *Pullback by an affine map preserves each of strong, ordinary and weak positivity.*

Proof. Let $F : W \to V$ have linear part L. On exterior covectors the pullback is $\bigwedge^p L^*$. It takes a square aa^t to the square of the pulled-back covector; decomposable covectors stay decomposable or become zero. This proves the strong and ordinary assertions. For a decomposable p-vector z on W, the exterior image $(\bigwedge^p L)z$ is decomposable or zero. The quadratic form of $F^*\alpha_H$ evaluated at z is the quadratic form of H evaluated at that image, proving weak positivity. □

Proposition 7.5.2. *Products of positive forms are positive. Products of strongly positive forms are strongly positive. The product of a weakly positive form and a strongly positive form is weakly positive. Products in degrees exceeding the dimension are zero.*

Proof. For exterior covectors a, b of degrees p, r, respectively, the signs give

$$(s_p a \wedge Ja) \wedge (s_r b \wedge Jb) = s_{p+r}(a \wedge b) \wedge J(a \wedge b).$$

Indeed moving b past Ja gives $(-1)^{pr}$ and $s_p s_r (-1)^{pr} = s_{p+r}$. Decompose the positive matrices into squares and use this identity term by term. If both factors are strong, their generating covectors are decomposable, so their wedge is decomposable or zero.

For the last assertion it suffices to consider one decomposable square $s_r b \wedge Jb$. Evaluate the product on a decomposable $(p+r)$-vector and restrict to the subspace it spans. If b restricts to zero there is nothing to prove. Otherwise choose a basis of this subspace so that b is a scalar multiple of the wedge of the last r dual basis vectors. The value of the product is a nonnegative square factor times the value of the first form on the wedge of the first p basis vectors. Weak positivity makes the latter value nonnegative. □

For a smooth function, Proposition 7.1.4 identifies the matrix of $\mathrm{d}'\mathrm{d}''f$ with its Hessian. Thus positivity, strong positivity and weak positivity of $\mathrm{d}'\mathrm{d}''f$ are equivalent. On a convex open set they are equivalent to convexity of f: along every line the second derivative is nonnegative, which is the one-variable criterion for convexity; conversely, second differences of a smooth convex function give a nonnegative Hessian. This special agreement in degree $(1,1)$ must not be extrapolated to all degrees.

7.6 A homotopy formula for the real double complex

The two exterior factors of a superform have different roles when one studies only one differential. This gives an elementary Poincaré lemma on an ordinary real open set, but also explains why a statement about the double complex requires more than two separate exactness statements.

Let $\Omega \subset \mathbb{R}^n$ be open and star-shaped about 0. Write every superform uniquely as

$$\alpha = \sum_{|J|=q} \alpha_J \wedge d''x_J, \qquad \alpha_J \in \Omega^p(\Omega),$$

where the first factor is an ordinary differential form in the $d'x_i$. In this description, d' is the ordinary exterior derivative on each α_J, while the $d''x_J$ form a fixed coefficient space.

Definition 7.6.1. For $p > 0$, define $H' : A^{p,q}(\Omega) \to A^{p-1,q}(\Omega)$ coefficientwise by

$$(H'\alpha)_x(v_1, \dots, v_{p-1}; w_1, \dots, w_q) = \int_0^1 t^{p-1} \alpha_{tx}(x, v_1, \dots, v_{p-1}; w_1, \dots, w_q)\, dt.$$

Set $H' = 0$ for $p = 0$. Define $E'\alpha = 0$ for $p > 0$; for $p = 0$, let E' replace each scalar coefficient of α by its value at 0, retaining the fixed d''-basis.

The semicolon separates the primed and double-primed alternating slots. Only the primed slots are scaled in this formula. In particular, E' is not the full geometric pullback by the constant map: that pullback would also kill positive double-primed degree.

Proposition 7.6.2 (Radial homotopy)**.** *The operators satisfy*

$$d'H' + H'd' = \mathrm{id} - E'.$$

Consequently the d'-cohomology on Ω vanishes in positive primed degree, while in primed degree zero it is the space of constant double-primed forms $\bigwedge^q(\mathbb{R}^n)^*$.

Proof. It suffices to prove the usual homotopy identity for each α_J. Let $F : [0,1] \times \Omega \to \Omega$ be $F(t,x) = tx$. Decompose the ordinary pullback of a primed form as $F^*\alpha_J = dt \wedge \beta_t + \gamma_t$, where γ_t contains no dt. The coefficient of dt in the identity $d(F^*\alpha_J) = F^*(d\alpha_J)$ is

$$\partial_t \gamma_t - d_x \beta_t = \text{the } dt\text{-coefficient of } F^*(d\alpha_J).$$

Integrate from 0 to 1. The boundary term is $\alpha_J - F_0^*\alpha_J$, and $\int_0^1 \beta_t\, dt$ is precisely the coefficientwise operator in Definition 7.6.1. This proves the formula. Smoothness at $t = 0$ follows directly from the displayed pullback, or from the factor t^{p-1} when $p > 0$.

If $p > 0$ and $d'\alpha = 0$, the identity gives $\alpha = d'H'\alpha$. For $p = 0$, closedness says that every coefficient has zero differential. The star-shaped open set is connected, so those coefficients are constant. There are no forms of primed degree -1, giving the asserted degree-zero cohomology. □

There is an analogous statement for d'', obtained by exchanging the exterior factors with the fixed sign convention. The two statements concern separate complexes. The operator H' was not constructed to commute with d'', and no such commutation is needed for Proposition 7.6.2. This point matters when one tries to obtain a $d'd''$-potential by applying two homotopies successively.

7.7 The symmetric Hessian potential theorem

In bidegree $(1,1)$ the extra compatibility has a familiar form: the coefficient matrix must be symmetric. On a star-shaped open set this condition, together with closedness, suffices for a scalar potential.

Theorem 7.7.1. *Let $\Omega \subset \mathbb{R}^n$ be open and star-shaped, and let*

$$\alpha = \sum_{i,j=1}^{n} A_{ij}(x) d'x_i \wedge d''x_j$$

have smooth coefficients with $A_{ij} = A_{ji}$. If $d'\alpha = 0$, then there is a smooth function u with $\alpha = d'd''u$. The potential is unique up to addition of an affine function. If Ω is convex, α is positive if and only if u is convex.

Proof. For each j, the primed one-form $\alpha_j = \sum_i A_{ij} d'x_i$ is closed. Proposition 7.6.2, after translating the star centre to zero, gives a smooth function b_j with $\partial_i b_j = A_{ij}$. Symmetry implies $\partial_i b_j = \partial_j b_i$. Hence the ordinary one-form $\sum_j b_j dx_j$ is closed. Apply the same homotopy formula once more to obtain u with $\partial_j u = b_j$. Its Hessian is A, proving the asserted identity.

If u and v are two potentials, every first derivative of $u - v$ has zero differential and is therefore constant on the connected set Ω. Thus $u - v$ is affine. Finally, positivity of a symmetric $(1,1)$-superform is positive semidefiniteness of its matrix. On a convex domain, the second derivative of the restriction of u to every line segment is $\xi^{\mathsf{T}} A \xi$. Nonnegativity of these second derivatives is equivalent to convexity. □

The same calculation shows that symmetry and d'-closedness imply d''-closedness here: the constructed Hessian is closed under both differentials. It also identifies the ambiguity more precisely than saying only that it is harmonic. In ordinary real coordinates the smooth Hessian kernel consists of affine functions; on an analytic space, current-harmonic functions can form a much larger class.

Example 7.7.2 (Double closedness without a scalar potential)**.** On any open ball in $\mathbb{R}^2$, the constant form

$$\alpha = d'x_1 \wedge d''x_2 - d'x_2 \wedge d''x_1$$

is both d'-closed and d''-closed. It cannot equal $d'd''u$ for a smooth scalar function, because that would give $\partial_1\partial_2 u = 1$ and $\partial_2\partial_1 u = -1$. It cannot have a scalar distributional potential either, since distributional partial derivatives commute. Thus two separate Poincaré lemmas do not imply an unrestricted local $d'd''$-lemma for this real double complex.

7.7.1 The passage to a polyhedral or analytic space

The preceding results are on an open subset of one real vector space. A form on a polyhedral support is a quotient of ambient forms by tangential relations, and the tangent directions can change across faces. Before using a homotopy on that quotient, one must prove that it preserves those relations and has the required support and domain properties. An ambient homotopy formula alone does not do this.

On a Berkovich space there is a further compatibility under refinements of moment charts. A potential constructed on one tropical presentation must be independent of the presentation or be shown to glue with the permitted transition functions. These are precisely the additional issues in local-potential theorems for currents and in comparisons of tropical and analytic cohomology. The explicit Euclidean theorem is a local building block, not a theorem covering every analytic chart quotient.

7.8 A radial homotopy on a polyhedral star

The ordinary de Rham homotopy contracts all covectors of a differential form. For the d'' complex of superforms, the primed degree must remain fixed. The correct operator contracts only the double-primed variables. This produces a local Poincaré lemma and identifies its degree-zero kernel with the multitangent coefficient space.

Let $C \subset \mathbb{R}^r$ be the support of a finite fan, and let B be an open ball centred at its vertex 0. Superforms on $C \cap B$ are restrictions of smooth ambient superforms, with two forms identified when their evaluations on tangent vectors of every cone agree. It is enough to work with germs at 0, so a common ambient representative is available. Write a representative as

$$\alpha = \sum_{|I|=p} d'x_I \wedge \omega_I, \qquad \omega_I = \sum_{|J|=q} a_{IJ}(x)\, d''x_J.$$

With the ordered-wedge convention of this book, d'' acts on each ω_I as $(-1)^p$ times the ordinary exterior derivative. This sign is essential in the homotopy formula.

For an ordinary q-form ω with $q > 0$, set

$$(H\omega)_x(v_1, \dots, v_{q-1}) = \int_0^1 t^{q-1} \omega_{tx}(x, v_1, \dots, v_{q-1})\, dt.$$

For $q = 0$ set $H = 0$, and define

$$K_p\alpha = (-1)^p \sum_{|I|=p} d'x_I \wedge H\omega_I.$$

The operator K_p has bidegree $(0, -1)$. Define $E\alpha$ to be zero for $q > 0$, and for $q = 0$ define it to be the constant-coefficient primed form $\sum_I a_I(0)d'x_I$, restricted to the fan.

Proposition 7.8.1. *The operators descend to superforms on the fan, and satisfy*

$$d''K_p + K_p d'' = \mathrm{id} - E.$$

Consequently every d''-closed germ of bidegree (p, q) with $q > 0$ is d''-exact.

Proof. Suppose an ambient representative vanishes on tangent vectors of every cone. For x in a cone σ, all the points tx with $t > 0$ remain in σ, and $x \in L_\sigma$. Evaluating $K_p\alpha$ on primed and double-primed vectors in L_σ therefore evaluates α_{tx} only on vectors in L_σ, including the inserted vector x. The integrand is zero. Its value at $t = 0$ follows by continuity and does not affect the integral. Thus K_p preserves the relation defining forms on C.

For $q = 0$, if an ambient primed form vanishes on each cone, evaluate along an interior ray of a cone and let its parameter tend to zero. Its constant value at zero annihilates $\bigwedge^p L_\sigma$ for every σ, so E also preserves the relation. For $q > 0$ there is nothing to check for E.

On ambient ordinary forms, differentiation under the integral gives $dH + Hd = \mathrm{id} - e_0^*$, where e_0^* keeps only degree-zero values at the origin. One can check this on a coefficient $a_J(x)$: the surviving integrand is the derivative in t of $t^q a_J(tx)$, with the evaluation on the appropriate vectors. The endpoint at $t = 1$ gives ω; the endpoint at $t = 0$ is zero for $q > 0$ and is $\omega(0)$ for $q = 0$. The two factors $(-1)^p$ from d'' and K_p cancel. Applying this identity to every ω_I proves the displayed formula on ambient forms, hence on the quotient. If $d''\alpha = 0$ and $q > 0$, it gives $\alpha = d''K_p\alpha$. □

7.8.1 The kernel is a multi-tangent dual

Put

$$F_p(0) = \sum_{\sigma \in C} \bigwedge^p L_\sigma \;\subset\; \bigwedge^p \mathbb{R}^r, \qquad F^p(0) = \mathrm{Hom}_{\mathbb{R}}(F_p(0), \mathbb{R}).$$

Corollary 7.8.2. *The space of d''-closed $(p,0)$ germs at the vertex is naturally $F^p(0)$. The local sequence*

$$0 \longrightarrow F^p \longrightarrow A^{p,0} \xrightarrow{d''} A^{p,1} \xrightarrow{d''} A^{p,2} \longrightarrow \cdots$$

is exact on finite polyhedral charts, with the natural face restrictions on F^p.

Proof. For a closed $(p,0)$ form, Proposition 7.8.1 gives $\alpha = E\alpha$ on a sufficiently small star. Thus it is represented by a constant ambient p-covector. Such a covector represents zero exactly when it annihilates every $\bigwedge^p L_\sigma$, equivalently their sum $F_p(0)$. The quotient of $\bigwedge^p(\mathbb{R}^r)^*$ by this annihilator is $F_p(0)^*$, since every linear functional on a subspace of a finite-dimensional vector space extends to the whole space. Exactness in positive double-prime degree is the proposition. Around a point in any face, translate the point to the origin and use the local star. Restriction to a smaller face star induces the indicated restrictions on the coefficient spaces. □

The sum in F_p must be formed after taking exterior powers. Replacing it by $\bigwedge^p(\sum_\sigma L_\sigma)$ changes the coefficient sheaf in general. The two-plane example in Section 24.7 makes the difference explicit. The homotopy proof identifies the correct space directly from the vanishing relation for actual forms.

Remark 7.8.3. If an integral linear map sends one fan to another, the homotopy commutes with its ordinary superform pullback: linear maps commute with radial scaling, and contraction by x corresponds to contraction by its image. Translation gives the same assertion for affine charts with specified centres. This compatibility is enough for the local polyhedral argument. A global homotopy on a Berkovich space would additionally require compatible analytic domains and centres; it is not supplied by this computation.

7.8.2 From a local resolution to global cohomology

On a paracompact polyhedral space with finite-part charts, the sheaves $A^{p,q}$ admit smooth partitions of unity. They are therefore fine. The exact sequence above is a fine resolution of F^p, so its global cohomology computes $H^q(X, F^p)$. Here we use the standard fine-resolution theorem of sheaf cohomology; its mechanism can be seen in a Čech double complex.

For a locally finite cover (U_i) and subordinate partition (ρ_i), a Čech cochain c with values in $A^{p,q}$ has a homotopy

$$(hc)_{i_0\dots i_{k-1}} = \sum_j \rho_j c_{j i_0\dots i_{k-1}}, \qquad \delta h + h\delta = \mathrm{id} \quad (k > 0).$$

The sum is locally finite; extension by zero is valid because the multiplier is supported in U_j. Thus the rows have no positive Čech cohomology. In the other direction, local exactness of the d'' complex leaves only F^p. Passing to sufficiently fine covers, the two computations of the total complex give the desired comparison.

This proves the finite-part local mechanism behind the tropical Dolbeault comparison. Extended tropical spaces also have sedentary boundary strata. Their form definition includes compatibility with projections to those strata, and their local Poincaré lemma requires that additional structure. The full comparison, including compact supports and extended charts, is the theorem of Jell–Shaw–Smacka stated in Chapter 24. The present radial proof does not remove the boundary hypotheses from that theorem.

References. [JSS, Sections 2–3]; the explicit homotopy above is the radial de Rham argument applied only to the double-primed variables, with the quotient and sign checks included.

Chapter 8

Intrinsic smooth forms, integration and Stokes

8.1 Original smooth forms on an analytic space

Let K be a complete non-Archimedean field. For the current constructions below, let X be a good Hausdorff K-analytic space of pure dimension n. When discussing the algebraic presentation, we impose Gubler's hypotheses: K is algebraically closed and nontrivially valued, and X is the analytification of an algebraic variety over K. Statements on that presentation will not be used to enlarge its original hypotheses silently.

Definition 8.1.1 (Tropical presentation of an original smooth form)**.** An analytic moment map on an analytic domain W is an analytic morphism

$$f = (f_1, \ldots, f_r) : W \longrightarrow (\mathbb{G}_m^r)^{\mathrm{an}}.$$

The functions f_i are invertible. We use the tropicalization convention

$$f_{\mathrm{trop}}(x) = (-\log|f_1(x)|, \ldots, -\log|f_r(x)|).$$

An original smooth (p,q)-form is locally represented by a pullback $f_{\mathrm{trop}}^*\alpha$, where α is a smooth superform on the tropical image. Presentations are compared after common refinement, and compatible local presentations are glued as a sheaf. Denote the resulting sheaf by $A_{\mathrm{sm}}^{p,q}$.

Here a superform on a polyhedral subset is the restriction of a superform from a real open neighborhood, modulo forms whose restrictions to tangent

spaces of all its cells vanish. The tangent-space quotient is essential: an ambient form can be nonzero while its restriction to the polyhedral set is zero. Subdivision does not change this quotient. Likewise, the analytic definition requires common refinements and sheafification; it is not the choice of one permanent tropical coordinate system on X.

Remark 8.1.2 (Functions and sheafification). The sheaf $A^{0,0}_{\mathrm{sm}}$ consists of functions locally of the form $\varphi(-\log|f_1|,\dots,-\log|f_r|)$ with φ smooth. On a fixed tropical chart, an arbitrary smooth form need not be represented by a single superform on that same chart; see [Gubler, Remark 5.7]. This is why a definition that merely says "pull back superforms along a tropicalization" is incomplete.

Example 8.1.3 (Continuous and tropically continuous functions)**.** On $\mathbb{A}^{1,\mathrm{an}}_K$, the function $x \mapsto |T(x)|$ is continuous. Near the rigid point $T = 0$, every analytic invertible function has constant absolute value on a sufficiently small neighborhood. Every function given near 0 by a continuous function of finitely many logarithms of invertible functions is therefore locally constant there. The function $|T|$ is not locally constant near 0 and has no such tropical presentation.

Consequently, continuity of a function or metric on X does not mean continuity of its coefficients in one local tropical presentation. This is the distinction made in [CLD12, §3.1.6].

Theorem 8.1.4 (Gubler's algebraic presentation)**.** *Let K be complete, algebraically closed, and nontrivially valued, and let Z be an algebraic variety over K. The smooth-form sheaf constructed on Z^{an} using algebraic moment maps on very affine Zariski open subsets agrees with the original smooth-form sheaf constructed using analytic moment maps. The identification respects d', d'', J, and wedge product.*

Proof. Use the common variety class of NAG-003062 and NAG-003078: integral separated schemes of finite type over the stated field. Use the same coordinates $t_f = -\log|f|$ and the same slot interchange J_0 in both presentations. Restricting algebraic units to an ordinary affinoid neighbourhood gives an analytic moment germ (NAG-002423–002424).

Conversely choose an ordinary affinoid witness for an analytic germ and work inside its ordinary interior. Include in one tuple all units of any finite germ comparison under consideration. By NAG-003062 there are a compact

Laurent affinoid neighbourhood C, a very affine algebraic open U and algebraic units g_i on U such that $C \subset U^{\mathrm{an}}$ lies in the witness and $|g_i| = |f_i|$ throughout C. The construction truncates in the actual Laurent quotient with error smaller than $\min_i \|f_i^{-1}\|^{-1}$ for a nonempty tuple (NAG-003061). It gives an ordinary open chart about the point inside C. On this chart the g_i are constants times characters for the algebraic unit lattice, and their tropicalizations are integral affine functions of its coordinates (NAG-003077).

On the actual joint image of the two tuples the projections agree both on points and on every cell tangent. Thus they induce the same pullback in both alternating factors, including after adjoining further coordinates (NAG-003065). Two choices can be compared on an ordinary common intrinsic refinement inside both witnesses (NAG-003077). Consequently replacement descends through the germ relation and gives an inverse on stalks to the restriction of algebraic presentations; this is the algebraic identification in NAG-003078. Wedge, the two differentials and J_0 commute with these affine pullbacks on the ambient superforms and descend by NAG-002422 and NAG-002424. □

The theorem is an equality of two presentations of the original sheaf. Enlarging the allowed tropicalizations to harmonic ones, or adjoining polyhedral currents, makes a different construction and requires a separate comparison theorem.

8.2 Local structure near classical points

8.2.1 Smooth real functions can be locally constant at rigid points

Let $x \in X(K)$ and let f be an analytic unit near x. Continuity of f gives a smaller neighbourhood on which

$$|f(y) - f(x)| < |f(x)|.$$

The strong triangle inequality then implies $|f(y)| = |f(x)|$. For finitely many units $f_1, \dots, f_r$, choose one neighbourhood where all these equalities hold. Every original smooth function presented as $F(-\log|f_1|, \dots, -\log|f_r|)$ is constant there.

The corresponding positive-degree forms presented by these coordinates vanish there because the local tropical image is a point. This is a distinctive

feature of the original real differential calculus. The word smooth should not evoke a real manifold chart of dimension $2n$ around a rigid point. The useful differential directions occur at nonclassical analytic points and their tropical neighbourhoods.

This does not force a global smooth function on a connected analytic space to be constant. The neighbourhoods on which it is constant can vary with the rigid point, and nonclassical points lie between them. On the projective line, a smooth regularization of the standard metric weight varies near the Gauss point while being constant in appropriate rigid neighbourhoods after local trivialization.

8.2.2 A direct consequence for singular potentials

On a curve, the point mass δ_x at a rigid point is a perfectly meaningful top-degree current: it evaluates smooth scalar tests at x. But it cannot be the curvature of an original smooth function near x, because such a function has zero Hessian on some neighbourhood. Its logarithmic potential is singular. Poincaré–Lelong supplies precisely this kind of potential.

The distinction between a smooth form and a current is therefore already necessary in dimension one. A smooth positive metric can approximate a model metric whose mass concentrates at a nonclassical point; a logarithmic Green function can create a divisor mass at a rigid point. These are different limiting and singularity phenomena.

8.2.3 Restriction of a form and vanishing of its germ

In ordinary differential geometry, a positive-degree form can restrict to zero on a point while remaining nonzero arbitrarily near it. CLD's tropical-dimension support lemmas give stronger vanishing statements for certain degrees at analytic points of small tropical dimension. Their proofs use the local structure of moment maps, not just the formal rule that a form restricts to zero on a lower-dimensional space.

This distinction is used in the higher-codimension extension arguments of CLD25. If a test has degree larger than the relevant tropical dimension along a cycle, the source's support theorem can force its support away from that cycle. This is why a singular expression on the complement may still pair with the differentiated test in a canonical way. One must cite the support

theorem; merely observing that the restriction to the cycle is zero would not justify the extension.

8.2.4 Smooth density does not imply positive density

Original smooth scalar functions are dense in continuous functions in the appropriate local or compact-support topology. This permits the measure-theoretic extension used in defining uT. The approximation can be made with prescribed scalar bounds under the relevant cutoff construction.

However, density of a vector space of smooth functions does not imply density of its psh cone in the continuous current-psh cone. An approximation may develop negative Hessian terms. This is analogous to the difference between approximating an arbitrary continuous function by polynomials and approximating a convex function by convex polynomials on a specified domain: the positivity constraint is an additional problem.

In real convex geometry, convolution on a smaller fixed open set preserves convexity. On an analytic space, a given continuous potential may not be represented by a convex function in one fixed moment chart, and refining charts introduces relations that the approximation must respect. This is the precise reason a local smooth-density theorem is insufficient to prove general psh regularization.

Example 8.2.1 (Local constancy of a unit logarithm)**.** For $f(T) = a + T$ with $a \neq 0$, the strict ultrametric inequality gives $|f| = |a|$ on $|T| < |a|$. Hence $\log|f|$ is constant on that disc. At the Gauss point of radius $r > |a|$, however, $|f| = \max(|a|, r) = r$. These two descriptions are the two regions of $v(f) = \min(v(a), -\log r)$ along the Gauss points. Local constancy near a classical point is compatible with variation along an annular skeleton.

References. [CLD12, §§3.1–3.3, 5.4–5.6]; [CLD25, §§7.1, 15.6, 16.8 and 18.7].

8.3 Integration on weighted polyhedral spaces

8.3.1 The data needed for a top-degree integral

Suppose a pure n-dimensional polyhedral complex has an integral tangent lattice N_σ on every maximal cell σ and a weight m_σ. Choose a lattice basis $e_1, \dots, e_n$ of N_σ. The basis fixes a normalized volume; replacing it by another

lattice basis changes the determinant by ± 1 and does not change the resulting density. A superform of degree (n, n) can be contracted against this volume and integrated over the cell. The total integral is the sum over maximal cells with the weights m_σ.

The three ingredients have distinct roles. The polyhedral support says where integration occurs. The lattice fixes a unit of volume. The weight records the multiplicity of the analytic or algebraic object over that support. The Euclidean volume inherited from an arbitrary ambient embedding generally has the wrong normalization.

Example 8.3.1 (A parametrization of index two)**.** Consider the map $\mathbb{G}_m \to \mathbb{G}_m$, $z \mapsto z^2$. On tropical coordinates it is $t \mapsto 2t$. Its generic algebraic degree is two, including the inseparable case when appropriate scheme-theoretic multiplicities are retained. If one integrates on the target using its standard lattice, the degree formula has a factor two. If instead one treats the source coordinate as a primitive target coordinate, the factor is lost. This simple calculation is the prototype for the lattice indices in tropical pushforward and the calibrated degree formulas of CLD25.

8.3.2 The boundary calculation behind balancing

For a compactly supported $(n-1, n)$-superform, apply ordinary Stokes on every maximal cell. Each codimension-one face receives a sum of contractions with its outgoing primitive normals. The sum vanishes precisely when the weighted normal vectors balance in the quotient by the tangent space of that face. Thus balancing turns a cellwise boundary formula into a global closedness statement.

This also explains why subdividing a cell changes no integral or current. Newly introduced interior faces occur in opposite pairs with compatible lattice volumes. Their boundary contributions cancel. A genuine corner of a function behaves differently: the normal derivatives on the two sides need not agree, and their difference survives as the divisor weight. The same integration-by-parts calculation therefore proves both closedness of a tropical cycle and the corner formula for a PL function.

Example 8.3.2 (Distributional curvature and outgoing slopes)**.** For $f, \eta \in C^2([a,b])$, integration by parts twice gives

$$\int_a^b f''\eta\,dt = \int_a^b f\eta''\,dt + [f'\eta - f\eta']_a^b.$$

If $a < 0 < b$ and η is compactly supported in (a,b), then

$$\int_a^b \max(0,t)\eta''(t)\,dt = \eta(0).$$

The distributional second derivative is therefore δ_0, although the classical second derivative vanishes off the origin. On the balanced tropical line, let the test be induced by an ambient smooth function and compactly supported away from the outer ends. The terms involving the first derivative of the test cancel by balancing. The calculation on the outgoing parameters then gives $(a_1+a_2+a_3)\eta(0)$ for a continuous affine-on-rays function with outgoing slopes a_1, a_2, a_3. The vertex coefficient is the sum of those slopes.

8.3.3 The tangent quotient

A curve can be embedded in a torus of arbitrarily large dimension. Its tropical image remains one-dimensional. A superform on the ambient real vector space is used only through its restrictions to tangent directions of the tropical image. Ambient components that vanish on all relevant tangent spaces represent the zero form on the image. This quotient is essential in the definition of CLD forms; the pullback notation alone does not specify equality of forms.

The example also explains why one cannot prove positivity by examining an arbitrary matrix in arbitrary ambient coordinates without passing to the quotient. Positivity is a condition on the actual tangent directions tested by the space. Refinement by additional moment coordinates must preserve that condition. In the intrinsic construction below, equality and positivity are formulated on common refinements so that the resulting forms do not depend on the original embedding.

8.4 Integration, canonical weights, and Stokes' theorem

8.4.1 Compatibility of integral presentations

Let X have pure dimension n and let α be a compactly supported smooth (n,n)-form. Locally write $\alpha = f_{\mathrm{trop}}^{*}\beta$ for an analytic moment map f. The image $f_{\mathrm{trop}}(X)$ can have singular faces, several analytic branches can map to the same face, and an integral-affine map between two presentations can have a nontrivial lattice index. Therefore the unweighted integral of β on the set-theoretic image cannot be the definition.

CLD assigns a canonical calibration to suitable tropical presentations. In the algebraic torus case, it agrees with the usual tropical multiplicity. In the analytic construction, finite maps and the degrees of the relevant analytic fibres supply the local weights. The compatibility theorem says that passing to a refined moment map and then projecting gives the same weighted integral. This is the essential geometric theorem behind the definition, not a consequence of the notation f_{trop}^{*}.

8.4.2 The construction in three steps

First, on a calibrated polyhedral cell, contract a top-degree superform with its calibrated tangent volume and integrate the resulting density. Second, sum over cells. Lower-dimensional overlaps do not contribute to a top-degree integral, and subdivision creates cancelling interior faces for boundary integrals. Third, use a smooth partition of unity on the analytic space to reduce a general compactly supported form to finitely many tropical presentations.

Independence must be proved at both transitions: common refinements compare local tropical presentations, while the partition-of-unity identity compares decompositions of the global form. For a paracompact good analytic space, the smooth partitions of unity constructed in CLD12 make this procedure available. Compact support is important because the actual integral is a finite local calculation even when the space itself is not compact.

Example 8.4.1 (The annular integral)**.** On an annulus with tropical coordinate $t = -\log|T|$, take a compactly supported form

$$\alpha = f(t)\, d't \wedge d''t.$$

The canonical weight is one, and $\int_X \alpha = \int f(t)\, dt$ in the chosen lattice normalization. Replacing T by T^m changes the real coordinate to $s = mt$. The pullback of $d's \wedge d''s$ has two factors of m, while the calibrated volume and the image parametrization account for the corresponding degree and Jacobian. One cannot change the coordinate and leave the integration normalization unchanged.

8.4.3 Stokes on the analytic space

On every cell, Stokes is ordinary integration by parts. At an interior codimension-one face, the sum of boundary contractions vanishes by the calibrated balancing condition. The remaining contributions lie on the analytic boundary. Thus, when the support of η avoids that boundary,

$$\int_X d'\eta = 0 \quad \text{for } \eta \in A_c^{n-1,n}(X), \qquad \int_X d''\xi = 0 \quad \text{for } \xi \in A_c^{n,n-1}(X).$$

For a compact analytic domain with boundary, there are corresponding boundary integrals. Deleting those terms is justified by the support condition, not by compactness of the domain.

This is the bridge from the balancing equation of Chapter 5 to the closedness of an integration current. It also explains why arbitrary piecewise smooth functions cannot be differentiated cellwise and then treated as smooth global functions. Their internal interfaces contribute additional terms.

8.4.4 The signed boundary measure for ordinary native forms

Theorem 8.4.2. *Let K be complete for a specified real-valued non-Archimedean absolute value, possibly trivial. Let X be a good Hausdorff K-analytic space in the norm-admissible generalized affinoid atlas, analytically pure of dimension $d \geq 1$, and let $\mathcal{F}$ be coherent on X_G. Take an ordinary real smooth moment form β with compact ordinary germ support S. Put $D = d'$ for type $(d-1, d)$, or $D = d''$ for type $(d, d-1)$. There is a unique finite signed Radon measure $\mu^D_{X,\mathcal{F},\beta}$ supported in $S \cap \partial_K X$, where*

$\partial_K X = X \setminus \mathrm{Int}(X/K)$, *such that, for every ordinary smooth real scalar* u *on* X,

$$\int_X u\, d\mu^D_{X,\mathcal{F},\beta} = I^N_{X,\mathcal{F}}\big(D(u\beta)\big). \tag{S1}$$

Here I^N *is the native integral of NAG-002425. In particular the mass is* $I^N_{X,\mathcal{F}}(D\beta)$. *Here support of a signed measure means support of its total variation. For support contained in the analytic interior the measure vanishes.*

We give the proof first for $D = d'$. All charts use $-\log|f|$ and the interleaved positive lattice top generator. The strict integral structure-coefficient estimate (B) below is used only for a single global representative of the primitive. The multiplier in (B) is any ordinary smooth scalar.

The native bulk comparison under free extension. On a pure compact affinoid $Y = \mathcal{M}(B)$, let M be a finite B-module. For a finite free polyradius let $L = K_r$, $B_L = B\widehat{\otimes}_K L$, $M_L = B_L \otimes_B M$, and $p : Y_L \to Y$. For any ordinary top section ω we have

$$I^N_{Y_L,M_L}(p^\natural\omega) = I^N_{Y,M}(\omega). \tag{S2}$$

To see this, extend the units and use the same ambient superform. The full inverse image of an affinoid domain with algebra C has algebra $C_L = C\widehat{\otimes}_K L$: the relative completed tensor with B_L identifies with this algebra by multiplying simple tensors, with inverse $c \otimes l \mapsto c \otimes (1 \otimes l)$ (NAG-001616, 001688, 001846). The free Gauss section is a right inverse on each domain (001569); unit and joint images agree. The inverse image of the ordinary interior of a germ witness is open. Thus the same joint-image equality gives a germ witness upstairs, including at a point of the absolute boundary. Restrictions, products, both differentials and the Lagerberg slot swap commute with this rule, and $p^\natural u = u \circ p$ (002422–24). Finite module restrictions are the tensors with their canonical norms (001565, 001686, 002414). Purity is preserved by free extension (002291).

For a single unit chart on a presenting piece with algebra C and restricted module P, subdivide the common moment image with the boundary images over both fields (002358, 002405). In a top cell, at a relative-interior weight w, both whole levels are interior. Choose an integral projection Q injective on the saturated tangent lattice N. Take the algebra of the whole w-level cut out by all original equalities of moduli, before applying Q. Those Laurent conditions base change, and 002306 gives a finite algebra over the exact-radius

polytorus at Qw. If H, H' are the Gauss fields for that polytorus over K, L, and E, E' the respective Gauss fibre algebras, 002301 gives $E' = E \otimes_H H'$ with $H \hookrightarrow H'$ isometric. In particular

$$(C_L \otimes_C P) \otimes_{C_L} E' \simeq (P \otimes_C E) \otimes_H H'.$$

The dimension over H' equals the original dimension over H; the weight divides that dimension by the same index $[\mathbb{Z}^d : Q(N)]$ over each field (002362). This retains the whole Artinian fibre including its lengths and residue degrees. Lower-dimensional images carry zero top integral. Pairing with the same tangential coefficient and volume proves the single-chart identity. On a finite presentation all actual intersections base change; summing their equalities with the same signs proves (S2) by 002416 and 002425.

For B integral and $\beta = t_f^* \alpha$ on all of Y, choose L nontrivially valued with B_L strict by 001840 (identity domain). It is integral of the same analytic dimension by 002291. Apply (B) upstairs and (S2) to the full derivative downstairs:

$$|I^N_{Y,B}(d'(u\beta))| = |I^N_{Y_L,B_L}(d'((u \circ p)p^\natural \beta))| \le C_{Y_L, p^\natural \beta} \|u\|_{C^0(Y)}. \tag{S3}$$

Choosing this one field gives a bound on the fixed original chart. There is no strictness condition on the local presentations of u. This argument changes the field for the bulk integral, without a comparison of the inverse images of the two absolute boundaries.

Coherent coefficients and finite cuts. On a compact pure Y, put $Y_\mathfrak{p} = \mathcal{M}(B/\mathfrak{p})$ and $\ell_\mathfrak{p}(M) = \operatorname{length}_{B_\mathfrak{p}}(M_\mathfrak{p})$ for $\mathfrak{p} \in \operatorname{Min} B$. The complete quotient is an integral pure affinoid of dimension d by 001562 and analytic purity. For every ordinary top section,

$$I^N_{Y,M}(\omega) = \sum_{\mathfrak{p} \in \operatorname{Min} B} \ell_\mathfrak{p}(M)\, I^N_{Y_\mathfrak{p}, \mathcal{O}}(\omega|_{Y_\mathfrak{p}}). \tag{S4}$$

Here restriction to a closed component is defined by restricting units and ambient coefficients: inverse images of the presenting neighbourhoods are neighbourhoods, and tangential equality restricts to their joint subimages (001846, 002406, 002422–24).

For the coefficient check, use a finite presentation and a nonempty chart intersection $V = \mathcal{M}(C)$. On its inverse image in $Y_\mathfrak{p}$ the algebra is $C/\mathfrak{p}C$, and the tuple is the restricted tuple. Subdivide with all these actual images

and all their boundary images. At w generic in a top cell, every contributing image locally has that same tangent plane, and its whole level is interior. Empty inverse levels give zero. The level with algebra D_w in V is also an affinoid domain of Y by domain transitivity (001609). It is finite over the chosen polytorus algebra R by 002306. Put $E = D_w \otimes_R H$ as above. The whole level on the inverse image in $Y_{\mathfrak{p}}$ is cut out by the same equalities, hence has algebra $D_w/\mathfrak{p}D_w$ and Gauss fibre $E/\mathfrak{p}E$ (001565, 001702, 001846). NAG-002360 applied to $B \to D_w$ gives the exact numerator

$$\dim_H(M \otimes_B E) = \sum_{\mathfrak{p}} \ell_{\mathfrak{p}}(M) \dim_H(E/\mathfrak{p}E).$$

In particular the component lengths refer to the original B even when the multiplier varies through finer charts. Divide by the same index of $Q(N)$ on each contributing image. Integrate the common tangential coefficient and sum on the presenting intersections using 002401, 002405 and 002416. This proves (S4).

For an arbitrary compact affinoid primitive, choose a finite cover by affinoid neighbourhoods whose ordinary interiors cover Y, each carrying one representative of β on the whole domain (002424). On a nonempty intersection $Y_J = \mathcal{M}(B_J)$ choose one restricted representative, and put $M_J = B_J \otimes_B M$. These domains are affinoid and pure (002411, 002413). Apply (S4) to $d'(u\beta)|_{Y_J}$ and then (S3) to the restricted single-chart primitive on each integral component. Every restricted multiplier has norm at most $\|u\|_{C^0(Y)}$. The signed cover formula for this variable top test follows by intersecting the cover with a presenting cover for the top test, applying 002401 on each single chart and interchanging finite sums (002416, 002425). Every nonempty intersection enters, including ones with empty ambient ordinary interior. We get a bound independent of u by summing the finitely many constants with their nonnegative component lengths. This proves boundedness of $T(u) = I^N_{Y,M}(d'(u\beta))$ on the entire compact affinoid.

Compact Radon representation within the ordinary calculus. For $F \in C(Y, \mathbb{R})$ a finite ordinary open cover of oscillation $< \epsilon$ and a smooth partition from 002428 approximate F uniformly by the partition-weighted sum of one value of F per nonempty open. Thus T extends uniquely and boundedly to $C(Y)$ by taking limits. For continuous $f \geq 0$ set $P(f) = \sup_{0 \leq g \leq f} T(g)$ over continuous g. It is finite, nonnegative and positively homogeneous. Split each $0 \leq g \leq f_1 + f_2$ as $\min(g, f_1) + (g - \min(g, f_1))$; in

the reverse direction add two approximate maximizers. This gives additivity. Extending by differences is well defined by additivity; both P and $P - T$ are positive linear functionals.

For each of these two functionals H_0 put

$$m(U) = \sup\{H_0(a) : a \in C(Y),\ 0 \le a \le 1,\ \operatorname{supp} a \subset U\}, \qquad m^*(E) = \inf_{U \supset E,\ U \text{ open}} m(U).$$

Then $m(Y) = H_0(1)$. For a countable union of opens, a test supported in the union has its compact support covered by finitely many of them. Splitting it by a partition near that support proves subadditivity on opens. Open supersets within errors $\epsilon 2^{-i}$ give countable subadditivity of m^*; it is monotone, vanishes at the empty set and agrees with m on opens. For an open U and an open $V \supset E$, choose a test a supported on a compact $K_0 \subset U \cap V$ with $H_0(a) > m(U \cap V) - \epsilon$. Adding any test on $V \setminus K_0$ keeps the sum at most one. Hence $m(V) \ge m^*(E \cap U) + m^*(E \setminus U) - \epsilon$. Infimizing and using subadditivity proves that U splits m^* additively on every set. Such splitting sets are closed under complement and finite union by successive splitting. For disjoint splitting sets B_i, successive splittings bound $m^*(E)$ below by each partial sum of $m^*(E \cap B_i)$ plus $m^*(E \setminus \bigcup_i B_i)$. Subadditivity gives splitting for their union and countable additivity. Disjointification gives closure under countable unions. Restriction to Borel sets therefore gives a finite measure ν, outer regular by construction; outer approximation of complements in compact Y gives compact inner approximation for Borel sets.

For $0 \le f \le N\delta$, approximate each $m(\{f > j\delta\})$ by tests a_j, $j = 1, \dots, N$. Pointwise $\delta \sum_j a_j \le f$, so $H_0(f) \ge \delta \sum_j \nu(\{f > j\delta\})$. The corresponding simple function differs from f by at most δ. Let δ tend to zero with $N\delta \ge \max f$. Apply the resulting lower bound also to $\max f - f$, using $\nu(Y) = H_0(1)$, to obtain $H_0(f) = \int f \, d\nu$. The difference of the measures for P and $P - T$ represents T.

We use the following uniqueness and localization test here and below. If a finite signed Radon measure annihilates smooth scalars compactly supported in an ordinary open U, it vanishes there. For compact $C \subset U$ take an open $C \subset V \subset U$ with variation on $V \setminus C$ as small as desired. A cutoff between zero and one supported compactly in V and equal to one near C exists by 002428. Its zero integral bounds the absolute measure of C by that variation. Compact inner approximation then gives zero on Borel subsets of U. A finite sum of the positive Radon measures in the construction dominates the variation, so the same regular approximations apply to

their signed differences. This proves uniqueness. Tests supported outside $\operatorname{supp}\beta$ give zero. For a scalar compactly supported in $\operatorname{Int}(Y/K)$, the product primitive is compactly supported there, and NAG-002577 annihilates its derivative. We have proved (S1) and the asserted support on Y. In particular (S2) and uniqueness identify the compact scalar-extension comparison on measures with $p_*\mu^{d'}_{Y_L,M_L,p^\natural\beta} = \mu^{d'}_{Y,M,\beta}$: the pushforward is Radon as the map is continuous between compact Hausdorff spaces. This comparison needs no boundary-fibre equality.

Ordinary localization and the closed-cut formula. On X take finitely many compact affinoid neighbourhoods Y_i whose ordinary interiors cover S, and smooth cutoffs χ_i compactly supported in those interiors, summing to one near S (002428). Goodness gives these neighbourhoods, purity holds by 002413, and 002414 gives finite coefficient modules on all their intersections. With $\iota_i : Y_i \hookrightarrow X$, set

$$\mu = \sum_i (\iota_i)_*\big((\chi_i|_{Y_i})\,\mu^{d'}_{Y_i,\mathcal{F}|_{Y_i},\beta|_{Y_i}}\big).$$

Compact-domain pushforwards are finite Radon measures: compact inner approximants have compact images, and outer regularity follows by approximating the complement from inside. Formula (S1) on Y_i, tested on $u\chi_i$, gives the integral of the derivative of the entire product. Its compact collar in the ordinary interior makes both derivative and integral compatible with zero extension by 002580. The sum equals $I^N_{X,\mathcal{F}}(d'(u\beta))$ since $\sum_i \chi_i = 1$ near S. The uniqueness test on ordinary opens now proves independence and gives support in $S \cap \partial_K X$: outside S the product vanishes and on the analytic interior use 002577 on compactly supported scalar multiples. The ambient space itself need not be compact or paracompact.

In particular, on an ordinary open U the restriction of μ is characterized on compactly supported smooth scalars a there by $I^N_{U,\mathcal{F}|_U}(d'(a\beta|_U))$. Indeed a extends by zero and both integrals use the same carrier inside U by 002425. This also works on the ambient ordinary interior of a closed affinoid domain. More generally, for a finite ordinary cover of S, local primitives with germ supports in $S \cap U_i$ have a common measure supported in S precisely when their evaluations on compact smooth overlap scalars agree after the two zero extensions. Necessity is restriction. For sufficiency take the sum of the zero extensions of the measures of $\chi_i\beta_i$ on U_i. Each product is compactly supported there. Testing against a scalar compactly supported in U_k, the products

of that scalar with χ_i are compactly supported in the overlaps; replace their local evaluations by the ones on U_k and sum. This recovers the evaluation on U_k. Uniqueness on each open proves the assertion. True boundary support follows as above locally: an ordinary open has the same structure-map interior on that open (001631). Equality of ordinary primitive germs on the overlaps suffices for the evaluation condition by the common-carrier argument.

There is also an expression without cutoffs. For any finite such closed affinoid cover of the support by ordinary interiors write $D_J = \bigcap_{j\in J} Y_j$, $B_J = \mathcal{O}(D_J)$, $M_J = \Gamma(D_J, \mathcal{F})$, and write i_{Jp} for the inclusion of $\mathcal{M}(B_J/\mathfrak{p})$ into X. Then

$$\mu^{d'}_{X,\mathcal{F},\beta} = \sum_{\emptyset\neq J,\ D_J\neq\varnothing} (-1)^{|J|+1} \sum_{\mathfrak{p}\in \operatorname{Min} B_J} \ell_{\mathfrak{p}}(M_J)(i_{Jp})_* \mu^{d'}_{\mathcal{M}(B_J/\mathfrak{p}),\,\mathcal{O},\,i_{Jp}^*\beta}. \tag{S5}$$

Testing against u, apply (S4) on each intersection and the finite cover formula on their compact union (002416, 002425). This yields (S1), hence equality by uniqueness. Each cut has its own absolute boundary, potentially including an artificial frontier. The support bound on X is for the signed sum. Neither raw restriction to a closed overlap nor addition of the total variations is used.

Finally let J be the native Lagerberg involution. It fixes scalars, interchanges d' and d'', and acts on interleaved top degree by $(-1)^d$, since each of the d pairs is reversed (002388, 002424, 002577). For a compact primitive γ of type $(d, d-1)$ this gives

$$\mu^{d''}_{X,\mathcal{F},\gamma} = (-1)^d \mu^{d'}_{X,\mathcal{F},J\gamma}.$$

Thus (S1) uses the derivative of $u\gamma$ with its own plus convention, and the signed sum and localization statements follow in this bidegree as well.

A uniform scalar bound from the fixed moment image

Let K be complete and nontrivially valued for its specified real non-Archimedean absolute value. Let $Z = \mathcal{M}(A)$ be a nonempty integral strict affinoid of analytic dimension $d \geq 1$, with structure coefficient. Fix analytic units $f = (f_1, \ldots, f_n)$, $t = t_f$ with coordinates $-\log|f_i|$, and a smooth real ambient form α of type $(d-1, d)$ on a neighbourhood of $P = t(Z)$. Write $\theta = t^*\alpha$, and I for the native ordinary top integral. Then there exists $C < \infty$, independent of every chart and cut used for the scalar, such that

$$|I(d'(u\theta))| \leq C\|u\|_{C^0(Z)} \qquad (u \in \mathcal{A}^{0,0}_Z(Z)). \tag{B}$$

The proof estimates a positive bulk integral in the fixed chart. No boundary measure or lifting of boundary sheets is a premise.

A cutoff in the original target. Put $H = t(\partial_K Z)$. By NAG-002358 and 002405, P is compact finite polyhedral of dimension at most d, H is finite polyhedral of dimension at most $d-1$, and the structure chain has fixed finite nonnegative weights m_σ on its d-cells. If H is empty, (B) follows with $C = 0$ from NAG-002577. If $\dim P < d$, the entire second slot of θ vanishes tangentially, giving the same conclusion. Take a finite subdivision of P adapted to H.

For each cell of H, choose a real affine hyperplane $\ell_j = 0$ containing it and containing none of the affine spans of the d-cells of P. Such choices exist: the space of affine functions vanishing on the span of that cell is not contained in the subspace of functions vanishing on any specified d-plane, by the dimension inequality. A finite union of proper linear subspaces cannot cover a real vector space (choose a nonzero linear equation for each and take their product, a nonzero polynomial). Use a smooth $0 \le b \le 1$ equal to one on $[-1,1]$ and zero outside $(-2,2)$, from the elementary smooth profile in the proof of NAG-002427. For $0 < \epsilon < 1$, put

$$\chi_\epsilon = 1 - \prod_j (1 - b(\ell_j/\epsilon)), \qquad \psi_\epsilon = \sum_j b(\ell_j/(2\epsilon)).$$

Then $0 \le \chi_\epsilon \le 1$, it is one on an open neighbourhood of H, and it vanishes off the union of the slabs $|\ell_j| \le 2\epsilon$. On the support of its derivative $\psi_\epsilon \ge 1$ and $\|\nabla \chi_\epsilon\| \le c/\epsilon$, with c fixed. Smoothness here is in real moment coordinates; the hyperplanes do not define new analytic domains.

For every coarse d-cell σ and every j, the lattice volume in σ of $\{|\ell_j| \le 4\epsilon\}$ is at most $c_{\sigma j}\epsilon$ for sufficiently small ϵ. If the restriction is constant nonzero the intersection is empty eventually. Otherwise in integral affine coordinates on the cell span some coordinate has nonzero coefficient a in ℓ_j. Enclose the compact cell in a fixed box and slice parallel to that axis. Each slice has length at most $8\epsilon/|a|$; the remaining box has fixed volume (NAG-002389). This proof applies equally on a compact fine d-cell whenever its projection has rank d and lands in σ, with a constant then allowed to depend on the fine cell.

A positive majorant independent of the multiplier. Order the ambient coordinates once and put

$$\Omega = \sum_{i_1<\cdots<i_d} \bigwedge_{k=1}^{d} (d' x_{i_k} \wedge d'' x_{i_k}).$$

In lattice coordinates on a d-plane its interleaved top coefficient is the sum of the squares of its coordinate d-minors, which is strictly positive. Use $a_\sigma > 0$ for this constant on $\mathrm{aff}(\sigma)$, and ω_σ for the interleaved top generator there. On σ write $d'\chi_\epsilon \wedge \alpha = q_{\epsilon,\sigma}\omega_\sigma$. Compactness, the gradient bound and the finitely many coarse tangent planes give one c_1 such that

$$|q_{\epsilon,\sigma}| \le (c_1/\epsilon)\psi_\epsilon a_\sigma.$$

Indeed $q_{\epsilon,\sigma}$ is linear in the first derivatives of χ_ϵ with bounded coefficients, and vanishes off their support. Define $\Lambda_\epsilon = (c_1/\epsilon)\psi_\epsilon\Omega$. By the coarse chart integral (NAG-002405) and the slab estimate,

$$I(t^*\Lambda_\epsilon) = \frac{c_1}{\epsilon}\sum_\sigma m_\sigma a_\sigma \int_\sigma \psi_\epsilon \, d\mathrm{vol}_\sigma \le c_1 \sum_{\sigma,j} m_\sigma a_\sigma c_{\sigma j} =: C. \tag{1}$$

We claim both top sections

$$\|u\|_\infty \, t^*\Lambda_\epsilon \;\pm\; u\, t^*(d'\chi_\epsilon \wedge \alpha)$$

are positive in the ordinary native sense. At each point use an ordinary affinoid scalar representative $u = v(t_g)$ and adjoin f. Subdivide the joint image so projection π to the f-coordinates maps each cell into a coarse cell. If π on a d-cell has rank $< d$, both pulled-back tops vanish. Otherwise the two coarse top coefficients are pulled back with the same squared determinant in integral coordinates. This is positive whatever the orientation. Use $|v(t_g)| \le \|u\|_\infty$ on the actual joint image and the preceding coefficient inequality. This proves positivity on all local representatives so chosen. NAG-002425, whose positivity includes varying finite domain presentations, now gives

$$|I(u\, t^*(d'\chi_\epsilon \wedge \alpha))| \le C\|u\|_\infty. \tag{2}$$

It is essential to take this inequality for the ordinary integral. Applying an absolute value separately to all signed terms of an arbitrary closed cover would not give (1).

The term containing the derivative of the multiplier. For each fixed u,

$$I((\chi_\epsilon \circ t)d'(u\theta)) \to 0. \tag{3}$$

Choose once for this u a finite cover by ordinary affinoid presentation neighbourhoods, with their interiors covering Z. On every actual nonempty closed intersection use one restricted scalar presentation and its joint tuple with f. The restriction and finite-cover formulae NAG-002416, 002424–25 allow the same finite family for all ϵ. On its joint image the top representative before multiplication by $\chi_\epsilon \circ \pi$ is

$$d'(v\,\pi^*\alpha) = d'v \wedge \pi^*\alpha + v\,\pi^*(d'\alpha),$$

where v is composed with the scalar-coordinate projection. On each d-cell on which π has rank $< d$, this is zero: all d double-primed covectors in both terms come from the coarse target. On each remaining cell the coefficient is bounded on the compact polytope for this fixed presentation. The portion on which $\chi_\epsilon \circ \pi \neq 0$ has lattice volume $O(\epsilon)$ by the fine-cell slab estimate above. All weights are finite and fixed for this u, and the finite signed sum tends to zero. This argument requires no uniform control on the derivatives, weights or number of charts of u.

Finally $(1-\chi_\epsilon \circ t)u\theta$ is identically zero on the inverse image of an ordinary neighbourhood of H. Its support is compact in $\mathrm{Int}(Z/K)$; hence NAG-002577 gives $I(d'((1-\chi_\epsilon \circ t)u\theta)) = 0$. The product rule with the scalar in first position yields

$$I(d'(u\theta)) = I(u\,t^*(d'\chi_\epsilon \wedge \alpha)) + I((\chi_\epsilon \circ t)d'(u\theta)).$$

Take $\epsilon \downarrow 0$ in (2)–(3). This proves (B) with the constant in (1), for multipliers with arbitrary ordinary local charts.

Normal and multiplicity check. For comparison, on a single global joint tuple, take subdivisions adapted to the projection and boundary images. For an incident fine cell–facet flag mapping with full ranks to a coarse flag, let e be the tangent-lattice index on the facet. Modulo the coarse facet lattice the primitive inward normal maps to k times the coarse primitive inward normal, with $k > 0$. In bases completed by these normals the full index is ek (block triangular determinant). In the normal-contraction formula of NAG-002389 the two lists of facet tangent vectors give e^2; the remaining normal gives k, with its tangent component killed by alternation. Scalar facet volume changes by $1/e$ upon passage from fine to coarse, leaving ek times

the original fine weight, with the common inward convention $(-1)^{d(d+1)/2}$. For a facet mapping to smaller facet dimension the tangent lists lose rank, and for a contracted incident d-cell the entire second slot loses rank. Each such flag has zero contraction of the pulled-back primitive. For $d = 1$ the empty facet tangent lists have index one. The generic pushforward of top cells in NAG-002405 counts their full weights times the full index: NAG-002306 and 002362 take the whole Gauss fibre degree with all local lengths and residue degrees and divide by the chosen projection index. Formula (1) uses those resulting coarse weights once. A coarse signed normal can vanish even when the individual incidences have nonzero absolute contractions. The bulk majorant avoids any identification of individual boundary sheets or any assumption that absolute normal masses commute with projection.

8.4.5 A Green identity and its use

Let u, v be smooth real functions, and let T be a closed current of suitable degree. Assume all products under consideration have compact support away from the analytic boundary. Applying the two Leibniz rules and Stokes yields the integration-by-parts symmetry

$$\int u \, \mathrm{d}'\mathrm{d}''v \wedge T = \int v \, \mathrm{d}'\mathrm{d}''u \wedge T$$

when the remaining degrees make these top-degree integrals. A convenient sufficient support assumption is that one factor is compactly supported and the other is defined on a neighbourhood of that support. More general formulations use cutoffs.

The identity is the engine behind several later results: it moves derivatives away from an approximating potential, proves symmetry of mixed expressions, and gives estimates in terms of the sup norm of the potential. However, its use with a nonsmooth potential requires a limiting argument. The formula itself does not authorize a product that has not yet been defined.

Example 8.4.3 (Boundary terms in Green's identity)**.** For $u, v \in C^2([a, b])$,

$$\int_a^b (uv'' - vu'') \, dt = [uv' - vu']_a^b.$$

For $u = 1$ and $v = t^2$, this is $2(b-a)$, which is nonzero when $a < b$. Compactness does not remove boundary terms. If one factor is compactly supported

in (a, b), it and its first derivative vanish near the endpoints, and the displayed boundary term is zero. This is the support mechanism in the current identity.

References. [CLD12, §§3.3–3.12]; the algebraic presentation and integration comparison in [Gubler].

8.4.6 The signed measure of an ordinary top form

Proposition NAG-002970 (Native top-form measures; F8CR-M)**.** *Let Y be good, Hausdorff, and pure of dimension d. For every real ordinary compact (d, d)-form ω, there is a unique finite signed Radon measure $\mu_{Y,\omega}$, supported in $S = \operatorname{Supp}\omega$, with*

$$\int_Y \phi \, d\mu_{Y,\omega} = \int_Y \phi\omega \quad \text{for every ordinary smooth scalar } \phi. \tag{8.4.1}$$

The construction is linear in ω and positive for positive ω. If its compact support is contained in an open subspace, the measure on Y is the zero extension of the measure there. It defines a separate Borel-measure extension; it does not enlarge the class of coefficients admitted by the frozen grid integral. Empty support gives the zero measure.

Proof. We spell out the boundedness and representation steps.

Density and cutoffs on a compact set. For $h \in C(S, \mathbb{R})$ and $\epsilon > 0$, take finitely many ordinary open sets covering S on each of which the oscillation of h on S is less than ϵ. Choose a point x_j of each nonempty piece. By `NAG-002428` there are nonnegative compact ordinary smooth χ_j subordinate to these sets, summing to one near S. Then $\sum_j h(x_j)\chi_j$ approximates h uniformly on S to error ϵ, and is nonnegative on all of Y when $h \geq 0$. Thus ordinary smooth scalar restrictions are dense in $C(S)$. Moreover for a compact $F \subset U \subset S$ with U relatively open, there is a continuous function on S equal to one on F, between zero and one, and with support contained in U. Lift U to an open set of Y, use finitely many point cutoffs from `NAG-002427`, and take $1 - \prod_j (1 - \chi_j)$. Finite partitions on compact supports follow from the same frozen partition theorem restricted to S.

A positive domination. Choose finitely many moment representatives α_j near the compact support and a nonnegative ordinary partition ρ_j with com-

pact supports inside those chart opens, summing to one near S. In the ambient real coordinates $t_1, \dots, t_r$ of a chart, put

$$\Theta_j = \frac{1}{d!} \left(\sum_{k=1}^{r} d' t_k \wedge d'' t_k \right)^d .$$

For $d = 0$ take $\Theta_j = 1$. On a d-plane the coefficient of Θ_j is the Euclidean Gram square and is positive. Replace α_j by $(\alpha_j + (-1)^d J\alpha_j)/2$; this does not change its restriction to any d-plane, since the top generator has J-sign $(-1)^d$. On the compact moment image of $\operatorname{Supp} \rho_j$, its coefficient matrix on $\bigwedge^d \mathbb{R}^r$ is bounded. The elementary finite-dimensional inequality $|v^T C v| \le c_j |v|^2$, for a sufficiently large constant c_j, therefore shows that $c_j \Theta_j \pm \alpha_j$ are nonnegative on all d-planes there. This remains true on the tangent planes of every joint moment refinement, including rank-deficient ones. If $r < d$, both restrictions are zero.

The zero extensions of $\rho_j c_j t_g^* \Theta_j$ define a positive compact form θ satisfying $-\theta \le \omega \le \theta$. For $M = \sup_S |\phi|$, the forms $M\theta \pm \phi\omega$ are positive: on S use the displayed bound and outside S use $\omega = 0$. Positivity of `NAG-002425` gives

$$\left| \int_Y \phi\omega \right| \le C_\omega \sup_S |\phi|, \qquad C_\omega = \int_Y \theta < \infty. \tag{8.4.2}$$

Hence the native functional depends only on $\phi|_S$ and extends uniquely by uniform limits to a bounded linear functional $\Lambda_\omega : C(S) \to \mathbb{R}$. For positive ω the positive approximants above prove that this extension is positive.

Representation of a positive functional, with the construction. For any positive bounded $\Lambda : C(S) \to \mathbb{R}$, define, for open $U \subset S$ and arbitrary $E \subset S$,

$$\begin{aligned} a(U) &= \sup\{\Lambda(g) : g \in C(S),\ 0 \le g \le 1,\ \operatorname{Supp} g \subset U\}, \\ a^*(E) &= \inf\{a(U) : E \subset U,\ U \text{ open in } S\}. \end{aligned}$$

Here $a(S) = \Lambda(1)$, and every quantity is finite. If an open set is covered by countably many open sets, the compact support of any test g is covered by finitely many of them. A finite partition on that support splits g into admissible summands, proving countable subadditivity for a on open sets. Choosing open supersets with errors $\epsilon 2^{-j}$ proves that a^* is an outer measure; it agrees with a on open sets.

Every open U is measurable for this outer measure. Indeed, fix E and choose open $V \supset E$ with $a(V) \le a^*(E) + \epsilon$. Choose a test g supported on

a compact $F \subset V \cap U$ with $\Lambda(g) \geq a(V \cap U) - \epsilon$. For a test h supported in $V \setminus F$, the function $g + h$ is still between zero and one and supported in V. Thus $a(V \setminus F) \leq a(V) - \Lambda(g)$, and

$$a^*(E \cap U) + a^*(E \setminus U) \leq a(V \cap U) + a(V \setminus F) \leq a^*(E) + 2\epsilon.$$

Outer subadditivity supplies the reverse inequality. Let $\epsilon \downarrow 0$. The sets satisfying this splitting equality form a sigma-algebra. Complements are immediate. For disjoint measurable E_j, finite iteration gives, for every H,

$$a^*(H) \geq \sum_{j=1}^{N} a^*(H \cap E_j) + a^* \left(H \setminus \bigcup_{j \geq 1} E_j \right).$$

Pass to the supremum over N and use outer subadditivity on $H \cap \bigcup E_j$ to obtain measurability of the union. General countable unions reduce to disjoint ones by differences. The same finite-iteration inequality and outer subadditivity prove countable additivity of the restricted outer measure. Hence $\mu = a^*|_{\mathrm{Borel}(S)}$ is a finite Borel measure.

For a test g above, $\Lambda(g) \leq \mu(\operatorname{Supp} g)$, since every open set containing that support admits g in its defining supremum. It follows that the measure of an open set is the supremum of the measures of its compact subsets. Outer regularity is built into a^*. Inner regularity for every Borel set follows by taking complements and using finiteness of $\mu(S)$. Thus μ is Radon. Integrals of nonnegative Borel functions are defined by the supremum over dominated nonnegative simple functions; bounded functions may equivalently be approximated uniformly by their finite dyadic step functions.

To verify representation without importing it, let $u \geq 0$ be continuous, $0 \leq u \leq m\delta$. Tests supported in $\{u > k\delta\}$, $k = 1, \ldots, m$, give

$$\delta \sum_{k=1}^{m} \mu\{u > k\delta\} \leq \Lambda(u).$$

For the upper bound put $v_k = \min(1, \max(0, (u - k\delta)/\delta))$, $k = 0, \ldots, m-1$. Then $u = \delta \sum v_k$, and for $k \geq 1$ the support of v_k is contained in $\{u > (k-1)\delta\}$. Consequently

$$\Lambda(u) \leq \delta\mu(S) + \delta \sum_{k=1}^{m-1} \mu\{u > (k-1)\delta\}.$$

The lower simple sum differs from u by at most δ, and the upper one by at most 2δ. Let $\delta \to 0$, choosing $m\delta \geq \|u\|_\infty$. Both bounds tend to $\int u\, d\mu$. Linearity gives $\Lambda(u) = \int u\, d\mu$ for all continuous u. Uniqueness follows from the cutoff formula for open sets and Radon inner regularity.

Signed functionals and locality. For a bounded real Λ, define on $u \geq 0$

$$\Lambda^+(u) = \sup_{0 \leq v \leq u} \Lambda(v).$$

This is finite, nonnegative, positively homogeneous, and additive: for one inequality split any $v \leq u_1 + u_2$ into $\min(v, u_1)$ and $v - \min(v, u_1)$; for the other add two nearly maximizing choices. Extend linearly to all of $C(S)$. It is bounded, since $\Lambda^+(u) \leq \|\Lambda\|\|u\|_\infty$ for $u \geq 0$. The functional $\Lambda^- := \Lambda^+ - \Lambda$ is positive and bounded. The positive construction gives $\mu = \mu^+ - \mu^-$. Any two finite signed Radon representations are equal by comparing the positive measures in their two decompositions and using positive uniqueness. Apply this construction to Λ_ω on S, then extend by zero along the closed inclusion $S \hookrightarrow Y$. This yields (8.4.1). The same uniqueness, density on a compact union of supports, and locality of the frozen integral prove linearity and compatibility with open restrictions wherever the form has compact support.

Cutoffs and the order-zero bound (lines 296–348). For compact $F \subset O \subset S$ with O relatively open, lift O to an ordinary open of Y. Finitely many point cutoffs there give $1 - \prod_j (1 - \chi_j)$ restricted to S, equal to one on F and supported in O. More generally a finite relatively open cover of F lifts to ambient opens; apply 002428 on F and restrict. For $u \in C(S)$, a finite such cover of S by sets of oscillation less than ϵ gives an approximant $\sum_j u(x_j)\chi_j$ of uniform error at most ϵ. If $u \geq 0$ the approximant is nonnegative. No continuous extension of u to Y is required.

In each moment presentation on a neighbourhood of S, use the interleaved form $\Theta = (\sum_k d't_k \wedge d''t_k)^d/d!$. In interleaved coordinates a tangent d-frame with Plücker vector v evaluates Θ to $\sum_I v_I^2$ and the given coefficient matrix to $v^{\mathrm{t}}Av$ (absorbing the fixed coordinate sign in A). The latter has absolute value bounded by $c\sum_I v_I^2$ on the compact chart region used by its cutoff. Projection of a frame under refinement uses the same calculation, including $v = 0$. In dimension zero take $\Theta = 1$. With the finite nonnegative partition, sum the cutoff multiples of $c\Theta$, extended by zero, to get a compact positive form θ bounding $\pm\omega$.

Here is a way to use precisely $M = \sup_S |\phi|$ even if ϕ is much larger just outside S. For $\epsilon > 0$ choose a smooth real scalar truncation β equal to the identity on $[-M - \epsilon/2, M + \epsilon/2]$ and bounded in modulus by $M + \epsilon$. For example multiply the identity by the cutoff profile of 002427 applied to $2(M + \epsilon - |t|)/\epsilon$; this factor is constant near zero. Then $\beta(\phi)\omega = \phi\omega$ as sections. Nonnegativity in common charts and 002425 give

$$\left| \int_Y \phi\omega \right| \leq (M + \epsilon) \int_Y \theta.$$

Let ϵ decrease to zero. For empty S take the zero measure. Otherwise the estimate and the approximants above extend the functional uniquely to a bounded linear Λ on $C(S)$: use limits of the real integrals of uniform approximants. Positivity is preserved for positive ω.

Positive representation (lines 350–425). For $\Lambda \geq 0$ let $a(O)$ be the supremum of $\Lambda(g)$ over $0 \leq g \leq 1$ continuous with support in an open $O \subset S$, and put $m(H) = \inf_{O \supset H} a(O)$ over open supersets. An open cover of $\operatorname{supp} g$ admits a finite subordinate partition there as above. Splitting g by the partition gives countable subadditivity of a on open covers. Taking open supersets of H_j within $\epsilon 2^{-j}$ of $m(H_j)$ gives countable subadditivity of m. It is monotone, $m(\varnothing) = 0$, $m(O) = a(O)$ for open O, and $m(S) = \Lambda(1) < \infty$. For arbitrary H and open O, choose open $V \supset H$ within ϵ of $m(H)$ and an admissible g in $V \cap O$ within ϵ of $a(V \cap O)$. Put $F = \operatorname{supp} g$. Any admissible test in $V \setminus F$ can be added to g without exceeding one. Thus $m(H \cap O) + m(H \setminus O) \leq m(H) + 2\epsilon$. Subadditivity gives the reverse inequality after passage to the limit.

The finite-intersection step implicit at lines 379–392 is as follows. Call A splitting if $m(H) = m(H \cap A) + m(H \setminus A)$ for every H. Complements split. For splitting A, B, split H by A, then $H \cap A$ by B. The last two pieces $(H \cap A) \setminus B$ and $H \setminus A$ have union $H \setminus (A \cap B)$. Subadditivity yields the splitting inequality for $A \cap B$, and the opposite inequality always holds. Hence splitting sets form an algebra. For disjoint splitting A_j, $A = \bigcup_j A_j$, finite iteration and monotonicity give

$$m(H) \;\geq\; \sum_{j \leq N} m(H \cap A_j) + m(H \setminus A).$$

Taking the supremum over N and using countable subadditivity on $H \cap A$ proves A splits. Finite differences in the algebra now disjointify any countable

union. Taking $H = A$ gives countable additivity on disjoint splitting sets by the same inequalities. Since all opens split, $\mu = m|_{\mathrm{Borel}}$ is a finite measure.

For an admissible g, $\Lambda(g) \le m(\operatorname{supp} g)$ by taking the infimum over open supersets of its support. This gives inner regularity on opens; outer regularity is by construction. For any Borel A, an open outer approximation to $S \setminus A$ has compact complement inside A, giving inner regularity there too. To check representation, fix $u \ge 0$ continuous, $0 \le u \le m_0\delta$. Put $O_k = \{u > k\delta\}$. For $1 \le k \le m_0$ choose near-maximizing tests g_k in O_k. The pointwise inequality $\delta \sum_k g_k \le u$ gives the lower sum $\delta \sum_{k=1}^{m_0} \mu(O_k) \le \Lambda(u)$ by finite approximation. For the upper sum, $u = \delta \sum_{k=0}^{m_0-1} \min(1, \max(0, u/\delta - k))$. For $k \ge 1$ each summand has support in O_{k-1}. Thus the upper sum is $\delta\mu(S) + \delta \sum_{k=1}^{m_0-1} \mu(O_{k-1})$. The two simple functions approximate u uniformly to within 2δ. Finiteness gives $\Lambda(u) = \int u \, d\mu$; differences give the identity on $C(S)$. Any positive Radon representation λ satisfies $\lambda(O) = a(O)$: use compact inner approximation and a cutoff equal to one on each chosen compact. Outer regularity then gives uniqueness on Borel sets.

Signed measures and locality (lines 427–446). For arbitrary bounded real Λ use the source construction $\Lambda^+(u) = \sup_{0 \le v \le u} \Lambda(v)$ on continuous $u \ge 0$. Splitting $v \le u_1 + u_2$ into $\min(v, u_1)$ and the remainder, and adding nearly maximizing tests in the other direction, prove additivity. Positive homogeneity gives a linear extension by differences, since $u - v = u' - v'$ implies $u + v' = u' + v$. Both Λ^+ and $\Lambda^- = \Lambda^+ - \Lambda$ are positive and bounded; for the first use $|\Lambda^+(w)| \le \Lambda^+(|w|) \le \|\Lambda\| \|w\|_\infty$. Let $\mu_\pm$ be their positive Radon representatives. Then $\sigma = \mu_+ - \mu_-$ represents Λ.

For conventions requiring regular *total variation*, the following argument is internal to this proof. Put $\nu = \mu_+ + \mu_-$. It is regular: for any Borel A, take the union of two compact inner approximants and the intersection of two open outer approximants. Define $v(A)$ by the supremum of $\sum_i |\sigma(A_i)|$ over finite Borel partitions of A. Then $v \le \nu$. On disjoint Borel A_j, intersecting any finite partition of their union with each A_j gives $v(\bigcup_j A_j) \le \sum_j v(A_j)$ by countable additivity of σ and the triangle inequality. Conversely combine near-optimal finite partitions on the first N pieces, adding the remainder as one block; take the supremum and then let N increase. Thus v is a finite measure. Compact and open approximants $F \subset A \subset O$ with $\nu(O \setminus F) < \epsilon$ give $v(O \setminus F) < \epsilon$, so $v = |\sigma|$ is Radon. Conversely, any candidate signed measure with regular finite variation decomposes as $(|\sigma| + \sigma)/2 - (|\sigma| -$

$\sigma)/2$ into positive measures dominated by its variation and hence regular by the same estimate. Signed uniqueness therefore reduces, by adding the opposite positive parts, to positive uniqueness. This proves what is needed here without borrowing the downstream NAG-002973.

The compact carrier S is closed in Hausdorff Y. Push each positive measure forward along $S \hookrightarrow Y$; compact inner approximants stay compact, and an outer open of S lifts to an open of Y to which $Y \setminus S$ may be added. This gives the claimed signed Radon measure on Y. For linearity compare on the compact union of the carriers by density. For support inside an open, the local integral equals the ambient integral on every multiplier by 002425 and 002428. Uniqueness on the same compact carrier proves the zero-extension assertion. □

8.5 The tropical infrastructure: charts, calibration, and smoothness

8.5.1 Paralinéaire means piecewise affine geometry

The French term *paralinéaire*, abbreviated pl, refers to piecewise affine geometry. Local polyhedral pieces carry affine structures, and subdivisions allow maps and functions to be described affinely on smaller cells. This is not yet the integral structure supplied by the character lattice of an algebraic torus. The abstract real framework separates what is needed for differential calculus from the additional rational or integral structure needed for arithmetic applications.

The reason for this separation can be seen in an ordinary PL function. Its graph has an honest piecewise affine structure, even at a corner where the graph is not the graph of one affine function. A cellwise derivative exists. A global distributional derivative has an additional interface term. CLD25 arranges the definitions so that these two operations remain distinguishable.

8.5.2 G-local descriptions and ordinary smooth descriptions

A G-tropical chart may be defined on an analytic or polyhedral domain, and a G-cover may assemble several such domains along interfaces. Pulling back ambient smooth superforms through these charts gives G-forms. Ordinary smooth forms require the refined affine structure to provide suitable descrip-

tions on ordinary neighbourhoods. A function can therefore be G-smooth without being ordinary smooth.

Take $u(t) = |t|$. On the two half-lines it is affine; as a G-local object it has zero cellwise Hessian away from the interface. As a function on the ordinary neighbourhood of 0, its current Hessian is $2\delta_0$. The distinction persists in higher degree and on analytic spaces. It is not removed by choosing a finer subdivision: refinement moves the interface into the combinatorics but does not eliminate its contribution to the current derivative.

This example also explains why a current originally defined on ordinary smooth tests cannot automatically be evaluated on every G-form. The larger testing operation requires either an extension theorem, measure coefficients with a compatible G-local construction, or a structured current for which the action is defined geometrically.

8.5.3 Calibrations are the replacement for an orientation and a density

On a smooth real manifold, integration uses an orientation and a top-degree density. On a tropical space there are several tangent pieces at a point, each possibly carrying a multiplicity. A calibration packages the appropriate tangent-volume data so that forms can be contracted and integrated. Positivity of a calibration corresponds to nonnegative weights in familiar integral polyhedral examples.

The calibration must be compatible under changes of chart. In a tropicalization of an analytic space, these compatibilities are degree formulas. In an abstract tropical space they are part of the structure and its axioms. The definitions involving skeletal subsets and their volume forms are designed to let integration depend on the intrinsic space instead of on a chosen embedding into $\mathbb{R}^r$.

8.5.4 Richness and harmonic compatibility

The vocabulary of CLD25 Chapters 5–6 can initially obscure a simple requirement: there must be enough calibrated polyhedral pieces to detect forms and enough balancing to prove Stokes. Richness provides sufficient skeletal geometry to represent the local differential data. The stronger interior hypotheses ensure that the relevant top-dimensional data are visible. Differential sepa-

ration ensures that the chosen functions and forms distinguish the directions needed by the calculus. Harmony expresses the cancellation of calibrated boundary contributions in tropical charts.

These descriptions explain the role of the axioms; they do not replace their exact definitions. In Definition 6.6.2, a tropical space of level n is a calibrated and refined G-tropical space satisfying richness, together with a specified closed boundary and further interior conditions. The traces of boundaries on skeletal subsets must also be negligible in the required dimension. Thus the level and boundary belong to the structure.

An ordinary balanced weighted polyhedral complex is a useful model, but the framework is designed to include analytic spaces whose polyhedral descriptions vary with the chart. A single global complex is not required. CLD25 Chapters 12–14 prove the necessary analytic facts rather than assuming that a Berkovich space is literally a polyhedron.

8.5.5 Stokes in the abstract tropical framework

Once the calibration and harmony conditions hold, Stokes follows by applying the cellwise formula and cancelling interior faces. Partitions of unity pass from one chart to arbitrary compactly supported forms. The later analytic proof consists in identifying the canonical analytic calibration with this abstract one and verifying the hypotheses. This division makes the theory modular: a new class of spaces or charts can be tested against the same integration requirements.

The useful question when reading a new result is therefore: which part is formal tropical calculus, and which part verifies an analytic hypothesis? A proof of a formula for calibrated polyhedra does not by itself prove independence from analytic presentations. Conversely, once the analytic compatibility theorem is established, repeating every polyhedral calculation on Berkovich spaces is unnecessary.

References. [CLD25, Chapters 2–6, especially Definition 6.6.2].

8.6 Analytic realization of the tropical framework

8.6.1 Graded algebra and residual intersection theory

CLD25 Chapter 10 supplies the algebra needed to carry residual geometry through changes of value group and field. A normed object has reductions at many radii, so the graded setting is the natural one. The chapter also treats Zariski–Riemann spaces and intersection numbers of residual bundles. Such a space organizes compatible data over modifications rather than choosing one preferred model.

The point of this machinery becomes visible later: a PL metric can have a residual line bundle at an analytic germ, and a curvature product can be compared with a residual intersection number. These numbers must be independent of auxiliary models and compatible with scalar extension. The descent results in the graded chapter are what makes that numerical comparison meaningful in the non-strict setting.

This is a different use of models from defining the entire analytic calculus as an inverse system of algebraic cycles. The ordinary test forms and currents are already analytic objects. Residual models are used to calculate and characterize particular currents arising from geometric metrics.

8.6.2 Analytic dimension and the source of polyhedra

CLD25 Chapter 11 collects the analytic dimension, finite-morphism, and reduction facts used later. CLD25 Chapter 12 studies the skeletal loci of moment maps and their characteristic polyhedra. The tropical dimension of a point is tied to how many independent real directions its analytic functions can expose. The Abhyankar invariants from Section 3.7 control this dimension.

The richness theorems in CLD25 Chapter 12 say, in the source's precise sense, that there are enough analytic skeletal subsets to detect the differential data; the boundaryless case supplies the stronger interior properties. These are existence and compatibility theorems. They do not assert that every analytic neighbourhood is homeomorphic to one finite polyhedron or that one fixed skeleton detects every current.

8.6.3 Degrees, multiplicities, and canonical calibration

CLD25 Chapter 13 defines the relevant degrees of quasi-finite morphisms, incorporating generic multiplicities and residue-field extensions. CLD25 Chapter 14 uses these degrees to calibrate suitable tropical cells. A local analytic map can be many-to-one even when the real tropical map is injective on a cell; the calibration records the missing degree.

A proof of a canonical integral must show compatibility under three operations: subdivision of cells, refinement of the moment map, and passage through finite analytic maps. The calibrated degree formula handles the last operation; the variation results ensure that these weights behave correctly in analytic families. This is the analytic counterpart of the lattice-index pushforward formula used in ordinary tropical intersection theory.

8.6.4 A family calculation and why it matters for Green currents

Suppose a family of fibres is cut out by analytic functions and a fixed compactly supported test is tropicalized by additional units. Under the source's equidimensional hypotheses, the calibrated tropical images of nearby fibres stabilize in the relevant sense. The integral of that fixed test over the fibres can therefore be compared with its integral over the central fibre.

This mechanism is used in the proof of the Bochner–Martinelli formula. A truncated logarithmic maximum has curvature concentrated over a small Gauss point in the parameter space. As the truncation level tends to zero, the calibrated fibre calculation identifies its action on the fixed test with integration over the complete intersection. Thus the higher-codimension formula depends on the analytic degree and variation machinery, not only on a formal differentiation of a maximum.

8.6.5 Analytic forms and finite pushforward

CLD25 Chapter 15 realizes the abstract differential calculus on analytic spaces, including scalar extension, finite maps, support, and integration. CLD25 Chapter 16 develops currents in this realization. Proper pushforward is defined by duality with smooth pullback, with the appropriate degree shift. Projection formulas are then proved on their stated domains.

One should keep finite-map trace and arbitrary proper pushforward separate. A finite map can preserve dimension and carries a degree normalization. A general proper map may change dimension and collapse directions. Neither operation supplies a general descent theorem for actual delta-form representatives merely because a pushforward current exists.

8.7 Tropical geometry in the analytic calculus

The same tropical constructions enter several analytic theories, but at different stages. In the CLD construction, the integral tangent lattice and canonical multiplicities normalize integration. Balancing then supplies Stokes' theorem, so that differentiation of currents is independent of a cell decomposition. The passage to the analytic space still requires compatibility under refinements of moment maps; a formula on one polyhedron is only a presentation of that construction. These steps are developed in Sections 8.1 and 8.4.

In the GK calculus, a tropical cycle also occurs as a factor in a form. The corner formula becomes a differential identity with a cycle-valued contribution. Stable intersection supplies the multiplication of these contributions under the hypotheses of the delta-calculus. Thus GK uses more than the set-theoretic tropicalization: tangent lattices, multiplicities, refinements, and intersection products all enter. Sections 16.1 and 15.7 explain how the resulting objects descend through algebraic tropical charts.

Skeleta provide a different kind of comparison. Their integral-affine structures and slope formulas relate model divisors to tropical functions. Faithful tropicalization controls a chosen skeleton, while the multiplicity-one section selects analytic points over a suitable tropical locus. Neither assertion identifies every analytic form with data on one preassigned skeleton. The precise geometric statements are used in Section 6.3; the distinction between topological reconstruction and differential comparison is taken up in Section 24.1.

Extended tropicalizations retain the boundary strata required by Green functions. Their role in the BGK theory is therefore complementary to the affine corner calculus: they allow compatible data on toric compactifications and local-height calculations on a common tropicalization. The valuation convention remains the same throughout. A lattice index is an integer multiplicity, whereas a logarithmic distance records the scale of the absolute value.

References. For tropicalization and multiplicities, see [Gub13, MS15]. For the analytic integration construction, see [CLD12, Chapters 2–4] and [CLD25]. The intersection and skeletal developments are [GK17, GRW16, GRW17]; the extended tropical intersection theory is [BGK25].

8.8 Scope and geometric inputs

Throughout the detailed construction below, K is complete, algebraically closed and nontrivially valued, and X is a separated integral algebraic variety of dimension n over K. We work on a paracompact open subset $W \subset X^{\mathrm{an}}$. The statements below concern the original smooth forms, not harmonic or weakly smooth forms. These assumptions are retained in every analytic consequence.

A very affine variety is a closed subvariety of a split algebraic torus. For a very affine open $U \subset X$, the group of units modulo constants gives a canonical torus T_U and a closed immersion $\varphi_U : U \to T_U$, determined up to translation. Its tropicalization is

$$\tau_U : U^{\mathrm{an}} \longrightarrow N_{U,\mathbb{R}}, \qquad x \longmapsto (-\log|f_1(x)|, \dots, -\log|f_r(x)|).$$

Here the chosen units give coordinates; their tropical image is denoted $\mathrm{Trop}(U)$. A tropical chart is $V = \tau_U^{-1}(\Omega)$, where Ω is relatively open in $\mathrm{Trop}(U)$.

Theorem 8.8.1 (Geometric inputs from Gubler)**.** *The following results are used with the hypotheses above.*

1. *Tropical charts form a basis. Intersections have common refinements coming from intersections of very affine opens; transition maps of tropicalizations are integral affine.*
2. *A superform on the image of a tropical chart defines zero on the analytic chart if and only if it is zero on that tropical image.*
3. *The algebraic-chart sheaf agrees with the original CLD sheaf using analytic moment maps.*
4. *Smooth partitions of unity subordinate to open covers exist, with locally finite compact supports.*

Open proof obligation (RC2). Supply native proofs of the geometric moment-image, boundary and degree inputs. The historical source expressly imports the listed Gubler16 results.

8.9 The quotient on a polyhedral image

Let P be a locally finite polyhedral subset of a real affine space and let $\Omega \subset P$ be relatively open. Choose an ambient open neighbourhood. Restrict ambient superforms to each face using its tangent space, and declare two representatives equal when all these restrictions agree. Subdivision does not change the resulting space.

Proposition 8.9.1. *The kernel of restriction is stable under d', d'', J and wedge multiplication. Thus these operations descend to the polyhedral quotient. Pullbacks along affine maps mapping faces into faces also descend.*

Proof. On the relative interior of a face, restriction is ordinary affine pullback. It commutes with the two differentials by their coordinate formulas, and with J and wedge by their algebraic definitions. A form restricting to zero has zero tangential coefficients; their tangential derivatives vanish on that relative interior. Continuity gives the same conclusion on its closure. Applying this argument face by face proves the kernel assertions. For an affine map, a representative vanishing on a target face pulls back to zero on every source face mapping to it. □

One must quotient by tangent restrictions, not merely by the values of the scalar coefficients on P. For example, on the line $x_2 = 0$ in $\mathbb{R}^2$, the form $d'x_2$ vanishes after restriction although its ambient coefficient is the constant one. The two cotangent copies are both restricted.

8.10 Gluing the analytic sheaf

Definition 8.10.1. A smooth (p, q)-form on W is a family of superforms on the images of a tropical-chart cover, whose pullbacks agree on common refinements of overlaps. Two families are identified if they agree after refining their covers. Write this sheaf as $A_X^{p,q}$.

Proposition 8.10.2. *This defines a sheaf. The operations d', d'', J and wedge descend to it and satisfy the identities of Chapter 7.*

Proof. A common refinement of three covers is obtained by successive intersections, so agreement after refinement is transitive. Restriction is restriction of each family. If forms on an open cover agree on overlaps, take the union of their chart families; the agreement condition makes this a global compatible family. Uniqueness is tested on the same covering charts. Proposition 8.9.1 proves that applying an operation to equal representatives preserves equality. Its algebraic identities therefore pass to the quotient and then to the glued sheaf. □

The support is the complement of the locus where the germ vanishes. The chart injectivity in Theorem 8.8.1 identifies vanishing of a form presented on one chart with vanishing of its superform on that chart's image. It does not say that every form on a fixed chart has a single presentation on that chart; a further cover may be needed [Gubler, Remark 5.7].

Example 8.10.3 (Behaviour at a classical point)**.** If $x \in X(K)$ and f is an analytic unit near x, then on a sufficiently small neighbourhood $|f(y) - f(x)| < |f(x)|$. The strong triangle inequality gives $|f(y)| = |f(x)|$. For finitely many units, intersect these neighbourhoods. A smooth function expressed through their logarithms is constant there, and positive-degree forms in these tropical coordinates vanish there. Thus these coordinates are not real-manifold coordinates near a classical point.

8.11 Compactly supported tests

Write $A_c^{p,q}(W)$ for forms with compact support. A fixed presentation consists of finitely many tropical charts and compact polyhedral subsets of their images, together with compatible coefficient representatives. Tropicalization is proper in the very affine setting, so their inverse images are compact. On each such presentation use the coefficient C^∞ seminorms on its fixed compact sets, with tangential restrictions on the faces. The actual test space has the final locally convex topology of all these finite presentations, as in [Gubler, §6.1–6.2]. Equivalently, a current is a linear functional continuous on each of the presentation spaces. In this book a functional on (q,q)-tests has bidimension (q,q) and bidegree $(n-q, n-q)$. Gubler's notation in §6.2

uses the test degree as the current index; this indexing difference changes no pairing. This topology is essential: a sequence that is smooth in a different chart at every stage is not automatically convergent in a fixed presentation. Arguments below that use uniform convergence of scalar functions will be justified by measures, rather than by pretending that the scalar functions all have a common chart.

8.12 Cutoffs and scalar density

Lemma 8.12.1. *For compact $C \subset W$ and an open neighbourhood O of C, there is $0 \le \chi \le 1$ smooth, compactly supported in O, and equal one near C.*

Proof. Choose $C \subset O' \Subset O$ using local compactness and compactness of C. Apply the partition theorem to the cover $\{O, W \setminus \overline{O'}\}$. Only finitely many members of the locally finite partition meet C. Sum all members meeting a smaller neighbourhood of C, chosen to avoid the other supports and to lie in O'. Every selected support is compact and lies in O. Their sum is one on that smaller neighbourhood and lies between zero and one everywhere. □

Proposition 8.12.2. *Every $f \in C_c(W)$ admits smooth approximants converging uniformly on W, with all supports in one compact subset. If $f \ge 0$, the approximants may be chosen nonnegative.*

Proof. Take a compactly supported cutoff χ equal one near $\operatorname{supp} f$, and put $L = \operatorname{supp} \chi$. Restrictions of global smooth functions form a real algebra containing constants on L. They separate points: a cutoff supported in a neighbourhood of one point and avoiding another separates those points. The real Stone–Weierstrass theorem gives smooth a_j converging uniformly to f on L. Then $\chi a_j \to f$ uniformly on W because $\chi f = f$, with supports in L. For nonnegative f, approximate $\sqrt{f}$ on L by smooth b_j and use χb_j^2 instead. □

This theorem is scalar density only. It does not preserve a positive Hessian and gives no general regularization theorem for current-psh functions. The approximation class in Chapter 11 remains a separate hypothesis.

8.13 Ordinary-positive domination on actual tests

An actual symmetric form is positive if locally it has a chart presentation whose restrictions to the faces are ordinary positive. This property is preserved by refinement because affine pullback preserves positivity. Multiplication by a nonnegative smooth scalar also preserves it, even if that scalar requires a further chart.

Proposition 8.13.1. *For every fixed finite test presentation and compact supports, there are a compactly supported actual positive form θ and a constant c such that every symmetric test η in that presentation satisfies*

$$c\|\eta\|_{D,0}\theta + \eta \geq 0, \qquad c\|\eta\|_{D,0}\theta - \eta \geq 0.$$

Here $\|\eta\|_{D,0}$ is a coefficient seminorm in that fixed presentation.

Proof. First localize by a fixed finite nonnegative smooth partition over the prescribed supports. For one chart, choose an ambient Euclidean norm and the constant form $\beta_q = \omega^q/q!$. On every tangent space its restriction has identity coefficient matrix in an orthonormal exterior basis. Restricting a coefficient tensor to that tangent space has uniformly bounded norm: the chosen Euclidean metric makes orthogonal restriction contractive. Thus the matrix estimate of Proposition 7.3.2 gives positive domination on every face by a common multiple of β_q. Multiply this control form by a cutoff equal one on the localized support and compactly supported inside the chart, then extend by zero. It is an actual positive form. Summing the finitely many control forms and constants gives the assertion. Tangent spaces of dimension less than q contribute zero to both the test and control form. □

8.14 Intrinsic continuous multiplication

The next construction avoids taking a quotient of arbitrary continuous chart coefficients. Each measure is attached directly to an actual smooth test form.

Theorem 8.14.1. *Let T be a symmetric linear functional on $A_c^{q,q}(W)$ nonnegative on ordinary positive tests. Then T is a current, of order zero in every fixed presentation. For every actual smooth compactly supported test η, there is a signed Radon measure $\mu_{T,\eta}$ determined by*

$$\int \phi \, d\mu_{T,\eta} = T(\phi\eta), \qquad \phi \in C_c^\infty(W).$$

Its support lies in $\operatorname{supp}\eta$*. For* $f \in C^0(W)$ *the formula*

$$(fT)(\eta) = \int f \, d\mu_{T,\eta}$$

defines an intrinsic current. It is positive when $f \geq 0$*, and agrees with smooth multiplication when* f *is smooth.*

Proof. Actual domination gives current continuity exactly as in Proposition 9.9.1; that proposition is purely functional analytic and does not use this theorem. If η is positive, then $\phi \mapsto T(\phi\eta)$ is nonnegative for nonnegative smooth ϕ, and

$$|T(\phi\eta)| \leq \|\phi\|_{\operatorname{supp}\eta,\infty}\, T(\eta).$$

Scalar density and the Riesz theorem give a positive Radon measure. It is supported in $\operatorname{supp}\eta$ because all tests outside that set pair to zero. For symmetric η, Proposition 8.13.1 writes η as a difference of two actual positive tests. Take the difference of their measures. Its values on smooth scalar tests determine it uniquely by scalar density; hence it is independent of the decomposition. Symmetrization handles all tests. This also proves linearity in the actual test η, including every relation among its chart presentations.

It remains to check continuity of fT. In a fixed presentation put $M = c\|\eta\|_{D,0}$ and use its fixed control form θ. The positivity of $M\theta \pm \eta$ implies

$$-M\mu_{T,\theta} \leq \mu_{T,\eta} \leq M\mu_{T,\theta}.$$

Consequently $|\mu_{T,\eta}| \leq M\mu_{T,\theta}$ and

$$|(fT)(\eta)| \leq c\|\eta\|_{D,0} \int |f| \, d\mu_{T,\theta}.$$

The final integral is finite because θ has compact support. This proves continuity on every presentation. Positivity and agreement with smooth multiplication follow from the defining measure formula. □

Proposition 8.14.2. *The construction commutes with restriction and satisfies* $f(gT) = (fg)T$ *for continuous* $f, g \geq 0$*. The same equality holds for signed continuous multipliers by linearity.*

Proof. For smooth compactly supported ϕ, uniqueness of the representing measures gives $\mu_{T,\phi\eta} = \phi\mu_{T,\eta}$. It follows that

$$(gT)(\phi\eta) = \int g\phi \, d\mu_{T,\eta}, \qquad \mu_{gT,\eta} = g\mu_{T,\eta}.$$

Integrating f proves associativity. The same uniqueness argument on open subsets proves restriction compatibility. Decompose a signed continuous multiplier into its positive and negative parts to obtain the signed version. □

Thus the relation problem for multiplication by a continuous scalar is resolved in the stated original algebraic CLD setting. This does not resolve the different kernel problem $\mathrm{d}'\mathrm{d}''[h] = 0 \Rightarrow \mathrm{d}'\mathrm{d}''(hT) = 0$: the latter concerns the action of curvature-equivalent potentials, not relations between presentations of one actual smooth test.

8.15 The degree in change of tropical coordinates

The foundations identify the origin of the three factors in an integral: the analytic fibre degree, the tangent lattice, and the Euclidean density. We now prove the elementary change-of-variables calculation and isolate the geometric comparison needed to apply it to analytic charts. This section uses the algebraic CLD setting of Section 8.8.

Lemma 8.15.1. *Let $F : \sigma \to \tau$ be an affine isomorphism between n-dimensional polyhedra whose linear part is integral and injective on their tangent lattices. Put $e = [N_\tau : F(N_\sigma)]$. For a compactly supported top-degree superform β on τ, in lattice-normalized integration,*

$$\int_\sigma F^*\beta = e \int_\tau \beta.$$

Proof. Choose integral bases of the two lattices, with real coordinates x, y. Write $y = Mx + b$. The positive top-degree generator has two cotangent factors. Its pullback acquires $(\det M)^2$, whereas ordinary real change of variables contributes $|\det M|^{-1}$. The resulting factor is $|\det M| = e$. This calculation is independent of the chosen bases, whose transition determinants have absolute value one. □

After subdividing a polyhedral map into cells on which it is injective, apply the lemma cell by cell. If source weights are m_σ, the target weight is

$$m_\tau = \sum_{F(\sigma)=\tau} m_\sigma [N_\tau : F(N_\sigma)].$$

Cells whose images have dimension below n contribute zero because top-degree pullback vanishes there. For a finite collection of cells and a form

whose relevant pullback has compact support, summing gives the projection formula for the weighted integrals.

Example 8.15.2. On an annulus write $t = -\log|T|$ and replace the moment coordinate T by T^m, with integer $m \geq 1$. Then $s = mt$ and

$$\int m^2 b(mt)\, dt = m \int b(s)\, ds.$$

The tropical image for this presentation has calibration m. The algebraic map $T \mapsto T^m$ is finite free of rank m, with basis $1, T, \ldots, T^{m-1}$, also when the residue characteristic divides m. The factor records algebraic degree, not the number of visibly distinct points of a fibre. A top form $a(t)d't \wedge d''t$ is represented in the s coordinate by $m^{-2}a(s/m)d's \wedge d''s$; integrating it with weight m gives the original integral.

8.15.1 Independence of the analytic integral

For two analytic presentations, adjoining all their unit coordinates gives a common moment map. The geometric input says that its calibrated weights push forward to the weights of either presentation. In the algebraic chart construction this is the tropical degree comparison used in [Gubler, Proposition 4.11 and Lemma 5.15]; the analytic construction is in [CLD12, §§3.3–3.8]. The lemma proves equality of the two local integrals once this geometric weight identity is known. It does not prove the weight identity itself.

The remaining partition argument is formal but useful to record. Let (ρ_i) and (σ_j) be two subordinate smooth partitions near the compact support of a form α, and denote the corresponding local integral functionals by I_i and J_j. Only finitely many terms meet the support. Refining both presentations where $\rho_i\sigma_j\alpha$ is supported, the local comparison gives

$$\begin{aligned}\sum_i I_i(\rho_i\alpha) &= \sum_{i,j} I_i(\rho_i\sigma_j\alpha) \\ &= \sum_{i,j} J_j(\rho_i\sigma_j\alpha) = \sum_j J_j(\sigma_j\alpha).\end{aligned}$$

Thus the global integral is independent of charts and partitions. The analytic-domain and sheaf constructions of Chapter 2 make the refinements meaningful; the tropical degree theorem supplies the nonformal part. For Stokes one further uses balancing and support away from the analytic boundary, as defined in Section 2.7. None of these steps is supplied by the homeomorphism type of a skeleton alone.

8.15.2 Ordinary presentation topologies on compact tests

Fix a complete non-Archimedean field K with its original real-valued absolute value. Let X be good and Hausdorff. We use the ordinary smooth moment sheaf $\mathcal{A}_X^{p,q}$ of `v037:forms:ordinarysmoothgerm` and `v037:forms:germsheaf`, and write $\mathcal{E}^{p,q}(X) = \mathcal{A}_{X,c}^{p,q}(X)$. All supports in this note are ordinary compact supports. A version with boundary-avoiding tests is obtained by applying the construction to a specified ordinary open subspace of the analytic interior. There is no boundary vanishing assertion in the topology construction. All one-form generators anticommute, and the positive top generator is $d'x_1 \wedge d''x_1 \wedge \cdots \wedge d'x_n \wedge d''x_n$, as in `v037:integration:conventions`.

For an open $\Omega \subset \mathbb{R}^N$, let $E^{p,q}(\Omega)$ be the space of smooth ambient superforms with the coefficient seminorms

$$p_{L,m}(\eta) = \max_{I,J,\,|a|\le m} \sup_{u\in L} |\partial^a \eta_{I,J}(u)|, \qquad L \Subset \Omega.$$

Empty coefficient slots give the zero space.

Definition 8.15.3 (The two generating families)**.** A cutoff presentation $i = (V, f, \Omega, \chi)$ consists of an affinoid domain $V \subset X$, a unit tuple f on V, an open set $\Omega \supset t_f(V)$, and $\chi \in \mathcal{A}_c^{0,0}(X)$ with $\operatorname{supp}\chi \subset \operatorname{Int}_{\mathrm{top}}(V, X)$. Here $t_f = -\log|f|$. Its map is

$$Q_i : E^{p,q}(\Omega) \longrightarrow \mathcal{E}^{p,q}(X), \qquad Q_i\eta = \operatorname{ext}_0(\chi t_f^*\eta).$$

Let τ_{cut} be the locally convex topology generated by all seminorms P on $\mathcal{E}^{p,q}(X)$ for which PQ_i is continuous for every i. This is the definition F8-T1 in the supplied candidate.

An ordinary presentation s consists of finitely many such charts (V_j, f_j, Ω_j), without cutoffs, and a compact set $K_s \subset \bigcup_j U_j$, where $U_j = \operatorname{Int}_{\mathrm{top}}(V_j, X)$. Let

$$H_s \subset \prod_j E^{p,q}(\Omega_j)$$

be the linear subspace of tuples (η_j) for which the induced ordinary sections agree on $U_j \cap U_k$ and vanish on $U_j \setminus K_s$. Give H_s its coefficient subspace topology. The sections glue with zero on $X \setminus K_s$ to a section $R_s\eta \in \mathcal{E}^{p,q}(X)$. Give $B_s = H_s/\ker R_s$ its quotient topology and let τ_{ord} be the final locally convex topology for the maps $B_s \to \mathcal{E}^{p,q}(X)$, equivalently for all R_s.

Both definitions refer to ordinary sections. In particular, the relations in H_s are not compatibility solely on a finite closed affinoid cover. Every compactly supported section belongs to some $\operatorname{im} R_s$: choose finitely many ordinary representing neighbourhoods over its support and take K_s to be that support. The cutoff maps also span the test space, by the finite partition of `v037:forms:compactpartition` on these same neighbourhoods.

Theorem 8.15.4 (Comparison of the two final topologies)**.** *In every bidegree,*

$$\tau_{\mathrm{cut}} = \tau_{\mathrm{ord}}.$$

More precisely, each R_s is a finite sum of cutoff presentation maps composed with continuous linear maps, and each Q_i factors continuously through one ordinary presentation space H_s. The charts, cutoffs and factorizing maps in each assertion are fixed before the variable ambient forms are chosen.

Proof. Fix s. Apply `v037:forms:compactpartition` to K_s and (U_j), obtaining fixed χ_j supported compactly in U_j, with sum one near K_s. If $\eta = (\eta_j) \in H_s$, then

$$R_s\eta = \sum_j Q_{(V_j,f_j,\Omega_j,\chi_j)}\eta_j.$$

On K_s this follows from the partition and compatibility; off K_s all relevant sections are zero. The coordinate projections from H_s are continuous. Thus every R_s is continuous for τ_{cut}.

Conversely, fix $i = (V, f, \Omega, \chi)$ and put $K_s = \operatorname{supp}\chi$. The empty-support case is zero. Otherwise choose finitely many affinoid neighbourhoods $W_j \subset \mathrm{Int}_{\mathrm{top}}(V, X)$ whose interiors cover K_s, such that χ is represented on W_j by a fixed smooth scalar coefficient c_j in a fixed unit tuple g_j. To do this, shrink inside an ordinary representing neighbourhood of χ before taking W_j; the neighbourhood basis and restriction assertion are part of the proof of `v037:forms:germsheaf`. On W_j use the joint tuple $h_j = (f|_{W_j}, g_j)$ and the ambient open set $\Omega \times \Omega_{g_j}$, where $\Omega_{g_j} \supset t_{g_j}(W_j)$. Set

$$L_j\eta = \pi_g^* c_j \, \pi_f^*\eta.$$

For every compact L in this product and every m, the product and chain rules give

$$p_{L,m}(L_j\eta) \le C_{j,L,m}\, p_{\pi_f(L),m}(\eta).$$

The derivatives of the fixed coefficient c_j are absorbed in $C_{j,L,m}$. The tuple $(L_j\eta)_j$ belongs to the fixed space H_s: its components represent the same

ordinary section $\chi t_f^* \eta$ and vanish off K_s. It depends continuously on η, and its glued section is $Q_i \eta$. Therefore Q_i is continuous for τ_{ord}.

A seminorm is continuous for either final topology exactly when its composites with that topology's generating maps are continuous. The two factorizations prove equality of these families of seminorms. □

Proposition 8.15.5 (The available direction of the tangential comparison)**.** *Let $P \subset \Omega \subset \mathbb{R}^N$ be a compact finite polyhedral set, and let N_P be the ambient smooth forms with zero tangential restriction on every face. Give $E^{p,q}(\Omega)/N_P$ its quotient topology. For each face σ, fix an affine tangent basis. The suprema, on compact subsets of σ, of tangential coefficient derivatives of any fixed finite order are well-defined continuous seminorms on this quotient. Consequently the topology they generate is no finer than the ambient quotient topology.*

Proof. In a tangent basis, restriction of each covector factor is a fixed linear map. Each tangential derivative is a fixed linear combination of ambient partial derivatives. Its coefficient supremum is bounded by a constant times the ambient seminorm of the same order on the same compact set. If a form belongs to N_P, its restricted coefficients vanish on the relative interior of every face. Their derivatives along that face vanish there and, by continuity, on its boundary. Thus the seminorms annihilate N_P and descend. The defining property of the quotient gives continuity of each descended seminorm. □

The intrinsic tangential comparison here gives continuity from the ambient quotient for each fixed compact polyhedral image. A converse would require a bound for ambient representatives. A comparison with general chart and support families would also require matching the underlying sheaves and the systems of presentations.

8.15.3 Multiplication in chart and support topologies

Fix bidegrees and an order $r \in \mathbf{N} \cup \{\infty\}$. The coefficients are smooth real scalars. The order r specifies which coefficient derivatives define the topology. For $r = \infty$, use every finite derivative order.

Let X be a Hausdorff G-tropical space with the following chart and sheaf data. Smooth forms on a compact chart image are tangential restrictions

of local ambient smooth forms on a finite polyhedral set. For every compact domain W in a chart $f : V \to E$, the image $f(W)$ is compact and finite polyhedral, and a form on it is uniquely determined by its pullback to W. Compact domains cover chart domains in the G-topology. The smooth G-form sheaf has restrictions and exterior products compatible with the corresponding affine pullbacks and is determined on G-covers. Domains restrict to chart intersections, with joint coordinates available for common refinements.

Definition 8.15.6 (Intrinsic seminorm on a compact image)**.** Let C be a compact cell, use affine coordinates on its affine span, and write a tangential form as $\alpha = \sum_{|I|=p,|J|=q} \alpha_{IJ}\, d'x_I \wedge d''x_J$. Put

$$\|\alpha\|_{C,m} = \max_{I,J,\,|a|\le m} \sup_{C} |\partial^a \alpha_{IJ}|, \qquad m \ge 0.$$

The derivatives at the boundary are the continuous restrictions of those on the relative interior. Coefficient slots with degree exceeding the dimension are zero. For a compact finite polyhedral set P, choose a finite cell decomposition $\mathcal{C}$ and affine coordinates on each cell; a standard seminorm is

$$\|\alpha\|_{P,m} = \max_{C\in\mathcal{C}} \|\alpha|_C\|_{C,m}.$$

Use the seminorms for $m \le r$, all finite m if $r = \infty$. This is the intrinsic cell topology on the space of smooth forms on P.

Different affine coordinates give equivalent seminorms: the tangential covectors change by fixed minors, and their derivatives by fixed linear combinations. Passing to a common finite subdivision gives the same cell topology. On a subdivided cell of full relative dimension the smooth coefficient derivatives are restrictions of the old ones; conversely these subcells cover a dense subset of the original cell, so the original suprema follow by continuity. Lower-dimensional cells are handled by their tangent restrictions. This uses only finitely many coordinate changes. It asserts no bounded ambient extension across faces.

For an affine map $a : P \to Q$ of compact polyhedral sets, subdivision of the source and target gives, for each m, a finite constant $c_{a,m}$ with

$$\|a^*\alpha\|_{P,m} \le c_{a,m}\|\alpha\|_{Q,m}. \tag{8.15.1}$$

To see the estimate, restrict to a source cell mapped into a target cell. The coefficients of pullback are sums with coefficients given by the p- and q-minors

of the fixed tangent matrix. A derivative of order $\ell \le m$ adds products of ℓ matrix entries, by the chain rule. Take the maximum of these finitely many constants. Translation contributes no derivatives. This proves the forward continuity needed below; no strictness assertion for a surjective affine map is used.

Definition 8.15.7 (A single-chart space)**.** Let $f : V \to E$ be a chart on a domain of X. Define $\mathcal{A}^{p,q}_{f,G}(X)$ to be the subspace of smooth G-forms ω on X for which, for *every* compact domain $W \subset V$, there is $\alpha_{f,W} \in \mathcal{A}^{p,q}(f(W))$ satisfying

$$\omega|_W = (f|_W)^* \alpha_{f,W}.$$

The compact-nullity prerequisite makes $\alpha_{f,W}$ unique. Transport the intrinsic topology from $\mathcal{A}^{p,q}(f(W))$ to its pullback space on W, and give $\mathcal{A}^{p,q}_{f,G}(X)$ the initial topology for all these restriction maps. Explicitly it is generated by

$$\rho_{f,W,m}(\omega) = \|\alpha_{f,W}\|_{f(W),m}, \qquad W \subset V \text{ compact}, \quad m \le r.$$

Definition 8.15.8 (A fixed chart family)**.** For a family $\mathcal{D} = \{f_i : V_i \to E_i\}$ put

$$\mathcal{A}^{p,q}_{\mathcal{D},G}(X) = \bigcap_i \mathcal{A}^{p,q}_{f_i,G}(X),$$

and give it the initial locally convex topology $T_{\mathcal{D},r}$ for all the inclusions into the single-chart spaces. Thus its seminorms are the restrictions of all $\rho_{f_i,W,m}$; neighbourhoods require only finitely many of these seminorms at a time. Families and compact domains need not be finite or countable. If the domains form a G-cover, compact nullity and the sheaf property imply that this topology is Hausdorff: vanishing of every seminorm gives zero on every chart domain and hence on X.

We use the word *refinement* with its geometric meaning. A chart $g : V' \to F$ refines $f : V \to E$ if $V' \subset V$ and $f|_{V'} = a \circ g$ for a fixed affine map $a : F \to E$. A family refines another if every chart in it has such a factorization through a chart of the latter. From a coarse family $\mathcal{D}$ to a refining family $\mathcal{R}$ one obtains an inclusion

$$\mathcal{A}^{p,q}_{\mathcal{D},G}(X) \longrightarrow \mathcal{A}^{p,q}_{\mathcal{R},G}(X).$$

It is continuous. For a compact W in the domain of $g \in \mathcal{R}$, the representative on $g(W)$ is $a^* \alpha_{f,W}$. Estimate (8.15.1) bounds each target seminorm by one source seminorm, proving continuity of the inclusion.

Definition 8.15.9 (Support and chart families)**.** The support of a G-form is the complement of the largest ordinary open set on which it is zero. A support family Φ is a family of ordinary closed subsets of X containing $\varnothing$, closed under finite unions and under taking ordinary closed subsets. Let Γ be a nonempty collection of G-covering chart families, admitting a common refining family in Γ for any two members.

For $\mathcal{D} \in \Gamma$ and $F \in \Phi$, let

$$\mathcal{A}^{p,q}_{\mathcal{D},G,F}(X) = \{\omega \in \mathcal{A}^{p,q}_{\mathcal{D},G}(X) : \operatorname{supp}\omega \subset F\},$$

with its subspace topology from $T_{\mathcal{D},r}$. Define

$$\mathcal{A}^{p,q}_{\Gamma,\Phi}(X) = \bigcup_{\mathcal{D}\in\Gamma,\, F\in\Phi} \mathcal{A}^{p,q}_{\mathcal{D},G,F}(X).$$

Common refinements and unions of supports make this a vector space. Its C^r topology is the locally convex final topology of all these stage inclusions. Equivalently its continuous seminorms are exactly those seminorms whose restrictions to every stage are continuous.

Consequently a linear map from this space to a locally convex space is continuous exactly when its restriction to each stage is continuous. This follows from the defining seminorms by composing with every continuous seminorm of the target.

Definition 8.15.10 (Ordinary smooth specialization)**.** In the ordinary affine specialization, the specified affine coordinates locally present the ordinary smooth forms on domain interiors. Use the nonempty family Γ_X of all affine chart families whose domain interiors cover X. A common refinement is obtained on the intersections using joint coordinates. For a support family Φ the ordinary test space here is $\mathcal{A}^{p,q}_{\Gamma_X,\Phi}(X)$ with the topology of Definition 8.15.9.

Proposition 8.15.11 (Multiplication by a fixed smooth form)**.** *Fix $r \in \mathbf{N} \cup \{\infty\}$. Let Γ and Φ, Ψ be as in Definition 8.15.9. Suppose the fixed smooth G-form α of bidegree (p', q') belongs to $\mathcal{A}^{p',q'}_{\Gamma}(X)$, using all ordinary closed supports. Assume that every ordinary closed subset of* $\operatorname{supp}\alpha$ *belonging to Φ also belongs to Ψ. Then*

$$M_\alpha : \mathcal{A}^{p,q}_{\Gamma,\Phi}(X) \longrightarrow \mathcal{A}^{p+p',q+q'}_{\Gamma,\Psi}(X), \qquad \omega \longmapsto \alpha \wedge \omega,$$

is well-defined, linear, and continuous for the C^r topologies. In particular this holds for ordinary smooth forms with $\Gamma = \Gamma_X$.

Proof. Choose once a family $\mathcal{E} \in \Gamma$ that tropicalizes α. Fix a source stage $(\mathcal{D}, F)$. A common refining family $\mathcal{R} \in \Gamma$ tropicalizes both α and every form in the source stage. Refinement gives a continuous linear inclusion from $\mathcal{A}_{\mathcal{D},G}$ to $\mathcal{A}_{\mathcal{R},G}$ by the preceding affine estimate.

On a compact domain W of a chart $h \in \mathcal{R}$ write $\alpha|_W = h^*a$ and $\omega|_W = h^*b$. In tangential cell coordinates, each coefficient of $a \wedge b$ is a finite signed sum of products of coefficients. For each finite $m \le r$, the product rule gives

$$\|a \wedge b\|_{h(W),m} \le N_{p,q,p',q',h(W)}\, 2^m \|a\|_{h(W),m}\, \|b\|_{h(W),m}. \tag{8.15.2}$$

Here N is a finite upper bound for the number of coefficient products in a fixed output slot on these finitely many cells. No ambient normal derivatives occur. The coefficient $\|a\|_{h(W),m}$ is fixed and finite. Thus for every target chart seminorm, multiplication is bounded by a constant times a continuous source seminorm after refinement. For $r = \infty$ apply this argument separately at each finite m.

For every source form its product has support in $G_F = F \cap \operatorname{supp}\alpha$. This is an ordinary closed subset of both F and $\operatorname{supp}\alpha$; hence $G_F \in \Phi$ and the support hypothesis gives $G_F \in \Psi$. The stage map therefore has its image in $\mathcal{A}^{p+p',q+q'}_{\mathcal{R},G,G_F}(X)$, equipped with the subspace topology. The continuous map into $\mathcal{A}_{\mathcal{R},G}$ just constructed is continuous into that subspace. Composing with its canonical continuous inclusion into the target final space proves continuity on the source stage. The linear universal property of the source final topology proves continuity of M_α. All choices of $\mathcal{R}$ and G_F depend only on the source stage and the fixed α, not on ω. □

In particular a fixed ordinary smooth cutoff acts continuously on the compact test space. If the compact support family is taken inside a specified ordinary open of the analytic interior, the same argument gives an endomorphism for that family because the support can only decrease.

8.15.4 Estimates on bounded sets in final test spaces

For an arbitrary locally convex space E, write E'_b for its continuous dual with the topology of uniform convergence on bounded subsets of E. Bounded means bounded for every continuous seminorm. No separation or completeness assumption will be needed. In this section E is spanned by the images of linear maps $Q_i : E_i \to E$, and has the locally convex final topology for these maps.

Theorem 8.15.12 (Uniform eventual equality on bounded sets)**.** *Let $D_M : E \to \mathbb{R}$ be linear, for real $M \geq 0$. Assume that every $D_M Q_i$ is continuous, and that for every i there is $H_i < \infty$ such that*

$$D_M Q_i = 0 \qquad \text{for all } M \geq H_i. \tag{8.15.3}$$

Then each D_M is continuous. For every bounded set $B \subset E$ there exists $H_B < \infty$ such that

$$D_M(b) = 0 \qquad (M \geq H_B,\ b \in B). \tag{8.15.4}$$

In particular $D_M \to 0$ in E_b'. No bounded-set lifting, countable cofinality, regular inductive limit, or uniform bound on the H_i is assumed.

Proof. Continuity of D_M is exactly the final seminorm criterion. Every $x \in E$ is a finite sum of elements of presentation images, so (8.15.3) makes $D_M(x)$ eventually zero. Suppose (8.15.4) fails. Choose $M_j \geq j$ and $b_j \in B$ with $D_{M_j}(b_j) \neq 0$, and put

$$c_j = \frac{j}{|D_{M_j}(b_j)|}, \qquad P(x) = \sum_{j \geq 1} c_j |D_{M_j}(x)|.$$

The sum is finite for every x, so P is a finite-valued seminorm. For each fixed i all terms with $j \geq H_i$ vanish on $Q_i(E_i)$. Thus PQ_i is a finite sum of continuous seminorms and is continuous. The final topology makes P continuous on E. But $P(b_j) \geq j$, contradicting boundedness of B. This contradiction proves (8.15.4), for the full real parameter family. □

For the following estimate, let $D_M : E \to \mathbb{R}$ be linear for each real $M \geq 0$.

Proposition 8.15.13 (A quantitative estimate across presentations)**.** *Let p_i be a continuous seminorm on E_i, let $w_i > 0$, and let $\varepsilon(M) > 0$. Suppose*

$$|D_M Q_i(\eta)| \leq \varepsilon(M) w_i p_i(\eta) \quad \text{for all } i, M, \eta. \tag{8.15.5}$$

Then the finite-decomposition gauge

$$P_w(x) = \inf_{x = \sum_{i \in F} Q_i \eta_i} \sum_{i \in F} w_i p_i(\eta_i), \qquad F \text{ finite}, \tag{8.15.6}$$

is a continuous seminorm on E, and

$$\sup_{x \in B} |D_M(x)| \leq \varepsilon(M) \sup_{x \in B} P_w(x) \tag{8.15.7}$$

for every bounded set B. Hence $\varepsilon(M) \to 0$ gives strong-dual convergence, even when no presentation is fixed for the members of B. The premise must hold with the actual generators of the topology whose bounded sets are used.

Proof. The spanning assumption makes the infimum finite. Concatenating finite decompositions proves subadditivity (with arbitrary small errors before taking infima); scaling proves absolute homogeneity. Repeated indices can be merged by the seminorm triangle inequality. Thus P_w is a seminorm, possibly with a kernel. The one-term decomposition gives $P_w Q_i \le w_i p_i$, so every composite is continuous. The final criterion gives continuity of P_w. Apply (8.15.5) to any finite decomposition, sum, and take its infimum to get $|D_M(x)| \le \varepsilon(M) P_w(x)$. Boundedness now proves (8.15.7). □

Chapter 9

Currents, order zero and continuous multipliers

9.1 Test spaces, currents, and their operations

Definition 9.1.1 (Test forms)**.** Write $A^{p,q}_{\mathrm{sm},c}(X)$ for the original smooth forms whose support is compact and disjoint from the analytic boundary ∂X. The topology is determined by finite tropical presentations: on a fixed finite family of tropical charts, with support in fixed compact subsets, the coefficients of smooth representatives and every derivative of their coefficients converge uniformly. Equivalent representatives are given the quotient topology, and the test space is given the final locally convex topology over these presentation spaces.

In the algebraic setting, this is the stagewise continuity formulation in [Gubler, §6.1–§6.2]; compare [CLD12, §4.1.1]. The description includes support control, fixed presentations, and all derivatives. Uniform convergence of functions alone is a different topology. No claim that this space is a strict countable LF limit is needed here.

The support convention is also substantive. In [CLD12, §4.2.1], "support propre" means compact support avoiding ∂X. If X has no analytic boundary, this reduces to ordinary compact support. On a space with boundary, a measure supported on ∂X vanishes on all these test functions and is not detected by the corresponding currents.

Definition 9.1.2 (Bidimension and bidegree)**.** A current of bidimension (p,q) is a continuous linear functional on $A^{p,q}_{\mathrm{sm},c}(X)$. Put

$$D^{\mathrm{sm}}_{p,q}(X) = \big(A^{p,q}_{\mathrm{sm},c}(X)\big)', \qquad D^{p,q}_{\mathrm{sm}}(X) = D^{\mathrm{sm}}_{n-p,n-q}(X).$$

The latter notation records bidegree. Thus a current of bidegree (p,q) acts on test forms of bidegree $(n-p,n-q)$.

For example,

$$[\omega](\alpha) = \int_X \omega \wedge \alpha, \qquad \omega \in A^{p,q}_{\mathrm{sm}}(X),$$

has bidegree (p,q). Integration over a d-dimensional algebraic cycle has bidimension (d,d) and bidegree $(n-d,n-d)$. A Radon measure acts on functions, so its bidimension is $(0,0)$ and its bidegree is (n,n). These examples prevent an ambiguous use of the phrase "a current of degree zero." See [CLD12, §4.2.2] and [Gubler, Examples 6.3, 6.5, and 6.7].

Definition 9.1.3 (Differentiation and symmetry of currents)**.** For T of bidegree (p,q), define

$$\langle d'T, \alpha\rangle = (-1)^{p+q+1}\langle T, d'\alpha\rangle,$$
$$\langle d''T, \beta\rangle = (-1)^{p+q+1}\langle T, d''\beta\rangle,$$

where α and β have the complementary test degrees. Set

$$\langle JT, \gamma\rangle = (-1)^n\langle T, J\gamma\rangle.$$

A current of bidegree (p,p) is symmetric if $JT = (-1)^pT$.

Proposition 9.1.4 (Compatibility with smooth forms)**.** *For every original smooth form ω,*

$$d'[\omega] = [d'\omega], \qquad d''[\omega] = [d''\omega], \qquad J[\omega] = [J\omega].$$

Moreover, for any current and a test form of the appropriate degree,

$$\langle d'd''T, \alpha\rangle = \langle T, d'd''\alpha\rangle. \tag{9.1.1}$$

Proof. Stokes' formula applies because test supports avoid the analytic boundary. If ω has total degree k, integrating $d'(\omega \wedge \alpha) =$

$d'\omega \wedge \alpha + (-1)^k \omega \wedge d'\alpha$ gives the first identity. The second is identical. On top-degree forms J acts by $(-1)^n$, which gives the third identity with the normalization in Definition 9.1.3. The two transpose signs in $d'd''$ multiply to -1; reversing $d''d'$ to $d'd''$ supplies the second minus sign and proves (9.1.1). □

The operations supplied by the definitions have precise domains:

1. A morphism pulls back smooth forms.
2. A proper morphism pushes currents forward by
$$\langle f_* T, \alpha \rangle = \langle T, f^* \alpha \rangle.$$
Properness ensures that the pulled-back test support is admissible.
3. A current can be multiplied by a fixed smooth form, using
$$\langle T \wedge \omega, \alpha \rangle = \langle T, \omega \wedge \alpha \rangle.$$

These formulas do not define a pullback of an arbitrary current or a product of two arbitrary currents. They are the operations in [CLD12, §4.2–§4.4] and [Gubler, §6.4–§6.6].

Definition 9.1.5 (Positive currents)**.** A symmetric current of bidegree (p, p) is weakly positive, positive, or strongly positive if it is nonnegative, respectively, on strongly positive, positive, or weakly positive test forms of bidegree $(n-p, n-p)$.

Symmetry is explicit in this definition. Testing only against symmetric forms must not be used as a substitute for specifying symmetry of the current itself. In bidegree $(1, 1)$ the three positivity conditions agree, because the complementary test degree is $(n-1, n-1)$. In middle degree, Example 7.4.3 prevents importing arguments that require strong positive tests to span all symmetric tests.

9.1.1 Positivity, order zero, and multiplication by a function

The test topology controls derivatives of all orders. Ordinary positivity supplies estimates involving coefficient values alone. We first establish those estimates, and then distinguish them from the stronger scalar-multiplier construction needed for arbitrary continuous functions.

Definition 9.1.6 (Order zero on a presentation)**.** A current T has order zero if, on each fixed finite tropical presentation and fixed compact support B, its value satisfies an estimate

$$|\langle T, \alpha\rangle| \leq C_B \|\alpha\|_{B,0}, \tag{9.1.2}$$

where the seminorm uses the suprema of coefficient values, with quotient seminorms for representatives, and does not use their derivatives. On a real coordinate domain, this is equivalent to the distributional coefficients of T being signed Radon measures.

Proposition 9.1.7 (The usable positive-current estimate)**.** *On a real coordinate domain, an ordinary positive current is of order zero. The same argument on finite tropical presentations gives order zero for ordinary positive currents on the original analytic test sheaf. It also applies to strongly positive currents. Weak positivity alone does not suffice in middle degree.*

Open proof obligation (RC2). Supply the analytic positive-form decomposition and quotient estimate used after passing from the coordinate calculation to the original test sheaf; GJR Lemma 7.12. The real coordinate calculation is retained in historical evidence.

Definition 9.1.8 (Continuous multiplication of an ordinary positive current)**.** Let T be an ordinary positive current and β an ordinary positive smooth test of complementary degree. The positive functional $\phi \mapsto \langle T, \phi\beta\rangle$ on smooth scalar functions satisfies

$$|\langle T, \phi\beta\rangle| \leq \|\phi\|_{\operatorname{supp}\beta} \langle T, \beta\rangle.$$

Uniform density of original smooth scalar functions on compact supports extends this functional to a finite Radon measure $\mu_{T,\beta}$ supported in $\operatorname{supp}\beta$. If a symmetric test is written $\alpha = \beta_1 - \beta_2$ with ordinary positive smooth tests, define

$$(uT)(\alpha) = \int u \, d\mu_{T,\beta_1} - \int u \, d\mu_{T,\beta_2}$$

for a continuous function u, and set its value on antisymmetric tests to zero.

The construction is independent of the decomposition: a linear relation among positive smooth tests gives the same relation among the measures, first on smooth scalar functions and then by uniform density. Finite-presentation

domination proves continuity in the current topology. The density statement is [CLD12, Corollaries 5.4.7–5.4.8]; the ordinary-positive decomposition is [CLD25, Proposition 6.5.2], also used in [GJR, Lemma 7.12]. The scalar-multiplier argument is [CLD25, Lemma 8.5.1]. Thus the definition applies even when u is not a function of the coordinates of one fixed tropical chart. A chartwise order-zero estimate alone would not justify that extension for an arbitrary current; [CLD25, Example 7.2.14] gives an order-zero scalar current on an infinite tropical star which is not a Radon measure.

If $u_j \to u$ uniformly near a test support, then $u_j T \to uT$ there. For $u \geq 0$, the product is positive. It need not be closed, even when T is closed; this distinction is essential in Section 13.1.

9.2 Tests, degrees and symmetry

Fix an open $\Omega \subset \mathbb{R}^n$. For a compact $K \Subset \Omega$, let $A^{p,q}_K$ consist of smooth forms supported in K, with seminorms

$$\|\eta\|_{K,r} = \max_{I,J,\ |\nu|\leq r} \sup_{x\in K} |\partial^\nu \eta_{IJ}(x)| \quad (r = 0, 1, \ldots).$$

The space $A^{p,q}_c$ has the usual test-function inductive-limit topology. A linear functional is a current if its restriction to every $A^{p,q}_K$ is continuous. Equivalently, for each K there are C_K and an integer r_K giving a bound by $C_K \|\eta\|_{K,r_K}$.

Definition 9.2.1. A current of bidimension (q, q) acts on $A^{q,q}_c$; its bidegree is $(n-q, n-q)$. It has order zero if r_K can be chosen to be zero for every K. It is symmetric if

$$T(J\eta) = (-1)^q T(\eta).$$

Put $\eta_s = (\eta + (-1)^q J\eta)/2$. A symmetric current satisfies $T(\eta) = T(\eta_s)$ and vanishes on the complementary antisymmetric part. In matrix notation symmetrization replaces H by $(H + H^t)/2$.

Definition 9.2.2. A symmetric current is weakly positive if it is nonnegative on strongly positive tests; positive if it is nonnegative on positive tests; strongly positive if it is nonnegative on weakly positive tests.

Thus the test cones are reversed: strong positivity of a current is the most restrictive condition. We use positive without an adjective for the middle

condition. Weak convergence $T_j \rightharpoonup T$ means convergence on every fixed smooth compactly supported test. It initially says nothing about continuous or varying tests.

Example 9.2.3. A locally integrable form defines $[\alpha](\eta) = \int \alpha \wedge \eta$. In particular [1] has bidegree $(0,0)$ and tests top-degree forms, while a scalar Radon measure has bidegree (n,n) and tests functions. By wedge duality a smooth positive form gives a positive current.

9.3 Order zero and measure coefficients

The scalar argument below needs no continuity assumption. Positivity itself supplies the required bound.

Lemma 9.3.1. *A linear functional $L : C_c^\infty(\Omega) \to \mathbb{R}$ nonnegative on nonnegative functions extends uniquely to a positive Radon measure.*

Proof. For tests supported in $K \Subset \Omega$, choose $0 \le \chi \in C_c^\infty$ equal to one near K. Then

$$-\|f\|_\infty \chi \le f \le \|f\|_\infty \chi, \qquad |L(f)| \le L(\chi)\,\|f\|_\infty .$$

Every continuous compactly supported function has uniformly converging smooth approximants supported in one larger compact subset: extend by zero and convolve with a smooth compactly supported approximate identity. The bound defines an extension independent of the approximants; compare any two choices on a common compact set. Nonnegative functions admit nonnegative approximants, so the extension is positive. The Riesz representation theorem on $C_c(\Omega)$ gives the unique locally finite Radon measure. □

Theorem 9.3.2 (Positive functionals have order zero)**.** *Let $T : A_c^{q,q}(\Omega) \to \mathbb{R}$ be symmetric and linear, nonnegative on ordinary positive tests. Then T is a current of order zero. There is a positive Radon measure μ_T characterized by*

$$\mu_T(\phi) = T(\phi\beta_q), \qquad \phi \in C_c^\infty(\Omega). \tag{9.3.1}$$

For a symmetric test $\eta = \alpha_H$ supported in K and χ as in Proposition 7.3.2,

$$|T(\eta)| \le \|H\|_{\infty,\mathrm{op}}\ \mu_T(\chi). \tag{9.3.2}$$

Proof. Since β_q is positive, $\phi \mapsto T(\phi\beta_q)$ is a positive scalar functional. Lemma 9.3.1 gives μ_T. Apply T to the two positive forms in (7.3.1); this proves the estimate. Replace an arbitrary test by its symmetric part and use the coefficient norm comparison in Remark 7.3.3. A C^0 bound holds on every compact set, proving both current continuity and order zero. □

The proof uses ordinary-positive domination, not strong-cone spanning. The trace measure depends on coordinates: it is a control measure, not an additional intrinsic volume form.

Theorem 9.3.3 (Matrix of coefficient measures)**.** *Put $m = \binom{n}{q}$. There are unique signed Radon measures $M_{ij} = M_{ji}$ with*

$$T(\alpha_H) = \sum_{i,j=1}^{m} \int H_{ij}\, dM_{ij}, \qquad \mu_T = \sum_i M_{ii}. \tag{9.3.3}$$

There is a measurable symmetric matrix A such that

$$dM_{ij} = A_{ij}\, d\mu_T, \qquad A \geq 0, \quad \operatorname{tr} A = 1 \quad \mu_T\text{-almost everywhere.} \tag{9.3.4}$$

Consequently

$$|T(\alpha_H)| \leq \int \|H(x)\|_{\mathrm{op}}\, d\mu_T(x). \tag{9.3.5}$$

Proof. Use the diagonal matrix units E_{ii} and put $F_{ij} = (E_{ij} + E_{ji})/2$ for $i \neq j$. Define

$$M_{ii}(\phi) = T(\alpha_{\phi E_{ii}}), \qquad M_{ij}(\phi) = T(\alpha_{\phi F_{ij}}).$$

The diagonal functionals are positive measures. The others are signed measures, since

$$F_{ij} = \tfrac{1}{4}\big((e_i + e_j)(e_i + e_j)^t - (e_i - e_j)(e_i - e_j)^t\big),$$

and each square gives a positive measure by the scalar lemma. Expanding H proves (9.3.3). The factor $1/2$ is necessary because the sum includes both ij and ji.

For any constant vector v, the measure $\sum v_i v_j M_{ij}$ is positive. Applying this to $e_i + e_j$ and $e_i - e_j$ gives

$$-(M_{ii} + M_{jj}) \leq 2M_{ij} \leq M_{ii} + M_{jj}.$$

Thus $|M_{ij}| \le (M_{ii} + M_{jj})/2$ and every coefficient measure is absolutely continuous with respect to μ_T. The Radon–Nikodym theorem gives A_{ij}. For each rational vector v, $v^t A v \ge 0$ outside a null set. Remove the union of these countably many null sets; continuity in v then yields $A \ge 0$. The trace identity follows from $\mu_T = \sum_i M_{ii}$. Finally

$$-\|H\|_{\mathrm{op}} \operatorname{Id} \le H \le \|H\|_{\mathrm{op}} \operatorname{Id} \quad \Longrightarrow \quad |\operatorname{tr}(AH)| \le \|H\|_{\mathrm{op}}.$$

Integrating proves the estimate. Testing the matrix units proves uniqueness. □

Corollary 9.3.4. *If both T and $-T$ are positive, then $T = 0$. More generally $\mu_T = 0$ implies $T = 0$.*

Proof. In the first case μ_T and its negative are positive, so $\mu_T = 0$. In either case (9.3.5) makes every test pairing zero. □

Corollary 9.3.5 (Coefficient Cauchy–Schwarz inequality)**.** *For every relatively compact Borel set $E \subset \Omega$,*

$$|M_{ij}|(E) \le \sqrt{M_{ii}(E) M_{jj}(E)}.$$

For any $a, b \ge 0$ there is also the measure inequality

$$ab|M_{ij}| \le \tfrac{1}{2}(a^2 M_{ii} + b^2 M_{jj}).$$

Proof. Positive semidefiniteness of A gives $|A_{ij}|^2 \le A_{ii} A_{jj}$, by taking the determinant of its two-by-two principal minor. Integrate $|A_{ij}| \le \sqrt{A_{ii} A_{jj}}$ over E and apply scalar Cauchy–Schwarz with respect to μ_T. The second assertion follows by integrating $ab|A_{ij}| \le (a^2 A_{ii} + b^2 A_{jj})/2$ on Borel sets. □

The weighted inequality is the real-chart form of [BGJK, Proposition 5.2.6]. Their setting also includes boundary strata, which are not treated here. Conversely, any locally finite positive measure μ and measurable positive semidefinite A with $\operatorname{tr} A = 1$ define a positive current by $T(\alpha_H) = \int \operatorname{tr}(AH)\, d\mu$. The estimate above proves continuity, and the trace measure is μ. Thus the matrix description characterizes positive currents in the present Euclidean setting.

9.4 Why weak positivity is insufficient

Weak positivity can allow derivatives of point masses. Thus it changes more than the constant in an estimate.

Theorem 9.4.1. *There is a nonzero symmetric weakly positive current on $\mathbb{R}^4$ of bidimension $(2,2)$ which is not of order zero. Its negative is also weakly positive.*

Proof. With B from Section 7.4, define on any coefficient matrix H

$$T_B(\alpha_H) = -\partial_{x_1} \operatorname{tr}(BH)(0). \tag{9.4.1}$$

This is a continuous distribution of order at most one. Since B is symmetric, it kills antisymmetric matrices. If α_H is strongly positive pointwise, then $\operatorname{tr}(BH(x)) = 0$ everywhere by Theorem 7.4.2. Hence both T_B and $-T_B$ vanish on all strong tests and are weakly positive.

Choose $\chi \in C_c^\infty(\mathbb{R}^4)$ equal one near zero, and $g \in C_c^\infty(\mathbb{R})$ with $g'(0) = 1$. Set

$$\phi_\varepsilon(x) = \varepsilon g(x_1/\varepsilon)\chi(x), \qquad H_\varepsilon = \phi_\varepsilon B.$$

All supports lie in $\operatorname{supp}\chi$ and $\|H_\varepsilon\|_\infty \to 0$. But $\partial_1\phi_\varepsilon(0) = 1$ and $\operatorname{tr}(B^2) = 6$, whence

$$T_B(\alpha_{H_\varepsilon}) = -6.$$

This contradicts any order-zero estimate on that fixed compact set. □

Also $T_B(\phi\beta_2) = 0$ since $\operatorname{tr} B = 0$. Its formal trace vanishes although the current is nonzero. The trace estimate for ordinary positive currents therefore cannot be applied to weakly positive currents. This is a distributional consequence of the nonpointed cone in [BGJK, Examples 2.3.6 and 5.2.8].

Remark 9.4.2 (The CLD25 order-zero assertion). [CLD25, Proposition 7.2.9] deduces order zero from positivity on strongly positive tests, using strong domination in its proof. On the standard real tropical space $\mathbb{R}^4$, (9.4.1) satisfies its symmetry and positivity assumptions and violates its conclusion. In bidegree $(2,2)$ the symmetric matrix condition agrees in the two conventions. Equip $\mathbb{R}^4$ with its usual affine functions and constant volume calibration in dimension four. Its standard tropical structure has level four; locally one may use the interior of a constantly calibrated four-dimensional cube, as in [CLD25, §6.6.3]. Coordinate functions separate points, hence it is affinely and

differentially separated in the sense of [CLD25, §6.2.3]. The counterexample is in this real tropical setting, not claimed on an analytic Berkovich space. Theorem 9.3.2 provides the ordinary-positive replacement on real charts.

9.5 Multiplication by continuous functions

Proposition 9.5.1. *A positive current has a unique extension to compactly supported continuous coefficient forms, continuous for uniform coefficient convergence on a common compact set. On symmetric tests it is (9.3.3); it vanishes on antisymmetric tests.*

Proof. The coefficient measures are finite on compact sets, so the integrals exist and converge under uniform convergence with fixed compact support. Uniqueness follows by uniformly approximating the finitely many coefficients by smooth functions supported in one larger compact set. Changing coordinates gives the same extension: the extensions agree on smooth forms and the fixed smooth transition matrix preserves uniform convergence on compact sets. □

Definition 9.5.2. For $f \in C^0(\Omega)$ define $(fT)(\eta) = T(f\eta)$ using this extension.

Proposition 9.5.3. *The current fT has order zero, and*

$$|(fT)(\alpha_H)| \le \int |f| \, \|H\|_{\mathrm{op}} \, d\mu_T.$$

If $f \ge 0$, it is positive and $\mu_{fT} = f\mu_T$. Multiplication is associative, commutes with restriction, and satisfies

$$\operatorname{supp}(fT) \subset \operatorname{supp} T \cap \operatorname{supp} f.$$

Proof. The inequality is (9.3.5) applied to fH. If $f, H \ge 0$, then $\operatorname{tr}(AfH) \ge 0$ almost everywhere. Testing $\phi\beta_q$ gives the trace identity. Associativity and restriction follow by multiplying each coefficient measure. On open sets where $T = 0$ its coefficient measures vanish, and on open sets where $f = 0$ the multiplied coefficients vanish. This proves the support statement. □

Remark 9.5.4. Locally bounded Borel functions also multiply the coefficient measures; the relevant equivalence is then equality μ_T-almost everywhere. Continuous multipliers are better suited to limits of varying currents because they admit uniform approximation. For an arbitrary distribution, coefficient distributions need not be measures, so this argument does not define continuous multiplication.

9.6 Weak convergence and varying multipliers

Lemma 9.6.1. *If positive currents of fixed bidimension satisfy $T_j \rightharpoonup T$, then T is positive and for every compact $K \Subset \Omega$,*

$$\sup_j \mu_{T_j}(K) < \infty.$$

Proof. Symmetry and nonnegativity on each positive test pass to the limit. Choose $0 \le \chi \in C_c^\infty$ equal one near K. Then

$$0 \le \mu_{T_j}(K) \le T_j(\chi \beta_q) \longrightarrow T(\chi \beta_q) < \infty.$$

A convergent real sequence, including its initial terms, is bounded. □

Proposition 9.6.2. *Under these hypotheses, $T_j(\eta) \to T(\eta)$ for every fixed continuous compactly supported coefficient form η.*

Proof. Approximate η uniformly by smooth η_ν supported in one compact K'. Theorem 9.3.3 and Lemma 9.6.1 bound $|T_j(\eta - \eta_\nu)|$ uniformly in j by a constant times the coefficient error. The same estimate holds for T. For fixed ν, weak convergence applies to η_ν. First choose ν to control both approximation errors, then let j tend to infinity. □

Theorem 9.6.3 (Joint continuity)**.** *If $T_j \rightharpoonup T$ are positive currents of fixed bidimension and $f_j \to f$ locally uniformly with f_j, f continuous, then*

$$f_j T_j \rightharpoonup fT.$$

Proof. For a symmetric smooth test $\eta = \alpha_H$ supported in K, write

$$T_j(f_j \eta) - T(f\eta) = T_j((f_j - f)\eta) + (T_j - T)(f\eta).$$

The first term is bounded in absolute value by

$$\|f_j - f\|_{K,\infty} \, \|H\|_{\infty,\mathrm{op}} \, \mu_{T_j}(K),$$

which tends to zero by the uniform mass bound. The second tends to zero by Proposition 9.6.2, since $f\eta$ is a fixed continuous test. Antisymmetric components vanish. □

Pointwise convergence of multipliers is insufficient. On $\mathbb{R}$, take $T_j = \delta_{1/j}$ and $f_j(x) = g(jx)$ for $g \in C_c^\infty((1/2, 3/2))$ with $g(1) = 1$. At each fixed x, $f_j(x) \to 0$, and $T_j \rightharpoonup \delta_0$, but $f_j T_j = \delta_{1/j} \rightharpoonup \delta_0$, not zero.

9.7 Differentiation and closedness

For a current S of total degree k, use the distributional convention

$$(d'S)(\eta) = (-1)^{k+1} S(d'\eta), \qquad (d''S)(\eta) = (-1)^{k+1} S(d''\eta).$$

Closed means both derivatives vanish. These signs agree with derivatives of integration currents. Indeed coefficients of a compactly supported top-degree derivative are sums of partial derivatives; Fubini and the one-variable fundamental theorem of calculus make each integral zero.

Lemma 9.7.1. *For a current of diagonal bidegree,*

$$(\mathrm{d}'\mathrm{d}''T)(\eta) = T(\mathrm{d}'\mathrm{d}''\eta). \tag{9.7.1}$$

Differentiation is continuous for weak convergence.

Proof. The total degree k is even. Applying the definitions twice gives

$$(-1)^{k+2}(-1)^{k+1} T(d''d'\eta) = T(d'd''\eta).$$

For continuity, the derivative of a fixed smooth compactly supported test is another fixed test. □

For a smooth form γ and a current S of total degree k, left multiplication is defined by

$$(\gamma \wedge S)(\eta) = (-1)^{k \deg \gamma} S(\gamma \wedge \eta).$$

This convention agrees with left multiplication of locally integrable forms and fixes the signs in the next statement.

Proposition 9.7.2. *If T is closed and f smooth, then*

$$d'(fT) = d'f \wedge T, \qquad d''(fT) = d''f \wedge T, \qquad \mathrm{d'd''}(fT) = \mathrm{d'd''}f \wedge T.$$

Proof. Insert $d'(f\eta) = d'f \wedge \eta + f\, d'\eta$ into $d'T = 0$ to obtain the first formula; exchange primes for the second. In general, the graded Leibniz rules give

$$\begin{aligned}\mathrm{d'd''}(fT) = {} & \mathrm{d'd''}f \wedge T - d''f \wedge d'T \\ & + d'f \wedge d''T + f\,\mathrm{d'd''}T.\end{aligned}$$

Closedness removes the last three terms. This argument needs smoothness of f: derivatives of a continuous multiplier cannot be inserted as ordinary smooth forms. □

Example 9.7.3 (Multiplication can destroy closedness)**.** On $\mathbb{R}$, $T = [1]$ is positive and closed. For $f(x) = e^x > 0$, the current fT is positive, but

$$d'[e^x](\phi\, d''x) = -\int e^x \phi'(x)\, dx = \int e^x \phi(x)\, dx.$$

This is positive for nonzero nonnegative $\phi \in C_c^\infty$, so fT is not closed.

Remark 9.7.4 (CLD25 Corollary 8.6.7). Definition 8.6.2 of [CLD25] requires a Bedford–Taylor current to be positive and closed and to satisfy a curvature-kernel condition. Corollary 8.6.7 asserts preservation under multiplication by an arbitrary positive continuous function. Example 9.7.3 contradicts closedness even for a smooth multiplier. The starting current [1] has the kernel property since $\mathrm{d'd''}(u[1]) = \mathrm{d'd''}[u]$ by definition. The valid replacement here is Proposition 9.5.3: positivity and order zero are preserved. Closedness needs extra assumptions. This does not invalidate the distinct operation $\mathrm{d'd''}(uT)$, whose closedness follows from the differential identities.

9.8 A first curvature product

For positive T and continuous u, $\mathrm{d'd''}(uT)$ is always a distributional current. Positivity requires more. We prove it under a stated approximation hypothesis.

Theorem 9.8.1. *Let T be positive and closed of bidimension (q,q), $q \geq 1$. Suppose u is continuous and every point has a neighbourhood U on which smooth u_j satisfy $\mathrm{d'd''}u_j \geq 0$ and converge to u locally uniformly on the same U. Then $R = \mathrm{d'd''}(uT)$ is positive and closed of bidimension $(q-1,q-1)$, and locally*

$$\mathrm{d'd''}u_j \wedge T \rightharpoonup R.$$

The limit is independent of the approximating sequence.

Proof. Theorem 9.6.3 with fixed T gives $u_j T \rightharpoonup uT$. Lemma 9.7.1 and Proposition 9.7.2 give the displayed convergence. For a positive test η of bidegree $(q-1,q-1)$, Proposition 7.5.2 makes $\mathrm{d'd''}u_j \wedge \eta$ positive. Thus $(\mathrm{d'd''}u_j \wedge T)(\eta) \geq 0$, and positivity and symmetry pass to the limit. A nonnegative smooth partition of unity on a test's compact support transfers local positivity to global positivity. The identities $(d')^2 = (d'')^2 = 0$ and $d'd'' = -d''d'$ give $d'R = d''R = 0$. The limit equals the already defined $\mathrm{d'd''}(uT)$, so it is independent of approximants. □

Proposition 9.8.2. *A continuous convex function on a convex open subset $\Omega \subset \mathbb{R}^n$ satisfies this approximation hypothesis locally.*

Proof. Choose a ball U with compact closure in Ω and a nonnegative smooth kernel ρ supported in the unit ball, of integral one. For sufficiently small ε put

$$u_\varepsilon(x) = \int u(x - \varepsilon y)\rho(y)\,dy, \qquad x \in U.$$

All functions are defined on the same U. Convolution gives smoothness. Apply convexity inside the integral to see that u_ε is convex. Uniform continuity of u on a slightly larger compact neighbourhood gives uniform convergence on $\overline{U}$. Convexity implies a positive semidefinite Hessian by the criterion in Chapter 7. □

Example 9.8.3 (The corner mass of an absolute value)**.** On $\mathbb{R}$, one has $\mathrm{d'd''}[|x|] = 2\delta_0$. For $\phi \in C_c^\infty(\mathbb{R})$, integration by parts on the two half-lines gives

$$\int_{-\infty}^{0} (-x)\phi''(x)\,dx + \int_{0}^{\infty} x\phi''(x)\,dx = 2\phi(0).$$

There is no smooth Hessian density off zero, but the full curvature is nonzero. The smooth convex $u_\varepsilon = \sqrt{x^2 + \varepsilon^2}$ converge locally uniformly to $|x|$, and their second derivatives converge weakly to this mass by Theorem 9.8.1.

Remark 9.8.4. Sequence independence for a fixed u is not curvature-class independence. If $\mathrm{d}'\mathrm{d}''[u] = \mathrm{d}'\mathrm{d}''[v]$, identifying the products still requires

$$\mathrm{d}'\mathrm{d}''((u-v)T) = 0.$$

For smooth $h = u - v$ with $\mathrm{d}'\mathrm{d}''h = 0$ as a form, Proposition 9.7.2 proves this. It does not apply to nonsmooth h. No general kernel assertion follows from Theorem 9.8.1. Symmetry of iterated products is likewise a further theorem, not a consequence of being able to repeat a definition.

9.9 Transfer through charts

The estimates are finite-dimensional in their coefficient directions. They give a transfer statement with explicit hypotheses, but relations among chart presentations must remain visible.

Proposition 9.9.1 (Transfer of order zero)**.** *Let $\mathcal{A}_c$ have a symmetry projection and local linear presentation maps P_D from smooth matrix-valued coefficients on Euclidean chart domains. Assume:*

1. *Tests are finite sums of chart-presented tests after localization.*
2. *For each fixed chart and compact coefficient support there are an actual positive test θ_D and c_D such that, for symmetric H,*

$$c_D \|H\|_\infty \theta_D \pm P_D(H) \quad \text{are actual positive tests.}$$

3. *The topology is the final locally convex topology of these presentation spaces, and presentations respect symmetrization.*

Then a symmetric linear functional S nonnegative on actual positive tests is a current, of order zero on every fixed presentation.

Proof. Applying S to the two tests gives

$$|S(P_D(H))| \le c_D S(\theta_D) \|H\|_\infty .$$

Use symmetrization for the remaining coefficient part. Each composite with a presentation is continuous even in the coefficient C^0 topology. The final topology gives continuity on $\mathcal{A}_c$, and the bound is the asserted order-zero estimate. □

The abstract transfer proposition alone does not define fS for arbitrary continuous f on an analytic space. Such an f need not come from a fixed tropical chart. Nor does separate extension in each presentation show that new continuous relations vanish. The next elementary criterion isolates this issue.

Lemma 9.9.2 (Extension across a quotient)**.** *Let E be a normed space of smooth presentations, $\widehat{E}$ its completion, $N \subset E$ the relation subspace and $\widehat{N} \subset \widehat{E}$ a proposed closed relation space. A bounded functional ℓ vanishing on N extends to $\widehat{E}/\widehat{N}$ if $\widehat{N} = \overline{N}$. In general it extends if and only if its continuous extension vanishes on all of $\widehat{N}$.*

Proof. The functional extends uniquely to $\widehat{E}$ and vanishes on $\overline{N}$ by continuity. A functional factors through a quotient precisely when it vanishes on the relation subspace. □

A chart construction must prove this relation statement, or use an intrinsic extension theorem that already contains it. Positivity provides continuity, not automatically descent through an enlarged relation space. In the algebraic CLD setting, Theorem 8.14.1 supplies a different route: measures attached directly to actual smooth tests.

9.10 Source concordance and subsequent use

The following list records the exact replacements proved in this instalment.

1. [CLD12, Lemma 5.2.3]: Theorem 7.4.2 disproves strong spanning on real charts; Proposition 7.3.2 supplies ordinary-positive domination.
2. [GK19, Remark 1.12(i)]: the weak cone is not pointed. Corollary 9.3.4 is an ordinary-positive statement, not a restoration of its weak analogue.
3. [CLD25, Proposition 7.2.9]: Theorem 9.4.1 gives a real tropical counterexample; Theorems 9.3.2 and 9.3.3 prove the ordinary-positive result.
4. [CLD25, Corollary 8.6.7]: Example 9.7.3 gives the closedness obstruction; Proposition 9.5.3 is the valid multiplier result here.

Chapter 8 constructs intrinsic scalar multiplication, and Chapter 13 treats mixed products in the stated algebraic CLD setting. The next comparison concerns generated potential classes and the curvature kernel, beyond the

admitted approximation class. No general continuous current-psh regularization theorem is claimed.

9.11 Compactness from a uniform trace bound

Order zero makes continuous multiplication possible. A uniform order-zero bound gives a second useful conclusion: compactness for weak convergence. These statements use different quantifiers. Every positive current has a finite trace on a fixed compact set, but a family of positive currents need not have a common bound there.

Theorem 9.11.1 (Local compactness of positive currents)**.** *Let $\Omega \subset \mathbb{R}^n$ be open. For each compact $K \Subset \Omega$, fix a finite number C_K. Let $\mathscr{T}$ be a family of ordinary positive currents of a fixed bidimension, and assume their trace measures in the fixed coordinates satisfy*

$$\sup_{T \in \mathscr{T}} \mu_T(K) \le C_K.$$

Every sequence in $\mathscr{T}$ has a subsequence converging weakly on smooth compactly supported tests to a positive current. If all members are d'-closed, or d''-closed, or both, so is the limit. The same conclusions hold for the closure of $\mathscr{T}$.

Proof. Choose a compact exhaustion $K_j \Subset K_{j+1}^{\circ}$ of Ω. By Theorem 9.3.3, the coefficient measures M_{ab} of every member are bounded in total variation on K_j by a constant controlled by C_{K_j}. Indeed, their densities satisfy $|A_{ab}| \le 1$ relative to μ_T. On the space of continuous functions supported in K_j, choose a countable set dense in the uniform norm. Such a set exists because K_j is compact metrizable; one can equivalently work with $C_0(K_j^{\circ})$ and its separability.

Diagonal extraction over these countable sets, all coefficient pairs and all j gives a subsequence for which every selected coefficient pairing converges. The uniform total-variation bounds extend this convergence to every compactly supported continuous coefficient. For example, if $f_\ell \to f$ uniformly with common compact support, then

$$\left| \int (f - f_\ell)\, dM_{ab,T} \right| \le C_K \|f - f_\ell\|_\infty$$

uniformly in T. Taking first the subsequence limit and then $\ell \to \infty$ gives a well-defined bounded functional on each coefficient space. The resulting current has order zero and the expected compatibility on overlapping compact sets.

For every ordinary positive test η, the limit of $T(\eta) \geq 0$ is nonnegative. Symmetry passes to the limit as well. This proves positivity. Closedness also passes to the limit: for every test ξ, the value on $d'\xi$ or $d''\xi$ is the limit of zero. No continuity of a differential on a space of measures is needed; the differential is applied to the fixed test. The estimates and the same argument remain valid for the weak closure, with the local mass bounds interpreted using compactly supported cutoffs. □

Remark 9.11.2. A weak limit need not satisfy the literal bound $\mu(K) \leq C_K$ for each originally chosen compact K: mass can move into its boundary. What is inherited without ambiguity is the bound on each fixed nonnegative continuous cutoff. For compactness it is enough to retain bounds on slightly larger compact neighbourhoods. This is why the theorem asserts relative compactness and local boundedness, rather than preserving an arbitrary list of sharp compact-set constants.

9.11.1 What changes for an analytic test space

An analytic current is not defined using one ambient Euclidean coordinate space. Its tests are presented in finitely many moment charts, modulo analytic relations. The following formulation isolates the exact functional-analytic input and avoids a countability assumption on the field.

Proposition 9.11.3. *Let E be a real test space with a locally convex final topology from presentation spaces E_a. Suppose a family $\mathscr{T} \subset E'$ satisfies, for every a, a common estimate*

$$|T(\iota_a \eta)| \leq C_a p_a(\eta), \qquad T \in \mathscr{T},$$

where p_a is a continuous seminorm and the images of the E_a span E. Then $\mathscr{T}$ is relatively compact for pointwise convergence on E, in the sense of convergent subnets. Every limit is a continuous functional on E. Positivity on a fixed test cone, vanishing on specified relations, and closedness under fixed continuous test differentials are preserved.

Proof. Each $e \in E$ is a finite sum of presented elements, so the estimates give a finite bound M_e for $|T(e)|$, uniform in T. Embed $\mathscr{T}$ in the compact product $\prod_{e\in E}[-M_e, M_e]$. Every net has a subnet converging in this product. The identities expressing linearity are closed conditions on finitely many coordinates, so the limit is linear. On E_a it still satisfies the displayed estimate, hence is continuous. The defining final topology makes the resulting linear functional continuous on E. Each positivity inequality or differential identity is another closed condition on evaluations. If tests are first defined in a presentation space and then quotiented, the equations annihilating the relation subspace are preserved in exactly the same way. □

A sequential statement requires a countable determining family with uniform approximation control, as in the Euclidean proof. Without such a family, the product compactness theorem gives subnets, not automatically subsequences. Conversely, convergence on a countable family alone need not imply convergence on all analytic tests unless the density and uniform bounds have been established.

9.12 Testing moving coefficients against a convergent current

Proposition 9.12.1. *Suppose $T_i \to T$ on all tests and the currents satisfy a common order-zero bound on each fixed compact presentation: $|T_i(v\eta)| \le C_\eta \|v\|_\infty$ for smooth scalar multipliers v, with C_η independent of i. Let u_i, u be continuous functions on a fixed compact neighbourhood of the support of a test η, with $u_i \to u$ uniformly. Assume the chartwise order-zero extension and continuous approximation for $u\eta$ used in Chapter 9. Then*

$$T_i(u_i\eta) \longrightarrow T(u\eta).$$

Proof. Choose smooth multipliers $v_k \to u$ uniformly on that compact neighbourhood, with a common cutoff where needed. The difference splits as

$$(T_i - T)(v_k\eta) + T_i((u_i - u)\eta) + (T_i - T)((u - v_k)\eta).$$

The first term tends to zero for fixed k. The second is bounded by a fixed constant times $\|u_i - u\|_\infty$. The last is bounded by a fixed constant times $\|u - v_k\|_\infty$, using the uniform order-zero estimates for T_i and T. First let i tend to its limit and then $k \to \infty$. □

Example 9.12.2 (Weak convergence is weaker than variation convergence)**.** On $\mathbb{R}$, $\delta_{1/j} \to \delta_0$ on every continuous compactly supported function, but $\|\delta_{1/j} - \delta_0\|_{\mathrm{TV}} = 2$. Moreover $\delta_j \to 0$ vaguely on $\mathbb{R}$ while every δ_j has total mass one. For a direct moving-test counterexample, take continuous functions u_j supported in small intervals around $1/j$ that avoid zero, with $0 \le u_j \le 1$, $u_j(1/j) = 1$, and interval lengths tending to zero fast enough that $u_j \to 0$ pointwise. Then $\delta_{1/j}(u_j) = 1$ while $\delta_0(0) = 0$. Thus pointwise convergence of moving multipliers does not suffice. The second shows why compactness of the underlying space, or a separate tightness argument, is needed to retain total mass.

The current compactness argument supplies limits, not uniqueness of nonlinear products. In Bedford–Taylor theory, uniqueness comes from the recursion and its estimates; compactness alone could leave several cluster currents. Likewise, positivity and closedness of a limit do not identify its cohomology class unless a class-detecting pairing is continuous and has been retained. These are separate uses of the same weak topology.

9.13 Radial mass monotonicity for closed supercurrents

Closedness imposes a dimension constraint on positive currents. A positive measure can be concentrated at any point, but a positive closed current of positive bidimension cannot place coefficient mass on a set of too small Hausdorff dimension. The real proof uses the convex radial function $|x|$, not the complex logarithmic radial weight. We begin with a regularized formula which avoids multiplying a current by a singular form at the centre.

Fix Euclidean coordinates on $\Omega \subset \mathbb{R}^n$ and put $\beta = \sum_i d'x_i \wedge d''x_i = \frac{1}{2} DD|x|^2$. For a weakly positive current T of bidimension (p,p), its scalar trace measure is

$$\mu_T = T \wedge \frac{\beta^p}{p!}.$$

It is positive because β^p is strongly positive. For weak positivity alone, this scalar measure need not dominate all coefficients of T. Ordinary positivity supplies that additional estimate by Theorem 9.3.3.

Lemma 9.13.1 (The regularized radial identity)**.** *Suppose T is smooth, closed under both differentials, and has bidimension (p,p) with $p \geq 1$. If $\overline{B}(a,r) \subset \Omega$ and $\rho_\varepsilon(x) = (|x-a|^2+\varepsilon^2)^{1/2}$, then*

$$\frac{\mu_T(B(a,r))}{(r^2+\varepsilon^2)^{p/2}} = \frac{1}{p!}\int_{B(a,r)} T \wedge (DD\rho_\varepsilon)^p.$$

Proof. One has $\beta = \rho_\varepsilon DD\rho_\varepsilon + d'\rho_\varepsilon \wedge d''\rho_\varepsilon$. Stokes and closedness give

$$\begin{aligned}\int_{B(a,r)} T \wedge \beta^p &= \int_{\partial B(a,r)} T \wedge d''(\rho_\varepsilon^2/2) \wedge \beta^{p-1} \\ &= \int_{\partial B(a,r)} \rho_\varepsilon^p T \wedge d''\rho_\varepsilon \wedge (DD\rho_\varepsilon)^{p-1} \\ &= (r^2+\varepsilon^2)^{p/2}\int_{B(a,r)} T \wedge (DD\rho_\varepsilon)^p.\end{aligned}$$

In the second line, every other term contains $d''\rho_\varepsilon$ twice and vanishes. In the third line, ρ_ε is constant on the boundary sphere and Stokes is used again. Dividing by $p!$ gives the formula. All forms used here are smooth at a because $\varepsilon > 0$. □

Theorem 9.13.2 (Real mass monotonicity)**.** *If T is weakly positive and d'- and d''-closed of bidimension (p,p), then*

$$r \longmapsto \frac{\mu_T(B(a,r))}{r^p}$$

is nondecreasing for balls compactly contained in Ω. For $p = 0$ this is simply monotonicity of the mass of increasing balls.

Proof. First suppose T smooth and $p \geq 1$. The function ρ_ε is convex, so $(DD\rho_\varepsilon)^p$ is strongly positive. The right side of Lemma 9.13.1 is the mass of a positive measure on the ball and is nondecreasing in r. For $r < s$, let $\varepsilon \downarrow 0$ in the corresponding inequality. This gives the asserted inequality between the two radius ratios. No singular product with $DD|x-a|$ is needed.

For a general current, convolve its coefficient distributions on a slightly smaller domain with a nonnegative smooth approximate identity. The currents T_δ are smooth, weakly positive and closed. Their trace measures converge weakly to μ_T. For radii r, s whose boundary spheres have zero μ_T mass, the ball masses converge, so the inequality passes to the limit. There

are at most countably many exceptional radii on each bounded interval, since the spheres are disjoint and the trace measure is locally finite. For arbitrary $r < s$, approach both radii from below by nonexceptional radii. Continuity from below of a measure on open balls finishes the proof. □

Example 9.13.3 (Closedness cannot be omitted)**.** On $\mathbb{R}$, let T be a nonzero nonnegative smooth scalar function supported in $[-1, 1]$, viewed as a current of bidimension $(1, 1)$. Its trace is $T(x)\,dx$. For $r > 1$, the mass $\mu_T((-r, r))$ is constant, so its ratio to r is strictly decreasing. The current is positive but not closed. Thus the literal statement of Proposition 3.4 in the consulted arXiv v1 of Lagerberg requires a closedness hypothesis; its Stokes proof uses exactly that hypothesis. Corollary 3.5 in that source does include closedness.

9.14 Hausdorff noncharging and support dimension

Corollary 9.14.1. *For a compact $K \Subset \Omega$, there are $R > 0$ and $C_K < \infty$ such that*

$$\mu_T(B(a, r)) \le C_K r^p \qquad (a \in K,\ 0 < r < R).$$

Proof. Choose R so that all closed balls $\overline{B}(a, R)$ with $a \in K$ lie in a fixed compact subset $K' \Subset \Omega$. Monotonicity gives $\mu_T(B(a, r)) \le (r/R)^p \mu_T(K')$. Take $C_K = R^{-p}\mu_T(K')$. □

Theorem 9.14.2. *Let T be an ordinary positive closed current of bidimension (p, p) on Ω. If a Borel set $E \subset \Omega$ has zero p-dimensional Hausdorff measure, then every coefficient measure of T has zero total variation on E. In particular, if the support of T has zero p-dimensional Hausdorff measure, then $T = 0$.*

Proof. Work first with E in a fixed compact subset K. For every $\varepsilon > 0$, the Hausdorff assumption provides a countable ball cover $E \subset \bigcup_j B(a_j, r_j)$ with centres in K, radii below the R of Corollary 9.14.1, and $\sum_j r_j^p < \varepsilon$ after adjusting an inessential dimensional constant. Therefore

$$\mu_T(E) \le \sum_j \mu_T(B(a_j, r_j)) \le C_K \sum_j r_j^p < C_K \varepsilon.$$

Letting $\varepsilon \downarrow 0$ gives zero trace mass. Ordinary positivity gives the coefficient total-variation bounds by the trace measure, so every coefficient vanishes on E. Exhaustion by compact subsets proves the general assertion. If all coefficients are supported on such an E, they are zero. □

The last step is exactly where ordinary positivity is used. The monotonicity of a scalar trace alone cannot recover coefficients invisible to that trace. Lagerberg's Remark 3.3 already records this real-versus-complex distinction. It is the same distinction behind the strong-cone correction in Chapter 7.

9.14.1 Lelong numbers and the metric normalization

If ω_p is the volume of the Euclidean unit ball in $\mathbb{R}^p$, the limit

$$\nu(T, a) = \lim_{r \downarrow 0} \frac{\mu_T(B(a, r))}{\omega_p r^p}$$

exists and is finite by monotonicity and local finiteness. For a smooth current of positive codimension, its trace has locally bounded density in n dimensions and $p < n$, so $\nu(T, a) = 0$. A coordinate p-plane with multiplicity m has $\nu = m$ in the standard coordinates.

Example 9.14.3. In $\mathbb{R}^2$, let $f(x, y) = \max(0, x + 2y)$. Its Hessian matrix current is $aa^{\mathsf{t}}\delta(\langle a, (x, y)\rangle)$ with $a = (1, 2)$. The trace is $5\delta(x + 2y)\, dx\, dy = \sqrt{5}\, ds$ on the line, where ds is Euclidean arclength. Hence $\nu(DDf, 0) = \sqrt{5}$. The tropical divisor has lattice weight one. There is no contradiction: Euclidean radial density and lattice multiplicity use different normalizations.

To obtain a support theorem on an analytic space, one must relate these real trace estimates to the measures attached to every actual analytic test. A single tropical pushforward may lose directions and even vanish when the map has insufficient rank. The analytic calibration and chart-detection arguments in CLD25 are therefore substantive extra steps. Section 19.14 uses the resulting all-test noncharging property, while the theorem above supplies its real measure-theoretic model.

References. [Lag12, Definition 3.1, Remark 3.3, Proposition 3.4 and Corollary 3.5]; compare [CLD25, Section 16.8]. Closedness and the distinction between weak and ordinary positivity have been retained explicitly.

9.15 Local equality, gluing and the full test space

A current identity is local because tests can be localized by a smooth partition of unity. This is different from checking the identity on a smaller chosen family of tests. Localization keeps the test category fixed. The distinction is particularly useful when a delta-form identity is proved in different local presentations.

Proposition 9.15.1. *Let X have a test-form sheaf admitting smooth partitions of unity, with continuous multiplication by fixed smooth functions. Suppose T_i are currents of one degree on an open cover U_i, and their restrictions agree on overlaps. Then they glue uniquely to a current T on X. Closedness is local. Positivity is local for a test cone stable under restriction and multiplication by nonnegative smooth functions.*

Proof. Choose a locally finite nonnegative smooth partition ρ_i subordinate to the cover and set

$$T(\eta) = \sum_i T_i(\rho_i \eta).$$

Only finitely many summands meet the compact support of a test. Each product is supported compactly inside U_i. If σ_j is another partition, insert $\sum_j \sigma_j = 1$ and use agreement on overlaps:

$$\sum_i T_i(\rho_i \eta) = \sum_{i,j} T_i(\rho_i \sigma_j \eta) = \sum_{i,j} T_j(\rho_i \sigma_j \eta) = \sum_j T_j(\sigma_j \eta).$$

Thus the definition is independent of the partition. The same calculation for a test supported in one U_i shows $T|_{U_i} = T_i$. On each fixed presentation space, the sum is finite and multiplication by each ρ_i is continuous after a common refinement, so the resulting functional is continuous in the final test topology. Uniqueness follows by applying any proposed global current to the partitioned test.

Multiplication by a nonnegative scalar preserves the chosen positive cone, so local positivity implies global positivity. Differentiation commutes with restriction, hence $d'T$ and $d''T$ restrict to the corresponding local derivatives. If these vanish on the cover, uniqueness of gluing makes them zero globally. The converse implications are immediate by restriction. □

The proposition applies to the original CLD test theory on its usual paracompact domains. It also applies to an enlarged test theory when that theory

is a module over the smooth scalar functions and has the stated topology. To prove an identity on all actual delta-tests, one still needs the local identity on every localized actual delta-test. A partition of unity does not turn an arbitrary delta-test into an external product or an original smooth test.

9.16 Zero extension is not automatic closed extension

An order-zero current on $X \setminus E$ may extend by zero if its coefficient measures have finite local mass near E. Positivity survives this operation when the coefficient cone is preserved, but closedness can fail because differentiation detects a boundary term. A simple real example is the scalar current $1_{(0,\infty)}$ on $\mathbb{R}\setminus\{0\}$. It is closed on that open set; its zero extension to $\mathbb{R}$ has derivative $\delta_0 d'x$.

The following sufficient criterion exhibits the estimate needed for a removable set. It is an elementary consequence of coefficient domination and a Hausdorff covering argument. Its growth hypothesis is explicit; closedness only on the complement does not supply that hypothesis at centres lying in the removed set.

Theorem 9.16.1 (An elementary removal criterion)**.** *Let E be relatively closed in $\Omega \subset \mathbb{R}^n$, and let T be an ordinary positive closed current of bidimension (p,p) on $\Omega \setminus E$, where $p \geq 2$. Assume its trace has finite mass on compact subsets up to E. Assume also that on each fixed compact neighbourhood there is a uniform estimate*

$$\mu_T(B(a,r) \setminus E) \leq Cr^p \quad (a \in E,\ 0 < r < r_0).$$

If $\mathcal{H}^{p-1}(E) = 0$, the zero extension $\widetilde{T}$ is positive and closed on Ω.

Proof. The coefficient total variations are bounded by a fixed multiple of the trace locally. Their finite-mass zero extensions therefore define an order-zero current, and integration against positive tests remains nonnegative. Fix a smooth test η with compact support, and work in a slightly larger compact neighbourhood. For arbitrary small $\varepsilon, \delta > 0$, cover the portion of E there by finitely many balls $B(a_j, r_j)$ centred on E, with $r_j < \delta$ and $\sum_j r_j^{p-1} < \varepsilon$. Compactness allows a finite subcover of a Hausdorff cover.

Choose smooth bumps χ_j, equal to one on $B(a_j, r_j)$, supported in $B(a_j, 2r_j)$, with $|\nabla \chi_j| \leq C_0/r_j$. Apply a fixed smooth truncation to $\sum_j \chi_j$

to obtain $0 \le \chi \le 1$, equal to one near the relevant part of E, supported in the union of the doubled balls, and satisfying $|\nabla\chi| \le C_1 \sum_j r_j^{-1} 1_{B(a_j,2r_j)}$. Choose radii small enough that all doubled balls remain in the fixed neighbourhood.

The test $(1-\chi)\eta$ is supported away from E, so closedness of T applies. Up to the harmless transpose sign, the pairing of $d'\widetilde{T}$ with η is bounded by

$$\begin{aligned} |\widetilde{T}(d'\eta)| &\le |\widetilde{T}(\chi d'\eta)| + |\widetilde{T}(d'\chi \wedge \eta)| \\ &\le C_\eta \sum_j \bigl(r_j^{p} + r_j^{p-1}\bigr) \le C_\eta(1+\delta)\varepsilon. \end{aligned}$$

The constants absorb the doubled radii and the coefficient estimate. Letting $\varepsilon \downarrow 0$ gives $d'\widetilde{T} = 0$. The same argument with the double-primed differential gives $d''\widetilde{T} = 0$. □

The threshold is dictated by the derivative of the cutoff: a ball mass of order r^p loses one power of r under differentiation. This explains why the noncharging theorem of Section 9.14 and the removal criterion have different Hausdorff exponents. Noncharging concerns the current itself; removable closedness concerns its first derivative. Stronger extension theorems, such as the toric Skoda–El Mir comparison, use additional geometry and should be cited with their own hypotheses.

References. The gluing construction is the standard duality argument for the CLD test sheaf, [CLD12, Section 4]. The removal estimate combines the coefficient control and ball-growth bounds developed above; it does not replace [BGJK, Theorem 7.2.4].

Chapter 10

Poincaré–Lelong and corner formulas

10.1 Cycles, logarithmic singularities, and the divisor formula

10.1.1 An integration current retains cycle multiplicities

For a closed analytic subspace Z of pure codimension r in a pure n-dimensional space, its integration current acts on a complementary test by integration over the associated cycle:

$$\langle \delta_Z, \eta \rangle = \int_{Z^{\mathrm{an}}} \eta, \qquad \eta \in A_c^{n-r,n-r}(X).$$

Here the notation includes the generic multiplicities of Z. For a nonreduced hypersurface defined by f^m, the integration current is $m\delta_{\mathrm{div}(f)}$ when the usual Cartier-divisor hypotheses hold. This is the analytic counterpart of the doubled tropical hypersurface in Chapter 5.

The current is closed on the boundaryless locus by Stokes. This does not mean every current supported on the same analytic set equals a scalar multiple of δ_Z. Support and degree constrain a current, but the coefficient structure and differential conditions still matter. The stronger support theorems in CLD25 have additional hypotheses.

10.1.2 The sign and the content of Poincaré–Lelong

For a nonzero meromorphic function under the divisor hypotheses of CLD, the formula is

$$\mathrm{d}'\mathrm{d}''[\log|f|] = \delta_{\mathrm{div}(f)}.$$

Equivalently, the logarithmic Green function $g = -\log|f|$ satisfies

$$\mathrm{d}'\mathrm{d}''[g] + \delta_{\mathrm{div}(f)} = 0.$$

There are two assertions here. First, $\log|f|$ is locally integrable in the sense needed to define a current. Second, its distributional Hessian is the cycle with the correct multiplicities. A computation away from the zero and pole set proves only that the Hessian vanishes there; it does not determine the singular term.

The proof reduces local analytic geometry to finite maps and one-dimensional slope calculations, while retaining the degree normalization. The zeros and poles contribute precisely their orders. Compatibility with tropical refinements then turns the local computation into an identity on every actual test form. This is why the divisor theorem requires more than the observation that $\log|f|$ is affine wherever f is a unit.

10.1.3 A projective-line check

On $\mathbb{P}^{1,\mathrm{an}}$ with coordinate z, let ξ_G be the Gauss point and give $\mathcal{O}(1)$ its standard model metric. On the torus put $t = -\log|z|$. The Green functions of 0 and ∞ can be written

$$g_0 = \max(0, t), \qquad g_\infty = \max(0, -t).$$

Their current identities are

$$\mathrm{d}'\mathrm{d}''[g_0] = \delta_{\xi_G} - \delta_0, \qquad \mathrm{d}'\mathrm{d}''[g_\infty] = \delta_{\xi_G} - \delta_\infty.$$

On the finite tropical line, one sees the positive corner at $t = 0$. The missing negative term lies at the appropriate analytic endpoint of the compactification. This is an effective sign check: on a compact boundaryless curve, a global distributional Hessian has total mass zero, whereas the Chern measure of $\mathcal{O}(1)$ has total mass one.

10.1.4 Weak convergence: fixed tests and varying tests

For currents, $T_j \rightharpoonup T$ means $\langle T_j, \eta\rangle \to \langle T, \eta\rangle$ for each fixed test η. It does not by itself permit replacing η by an arbitrary sequence η_j. If T_j are positive, the coefficient estimates proved above often provide the missing uniform bound for continuous scalar multipliers.

For example, if $u_j \to u$ uniformly on a compact neighbourhood of a fixed test support, then

$$\langle \mathrm{d}'\mathrm{d}''[u_j] - \mathrm{d}'\mathrm{d}''[u], \eta\rangle = \int (u_j - u)\mathrm{d}'\mathrm{d}''\eta \longrightarrow 0.$$

The estimate uses the total variation of the signed measure represented by $\mathrm{d}'\mathrm{d}''\eta$. No derivatives of u_j need converge. Conversely, weak convergence of positive measures does not imply convergence in total variation: on a real interval, $\delta_{1/j} \rightharpoonup \delta_0$, while $\|\delta_{1/j} - \delta_0\|_{\mathrm{TV}} = 2$ for every j.

10.1.5 Curvature currents and top-degree measures

On a real two-dimensional chart, consider the positive $(1,1)$-forms

$$\alpha = d'x_1 \wedge d''x_1, \qquad \beta = d'x_2 \wedge d''x_2.$$

They are different, but their squares both vanish. Thus even the self-top-degree measure need not determine a positive lower-degree current. More refined uniqueness statements, such as uniqueness of a normalized global solution of a Monge–Ampère equation in an ample class, use global geometry and a specified class of potentials. They cannot be inferred from the notation $T \mapsto T^n$ alone.

References. [CLD12, §§4.3–4.6]; [Gubler, Proposition 6.10].

10.1.6 Logarithmic clipping on bounded ordinary tests

The finite localization estimate uses the signed top-form measure of Proposition NAG-002970, the native topology of NAG-002472/002473, and the top-degree vanishing of NAG-002806. The clipping argument below is an application under the native fixed-test Poincaré–Lelong and current hypotheses of NAG-003103 and NAG-003106. Its setting is a specified compact support and a finite ordinary Cartier cover.

Test topology. Let k be a complete non-Archimedean field with its given absolute value, X a good Hausdorff pure n-dimensional k-analytic space, $n \geq 1$, and U an ordinary open in $\mathrm{Int}(X/k)$. Use real ordinary smooth moment forms, the structure-sheaf integral, and ordinary compact supports. Write $E = \mathcal{A}_c^{n-1,n-1}(U)$. For an affinoid domain $V \subset U$, a finite tuple g of units on V, an open $\Omega \supset t_g(V)$ with $t_g = -\log|g|$, and an ordinary smooth χ compactly supported in the topological interior of V in U, put

$$Q_i\alpha = \mathrm{ext}_0(\chi\, t_g^*\alpha), \qquad p_{L,m}(\alpha) = \max_{I,J,|\gamma|\leq m} \sup_{v\in L} |\partial^\gamma \alpha_{IJ}(v)| \quad (L \Subset \Omega).$$

An empty coefficient space has zero seminorms. NAG-002472 defines τ_{cut} by all seminorms P for which PQ_i is continuous for every i. The ordinary stages instead glue finitely many ambient representatives agreeing on pairwise intersections of the ordinary chart interiors and vanishing on each interior outside a fixed compact contained in their union. Each coefficient subspace has the induced finite product topology and is divided by the kernel of gluing with its quotient topology. Their final locally convex topology τ_{ord} equals τ_{cut} by NAG-002473. A set $B \subset E$ is bounded if $\sup_{\eta\in B} P(\eta) < \infty$ for every continuous seminorm P. The proof uses this definition directly on E.

Finite localization lemma. Fix a compact $S \subset U$ and a finite ordinary open cover $O_1, \dots, O_r$ of S in U. On each O_j fix analytic a_j, b_j which are non-zero-divisors at every stalk. Write $Z_j = V(a_j) \cup V(b_j)$. Choose ordinary smooth nonnegative ρ_j on U with compact support in O_j and $\sum_j \rho_j = 1$ near S (NAG-002428). Define four compact top forms on O_j, restricting all products there, by

$$T_{j0}\eta = \rho_j d'd''\eta, \qquad T_{j1}\eta = d'\rho_j \wedge d''\eta,$$
$$T_{j2}\eta = -d''\rho_j \wedge d'\eta, \quad T_{j3}\eta = d'd''\rho_j \wedge \eta.$$

Let $\mu_{jt}(\eta)$ be their respective signed native Radon measures (NAG-002970). For every bounded family $B \subset E$ whose tests are supported in S, there are $C, H < \infty$ such that for all $\eta \in B$,

$$\sum_{j,t} |\mu_{jt}(\eta)|(O_j) \leq C, \qquad |\mu_{jt}(\eta)|\big(\{\min(|a_j|,|b_j|) < e^{-H}\}\big) = 0 \quad \text{for every } j,t.$$

Hence clipping either logarithm by any Borel profile equal to the identity above $-H$, including at $-H$, leaves each weighted signed measure unchanged on the complement of Z_j; extend it by zero on Z_j. The same holds for the difference of the two weighted measures. These are assertions about variation of localized top-form measures. The original tests can meet the divisor.

Proof. Every output and every derivative used in its definition is local. In particular the support of $T_{jt}\eta$ is contained in $\operatorname{supp}\rho_j \cap \operatorname{supp}\eta$. By NAG-002806 a compact ordinary top form on O_j vanishes on an ordinary neighbourhood of the zeros of each regular factor. Thus its compact support is disjoint from Z_j. This assertion so far is per form.

Fix an incoming cutoff stage i and an index j. The compact set $D = \operatorname{supp}\chi \cap \operatorname{supp}\rho_j$ is contained in $\operatorname{Int}_{\mathrm{top}}(V,U) \cap O_j$. Cover D by finitely many ordinary interiors of affinoid neighbourhoods in that intersection on which χ, ρ_j have fixed ordinary moment representatives. Adjoin the restricted tuple g to their coordinates. Use NAG-002428 on O_j for a compactly supported partition subordinate to these interiors, summing to one near D. Apply that partition *after* differentiating. NAG-002424 and the coefficient product rule then give, for each t, a finite expansion with terms

$$((\partial^{\gamma_\lambda}\alpha_{I_\lambda J_\lambda}) \circ t_g)\,\beta_{ijt\lambda}, \qquad |\gamma_\lambda| \le 2.$$

Indeed the undifferentiated variable form is a sum of coefficients times wedges of coordinate differentials. Each differentiation of a variable coefficient inserts a coordinate differential and an ambient partial derivative. Differentiations of the fixed cutoffs only change fixed factors. All those factors, signs, coordinate wedges and the partition element belong to the fixed compact ordinary top form $\beta_{ijt\lambda}$, extended by zero in O_j. If D is empty take the empty sum. The expansion uses one fixed cover for the whole incoming coefficient space. The tangential relations are preserved by the ordinary operations on the forms.

For a smooth scalar h defined near $\operatorname{supp}\beta$, $\mu_{h\beta} = h\mu_\beta$. Indeed cut h off by a fixed ordinary function equal to one near the compact support, using NAG-002428 there (include the complementary open away from the support if needed). The two measures give the same values on every smooth scalar test, by multiplication and uniqueness in NAG-002970. Multiplication by a bounded continuous scalar on a compact carrier preserves Radon regularity, since errors on Borel differences are bounded by its supremum times the old variation error. The finite-partition definition of variation gives $|h\mu| \le \|h\|_{\infty,\operatorname{supp}\mu}|\mu|$ by integrating on each piece and summing, and gives the triangle inequality for sums. Consequently, with $L_i = t_g(V)$,

$$|\mu_{jt}(Q_i\alpha)| \le p_{L_i,2}(\alpha)\,\nu_{ijt}, \qquad \nu_{ijt} = \sum_\lambda |\mu_{\beta_{ijt\lambda}}|.$$

The carrier $C_{ijt} = \bigcup_\lambda \operatorname{supp}\beta_{ijt\lambda}$ is compact in $O_j \setminus Z_j$ by NAG-002806 separately on these finitely many fixed forms. This fixed carrier is the extra input needed for the passage to bounded families.

Take nonnegative locally bounded Borel weights w_j on $O_j \setminus Z_j$. The expression

$$P_w(\eta) = \sum_{j,t} \int_{O_j \setminus Z_j} w_j \, d|\mu_{jt}(\eta)|$$

is finite on all of E by the per-form support assertion, and subadditive and absolutely homogeneous by variation. The stage bound is

$$P_w(Q_i\alpha) \le p_{L_i,2}(\alpha) \sum_{j,t} \big(\sup_{C_{ijt}} w_j\big)\nu_{ijt}(O_j),$$

with zero for an empty carrier. The coefficient is finite for each i. Hence P_w is continuous in the original topology and is bounded on B, without a common presentation of B. The choice $w_j = 1$ proves the bound C.

On $O_j \setminus Z_j$ set $s_j = \max(0, -\log|a_j|, -\log|b_j|)$. If no common H worked, for every integer $k \ge 1$ we could choose $\eta_k \in B, j_k, t_k$ with $m_k = |\mu_{j_k t_k}(\eta_k)|(\{s_{j_k} > k\}) > 0$. Then put

$$w_j = \sum_{k:\, j_k = j} (k/m_k)\mathbf{1}_{\{s_j > k\}}.$$

These weights are locally bounded Borel functions since s_j is finite and continuous off Z_j. But $P_w(\eta_k) \ge k$, a contradiction. Together with the individual zero mass on Z_j this gives H; the upper logarithms are bounded on the fixed compact supports of the cutoffs. Clipping is therefore stationary in each measure separately. The auxiliary Borel weights occur only in integrals against the already constructed Radon variations.

Finally the graded Leibniz rule gives $d'd''(\rho_j\eta) = \sum_t T_{jt}\eta$ on O_j. After zero extension, summing in j gives $d'd''\eta$ for $\operatorname{supp}\eta \subset S$, since the partition sums to one near S. Thus all four summands in localizing a Cartier potential can be estimated before either the partition cancellation or numerator and denominator subtraction. This does not assert that local clipped potentials glue. The bound concerns the native structure-sheaf measure of these ordinary top tests only.

Exact consumer. NAG-003103, proof lines 240–254 of F8NM, applies scalar Poincaré–Lelong to $\rho_j\eta$ for each regular factor. Its left side before subtraction is $L_{c_j}(\mathrm{d}'\mathrm{d}''(\rho_j\eta))$, where $c_j = a_j$ or b_j. NAG-003106, lines 372–381, uses the positive transpose sign $(\mathrm{d}'\mathrm{d}''T)(\eta) = T(\mathrm{d}'\mathrm{d}''\eta)$ for a degree-zero

current in the native topology. We apply the reviewed cutoff lemma precisely to these operations.

Let k be complete non-Archimedean with its given absolute value, X good Hausdorff pure of dimension $n \geq 1$, and $U \subseteq \mathrm{Int}(X/k)$ ordinary open. Use the native structure-cycle integral, ordinary real smooth moment tests, and $\tau_a = \tau_{\mathrm{cut}} = \tau_{\mathrm{ord}}$ of NAG-002472/002473. Assume the native fixed-test PL inputs of NAG-003103, with their full structure and Cartier multiplicities. For an ambient regular meromorphic f, fix the finite localization of that proof near a specified compact $S \subset U$. Thus $f = a_j/b_j$ on ordinary opens O_j covering S, in the true interiors of the chosen affinoid neighborhoods as in lines 240–245. Each factor is an ambient non-zero-divisor. The fixed smooth ρ_j have compact support in O_j and sum to one near S. Let E_S be the $(n-1,n-1)$-tests on U supported in S, with the topology induced by τ_a.

The actual operation. Use the smooth profile of NAG-002480. For $M \geq 0$ set $\lambda_M(t) = -M + (t+M)\theta(t+M+1)$, where θ is smooth, between zero and one, zero on $(-\infty, 0]$ and one on $[1, \infty)$. Write $\ell_M(c) = \lambda_M(\log|c|)$ off its zero set and $-M$ on it. This is ordinary smooth, locally constant near the zeros, and equals $\log|c|$ wherever $\log|c| \geq -M$. Put $u_{j,M} = \ell_M(a_j) - \ell_M(b_j)$. Consider the two continuous linear functionals

$$A_M(\eta) = \sum_j \int_{O_j} u_{j,M}\, \mathrm{d}'\mathrm{d}''(\rho_j \eta), \qquad P_M(\eta) = \sum_j \int_{O_j} u_{j,M}\, \rho_j \mathrm{d}'\mathrm{d}''\eta.$$

Here A_M regularizes the local proof terms. The other operation is $P_M = (\mathrm{d}'\mathrm{d}''T_{v_M})|_{E_S}$, where $v_M = \sum_j \mathrm{ext}_0(\rho_j u_{j,M})$ and $T_v(\alpha) = \int_U v\alpha$. This is the derivative of a specified partition-dependent smooth potential. Raw local clips need not agree on overlaps. For every bounded $B \subset E_S$ there is H_B such that for every real $M \geq H_B$ and every $\eta \in B$,

$$A_M(\eta) = P_M(\eta) = L_f(\mathrm{d}'\mathrm{d}''\eta) = \delta_{\mathrm{div}\, f}(\eta).$$

In particular both sequences of functionals (or real-parameter families) converge on bounded sets of the indicated supported space.

Proof. Let $\mu_{jt}(\eta)$, $t = 0,1,2,3$, represent the top forms $\rho_j \mathrm{d}'\mathrm{d}''\eta$, $d'\rho_j \wedge d''\eta$, $-d''\rho_j \wedge d'\eta$, $\mathrm{d}'\mathrm{d}''\rho_j \wedge \eta$, respectively. Their sum represents $\mathrm{d}'\mathrm{d}''(\rho_j\eta)$ on O_j. Boundedness in E_S implies boundedness under its inclusion in the full test space. The independently reviewed finite cutoff lemma supplies

$C_B, H_B \geq 0$, with total variation summed over j, t at most C_B, and zero variation for each measure where either factor has absolute value $< e^{-H_B}$. The original lower-degree test can have support meeting the divisor.

Choose $L \geq 0$ so both factors on every $\operatorname{supp} \rho_j$ have absolute value at most e^L, using compactness and finiteness. For either factor separately the logarithm lies in $[-H_B, L]$ almost everywhere for every indicated variation (values on the zero set are immaterial). Thus with $D = \max(H_B, L)$, the sum of the absolute logarithmic integrals for one chosen factor per chart is at most DC_B; the absolute-log tails at levels $R > D$ are zero, uniformly. In particular for every $M \geq H_B$,

$$\sum_{j,t} \sum_{c \in (a_j, b_j)} \int_{O_j \setminus (V(a_j) \cup V(b_j))} \big|\ell_M(c) - \log|c|\big| \, d|\mu_{jt}(\eta)| = 0.$$

The inner sum counts the two factors with repetition. This proves the required uniform integrability in the varying test measures before any signed cancellation. For each native integral this is indeed its Radon realization: the fixed top test vanishes near both zero loci by NAG-002806, so the genuine logarithm on its compact support can be cut off smoothly within the common unit locus. NAG-002970 identifies its smooth-multiplier pairing with the Radon integral. No Borel extension of the native coefficient integral enters.

For fixed M, the reviewed stage estimate bounds each clipped term by the order-two input coefficient seminorm times the finite mass of a fixed stage measure times the supremum of the fixed multiplier on its compact carrier. The same works for the true logarithm since that carrier avoids the zero loci. Finality gives continuity, hence continuity on E_S by restriction. For the limit of A_M, all four true-log terms sum to $L_f(\sum_j \mathrm{d}'\mathrm{d}''(\rho_j \eta))$, by linearity and the ordinary restriction of the meromorphic pairing. The inner sum equals $\mathrm{d}'\mathrm{d}''\eta$. For P_M only term zero occurs and $\sum_j \rho_j \mathrm{d}'\mathrm{d}''\eta = \mathrm{d}'\mathrm{d}''\eta$. NAG-003103 now identifies the common value with the full Cartier functional. Unit transition factors, if different representatives of the same quotient are used, leave the true pairing unchanged. No equality of these two localized expressions at shallow clipping depth is required.

For the smooth-metric clause of the same consumer take ordinary frames, $s = f_j e_j$, $h_j = -\log \|e_j\|$. Replace both multipliers $u_{j,M}$ by $h_j - u_{j,M}$. The h_j terms are unchanged throughout the limit. Both resulting functionals eventually equal $G_s(\mathrm{d}'\mathrm{d}''\eta) = (T_{c_1} - \delta_{D_s})(\eta)$ on B, by the frame gluing and Stokes calculation in NAG-003103, lines 259–285. The sign is also fixed by the norm-one trivial frame.

10.1.7 Native fixed-test identities and the current transfer

The following native proof excerpts use the stated hypotheses on the actual regular factors, on the compact interior tests and on the ordinary smooth metric. The closed-immersion pullback uses the native cutoff topology. The current identities require the fixed-test inputs as stated. Their transitive prerequisite cone is still under review. References to the detached quotient comparison and its two formulae are printed by their permanent identifiers.

Let K be a complete non-Archimedean field with its specified absolute value, including trivial valuation. Generalized affinoids are allowed. Let X be good, Hausdorff and pure of analytic dimension $n \geq 1$, and fix an ordinary open $U \subseteq \mathrm{Int}(X/K)$. There is no ambient S_1 assumption. All test supports are ordinary compact subsets of U. Use the frozen ordinary moment sheaf, native integration and positive interleaved convention of `NAG-002424`, `002425` and `002388`. Thus the scalar logarithm has the positive sign $L_a(\mathrm{d}'\mathrm{d}''\eta) = \delta_{\mathrm{div}(a)}(\eta)$, without an extra constant.

A function is *ambient regular* when multiplication on $\mathcal{O}_{U_G}$ is injective; on actual affinoid restrictions this is regularity in their coordinate rings. A regular meromorphic function is a unit of the local total quotient sheaf in the ordinary good-space scope, represented locally by a quotient of ambient regular functions. A line bundle has ordinary local analytic frames, a smooth metric has ordinary smooth weights, and a regular meromorphic section is given data. No existence theorem for such a section is asserted.

The proof inputs are distinguished as follows.

(1) The exact component source has SHA-256

`34cea90b0be25d9bcbc565721888844e328453d64137a13c9bf79097495005af`.

F8CR-S gives reduced-component S_1, compact analytic-interior pullback and differential compatibility. F8CR-N/I give weighted native integration and absolute logarithmic integration. F8CR-C gives full Cartier comparison under its written $\mathrm{ZP}_{\mathrm{loc}}$ premise; F8CR-P gives the analytic scalar identity under $\mathrm{ZP}_{\mathrm{loc}}$ and RPL. The separate nonauthor interface-fit receipt discharges those premises on the actual pure affinoids and reduced components used there. These are reviewed applications, still separate from canonical admission.

(2) Frozen `NAG-002772/002778` retain their actual quotient and pure-zero-space scopes. Their effective length comparison and regular-quotient

construction do not require ambient S_1. The reviewed ZP input supplies empty-or-pure dimension $n-1$ on all actual common refinements, including regular cross-products.

(3) Frozen `NAG-002806` gives neighbourhood vanishing of ordinary top-degree tests near zeros of an ambient regular function. Frozen `NAG-002577` supplies restricted Stokes for genuine analytic-interior compact supports. Neither is replaced by a component boundary argument.

(4) The exact reviewed meromorphic source has SHA-256

`b211ebdb9ad232a3f66d7e80feac6be31857f66a1415c91a034b599ca615aa2b`.

Its F8MG-O/T continuity results will be used only on the actual reduced components, where their stated S_1 hypothesis holds. The new ambient extension is proved below rather than read into those bytes.

For an affinoid $V = \mathcal{M}(A)$ pure of dimension n, set

$$\operatorname{Min} A = \{\mathfrak{q}_i\}_{i=1}^{r}, \qquad V_i = \mathcal{M}(A/\mathfrak{q}_i), \quad j_i : V_i \hookrightarrow V, \qquad m_i = \operatorname{length}_{A_{\mathfrak{q}_i}} A_{\mathfrak{q}_i}.$$

All m_i are positive integers. They are not replaced by one. A general X is treated on finitely many such neighbourhoods for each compact localization, not by an assumed finite global component list.

Theorem NAG-003103 (Non-S_1 fixed-test identities; F8NM-G)**.** *Under the exact reviewed component/scalar and ZP inputs above, every ambient regular meromorphic f on U satisfies*

$$L_f(\mathrm{d}'\mathrm{d}''\eta) = \delta_{\operatorname{div}(f)}(\eta) \qquad (\eta \in \mathcal{A}_c^{n-1,n-1}(U)). \tag{10.1.1}$$

Let $\overline{\mathcal{L}}$ have ordinary smooth metric weights and a given ambient regular meromorphic section s. Its potential pairing G_s, full Cartier divisor D_s, and ordinary curvature satisfy

$$G_s(\mathrm{d}'\mathrm{d}''\eta) + \delta_{D_s}(\eta) = \int_U c_1(\overline{\mathcal{L}}) \wedge \eta. \tag{10.1.2}$$

These are fixed-test statements before any topology is invoked.

Proof. Choose a finite ordinary quotient atlas near $\operatorname{Supp}\eta$ and a compact smooth partition ρ_j subordinate to it. Each $\eta_j = \rho_j\eta$ is supported in

the true analytic interior of a chosen affinoid neighbourhood: use $O_j \subset \mathrm{Int}_{\mathrm{top}}(V_j, X) \cap \mathrm{Int}(X/K)$ and frozen NAG-001630/001704, exactly as in reviewed F8CR-P. Apply that reviewed scalar implication to a_j and b_j, then subtract. Proposition NAG-003100 identifies the full Cartier right side. Since $\eta = \sum_j \rho_j \eta$, their mixed derivatives sum to $\mathrm{d}'\mathrm{d}''\eta$. In particular all four terms

$$\mathrm{d}'\mathrm{d}''(\rho_j\eta) = \rho_j \mathrm{d}'\mathrm{d}''\eta + d'\rho_j \wedge d''\eta - d''\rho_j \wedge d'\eta + \mathrm{d}'\mathrm{d}''\rho_j \wedge \eta$$

are retained. The finite sum proves (10.1.1). The component tests are admissible by reviewed F8CR-S, and its pullback commutes with the ordered $d'd''$. This supplies exactly the use of the reduced scalar input; ambient S_1 is never inferred.

For the metric identity, choose frames e_i with $e_j = u_{ij}e_i$, put $h_i = -\log\|e_i\|$, and write $s = f_i e_i$. Then

$$h_j = h_i - \log|u_{ij}|, \qquad f_j = f_i/u_{ij}.$$

The unit logarithm is a linear moment coordinate, so its $d'd''$ is zero without using PL. Consequently $c_1|_{U_i} = \mathrm{d}'\mathrm{d}''h_i$ glues, as do $D_s|_{U_i} = \mathrm{div}(f_i)$ and

$$G_{s,i}(\alpha) = \int h_i\alpha - L_{f_i}(\alpha).$$

The displayed transition equalities cancel on every top test; finite product partitions again prove independence.

For a compact interior test in one frame, frozen restricted Stokes applies to $h_i d''\eta$ and $d'h_i \wedge \eta$. Writing $A = \int d'h_i \wedge d''\eta$, its two equations are

$$0 = A + \int h_i \mathrm{d}'\mathrm{d}''\eta, \qquad 0 = -\int \mathrm{d}'\mathrm{d}''h_i \wedge \eta - A.$$

Thus $\int h_i \mathrm{d}'\mathrm{d}''\eta = \int \mathrm{d}'\mathrm{d}''h_i \wedge \eta$, with the positive sign. Subtract (10.1.1) for f_i and use a finite partition to obtain (10.1.2). The support and all boundary hypotheses of Stokes remain in force. The norm-one trivial frame gives $G_s = -L_f$, independently checking the divisor sign. □

Write $\tau_a = \tau_{\mathrm{cut}} = \tau_{\mathrm{ord}}$ for exactly the frozen final topology of NAG-002472/002473. No intrinsic or external topology is identified here.

Proposition NAG-003106 (Closed pullback and non-S_1 current transfer; F8NM-T)**.** *For a closed immersion $j : Y \hookrightarrow X$ of good Hausdorff spaces, pullback is continuous on native ordinary compact test spaces with τ_a, including after restriction to an ordinary open $U \subseteq \mathrm{Int}(X/K)$ and its inverse image. It preserves compact interior support and commutes with d' and d''.*

Under the reviewed inputs of Theorem NAG-003103, L_f, G_s, $\delta_{\mathrm{div}(f)}$, δ_{D_s} and the smooth curvature pairing are continuous for this topology on U. The fixed-test identities are therefore native current identities

$$\mathrm{d}'\mathrm{d}''L_f = \delta_{\mathrm{div}(f)}, \qquad \mathrm{d}'\mathrm{d}''G_s + \delta_{D_s} = T_{c_1(\overline{\mathcal{L}})}.$$

This extends ambient scope without enlarging the written S_1 statements of the reviewed meromorphic source.

Proof. For a cutoff stage $Q = (V, f, \Omega, \chi)$ on X, its inverse image $W = Y \times_X V$ is the actual affinoid closed subspace of V. The pulled-back tuple is a unit tuple. Closedness of the immersion makes $j^{-1}\operatorname{Supp}\chi$ compact, and continuity gives

$$j^{-1}\operatorname{Int}_{\mathrm{top}}(V, X) \subseteq \operatorname{Int}_{\mathrm{top}}(W, Y).$$

Hence $j^*\chi$ is an admissible cutoff on Y, and

$$j^*Q\eta = Q_{(W, f|_W, \Omega, j^*\chi)}\eta.$$

The empty inverse image gives zero. This is a continuous single-stage factorization, so finality proves pullback continuity. Reviewed F8CR-S gives the analytic-interior support and differential statements. Applying this argument to the base-changed closed immersion $j^{-1}U \hookrightarrow U$ gives the restricted assertion.

First suppose U is an ordinary open in a pure affinoid V. Its finitely many reduced components restrict to closed immersions $j_i^{-1}U \to U$, each in the exact S_1 scope of F8CR-S. The reviewed F8MG-O/T are therefore applicable to their logarithmic, Cartier and metric pairings. Formulae (`NAG-003101`) and (`NAG-003102`) express the ambient logarithmic and Cartier pairings as finite sums $\sum_i m_i (j_i)_* T_i$. Each summand is continuous by the pullback factorization just proved. The same decomposition for G_s follows locally from F8CR-N for $h_i\alpha$ and (`NAG-003101`), then from a finite frame partition. Smooth scalar and curvature pairings also have their direct fixed-chart coefficient bounds from the frozen native integrator.

For a general U, test continuity on a fixed cutoff stage. Its support has a fixed finite cover by ordinary opens contained in the interiors of pure affinoid neighbourhoods and by quotient/frame opens. Choose a fixed compact partition there. Multiplying this stage by each partition member and regarding it as a test in its smaller open is continuous: cover its compact support by affinoids inside that open, adjoin the fixed cutoff moments to the original unit tuple, and write the variable coefficient as a fixed smooth coefficient times its pullback by a fixed projection. The product and chain rules give the coefficient bounds. This is precisely the finite-stage factorization used in frozen `NAG-002473`, without an S_1 assumption. On each such open the preceding finite-component argument applies. Their finite sum proves continuity on the original stage, and finality proves continuity on U. No global finite component decomposition or common presentation for all bounded sets is asserted.

For completeness, the native test differentials are continuous in this same non-S_1 category. On a cutoff image use $d'(\chi t_f^*\eta) = d'\chi \wedge t_f^*\eta + \chi t_f^* d'\eta$. The first term has the fixed finite joint-chart factorization just used; the second uses one extra ambient derivative. The identical argument applies to d''. It uses only the frozen ordinary sheaf. With the usual dual sign $(-1)^{p+q+1}$, their composition on a degree-zero current is $\mathrm{d}'\mathrm{d}''T(\eta) = T(\mathrm{d}'\mathrm{d}''\eta)$. The equalities on every test from Theorem NAG-003103 now give exactly the displayed current identities. □

10.2 The distributional calculation on an interval

A corner formula is an integration-by-parts identity. Keeping its endpoint terms visible settles the signs and explains why convexity on individual faces is insufficient. We use the normalization $\int a(t)d't \wedge d''t = \int a(t)dt$.

Theorem 10.2.1. *Let $I \subset \mathbb{R}$ be open and let u be continuous, with a locally finite subdivision into intervals on whose closures it is C^2 from each side. Then*

$$\mathrm{d}'\mathrm{d}''[u] = u''(t)\,dt + \sum_p \big(u'_+(p) - u'_-(p)\big)\delta_p.$$

The density is taken off the subdivision vertices. The current is positive if and only if the density is nonnegative on every open interval and every displayed jump is nonnegative.

Proof. A compact test support meets finitely many pieces. On one piece $[a, b]$ two integrations by parts give

$$\int_a^b u\phi'' \, dt = \int_a^b u''\phi \, dt + [u\phi' - u'\phi]_a^b.$$

At an internal vertex p, continuity of u cancels the two terms containing $\phi'(p)$. The remaining contribution is $(u'_+(p) - u'_-(p))\phi(p)$. The outer endpoints lie off the test support. This proves the formula. For positivity, tests supported inside a single interval detect its density. Shrinking nonnegative tests around one vertex detect its atomic coefficient, because the integral of the locally bounded density tends to zero. Conversely both kinds of coefficients nonnegative make the displayed Radon measure positive. □

Example 10.2.2. The function $u = -|t|$ is affine on both half-lines, so every facewise Hessian is zero. Its derivative jump is -2, and $\mathrm{d}'\mathrm{d}''[u] = -2\delta_0$. The function $v = -t^2$ has no corners and $\mathrm{d}'\mathrm{d}''[v] = -2dt$. Thus a criterion using only facewise convexity misses the first obstruction, and one using only slope sums at vertices misses the second.

10.3 Balanced graphs and endpoint conventions

Let C be a locally finite rational polyhedral curve in $N_{\mathbb{R}}$, with constant positive edge weights m_e. An edge coordinate is measured in its primitive lattice direction, not Euclidean arclength. Assume balancing at all vertices where tests may meet the curve. Tests are restrictions of ambient smooth functions; their directional derivatives at a vertex therefore come from one ambient differential.

Theorem 10.3.1. *For a continuous edgewise C^2 function u with one-sided derivatives at vertices, one has*

$$\mathrm{d}'\mathrm{d}''[u] = \sum_e m_e u''_e \, dt_e + \sum_v \left(\sum_{e \ni v} m_e \partial_e u(v) \right) \delta_v.$$

Here ∂_e is the outgoing primitive derivative. The identity is on tests supported away from any designated graph boundary.

Proof. Apply the interval calculation on each edge. At a vertex the endpoint contribution is

$$-u(v)\sum_{e\ni v} m_e\partial_e\phi(v) + \phi(v)\sum_{e\ni v} m_e\partial_e u(v).$$

The first sum is $d\phi_v(\sum_e m_e n_{e,v}) = 0$ by balancing. This leaves exactly the asserted mass. Local finiteness reduces the calculation to finitely many edges for each test. □

If balancing is absent at an endpoint and tests are allowed to reach it, the term involving the derivative of the test remains. One must then formulate a boundary-current identity or impose a boundary test convention. It is incorrect to omit this term merely because the underlying graph is compact. In the analytic theory the relevant support condition is avoidance of the analytic boundary.

Corollary 10.3.2. *On a compact connected metric graph without boundary, equipped with the usual Kirchhoff test condition, an edgewise C^2 continuous function with nonnegative Laplacian is constant.*

Proof. Pair the Laplacian with the constant function one. Its total mass is zero. Positivity implies that the measure is zero, hence $u'' = 0$ on edges and every outgoing slope sum vanishes. Edgewise integration by parts now gives

$$\sum_e m_e\int_e (u_e')^2 dt_e = -\sum_e m_e\int_e u_e u_e'' dt_e - \sum_v u(v)\sum_{e\ni v} m_e\partial_e u(v) = 0.$$

All derivatives vanish, and connectedness identifies the constants. □

10.4 An annulus as a sign check

On an annulus use $t = -\log|T|$. For a real number a in its skeletal interval and a function pulled back from that interval,

$$\mathrm{d}'\mathrm{d}''\max(0, t-a) = \delta_a, \qquad \mathrm{d}'\mathrm{d}''\min(0, t-a) = -\delta_a$$

in the interval calculation. The analytic interpretation uses the annular tropicalization and its calibration. At this stage the formulas describe the tropical measure; identifying it with an analytic current on every analytic test additionally uses the comparison theorem.

For a meromorphic function f, the analytic Poincaré–Lelong normalization in this book is

$$\mathrm{d}'\mathrm{d}''[\log|f|] = \delta_{\mathrm{div}(f)}, \qquad \mathrm{d}'\mathrm{d}''[-\log|f|] = -\delta_{\mathrm{div}(f)}.$$

Consequently, for a meromorphic section $s = fe$ and a metric weight $\varphi_e = -\log\|e\|$,

$$-\log\|s\| = \varphi_e - \log|f|, \qquad \mathrm{d}'\mathrm{d}''[-\log\|s\|] + \delta_{\mathrm{div}(s)} = c_1(L, \|\cdot\|).$$

This calculation checks the sign of the Green identity. It uses the analytic divisor theorem; the graph calculation alone is not a proof of Poincaré–Lelong in higher dimension.

10.5 Subdivision and normal lifts

The higher-dimensional corner weight has two terms because a primitive normal is a class modulo tangent directions. Let C be balanced with constant weights and let u be continuous and affine on its cells. For a facet τ, write ℓ_σ and ℓ_τ for the linear parts. Then

$$c_\tau(u) = \sum_{\sigma \supset \tau} m_\sigma \ell_\sigma(n_{\sigma,\tau}) - \ell_\tau\left(\sum_{\sigma \supset \tau} m_\sigma n_{\sigma,\tau}\right).$$

Proposition 10.5.1. *This expression is independent of all normal lifts, vanishes for an ambient affine function, and gives zero on a new facet introduced inside a cell where u was already affine.*

Proof. Balancing puts the argument of ℓ_τ in its domain. Changing a lift by $a \in N_\tau$ changes the first term by $m_\sigma \ell_\sigma(a)$ and the second by $m_\sigma \ell_\tau(a)$. Continuity gives equality of these restrictions, so the changes cancel. For an ambient affine function all linear parts are restrictions of one functional, and cancellation is immediate. At a subdivision facet inside one cell the two normal classes are opposite, the weights agree, and the linear parts are the same. Choose opposite lifts; both sums then vanish. □

This proves the elementary independence statements used in calculations. Balancing of the resulting codimension-one cycle does not follow from lift independence alone. Theorem 15.5.2 supplies the missing argument in the constant-weight PL setting, using the current identity and the Stokes characterization of balancing.

10.6 The two positive parts of a piecewise smooth Hessian

A piecewise smooth function has an interior Hessian and a corner contribution. Positivity of their sum can be separated because these two measures live in different dimensions. This is stronger than merely testing the Hessian on the interiors of maximal cells. It is the first step in the maximum principle of Gubler–Rabinoff.

Proposition 10.6.1. *Let C be a pure d-dimensional rational tropical cycle with positive constant weights, and let f be continuous and smooth on the cells of a locally finite subdivision. Then*

$$DD(f\delta_C) = [d'_P d''_P f] \wedge \delta_C + \delta_{f\cdot C}.$$

This current is positive if and only if the cellwise Hessian is positive on every maximal cell and every smooth weight of the corner cycle $f \cdot C$ is nonnegative.

Proof. The identity is the piecewise smooth Poincaré–Lelong formula: twice integrating by parts leaves the interior second derivatives and the normal slope jumps, with the tangential correction at each facet. This is the same computation as the PL formula, with the cellwise second derivative retained.

The sufficiency of the two positivity conditions follows by summing positive pairings. For necessity, positivity against tests supported in the interior of one maximal cell gives its Hessian positivity. To isolate the corner, fix a positive compactly supported test η and let Z be the union of the facets meeting its support. Choose smooth cutoffs $0 \le \chi_j \le 1$, equal to one on those facets near the support and tending to zero away from them. Then

$$0 \le DD(f\delta_C)(\chi_j\eta) = ([d'_P d''_P f] \wedge \delta_C)(\chi_j\eta) + \delta_{f\cdot C}(\eta).$$

The first term tends to zero. Indeed, it is integration against a signed measure with smooth densities on the d-cells, and its total variation gives zero mass to their lower-dimensional faces. Dominated convergence for that total variation applies. This argument does not require applying monotone convergence directly to a signed measure. Thus the corner current is positive. Tests near the relative interior of a facet detect its scalar weight, proving nonnegativity of every weight; continuity extends the conclusion to facet boundaries. □

10.7 The tropical strong maximum principle

Theorem 10.7.1. *In the setting of Proposition 10.6.1, if $DD(f\delta_C) \geq 0$ and f has a local maximum at an interior point x of the cycle, then f is constant in a neighbourhood of x in C.*

Proof. Translate x to zero and restrict to a small ball. The local cycle is the intersection of a rational fan with the ball, and we may assume $f \leq f(0)$ throughout it. We prove the stronger claim that f is constant on this entire small fan neighbourhood, and argue by induction on d. Dimension zero is immediate. In dimension one, write s_e for the outgoing derivative on a ray and $m_e > 0$ for its weight. The maximum gives $s_e \leq 0$, while positivity of the corner gives $\sum_e m_e s_e \geq 0$. Hence every $s_e = 0$. The nonnegative second derivative on each ray makes its derivative nondecreasing from zero. Thus $f \geq f(0)$ on each ray, while the maximum gives the reverse inequality. The function is constant.

Suppose $d \geq 2$ and f is not constant. By continuity and density of rational points in the rational fan, choose a rational point y with $f(y) < f(0)$, in the relative interior of a maximal cone and away from the finitely many unwanted face spans. Choose a rational linear hyperplane H through 0 and y, otherwise generic for the fan. The stable intersection $C' = C \cdot H$ is an effective cycle of pure dimension $d-1$, with support equal to $C \cap H$ near these points. This is the proper-intersection case of the tropical displacement theorem.

The restricted cellwise Hessian stays positive. The divisor-intersection identity gives

$$(f|_{C'}) \cdot C' = (f \cdot C) \cdot H.$$

The right side has nonnegative weights: the corner cycle is effective and the intersection coefficients with the linear hyperplane are nonnegative lattice indices. Proposition 10.6.1 therefore shows that $f|_{C'}$ is weakly tropically psh. The induction hypothesis, applied on the same fan neighbourhood, makes it constant there. This contradicts $f(y) < f(0)$ and proves the result. □

The positive weights and balancing have different jobs. Positive weights force each outgoing derivative to vanish in dimension one. Balancing makes the corner formula and the restriction by a tropical hyperplane compatible. A collection of convex functions on separate cells, without the corner inequalities, need not satisfy this maximum principle.

10.8 Separating an analytic point from the boundary

To apply the tropical theorem to an analytic function, it is not enough that the chosen analytic point is interior. Its tropical value could also be attained at a boundary point. Additional coordinates remove this overlap.

Lemma 10.8.1. *Let X be a strict affinoid space and let $x \in \mathrm{Int}(X)$. There are a strict affinoid neighbourhood U of x and analytic units $g_1, \dots, g_s$ on U such that the tropicalization $b = (-\log|g_i|)_i$ satisfies*

$$b(x) \notin b(\partial U).$$

Proof. The affinoid boundary lemma of CLD12 supplies finitely many analytic functions g_i on X with $|g_i| \le 1$, $|g_i(x)| < 1$, and with at least one $|g_i(y)| = 1$ at every $y \in \partial X$. If $g_i(x) = 0$, add a nonzero constant of norm less than one; this preserves all three properties and makes the value at x nonzero. This boundary lemma is the geometric input.

Choose r in the positive value group, larger than every $-\log|g_i(x)|$, and set $U = \bigcap_i \{|g_i| \ge e^{-r}\}$. It is a strict Laurent domain and a neighbourhood of x. The functions g_i are units on U, and $b(U) \subset [0, r]^s$. Points inherited from ∂X have at least one coordinate zero. Any new boundary point of the Laurent domain has at least one coordinate r, by the relative-boundary formula. Hence $b(\partial U)$ lies in the boundary of the box, while $b(x) \in (0, r)^s$. □

Theorem 10.8.2 (The analytic maximum principle in the piecewise smooth class)**.** *Let X be a good strict analytic space of pure dimension over a complete nontrivially valued field. If a piecewise smooth function h is current-psh for the original CLD tests and has a local maximum at an interior point x, then it is locally constant at x.*

Proof. Choose a strict affinoid neighbourhood on which $h = F \circ a$ for a moment tropicalization a and a piecewise smooth F, and shrink so that $h \le h(x)$. Adjoin the boundary-separating coordinates of Lemma 10.8.1; the new chart is $a' = (a, b)$, and F is pulled back from the first factor. Write $\omega = a'(x)$. Then $\omega \notin a'(\partial U)$.

By Theorem 6.10.2, the image has pure dimension $\dim X$ near ω. The analytic calibration and balancing theorem equips it there with positive tropical multiplicities. For a positive compactly supported superform η on this image, with support avoiding the tropicalized boundary, its pullback is an admissible positive CLD test. The integration formula gives

$$DD(F\delta_{a'(U)})(\eta) = DD[h]((a')^*\eta) \geq 0.$$

The tropical function has a maximum at ω, because $h \leq h(x)$ on all of U. Theorem 10.7.1 makes F constant near ω in the tropical image. Pulling back proves the assertion. □

The proof uses neither a converse from current positivity to model semipositivity nor a general smooth regularization theorem. It applies directly to the piecewise smooth class. On a compact connected boundaryless space in this setting, every such psh function is constant: its maximum set is nonempty and closed, and the local theorem makes that set open. This conclusion concerns scalar functions; a semipositive metric on a nontrivial line bundle has transition functions and is not a single global psh scalar.

References. [GRPL, Section 7, Proposition 7.4 and Theorem 7.7]; the boundary-separating input is [CLD12, Lemma 3.4.3]. The pure-dimensional analytic case is stated here so that no restriction-to-components step is implicit.

Part III

Potential theory and products

Chapter 11

Psh functions and generated classes

All continuity and local-uniform approximation in this chapter use the ordinary Berkovich topology, not the path metric on a skeleton. Example 4.2.1 shows why these topologies cannot be interchanged. A neighborhood on which an approximating family is defined is fixed before taking its limit; restricting to an analytic domain is not a substitute for proving such a neighborhood exists.

11.1 Continuous potentials, smooth approximation, and curvature

Assume for this section that X has no analytic boundary. A function is locally integrable in the current formalism if it is locally integrable against the measures represented by smooth top-degree test forms. There is no unspecified ambient Lebesgue measure on X in this definition. For a continuous real function u, the current

$$\langle [u], \alpha \rangle = \int_X u\alpha$$

is well-defined and has bidegree $(0, 0)$.

Definition 11.1.1 (The continuous psh condition in CLD 2012)**.** A continuous real function u is psh in the sense of [CLD12, Definition 5.5.1] if $d'd''[u]$ is a positive current. It is locally smoothly psh-approximable if each point has a neighborhood on which u is a uniform limit of original smooth psh functions.

The second condition is Definition 5.6.3 of the same source. It has an existential approximation requirement that does not occur in the first condition.

Proposition 11.1.2 (Stability under locally uniform limits)**.** *Let u_j be continuous current-psh functions converging locally uniformly to a finite continuous function u. Then u is current-psh. In particular, local smooth psh approximation implies the continuous psh condition of Definition 11.1.1.*

Proof. Let α be a positive smooth test form of bidegree $(n-1, n-1)$, and choose a compact neighborhood C of its support. The smooth top-degree form $d'd''\alpha$ represents a signed measure μ_α of finite total variation on C. Hence

$$\left|\langle d'd''[u_j - u], \alpha\rangle\right| = \left|\int_C (u_j - u)\, d\mu_\alpha\right| \leq \|u_j - u\|_C\, |\mu_\alpha|(C) \longrightarrow 0.$$

Each $\langle d'd''[u_j], \alpha\rangle$ is nonnegative, so its limit is nonnegative. Positivity is local and the test form was arbitrary. □

Remark 11.1.3 (What the 2012 construction does not establish). The converse to Proposition 11.1.2 is not proved in CLD 2012; [CLD12, Remark 5.6.4] explicitly records this difficulty. Consequently, a theorem whose hypotheses require local smooth psh approximation cannot be invoked merely because a continuous function has positive $d'd''$-current.

The sentence following [CLD12, Definition 5.5.1] also asserts functoriality for all continuous current-psh functions. Later work distinguishes current-psh from functorial psh and does not supply that general assertion as an automatic consequence of the current definition. It must therefore be cited with a separately verified hypothesis, rather than proved by an undefined pullback of currents. Smooth potentials and local uniform limits of smooth psh potentials are legitimate settings for the corresponding pullback argument; the smooth comparison is also covered by [GK19, Corollary 8.9].

Example 11.1.4 (A corner contributes to the current)**.** On an interval around 0, set $u(t) = -|t|$. Both restrictions to $t < 0$ and $t > 0$ are affine, hence convex. Nevertheless, for a smooth compactly supported test function χ,

$$\langle d'd''[u], \chi\rangle = \int_{\mathbb{R}} -|t|\,\chi''(t)\,dt = -2\chi(0).$$

Thus $d'd''[u] = -2\delta_0$ and u is not psh. The identical calculation applies in a tropical coordinate on an annulus. The facewise Hessians vanish; the negative contribution lies at the corner.

For a tropical curve with three outgoing primitive directions $(1,0)$, $(0,1)$, $(-1,-1)$, a continuous piecewise affine potential with outgoing slopes a, b, c has vertex mass $a+b+c$. Choosing $a+b+c<0$ gives the counterexample in [GK19, Example 3.3] to the continuous version of [CLD12, Lemma 5.5.3]. The smooth restriction of that criterion is the one retained in [GK19, Remark 3.2].

Theorem 11.1.5 (The mixed product constructed in CLD 2012)**.** *Let $u_1, \dots, u_m$ be locally smoothly psh-approximable continuous functions, where $0 \le m \le n$. There is a positive current of bidegree (m,m) denoted by*

$$d'd''u_1 \wedge \cdots \wedge d'd''u_m.$$

On an open subset where $u_{i,j}$ are smooth psh functions converging uniformly to u_i, its value on a test form α is

$$\lim_{j\to\infty} \int_X d'd''u_{1,j} \wedge \cdots \wedge d'd''u_{m,j} \wedge \alpha.$$

The limit is independent of the approximating sequences and is compatible with restriction. If $m = n$, it is a positive measure.

Open proof obligation (RC2). Prove the native mixed-current assertion under its local uniform approximation hypothesis. CLD12 Corollary 5.6.5 is supplied here; the described internal route also has open multiplier convergence.

Definition 11.1.6 (Metric weights and curvature)**.** Let L be a line bundle with a finite continuous metric. For a local nonvanishing frame e, write

$$\varphi_e = -\log\|e\|, \qquad c_1(L, \|\cdot\|)|_U = d'd''[\varphi_e].$$

The metric is current-psh if its local weights satisfy Definition 11.1.1.

Proposition 11.1.7 (Frame independence and the Poincaré–Lelong sign)**.** *The local curvature currents in Definition 11.1.6 agree on overlaps. If Z is an algebraic variety and s a nonzero meromorphic section of L, then*

$$c_1(L, \|\cdot\|) = d'd''[-\log\|s\|] + \delta_{\mathrm{div}(s)}.$$

Open proof obligation (RC2). Prove the scalar divisor/current identity with the stated sign and scope; CLD12 Theorem 4.6.5 or the cited Gubler alternative is the premise.

A local smooth approximation of each weight, a global smooth approximation of a metric on a fixed line bundle, and an approximation by relatively nef algebraic models are three different requirements. The first two are already separated in [CLD12, §6.3.1]; the third belongs to model positivity. The weak convergence of curvature currents proved in Proposition 11.1.2 does not identify these approximation categories. Likewise, adding a constant to every local weight leaves the curvature unchanged but changes the metric. Curvature comparison alone does not identify metrized line bundles.

11.2 The change of psh definition from 2012 to 2025

We work on a good Hausdorff boundaryless analytic space of pure dimension n and use the original smooth-form sheaf. An original smooth function is called psh when the smooth $(1,1)$-form $d'd''u$ is positive. The smooth-form/current comparison identifies this with positivity of $d'd''[u]$ in this regularity class; see [CLD25, Corollary 7.3.10 and Theorem 12.11.1]. The issue in this section is the passage to nonsmooth functions.

11.2.1 The classes that must remain separate

Definition 11.2.1 (Local approximation classes)**.** For an open set U, let $\mathcal{U}(U)$ be the continuous real functions u such that every $x \in U$ has an open neighbourhood $V \subset U$ on which u is the locally uniform limit of a sequence of original smooth psh functions defined on that same V. Let $\mathcal{M}(U)$ be the finite continuous functions which, locally on one fixed open neighbourhood, are decreasing limits of sequences of original smooth psh functions. Write $\mathrm{Psh}_{12}^{\mathrm{cont}}(U)$ for the continuous current-psh functions of Definition 11.1.1.

The neighbourhood in the definition is fixed for the whole approximating sequence. Neither definition is satisfied merely by finding, for every accuracy parameter, a function which is psh on a different shrinking neighbourhood of the point.

Definition 11.2.2 (The two generated sheaves in CLD25)**.** The sheaf $\mathcal{P}_{25}$ is the smallest subsheaf of functions with values in $\mathbb{R} \cup \{-\infty\}$ which contains the original smooth psh functions and is stable under decreasing limits of sequences on every open set. The sheaf $\mathcal{C}_{25}$ is the smallest subsheaf of finite continuous functions which contains the same smooth functions and is stable under uniform limits. These are [CLD25, Definitions 8.1.1 and 8.1.7].

The ordinal stages of the first construction are defined by

$$\mathcal{P}_{25,0} = \{\text{original smooth psh functions}\},$$

with $\mathcal{P}_{25,\alpha+1}$ consisting of local decreasing limits of sequences from $\mathcal{P}_{25,\alpha}$; at a limit ordinal one takes the sheaf of functions which locally belong to an earlier stage. In particular, the first stage $\mathcal{P}_{25,1}$ has an explicit local decreasing smooth approximation. The generated sheaf includes the later stages as well.

The word "smallest" refers to intersection among subsheaves satisfying the two defining properties. It is an exact definition and does not assert that every resulting section is a one-step limit of smooth functions. The transfinite description in [CLD25, Lemma 8.1.4] makes this distinction visible.

Proposition 11.2.3 (The continuous first stage)**.** *As sheaves of finite continuous functions,*

$$\mathcal{M} = \mathcal{U} = C^0 \cap \mathcal{P}_{25,1}. \tag{11.2.1}$$

Proof. If $u_j \downarrow u$ on a fixed neighbourhood and u_j, u are continuous, Dini's theorem gives uniform convergence on every compact subset. This proves $\mathcal{M} \subset \mathcal{U}$.

Conversely, fix a smaller open neighbourhood V with compact closure inside a common approximation neighbourhood. Choose smooth psh functions v_j such that

$$\|v_j - u\|_{\overline{V}} \leq 4^{-j}.$$

Put $w_j = v_j + 2 \cdot 4^{-j}$. Addition of a constant preserves smooth psh. On V,

$$u + 4^{-j} \leq w_j \leq u + 3 \cdot 4^{-j}, \qquad w_{j+1} \leq u + \tfrac{3}{4}\, 4^{-j} < w_j.$$

Hence $w_j \downarrow u$ on V. This proves the reverse inclusion. The last equality is the definition of the first generated stage, with the continuous endpoint required explicitly. □

Proposition 11.2.4 (Implications obtained without identifying the higher stages)**.** *In the setting above one has*

$$\mathcal{U} \subseteq \mathcal{C}_{25} \subseteq C^0 \cap \mathcal{P}_{25} \subseteq \mathrm{Psh}_{12}^{\mathrm{cont}}. \tag{11.2.2}$$

More generally, a locally integrable section u of $\mathcal{P}_{25}$ has positive original current $d'd''[u]$. None of the reverse inclusions for general analytic spaces is inferred here from the definitions.

Proof. The first inclusion follows by restricting to relatively compact neighbourhoods and applying the uniform-limit closure property of $\mathcal{C}_{25}$. The generated psh sheaf is a cone, is stable under addition of constants, and is local. A locally uniform limit of finite members can therefore be replaced, on a smaller relatively compact neighbourhood, by a decreasing subsequence after adding positive constants, exactly as in Proposition 11.2.3. Thus $C^0 \cap \mathcal{P}_{25}$ contains smooth psh functions and is stable under uniform limits; the minimality of $\mathcal{C}_{25}$ proves the second inclusion.

For the last assertion, use transfinite induction on the decreasing-limit construction. Suppose $u_j \downarrow u$, where u is locally integrable and each u_j belongs to an earlier stage. All generated functions are upper semicontinuous, so u_j is bounded above on compact sets. The inequalities $u \le u_j \le u_1$ imply local integrability of u_j; Proposition 11.6.7 records this inheritance at every ordinal. If α is a positive top-degree test form, monotone convergence applied after subtracting a local upper bound gives

$$\int u_j \alpha \longrightarrow \int u\alpha.$$

A general top-degree test is a difference of positive tests. Hence $[u_j] \to [u]$, and

$$d'd''[u_j] \longrightarrow d'd''[u].$$

Positivity is preserved under this convergence because each positive test has a nonnegative value at every stage. Locality treats sheafification and limit ordinals. The smooth initial stage follows from integration of positive smooth forms. This supplies the usable argument for [CLD25, Proposition 8.3.1(a)] without invoking Proposition 8.1.11 or the false weak-positive order-zero statement. □

Remark 11.2.5 (The equality printed in Proposition 8.1.11). CLD25 states the stronger equality

$$\mathcal{C}_{25} = C^0 \cap \mathcal{P}_{25}$$

and an equality at every ordinal stage. Its printed successor-stage argument starts with approximants in the continuous stage, whereas the decreasing-limit definition provides approximants in the possibly discontinuous stage. The missing replacement is discussed in Remark A.1.4. We use (11.2.2) and the proved first-stage equality (11.2.1); no argument in this book requires the stronger printed equality. This is a limitation of the available argument, not a counterexample to that equality. Sections 11.6.1–11.6.3 prove the compact-domain barrier theorem, the indexed forward inclusion and the locally integrable comparison, and isolate the additional domain requirement. A projective route, using Wu's comparison and envelope inputs, is given in Section 11.7. Theorem 11.7.5 proves that bounded first-stage decreasing stability is sufficient for the full continuous-stage equality.

Remark 11.2.6 (Translation of the word *approchable*). In [CLD12, Definition 5.6.3], "locally psh-approachable" is the positive approximation condition represented here by $\mathcal{U}$, whereas "locally approachable" denotes a difference of two such functions. In [CLD25, Definition 8.1.13], a psh function is "locally approachable" when it belongs to $\mathcal{P}_{25,1}$. Thus the latter expression is on the positive side of the theory. For finite continuous endpoints, Proposition 11.2.3 compares the positive classes; it does not identify the old difference class with the new positive class.

11.2.2 Pulling back functions and pulling back currents

Proposition 11.2.7 (Pullback for generated and approximable psh functions)**.** *Let $f : Y \to X$ be an analytic morphism in the preceding category. Pullback of functions preserves $\mathcal{P}_{25}$ and $\mathcal{U}$. In the generated class it preserves each ordinal stage.*

Proof. Pullback of an original smooth function is original smooth, and pullback of its positive $(1,1)$-form is positive. It commutes with composition and with pointwise decreasing limits. Transfinite induction, followed by locality, proves the assertion for $\mathcal{P}_{25}$, as in [CLD25, Proposition 8.1.5]. For $\mathcal{U}$, pull back the smooth approximants on the inverse image of a common neighbourhood. A compact subset of that inverse image has compact image, so local uniform convergence is preserved. □

The proof constructs $u \circ f$ and, whenever this function is locally integrable, its curvature current. Local integrability holds for continuous u; a singular psh pullback may instead be identically $-\infty$. The proof never defines a pullback of an arbitrary current. In particular, the notation

$$d'd''[u \circ f]$$

does not by itself justify an operation $f^*(d'd''[u])$ on the whole current space. The general pullback assertion following [CLD12, Definition 5.5.1] cannot be used as a proof of this operation or as a comparison with all current-psh functions; see Section A.1 and the functorial conditions in the delta-form section.

11.2.3 A complete comparison on real convex domains

Proposition 11.2.8 (Continuous distributional convexity)**.** *Let $\Omega \subset \mathbb{R}^r$ be open and convex, and let $u : \Omega \to \mathbb{R}$ be continuous. The following are equivalent:*

1. *u is convex;*
2. *$d'd''[u]$ is a positive supercurrent;*
3. *u is locally a uniform limit of smooth convex functions;*
4. *u is locally a decreasing limit of smooth convex functions.*

Proof. Convolution on a slightly larger relatively compact domain preserves convexity and gives smooth functions converging uniformly on smaller compact sets. Thus (1) implies (3). The shift construction in Proposition 11.2.3 proves (3) implies (4), and local uniform convergence of the corresponding distributions gives (3) implies (2).

If (2) holds, convolve the distributional Hessian with a nonnegative smooth kernel. The resulting matrix is positive semidefinite, so each regularization u_ε is convex by Proposition 7.2.4. Continuity gives $u_\varepsilon \to u$ uniformly on compact subsets. Passing to the limit in the convexity inequality proves local convexity. Restricting to a line segment in Ω then gives global convexity on the convex domain. Finally, (4) implies (3) by Dini's theorem. □

Remark 11.2.9 (Why continuity cannot be omitted from a pointwise statement). On $(-1,1)$ define $u(0)=1$ and $u(t)=0$ for $t\neq 0$. Then u is locally integrable and even upper semicontinuous, and $[u]=[0]$. Thus $d'd''[u]=0$, but u is not convex:

$$u(0)=1>\tfrac{1}{2}u(-a)+\tfrac{1}{2}u(a)=0 \qquad (0<a<1).$$

The implication from the locally integrable current condition to pointwise convexity in [CLD25, Proposition 8.3.2] is therefore false literally. A distribution determines an almost-everywhere class; a convex representative must be specified, or continuity must be assumed as in Proposition 11.2.8. Upper semicontinuity alone does not repair this example.

11.3 Positive distributional Hessians and convex representatives

A current defined by a locally integrable function only remembers its almost-everywhere class. To repair a pointwise convexity assertion, one must first construct the canonical representative of that class. The continuous comparison in Section 11.2 is a special case of the result below; the proof also explains precisely why changing one point can invalidate the unrepaired statement.

Let $\Omega\subset\mathbb{R}^n$ be a nonempty convex open set. For $u\in L^1_{\mathrm{loc}}(\Omega)$, say that its distributional Hessian is positive if

$$\sum_{i,j}\xi_i\xi_j\,\partial_i\partial_j u\geq 0 \quad \text{as a distribution for every } \xi\in\mathbb{R}^n.$$

This is equivalent to ordinary positivity of $DD[u]$ in the real superform convention. It is a condition on a distribution, so it is unchanged by modifications on Lebesgue-null sets.

Lemma 11.3.1 (An interior bound from an L^1 bound)**.** *Let v be a smooth convex function on $B(0,4r)$ and put $M=\int_{B(0,4r)}|v|\,dx$. With $V_r=\mathrm{vol}(B(0,r))$, one has*

$$-3M/V_r\leq v(x)\leq M/V_r \quad (x\in B(0,r)).$$

The functions with a common M bound are uniformly Lipschitz on $B(0,r/2)$, with a constant depending only on M,r,n.

Proof. For $z \in B(0,3r)$, the average of points in $B(z,r)$ is z. Jensen's inequality gives $v(z) \le M/V_r =: A$. For $x \in B(0,r)$ and $y \in B(x,r)$, the point $2y - x$ lies in $B(0,3r)$; convexity gives

$$v(y) \le \tfrac{1}{2}v(x) + \tfrac{1}{2}v(2y - x) \le \tfrac{1}{2}(v(x) + A).$$

If $v(x) \ge -A$ there is nothing to prove. Otherwise the right side is negative throughout that ball, and integration of $|v|$ yields $M \ge V_r(-v(x) - A)/2$. Thus $v(x) \ge -3A$. For $x \in B(0,r/2)$ and a unit vector e, the two points $x \pm (r/2)e$ lie in $B(0,r)$ after an arbitrarily small inward adjustment. The tangent-line inequality for a convex function bounds $|\partial_e v(x)|$ by $8A/r$. This gives a uniform gradient bound and the Lipschitz assertion. □

Theorem 11.3.2 (The convex representative theorem)**.** *A function $u \in L^1_{\mathrm{loc}}(\Omega)$ has positive distributional Hessian if and only if its almost-everywhere class contains a finite convex function v. Such a representative is unique. Every standard compactly supported nonnegative smooth approximate identity ρ_ε gives*

$$u * \rho_\varepsilon \longrightarrow v \quad \textit{locally uniformly on } \Omega.$$

Proof. Assume first that the Hessian is positive. On the inner domain where convolution is defined, $u_\varepsilon = u * \rho_\varepsilon$ is smooth and its Hessian is positive semidefinite: for each vector ξ, its quadratic Hessian coefficient is the convolution of the nonnegative distribution $\partial_\xi^2 u$ with a nonnegative kernel. Thus u_ε is convex on every convex inner subdomain.

The convolutions converge to u in L^1 on every relatively compact open subset. Their L^1 norms are therefore uniformly bounded there for sufficiently small ε. Cover a given compact subset by finitely many balls whose fourfold enlargements remain in Ω. Lemma 11.3.1 gives uniform bounds and equicontinuity on the compact subset. Arzelà–Ascoli and a diagonal argument over an exhaustion give a subsequence converging locally uniformly to a finite continuous function v. The convex inequality passes to the limit on each compact line segment, so v is convex. Its L^1 limit is also u, hence $v = u$ almost everywhere.

Two finite convex representatives are continuous. If they differed at one point, they would differ on a nonempty open set, which has positive Lebesgue measure. Hence they coincide. This uniqueness and the same local compactness argument show convergence of the whole family u_ε, not just the selected

subsequence: any sequence violating uniform convergence on a compact set would have a locally uniformly convergent subsequence with the same unique limit.

Conversely, if $u = v$ almost everywhere for a finite convex v, its convolutions are smooth convex functions and converge locally uniformly to v. Their positive Hessian distributions converge to the Hessian distribution of u. Positivity is closed under convergence on tests, proving the converse. □

11.3.1 Which pointwise representative is admissible

Corollary 11.3.3. *The representative in Theorem 11.3.2 is characterized by*

$$v(x) = \lim_{r\downarrow 0} \frac{1}{\operatorname{vol} B(x,r)} \int_{B(x,r)} u(y)\, dy = \lim_{r\downarrow 0} \operatorname*{ess\,sup}_{B(x,r)} u.$$

If the chosen pointwise function u is strongly upper semicontinuous, in the sense that $u(x) = \limsup_{y\to x,\, y\notin E} u(y)$ for every Lebesgue-null set E, then $u = v$ everywhere. Ordinary upper semicontinuity alone does not suffice.

Proof. The integral and essential supremum are unchanged by replacing u with v. Continuity of v makes both limits equal to $v(x)$. For the last assertion take $E = \{u \neq v\}$, which is null. Its complement is dense in every open ball. Strong upper semicontinuity and continuity of v therefore give $u(x) = v(x)$. To see the failure for ordinary upper semicontinuity, put $u(0) = 1$ and $u = 0$ elsewhere on a ball containing zero. This function is upper semicontinuous and defines the zero distribution, but is not convex. □

This gives two precise replacements for the pointwise claim in CLD25 Proposition 8.3.2: either work with a continuous function from the outset, or work with an L^1 class and its unique convex representative. Strong upper semicontinuity selects that representative. The approximation converges uniformly to the representative, not to an arbitrarily modified pointwise version.

11.4 The Hessian kernel and potential independence on a real domain

Proposition 11.4.1. *If a distribution w on a connected open subset of $\mathbb{R}^n$ satisfies $\partial_i \partial_j w = 0$ for all i, j, then w is the distribution of an affine function. Consequently two finite convex functions on a convex open set with the same Hessian current differ by an affine function.*

Proof. On any ball compactly contained in the domain, small convolutions w_ε have zero Hessian, hence are affine on a smaller concentric ball. The finite-dimensional space of affine distributions on that smaller ball is closed in the distribution topology. To see this directly, choose $n+1$ test functions whose integrals against $1, x_1, \dots, x_n$ form an invertible matrix; convergence on these tests forces convergence of all affine coefficients. Since $w_\varepsilon \to w$ as distributions, w is affine on the smaller ball. On overlapping balls the affine polynomials agree as distributions and hence as functions on an open set, so their coefficients agree. Connectedness glues them to one affine function. Apply this to the difference of the two functions and then use their continuity. □

Corollary 11.4.2. *Let S be an ordinary positive current closed under both differentials on a convex real domain. If finite convex functions u, v have $DD[u] = DD[v]$, then*

$$DD(uS) = DD(vS).$$

Proof. Both functions are continuous, so multiplication is defined by the order-zero theorem. Their difference $a = u - v$ is affine by Proposition 11.4.1. The smooth Leibniz formula and closedness give $DD(aS) = (DDa) \wedge S = 0$. Subtracting the two products proves the result. □

On a singular tropical support, the kernel of the current Hessian can be larger than the affine functions inherited from the ambient space. The cross example in Chapter 14 exploits exactly that enlargement. Thus the Euclidean argument proves potential independence on its own domain and simultaneously identifies the step that cannot simply be carried over to an arbitrary quotient of charts.

References. This section supplies the representative-sensitive repair of [CLD25, Proposition 8.3.2]. The distinction between strong upper semicontinuity and a mere distributional class is also used in [BGJK25, Section 3.1].

11.5 A convex potential for a positive closed real current

On a convex real domain, every positive closed $(1,1)$ supercurrent has a convex potential. The assertion concerns an ambient Euclidean domain. It is a useful comparison theorem for the local analytic problem, but its proof uses linear coordinates and cannot be transferred to a singular tropical quotient without checking those operations. We include the distributional step so that regularizing the current does not conceal a missing convergence argument for its potentials.

Lemma 11.5.1 (Positive Laplacian gives local integrability)**.** *Let u be a distribution on an open subset of $\mathbb{R}^n$. If Δu is a positive distribution, then u is represented by a locally integrable function.*

Proof. A positive distribution is a positive Radon measure; write $\mu = \Delta u$. Choose a compactly supported smooth cutoff χ equal to one on a smaller open set. A fundamental solution E_n for Δ is

$$E_1(x) = \tfrac{1}{2}|x|, \qquad E_2(x) = \tfrac{1}{2\pi}\log|x|,$$
$$E_n(x) = -\frac{|x|^{2-n}}{(n-2)\ \mathrm{area}(S^{n-1})} \quad (n \geq 3).$$

Each kernel is locally integrable. The convolution $v = E_n * (\chi\mu)$ is locally integrable by Fubini: on a compact set, the integral of $|E_n(x-y)|$ in x is uniformly bounded for y in the compact support of $\chi\mu$. In distributions, $\Delta v = \chi\mu$. Thus $u - v$ is a harmonic distribution on the smaller open set.

For completeness, a harmonic distribution is smooth. Its small convolutions are smooth harmonic functions. Fix a smooth radial averaging kernel supported in a ball small enough for the harmonic mean-value identity. Every such convolution equals its convolution with this fixed kernel on a smaller ball. Passing to distributions shows that the original distribution equals its convolution with the fixed kernel there, which is smooth. This is the distributional Weyl lemma. Consequently $u = v + (u - v)$ is locally integrable. □

Corollary 11.5.2. *A distribution whose Hessian is positive semidefinite in the distributional sense is represented by a unique finite convex function on a convex domain.*

Proof. Its Laplacian is a sum of positive distributions, hence positive. Lemma 11.5.1 gives a locally integrable representative, and Theorem 11.3.2 gives the unique finite convex representative. □

Theorem 11.5.3 (The real DD potential theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be convex and open. A symmetric ordinary positive $(1,1)$ supercurrent T with $d'T = d''T = 0$ has a finite convex potential u satisfying $DD[u] = T$. The potential is unique up to addition of an affine function.*

Proof. Write $T = \sum_{i,j} T_{ij}\, d'x_i \wedge d''x_j$. Symmetry gives $T_{ij} = T_{ji}$, and closedness gives

$$\partial_k T_{ij} = \partial_i T_{kj}.$$

The trace $\mu = \sum_i T_{ii}$ is a positive Radon measure. Commuting distributional derivatives and using symmetry gives

$$\Delta T_{ij} = \sum_k \partial_k \partial_i T_{kj} = \partial_i \partial_j \sum_k T_{kk} = \partial_i \partial_j \mu.$$

First work on a relatively compact convex subdomain $U \Subset \Omega$. Choose $\chi = 1$ near its closure and form $v = E_n * (\chi\mu)$. The preceding lemma's argument makes v locally integrable. On U, the distributions

$$S_{ij} = T_{ij} - \partial_i \partial_j v$$

are harmonic, hence smooth. They remain symmetric and satisfy $\partial_k S_{ij} = \partial_i S_{kj}$. Translate an interior base point of U to zero and define

$$a_j(x) = \int_0^1 \sum_i x_i S_{ij}(tx)\, dt.$$

Differentiating and using the closed-column identity gives

$$\partial_k a_j(x) = \int_0^1 \left(S_{kj}(tx) + t \sum_i x_i \partial_i S_{kj}(tx) \right) dt = S_{kj}(x).$$

Symmetry now says that the smooth one-form $\sum_j a_j dx_j$ is closed. A second radial integration,

$$b(x) = \int_0^1 \sum_j x_j a_j(tx)\, dt,$$

gives $\partial_j b = a_j$ and $D^2 b = S$. Hence the locally integrable function $v + b$ has Hessian T on U. Positivity and Theorem 11.3.2 replace it by a finite convex representative u_U.

Exhaust Ω by nested relatively compact convex open subdomains U_j. Such an exhaustion is obtained by intersecting large balls with inner parallel convex domains. On U_j, the difference $u_{U_{j+1}} - u_{U_j}$ has zero Hessian, hence is affine by Proposition 11.4.1. Subtract this affine function from $u_{U_{j+1}}$ and proceed inductively. Affine subtraction preserves convexity, and the adjusted potentials agree on overlaps. They glue to the required finite convex function on Ω. The same Hessian-kernel proposition gives uniqueness modulo affine functions. □

This proof makes two uses of the ambient real structure explicit. Coordinate derivatives commute as distributions, allowing the Laplacian calculation; and straight segments remain in the convex domain, allowing the two radial integrations. The equation $DD[u] = T$ on a tropical support is an equality modulo tangential relations. It need not furnish an ambient matrix T_{ij} satisfying these unrestricted coordinate identities. This is the precise obstruction to treating the real potential theorem as a local-potential theorem on every Berkovich chart.

Corollary 11.5.4. *For a positive closed* $(1,1)$ *current on all of* $\mathbb{R}^n$*, every connected component of the complement of its support is convex.*

Proof. Let u be its global convex potential and let V be such a component. The potential agrees there with one affine function a. At any point of V, that affine function is a supporting plane for u, so $u \geq a$ globally. The zero set of the convex nonnegative function $u - a$ is convex. It contains V, hence its convex hull. Since V is open, its convex hull is open as well; u is affine on that open hull, so the current vanishes there. Connectedness and maximality of the component then imply $\mathrm{conv}(V) = V$. □

References. The potential theorem and the support-complement consequence occur in [Lag12, Propositions 1.13–1.14]. The proof here supplies the distributional

integrability and gluing steps explicitly, using the representative theorem of Section 11.3.

11.6 Which approximation statement is available

The first-stage theorem and the generated-sheaf comparison serve different purposes. The former constructs a single smooth sequence on a fixed neighbourhood and is the input for Theorem 13.6.3. The latter allows repeated local limiting operations. A section obtained after several such operations must not be used as a first-stage section without an additional argument.

Lemma 11.6.1 (The upper-semicontinuous form of Dini's argument)**.** *Let K be compact, let u_j be finite upper semicontinuous functions decreasing to a continuous finite function u. Then $u_j \to u$ uniformly on K.*

Proof. For $\epsilon > 0$, the sets $O_j = \{u_j - u < \epsilon\}$ are open, increase with j, and cover K. Compactness gives $K = O_N$ for some N. Thus $0 \le u_j - u < \epsilon$ on K for all $j \ge N$. □

This observation locates the difficulty in CLD25 Proposition 8.1.11 more precisely. The obstacle is not the uniformity of a decreasing limit with continuous endpoint: upper semicontinuity already suffices for that. It is the replacement of the possibly discontinuous predecessors in $\mathcal{P}_{25,\alpha}$ by members of $\mathcal{C}_{25,\alpha}$. Uniform convergence of discontinuous predecessors does not make those predecessors continuous. Proposition 11.2.3 avoids this issue because its predecessors are smooth.

Assertion	Usable justification
$\mathcal{U} = \mathcal{M} = C^0 \cap \mathcal{P}_{25,1}$	Proposition 11.2.3, with fixed local approximation domain.
$\mathcal{U} \subseteq \mathcal{C}_{25} \subseteq C^0 \cap \mathcal{P}_{25}$	Proposition 11.2.4; no reverse inclusion is used.
$\mathcal{C}_{25} = C^0 \cap \mathcal{P}_{25}$	Stronger source assertion; the predecessor replacement remains unproved here.
$R_X(u_1, \dots, u_k; T)$	Theorem 13.6.3 for $u_i \in \mathcal{U}_X$ and the stated ordinary positive closed T.

The kernel question is independent of this existence statement. The cross calculation in Proposition 14.2.2 refutes its unrestricted abstract version. A

positive result on an analytic space must specify the harmonic kernel, the test quotient and the permitted background currents.

References. CLD25 Definitions 8.1.1, 8.1.7 and Proposition 8.1.11; the compactness argument above is elementary.

11.6.1 A continuous upper barrier on a fixed compact set

To understand the missing step in Proposition 8.1.11, we first remove one difficulty from the problem. Repeated decreasing limits on a fixed compact set can be handled without making every predecessor continuous. A continuous upper barrier allows us to select one predecessor at a time. The distinction between this construction and a sheaf construction will then be explicit.

Definition 11.6.2 (The decreasing hierarchy on a fixed compact set)**.** Let K be a compact Hausdorff space and let $\mathscr{S} \subset C(K, \mathbf{R})$ be nonempty. Define classes of actual functions $K \to \mathbf{R} \cup \{-\infty\}$ by

$$\begin{aligned} D_0(\mathscr{S}) &= \mathscr{S}, \\ D_{\alpha+1}(\mathscr{S}) &= \{f : \text{there are } f_j \in D_\alpha(\mathscr{S}) \text{ with } f_j \downarrow f \text{ on } K\}, \\ D_\lambda(\mathscr{S}) &= \bigcup_{\beta<\lambda} D_\beta(\mathscr{S}) \quad (\lambda \text{ a limit ordinal}). \end{aligned}$$

Write $D(\mathscr{S})$ for the class obtained after stabilization. All functions at all stages have domain K. There is no sheafification or passage to a smaller domain in this definition.

The classes increase because constant sequences are allowed. They stabilize since they are subsets of the fixed set of all functions on K. Every member is upper semicontinuous: this is preserved by a decreasing limit and is unchanged by taking a union of classes. These observations alone do not give first-stage stabilization for discontinuous functions. Corollary 11.24.6 below proves it by a separate compact diagonal argument.

Lemma 11.6.3 (Selection below a continuous barrier)**.** *If $f \in D(\mathscr{S})$ and $b \in C(K, \mathbf{R})$ satisfy $f < b$ everywhere on K, then there is $s \in \mathscr{S}$ such that*

$$f \le s < b \quad \text{on } K. \tag{11.6.1}$$

Proof. We prove the assertion by induction on the stage containing f. At stage zero, take $s = f$. At a successor stage, write $f_j \downarrow f$ with all f_j in the preceding stage. The sets

$$O_j = \{z \in K : f_j(z) < b(z)\}$$

are open, increase with j, and cover K. Openness uses upper semicontinuity of $f_j - b$, and the covering property uses $f < b$. By compactness, $O_N = K$ for some N. Apply the induction hypothesis to f_N and the same barrier b. It gives $f \le f_N \le s < b$. At a limit ordinal, f already belongs to one earlier stage by definition, so the induction hypothesis applies directly. □

Only one predecessor is retained at each successor step. The selected predecessor may be discontinuous; its upper semicontinuity is enough for the next selection. All selections use the same compact set and the same barrier. This is the point that a direct appeal to continuity of the predecessors would miss.

Theorem 11.6.4 (Continuous endpoints of the compact hierarchy)**.** *Suppose $\mathscr{S}$ is stable under addition of real constants. Then*

$$C(K, \mathbf{R}) \cap D(\mathscr{S}) = \overline{\mathscr{S}}^{\|\cdot\|_\infty} = C(K, \mathbf{R}) \cap D_1(\mathscr{S}). \tag{11.6.2}$$

Consequently, the continuous parts of $D_\alpha(\mathscr{S})$ agree for every $\alpha \ge 1$.

Proof. Let u be a finite continuous member of $D(\mathscr{S})$. Apply Lemma 11.6.3 with $b = u + \epsilon$. For every $\epsilon > 0$ it gives $s_\epsilon \in \mathscr{S}$ with $0 \le s_\epsilon - u < \epsilon$ on all of K. Thus u is in the uniform closure of $\mathscr{S}$.

Conversely, let u be in that uniform closure. Choose $s_j \in \mathscr{S}$ with $\|s_j - u\|_\infty \le \delta_j$, where $\delta_j = 4^{-j}$, and set $t_j = s_j + 2\delta_j$. Then $t_j \in \mathscr{S}$ and

$$u + \delta_j \le t_j \le u + 3\delta_j, \qquad t_{j+1} \le u + \tfrac{3}{4}\delta_j < t_j.$$

Hence $t_j \downarrow u$ uniformly on K, so $u \in D_1(\mathscr{S})$. The remaining inclusion follows from $D_1 \subset D$. The assertion for every positive ordinal follows by inserting $D_1 \subset D_\alpha \subset D$ into (11.6.2). □

This is an elementary statement about monotone closure on a fixed compact domain. It does not use positivity, tropical charts or a regularization theorem. Its application to the generated psh sheaf requires an additional representation property, stated next.

11.6.2 The remaining domain requirement

Corollary 11.6.5 (A compact representation gives first-stage approximation)**.** *Let K be a compact neighbourhood of x in a Hausdorff locally compact analytic space X. Let $\mathscr{S}_K$ consist of the restrictions to K of original smooth psh functions defined on open neighbourhoods of K. These neighbourhoods may depend on the function. Let u be finite and continuous near K. If*

$$u|_K \in D(\mathscr{S}_K), \tag{11.6.3}$$

then u has a decreasing original smooth psh approximation on the single open neighbourhood $\mathrm{Int}_X(K)$ of x, uniformly on compact subsets there. In particular, u is locally in $\mathcal{U}_X$ near x.

Proof. Theorem 11.6.4 gives a decreasing sequence $t_j \in \mathscr{S}_K$ converging uniformly to $u|_K$. Choose smooth psh representatives near K. Every representative is defined on $\mathrm{Int}_X(K)$, even though their larger neighbourhoods may differ. Their restrictions to this fixed interior therefore give the required sequence. The inequalities and the uniform bounds on K remain valid there. □

Condition (11.6.3) is the substantive hypothesis. Membership in $\mathcal{P}_{25,\alpha}$ allows local representations at every successor stage and sheafification at limit stages. Restricting one section to K does not by itself give a representation in $D_\alpha(\mathscr{S}_K)$: the lower-stage sections may only have representations on a cover of K, and their smooth predecessors need not extend across that cover. The compact hierarchy has no such local operation.

In particular, approximation on a neighbourhood depending on the error is insufficient. On the real tropical line, the function $u(t) = -t^2$ is not psh on any interval, since $u'' = -2$. Nevertheless, for every $\epsilon > 0$, the smooth psh constant zero satisfies

$$|0 - u(t)| < \epsilon \quad \text{for } |t| < \sqrt{\epsilon}.$$

Indeed, every continuous function admits such shrinking-domain constant approximations at a point. Thus this condition carries no psh information. In Corollary 11.6.5, the compact neighbourhood is fixed before the error tends to zero.

The conclusion of the corollary is stronger than the unindexed equality asserted in CLD25 Proposition 8.1.11: it gives first-stage approximation. Consequently, constructing compact representations for every continuous generated psh function would settle more than that equality. Failure to construct them does not refute the weaker equality. For that weaker target, continuous replacements already in $\mathcal{C}_{25}$ on one fixed domain would suffice; their ordinal need not match the ordinal of the original predecessor. The same-ordinal assertion in the source imposes this further constraint.

11.6.3 What survives at every ordinal

We record the valid direction of the continuous comparison with its ordinal intact. Define $\mathcal{C}_{25,0} = \mathcal{P}_{25,0}$, let $\mathcal{C}_{25,\alpha+1}$ be the sheaf of local uniform limits of sections of $\mathcal{C}_{25,\alpha}$, and at a limit ordinal take the sheaf of functions locally in an earlier continuous stage. All its sections are finite and continuous.

Proposition 11.6.6 (The indexed inclusion)**.** *For every ordinal α,*

$$\mathcal{C}_{25,\alpha} \subset C^0 \cap \mathcal{P}_{25,\alpha}.$$

Equality holds for $\alpha = 0, 1$. No reverse inclusion at later stages is asserted here.

Proof. First, addition of a constant preserves every $\mathcal{C}_{25,\alpha}$: prove this by induction, using the assertion for smooth psh functions, compatibility of translation with uniform limits, and locality at limit ordinals. Now argue by induction for the displayed inclusion. The zero stage is immediate. At a successor stage, work on an open set where $u_j \in \mathcal{C}_{25,\alpha}$ converges uniformly to u. After passing to a subsequence, assume $\|u_j - u\| \leq 4^{-j}$. The functions $u_j + 2 \cdot 4^{-j}$ remain in $\mathcal{C}_{25,\alpha}$ and decrease to u, by the estimate in Proposition 11.2.3. By induction they lie in $\mathcal{P}_{25,\alpha}$; hence $u \in \mathcal{P}_{25,\alpha+1}$. If the convergence is only locally uniform, first restrict to a relatively compact open neighbourhood. Sheafification and limit ordinals are treated by restriction to the corresponding local representations. Equality at stage one is Proposition 11.2.3. □

Local integrability behaves differently under the order relation defining a decreasing limit. If u is locally integrable, v is upper semicontinuous, and $u \leq v$, then on a compact set where $v \leq M$ one has

$$|v| \leq |u| + \max(M, 0) \quad \text{almost everywhere.} \tag{11.6.4}$$

Thus v is integrable for each locally finite measure used in the definition. In CLD25 paragraph 5.7.2 this can be checked on skeletal cells with their Lebesgue measures, or against the specified locally bounded volume forms. It does not mean integrability against every arbitrary background current.

Proposition 11.6.7 (The locally integrable hierarchy)**.** *Let $\mathcal{I}_{25,\alpha}$ be the hierarchy obtained from the original smooth psh functions by the same local decreasing-limit and limit-ordinal operations as $\mathcal{P}_{25,\alpha}$, but retaining only locally integrable functions and requiring locally integrable endpoints. Then, for every ordinal α,*

$$\mathcal{I}_{25,\alpha} = L^1_{\mathrm{loc}} \cap \mathcal{P}_{25,\alpha}. \tag{11.6.5}$$

These are classes of actual functions; the order and limits are pointwise, not operations on equivalence classes modulo null sets. Moreover, if $u_j \downarrow u$ are generated psh functions and u is locally integrable, then every u_j is locally integrable and $u_j \to u$ in local L^1 for each of the measures in that definition.

Proof. This is the argument of CLD25 Proposition 8.1.12, with the inherited integrability made explicit. The inclusion from left to right in (11.6.5) follows from the construction. For the reverse inclusion, use induction on α. At a successor stage, write a locally integrable u locally as $u_j \downarrow u$ with $u_j \in \mathcal{P}_{25,\alpha}$. The functions u_j are upper semicontinuous and bounded above on compact sets. Since $u \le u_j$, estimate (11.6.4) shows that each u_j is locally integrable. The induction hypothesis puts them in $\mathcal{I}_{25,\alpha}$, so their locally integrable endpoint belongs to $\mathcal{I}_{25,\alpha+1}$. The zero and limit stages follow directly from the definitions and locality.

For the convergence assertion, work on a compact set for one of the defining measures. The nonnegative integrable function $u_1 - u$ dominates $u_j - u$, which decreases to zero almost everywhere. Dominated convergence gives

$$\int |u_j - u| \, d\mu \longrightarrow 0.$$

Apply this on the local cells or to each required volume measure. Thus the currents $[u_j]$ converge to $[u]$ against the original smooth tests wherever these currents are defined, as used in Proposition 11.2.4. No continuous predecessor replacement is needed. □

Remark 11.6.8 (Consequences for the approximation audit). The compact barrier theorem resolves the transfinite selection problem when one common compact representation is supplied. It does not prove that sheaf-generated sections admit such a representation. The indexed inclusion and the locally integrable equality above are valid without that representation. Consequently, the gap identified in Proposition 8.1.11 does not invalidate Proposition 8.1.12 or the locally integrable current-positivity argument. Neither local L^1 convergence nor membership in a generated class alone defines products of unbounded potentials with arbitrary currents. Such products require their own integrability and continuity statements.

References. CLD25 v2, paragraph 5.7.2, Definitions 8.1.1 and 8.1.7, Lemmas 8.1.4, 8.1.8 and 8.1.10, Propositions 8.1.11–8.1.12. The compact hierarchy and its barrier proof are elementary deductions stated separately from the source's sheaf hierarchy. The indexed forward inclusion and the L^1 equality expand valid portions of the source argument; no counterexample to its continuous equality is asserted.

11.7 Envelope methods and bounded localization

The compact argument of Section 11.6.1 leaves a geometric question: how can approximants on different local domains be compared on one neighbourhood? A projective ambient space provides another route. One first extends bounded local potentials to a common global background, uses an envelope comparison there, and then restricts back. This section gives the order estimates needed for that route and identifies the additional theorems it uses.

11.7.1 Three classes in the recent local theory

We fix the preprint [Wu26], arXiv version 1 of 23 September 2026. Its local theory is formulated on opens in algebraic analytifications over a complete discretely or trivially valued field. Write $\mathcal{P}_W$ for its class of decreasing limits of tropically convex PL functions, and $\mathcal{R}_W$ for its smooth-approximable class. These are first-stage local approximation conditions. They must be distinguished from the sheaf $\mathcal{P}_{25}$ obtained by iterating decreasing limits and localization.

For the comparisons below we use locally bounded functions. They are finite, so the exclusion of functions identically $-\infty$ on a component causes no ambiguity. The smooth generators are the original CLD generators; re-

placing them by weakly smooth or harmonic-chart generators would change the comparison.

Class	Local requirement	Relevant source
$\mathcal{P}_W$	One decreasing PL sequence on one open neighbourhood	[Wu26], Definition 3.8
$\mathcal{R}_W$	One decreasing original s-mooth convex sequence	[Wu26], Definition A.1
$\mathcal{P}_{25}$	The generated sheaf, includ-ing further decreasing limits	[CLD25], Definition 8.1.1

Theorem A.3 of [Wu26] asserts $\mathcal{P}_W = \mathcal{R}_W$. Corollary 4.18(1) supplies decreasing-limit stability of $\mathcal{P}_W$ when the projective ambient variety has the envelope property. Remark A.2 explicitly retains this hypothesis in the comparison with the generated CLD definition. These are substantial source inputs, not consequences of the terminology "psh". Their implication for continuous generated functions is proved in Section 11.7.4.

In the convex terminology here, the supporting affine function touches the function at the point. Example A.8.1 records a defect in the source's displayed definition and its repair. For the explicit rational-generator class of Section 11.9.1, Theorem 11.9.5 supplies one direction of approximation independently of Theorem A.3, over a nontrivially valued field.

11.7.2 Continuous obstacles on a compact space

An envelope argument needs continuous majorants of a local potential. Compactness alone does not make every upper semicontinuous function a decreasing limit of a sequence of continuous functions on an arbitrary Hausdorff space. The local approximation hypothesis supplies precisely the missing information here.

Lemma 11.7.1 (Continuous majorants from local sequences)**.** *Let X be compact Hausdorff and let $u : X \to \mathbf{R}$. Suppose every point has an open neighbourhood on which u is a decreasing limit of finite continuous functions. Then there are $F_j \in C(X, \mathbf{R})$ with $F_j \downarrow u$ pointwise on X. No metrizability assumption is needed.*

Proof. Choose a finite open cover $U_1, \dots, U_r$ with continuous sequences $u_{i,j} \downarrow u|_{U_i}$. By normality of a compact Hausdorff space, choose a continuous par-

tition of unity ρ_i with $\operatorname{supp}\rho_i \subset U_i$. Extend $\rho_i u_{i,j}$ by zero outside U_i and put

$$F_j = \sum_{i=1}^{r} \rho_i u_{i,j}.$$

Each summand is continuous: its support stays inside its defining open set. Nonnegative coefficients give $F_{j+1} \le F_j$. At every point only finitely many terms occur, and their limit is $\sum_i \rho_i u = u$. □

The functions F_j are continuous obstacles; the proof does not assert that they are psh. Applying a partition of unity directly to psh potentials would introduce uncontrolled derivatives of the coefficients. Positivity is recovered by an envelope, under the following explicit hypotheses.

Proposition 11.7.2 (The bounded envelope comparison)**.** *Fix a projective ambient space and an ample semipositive model background ω. Let $\mathscr{G}_\omega$ and $\mathscr{L}_\omega$ be respectively its global and local classes of bounded ω-psh potentials, with $\mathscr{G}_\omega \subset \mathscr{L}_\omega$. Assume the following inputs:*

1. *Every $u \in \mathscr{L}_\omega$ is locally a decreasing limit of continuous functions.*
2. *For $F \in C(X^{\mathrm{an}}, \mathbf{R})$, the envelope $P_\omega F$ is a continuous member of $\mathscr{G}_\omega$, satisfies $P_\omega F \le F$, and its Monge–Ampère measure is supported on $\{P_\omega F = F\}$.*
3. *Domination holds: for $u \in \mathscr{L}_\omega$ and $v \in \mathscr{G}_\omega$, the inequality $u \le v$ almost everywhere for $\mathrm{MA}_\omega(v)$ implies $u \le v$ everywhere.*
4. *$\mathscr{G}_\omega$ is closed under decreasing limits with bounded endpoints.*

Then $\mathscr{L}_\omega = \mathscr{G}_\omega$.

Proof. Take $u \in \mathscr{L}_\omega$. Lemma 11.7.1 gives $F_j \downarrow u$ with F_j continuous. Put $p_j = P_\omega F_j$. The definition of an envelope is order preserving, so $p_{j+1} \le p_j$. On the support of $\mathrm{MA}_\omega(p_j)$ one has $u \le F_j = p_j$. Domination gives $u \le p_j$ on the whole space. Thus

$$u \le p_j \le F_j, \qquad p_j \downarrow u.$$

The endpoint is bounded, so the last hypothesis gives $u \in \mathscr{G}_\omega$. □

This is the bounded part of the mechanism in [Wu26], Theorem 4.17. Envelope continuity, orthogonality, and domination remain imported inputs;

none follows from Lemma 11.7.1. The lemma makes the continuous-obstacle construction independent of a countable dense subfield. It does not remove countability hypotheses from other theorems in that paper.

11.7.3 An order-preserving extension on a fixed neighbourhood

The next construction explains why the global comparison is relevant to a local decreasing sequence. The extensions must preserve order; choosing an unrelated global extension for every term would not suffice. We use a local potential class closed under finite maxima, positive scaling and addition of constants. Model backgrounds in this subsection are curvature classes of metrics, not automatically original smooth CLD forms.

Proposition 11.7.3 (A fixed bounded extension formula)**.** *Let X be projective. Let ω, θ be semipositive model backgrounds on the same $\mathbf{Q}$-line bundle, admitting local psh potentials in the chosen class, and choose a global continuous PL function b with $\theta = \omega + d'd''b$. On an open $Y \subset X^{\mathrm{an}}$ let v be a continuous psh potential of θ. Assume*

$$Q = \{v \le 0\} \Subset Y \cap U, \qquad P = \{v \le -\epsilon\}, \qquad x \in \mathrm{Int}(P), \qquad \epsilon > 0,$$

where U is an open set on which the local potentials will be given. Fix $M > 0$ and a positive rational number A with $A\epsilon > 2M + 1$, and set $h = Av + M + 1$. For any psh u on U with $|u| \le M$ on Q, define

$$\Psi(u) = \begin{cases} \max\{u - h, 0\}, & \text{on } Q, \\ 0, & \text{on } X^{\mathrm{an}} \setminus Q, \end{cases} \qquad \Phi(u) = A^{-1}\Psi(u) + b. \quad (11.7.1)$$

Then $\Phi(u)$ is a bounded local ω-psh potential. On the same open neighbourhood $V = \mathrm{Int}(P)$ of x, for all such u,

$$\Phi(u) + g = A^{-1}u, \qquad g = v - b + A^{-1}(M + 1). \quad (11.7.2)$$

The map Φ preserves order and decreasing pointwise limits within this bounded family. Moreover,

$$\|\Phi(u) - \Phi(w)\|_{X^{\mathrm{an}}} \le A^{-1}\|u - w\|_Q. \quad (11.7.3)$$

Proof. On P one has $h < -M \le u$. On the compact collar $C = \{-1/(2A) \le v \le 0\}$ one has $h \ge M + 1/2 > u$. Consequently $\Psi(u)$ vanishes on a neighbourhood of the boundary of Q, and the piecewise formula is well defined without a boundary jump. On $\mathrm{Int}(Q)$,

$$\Psi(u) + h = \max\{u, h\}$$

is psh. Outside a smaller compact sublevel set, $\Psi(u) = 0$. Since θ is semi-positive with admitted local potentials, these statements show that $\Psi(u)$ is locally $A\theta$-psh. After scaling and changing the background,

$$\omega + d'd''\Phi(u) = \theta + A^{-1}d'd''\Psi(u) \ge 0$$

in the local-potential sense. Boundedness follows from the bounds on u, continuity of h on Q, and continuity of b on the compact space. On P the maximum selects $u - h$, which proves (11.7.2). Finally, the scalar map $t \mapsto \max(t - h, 0)$ is increasing, continuous, and 1-Lipschitz. This proves the order, limit, and norm assertions using the same A, h, b throughout. □

Lemma 11.7.4 (One collar for a whole decreasing tail)**.** *In the preceding setting, suppose u_j are finite upper semicontinuous functions on U with $u_j \downarrow u$ and $|u| \le M$ on Q. Then, for all sufficiently large j, $u_j < h$ on the fixed collar C. If the u_j are continuous, their extensions by (11.7.1) are continuous for this entire tail. On P the identity (11.7.2) holds for every j.*

Proof. The open sets $\{u_j < M + 1/4\}$ increase and cover the compact set C, since $u \le M$. One index therefore works on all of C, and every later index works as well. There $u_j < M + 1/4 < h$. On P, the inequalities $u_j \ge u \ge -M > h$ give the claimed identity. □

These formulas reconstruct the bounded localization mechanism of [Wu26], Proposition 3.28, while recording the background change and the common collar. The existence of the projective barrier data is a separate geometric input, constructed for integral projective varieties in Theorem 11.8.9. In a weighted Fubini–Study presentation its local shape can be

$$v = \max\{-a, \ell_1, \dots, \ell_r\}, \qquad a > \epsilon > 0, \quad \ell_i(x) < -\epsilon. \tag{11.7.4}$$

Here the ℓ_i are normalized logarithmic section ratios with their weights, and the outer sublevel set must be chosen compactly inside U. The negative

constant branch is essential. A formula $\max\{0, \ell_1, \dots, \ell_r\}$ is nonnegative and cannot satisfy $v(x) < 0$. Also, equality of underlying line bundles does not identify their model backgrounds; the term b in (11.7.1) performs the required conversion.

11.7.4 What bounded decreasing stability would settle

We now isolate a sufficient input that is weaker than constructing the raw compact representation of Corollary 11.6.5. Let $\mathcal{R} = \mathcal{P}_{25,1}$ denote the original first-stage sheaf, allowing discontinuous functions.

Theorem 11.7.5 (Bounded stability implies collapse of continuous stages)**.** *Assume that on every open set a decreasing sequence of locally bounded sections of $\mathcal{R}$, whose endpoint is locally bounded, again has its endpoint in $\mathcal{R}$. Then*

$$\mathcal{P}_{25} \cap L^{\infty}_{\mathrm{loc}} = \mathcal{R} \cap L^{\infty}_{\mathrm{loc}}. \tag{11.7.5}$$

For every ordinal $\alpha \geq 1$, one moreover has

$$\mathcal{C}_{25,\alpha} = C^0 \cap \mathcal{P}_{25,\alpha} = \mathcal{U}. \tag{11.7.6}$$

All boundedness conditions here concern actual pointwise functions.

Proof. We prove by induction on α that each locally bounded section of $\mathcal{P}_{25,\alpha}$ belongs to $\mathcal{R}$. This holds at stage zero. At a successor stage, work on an open set where $u_j \downarrow u$, with $u_j \in \mathcal{P}_{25,\alpha}$ and u locally bounded. Since $u_j \geq u$ and u_j is upper semicontinuous with no value $+\infty$, every u_j is locally bounded. The induction hypothesis puts every predecessor in $\mathcal{R}$. The assumed bounded decreasing stability now applies on this open set. At a limit ordinal use the local earlier-stage representations and the sheaf property of $\mathcal{R}$. This proves (11.7.5).

A finite continuous $u \in \mathcal{P}_{25,\alpha}$ is locally bounded, hence belongs to $\mathcal{R}$. Dini's argument on a smaller relatively compact neighbourhood turns its decreasing smooth approximation into local uniform approximation. Thus $u \in \mathcal{U}$. Conversely, $\mathcal{U} = \mathcal{C}_{25,1}$ by Proposition 11.2.3, and the continuous hierarchy is increasing. Combine these inclusions with Proposition 11.6.6 to obtain (11.7.6). □

The hypothesis is a genuine stability statement on a common open set. It does not follow merely by choosing a separate regularization neighbourhood

for each predecessor. Once it is known, no additional ordinal estimate is needed for continuous functions: all positive stages already coincide. This strengthens the earlier repair plan, which treated ordinal control as a further possible task.

11.7.5 The projective scope and its source dependencies

Corollary 11.7.6 (Application of the envelope comparison inputs)**.** *Let K be complete and discretely or trivially valued, let X be projective over K, and suppose X has the envelope property over K in the sense of [Wu26]. Import Theorem A.3 and Corollary 4.18(1) of that source, with its comparison of analytic and algebraic smooth generators. Then on opens of X^{an}, the conclusions* (11.7.5) *and* (11.7.6) *hold.*

Proof. For locally bounded functions, Theorem A.3 identifies $\mathcal{R}$ with $\mathcal{P}_W$. A locally bounded decreasing endpoint is generically finite, so Corollary 4.18(1) gives the bounded stability assumed in Theorem 11.7.5. Apply that theorem. □

Wu's Theorem 7.1 supplies the envelope property for projective geometrically unibranch varieties over discretely or trivially valued fields of equicharacteristic zero; normal projective varieties are included. Accordingly, this is a source-dependent route to the continuous-stage comparison in that category. The paper's Fornæss–Narasimhan step, Theorem 6.5, includes a countable-dense-subfield hypothesis; the proof of Theorem 7.1 contains a further descent argument to remove it. Neither hypothesis nor descent step should be suppressed when reconstructing that proof.

Remark 11.7.7 (What has been proved here and what is imported). The continuous-majorant lemma, the fixed extension estimates, and the transfinite bounded-stability implication have complete proofs above. The projective corollary uses the indicated preprint theorems. It is not an independent reconstruction of the full proof of Theorem A.3, the domination principle, or the envelope theorem. In particular, Appendix A of [Wu26] invokes two Amini–Piquerez–Wu preprints and passes from approximation of generators to approximation classes. Section 11.8 matches the localization and PL-approximation locators, corrects a false whole-domain inequality, and proves fixed-domain estimates. The general generator-to-class passage and the separate mixed-calculus input remain to be verified.

The general analytic question from Proposition 8.1.11 remains separate. The cited envelope theorem does not cover arbitrary complete fields or arbitrary Berkovich spaces; in particular, its equicharacteristic-zero statement is not a theorem for every finite extension of $\mathbf{Q}_p$. Nor does a comparison of functions establish an arbitrary-background-current Bedford–Taylor calculus. The next task is to verify the source comparison chain in the projective category, while retaining bounded first-stage stability as the precise remaining input outside it.

References. Wu, arXiv:2609.27610v1, Definitions 3.8 and A.1, Proposition 3.28, Theorem 4.17, Corollary 4.18, Theorems 6.5 and 7.1, Remark A.2 and Theorem A.3. The present bounded extension is a corrected reconstruction of the localization mechanism; the continuous-majorant and transfinite arguments are elementary deductions. The external comparison theorems remain attributed inputs.

11.8 Common domains in local approximation

A comparison of generators becomes a comparison of first-stage classes only when the approximations can be made on one neighbourhood. We first give the required diagonal argument. We then reconstruct two localization procedures that produce such neighbourhoods: one on a tropical compactification and one on a projective analytic space. The tropical argument also identifies an incorrect extra inequality in the printed source. The local conclusion survives after a precise restriction of its domain.

11.8.1 A diagonal sequence on a prescribed domain

Lemma 11.8.1. *Let V be a fixed set, let $u_j : V \to \mathbb{R}$ decrease pointwise to $u : V \to \mathbb{R} \cup \{-\infty\}$, and let $\mathcal{D}(V)$ be a class stable under addition of rational constants. Suppose that, for every $j \geq 1$, there is $a_j \in \mathcal{D}(V)$ with*

$$\|a_j - u_j\|_V \leq \epsilon_j, \qquad \epsilon_j = 4^{-j}.$$

Then $b_j = a_j + 2\epsilon_j$ belongs to $\mathcal{D}(V)$ and decreases pointwise to u on this same V.

Proof. The errors give

$$u_j + \epsilon_j \leq b_j \leq u_j + 3\epsilon_j, \qquad b_{j+1} \leq u_{j+1} + 3\epsilon_{j+1} \leq u_j + \epsilon_j \leq b_j.$$

The first pair of inequalities gives the pointwise limit, including at points where $u = -\infty$. No boundedness of u and no continuity of the endpoint are required. □

Proposition 11.8.2 (A sufficient criterion for transferring first-stage classes)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be classes of finite continuous functions on ordinary open sets of a locally compact Hausdorff space, stable under restriction, and suppose $\mathcal{B}$ is stable under addition of rational constants. Assume that whenever $V \Subset U$ are open and $a \in \mathcal{A}(U)$, then for every $\delta > 0$ there exists $b \in \mathcal{B}(V)$ with $\|a - b\|_V < \delta$. Every function locally given by a decreasing sequence of $\mathcal{A}$-functions is then locally given by a decreasing sequence of $\mathcal{B}$-functions. Any separate endpoint condition, such as generic finiteness, is unchanged.*

Proof. At a point x, choose a representation $u_j \downarrow u$ on one open U. Choose $x \in V \Subset U$ once. The hypothesis gives an approximation to $u_j|_V$ with error at most 4^{-j} for every j, without changing V. Lemma 11.8.1 applies. □

The hypothesis is stronger than approximation near each point of each individual generator. From statements with domains V_j depending on j, one cannot simply form $\bigcap_j V_j$ and call it a neighbourhood. For example, the neighbourhoods $(-1/j, 1/j)$ of zero have intersection $\{0\}$. This logical distinction identifies a missing justification; it is not by itself a counterexample to equality of the two function classes.

Corollary 11.8.3 (Decreasing finite logarithmic maxima)**.** *Let V be an ordinary open set of a good analytic space in the original CLD category. Suppose u_j decreases pointwise on V and each term has a finite presentation*

$$u_j = \max\{c_j,\ a_{j1} \log |f_{j1}| + d_{j1}, \dots, a_{jr_j} \log |f_{jr_j}| + d_{jr_j}\}, \quad a_{ji} > 0,$$

where f_{ji} are analytic on all of V and c_j, d_{ji} are real constants. There is one decreasing sequence of original smooth psh functions on V with the same pointwise limit. Different terms may use different analytic functions; these functions may have zeros.

Proof. For each j, regularize its finite maximum by the compactly supported regularized maximum used in Theorem 20.8.1. The constant branch makes u_j finite and continuous. At a zero of one of the f_{ji}, its logarithmic branch tends to $-\infty$ and is strictly below the constant branch on a neighbourhood. Exact deletion of inactive coordinates makes the regularization smooth there. At

other points the surviving functions are analytic units, and convexity of the regularized maximum gives a positive Hessian in a moment chart. The local formulas agree by exact deletion. Thus they define a smooth psh function on all of V, with a uniform error bounded by the regularization parameter. Choose that parameter at most 4^{-j} and apply Lemma 11.8.1. □

This is a common-domain result for the displayed presentations. It does not assert that an arbitrary locally tropically convex PL function has one such presentation on an arbitrarily prescribed open set. Log-sum-exp alone would not justify smoothness at zeros; the exact-deletion property is the reason for using this regularization. The unit-only formula in Theorem 19.18.15 has no such zero issue.

11.8.2 The local affine support calculation

Lemma 11.8.4. *Let Π be a locally finite polyhedral complex in a finite-dimensional real vector space E, let $t \in |\Pi|$, and let W be an open neighbourhood of t. There is $r > 0$, depending only on Π, W, t, such that every cell meeting $B(t,r)$ contains t and $B(t,r) \subset W$. If $F \in C^1(W)$ is convex on $\sigma \cap B(t,r)$ for every cell σ, then*

$$F(z) \geq \ell_t(z), \qquad \ell_t(z) = F(t) + dF_t(z-t), \qquad z \in B(t,r) \cap |\Pi|. \quad (11.8.1)$$

In particular, the same radius works for any family satisfying these hypotheses for the same complex and open set.

Proof. Local finiteness gives a neighbourhood meeting only finitely many cells. Each of those not containing t is closed and has positive distance from t. A sufficiently small ball avoids all of them and lies in W. If z is in the remaining ball and in a cell σ, then $t, z \in \sigma$ and their segment lies in $\sigma \cap B(t,r)$. Apply the supporting-line inequality to the convex function $s \mapsto F(t + s(z-t))$ on $[0,1]$. Its right derivative at zero is $dF_t(z-t)$. □

The affine expression must be centred at t. Interpreting dF_t as a linear functional on E, the expression $dF_t(z) + F(t)$ generally fails to take the value $F(t)$ at $z = t$. Also, the cells here contain the tropical point $t = \mathrm{Trop}(x)$, not the analytic point x. These observations correct the notation in the supporting-affine-function step of Wu's Appendix A. They prove local support for a fixed polyhedral presentation, not ambient convexity on all of W.

11.8.3 A false global inequality and the local replacement

We fix [APW26], arXiv:2610.25602v1. Its Proposition 5.19 contains both a local extension assertion and the additional inequality $\varphi + g \geq Cf$ on the whole original open set U. Only the latter is contradicted by the following example.

Example 11.8.5 (Failure of majorization on all of U)**.** For every $n \geq 1$, take the projective tropical toric variety $X = (\mathbb{TP}^1)^n$, with dense stratum $X^\circ = \mathbb{R}^n$ and product fan. Set

$$g(t) = \sum_{i=1}^{n} |t_i|, \qquad U = (-1, 1)^n, \qquad f(t) = \frac{1}{1 - t_1} + \sum_{i=2}^{n} t_i^2.$$

The function g satisfies the positivity and conewise-linearity assumptions in that proposition, and f is smooth and strictly convex on U. Nevertheless, for $\omega = \partial\bar{\partial} g$, there are no $C > 0$ and $\varphi \in \mathrm{CPSH}_\omega(X)$ satisfying $\varphi + g \geq Cf$ throughout U.

Proof. The function g is convex, homogeneous, linear on each orthant, and positive away from zero. The Hessian of f is diagonal with entries $2/(1 - t_1)^3, 2, \ldots, 2$, hence positive definite. By Definition 4.3 of the source, every member of $\mathrm{CPSH}_\omega(X)$ is bounded. Since $g \leq n$ on U, the left-hand side of the proposed inequality is bounded above there. The right-hand side tends to $+\infty$ along $(t_1, 0, \ldots, 0)$ as $t_1 \uparrow 1$. This is impossible for $C > 0$. □

The printed proof first bounds f on an inner compact neighbourhood U' but then uses that bound to control the set where a global barrier lies below f on all of U. The example shows why that enlargement of the domain is invalid. The construction can instead be carried out on the inner neighbourhood alone. The following formulation makes the domain and all constants uniform for a bounded family.

Proposition 11.8.6 (Uniform tropical localization on an inner domain)**.** *Use the projective tropical toric setting and the bounded-growth correspondence of [APW26, Example 4.9]. Write $Y = X^\circ \subset E = \mathbb{R}^N$, let g be convex homogeneous PL with $g > 0$ away from zero, put $h = g|_Y$, and let $\omega = \partial\bar{\partial} h$. Fix $x \in Y$, an open $U \subset Y$, and $r > 0$ such that*

$$Q = Y \cap \overline{B}(x, r) \Subset U.$$

Fix $M > 0$. There are one neighbourhood V of x in Q and one $\lambda > 0$ such that every $f \in \mathrm{LConv}(U)$ with $|f| \le M$ on Q admits $\Phi_f \in \mathrm{CPSH}_\omega(X)$ satisfying

$$\Phi_f + h = \lambda^{-1} f \text{ on } V, \qquad \Phi_f + h \ge \lambda^{-1} f \text{ on } Q. \tag{11.8.2}$$

The assignment preserves order and is λ^{-1}-Lipschitz for the uniform norms on Q and X. If f is PL, the extension is PL in the source's category. No inequality is asserted on $U \setminus Q$.

Proof. Continuity on the unit sphere gives $c > 0$ with $g(z) \ge c\|z\|$. Choose λ with $\lambda cr > 2M + 2$ and put

$$g_x(y) = g(y - x), \qquad H = \lambda g_x - M - 1,$$
$$F_f(y) = \begin{cases} \max\{f(y), H(y)\}, & y \in Q, \\ H(y), & y \in Y \setminus Q. \end{cases}$$

On a fixed inner collar of ∂Q, the inequality $H > M$ holds, so the two formulas agree on a neighbourhood of that boundary. Thus F_f is continuous and locally convex on Y: use the maximum of locally convex functions inside, and H outside. On the fixed neighbourhood

$$V = Y \cap \{g_x < 1/(2\lambda)\} \subset \mathrm{Int}_Y Q$$

one has $H < -M - 1/2 < f$, so $F_f = f$. Set $\Phi_f = \lambda^{-1} F_f - h$ on Y. Outside Q this is $g_x - g - (M + 1)/\lambda$. A finite convex homogeneous PL function is globally Lipschitz, hence $|g_x - g| \le L\|x\|$. On Q boundedness follows from the hypotheses. Example 4.9 of the source therefore gives the bounded ω-psh extension to X. In the PL case, the translated background extends as a PL potential difference as in the proof of its Proposition 5.19; the modification has compact support in Y and is locally a maximum of PL functions. This preserves PL.

The two identities follow directly from the maximum. On Q, this maximum is increasing and 1-Lipschitz as a function of f; outside Q the extension is independent of f. The norm bound follows on Y, hence on X by continuity and density. These observations prove order preservation as well. If a decreasing family stays within the stated bounded class, the same formulas also commute with its pointwise limit. □

This proves the uniformity needed to localize a sequence already uniformly bounded on a prescribed compact tropical domain. Combined with the

source's uniform PL density theorem, it repairs the localization input in Proposition 5.21. That proof uses the local identity, not majorization on all of U. The correction consequently does not refute its local PL approximation conclusion. Neither the tropical extension nor uniform PL density, however, supplies the analytic pullback comparison invoked separately in Wu's Appendix A.

11.8.4 Projective Fubini–Study barriers

We now supply the geometric data assumed in Proposition 11.7.3, first as an explicit construction of weighted metrics. Let K be any complete non-Archimedean field, let X be an integral projective K-variety, and let L be ample. A $\mathbf{Q}$-line bundle is handled by first taking an actual tensor power. All topological neighbourhoods below are ordinary neighbourhoods in X^{an}.

Lemma 11.8.7 (An affine compact buffer)**.** *For $x \in U \subset X^{\mathrm{an}}$ with U open, there are $d > 0$, a nonzero $s \in H^0(X, L^d)$, the affine open $Y = D(s)$ containing x, finitely many $f_i \in K[Y]$, and rational constants c_i, such that*

$$\begin{aligned} K_0 = \{y \in Y^{\mathrm{an}} : |f_i(y)| \le e^{-c_i} \text{ for all } i\} \Subset U \cap Y^{\mathrm{an}}, \\ |f_i(x)| < e^{-c_i} \qquad \text{for all } i. \end{aligned} \tag{11.8.3}$$

Proof. A globally generated power of L has a section s_0 nonvanishing at x; its principal open $Y_0 = D(s_0)$ is affine. The topology of its analytification is generated by absolute values of regular functions, as in Section 2.10. Thus a finite intersection of strict inequalities $|p_i| < R_i$ and $|q_j| > r_j > 0$ contains x and is contained in $U \cap Y_0^{\mathrm{an}}$. On $Y = Y_0 \setminus \bigcup_j Z(q_j)$ the lower inequalities become upper inequalities for q_j^{-1}. This is again a principal affine open of the required type: clear the denominators of the finite product of the q_j using a power of s_0, then multiply its extending section by s_0 to ensure nonvanishing only inside Y_0. The resulting nonzero section of a power of L defines exactly Y.

Replace each strict upper bound by a closed bound strictly between the value at x and that upper bound. These bounds may be chosen in $\exp(\mathbf{Q})$, which is dense in $\mathbb{R}_{>0}$. Add bounds, again strict at x, for a finite set of algebra generators of $K[Y]$. Those latter bounds cut out the intersection of a closed polydisc with a closed affine embedding of Y, hence a compact affinoid domain; arbitrary positive radii are allowed. Intersecting with the other closed inequalities gives a compact subset K_0 of it. All original strict

inequalities hold on K_0, so $K_0 \subset U$. Their strictness at x supplies its ordinary interior neighbourhood. This argument applies also to a trivially valued field and to non-strict affinoid radii. □

Lemma 11.8.8 (One denominator for finitely many functions)**.** *Let s be a nonzero section of L^d on an integral projective variety, and let $f_1, \dots, f_m$ be regular on $Y = D(s)$. For every sufficiently large N, each $f_i s^N$ extends to a section of L^{dN} on X.*

Proof. Cover X by finitely many affine opens trivializing L^d. On such an open $\operatorname{Spec} A$, the section s is a nonzero element $a \in A$, and $f_i|_{D(a)} \in A_a$. Clearing the finitely many denominators gives a common exponent for every i and every open in the cover. The resulting local sections agree on the dense open $D(s)$ and hence on overlaps, since X is integral. They glue. A larger exponent is obtained by multiplication by a further power of s. This also justifies the denominator-clearing step in Lemma 11.8.7. □

Theorem 11.8.9 (A negative Fubini–Study barrier)**.** *In the projective setting above, given $x \in U$ there exist a rational-weighted finite Fubini–Study metric θ on L, an affine open $Y = D(s)$ containing x, and a continuous local weight v for θ on Y^{an} such that, for some $\epsilon > 0$,*

$$Q = \{v \le 0\} \Subset Y^{\mathrm{an}} \cap U, \qquad x \in \operatorname{Int}_{X^{\mathrm{an}}}\{v \le -\epsilon\}. \tag{11.8.4}$$

The weight is locally tropically convex and is locally uniformly approximable by original smooth psh functions. These assertions concern a metric and its local weights. Identifying its curvature with a model background in a narrower source category requires that category's Fubini–Study/model comparison.

Proof. Take Y, s, f_i, c_i, K_0 from Lemma 11.8.7. By Lemma 11.8.8, choose one N such that all $t_i = f_i s^N$ extend to sections of L^{dN} and that line bundle is globally generated. Choose a finite globally generating list $r_1, \dots, r_a$ of its sections. Every r_k/s^N is regular on Y. Choose rational b_k so negative that $\log|r_k/s^N|(x) + b_k < 0$, with $\log 0 = -\infty$. Choose also a rational constant $c_0 < 0$. On Y^{an} put

$$v = \frac{1}{dN} \max\left\{c_0,\ (\log|f_i| + c_i)_i, (\log|r_k/s^N| + b_k)_k\right\}. \tag{11.8.5}$$

This is the weight, relative to the nonvanishing section s^N of L^{dN}, of the metric

$$\theta = \frac{1}{dN} \max \left\{ \log |s^N| + c_0,\ (\log |t_i| + c_i)_i, (\log |r_k| + b_k)_k \right\}.$$

The last group of sections has no common zero, so this formula defines a continuous metric globally. Every branch of (11.8.5) is strictly negative at x, and the maximum is finite because of c_0. Hence $v(x) < 0$ and v is continuous. Choose $0 < \epsilon < -v(x)$; a neighbourhood of x then lies in $\{v < -\epsilon\}$. Moreover, $v \leq 0$ implies all inequalities defining K_0, so $Q \subset K_0$. The set Q is closed in Y^{an} and therefore closed in the compact set K_0. This proves the required compact containment.

Locally, branches vanishing at a point can be deleted because the constant branch dominates them nearby. The remaining branches are logarithms of analytic units with constants; their maximum is a convex PL function of moment coordinates. This proves local tropical convexity. Theorem 20.8.1, or its local exact-deletion argument, supplies the original smooth psh approximations. The construction uses rational weights but does not require those weights to belong to the value group, nor does it change the field. □

In the normalized discrete/trivial framework of [Wu26], Definition 2.3 admits these rational-weighted metrics; Proposition 2.4 is the cited model identification. Thus the theorem reconstructs the barrier geometry for integral projective varieties in that framework. If ω is the initial model background on L, let b be the difference of the constructed metric and the initial metric. It is a global continuous PL function there, and $\theta = \omega + d'd''b$. Inserting this b and (11.8.4) into Proposition 11.7.3 gives its extension formula on a fixed neighbourhood, with the required background correction. The geometric construction is proved here; the model identification is an explicitly attributed input. Reducible varieties and more general analytic spaces have not been added to this statement.

11.8.5 The remaining comparison input

The preceding results settle the quantitative diagonal step, bounded-family tropical localization and projective barrier geometry. Section 11.9 now assembles these with model approximation to obtain the PL-to-smooth direction on one ordinary neighbourhood, for its explicitly stated nontrivially valued rational-generator category. Proposition 11.22.14 supplies the smooth-to-PL

direction for one original smooth psh input in this projective setting. The common-domain passage for arbitrary sequences, identification of the full source generator class and extension to trivial valuation remain separate.

There is also a source dependency that cannot be replaced by the elementary constructions above. Wu invokes Theorem 6.13 of Amini–Piquerez–Wu, *Mixed differential calculus and Monge–Ampère equations on tropical varieties*. Both inspected bibliographies list it as a 2026 preprint without an identifier. A targeted search of the title and the authors' publication pages did not locate its primary full text in this revision. Accordingly, the analytic comparison assigned to that theorem is recorded as an unverified imported input. This is an access limitation, not evidence that the paper or theorem does not exist. Corollary 11.7.6 retains its explicit source-dependent status.

References. Wu26, Definitions 2.3 and A.1, Propositions 2.4, 2.11 and 3.28, and Appendix A, pp. 52–54; APW26, Definition 4.3, Example 4.9 and Propositions 5.19 and 5.21, fixed October v1. The counterexample and the common-domain estimates are proved here. Tropical bounded-growth extension and the indicated model comparison remain cited framework inputs.

11.9 From PL sequences to smooth sequences on one neighbourhood

The projective barrier of Section 11.8.4 can be chosen before the functions to be approximated. The bound of an individual function then affects only a scalar in the construction, not the final neighbourhood. Combining this observation with the known approximation of nef model metrics gives one direction of the local comparison without an envelope hypothesis. We state the coefficient and model conventions explicitly: arbitrary real PL weights cannot be identified with rational model metrics without a further argument.

11.9.1 The model-compatible class of generators

Throughout this section K is complete and nontrivially valued, its absolute value is fixed, and X is an integral projective K-variety of dimension at least one. Put $\Gamma = -\log|K^\times|$ and $\Gamma_{\mathbf{Q}} = \mathbf{Q}\Gamma$. For an ordinary open $U \subset X^{\mathrm{an}}$, write $\mathcal{G}(U)$ for the finite continuous functions which locally have the form

$$u = \max_{1\le a\le r}\left\{ c_a + \sum_{i=1}^{m} q_{ai}\log|f_i| \right\}, \qquad c_a \in \Gamma_{\mathbf{Q}}, \quad q_{ai} \in \mathbf{Q}, \tag{11.9.1}$$

where the f_i are analytic units on the neighbourhood in question. Presentations and neighbourhoods may vary with the point. This is the local convex rational PL class used below. It is stable under finite maxima, positive rational scaling, and addition of constants in $\Gamma_{\mathbf{Q}}$. Products of the finite lists of units give common moment charts for these operations.

Every such local presentation is uniformly approximable by original smooth psh functions on that neighbourhood: apply the regularized maximum to its affine branches. This assertion concerns a single presentation. It does not yet give a common approximation domain for a sequence of functions, each with its own covering by presentations. A finite convex PL function on an ordinary real open set has the displayed form after shrinking to a convex neighbourhood. At a boundary point of an extended tropical space, the reduction to finite coordinates is supplied for ambient convex PL functions in Section 11.10.1. The present theorem assumes (11.9.1); the relative-image class is discussed separately in Remark 11.10.10.

Lemma 11.9.1 (Positivity at a continuous endpoint)**.** *On an ordinary open set, suppose $u_j \in \mathcal{G}(U)$ decreases to a finite continuous function u. Then $d'd''[u]$ is positive against the original CLD compactly supported smooth tests.*

Proof. Local regularization of (11.9.1) gives positivity of $d'd''[u_j]$. For a fixed test η, Dini's lemma gives uniform convergence on its compact support. The smooth top-degree form $d'd''\eta$ defines a signed measure of finite total variation there, so

$$\langle d'd''[u_j], \eta\rangle = \langle [u_j], d'd''\eta\rangle \longrightarrow \langle [u], d'd''\eta\rangle.$$

Nonnegativity on positive tests passes to the limit. No smooth approximating sequence on a common neighbourhood has been used. □

We use two model-theoretic inputs. First, on a proper algebraic analytification, integral PL metrics are formal/model metrics; rational versions follow by taking a common tensor power on a finite cover. The model/formal/PL comparison is [GM, Theorem 1.1]. Constants in $\Gamma_{\mathbf{Q}}$ cause no extension of K: after a tensor power they are logarithms of absolute values of elements of K. Second, a model metric whose curvature current is positive on the whole projective analytification is nef. This is the model criterion in [CLD25, Corollary 18.8.8], with the residual mechanism recorded in Theorem 19.18.12 and Section 19.18.13. These are named geometric inputs. For the specific maximum weights below, Proposition 11.10.7 also supplies a route through Gubler–Künnemann functorial positivity and Gubler–Martin descent; its proof uses

the displayed generators and does not depend on the present approximation theorem.

11.9.2 A fixed barrier with variable scales

Lemma 11.9.2. *For $x \in U$, the data L, Y, v, Q, ϵ of Theorem 11.8.9 can be chosen so that all weights of the Fubini–Study metric θ lie in $\Gamma_{\mathbf{Q}}$. In particular θ is a rational model metric and $v \in \mathcal{G}(Y^{\mathrm{an}})$.*

Open proof obligation (RC2). Prove the value-group barrier in the stated scope; BE21 Theorem 5.14 is invoked.

Proposition 11.9.3 (Localization with a domain independent of the bound)**.** *Fix the preceding barrier, put $P = \{v \le -\epsilon\}$ and $V = \mathrm{Int}_{X^{\mathrm{an}}} P$, and let $u \in \mathcal{G}(U)$. Choose $M, \tau \in \Gamma_{\mathbf{Q}}$ with $M > 0$, $\tau > 0$, $|u| \le M$ on Q, and a rational $A > 0$ with $A\epsilon > 2M + \tau$. Then there is a nef rational model metric m_u on the fixed line bundle L whose weight w_u relative to the same power-section frame as v satisfies*

$$Aw_u + M + \tau = u \quad \text{on } V. \tag{11.9.2}$$

The neighbourhood V depends only on x, U, L and the chosen barrier. The numbers M, A may depend on u.

Proof. Define a global function by

$$\phi_u = \begin{cases} A^{-1}\max\{u - Av - M - \tau, 0\}, & \text{on } Q, \\ 0, & \text{on } X^{\mathrm{an}} \setminus Q. \end{cases} \qquad m_u = \theta + \phi_u. \tag{11.9.3}$$

Every boundary point of Q lies in $Y^{\mathrm{an}} \cap U$ and has $v = 0$. There $u - Av - M - \tau \le -\tau$. Continuity gives an open collar on which the maximum vanishes, so the formulas glue. The collar may depend on u, A, which is harmless: it is used only to construct this one global metric. Since u, v are locally rational PL with constants in $\Gamma_{\mathbf{Q}}$, the glued function is rational PL. The model/PL comparison stated above makes m_u a rational model metric.

On the interior of Q, its local weight is

$$w_u = v + \phi_u = \max\{A^{-1}(u - M - \tau), v\}.$$

Both branches, and their maximum on a common moment chart, are in $\mathcal{G}$. Outside the modification the metric is θ. Hence its curvature current

is positive everywhere. The quoted current/model criterion gives nefness; alternatively apply Proposition 11.10.7 directly to these weights. Finally, on P,

$$Av + M + \tau \leq -A\epsilon + M + \tau < -M \leq u,$$

so the first branch is active and (11.9.2) follows. □

Unlike Proposition 11.7.3, this proposition does not claim that the global extensions preserve the order of an entire family. Different scales usually destroy that order. Order is restored only after restricting the approximants to the fixed V. This is why a uniform bound for the whole family is unnecessary.

11.9.3 Global approximation and the fixed-domain diagonal

Proposition 11.9.4 (A known global approximation input)**.** *A nef rational model metric m on an ample line bundle over a projective K-scheme is a uniform limit of pure Fubini–Study metrics. Consequently, for every $\delta > 0$ it admits a global original smooth semipositive metric $\widetilde{m}$ with $\|\widetilde{m} - m\|_{X^{\mathrm{an}}} < \delta$. Here K is complete and nontrivially valued.*

Open proof obligation (RC2). Prove the required FS density for the nef classes and its exact hypotheses; BE21 Theorem 7.8 is invoked.

This approximation theorem is already in the literature. Its role here is to supply functions on all of Y^{an}, so that the local domains no longer depend on the generator being approximated. No envelope continuity theorem is used in this step.

Theorem 11.9.5 (One neighbourhood for a decreasing PL sequence)**.** *Let $u_j \in \mathcal{G}(U)$ decrease pointwise to $u : U \to \mathbf{R} \cup \{-\infty\}$. For every $x \in U$ there is one ordinary open neighbourhood V of x and original smooth psh functions b_j on V such that $b_j \downarrow u|_V$. Neither a uniform bound on the family (u_j) nor local boundedness or continuity of u is required. Any generic-finiteness condition imposed in the definition of the endpoint class is preserved.*

Proof. Choose the barrier and V once by Lemma 11.9.2. Fix $\tau > 0$ in $\Gamma_{\mathbf{Q}}$. Each u_j is continuous on the compact set Q, so choose its own $M_j \in \Gamma_{\mathbf{Q}}$ and rational $A_j > 0$ as in Proposition 11.9.3. This gives nef model metrics m_j on the same ample L, with weights w_j satisfying

$$u_j = A_j w_j + M_j + \tau \quad \text{on } V.$$

Put $\epsilon_j = 4^{-j}$. Apply Proposition 11.9.4 to m_j with error less than ϵ_j/A_j. Let $\widetilde{w}_j$ be the resulting smooth weight in the same power-section frame on Y^{an}. The function

$$a_j = A_j\widetilde{w}_j + M_j + \tau \quad \text{satisfies} \quad \|a_j - u_j\|_V < \epsilon_j.$$

It is original smooth and psh because $A_j > 0$. A power-section frame defines a normalized weight by dividing by its tensor exponent; it does not require a root of that section over K. Finally set $b_j = a_j + 2\epsilon_j$. Lemma 11.8.1 proves $b_j \downarrow u$ on this same V. □

Corollary 11.9.6. *Under the hypotheses of Theorem 11.9.5, if u is finite continuous, then u is locally uniformly approximable by original smooth psh functions. Thus the continuous endpoints of this PL first-stage class belong to $\mathcal{U}$.*

Proof. Shrink to a relatively compact neighbourhood inside V and apply Dini's lemma to the decreasing continuous functions b_j and their continuous limit. □

Remark 11.9.7. This combines Wu's localization mechanism, a model semi-positivity criterion, Gubler–Martin's PL/model comparison and Boucksom–Eriksson's approximation theorem. The explicit choices are a fixed neighbourhood and errors ϵ_j/A_j; no priority claim is made. Wu's convention $\mathbf{Z} \subseteq \Gamma_K$ places his rational constants in $\Gamma_{\mathbf{Q}}$. The ambient analytic generators are identified in Theorem 11.10.3; the relative finite-selection presentation is supplied in Theorem 11.12.5, with its explicit convention. Arbitrary real PL weights and trivial valuation are not covered.

11.9.4 First-stage comparison and further stability

Let $\mathcal{P}_{\mathcal{G}}$ be the local decreasing class generated by $\mathcal{G}$, with the same endpoint convention as the original smooth first-stage class $\mathcal{R}$. Theorem 11.9.5 proves $\mathcal{P}_{\mathcal{G}} \subseteq \mathcal{R}$. Corollary 11.24.4 now supplies the reverse inclusion in the same nontrivially valued integral-projective category, by rational replacement on a fixed affinoid. Thus $\mathcal{P}_{\mathcal{G}} = \mathcal{R}$ in this precisely specified first-stage sense.

The following sufficient criterion is retained for its stability implication. Generator membership and first-stage equality are no longer the missing steps in this setting. Closure under decreasing limits of arbitrary sections of

the resulting sheaf remains additional: their row domains may vary. Lemma 11.24.5 handles common-domain rows, but does not provide a common domain from termwise germ membership.

Proposition 11.9.8 (A sufficient stability criterion)**.** *Suppose every original smooth psh function locally belongs to $\mathcal{P}_{\mathcal{G}}$, and $\mathcal{P}_{\mathcal{G}}$ is closed under decreasing limits of its sections on ordinary open sets whenever the endpoint is allowed. Then $\mathcal{R} = \mathcal{P}_{\mathcal{G}}$. In particular, its bounded decreasing stability implies the continuous-stage equalities of Theorem 11.7.5.*

Proof. On a neighbourhood representing a section of $\mathcal{R}$, each smooth predecessor is a section of the sheaf $\mathcal{P}_{\mathcal{G}}$ by the first hypothesis. The second hypothesis places their decreasing limit there on that same open set. This proves the reverse inclusion; combine it with Theorem 11.9.5. Apply Theorem 11.7.5 for the last assertion. □

11.10 Convex PL generators at the extended tropical boundary

The logarithms in the presentation of $\mathcal{G}$ are logarithms of units. An analytic tropicalization, however, may be defined by functions with zeros, so its image meets the boundary of $\mathbb{T}^k$, where $\mathbb{T} = \mathbf{R} \cup \{+\infty\}$. We explain why this distinction disappears locally for finite ambient convex PL functions. The argument expands [Wu26, Remark 3.2]; it also makes precise the absolute-value estimate used in Remark 3.9 of that paper. The word ambient is essential: the defining function is given on an open subset of $\mathbb{T}^k$, not merely on a tropical image.

11.10.1 Finite affine selections and boundary factorization

We use the finite-valued convention of [Wu26, Definition 3.1(1)–(2)]. On an ambient open $\Omega \subset \mathbb{T}^k$, an affine function is a finite continuous function $\ell(t) = c + \sum_i a_i t_i$ for which every summand is finite; in particular $a_i = 0$ if Ω contains a point with $t_i = +\infty$. A finite PL function is locally a continuous selection from a finite list of such affine functions. Locally convex means that at every point there is a touching affine lower support on a neighbourhood. Thus the contact condition is $f(z) = \ell(z)$ and $f \geq \ell$ there, with the correction explained in Example A.8.1.

Lemma 11.10.1 (Eliminating infinite coordinates)**.** *Let $z \in \mathbb{T}^s \times \mathbf{R}^d$ have its first s coordinates infinite and its remaining coordinates equal to y. Let f be finite PL on an ambient open neighbourhood of z. There are $R \in \mathbf{R}$, an open ball $B \subset \mathbf{R}^d$ about y, and a finite continuous PL function F on B such that*

$$N = (R, +\infty]^s \times B, \qquad f|_N = F \circ p, \tag{11.10.1}$$

where p is the projection onto the last d coordinates. The affine pieces of F are obtained by deleting the first s coordinates from one local affine list for f. If $d = 0$, the conclusion is that f is constant on N.

Proof. Choose one open patch containing z on which f selects from finitely many affine functions $\ell_1, \dots, \ell_r$, and choose the product N inside that patch. Finiteness of each summand at z forces all coefficients of the first s coordinates to vanish. Write the remaining affine functions as L_a on B. For fixed $b \in B$, the continuous function f on the connected fibre $(R, +\infty]^s \times \{b\}$ takes values in the finite set $\{L_a(b) : 1 \le a \le r\}$. It is therefore constant on that fibre. Evaluation along any fixed section of p defines a continuous F, and $F(b)$ is one of the $L_a(b)$. This proves the factorization and the PL assertion. □

Independence of the infinite coordinates is not yet a maximum presentation. The following elementary real argument supplies that remaining step, and keeps the original affine coefficients.

Lemma 11.10.2 (From affine selection to a convex maximum)**.** *Let $B \subset \mathbf{R}^d$ be an open ball and let F be continuous, with $F(b) \in \{L_1(b), \dots, L_r(b)\}$ for affine $L_a(b) = c_a + \langle a_a, b\rangle$. Then F is Lipschitz on B, with constant $\max_a \|a_a\|$ for the Euclidean norm. If F has a local touching affine lower support at every point, it is convex and there is a nonempty subset $I \subseteq \{1, \dots, r\}$ such that*

$$F = \max_{a \in I} L_a \quad \text{on } B. \tag{11.10.2}$$

Proof. On a line segment in B, restrict the finite list of affine functions and discard identical restrictions. Distinct restrictions intersect at at most one point. Between these finitely many intersection points, continuity forces F to follow one affine restriction. Its slope on each interval is bounded by $\max_a \|a_a\|$ times the speed of the segment. Summing the interval estimates proves the Lipschitz bound.

At an interior breakpoint of the segment, a touching affine lower support has slope m satisfying $m_- \le m \le m_+$, where m_- and m_+ are the left and

right slopes of F. Hence the slopes never decrease. The restriction to every segment is convex, so F is convex on B.

Discard duplicate affine functions on B. The hyperplanes $L_a = L_b$ cut a dense open subset of B into finitely many chambers. On each chamber the selected affine function is fixed. Let I consist of the indices selected on some chamber. If $a \in I$, choose b in that chamber. For any $c \in B$ and sufficiently small $t > 0$, the point $b + t(c - b)$ remains in the chamber. Convexity gives

$$(1 - t)L_a(b) + tL_a(c) = F(b + t(c - b)) \leq (1 - t)F(b) + tF(c),$$

so $L_a(c) \leq F(c)$. Equality with one of these supporting pieces holds on every chamber and hence, by continuity, throughout B. This proves (11.10.2). For $d = 0$ both assertions are immediate. □

If the function f of Lemma 11.10.1 is locally convex, then F has the lower supports required by Lemma 11.10.2: fix finite values of the first s coordinates and restrict each local support to that slice. Thus a finite ambient convex PL function near a boundary point is a maximum of finitely many affine functions of the finite coordinates alone. No convex-extension theorem for a tropical variety enters this argument.

11.10.2 Analytic pullbacks and the value group

Fix the absolute value of K and put $\Gamma = -\log|K^\times|$, $\Gamma_{\mathbf{Q}} = \mathbf{Q}\Gamma$. In this subsection a compatible PL function has rational slopes and constants in $\Gamma_{\mathbf{Q}}$ in its local affine lists. This is a coefficient convention, not a normalization of the field.

Theorem 11.10.3 (Ambient analytic generators and $\mathcal{G}$)**.** *Let W be an ordinary open in a K-analytic space, let $g_1, \dots, g_k$ be analytic functions on W, and write $\operatorname{trop}_g = (-\log|g_1|, \dots, -\log|g_k|)$, with $-\log 0 = +\infty$. Suppose f is a finite compatible locally convex PL function on an ambient open $\Omega \subset \mathbb{T}^k$ containing $\operatorname{trop}_g(W)$. Then $u = f \circ \operatorname{trop}_g$ locally has the unit-logarithm maximum presentation (11.9.1).*

Conversely, if K is nontrivially valued, every germ with that presentation can be written as $H \circ \operatorname{trop}_h$ for analytic units $h_1, \dots, h_r$ and a finite compatible locally convex PL function H on all of $\mathbb{T}^r$. Consequently these global-ambient analytic generators and $\mathcal{G}$ define the same sheaf of germs.

Proof. Fix $x \in W$. Reorder the coordinates so that the functions vanishing at x are precisely the first s functions. Apply Lemmas 11.10.1 and 11.10.2 at $\mathrm{trop}_g(x)$. On a product neighbourhood N of that point,

$$f(t) = \max_{a \in I} \left(c_a + \sum_{i>s} a_{ai} t_i \right), \qquad a_{ai} \in \mathbf{Q}, \quad c_a \in \Gamma_{\mathbf{Q}}.$$

On the inverse image of N, shrink about x so that the remaining g_i are analytic units. Substitution gives (11.9.1), with $q_{ai} = -a_{ai}$. The functions that vanish at x have been deleted exactly, not assigned logarithmic values by a limiting convention. If all $g_i(x)$ vanish, the pullback is constant near x.

For the converse write $u = \max_a(c_a + \sum_i q_{ai} \log |f_i|)$ on a neighbourhood of x. Choose a positive integer m with $mq_{ai} \in \mathbf{Z}$ and put $h_a = \prod_i f_i^{mq_{ai}}$. Negative exponents are allowed because the f_i are units. Nontrivial valuation makes $\Gamma_{\mathbf{Q}}$ dense in $\mathbf{R}$. Choose $c_0 \in \Gamma_{\mathbf{Q}}$ with $c_0 < u(x)$ and shrink so that $c_0 < u$. Define

$$H(t_1, \dots, t_r) = \max\{c_0, c_1 - t_1/m, \dots, c_r - t_r/m\}, \tag{11.10.3}$$

where a branch with $t_a = +\infty$ contributes $-\infty$ to this maximum. At such a coordinate, the inequality $t_a > m(c_a - c_0)$ deletes that branch on a neighbourhood. The remaining branches are finite affine functions. More explicitly, cover each coordinate by $\mathbf{R}$ and $(m(c_a - c_0), +\infty]$; their finitely many product sets give finite affine presentations after the inactive branches are deleted. Thus H is finite continuous PL on all of $\mathbb{T}^r$, with the stated coefficients. On each such neighbourhood it is a maximum of affine functions and hence has touching lower supports. Finally $H \circ \mathrm{trop}_h = \max(c_0, u) = u$ on the chosen neighbourhood. No extension of K or change of its absolute value has occurred. □

Remark 11.10.4. Wu assumes $\mathbf{Z} \subseteq \Gamma_K$ for a nontrivially valued field in [Wu26, Section 2.1, p. 5]. Under that convention, the rational constants in Definition 3.1 belong to $\Gamma_{\mathbf{Q}}$, so the preceding theorem applies to the ambient analytic generators in Remark 3.9, after correcting the touching-support formula. For a fixed absolute value without this convention, rational constants need not lie in $\Gamma_{\mathbf{Q}}$; our theorem retains the actual value-group condition. The boundary factorization itself allows real coefficients, but the rational model argument of Section 11.9 does not thereby acquire real coefficients.

11.10.3 Approximation through zeros

At a zero of g, uniform control of $\log|g|$ is unavailable. The useful coordinates for perturbing the analytic functions are their absolute values. Boundary factorization provides precisely the needed regularity in these coordinates.

Proposition 11.10.5 (A local Lipschitz estimate in absolute values)**.** *Let f be any finite PL function with real affine pieces on an ambient open $\Omega \subset \mathbb{T}^k$. Define $D = (-\log)^{-1}(\Omega) \subset [0,+\infty)^k$, with $-\log 0 = +\infty$. Then $h = f \circ (-\log)$ is locally Lipschitz on D for the Euclidean distance. Its restriction to every compact subset $C \subset D$ is Lipschitz. Convexity is not required.*

Proof. Fix $r \in D$, with its first s coordinates zero and the rest positive. Lemma 11.10.1 makes h independent of the first s coordinates on a neighbourhood of r. For the remaining coordinates choose a neighbourhood on which $r_i \geq a > 0$ and the logarithmic image lies in the ball B of that lemma. The first part of Lemma 11.10.2 gives a Lipschitz constant L for F on B. Since

$$|\log r_i - \log r_i'| \leq a^{-1}|r_i - r_i'|,$$

the resulting local constant for h is at most L/a. If there are no positive coordinates, h is locally constant.

To pass to a compact C, take a finite cover by relative neighbourhoods on which these pairwise Lipschitz estimates hold. A Lebesgue number $\delta > 0$ ensures that two points of C at distance less than δ lie in one member of the cover. Use the largest of the finitely many local constants for these pairs. For the other pairs, use the bound $\operatorname{osc}_C(h)/\delta$. The maximum of the two constants works on all of C. □

Corollary 11.10.6. *Let C be compact. Suppose that continuous maps r_j, r from C to $[0,+\infty)^k$ satisfy $r_j \to r$ uniformly and $r(C) \subset D$. For all sufficiently large j, the compositions are defined and*

$$\|f \circ (-\log) \circ r_j - f \circ (-\log) \circ r\|_C \leq L_C \|r_j - r\|_C \longrightarrow 0 \qquad (11.10.4)$$

for a constant L_C independent of j. In particular, this applies to uniformly approximating tuples of absolute values of analytic functions, including at their zeros.

Proof. The compact image $r(C)$ has a compact neighbourhood inside the relatively open set D. Uniform convergence places all sufficiently large images $r_j(C)$ in that neighbourhood. Apply Proposition 11.10.5 there. □

This proves the change-of-coordinates estimate used in [Wu26, Remark 3.9]. It does not construct the algebraic approximants to the analytic functions: that is a separate approximation assertion on the specified rational domain. Once absolute-value approximants on a fixed compact domain are supplied, the displayed estimate transfers their error without taking logarithms at zeros.

11.10.4 Model semipositivity through functorial positivity

For these generators the model-nefness step also follows from the earlier work of Gubler–Künnemann and Gubler–Martin. The reason is stronger than positivity of a single current: the same maximum presentation survives pullback. This gives another route for the metrics constructed in Proposition 11.9.3.

Proposition 11.10.7 (A criterion for the constructed model metrics)**.** *Let K be complete and nontrivially valued, let X be a proper K-scheme, and let m be a rational model metric on a line bundle L. Suppose its local weights have the maximum presentation* (11.9.1)*. Then m is a nef model metric.*

Open proof obligation (RC2). Establish the model-nef criterion and its functoriality in the stated category; the proof uses GK19 and GM.

This proof uses the published model comparison and descent theorems as inputs; it does not re-prove their lifting arguments. The passage through C_K descends a numerical property of a model, rather than an equality of currents or representatives. The local regularizations and the final sequence of Theorem 11.9.5 are still constructed over the original K. The criterion is specific to weights with the displayed presentations and does not assert that arbitrary positive currents are functorially positive.

11.10.5 The resulting regularization statement

Corollary 11.10.8 (Decreasing ambient PL generators)**.** *Let K be complete and nontrivially valued, let X be an integral projective K-variety of positive dimension, and let $U \subset X^{\mathrm{an}}$ be ordinary open. Suppose the finite continuous functions u_j decrease pointwise to u, and that each u_j locally admits an ambient analytic presentation as in Theorem 11.10.3, with rational slopes and constants in $\Gamma_{\mathbf{Q}}$. Then near each $x \in U$ there is one ordinary open V and original smooth psh functions b_j on V with $b_j \downarrow u|_V$. The analytic functions in the presentations may have zeros; their charts and neighbourhoods may vary with both j and the point. If u is finite continuous, the convergence is uniform on compact subsets of V.*

Proof. Theorem 11.10.3 gives $u_j \in \mathcal{G}(U)$ for every j as a local membership assertion on the original U. Apply Theorem 11.9.5. Its projective barrier construction, not an intersection of countably many presentation neighbourhoods, provides the common V. The last assertion is Dini's lemma on each compact subset. □

A relative touching-support condition on a polyhedral set must be distinguished from ambient convexity even before passing to analytic spaces. The following example explains why a same-chart extension cannot be taken for granted.

Example 11.10.9 (Relative supports do not give a same-chart convex extension)**.** In $\mathbf{R}^2$, let Σ be the one-dimensional fan with primitive ray vectors

$$v_1 = (1,0), \quad v_2 = (0,1), \quad v_3 = (1,1), \quad v_4 = (-1,-1),$$
$$(w_1, w_2, w_3, w_4) = (1,1,1,2).$$

It is positively weighted and balanced. Define a continuous rational PL function on its support by $F(tv_3) = t$ for $t \geq 0$, and $F(tv_i) = 0$ for $i = 1, 2, 4$. At every point of $|\Sigma|$, F has an affine lower support touching on a relative neighbourhood. Nevertheless its germ at the origin is not the restriction of an ambient convex function on any open convex neighbourhood of the origin.

Proof. Balancing is the identity $v_1 + v_2 + v_3 + 2v_4 = 0$. The function is continuous at the origin and affine on each ray; it selects from 0 and $(x+y)/2$. At the origin, the constant zero function is a touching lower support. At a nonzero point a sufficiently small relative neighbourhood meets only its ray,

where either 0 or $(x+y)/2$ agrees with F. Suppose that a convex ambient extension $\widetilde{F}$ existed near the origin. For sufficiently small $t>0$, its midpoint inequality would give

$$t=\widetilde{F}(t,t)\le \tfrac{1}{2}\widetilde{F}(2t,0)+\tfrac{1}{2}\widetilde{F}(0,2t)=0,$$

a contradiction. The obstruction depends on the presentation. Indeed, take the balanced fan $\widetilde{\Sigma}$ in $\mathbf{R}^3$ with rays generated by $e_1,e_2,e_3,-e_1-e_2-e_3$, all of weight one. The integral linear map

$$A(x,y,z)=(x+z,y+z)$$

maps these generators to $v_1,v_2,v_3,2v_4$, respectively. It maps the supports homeomorphically, and its ray lattice indices give the weights $(1,1,1,2)$ on Σ. On the lifted support one has

$$F\circ A=\max\{z,0\}.$$

This is an ambient convex PL presentation after changing the fan. This calculation is polyhedral; Proposition 11.16.1 realizes it by analytic units over fields with $|2|=1$. □

Remark 11.10.10 (The remaining source-class interface). The ambient analytic generators of [Wu26, Remark 3.9] are now covered, with the stated coefficient convention and corrected convexity. Definitions 3.6–3.7 there instead use PL and convex functions on $\mathrm{Trop}(W)$, and the two properties may initially be presented in different charts. Such an image need not be ambient open. A general identification therefore needs a presentation theorem with any necessary changes of chart, not a presumed same-chart convex extension. Example 11.10.9 refutes that elementary extension shortcut; Section 11.16 supplies its analytic realization and a smooth psh variant, but no obstruction after changing tropicalization. The unit-chart PL identification and the relative-support positivity step are proved in Section 11.11. Section 11.12 completes the extended-image PL identification for its explicit relative finite-selection convention. The prescribed-affinoid reverse replacement and the comparison of four explicit first-stage classes are proved in Section 11.24.3. Source-category identification and sheaf-level decreasing closure remain separate before identifying the full classes in Theorem A.3.

11.10.6 A transfer for locally semipositive rational PL functions

There is another way to use the projective barrier without demanding an ambient maximum presentation for every generator. Assume directly that the rational PL generators are locally semipositive in the model sense. The maximum theorem of Gubler–Rabinoff then supplies the positivity of the patched metric. We record the precise synthesis because it isolates a potentially weaker interface for comparison with another local psh theory.

Write $\mathcal{S}_{\mathbf{Q}}(U)$ for the finite continuous $\mathbf{Q}$-PL functions on an ordinary open U whose residues are nef at every point, in the sense of [GJR, Proposition 7.1 and Definition 7.2]. Equivalently, locally a tensor power of the metric $\|1\| = e^{-h}$ is given on a strictly analytic neighbourhood by a nef formal model. The equivalence is the cited proposition; it is not deduced merely from positivity of $d'd''[h]$. Rational PL here uses logarithms of analytic units on a strict analytic domain cover, so constants in $\Gamma_{\mathbf{Q}}$ are admitted.

Theorem 11.10.11 (Fixed-domain transfer from locally nef PL generators)**.** *Let K be complete and nontrivially valued, let X be an integral projective K-variety of positive dimension, and let $U \subset X^{\mathrm{an}}$ be ordinary open. If $u_j \in \mathcal{S}_{\mathbf{Q}}(U)$ decreases pointwise to $u : U \to \mathbf{R} \cup \{-\infty\}$, then every $x \in U$ has one ordinary open neighbourhood V carrying original smooth psh functions $b_j \downarrow u|_V$. No uniform bound on the family and no local boundedness of the endpoint are required. For a finite continuous endpoint the convergence is uniform on compact subsets of V.*

Open proof obligation (RC2). Prove the transfer between nef, PL and approximated metrics in the original hypotheses; the proof uses the listed GM, GRPL and GJR inputs.

Every member of $\mathcal{G}$ is locally in $\mathcal{S}_{\mathbf{Q}}$: its affine unit-logarithm branches are rationally harmonic, and repeated application of the same maximum theorem preserves semipositivity. The new theorem therefore gives a second proof of the earlier transfer. More generally, it suffices to verify model compatibility and local semipositivity of a proposed source generator; an ambient convex extension is unnecessary for this route. Theorem 11.11.6 completes the semipositivity step whenever the function is already model-compatible PL; Proposition 11.11.5 verifies that membership for unit-chart relative presentations. Theorem 11.12.5 now supplies extended relative-image membership under its common-branch convention. The rational coefficient restriction also

remains substantive: [GRPL, the paragraph following Corollary 5.5] warns that maxima need not remain Λ-PL in the strict domain category for larger groups Λ.

For a single $h \in \mathcal{S}_{\mathbf{Q}}(U)$, apply Theorem 11.10.11 to the constant sequence $u_j = h$. This gives decreasing original smooth psh approximants near each point. The related [GRPL, Theorem 1.5] applies to semipositive real PL functions on a boundaryless strictly analytic space and gives uniform approximation by locally Fubini–Study functions. The present argument keeps rational coefficients and a projective ambient space, while handling a whole decreasing family on one neighbourhood and choosing original smooth approximants. The locally Fubini–Study conclusion and the original smooth conclusion should not be identified without the smoothing step. Likewise, [GRPL, Theorem 1.4] gives semipositivity of a pointwise limit only when that limit is still real PL; it does not supply the decreasing closure of an arbitrary generated psh class needed in Section 11.9.4.

11.11 Relative supports and the curvewise route to nefness

The obstruction in Example 11.10.9 concerns extension in a fixed tropical chart. It does not obstruct a different route: pull a touching affine support back to an analytic curve, use harmonic comparison there, and then apply the curve criterion for semipositive PL functions. The two steps to distinguish are positivity and membership in the model-compatible PL class. This section proves the positivity step even at extended tropical boundary points, and proves the PL identification when the presentation uses analytic units.

11.11.1 A lower-contact criterion on curves

We first isolate a comparison argument that requires no polyhedral subdivision and no differentiability. Harmonicity in this subsection is Thuillier harmonicity on smooth strictly analytic curves over a nontrivially valued field. We use its vector-space property, its local maximum principle, and the comparison definition of subharmonicity; precise modern references are [GRPSH, Sections 3.2–3.3 and Proposition 3.14].

Lemma 11.11.1 (Harmonic lower contacts)**.** *Let C be a smooth strictly analytic curve over a complete nontrivially valued field, and let $u : C \to \mathbf{R}$ be continuous. Suppose that every $z \in C$ has an open neighbourhood N_z and a harmonic function h_z on N_z such that*

$$h_z \le u \quad \text{on } N_z, \qquad h_z(z) = u(z). \tag{11.11.1}$$

Then u is subharmonic.

Open proof obligation (RC2). Establish the comparison for the pulled-back function on the analytic curve; GRPSH 3.1(3) used.

The contact inequality is used on a whole neighbourhood. Convexity on individual cells would not give it at a branching point. The proof also tests every interior point of D; no restriction to skeleton vertices or to a chosen collection of outgoing directions has been made.

11.11.2 Relative tropical supports, including zeros

Here is the exact relative interpretation used below. For a subset $S \subset \mathbb{T}^r$, a finite continuous $F : S \to \mathbf{R}$ has relative affine lower supports if for every $t \in S$ there is a relative open neighbourhood $O_t \subset S$ and an expression $\ell_t(s) = c + \sum_i a_i s_i$, with real coefficients, such that every summand is finite on O_t, $F \ge \ell_t$ there, and $F(t) = \ell_t(t)$. This is the corrected touching convention of Example A.8.1, now explicitly on the relative image. We do not silently extend a definition stated only for ambient opens to arbitrary subsets.

Theorem 11.11.2 (Relative supports give classical psh functions)**.** *Let K be any complete non-Archimedean field and W a K-analytic space. Suppose locally on ordinary opens that*

$$u = F \circ \tau, \qquad \tau = (-\log|g_1|, \dots, -\log|g_r|), \qquad g_i \in \mathcal{O}(W),$$

where F is finite continuous on $\tau(W)$ and has relative affine lower supports in the convention just specified. Then u is classically psh in the sense of [GRPSH, Definition 4.1]. The functions g_i may vanish; no PL assumption on F is needed.

Open proof obligation (RC2). Prove the curvewise test in the specified relative setting; the proof uses GRPSH Proposition 3.14.

Historical proof-local display withheld. (11.11.2)

Corollary 11.11.3 (Decreasing limits in the curvewise class)**.** *A pointwise decreasing sequence of functions covered by Theorem 11.11.2, all defined on the same ordinary open U, has a classically psh limit $u : U \to \mathbf{R} \cup \{-\infty\}$. The presentations may vary with the sequence index.*

Open proof obligation (RC2). Prove the relevant limit property inside the curvewise psh class; the proof applies GRPSH Proposition 4.2(4).

11.11.3 Identifying PL functions on a unit chart

We now impose the nontrivial valuation needed for the strict model theory. Put $\Gamma = -\log|K^\times|$ and $\Gamma_{\mathbf{Q}} = \mathbf{Q}\Gamma$. A rational affine expression has rational slopes and its constant in $\Gamma_{\mathbf{Q}}$. This coefficient condition is essential for the strict analytic domains in the proof.

Lemma 11.11.4 (Selection on a finite polyhedral image)**.** *Let $S \subset \mathbf{R}^r$ be the support of a finite rational $\Gamma_{\mathbf{Q}}$-polytopal complex. Let $F : S \to \mathbf{R}$ be continuous and satisfy $F(t) \in \{L_1(t), \dots, L_m(t)\}$ for finitely many rational affine expressions L_a. There is a finite rational $\Gamma_{\mathbf{Q}}$-polytopal subdivision of S on each of whose closed cells F equals one L_a.*

Proof. Subdivide every cell by all hyperplanes $L_a = L_b$. On the relative interior σ° of any resulting cell, two branches either agree identically or are everywhere distinct, and their order is fixed. At a point of σ°, distinct values have a positive separation, so continuity of F makes the chosen equivalence class of branches locally constant. The set σ° is connected; hence one branch is selected throughout it. Continuity extends the identity to σ. Finiteness and rationality follow from the finite hyperplane arrangement. This argument uses connected polyhedral cells, not connected fibres of the tropicalization. □

Proposition 11.11.5 (Relative unit-chart PL functions are model-compatible)**.** *Let X be a good strictly K-analytic space of pure dimension, with K complete and nontrivially valued. Suppose that, locally on ordinary opens W, a finite continuous function u has a presentation $u = F \circ \tau$, where $\tau = (-\log|g_1|, \dots, -\log|g_r|)$ for analytic units g_i on W, and F is locally a continuous finite selection of rational affine expressions on the relative image $\tau(W)$. Then u is Q-PL in the sense of [GJR, Definition 5.1]. No convexity is assumed.*

Open proof obligation (RC2). Prove the polyhedral image and analytic sheaf gluing required by the local presentation; GJR section 3.2 and 5.2(4).

11.11.4 Nef residues and the fixed-domain consequence

Theorem 11.11.6 (The relative-support model criterion)**.** *Let X be good and strictly analytic over a complete nontrivially valued K. Let u be an R-PL function in the GJR sense. If u has the relative-support presentations of Theorem 11.11.2, then its residue line bundle $L_u(x)$ is nef at every $x \in X$. In particular, on a pure-dimensional X, it suffices that u have both a unit-chart rational PL presentation as in Proposition 11.11.5 and a relative-support presentation. These two presentations need not use the same tropicalization.*

Open proof obligation (RC2). Prove the model-nef comparison and the residual criterion; GRPSH Theorem 4.3 and GRPL Proposition 5.1/Corollary 5.4.

There is a version-numbering discrepancy worth recording for citation. The proof of [GRPSH, Theorem 4.3] in arXiv:2506.13548v2 refers to Proposition 3.1 and Corollary 3.5 of its companion paper. In the fixed arXiv:2503.05513v1 consulted here, the corresponding statements are Proposition 5.1 and Corollary 5.4. We use the latter locators for that file; the discrepancy is bibliographic and does not invalidate the comparison theorem.

Corollary 11.11.7 (Relative generators on one final neighbourhood)**.** *Let X be integral projective of positive dimension over a complete nontrivially valued K, and let $U \subset X^{\mathrm{an}}$ be ordinary open. Suppose $u_j : U \to \mathbf{R}$ decrease pointwise to u, each u_j has the relative-support presentations above, and each u_j is Q-PL in the GJR sense. For every $x \in U$ there are one ordinary open neighbourhood $V \subset U$ and original smooth psh functions b_j on V decreasing to $u|_V$. It is enough to verify the Q-PL hypothesis by the local unit-chart presentations of Proposition 11.11.5. If u is finite continuous, convergence is uniform on compact subsets of V.*

Proof. Theorem 11.11.6 gives nef residues for every u_j. Apply Theorem 11.10.11. In that theorem the barrier and final neighbourhood are fixed before the sequence terms are extended; their bounds and scaling factors may depend on j. Consequently no intersection of countably many chart neighbourhoods is taken here. The last assertion is Dini's theorem on each compact subset. □

Example 11.11.8 (Cellwise affine does not suffice)**.** Choose $a \in K$ with $0 < |a| < 1$, and on the open annulus $|a| < |T| < |a|^{-1}$ put $u = -|\log|T||$. This function is Q-PL and affine on each of the two skeletal intervals separated by $|T| = 1$, but it is not subharmonic. Indeed, choose $r \in |K^\times|^{\mathbf{Q}}$ with $|a| < r < 1$. On the strict closed annulus $r \le |T| \le r^{-1}$, the constant harmonic function $H = \log r < 0$ agrees with u at both boundary points, whereas $u = 0 > H$ at the unit-radius Gauss point. Thus harmonic comparison fails. Equivalently, a touching affine lower support at the central tropical point would require a slope simultaneously at least 1 and at most -1.

Example 11.11.9 (A monomial inequality at a common zero)**.** In the closed unit bidisc $B = \mathcal{M}(K\langle X, Y\rangle)$ over a complete nontrivially valued field, put $D = \{|X| \le |Y|\}$. There is no rational domain R of B with $0 \in R \subset D$. In particular D is not a finite union of rational domains of B. This explains why the Laurent-domain step in Proposition 11.11.5 cannot simply be repeated with vanishing coordinates.

Proof. Write a rational domain containing 0 in the form $|f_i| \le r_i|g|$, where $r_i > 0$ and $g, f_1, \dots, f_m$ generate the unit ideal. Necessarily $g(0) \neq 0$, since otherwise all $f_i(0)$ would vanish. An analytic function nonzero at the K-rational point 0 has constant absolute value on some ordinary neighbourhood of 0: use $|f - f(0)| < |f(0)|$. For the finitely many f_i vanishing at 0, continuity makes $|f_i| < r_i|g(0)|$ after shrinking. Hence every such rational domain contains an ordinary neighbourhood of 0. But, for any $a \in K$ with $0 < |a| < 1$, the points $(a^n, 0)$ converge to 0 and lie outside D. This proves the claim. □

11.11.5 Two local proof repairs in tropical regularization

The reverse approximation direction leads back to [APW26, Proposition 5.21]. We record two elementary repairs in its surrounding convexity and localization arguments. They concern the stated auxiliary claim and a displayed zero set, not counterexamples to the intended regularization theorem.

Proposition 11.11.10 (The missing hypothesis in the interval claim)**.** *A continuous $g : [0, 1] \to \mathbf{R}$ with a touching affine lower support at every interior point is convex. If an affine ℓ also touches g from below on $[0, \epsilon)$ at 0, then $g \ge \ell$ on all of $[0, 1]$. The contact condition at 0 alone does not imply this conclusion.*

Proof. On any interval $[a,b] \subset [0,1]$, let H be the affine interpolant of the endpoint values. If $g-H$ had a positive maximum M, its maximum set would be a nonempty compact subset of (a,b). At a maximum point, subtracting H from a touching affine support gives an affine function with an interior local maximum, hence a constant. As in Lemma 11.11.1, the maximum set is therefore open as well as closed in $[a,b]$, a contradiction. Thus $g \le H$, which proves convexity. If $g(t) < \ell(t)$ for some $t > 0$, convexity gives, for $0 < s < \min(t, \epsilon)$,

$$g(s) \le (1 - s/t)g(0) + (s/t)g(t) < (1 - s/t)\ell(0) + (s/t)\ell(t) = \ell(s),$$

contradicting the lower support near 0. Without the interior hypothesis, $g(t) = t - 2t^2$ and $\ell = 0$ give a counterexample: $g \ge 0$ near 0, but $g(1) = -1$. □

The auxiliary interval claim printed in the proof of [APW26, Proposition 3.3], p. 23, states only the condition at 0. Its subsequent proof uses a support at an interior maximum point. The proposition's actual local-convexity hypothesis supplies exactly that missing condition, so the corrected claim above repairs this inference.

Lemma 11.11.11 (The strict-support zero set)**.** *Let Q be a compact Hausdorff space, $x \in Q$, and let A, B be continuous nonnegative functions with $A^{-1}(0) = \{x\}$ and $B(x) = 0$. For positive decreasing sequences $C_j, \delta_j \to 0$, put*

$$V_j = \{A < C_j B + \delta_j\} \subset Q.$$

Then $x \in V_j$, the closures are nested, and

$$\bigcap_j \overline{V_j} = A^{-1}(0) = \{x\}. \tag{11.11.3}$$

For every neighbourhood O of x, eventually $\overline{V_j} \subset O$.

Proof. Positivity of δ_j gives $x \in V_j$, and nonnegativity of B makes V_j decreasing. Continuity and the uniform bound give

$$\overline{V_j} \subset \{A \le C_j B + \delta_j\}, \qquad \|C_j B + \delta_j\|_Q \le C_j \|B\|_Q + \delta_j \longrightarrow 0.$$

Thus their intersection lies in $A^{-1}(0)$ and contains x. If the final assertion failed, the nonempty nested compact sets $\overline{V_j} \cap (Q \setminus O)$ would have a common point, contradicting (11.11.3). □

In [APW26, Proposition 5.10], put $g = \varphi + h$, let ℓ_g be its strict local affine support at x, and let ℓ_f support f. Shrink to a compact neighbourhood on which both support inequalities hold, and take

$$A = g - \ell_g, \qquad B = f - \ell_f.$$

The strict support gives $A^{-1}(0) = \{x\}$. Thus the displayed intersection in the proof on p. 37 must use $\{g - \ell_g \leq 0\}$, not $\{f - \ell_f \leq 0\}$. The latter can be the whole neighbourhood when $f = \ell_f$. The shrinking argument is valid with the corrected zero set. In addition, the maximum modification has the exact pointwise increment

$$\psi - \varphi = \max\{0, CB + \delta - A\}, \qquad 0 \leq \psi - \varphi \leq C\|B\|_Q + \delta. \tag{11.11.4}$$

Taking the right side at most ϵ and the modification set compactly inside the support neighbourhood makes the function equal to φ near its boundary. Hence it glues to φ outside. On $V = \{A < CB + \delta\}$ one obtains $\psi + h = Cf + \ell$, where $\ell = \delta + \ell_g - C\ell_f$. Preservation of the tropical psh class under this maximum and gluing is the separate cited input [APW26, Lemma 4.13]. We have repaired its localization calculation, not re-proved the entire tropical regularization theorem.

Remark 11.11.12 (The coefficient issue in local PL density). In [APW26, Sections 2.1.1, 2.1.4–2.1.5], rationality of a polyhedron constrains the normals of its defining half-spaces, while their constants may be real; the affine functions used for PL functions are real affine. Proposition 5.21 is stated for this PL class. It does not state that its approximants have rational slopes and constants in the value group of an independently chosen analytic field. This extra conclusion cannot be inferred from the word "rational" attached to the polyhedral structure.

For example, the real convex PL function $F(t) = \max(1 - \pi t, 0)$ is allowed in that polyhedral category. Over a field whose fixed additive value group is $\mathbf{Q}$, its pullback $\max(1 + \pi \log|T|, 0)$ on $\mathbf{G}_m^{\mathrm{an}}$ is not R-PL in the strict GJR category: the bend at $t = 1/\pi$ cannot occur in a finite strict polyhedral presentation. This is the example of [GJR, Example 5.3], applied to the present comparison. It does not disprove the existence of better approximants. It shows that a coefficient-preserving approximation theorem, or an explicit replacement argument, is required before using APW local density as a Q-PL/model input.

There is nevertheless a simple replacement on a bounded unit chart when an ambient convex maximum is already available. This distinguishes a coefficient issue from the separate problem of extending a relatively convex function off its tropical image.

Proposition 11.11.13 (Coefficient-controlled approximation on a bounded chart)**.** *Let K be complete and nontrivially valued, with $\Gamma = -\log|K^\times|$. Let $S \subset \mathbf{R}^r$ be bounded and*

$$F(t) = \max_{1\le a\le m}\{c_a + \langle v_a, t\rangle\}, \qquad c_a \in \mathbf{R}, \quad v_a \in \mathbf{R}^r.$$

There are convex maxima H_j of affine functions with slopes in $\mathbf{Q}^r$ and constants in $\Gamma_{\mathbf{Q}}$ such that $H_j|_S$ decreases uniformly to $F|_S$. Consequently, on a good strict analytic unit chart with bounded tropical image, the pullback of F has a decreasing uniform approximation by GJR Q-PL functions having relative affine lower supports. No change of the absolute value or extension of K is required.

Open proof obligation (RC2). Prove the analytic Q-PL assertion for the finite maxima. The coefficient approximation is in the withheld proof; the non-Stacks use is GJR 5.2(7).

The monotonicity is asserted on S, not on all of $\mathbf{R}^r$. Every point of a unit chart has a neighbourhood with bounded tropical image. Thus the proposition resolves the local coefficient problem for finite ambient convex maxima, including the irrational-bend example above. It does not supply such a maximum for an arbitrary relatively convex tropical function; Example 11.10.9 prevents that shortcut on a fixed chart.

11.11.6 What remains in the source comparison

Under the stated relative interpretation, Wu's convexity condition gives classical psh, and known Q-PL membership then gives nef residues. The unit-chart membership is proved above. With vanishing coordinates, the ambient connected-fibre proof of Lemma 11.10.1 does not apply directly. Section 11.12 supplies the missing strict G-local unit-log presentation using finite polyhedral strata, with explicit common-branch and coefficient hypotheses.

Proposition 11.22.14 proves the reverse original-smooth-to-PL comparison for a single input in the stated projective setting. The arbitrary-family comparison and first-stage decreasing closure remain open; Corollary 11.11.3

proves only classical closure. Full Wu Theorem A.3, CLD25 Proposition 8.1.11 and arbitrary-current mixed products are not established here. The primary text of mixed-calculus Theorem 6.13 remains unavailable and is not used.

11.12 Finite selections at the extended tropical boundary

The remaining PL identification in Section 11.11.6 can be settled for a precise relative finite-selection class. The missing ingredient is finiteness of the polyhedral strata of an analytic tropical image. It permits a function to forget the coordinates tending to infinity, even though the fibres of the projection need not be connected. We first prove that assertion as a statement about polyhedra, and then identify its analytic hypotheses.

11.12.1 A recession argument

A closed polyhedron is an intersection of finitely many closed affine half-spaces in a finite-dimensional real vector space; the empty set is allowed. For $P = \{v : Av \le b\} \neq \varnothing$, its recession cone is $\operatorname{rec}(P) = \{w : Aw \le 0\}$. Thus $v + \lambda w \in P$ for every $v \in P$, $w \in \operatorname{rec}(P)$ and $\lambda \ge 0$.

Lemma 11.12.1 (Simultaneous escape)**.** *Let $P \subset \mathbf{R}^s \times \mathbf{R}^d$ be a nonempty closed polyhedron, where $s > 0$, and suppose that its projection to the second factor is bounded. Exactly one of the following alternatives holds:*

1. *There is $(w, 0) \in \operatorname{rec}(P)$ with $w_i > 0$ for every i.*
2. *There is $M \in \mathbf{R}$ such that $\min_i t_i \le M$ for every $(t, y) \in P$.*

Proof. We use the elementary finite-dimensional Farkas alternative in the following form: if a linear functional c is nonpositive on $\{w : Aw \le 0\}$, then $c = \lambda^t A$ for some column vector $\lambda \ge 0$. Here is a brief justification. The cone generated by the rows of A is closed: in each nonnegative representation, a linear dependence can eliminate one positive coefficient, until its support is linearly independent. There are finitely many such supports; on a fixed independent support convergence of the represented vectors forces convergence of the coefficients. If c were outside this closed convex cone, separation would give w with $Aw \le 0 < c(w)$, a contradiction.

It follows that a linear functional unbounded above on a nonempty polyhedron $Q = \{v : Av \le b\}$ is positive on some recession vector: otherwise $c = \lambda^t A$ gives $c(v) \le \lambda^t b$ on Q. Apply this observation to

$$Q = \{(t, y, z) : (t, y) \in P,\ z \le t_i \text{ for all } i\}.$$

If alternative (2) fails, z is unbounded above on Q. Hence Q has a recession vector (w, v, a) with $a > 0$. The inequalities give $w_i \ge a$. Boundedness of the y-projection forces $v = 0$, proving (1). Conversely, a ray with every $w_i > 0$ makes $\min_i t_i$ unbounded. □

Theorem 11.12.2 (Factorization on a polyhedral tail)**.** *Let $B \subset \mathbf{R}^d$ be bounded and let $S \subset [R_0, +\infty]^s \times B$ be compact, with $s > 0$. For each $I \subset \{1, \dots, s\}$, assume that the stratum*

$$S_I = \{(t, y) \in S : t_i = +\infty \iff i \in I\}$$

is, after deleting the infinite coordinates, a finite union of closed polyhedra. Let $F : S \to \mathbf{R}$ be continuous and suppose that there are finitely many functions $L_1, \dots, L_m : B \to \mathbf{R}$ such that $F(t, y) \in \{L_1(y), \dots, L_m(y)\}$ everywhere. Then there exists R such that, for every $(t, y) \in S$ with all $t_i \ge R$,

$$(+\infty, \dots, +\infty, y) \in S, \qquad F(t, y) = F(+\infty, \dots, +\infty, y). \tag{11.12.1}$$

The threshold depends on S and its finite polyhedral presentations, not on F or on the functions L_a.

Proof. For a polyhedron P in a stratum with at least one finite t-coordinate, apply Lemma 11.12.1 to those coordinates. If its second alternative holds, choose M_P as there. Finitely many strata and polyhedra occur, so choose R greater than all these bounds and at least R_0; if there are no such bounds, any $R \ge R_0$ suffices.

A point (t, y) in the stated tail either already has all $t_i = +\infty$, or lies in a polyhedron of the first type. In the latter case choose its recession vector $(w, 0)$. The ray obtained by increasing the finite t-coordinates by λw, and leaving the others infinite, lies in S for every $\lambda \ge 0$. Along this connected ray, F takes values in the fixed finite subset $\{L_a(y)\}_a$ of $\mathbf{R}$. Continuity therefore makes F constant on the ray. Its limit in the ambient extended space is $(+\infty, \dots, +\infty, y)$; compactness of S places that point in S, and continuity gives (11.12.1). □

No connectedness of a whole projection fibre was used. The proof instead connects each point in a sufficiently deep tail to the same boundary point in its fibre. This distinction is important for relative images.

Example 11.12.3 (Why finite polyhedral structure is needed)**.** In $[1, +\infty] \times [0, 1]$, let

$$S = \{(1/y, y) : 0 < y \leq 1\} \cup \{(2/y, y) : 0 < y \leq 1\} \cup \{(+\infty, 0)\}.$$

Set $F = y$ on the first curve, $F = -y$ on the second, and $F(+\infty, 0) = 0$. Both compactified curves are continuous images of $[0, 1]$, so S is compact. They meet only at their common endpoint, where both values tend to zero; hence F is continuous and selects from the two affine functions $y, -y$. It also has relative affine lower supports: use the active branch away from the endpoint, and $-y$ at the endpoint. For every R, small positive y gives two points in the tail with the same y and opposite values. Thus F does not factor through y on any tail. The finite-coordinate stratum is not a finite union of polyhedra. This is an abstract counterexample to dropping that hypothesis, not an asserted analytic tropical image.

11.12.2 The analytic image input and zero strata

We use the following named input from Martin: for a compact strictly K-analytic space over a complete nontrivially valued field, the positive-coordinate image of finitely many analytic functions is a finite polyhedral set. In logarithmic coordinates its normals are rational and its constants lie in $\Gamma_{\mathbf{Q}}$, where $\Gamma = -\log|K^\times|$. See [Mar13, Theorem 4.3.1, pp. 172–173; conventions in Section 4.1.1]. The published strict version is [Mar15]. We checked the thesis proof; the published article was available only through its abstract. No broader non-strict or trivially valued formulation is used.

Lemma 11.12.4 (Stratified polyhedral analytic images)**.** *Let V be strictly K-affinoid, with K complete and nontrivially valued. Let $g_1, \ldots, g_s$ be analytic functions and $h_1, \ldots, h_d$ analytic units on V. The image*

$$S = \{((-\log|g_i(z)|)_{i=1}^s, (-\log|h_j(z)|)_{j=1}^d) : z \in V\}$$

satisfies the compactness, boundedness and finite-stratum hypotheses of Theorem 11.12.2, for some R_0 and bounded box B.

Proof. Compactness follows from continuity and compactness of V. The functions g_i are bounded, so their negative logarithms have a common lower bound. Both h_j and h_j^{-1} are bounded, so the last coordinates lie in a bounded box. Fix $I \subset \{1, \dots, s\}$ and take the closed analytic subspace $Z_I \subset V$ defined by $g_i = 0$ for $i \in I$. It is strictly affinoid. Apply Martin's theorem on Z_I to the remaining g_i, together with all h_j, and take its positive-coordinate image. Positivity excludes exactly the additional zeros $g_i = 0$ for $i \notin I$. After taking negative logarithms this is precisely S_I with its infinite coordinates deleted, not its closure across other strata. Martin's finite union of closed polyhedra gives the assertion; empty strata cause no difficulty. If no coordinates remain, the stratum is empty or a single point, and the conclusion is immediate. □

11.12.3 From relative finite selections to model PL functions

Specify the relative convention before using it. On a relative neighbourhood O of a point in a tropical image, a rational finite selection is a continuous real-valued function selecting from one finite list

$$\ell_a(t) = c_a + \sum_i q_{ai} t_i, \qquad c_a \in \Gamma_{\mathbf{Q}}, \quad q_{ai} \in \mathbf{Q}, \qquad a = 1, \dots, m. \tag{11.12.2}$$

Every individual summand of every branch must be finite everywhere on O, with $0 \cdot (+\infty) = 0$. In particular, a branch cannot be defined only on the subset where it is selected. This is a substantive hypothesis.

Theorem 11.12.5 (Relative rational PL identification through zeros)**.** *Let X be a pure-dimensional good strictly K-analytic space, where K is complete and nontrivially valued. Suppose a function u is locally, on ordinary opens, the pullback under the negative logarithms of finitely many analytic functions of a relative rational finite selection in the convention* (11.12.2)*. Then u is $\mathbf{Q}$-PL in the strict GJR sense. More precisely, near each point it has such a finite-selection presentation using only analytic units. No convexity assumption is required.*

Proof. Fix x. Shrink to an ordinary neighbourhood on whose tropical image one finite list works. Reorder the coordinates as $g_1, \dots, g_s$, vanishing at x, and $h_1, \dots, h_d$, nonvanishing at x. Shrink again so that the latter are units. If $s = 0$, Proposition 11.11.5 applies directly. Otherwise finiteness of each summand at the image of x forces the coefficient of each g_i to vanish in every branch. Thus all branches have the form $L_a(y)$, where $y = (-\log |h_j|)_j$.

Choose a compact strict affinoid neighbourhood V of x inside this open, and put $S = \tau(V)$. Lemma 11.12.4 and Theorem 11.12.2 give a threshold R. Enlarge it to an element of $\Gamma_{\mathbf{Q}}$; this is possible since Γ contains a positive element. Then

$$V_R = \{z \in V : |g_i(z)| \le e^{-R} \text{ for all } i\}$$

is a strict affinoid domain containing an ordinary neighbourhood of x. Indeed, choose $N > 0$ and $a \in K^\times$ with $NR = -\log|a|$; its inequalities are $|g_i^N/a| \le 1$. At x they are strict, and x lies in the ordinary interior of V.

Define the boundary slice and its restricted function by

$$S_\infty = \{y : (+\infty, \dots, +\infty, y) \in S\},$$
$$F_\infty(y) = F(+\infty, \dots, +\infty, y).$$

The function F_∞ is continuous on the compact set S_∞. The tail theorem gives $y(V_R) \subset S_\infty$ and $u|_{V_R} = F_\infty \circ y$. It still selects from the same rational branches L_a. Restrict to the ordinary interior of V_R and apply Proposition 11.11.5. Varying x proves the Q-PL assertion by sheafiness. □

This proof avoids subdividing inequalities between vanishing monomials into rational domains at their common zero, the invalid shortcut excluded by Example 11.11.9. Only inequalities against a nonzero constant are used to select V_R; the subsequent Laurent subdivision involves units. The argument combines Martin's image theorem with an elementary recession calculation and the earlier unit-chart identification; no priority claim is made for that combination.

Corollary 11.12.6 (Relative generators in the fixed-domain transfer)**.** *Let X be an integral projective variety of positive dimension over a complete nontrivially valued field K. On one ordinary open $U \subset X^{\mathrm{an}}$, let u_j be a decreasing sequence of functions satisfying Theorem 11.12.5, and assume each has local relative affine lower supports as in Theorem 11.11.2. The presentations and their charts may depend on j. For every $x \in U$, there is one ordinary neighbourhood $V \subset U$ and a decreasing sequence of original CLD smooth psh functions b_j on V with $b_j \downarrow u = \inf_j u_j$. If u is finite continuous, this convergence is uniform on compact subsets of V.*

Proof. Theorem 11.12.5 gives Q-PL membership for each term on U. Theorem 11.11.6 gives nef residues, using the classical psh criterion and its cited

comparison theorem. Now apply Theorem 11.10.11. The varying neighbourhoods used to identify individual terms are not intersected: they establish a sheaf property on U before the fixed projective barrier is chosen. Its termwise scaling constants remain free to vary. Dini's theorem gives the final assertion. □

Remark 11.12.7 (One tube for a family of finite selections). The unit-reduction part of the proof does not require rational coefficients. On a fixed compact image S as above, every continuous finite selection from branches depending only on y factors on the same tail, since the threshold in Theorem 11.12.2 is independent of those branches. The finite lists may differ for different members of a family. The resulting boundary restriction preserves pointwise order, maxima and minima. This gives a uniform domain when the image and the vanishing-coordinate set are already fixed. It gives no uniform domain for an arbitrary family of different analytic charts, and real coefficients do not by themselves give strict GJR PL membership.

Example 11.12.8 (Rational constants and a fixed valuation)**.** Equip $K = \mathbf{Q}_p$ with the absolute value $|a| = \exp(-\sqrt{2}\, v_p(a))$. This complete nontrivially valued field has $\Gamma_{\mathbf{Q}} = \sqrt{2}\,\mathbf{Q}$. The constant function 1 on the analytic point $\mathcal{M}(K)$ is not Q-PL in the strict unit-log sense: such a presentation would give $N = \log|b| \in \sqrt{2}\,\mathbf{Z}$ for some positive integer N and $b \in K^{\times}$, which is impossible. The same obstruction holds at a K-rational point of an analytic space, by evaluating the unit-log presentation there. Nevertheless 1 is an original smooth pluriharmonic function. Thus this example separates coefficient categories; it is not a counterexample to smooth approximation or to psh comparison. No absolute value is changed during either argument.

11.12.4 Coefficient control for ambient convex functions

The remaining relative approximation problem should be distinguished from a coefficient problem that can be solved directly. Proposition 11.11.13 treats finite ambient maxima. Supporting hyperplanes extend that argument to every finite convex function on an ambient convex open set.

Proposition 11.12.9 (Decreasing rational approximation of ambient convex functions)**.** *Let $\Omega \subset \mathbf{R}^d$ be convex and open, let $F : \Omega \to \mathbf{R}$ be finite continuous and convex, and let $S \subset \Omega$ be nonempty compact. For every nonzero additive subgroup $\Gamma \subset \mathbf{R}$, there are finite maxima H_j of affine functions with rational slopes and constants in $\Gamma_{\mathbf{Q}}$ whose restrictions to S decrease uniformly to $F|_S$. Each H_j is defined and convex on all of $\mathbf{R}^d$, but monotonicity is asserted only on S.*

Proof. At each $x \in \Omega$ a supporting affine function ℓ_x satisfies $\ell_x \le F$ on Ω and $\ell_x(x) = F(x)$. To recall the input, separate the open convex epigraph $\{(t, r) : t \in \Omega,\ r > F(t)\}$ from its boundary point $(x, F(x))$. The separating functional has nonnegative last coefficient; that coefficient cannot be zero because x is interior to Ω. Division by its positive last coefficient gives ℓ_x.

Choose $\rho \in \Gamma$ positive and $\epsilon_j = \rho 8^{-j}$. For fixed j, continuity and compactness give finitely many supports with maximum M_j satisfying $F - \epsilon_j \le M_j \le F$ on S: the relative open sets $\{F - \ell_x < \epsilon_j\} \cap S$, for $x \in S$, cover S. Since S is bounded, approximate each selected slope by a rational vector and each constant by an element of $\rho\mathbf{Q}$ so that the resulting affine function differs by at most ϵ_j on S. Let G_j be the maximum of these rational affine functions. Then

$$F - 2\epsilon_j \le G_j \le F + \epsilon_j \quad \text{on } S.$$

Set $H_j = G_j + 3\epsilon_j$. The shift is still in $\Gamma_{\mathbf{Q}}$, and

$$F + \epsilon_j \le H_j \le F + 4\epsilon_j, \qquad H_{j+1} \le F + \tfrac{1}{2}\epsilon_j \le H_j \quad \text{on } S.$$

This proves both assertions. It is not necessary to choose nested sets of supporting hyperplanes: the positive shifts enforce monotonicity on the prescribed compact set. □

Corollary 11.12.10 (A coefficient-compatible local analytic class)**.** *On a pure-dimensional good strictly analytic space over a complete nontrivially valued K, suppose locally that $u = F \circ \tau$, where τ is given by analytic units and F is finite continuous convex on a convex ambient open containing the image. Near every point, u is a uniform decreasing limit of finite rational maxima of unit logarithms, with constants in $\Gamma_{\mathbf{Q}}$. These approximants are Q-PL and have relative affine lower supports. For an integral projective X of positive dimension, Corollary 11.12.6 consequently gives a decreasing original smooth psh approximation on one neighbourhood.*

Open proof obligation (RC2). Prove the Q-PL analytic presentation and stability under the maximum operation; the proof takes GJR 5.2(7) as input.

The supporting hyperplanes in this proof lie below F on the entire ambient convex open. Relative local supports on a polyhedral union need not have that property. Thus the proposition cannot be used to fill the remaining general relative-density step by replacing the ambient set with a tropical image.

11.12.5 A maximum statement that must retain PL membership

The coefficient distinction also corrects a literal closure assertion in the consulted GJR v3, also present in the published 2026 version, pp. 191–192. Remark 7.11 suggests that the maximum of two semipositive R-PL functions is again semipositive R-PL, and its proposed argument treats maxima of R-PL LFS functions as R-PL LFS. Example 5.3 of that same version prevents this without an additional membership hypothesis. The later [GRPL, Corollary 5.5] explicitly restricts the unconditional statement to $\Lambda = \mathbf{Z}$ or $\mathbf{Q}$.

Proposition 11.12.11 (Maximum with the correct coefficient scope)**.** *Let X be a good strictly analytic space over a complete nontrivially valued field. If h_1, h_2 are semipositive R-PL functions and $h = \max(h_1, h_2)$ is R-PL, then h is semipositive. For Q-PL or integral PL inputs, membership of the maximum is automatic and the conclusion holds in the same coefficient class. For arbitrary R-PL inputs, that membership can fail even when both inputs are harmonic.*

Open proof obligation (RC2). Prove the applicable positivity and closure implications for the specified maximum; the proof uses GRPSH, GJR and GRPL.

The obstruction also occurs in the proper ambient setting of GJR Remark 7.11, which asks for semipositivity at a point. On $\mathbf{P}^{1,\mathrm{an}}$ extend $q = \min\{0, \max\{-1, \log|T|\}\}$ by its constant values near 0 and ∞. This is integral PL when $\Gamma = \mathbf{Q}$. Then $1+\pi q$ and 0 are global R-PL functions, harmonic near the point of the standard skeleton with $\log|T| = -1/\pi$. Their maximum has the same irrational bend there and is not R-PL near that point. Thus properness of the ambient curve does not repair the omitted local membership hypothesis.

The other expectation in GJR Remark 7.11, concerning pointwise limits, has a different status. It is now supplied by [GRPL, Theorem 5.6]: a pointwise limit of semipositive R-PL functions is semipositive when the limit is itself R-PL. That final membership hypothesis must also be retained. It neither asserts that all such limits are PL nor proves first-stage original-smooth decreasing closure.

11.12.6 Gaussian regularization at a boundary stratum

The projective tropical localization used earlier rests on a Gaussian regularization of a finite convex conewise-linear function. The calculation at infinity in [APW26, Proposition 5.7, pp. 34–35] can be made independent of the chosen approach sequence. We give the elementary calculation, including the growth hypothesis and the direction of monotonicity.

Proposition 11.12.12 (Gaussian boundary limit for a finite maximum)**.** *Let E be a finite-dimensional Euclidean space, let $v_1, \dots, v_s$ be linearly independent with $s \geq 1$, and put $\delta = \sum_i \mathbf{R}_{\geq 0} v_i$, $T = \operatorname{span}(\delta)$ and $Q = T^{\perp}$. Let $p : E \to Q$ be orthogonal projection. Suppose*

$$h(x) = \max_{1 \leq a \leq m} \langle m_a, x \rangle \geq 0, \qquad h|_{\delta} = 0.$$

Set $A_\delta = \{a : \langle m_a, v_i \rangle = 0 \text{ for every } i\}$, and $h^\delta(z) = \max_{a \in A_\delta} \langle m_a, z \rangle$ for $z \in Q$. This set of active indices is nonempty. If ρ_E and ρ_Q are the standard normalized Gaussian densities and $H = h * \rho_E$, then

$$\lim_{\min_i t_i \to +\infty} H\left(\sum_i t_i v_i + z\right) = \int_Q h^\delta(z - b)\rho_Q(b)\, db. \tag{11.12.3}$$

The convergence is uniform for z in compact subsets of Q. No monotonicity of the approaching parameters is assumed. Moreover $H(x + v) \leq H(x)$ for $v \in \delta$. If $h(-v) > 0$ for a nonzero $v \in \delta$, the inequality is strict.

Proof. Every $\langle m_a, v_i \rangle$ is nonpositive. Since $h(\sum_i v_i) = 0$, a maximizing index there has all these coefficients zero, proving that A_δ is nonempty. For $y = \sum_i t_i v_i$, every inactive linear form in $h(y+z-w)$ tends to $-\infty$ as $\min_i t_i \to \infty$; active forms annihilate T. Consequently, for each fixed z, w,

$$h(y + z - w) \longrightarrow h^\delta(z - p(w)).$$

Write $L = \max_a \|m_a\|$. The function h is L-Lipschitz and subadditive. For $t_i \geq 0$, its nonnegativity and $h(y) = 0$ give the integrable bound

$$0 \leq h(y + z - w) \leq h(z - w) \leq L(\|z\| + \|w\|).$$

Dominated convergence now gives the limit of the integral. Orthogonal decomposition $E = T \oplus Q$ factors the normalized Gaussian into the product of the two Gaussian densities; integration in T gives exactly (11.12.3). Both sides, as functions of z, have Lipschitz constant at most L. A finite net on a compact subset of Q upgrades pointwise convergence to uniform convergence. Equivalently, any failure of uniform convergence would give a sequence of parameters and a convergent sequence of points contradicting that bound and the already proved pointwise limit.

Finally, $h(x+v) \leq h(x)+h(v) = h(x)$, and integration gives monotonicity. If $h(-v) > 0$, the difference $h(x-w)-h(x+v-w)$ is a nonnegative continuous function of w and is positive at $w = x+v$: its value there is $h(-v)-h(0) > 0$. It is therefore positive on a nonempty open set of positive Gaussian measure. Its integral is strictly positive. This proves strictness without a separate strict-convexity argument. □

Corollary 11.12.13 (Strict affine contact across all faces)**.** *In Proposition 11.12.12, suppose in addition that $h^{-1}(0) = \delta$. The Gaussian convolution extends continuously to $[0, +\infty]^s \times Q$ by its quotient formulas on all infinity strata. At each point $(+\infty, \dots, +\infty, z_0)$ it admits a strict affine lower support depending only on z. Here strictness means equality only at that point, including comparison with points on partial infinity strata.*

Proof. For $I \subset \{1, \dots, s\}$ put $T_I = \operatorname{span}\{v_i : i \in I\}$ and $Q_I = T_I^{\perp}$, with orthogonal projection p_I. Retain only the linear forms annihilating T_I, and let their maximum on Q_I be h^I. The preceding proposition, applied to the face generated by the v_i for $i \in I$, gives the boundary value $H_I = h^I * \rho_{Q_I}$, evaluated at $p_I(\sum_{i \notin I} t_i v_i + z)$. For $I = \varnothing$ this is the original convolution.

We check the zero cone, since it controls strictness at partially infinite points. For $I \neq \varnothing$, put $v_I = \sum_{i \in I} v_i$. If $h^I(q) = 0$, all active linear forms at v_I have value at most zero on q. Every inactive form is strictly negative on v_I, so for sufficiently large a all forms have value at most zero on $q + av_I$. Since

$h \geq 0$ and $h^{-1}(0) = \delta$, this gives $q + av_I \in \delta$. Conversely, if $q \in Q_I \cap (\delta + T_I)$, every active form is nonpositive at q, and $h^I(q) = 0$ by nonnegativity. Thus

$$(h^I)^{-1}(0) = p_I(\delta) = \sum_{i \notin I} \mathbf{R}_{\geq 0} p_I(v_i).$$

The projected generators are linearly independent, because the original generators are. In particular this cone is pointed. The formula is immediate also for $I = \varnothing$. If $I \subset J$, repeating the active-form construction on Q_I gives precisely h^J; orthogonal Gaussian pushforwards compose because $p_J p_I = p_J$. Proposition 11.12.12, with its locally uniform assertion, therefore proves continuity on approaching any higher stratum. There are only finitely many strata, so this proves continuity of the assembled extension.

Let I contain all indices, so $Q_I = Q$ and $(h^I)^{-1}(0) = \{0\}$. Gaussian convolution of a Lipschitz function is smooth, by differentiation of the Gaussian under the integral. It is strictly convex here. Indeed, for each nonzero direction d, the restriction $h^I(td)$ has different one-sided slopes: their difference is $h^I(d) + h^I(-d) > 0$. A translate of any nontrivial segment in direction d can cross this bend. The convexity gap is then positive on an open set of translates, and integration against the positive Gaussian makes the gap strictly positive. Hence the tangent affine function ℓ to H_I at z_0 lies strictly below $H_I(z)$ for $z \neq z_0$. If $Q = 0$, take its unique constant affine function instead.

On a proper infinity stratum indexed by J, move in the nonzero direction $\sum_{i \notin J} p_J(v_i)$. Its negative is outside the pointed zero cone, so Proposition 11.12.12 gives strict decrease after any positive step. Continuing toward the full infinity stratum yields $H_J > H_I(z) \geq \ell(z)$. On the full infinity stratum equality occurs only at $z = z_0$. This proves the stated strict contact, including all partial strata. □

Remark 11.12.14 (What the corrected calculation supplies). In the application of APW Proposition 5.7, $h = g - \ell$ is a finite maximum of linear forms with zero set the strongly convex cone δ. Thus $h(-v) > 0$ for every nonzero $v \in \delta$, and finite points lie strictly above the boundary limit: first move by a positive multiple of $\sum_i v_i$, then pass to infinity using monotonicity. The normalized quotient Gaussian in (11.12.3) is the pushforward of the ambient Gaussian; an arbitrary nonorthogonal coordinate splitting must instead retain that pushforward density.

On p. 35 the source initially allows an arbitrary sequence tending to infinity and then calls the translated functions decreasing. Arbitrary approach sequences need not be monotone; the dominated-convergence proof above removes that restriction. The integrand before taking the limit must be $h(y_i + z - w)$, with $h^\delta(z - p(w))$ appearing after passage to the limit. Later on the same page, for $y_{i+1} \geq y_i$, the printed strict inequality has the wrong direction: for a nonzero increment it is $H(y_{i+1}+z) < H(y_i+z)$. If two terms coincide, only equality is available. These corrections preserve the intended finite-point-to-boundary inequality. Corollary 11.12.13 supplies the same calculation on every partial infinity stratum, with its pointed zero cone verified explicitly. This proposition supplies the analytic limit calculation, not a new proof here of every global tropical-geometric input of APW Proposition 5.7.

Example 11.12.15 (Why Gaussian convolution needs a growth hypothesis)**.** The finite-valued formulation of [APW26, Lemma 5.8] needs an integrability qualification for arbitrary convex functions. On $\mathbf{R}$, $g(t) = \exp(t^4)$ is convex, since $g''(t) = (12t^2 + 16t^6)\exp(t^4) \geq 0$, but $(g * \rho)(x) = +\infty$ for every x: the quartic term $(x - w)^4$ dominates $w^2/2$ in the exponent as $|w| \to \infty$. Thus arbitrary finite convex input does not give finite Gaussian convolution. For the globally Lipschitz finite PL input actually used in Proposition 5.7, the bound in the preceding proof gives finiteness, and convolution preserves convexity by integrating its defining inequality. The local proof repair does not contradict that intended application.

11.12.7 Rational approximation on a fixed rational fan

There is a relative case beyond ambient convex extension in which coefficient control can also be proved. The essential restrictions are that the input is already piecewise linear and that its fixed fan is rational. The argument is a finite-dimensional rational-polyhedral calculation, not a general regularization theorem for continuous relatively convex functions.

Proposition 11.12.16 (Coefficient density for relatively convex fan functions)**.** *Let Σ be a finite rational polyhedral fan in $\mathbf{R}^n$, let $S \subset |\Sigma|$ be nonempty compact, and let $F : |\Sigma| \to \mathbf{R}$ be continuous, homogeneous and linear on each cone. Suppose that F has a relative affine lower support near every point of $|\Sigma|$. For any nonzero additive subgroup $\Gamma \subset \mathbf{R}$, there exist functions H_j on $|\Sigma|$, affine with rational slopes and constants in $\Gamma_{\mathbf{Q}}$ on the same cones, which have relative affine lower supports everywhere and decrease uniformly to F*

on S. The monotonicity is asserted only on S. No ambient convex extension of F is assumed.

Proof. Choose ambient covectors q_σ with $F|_\sigma = q_\sigma|_\sigma$. For every face τ choose a point in its relative interior. A local affine lower support at that point has a slope a_τ satisfying

$$\begin{aligned} q_\sigma|_{\sigma\cap\sigma'} &= q_{\sigma'}|_{\sigma\cap\sigma'}, \\ a_\tau|_\tau &= q_\tau|_\tau, \\ (q_\sigma - a_\tau)|_\sigma &\geq 0 \qquad (\sigma \supseteq \tau). \end{aligned} \tag{11.12.4}$$

Indeed, comparison in both tangent directions inside τ identifies the slope on $\operatorname{span}(\tau)$, and the support's constant is then zero by homogeneity. For $x \in \operatorname{relint}(\tau)$ and $v \in \sigma \supseteq \tau$, the point $x+tv$ lies in σ for $t \geq 0$. Applying the local support inequality for small t gives the last line. Conversely, these constraints imply a relative lower support at every point of $\operatorname{relint}(\tau)$: cones not containing that point can be excluded by shrinking, and on every remaining cone the covector a_τ is a lower support with contact.

Because the fan is rational and finite, the displayed conditions are finitely many homogeneous rational linear equalities and inequalities in the coordinates of all q_σ, a_τ. For the inequalities it suffices to test a finite rational generating set of each cone, including generators of its lineality space with both signs if necessary. Thus the tuples form a rational polyhedral cone C in a finite-dimensional vector space. Rational points are dense in C. To see this at a particular tuple c, impose as equalities the rational inequalities active at c, together with the original equalities. Their nullspace has a rational basis. Approximate the coordinates of c in that basis by rational numbers. The inactive inequalities remain strict for sufficiently close approximations, and the active equalities are preserved exactly.

Choose $\rho \in \Gamma$ positive and set $\epsilon_j = \rho 8^{-j}$. Approximate the tuple of F inside C so closely that its rational covectors define a continuous homogeneous function G_j with $|G_j - F| \leq \epsilon_j$ on S. This is possible because S is bounded and there are finitely many cones. The constraints give relative affine lower supports for G_j. Set $H_j = G_j + 2\epsilon_j$. Then

$$F + \epsilon_j \leq H_j \leq F + 3\epsilon_j, \qquad H_{j+1} \leq F + \tfrac{3}{8}\epsilon_j \leq H_j \quad \text{on } S.$$

The constants $2\epsilon_j$ belong to $\Gamma_{\mathbf{Q}}$, and shifting the supports by the same constants preserves contact. This proves the assertion. □

This argument combines a finite support criterion with rational-point density in a rational polyhedral cone; no priority claim is attached to the combination. On a unit chart whose image is contained in $|\Sigma|$, take S to be the image of a compact strict affinoid neighbourhood. The pullbacks of H_j are Q-PL and have relative lower supports. Thus, in the projective setting of Corollary 11.12.6, they yield a decreasing original smooth psh approximation to $F \circ \tau$ near the point. This does not require a same-chart ambient convex extension.

For a nonhomogeneous complex with vertices involving arbitrary elements of $\Gamma_{\mathbf{Q}}$, the displayed system need not have rational coefficients, and its affine constants require separate control. The argument above does not treat that case. Nor does it show that a general continuous relatively convex function is approximable by functions linear on one fixed rational fan. These are distinct remaining questions.

Example 11.12.17 (A fixed irrational-length subdivision can obstruct density)**.** Let $a = \sqrt{2}$, and let S be the boundary of the rectangle with vertices $A = (0,0)$, $B = (1,0)$, $C = (1,a)$ and $D = (0,a)$, with its four-edge polyhedral complex. Take $\Gamma = \mathbf{Z} + \sqrt{2}\mathbf{Z}$. All directions are rational and all vertex coordinates belong to $\Gamma_{\mathbf{Q}}$. Define F by the vertex values

$$F(A) = 0, \qquad F(B) = F(C) = F(D) = a,$$

and extend linearly on each edge. This is a continuous relatively convex real PL function, but any continuous function G affine with rational slopes on these same four edges satisfies

$$\|G - F\|_{C^0(S)} \geq \frac{\sqrt{2}}{4}. \tag{11.12.5}$$

The obstruction persists if the affine constants of G are allowed to be arbitrary real numbers.

Proof. At an interior point of an edge, its affine expression is a relative support after shrinking. At a vertex the two adjacent edge directions are linearly independent, so there is an ambient affine function agreeing with F on both incident edges. This supplies a relative support near that vertex. Thus F has the stated relative convexity; no ambient convex extension is being asserted.

For G let $p_b, p_t \in \mathbf{Q}$ be the derivatives in the positive horizontal direction on the bottom and top edges, and $q_l, q_r \in \mathbf{Q}$ the derivatives in the positive

vertical direction on the left and right edges. Continuity around the rectangle gives

$$p_b + aq_r - p_t - aq_l = 0.$$

Since 1 and a are linearly independent over $\mathbf{Q}$, necessarily $p_b = p_t$ and $q_r = q_l$. In particular

$$G(A) + G(C) - G(B) - G(D) = 0.$$

For F the same expression equals $-a$. Comparing the four vertex values proves (11.12.5). This calculation uses only the slopes and continuity, so there is no restriction on the constants. □

In this example refinement does remove the obstruction. Choose rational $r \to a$ and rational $q_1, q_2 \to 1$ with $q_1 < r/a < q_2$. Put $G = rx$ on the bottom edge, $G = r$ on the top and right edges, and on the left edge set

$$G(0, y) = \max\{q_1 y,\ q_2 y + r - q_2 a\}, \qquad 0 \le y \le a.$$

The values at the four vertices match. The new breakpoint is $(q_2 a - r)/(q_2 - q_1) \in \Gamma_{\mathbf{Q}}$, and all slopes are rational, with constants in $\Gamma_{\mathbf{Q}}$. The left-edge slopes increase from q_1 to q_2, so relative convexity holds there; at the vertices affine supports again exist by independence of the incident directions. The uniform error is at most $\max\{|r - a|, a|q_1 - 1|, a|q_2 - 1|\}$ and tends to zero. Choosing these errors geometrically small and adding the positive shifts used in Proposition 11.12.16 also makes the approximation decreasing. Thus the example isolates the fixed-subdivision obstruction rather than an obstruction to approximation itself.

Taking $S \times [0, 1]^{d-1}$ and pulling back F gives the same obstruction in every positive dimension d, for the product complex: restrict any proposed approximant to $S \times \{0\}$. Relative supports pull back by projection. This is a counterexample in the category of finite polyhedral complexes with rational directions. The rectangle boundary is not asserted to be a balanced tropical variety or an analytic tropical image. The example does not rule out approximation after subdivision, changing charts, or using other analytic presentations, and does not refute the unrestricted regularization conclusion of APW. It explains why a proof that rationalizes coefficients while freezing an arbitrary $\Gamma_{\mathbf{Q}}$-subdivision is insufficient. The distinction is sharpened in Section 11.19.4: the obstruction survives each fixed value-group refinement, but Example 11.19.9 gives an explicit decreasing approximation with a moving breakpoint.

11.12.8 Scope of the completed direction

The relative PL interface is now established with the stated common-branch and coefficient conventions. Wu's Definition 3.1(1) uses rational constants, and his standing convention in Section 2.1 is $\mathbf{Z} \subseteq \Gamma_K$ when the valuation is nontrivial. Under that convention, these constants do lie in $\Gamma_{\mathbf{Q}}$. Our formulation retains arbitrary fixed nontrivial absolute values by stating the coefficient condition explicitly; Example 11.12.8 explains why one must not omit it when leaving Wu's normalization. It does not refute a claim under his standing convention. Definition 3.6 also uses images that need not be ambient open sets; our relative convention is explicit, rather than an attribution of an unstated convention to the source.

The direction obtained is from these relative rational convex generators to original smooth approximation on a fixed analytic neighbourhood. The reverse direction, coefficient-compatible approximation for general relatively convex functions, and the first-stage decreasing-closure problem remain separate. Neither full Wu Theorem A.3 nor CLD25 Proposition 8.1.11 follows from this section. No enlargement to an unrestricted positive-current Bedford–Taylor calculus, change of actual tests or support, or field-descent theorem is asserted.

11.13 Rational approximation on a fixed residue tube

The coefficient problem has a second, geometric form. A function already known to be R-PL has a finite presentation by integral PL functions. Their residue line bundles can be placed on one projective model. This permits rational perturbation after adding a small ample class, without rationalizing the vertices of a tropical subdivision. We carry out this argument with one neighbourhood for the whole finite-dimensional family.

Throughout this section K is complete and nontrivially valued, $\Gamma = -\log|K^\times|$, and $\Gamma_{\mathbf{Q}} = \Gamma \otimes_{\mathbf{Z}} \mathbf{Q} \subset \mathbf{R}$. PL, Q-PL and R-PL have the strict analytic meaning of [GJR, Section 5]. In particular an arbitrary relative real PL presentation is not assumed to be an R-PL presentation in this sense. The geometric input below is the residue-model construction in [GJR, Lemma 6.14 and proof of Proposition 7.10]; the coefficient estimate and its ordered-family application are written out here.

11.13.1 A finite numerical estimate

Lemma 11.13.1 (Rational perturbation with an ample margin)**.** *Let V be a projective variety over a field, A an ample line bundle, and $L_1, \dots, L_m$ line bundles. Choose positive integers N_i such that $N_iA + L_i$ and $N_iA - L_i$ are ample, using additive notation in $\mathrm{Pic}(V)_{\mathbf{R}}$. If $L(a) = \sum_i a_i L_i$ is nef, $\varepsilon > 0$, and $r_i \in \mathbf{R}$ satisfy*

$$D := \sum_{i=1}^{m} N_i |r_i - a_i| < \varepsilon, \tag{11.13.1}$$

then $\sum_i r_i L_i + \varepsilon A$ is ample. In particular r_i and ε may be chosen rational, with r arbitrarily close to a as ε tends to zero. No rational polyhedrality of the nef cone is required.

Open proof obligation (RC2). Prove the ample margin used in the construction; the proof uses the GJR Proposition 7.10 argument.

11.13.2 Realizing a family before choosing its coefficients

Lemma 11.13.2 (One residue tube for a finite PL presentation)**.** *Let X be good and strictly K-analytic, let $x \in \mathrm{Int}(X)$, and let $h_1, \dots, h_m$ be integral PL functions on a neighbourhood of x. After shrinking once, there are a compact strictly analytic neighbourhood W of x, an ordinary open neighbourhood $U \subset W$ of x, an integral PL function b on W, a projective variety $V/\widetilde{K}$, line bundles L_i on V, and an ample line bundle A on V with the following properties.*

1. *L_i and A model the residues $L_{h_i}(x)$ and $L_b(x)$. For every $a \in \mathbf{R}^m$, $\sum_i a_i h_i$ is semipositive at x if and only if $\sum_i a_i L_i$ is nef on V.*
2. *If $\sum_i r_i L_i + tA$ is semiample, then $\sum_i r_i h_i + tb$ is locally Fubini–Study on U. If that class is nef, the function is semipositive on U. Here r_i, t are real and locally Fubini–Study means semiampleness of the residue at every point.*

All these data depend on the finite list (h_i) and x, not on a, r, t.

Open proof obligation (RC2). Prove the simultaneous neighbourhood/tube with the required residual class control; the proof applies the listed GJR inputs.

11.13.3 A decreasing Q-PL approximation

Theorem 11.13.3 (Rational approximation of a semipositive R-PL germ)**.** *Let X be good and strictly K-analytic, $x \in \mathrm{Int}(X)$, and let h be R-PL and semipositive at x. There are one ordinary open neighbourhood U of x and Q-PL locally Fubini–Study functions q_j on U which decrease uniformly to $h|_U$. More precisely, for any fixed $\rho \in \Gamma_{>0}$, the construction can be made to satisfy*

$$h + \eta_j \le q_j \le h + 3\eta_j \quad \text{on } U, \qquad \eta_j = \rho\, 8^{-j} \quad (j \ge 1). \tag{11.13.2}$$

The approximants need not be harmonic when h is harmonic. The claim is an explicit rational and ordered refinement of the construction of [GJR, Proposition 7.10], not an assertion that that proposition is false.

Proof. Write $h = \sum_i a_i h_i$ on one neighbourhood, with h_i integral PL, and apply Lemma 11.13.2. All h_i and b are continuous on the compact outer neighbourhood W, so put $M_i = \|h_i\|_W$ and $M_b = \|b\|_W$. Choose the integers N_i in Lemma 11.13.1. For each j choose $\varepsilon_j \in \mathbf{Q}_{>0}$ and $r_{ji} \in \mathbf{Q}$ such that

$$\varepsilon_j M_b < \frac{\eta_j}{2}, \qquad \sum_i M_i |r_{ji} - a_i| < \frac{\eta_j}{2}, \qquad \sum_i N_i |r_{ji} - a_i| < \frac{\varepsilon_j}{2}. \tag{11.13.3}$$

First choose the positive rational ε_j; then choose the rational vector sufficiently close to a. Zero values of some M_i or M_b impose no restriction. Since $L(a)$ is nef, the class $\sum_i r_{ji} L_i + \varepsilon_j A$ is ample. Consequently

$$p_j = \sum_i r_{ji} h_i + \varepsilon_j b$$

is Q-PL on W and locally Fubini–Study on U, while $\|p_j - h\|_W < \eta_j$. Set $q_j = p_j + 2\eta_j$. The added constant is in $\Gamma_{\mathbf{Q}}$, hence is Q-PL with zero residue: after clearing denominators it is a logarithm of an element of $K^\times$. Thus q_j is still locally Fubini–Study on U. Its error band is (11.13.2), and

$$q_{j+1} \le h + 3\eta_{j+1} = h + \tfrac{3}{8}\eta_j \le h + \eta_j \le q_j.$$

The bound $3\eta_j \to 0$ proves uniform convergence on all of U. No intersection of a sequence of shrinking neighbourhoods occurs. □

Corollary 11.13.4 (Ordered families in a fixed finite presentation)**.** *In the setting of Lemma 11.13.2, suppose $u_j = \sum_i a_{ji} h_i$ are semipositive at x and decrease pointwise on their common original neighbourhood. Then on one ordinary open neighbourhood U of x there are Q-PL locally Fubini–Study functions q_j such that*

$$u_j + \eta_j \le q_j \le u_j + 3\eta_j, \qquad q_{j+1} \le q_j, \qquad \inf_j q_j = \inf_j u_j. \tag{11.13.4}$$

No bound on the coefficient vectors a_j or on the family (u_j) is required. The common finite presentation is required.

Proof. Choose the data W, U, b, V, A, L_i, N_i once for (h_i). In (11.13.3) replace a_i by a_{ji} for the jth term. The constants M_i, M_b, N_i remain fixed; the rational choices may vary. The same estimate gives the first band, and

$$q_{j+1} \le u_{j+1} + 3\eta_{j+1} \le u_j + \tfrac{3}{8}\eta_j \le u_j + \eta_j \le q_j.$$

Since $q_j - u_j \to 0$ uniformly, their decreasing limits agree, also when the limit takes the value $-\infty$. □

11.13.4 Consequences for original smooth approximation

Corollary 11.13.5 (Semipositive R-PL functions in the projective case)**.** *Let X be an integral projective variety of positive dimension over K. If h is R-PL near $x \in X^{\mathrm{an}}$ and semipositive at x, then on one ordinary open neighbourhood of x there is a decreasing sequence of original CLD smooth psh functions converging to h, uniformly on compact subsets. The same assertion holds for $u = \inf_j u_j$ from Corollary 11.13.4, except that uniform convergence on compact subsets is asserted only when u is finite continuous.*

Proof. The preceding results give decreasing Q-PL functions on one ordinary open set, with nef residues at all its points. Apply Theorem 11.10.11 to that sequence. Its projective barrier and termwise rescaling supply the common smooth approximation domain. Dini's theorem gives the asserted compact uniform convergence for a finite continuous endpoint. The residue rationalization itself used neither a scalar extension nor a change of the chosen absolute value. □

Theorem 11.13.6 (A comparison confined to strict R-PL germs)**.** *Under the projective hypotheses of Corollary 11.13.5, let h be an R-PL germ at x. The following three conditions are equivalent: h is semipositive at x; h is classically psh on a neighbourhood of x in the curvewise sense of [GRPSH, Definition 4.1]; on some fixed neighbourhood h is a uniform decreasing limit of Q-PL LFS functions. They imply that, on a fixed neighbourhood, h is a decreasing limit of original CLD smooth psh functions, uniformly on compact subsets, and hence that $d'd''[h]$ is positive on the original CLD tests. Each neighbourhood here is ordinary open.*

Open proof obligation (RC2). Prove both comparisons in the R-PL scope; the proof applies GRPL Theorem 5.6 and GRPSH Theorem 4.3.

The last current-positivity conclusion is not used to reverse the argument. In GRPSH, "classically psh" requires subharmonic pullback to curves after every analytic field extension. Positivity on tests on X alone is a different condition. Neither a pullback of an arbitrary current nor an identification of these two definitions is hidden in this proof.

Remark 11.13.7 (What this coefficient argument does not identify). For an already R-PL function, classical psh is equivalent to semipositivity by [GRPSH, Theorem 4.3], so that condition may replace semipositivity on a neighbourhood in the preceding corollary. An arbitrary decreasing sequence of semipositive R-PL functions need not come with one finite presentation valid for all terms. Applying Theorem 11.13.3 separately then gives neighbourhoods depending on the term; Corollary 11.13.4 does not remove that problem without its additional hypothesis. Nor does the construction identify arbitrary relative real PL functions with GJR R-PL functions, or approximate every original smooth psh function by relative rational generators. Those are different interfaces.

The projective corollary concerns the original CLD smooth class and consequently its original tests; it does not replace those tests by weakly smooth or tropical tests. It introduces no current pushforward, altered support, or unverified descent. Full Wu Theorem A.3, CLD25 Proposition 8.1.11, and an unrestricted positive-current Bedford–Taylor calculus do not follow. The improvement over the preceding section is a proved coefficient conversion for already semipositive strict R-PL germs, including ordered families with a fixed finite presentation.

11.14 Ample extensions: a fixed-model obstruction and the cofinal repair

The approximation argument in Section 11.9 needs an ample model after passing to a suitable higher model. It does not need an ample extension on every prescribed projective model. These assertions are different. The published [GM, Lemma 4.12, p. 353] states the latter assertion; the example below disproves that literal statement. We then recover exactly the cofinal formal-model input needed for the metric approximation. This does not disprove [GM, Proposition 4.11] or [BE21, Theorem 7.8].

11.14.1 The missing generic-fibre factor

In that lemma, $f : \mathscr{Y} \to \mathscr{X}_1$ is a projective morphism of models inducing the identity on X, and an ample $\mathscr{L}_1$ extends L^{m_1}. The proof chooses an arbitrary f-ample bundle E from a projective embedding and forms $f^*\mathscr{L}_1^{m_2} \otimes E$. Its generic restriction is

$$\big(f^*\mathscr{L}_1^{m_2} \otimes E\big)|_X \simeq L^{m_1 m_2} \otimes E|_X.$$

The assumption $f|_X = \mathrm{id}_X$ does not trivialize $E|_X$: every line bundle is relatively ample for an identity morphism. The final identification in the published proof drops this factor. A missing justification alone would not refute the lemma. The following construction supplies the additional obstruction.

Example 11.14.1 (No ample extension on the prescribed model)**.** Let $R = k[[t]]$, $K = k((t))$, with $|t| = e^{-1}$, where k is any field. There are projective algebraic R-models $\mathscr{Y}, \mathscr{X}_1$ of $X = \mathbf{P}^1_K \times \mathbf{P}^1_K$ and a morphism $f : \mathscr{Y} \to \mathscr{X}_1$ inducing id_X such that $L = \mathcal{O}_X(1,1)$ extends to an ample line bundle on $\mathscr{X}_1$, but no L^m, $m > 0$, extends to an ample line bundle on $\mathscr{Y}$. The same obstruction exists in every generic dimension $n \geq 2$.

Proof. On $\mathbf{P}^1_R$, take the rank-two vector bundle F defined by the extension class $t\xi$, where ξ generates

$$\mathrm{Ext}^1(\mathcal{O}(2), \mathcal{O}) = H^1(\mathbf{P}^1_R, \mathcal{O}(-2)) \simeq R, \qquad 0 \longrightarrow \mathcal{O} \longrightarrow F \longrightarrow \mathcal{O}(2) \longrightarrow 0.$$

The middle term is locally free since the quotient is locally free. On the generic fibre the extension is nonzero, hence, by rescaling the one-dimensional extension class and using the Euler sequence, $F_K \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$. On the special

fibre the extension splits: $F_k \simeq \mathcal{O} \oplus \mathcal{O}(2)$. These are the classical deformation data relating the Hirzebruch surfaces $\mathbb{F}_0$ and $\mathbb{F}_2$; the argument uses them as an explicit model obstruction, not as a new deformation construction.

Use the quotient convention for $\mathscr{Y} = \mathbf{P}(F)$ and put $M = \mathcal{O}_{\mathbf{P}(F)}(1)$. This is a smooth projective R-scheme, hence regular and integral, with irreducible special fibre. Its generic fibre is X, and $M_K = L$. The special quotient $F_k \twoheadrightarrow \mathcal{O}$ defines a section $C \simeq \mathbf{P}^1_k$ on which M has degree zero.

The vanishing $H^1(\mathbf{P}^1_R, \mathcal{O}) = 0$ shows that all three standard global sections of $\mathcal{O}(2)$ lift to F. Together with the section of the subbundle $\mathcal{O}$, they generate F: locally a section can first be matched in the quotient and then in the kernel. Thus M is generated by four sections and defines $g : \mathscr{Y} \to \mathbf{P}^3_R$. Its generic restriction is the Segre embedding associated to $\mathcal{O}(1,1)$. Let $\mathscr{X}_1$ be the schematic image. It is projective and flat over R: its structure sheaf embeds into $g_*\mathcal{O}_{\mathscr{Y}}$, which has no t-torsion, and torsion-free modules over a DVR are flat. Its generic fibre is the Segre copy of X. The induced $f : \mathscr{Y} \to \mathscr{X}_1$ is projective (use the closed graph and a projective embedding of $\mathscr{Y}$ over R). With this identification of the generic fibres it induces the identity. The bundle $\mathscr{L}_1 = \mathcal{O}_{\mathbf{P}^3_R}(1)|_{\mathscr{X}_1}$ is ample and extends L, and $f^*\mathscr{L}_1 = M$.

Suppose that a line bundle N on $\mathscr{Y}$ extends L^m. Then $N \otimes M^{-m}$ is generically trivial. A generic trivialization gives a rational section whose divisor is vertical. Since $\mathscr{Y}$ is regular and its special fibre is irreducible and reduced, this divisor is $a\mathscr{Y}_k$ for an integer a. But $\mathscr{Y}_k = \operatorname{div}(t)$ is principal. Thus $N \simeq M^m$, and $\deg(N|_C) = 0$. An ample bundle has positive degree on every projective integral curve, so N is not ample. This proves the claimed failure for every $m > 0$.

For $n > 2$, take the products of both models with $\mathbf{P}^{n-2}_R$ and replace $L, M, \mathscr{L}_1$ by their external tensor products with $\mathcal{O}(1)$. Regularity and irreducibility of the special fibre persist. The same vertical-divisor argument forces any extension of L^m to be the indicated tensor power, whose degree on $C \times \{p\}$ is zero for a k-rational point p. The ample extension on the target is the external tensor product of two ample bundles. □

Proposition 11.14.2 (The obstruction over any nontrivially valued field)**.** *The conclusion of Example 11.14.1 holds over every complete nontrivially valued field K, with its fixed absolute value, and in every dimension at least two. No discrete valuation, characteristic assumption or field extension is needed.*

Proof. Put $R = K^\circ$ and choose $0 \neq a \in K^{\circ\circ}$. On $\mathbf{P}^1_R$, with homogeneous coordinates $[x : y]$, define F by

$$0 \longrightarrow \mathcal{O}(-1)^2 \xrightarrow{B} \mathcal{O}^4 \longrightarrow F \longrightarrow 0, \qquad B = \begin{pmatrix} -y & 0 \\ x & -y \\ 0 & x \\ 0 & ay \end{pmatrix}.$$

The minors include x^2 and y^2, so B is locally a split injection and F is locally free of rank two. On the special fibre its cokernel is $\mathcal{O}(2) \oplus \mathcal{O}$, as seen from the quotient map $(x^2, xy, y^2, 0)$ and the fourth basis vector. On the generic fibre, adding a^{-1} times the fourth row to the second row makes the matrix block diagonal, with columns $(-y, x)$ and (x, ay); each block has cokernel $\mathcal{O}(1)$. Thus $F_K = \mathcal{O}(1)^2$. This is an explicit form of the preceding classical deformation; see also [Addington21, Worksheet 2, part 3].

Set $\mathscr{Y} = \mathbf{P}(F)$ and $M = \mathcal{O}_{\mathbf{P}(F)}(1)$. Its four generating sections give a morphism to the quadric

$$\mathscr{X}_1 = \{z_0 z_2 - z_1^2 + a z_1 z_3 = 0\} \subset \mathbf{P}^3_R.$$

Indeed the incidence relations are $xz_1 - yz_0 = 0$ and $xz_2 - yz_1 + ayz_3 = 0$, whose determinant is the displayed equation. The homogeneous coordinate ring is free of rank two over $R[z_0, z_2, z_3]$, because the equation is monic in z_1. Hence $\mathscr{X}_1$ is flat and of finite presentation over R. The generic quadric is split and smooth: replacing z_3 by $az_3 - z_1$ gives the usual Segre equation. The generic morphism is the complete $\mathcal{O}(1,1)$ embedding, so, after identifying generic fibres, $f : \mathscr{Y} \to \mathscr{X}_1$ is a projective model morphism inducing the identity. The target has the ample extension $\mathcal{O}(1)$.

To replace the DVR divisor argument, use the projective-bundle Picard formula. Over a connected base S, this formula says that a line bundle on $\mathbf{P}(E)$ is uniquely $\pi^* N \otimes \mathcal{O}(d)$. Here is the needed general-base justification. At any $s \in S$, twist by the degree d_s on the projective-space fibre. Its fibre cohomology is then one-dimensional in degree zero and zero otherwise. Cohomology and base change, in the form [StacksPB, Tag 0B9S], gives locally a line bundle $N = \pi_* \mathcal{L}(-d_s)$; the evaluation map is an isomorphism on every fibre and hence an isomorphism after shrinking. Thus d_s is locally constant, and connectedness makes it constant. The local N glue by the evaluation isomorphisms, and uniqueness follows from $\pi_* \mathcal{O} = \mathcal{O}_S$. This argument applies without noetherian assumptions.

Since R is local, $\mathrm{Pic}(R) = 0$. Applying the formula twice gives $\mathrm{Pic}(\mathscr{Y}) = \mathbf{Z}[M] \oplus \mathbf{Z}[\pi^*\mathcal{O}(1)]$. On $X = \mathbf{P}^1_K \times \mathbf{P}^1_K$ these generators restrict to $(1,1)$ and $(1,0)$, a basis of $\mathrm{Pic}(X)$. Therefore restriction is injective, and any extension of L^m must be M^m. Its degree on the section defined by $F_{\widetilde{K}} \twoheadrightarrow \mathcal{O}$ is zero, so it cannot be ample. Products with projective space give every higher dimension, with injectivity of Picard restriction again supplied by the same formula. No renormalization of $|\cdot|$ has been made. □

In this example the failure has a precise geometric cause. Every generically trivial line bundle on $\mathscr{Y}$ is trivial, whereas an f-ample bundle must have positive degree on the contracted curve C. Hence no f-ample bundle can be generically trivial. Even though $f^*\mathscr{L}_1 = M$ is semiample, no perturbation within the same generic line bundle on this fixed model can make it ample. Passing to a higher model in the approximation theorem is therefore essential.

11.14.2 The sufficient vertical hypothesis

Lemma 11.14.3 (Ample extension along a vertical polarization)**.** *Let $f : \mathscr{Y} \to \mathscr{X}_1$ be as above. If there is an f-ample line bundle E with $E|_X \simeq \mathcal{O}_X$, then some positive power of L extends to an ample bundle on $\mathscr{Y}$. This hypothesis holds when f is the blowup of a coherent vertical ideal.*

Proof. For sufficiently large q, $f^*\mathscr{L}_1^q \otimes E$ is ample by the relative ample twisting theorem, and its generic restriction is now $L^{m_1 q}$. For a vertical blowup use the relatively ample tautological bundle $E = \mathcal{O}(1)$ of the Rees Proj. The ideal becomes the unit ideal on X, so this bundle is generically trivial. This is the original proof with its missing hypothesis supplied. □

Proposition 11.14.4 (Cofinal ample models for metric approximation)**.** *Let K be complete and nontrivially valued, X/K projective, L ample, and $\mathfrak{X}_0$ a formal model of X^{an}. There is a projective algebraic K°-model $\mathscr{Y}$ whose completion dominates $\mathfrak{X}_0$ and which carries an ample extension of some L^m, $m > 0$. No prescribed algebraic model is required to remain fixed.*

Open proof obligation (RC2). Prove the cofinal construction and its ample-model assertion; the proof applies the construction in GM Lemma 2.4.

Corollary 11.14.5 (Finitely many ample generic polarizations)**.** *In the setting of Proposition 11.14.4, let $L_1, \dots, L_r$ be ample line bundles on X. One can choose a single projective algebraic model whose completion dominates $\mathfrak{X}_0$ and on which every L_i has an ample Q-model.*

Open proof obligation (RC2). Prove the relative ampleness step for the simultaneous models; the proof uses the GM Proposition 4.13 argument.

Remark 11.14.6 (Effect on the approximation argument). For a nef model metric of an ample L, pull its formal model bundle to the completion in Proposition 11.14.4; nefness persists under pullback. Proper formal GAGA algebraizes the pulled-back formal line bundle on $\mathscr{Y}$, preserving its generic identification. Divide the ample extension by its exponent to obtain an ample Q-model of the same L. The rational convex combinations of these two model bundles are ample and their metrics converge uniformly to the original metric. Sufficient powers are generated by global sections, giving the pure Fubini–Study approximation used in Section 11.9. Thus the fixed-model counterexample removes an unnecessary stronger assertion without removing this metric approximation route. Corollary 11.14.5 also supplies the finite-polarization geometric input used in [GM, Proposition 4.14] and [BGM22, arXiv v1, Lemma 1.1]. The generic Picard relation and the small model-function bound require a further argument, supplied in Proposition 11.14.8 below with explicit formal-domination scope. The reduced-special-fibre input of BGM Lemma 1.1 is separate. No scalar extension or change of valuation is involved.

11.14.3 Preserving the generic Picard relation

There is one further distinction in the proof of [GM, Proposition 4.14]. A numerical equality on the generic fibre does not itself trivialize a line bundle. To obtain a model function from the difference of two model bundles, one needs equality in $\mathrm{Pic}(X)_{\mathbf{Q}}$. The ample generic class permits the following choice, which supplies that stronger relation before rational approximation.

Lemma 11.14.7 (Exact ample decomposition and rational relations)**.** *Let X be projective over a field and $L_1, \dots, L_m$ line bundles. Suppose $\Theta = \sum_i \lambda_i [L_i] \in \mathrm{Pic}(X)_{\mathbf{R}}$ has ample numerical class. There are ample line bundles $H_1, \dots, H_r$ and real numbers $\mu_j > 0$ such that*

$$\sum_i \lambda_i [L_i] = \sum_j \mu_j [H_j] \quad \text{in } \mathrm{Pic}(X)_{\mathbf{R}}.$$

Moreover (λ, μ) *can be approximated arbitrarily closely by rational vectors* (λ', μ') *with all* $\mu'_j > 0$ *and the same exact relation in* $\mathrm{Pic}(X)_{\mathbf{Q}}$.

Proof. The inverse image of the numerical ample cone under $\mathbf{R}^m \to N^1(X)_{\mathbf{R}}$ is open and contains λ. Choose a rational box containing λ in its interior and contained in this inverse image. Every vertex v gives an ample Q-line bundle $\sum_i v_i[L_i]$; multiplying by a positive integer d_v gives an ample integral line bundle H_v. Express λ as the convex combination of the vertices with its positive multilinear barycentric coefficients t_v. Then the required equality in the Picard group, not merely in the numerical quotient, holds with $\mu_v = t_v/d_v$. This is an explicit use of openness of the ample cone.

For the second assertion, map $\mathbf{Q}^{m+r}$ to the finite-dimensional Q-subspace of $\mathrm{Pic}(X)_{\mathbf{Q}}$ spanned by these bundles, sending (a, b) to $\sum_i a_i[L_i] - \sum_j b_j[H_j]$. Its kernel has a rational basis, and its realification contains (λ, μ). Rational linear combinations of that basis are dense. Choose one sufficiently close to preserve all strict inequalities $\mu'_j > 0$. Equality in $\mathrm{Pic}(X)_{\mathbf{Q}}$ means that, after clearing denominators and killing a possible torsion class, the corresponding integral bundles are isomorphic. This is exactly what is needed for a generic trivialization. □

Proposition 11.14.8 (Ample correction of a real model class)**.** *Let* K *be complete and nontrivially valued and* X/K *projective. Let* θ *be a model* $(1,1)$*-class in the sense of [GM, 4.4–4.6], represented by a finite real combination of model line bundles, with ample generic numerical class. Given a formal model carrying those data, there is a projective algebraic model whose completion dominates it and a rational model function* φ *such that* $\theta + dd^c\varphi$ *is represented by an ample real class on that model. If* θ *is semipositive, then for every* $\epsilon > 0$ *one can arrange* $-\epsilon \le \varphi \le 0$ *on* X^{an}*. This statement concerns GM model classes; it is not a new local theorem about arbitrary CLD currents.*

Open proof obligation (RC2). Prove the model-class correction and uniform comparison with the stated positivity prerequisites; the proof uses GM 4.5 and Proposition 4.14.

Section 11.15.1 now supplies the additional morphism-algebraization theorem and strengthens these statements to actual algebraic domination. Section 11.15.2 verifies the separate reduced-fibre input in the published BGM Lemma 1.1.

11.15 Algebraic domination and reduced models

The cofinal constructions of Section 11.14 were formulated for formal completions. To use them in algebraic intersection theory, one must recover actual morphisms of algebraic models. This passage is available over a complete rank-one valuation ring, but it requires the non-Noetherian form of formal GAGA. We give the hypotheses explicitly, then apply the result to the simultaneous ample-model construction. A second, independent step produces a reduced special fibre over an algebraically closed field. Its role is to turn an analytic supnorm into an algebraic lattice norm.

11.15.1 Algebraizing the morphism, not merely the line bundle

Throughout this subsection, K is complete and nontrivially valued, with its given real-valued absolute value, and $R = K^\circ$. Fix $\pi \in R$ with $0 < |\pi| < 1$. Completion always means π-adic completion. We work with proper finitely presented R-schemes. No discrete-valuation assumption is made.

Proposition 11.15.1 (Formal GAGA for model morphisms)**.** *For proper finitely presented R-schemes $\mathscr{Y}, \mathscr{Z}$, completion induces a bijection*

$$\mathrm{Hom}_R(\mathscr{Y}, \mathscr{Z}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Spf}\, R}(\widehat{\mathscr{Y}}, \widehat{\mathscr{Z}}).$$

In particular, if they are models of the same proper K-scheme X, a formal morphism inducing $\mathrm{id}_{X^{\mathrm{an}}}$ algebraizes uniquely to a morphism inducing id_X.

Open proof obligation (RC2). Prove formal existence and faithfulness of proper analytification in the stated formal model category; the proof uses the cited FK17 corollaries/proposition.

The distinction between sheaves and morphisms matters here. Formal GAGA for coherent sheaves, [FK17, I, Theorem 10.1.2], algebraizes a formal line bundle on an already algebraized proper model. Proposition 11.15.1 also algebraizes the map to the prescribed model. In the source, the latter result is obtained by algebraizing the ideal of the graph. This is the additional step needed for domination in the algebraic directed system. All FK locators in this section refer to the fixed arXiv v5 of 28 February 2017.

Corollary 11.15.2 (Simultaneous algebraic domination)**.** *Let X/K be projective, let $L_1, \dots, L_r$ be ample, and prescribe finitely many proper finitely presented algebraic R-models $\mathscr{X}_a$ of X. There is a projective model $\mathscr{Y}$ with morphisms $\mathscr{Y} \to \mathscr{X}_a$ extending id_X, on which every L_i has an ample Q-model. All prescribed commutative diagrams among the $\mathscr{X}_a$ remain commutative after pullback.*

Proof. Choose a common formal refinement of the finitely many completions. Corollary 11.14.5 supplies $\mathscr{Y}$ and the ample Q-models, with a formal map from $\widehat{\mathscr{Y}}$ to each $\widehat{\mathscr{X}_a}$. Proposition 11.15.1 algebraizes each map and identifies its generic restriction. Equality of the two composites in any diagram follows from faithfulness of completion. Pullback of model line bundles along these algebraic maps has the prescribed formal pullback, so the metric identifications used in Section 11.14 are unchanged. □

Corollary 11.15.3 (The corrected algebraic model-class perturbation)**.** *Proposition 11.14.8 holds with algebraic domination of any prescribed proper finitely presented model. In particular, its rational model function can be represented on one algebraic model carrying the pulled-back initial data. If the initial class is semipositive, the bound $-\epsilon \le \varphi \le 0$ holds on X^{an} with the original absolute value.*

Proof. Include the prescribed model and all finitely many representatives of the initial class in Corollary 11.15.2. The exact generic Picard relation and the rational perturbation are those of Lemma 11.14.7 and Proposition 11.14.8. The new algebraic maps realize exactly the formal pullbacks used there. Hence their numerical classes, model functions and uniform bounds agree. No additional approximation or limiting argument is required. □

These corollaries reconstruct the conclusions needed from GM Propositions 4.11 and 4.14, using a different cofinal model. They do not restore the false fixed-model assertion of GM Lemma 4.12. The counterexample in Section 11.14.1 continues to apply.

11.15.2 The reduced-special-fibre step

For the next theorem assume additionally that K is algebraically closed and that X is reduced. The closure used below is the integral closure of a model

in its given generic fibre; it is not a replacement of X by its normalization. This distinction preserves the model's generic identification even when X is singular.

Theorem 11.15.4 (Reduced cofinal models with finitely many polarizations)**.** *Let K be complete, algebraically closed and nontrivially valued. Let X/K be reduced and projective, and $L_1, \dots, L_r$ ample. Given finitely many algebraic models as in Corollary 11.15.2, there is a projective model $\mathscr{X}$ dominating all of them, with reduced special fibre and ample Q-models of all L_i.*

Open proof obligation (RC2). Prove finiteness of the normalization and the reduced-special-fibre model step from the stated hypotheses; the argument invokes BE21 and BGM22.

Remark 11.15.5 (The field hypothesis in the reduced fibre theorem). BE Theorem 4.20 also gives finiteness of the integral closure when K is discretely valued. It does not assert a reduced special fibre in that case. For an arbitrary densely valued field, even finiteness can fail; [BE21, Remark 4.21] explains the obstruction from values of spectral norms outside $|K^\times|$. Thus algebraic domination, proved above over every complete nontrivially valued K, and reduced cofinality, asserted here over algebraically closed K, have different scopes.

11.15.3 Exact norms and the error that disappears in volume

Let $\mathscr{L}$ be a line bundle on a projective flat R-model $\mathscr{X}$, with generic fibre L, and put $\Lambda = H^0(\mathscr{X}, \mathscr{L}) \subset H^0(X, L)$. The lattice is finitely generated and spans the generic space. Its norm and the supnorm of the induced metric are

$$\|s\|_\Lambda = \inf\{|a| : a \in K^\times,\ s \in a\Lambda\}, \qquad \|s\|_{\sup} = \sup_{x \in X^{\mathrm{an}}} |s(x)|_{\mathscr{L}}.$$

By [BE21, Lemma 6.3], $\|s\|_{\sup} \le \|s\|_\Lambda$, with equality when the special fibre is reduced. The equality is also the input of [BGM22, Lemma 1.3]. The next calculation shows why replacing the hypothesis by normality of the model would not justify the same conclusion.

Example 11.15.6 (A normal ramified model with unequal norms)**.** Let K be complete discretely valued, π a uniformizer, and let $e \geq 2$ be prime to the residue characteristic. Put $E = K(\alpha)$, where $\alpha^e = \pi$. The polynomial is Eisenstein, E/K is separable and totally ramified, and $R_E = R[\alpha]$ is the valuation ring of E. For any $n \geq 0$ take

$$\mathscr{X} = \mathbf{P}^n_{R_E}, \quad X = \mathbf{P}^n_E \quad \text{as an } R\text{-model and a } K\text{-scheme}, \qquad \mathscr{L} = \mathcal{O}(1).$$

Then $\mathscr{X}$ is normal, projective and flat over R, and X is smooth over K. Its special fibre over R is not reduced. For $m \geq 1$ and

$$s = \sum_{|\beta|=m} \sum_{i=0}^{e-1} b_{i\beta} \alpha^i T^\beta, \qquad b_{i\beta} \in K,$$

the two norms are

$$\|s\|_{\Lambda_m} = \max_{i,\beta} |b_{i\beta}|, \qquad \|s\|_{m\phi} = \max_{i,\beta} |b_{i\beta}|\,|\pi|^{i/e}, \qquad \Lambda_m = H^0(\mathscr{X}, \mathcal{O}(m)). \tag{11.15.1}$$

Consequently the norms have the same unit ball but are unequal, and

$$|\pi|^{(e-1)/e} \|s\|_{\Lambda_m} \leq \|s\|_{m\phi} \leq \|s\|_{\Lambda_m}.$$

Proof. The special fibre is $\mathbf{P}^n_{k[\alpha]/(\alpha^e)}$, whereas $\mathscr{X}$ is a projective space over the DVR R_E and hence is normal. The monomials $\alpha^i T^\beta$ form an R-basis of Λ_m, proving the first formula. Write a coefficient as $c_\beta = \sum_i b_{i\beta}\alpha^i$. Distinct nonzero summands have valuations in different cosets of $\mathbf{Z}$ in $e^{-1}\mathbf{Z}$, when $v(\pi) = 1$. Thus there is a unique term of least valuation and no cancellation at that valuation:

$$|c_\beta| = \max_i |b_{i\beta}|\,|\pi|^{i/e}.$$

For the standard projective model metric, the supnorm of a homogeneous polynomial equals the largest absolute value of its coefficients. The ultrametric inequality gives the upper bound, and the Gauss point in the chart $T_0 \neq 0$ gives equality. This proves the second formula. Finally, $v(b_{i\beta}) + i/e \geq 0$ holds for every i, β if and only if every integer $v(b_{i\beta})$ is nonnegative. Hence the unit balls agree. Taking $s = \alpha T_0^m$ gives supnorm $|\pi|^{1/e}$ and lattice norm 1, so the norms differ in every dimension. □

The discrepancy can be computed on the determinant line as well. Put $M_m = \binom{m+n}{n}$ and $\lambda = -\log|\pi|$. The eM_m basis elements in (11.15.1) are

orthogonal for both norms. The determinant norm of an orthogonal basis is the product of its norms: the ultrametric determinant inequality gives the lower bound for any decomposition of the same determinant vector, and the chosen basis attains it. Consequently,

$$\log \frac{\det \| \cdot \|_{\Lambda_m}}{\det \| \cdot \|_{m\phi}} = M_m \lambda \sum_{i=0}^{e-1} \frac{i}{e} = \frac{e-1}{2} M_m \lambda.$$

This grows like m^n, but multiplication by $n!/m^{n+1}$ makes it asymptotic to $(e-1)\lambda/(2m)$. Thus a persistent, nonzero norm discrepancy can disappear in the volume normalization. The pointwise comparison constant $|\pi|^{-(e-1)/e}$ is optimal, as the section $\alpha^{e-1} T_0^m$ shows. These exact calculations illustrate the two different scales in Proposition 11.15.8.

The absolute value on E in this example is the extension of the one on K: $|\alpha| = |\pi|^{1/e}$. Renormalizing the valuation of E so that α has value 1 would multiply its restriction to K by e and would change the problem. Also, equality of unit balls determines a norm when $|K^\times|$ is dense, by arbitrarily close scalar rescaling; it need not do so over a discretely valued field. This is why a lattice description by itself does not establish an isometry.

This failure of isometry does not refute the effectivity criterion for vertical Cartier divisors on normal models. GM Remark 4.10 gives that criterion without a reduced-special-fibre assumption. In the explicit model above, a vertical divisor $k \operatorname{div}(\alpha)$ is effective exactly when $k \geq 0$, and its model function is $-k \log |\alpha|$, with the same sign. Effectivity detects the unit-ball inequality; an isometry determines every radius.

Example 11.15.7 (Strict norm inequality after wild scalar extension)**.** Let K be a finite extension of $\mathbf{Q}_p$ containing a primitive pth root of unity ζ, let π be a uniformizer, and put $E = K(\alpha)$ with $\alpha^p = \pi$. Then E/K is cyclic of degree p. For $X = \operatorname{Spec} E$ and the trivial metric, the original supnorm is $|\cdot|_E$. On $E \otimes_K E$, the scalar extension of this norm is strictly larger than the supnorm of the pulled-back metric. More precisely,

$$q = \frac{1}{p} \sum_{i=0}^{p-1} \alpha^{-i} \otimes \alpha^i \quad \text{satisfies} \quad \|q\|_{\text{tensor}} = |p|^{-1}, \qquad \|q\|_{\sup} = 1. \tag{11.15.2}$$

The same discrepancy occurs for sections of $\mathcal{O}(m)$ on $\mathbf{P}^n_E$ regarded as a K-scheme, in every dimension n and every $m \geq 1$. This is an explicit instance of the obstruction in [BE21, Lemma 6.2], not a counterexample to that lemma.

Proof. Eisenstein's criterion gives $[E:K]=p$, and the automorphisms are $\sigma_j(\alpha)=\zeta^j\alpha$, $0\le j<p$. The distinct valuation classes of $1,\alpha,\dots,\alpha^{p-1}$ make this an orthogonal K-basis of the normed space E. Its scalar-extended norm, with the left tensor factor as coefficient field, is therefore

$$\left\|\sum_i c_i\otimes\alpha^i\right\|_{\mathrm{tensor}}=\max_i |c_i|\,|\alpha|^i.$$

For q all displayed terms have norm $|p|^{-1}$. On the other hand, the splitting isomorphism

$$E\otimes_K E\longrightarrow\prod_{j=0}^{p-1}E,\qquad a\otimes b\longmapsto(a\sigma_j(b))_j$$

sends q to $(1,0,\dots,0)$, by the finite geometric sum $p^{-1}\sum_i\zeta^{ij}$. The pulled-back metric has supnorm equal to the maximum of the component norms, so its value on q is 1. For projective space, take the section qT_0^m. The monomial basis together with the orthogonal basis of E/K gives the same tensor norm, and the Gauss point on the first split component gives the same supnorm. All norms use the unique extension of the original absolute value. □

Proposition 11.15.8 (Uniform norm comparison and relative volume)**.** *Let X be geometrically reduced and projective over complete nontrivially valued K, let L be ample, and let $\mathscr{L}_1,\mathscr{L}_2$ be integral models of L inducing metrics ϕ_1,ϕ_2. Put $N_m=h^0(X,L^m)$, $s_{i,m}=\|\cdot\|_{m\phi_i}$ and $\ell_{i,m}=\|\cdot\|_{H^0(m\mathscr{L}_i)}$. There are $C_i\ge1$, independent of m, such that*

$$s_{i,m}\le\ell_{i,m}\le C_i s_{i,m}.$$

With $\mathrm{vol}(u,v)=\log(\det v/\det u)$ *on the determinant line, one has*

$$|\mathrm{vol}(s_{1,m},s_{2,m})-\mathrm{vol}(\ell_{1,m},\ell_{2,m})|\le N_m\max\{\log C_1,\log C_2\}. \tag{11.15.3}$$

If $d=\dim X$, the difference multiplied by $d!/m^{d+1}$ tends to zero. Thus replacing supnorms by these lattice norms preserves relative volume, although it need not preserve any individual norm.

Open proof obligation (RC2). Prove the uniform bound and determinant-to-volume limit; BE21 Theorem 6.4 and Lemma 9.17 are used.

There is a further distinction in the scalar-extension step. For a complete extension F/K, the pulled-back metric has the same supnorm on sections originally defined over K. On all of $H^0(X_F, L_F)$, however, its supnorm is only bounded above by the scalar extension of the original norm in general; see [BE21, Lemmas 6.1 and 6.2]. Equality on the original sections is sufficient in the comparison proof. One must not replace it by an unproved isometry on the enlarged vector space. None of these model and norm results identifies an arbitrary positive CLD current with a nef metric, or establishes an unrestricted Bedford–Taylor product.

Geometric reducedness also has a concrete role in the extension step. Over $K = \mathbf{F}_p((\pi))$, put $E = K(\alpha)$ with $\alpha^p = \pi$. The scheme $\operatorname{Spec} E$ is reduced over K, but $E \otimes_K E \simeq E[\varepsilon]/(\varepsilon^p)$. The nonzero element $1 \otimes \alpha - \alpha \otimes 1$ has zero analytic sup-seminorm after extension, whereas its scalar-extended vector-space norm is positive. Thus the proof cannot replace geometric reducedness by reducedness when it requires a norm on every extended section space. This calculation is not a counterexample to BE Theorem 6.4 under its stated hypotheses.

11.16 An analytic realization of the fixed-chart obstruction

Example 11.10.9 separated relative convexity from ambient convex extension on an abstract fan. We now realize that fan by analytic units. The same chart also supports an original CLD smooth psh function with no ambient convex extension, even locally at its central vertex. This gives a concrete test for a proposed smooth-to-PL argument: it must allow relative convexity or a change of chart. It does not contradict such a comparison when these choices are allowed.

11.16.1 Four punctures and two tropical coordinates

Let K be complete and nontrivially valued, with $|2| = 1$. On

$$C = \mathbf{P}^1_K \setminus \{0, 1, -1, \infty\}$$

use the analytic units $f = t(t-1)$ and $g = t(t+1)$, and write

$$x = -\log|f|, \qquad y = -\log|g|, \qquad s = -\log|t|,$$
$$\tau = (x, y) : C^{\mathrm{an}} \longrightarrow \mathbf{R}^2.$$

The unit-disc Gauss point ξ belongs to C^{an} and maps to the origin. No extension of K is made.

Proposition 11.16.1 (The fan and its analytic refinement)**.** *The image of τ is the support of the fan in Example 11.10.9: the four rays generated by $(1,0),(0,1),(1,1),(-1,-1)$. The refinement by the three units $t-1, t+1, t$ has image the four rays generated by $e_1, e_2, e_3, -e_1-e_2-e_3$. Under*

$$(z_1,z_2,z_3) = (-\log|t-1|, -\log|t+1|, -\log|t|),$$
$$A(z_1,z_2,z_3) = (z_1+z_3, z_2+z_3).$$

one has $\tau = A\circ\widetilde{\tau}$. The relative PL function of that example pulls back to $\max\{s,0\}$. Thus its previously abstract convex presentation after refinement is realized by analytic units.

Proof. If $|t|<1$, the ultrametric inequality gives $|t-1| = |t+1| = 1$, so $(x,y) = (s,s)$ with $s>0$. If $|t|>1$, both $|t\pm 1|$ equal $|t|$, so $(x,y) = (2s,2s)$ with $s<0$. If $|t|=1$, both $|t\pm 1|\le 1$. They cannot both be strictly smaller than 1, because their difference is 2 and $|2|=1$. Thus the image lies on a nonnegative coordinate ray or at the origin. These arguments apply to the multiplicative seminorm of every Berkovich point.

Conversely, the Gauss points of discs centered at $0,1,-1$ with radius $r<1$, and of discs centered at 0 with radius $r>1$, realize every point on the four rays. Arbitrary positive real radii are allowed for Berkovich points; no value-group enlargement is needed. The same case distinction gives the three-coordinate image and the identity involving A. On the ray over 0 the PL function is s, and it is zero on the other rays, hence equals $\max\{s,0\}$. The orders of f,g at the four punctures are $(1,1),(1,0),(0,1),(-2,-2)$, explaining the primitive ray weights $1,1,1,2$ and their balancing identity. □

Lemma 11.16.2 (A smooth barrier does not remove the first-order obstruction)**.** *Let F_0 denote the relative PL function of Example 11.10.9, so $F_0(h,h) = h$ and $F_0(2h,0) = F_0(0,2h) = 0$. For every ambient C^2 function B near the origin, $F_0 + B|_\Sigma$ still has no ambient convex extension near the origin. If $\|D^2B\|_{\mathrm{op}} \le M$ on the relevant segments, its midpoint convexity defect is at least $h - Mh^2$.*

Proof. Put $a=(h,h)$ and $v=(h,-h)$. The affine part of the Taylor expansion cancels in $B(a)-\frac{1}{2}B(a+v)-\frac{1}{2}B(a-v)$. The remainder is bounded below

by $-M\|v\|^2/2 = -Mh^2$. Adding the exact defect h of F_0 gives the assertion, which is positive for all sufficiently small $h > 0$. Thus even an arbitrarily large fixed quadratic barrier cannot correct this first-order failure. □

For a convex smooth B, the pullback $\max\{s, 0\} + B(x, y)$ is nevertheless psh: the first summand is a locally uniform limit of smooth convex functions of the pluriharmonic unit logarithm s, and the second is an ambient convex pullback. The lemma therefore concerns a false extension shortcut even inside a positive class. It does not obstruct relative PL approximation or a change of analytic presentation.

11.16.2 A smooth psh function that is not ambient convex in this chart

Example 11.16.3 (A fixed-chart obstruction already for smooth psh functions)**.** The function $u = xy$ on C^{an} is original CLD smooth and psh on all original tests. Nevertheless, the function $F(a, b) = ab$ restricted to $\tau(C^{\mathrm{an}})$ has no ambient convex extension on any convex open neighbourhood of the origin. Moreover, on any subset containing $(h, h), (2h, 0), (0, 2h)$ with $h > 0$, every restriction G of an ambient convex function satisfies

$$\max_{z\in\{(h,h),(2h,0),(0,2h)\}} |G(z) - F(z)| \geq \frac{h^2}{2}. \tag{11.16.1}$$

Every ordinary open neighbourhood of ξ has a tropical image containing such a triple for some $h > 0$. Consequently a fixed-chart uniform approximation by ambient convex functions fails on every such neighbourhood.

Proof. Smoothness follows because x, y are logarithms of analytic units and multiplication is a smooth operation. The case distinction in Proposition 11.16.1 gives the additional identity

$$u = H(s), \qquad H(v) = (\max\{v, 0\})^2 + 4(\max\{-v, 0\})^2.$$

The real function H is convex: its derivative is $8v$ for $v < 0$ and $2v$ for $v > 0$, with common value zero at the origin. Choose smooth convex regularizations H_ϵ converging uniformly on compact real intervals to H. The function $s = -\log|t|$ is original smooth and pluriharmonic, since t is a unit. Therefore

$$d'd''H_\epsilon(s) = H_\epsilon''(s)\, d's \wedge d''s \geq 0.$$

For every compactly supported original CLD test, s is bounded on its support, so $H_\epsilon(s) \to u$ uniformly there. Passing to curvature currents preserves positivity on every original positive test. This proves psh without asserting that the ambient polynomial ab is convex. The relative function has touching affine lower supports: zero at the vertex and a tangent affine function along a ray away from the vertex.

Suppose the maximum error on the three points is ϵ. Since (h, h) is the midpoint of $(2h, 0)$ and $(0, 2h)$, convexity gives

$$h^2 - \epsilon \le G(h, h) \le \tfrac{1}{2}G(2h, 0) + \tfrac{1}{2}G(0, 2h) \le \epsilon.$$

This proves (11.16.1) and rules out an exact extension. The three Gauss-point paths toward $0, 1, -1$ are continuous and have common limit ξ as their radii tend to 1. An ordinary neighbourhood of ξ therefore contains initial segments of all three paths, giving the asserted triple for small h. The argument also rules out uniform convergence on a fixed compact set containing the corresponding three analytic points. □

Corollary 11.16.4 (The obstruction in every positive dimension)**.** *For every $d \ge 1$, the preceding smooth psh example occurs on an ordinary open subset of the integral projective variety $(\mathbf{P}^1_K)^d$. In the product moment chart the same fixed-chart obstruction persists, while a different analytic presentation makes the function ambient convex.*

Proof. Use $C \times (\mathbf{G}_m)^{d-1}$, pull u back from its first factor, and take the chart consisting of f, g and the torus coordinates. The smooth convex approximations $H_\epsilon(-\log|t|)$ still have positive original curvature, so their locally uniform limit is psh. Fix the remaining tropical coordinates to zero. Restricting any proposed ambient convex extension to that affine slice contradicts (11.16.1). On the other hand, the unit t alone presents the function as the pullback of the convex real function H. The coefficient-controlled approximation of Section 11.12.4 can be applied to this latter presentation on bounded tropical images. □

The example closes the analytic-realization question left by the earlier fan calculation, but not the general chart-selection problem. It proves failure of one fixed-chart shortcut, together with an explicit successful change of analytic presentation in the example itself. In particular, it is not a counterexample

to full Wu Theorem A.3, and it does not identify the general original smooth psh class with a prescribed tropical generator class.

For uniform approximation, the fixed-domain qualification is essential: the error bound concerns ambient convex functions defined at the three fixed points, or locally convex functions on one fixed ambient neighbourhood containing a convex ball around the origin. It does not rule out approximants whose ambient neighbourhoods shrink with the index, nor approximants defined only by relative supports on the tropical image.

11.16.3 Removing the residue-characteristic restriction

The four rational punctures above require $|2| = 1$ so that their residue directions remain distinct. That restriction can be removed by replacing one rational puncture with the zero locus of a polynomial whose reduction avoids both 0 and 1. The resulting function and its convex alternative presentation are still defined over the original field.

Theorem 11.16.5 (A fixed-chart smooth psh obstruction over every nontrivial field)**.** *Let K be any complete nontrivially valued non-Archimedean field, with its fixed absolute value. Set*

$$\begin{gathered} P(t) = t^2 - t + 1, \\ C = \mathbf{P}^1_K \setminus \big(\{0, 1, \infty\} \cup V(P)\big), \\ f = t(t-1), \qquad g = tP(t), \\ x = -\log|f|, \quad y = -\log|g|, \quad s = -\log|t|. \end{gathered}$$

Then $u = xy$ is original smooth psh on C^{an}, but it has no ambient convex presentation in the fixed chart (x, y) near the unit-disc Gauss point. The quantitative bound (11.16.1) still holds. The example extends to an ordinary open subset of $(\mathbf{P}^1_K)^d$ for every $d \geq 1$. In particular, residue characteristics 2 and 3 are included.

Proof. Both f and g are units on C. If $|t| < 1$, then $|t-1| = |P(t)| = 1$, giving $(x, y) = (s, s)$. If $|t| > 1$, then $|t-1| = |t|$ and $|P(t)| = |t|^2$, giving $(x, y) = (2s, 3s)$. If $|t| = 1$, put $f = t(t-1)$ and note the exact identity $P = f + 1$. The numbers $|f|, |P|$ are at most 1 and cannot both be smaller than 1. Since $|g| = |P|$ in this case, at least one of x, y is zero. Therefore

$$u = H(s), \qquad H(v) = (\max\{v, 0\})^2 + 6(\max\{-v, 0\})^2.$$

This function H is convex, with derivatives $12v$ for $v < 0$ and $2v$ for $v > 0$. The same convex regularization argument as above proves positivity against all original compactly supported tests. The formula $u = xy$ proves original smoothness.

The diagonal and horizontal ray segments at the origin are realized by Gauss points of small discs around 0 and 1. To exhibit the vertical segment, choose a root a of P in a finite extension K'/K, endowed with the unique extended absolute value. Both roots have absolute value 1, and $P(1) = 1$ implies $|a - 1| = 1$. Gauss points of K'-discs centered at a with radii $r < 1$ and close to 1 therefore have $|t| = |t - 1| = 1$, while $|P(t)| < 1$ approaches 1 as $r \uparrow 1$. More precisely, factor P over a finite splitting extension. On such a disc,

$$|P|_{a,r} = \prod_{P(b)=0} \max\{r, |a - b|\},$$

where roots are counted with multiplicity. For r sufficiently close to 1, this is r^q with $q \in \{1, 2\}$, so it realizes a whole vertical interval. Restricting these multiplicative seminorms to $K[t]$ gives actual points of C^{an} tending to its unit-disc Gauss point. This use of an auxiliary field constructs points over K; the function, the positivity proof and the approximation question have not been base-changed, and no current-descent assertion is used.

Every ordinary neighbourhood of that Gauss point consequently contains analytic points with tropical coordinates $(h, h), (2h, 0), (0, 2h)$ for small $h > 0$. Their function values are $h^2, 0, 0$, giving the same midpoint obstruction and error bound. The product construction of Corollary 11.16.4 gives every positive dimension. Repeated roots of P cause no difficulty: their multiplicities are already included in the displayed Gauss-norm formula. □

11.17 Relative approximation in a chart without convex extension

The obstruction in Section 11.16 concerns ambient convex functions. It leaves open whether relative convex PL functions suffice on the very same chart. We answer this question for the explicit family constructed there, including a perturbation with a uniform positive tangential Hessian. The proof keeps the value group visible: all affine slopes of the approximants are rational, and all constants belong to $\Gamma_{\mathbf{Q}}$. It gives a worked comparison in every positive dimension, rather than the general converse in Wu's Theorem A.3.

The source distinction behind the construction is already important for smooth functions. Gubler–Künnemann retain the smooth facewise-convexity criterion in [GK19, Remark 3.2], while their Example 3.3 disproves its extension to arbitrary continuous functions; see Example 11.1.4. For a smooth representative on a locally finite complex, Lemma 11.8.4 supplies a centred affine lower support at each point. Neither statement supplies an ambient convex extension, nor a lower bound on the contact radius independent of the point. The following calculation separates all three assertions.

11.17.1 Tangential positivity and the size of a contact neighbourhood

Fix a complete nontrivially valued field K with its given absolute value. Use the all-field construction of Theorem 11.16.5:

$$P(t) = t^2 - t + 1, \qquad C = \mathbf{P}^1_K \setminus \big(\{0, 1, \infty\} \cup V(P)\big),$$
$$Y = C \times (\mathbf{G}_m)^{d-1}, \qquad d \geq 1,$$
$$x = -\log|t(t-1)|, \quad y = -\log|tP(t)|, \quad z_i = -\log|w_i|,$$
$$\tau = (x, y, z_1, \dots, z_{d-1}).$$

Here w_i are the torus coordinates. The image is $S \times \mathbf{R}^{d-1}$, where S is the union of the four rays generated by $(1,1), (1,0), (0,1), (-2,-3)$. The unit-disc Gauss point in the first factor, together with $w_i = 1$ in the other factors, gives a point ξ_d with tropical image zero.

Proposition 11.17.1 (Positive tangential Hessian without ambient convex extension)**.** *Let $0 < \varepsilon < 1/2$, and put*

$$F_\varepsilon(x, y, z) = xy + \varepsilon\left(x^2 + y^2 + \sum_{i=1}^{d-1} z_i^2\right), \qquad u_\varepsilon = F_\varepsilon \circ \tau. \tag{11.17.1}$$

Then u_ε is original CLD smooth and psh. For each maximal face σ of $S \times \mathbf{R}^{d-1}$ and each vector v tangent to σ,

$$D^2 F_\varepsilon(v, v) \geq 2\varepsilon\|v\|^2. \tag{11.17.2}$$

Nevertheless, $F_\varepsilon|_{S \times \mathbf{R}^{d-1}}$ has no ambient convex extension near zero. If an ambient convex function G is defined on a convex set containing

$$p_h = (h, h, 0), \qquad q_h = (2h, 0, 0), \qquad r_h = (0, 2h, 0), \quad h > 0,$$

then

$$\max_{a\in\{p_h,q_h,r_h\}} |G(a) - F_\varepsilon(a)| \geq \frac{1-2\varepsilon}{2}h^2. \tag{11.17.3}$$

Every ordinary neighbourhood of ξ_d contains analytic points realizing such triples for all sufficiently small $h > 0$. Thus ambient convex approximation in this chart cannot converge uniformly on any fixed such neighbourhood.

Proof. Theorem 11.16.5 proves that xy is original smooth psh, by expressing it as the convex function $s_+^2 + 6s_-^2$ of $s = -\log|t|$ and using smooth regularization. The added quadratic is an ambient smooth convex function of unit logarithms. Its pullback is smooth psh, so their sum is psh on all original CLD tests. No pullback operation on arbitrary currents is involved.

Write a tangent vector as $v = (a\alpha, a\beta, v_z)$, where (α, β) is a generator of one of the four rays. In each case $\alpha\beta \geq 0$, and

$$D^2F_\varepsilon(v,v) = 2a^2\alpha\beta + 2\varepsilon\|v\|^2,$$

which proves (11.17.2). This is the precise tangential positivity assertion used here; it is not a claim about every possible convention for strict psh.

Since $p_h = (q_h + r_h)/2$, the values of F_ε give the midpoint defect

$$F_\varepsilon(p_h) - \frac{F_\varepsilon(q_h) + F_\varepsilon(r_h)}{2} = (1-2\varepsilon)h^2.$$

If the error at all three points is at most e, convexity of G bounds this defect above by $2e$. This proves (11.17.3). The Gauss-point paths in Theorem 11.16.5, with $w_i = 1$, converge to ξ_d and realize the three rays in question. Fixing one sufficiently small h in a proposed approximation domain gives a positive error lower bound independent of the approximant. □

The threshold in this quadratic family is exact: the ambient Hessian has eigenvalues $2\varepsilon + 1$, $2\varepsilon - 1$, and 2ε in the remaining directions. Thus F_ε itself is ambient convex when $\varepsilon \geq 1/2$. A positive lower bound on its restrictions to face tangent spaces is therefore strictly weaker than ambient convexity.

Corollary 11.17.2 (Monomial refinements and nonuniform contact radii)**.** *Replacing or augmenting the coordinates of τ only by constants times monomials in its original units does not remove the ambient convex obstruction. Also, any affine lower support ℓ for $F_\varepsilon|_{S\times\mathbf{R}^{d-1}}$ at p_h, valid on a relative ball $B(p_h, R) \cap (S \times \mathbf{R}^{d-1})$, must satisfy $R \leq \sqrt{2}h$. In particular, the contact radii cannot have a positive uniform lower bound as $p_h \to 0$.*

Proof. A monomial change has tropical form $b \mapsto Bb + c$, with an integral matrix B and $c \in \Gamma^m$. It preserves the midpoint relation of the three points. The composition of an ambient convex function with this affine map is convex on the inverse image of its convex domain, so the preceding defect argument applies unchanged. Injectivity of B is unnecessary. For the second assertion, if $R > \sqrt{2}h$, both q_h, r_h belong to the ball. An affine lower support with equality at p_h would give

$$F_\varepsilon(p_h) = \ell(p_h) = \frac{\ell(q_h) + \ell(r_h)}{2} \le \frac{F_\varepsilon(q_h) + F_\varepsilon(r_h)}{2},$$

contrary to the positive defect. □

There is no conflict with Lemma 11.8.4: that lemma fixes the contact point, and its radius is allowed to depend on the point. Nor does this corollary require a chart change for relative PL approximation. The construction below takes place in the original chart. New analytic units can change the ambient problem: the unit t used above is not a monomial in $t(t-1)$ and $tP(t)$, even after allowing a nonzero integral power of t. Indeed, in $t^m = c[t(t-1)]^a[tP(t)]^b$, orders at 1 force $a = 0$, orders at a zero of P force $b = 0$, and the order at 0 then forces $m = 0$.

11.17.2 An explicit decreasing approximation with controlled coefficients

The general ambient density argument in Section 11.12.4 can be made particularly explicit for the positive-part square. We give the formula because it allows composition with relative PL functions without introducing constants outside the value group.

Lemma 11.17.3 (Rational PL approximation of the positive-part square)**.** *Let $\Gamma \subset \mathbf{R}$ be a nonzero additive subgroup, fix $\lambda \in \Gamma_{>0}$ and $M > 0$, and put $\Gamma_{\mathbf{Q}} = \Gamma \otimes_{\mathbf{Z}} \mathbf{Q} \subset \mathbf{R}$. There are nonnegative, nondecreasing, convex PL functions $q_j : \mathbf{R} \to \mathbf{R}$, with rational slopes and constants in $\Gamma_{\mathbf{Q}}$, such that on $[-M, M]$*

$$q_{j+1} \le q_j, \qquad 0 \le q_j(t) - t_+^2 \le \frac{8}{3}\lambda 4^{-j}, \qquad j \ge 0. \tag{11.17.4}$$

The decrease in j is asserted on this fixed interval, not on all of $\mathbf{R}$.

Proof. Choose an integer $N_0 \geq 1$ with $(4N_0^2)^{-1} \leq \lambda$, and set $N_j = N_0 2^j$ and $\gamma_j = \lambda 4^{-j}$. For $0 \leq k \leq \lceil 2MN_j \rceil$, put

$$p_{j,k} = \frac{k}{N_j}, \qquad c_{j,k} = \gamma_j \left\lfloor -\frac{p_{j,k}^2}{4\gamma_j} \right\rfloor, \qquad a_j(t) = \max_k \{p_{j,k}t + c_{j,k}\}.$$

The slopes are rational and nonnegative. The rounded constants are in $\Gamma_{\mathbf{Q}}$; in particular we have not assumed that squares of value-group elements remain in that group. The $k = 0$ branch is zero. Every other branch lies below the tangent line $pt - p^2/4$ of t^2 at $p/2$, and hence below t_+^2 on $\mathbf{R}$. For $t \in [0, M]$, select $p = k/N_j$ with $|2t - p| \leq N_j^{-1}$. The two errors satisfy

$$t^2 - (pt - p^2/4) = (t - p/2)^2 \leq \frac{1}{4N_j^2} \leq \gamma_j,$$
$$0 \leq -p^2/4 - c_{j,k} < \gamma_j.$$

For $t < 0$, $a_j(t) = 0 = t_+^2$. Thus $0 \leq t_+^2 - a_j(t) \leq 2\gamma_j$ on $[-M, M]$. To obtain decrease without requiring the supporting lines to be nested, define

$$\delta_j = \sum_{k=j}^{\infty} 2\gamma_k = \frac{8}{3}\gamma_j, \qquad q_j = a_j + \delta_j.$$

Both δ_j and every affine constant of q_j belong to $\Gamma_{\mathbf{Q}}$. On the fixed interval,

$$q_j - q_{j+1} \geq -2\gamma_j + \delta_j - \delta_{j+1} = 0, \qquad \frac{2}{3}\gamma_j \leq q_j - t_+^2 \leq \frac{8}{3}\gamma_j.$$

These inequalities prove the claim. Nonnegativity and monotonicity as functions of t follow from the zero branch, the positive shift, and the nonnegative slopes. □

Proposition 11.17.4 (Composition with relative PL supports)**.** *Let T be a subset of a real vector space. Suppose that the continuous functions $\psi_1, \ldots, \psi_m : T \to [0, M]$ have local affine lower supports and finite-selection PL presentations with rational slopes and constants in $\Gamma_{\mathbf{Q}}$. For $b_i \in \mathbf{Q}_{\geq 0}$, the functions*

$$v_j = \sum_{i=1}^{m} b_i q_j(\psi_i)$$

have the same two properties. On all of T, they decrease uniformly to $\sum_i b_i \psi_i^2$, with error at most $(8/3)\lambda 4^{-j} \sum_i b_i$. If T is the image of a chart of analytic

units over K and $\Gamma = -\log|K^\times|$, their pullbacks are actual $\mathbf{Q}$-PL functions by Proposition 11.11.5, on its good strictly analytic pure-dimensional domain. They are classically psh by Theorem 11.11.2, and have nef residues by Theorem 11.11.6.

Proof. Fix $a \in T$ and an affine lower support ℓ for ψ_i at a. Choose an active affine branch $pt + c$ of q_j at $\psi_i(a)$. Since $p \geq 0$, near a one has

$$q_j(\psi_i(b)) \geq p\psi_i(b) + c \geq p\ell(b) + c,$$

with equality at $b = a$. Summing the resulting affine lower supports proves the support assertion. Finite selections remain finite after composition with a finite maximum and after a finite sum. Rational coefficients remain rational, and $\Gamma_{\mathbf{Q}}$ is closed under rational multiplication and addition. The uniform estimate and the decrease follow term by term from Lemma 11.17.3. The analytic conclusions use the three cited results with exactly these coefficient and support hypotheses; no ambient convex extension is required. □

Theorem 11.17.5 (Same-chart relative approximation of the obstructed family)**.** *In (11.17.1), assume also $\varepsilon \in \mathbf{Q}$. For every $M > 0$, on the single ordinary open set*

$$V = \tau^{-1}\big((-M, M)^{d+1}\big) \subset Y^{\mathrm{an}},$$

there is a decreasing sequence of relative convex rational PL functions in the original chart, with constants in $\Gamma_{\mathbf{Q}}$, whose pullbacks converge uniformly to u_ε. These pullbacks are actual $\mathbf{Q}$-PL functions with nef residues. No extension of the base field and no change of the analytic chart are needed.

Proof. On S, define

$$A(x, y) = \max\{0, \min(x, y)\}, \qquad B(x, y) = \max\{0, -x/2, -y/3\}.$$

The function A has local affine lower supports: at the vertex, zero is a support; away from it a sufficiently small relative neighbourhood lies on one ray, where A is affine. The same argument applies on $S \times \mathbf{R}^{d-1}$, since A is independent of z. The function B is ambient convex. Both have rational finite-selection PL presentations with zero constants. Evaluating on the four rays gives

$$xy = A^2 + 6B^2 \quad \text{on } S. \tag{11.17.5}$$

Indeed, the values of (A, B) are $(s, 0)$ on $(x, y) = (s, s)$ with $s \geq 0$, $(0, -s)$ on $(x, y) = (2s, 3s)$ with $s \leq 0$, and $(0, 0)$ on both nonnegative coordinate rays.

Choose $\lambda \in \Gamma_{>0}$, which is possible because the valuation is nontrivial, and use the functions q_j of Lemma 11.17.3. On the fixed tropical image of V, put

$$v_j = q_j(A) + 6q_j(B) + \varepsilon \left(q_j(|x|) + q_j(|y|) + \sum_{i=1}^{d-1} q_j(|z_i|) \right). \tag{11.17.6}$$

Every argument of q_j lies in $[0, M]$. The functions $|x|, |y|, |z_i|$ are ambient convex rational PL. Proposition 11.17.4 therefore applies to all summands, and (11.17.5) identifies the limit as F_ε. More precisely, throughout $\tau(V)$,

$$0 \le v_j - F_\varepsilon \le \big(7 + \varepsilon(d+1)\big)\frac{8}{3}\lambda 4^{-j}, \qquad v_{j+1} \le v_j. \tag{11.17.7}$$

The set V is an ordinary open subset of the smooth analytification Y^{an}, hence good, strictly analytic, and pure-dimensional. The coefficient, classical psh and nef-residue conclusions of Proposition 11.17.4 apply on V. All constants were constructed inside the original $\Gamma_{\mathbf{Q}}$; the use of its divisible hull means rational model data, not a change of the absolute value or an implicit field extension. □

This example gives both the failed ambient approximation and a successful relative approximation on the same fixed neighbourhood. The argument extends beyond squares. The extra point in the next statement is to preserve nonnegative slopes when rationalizing supporting hyperplanes: that sign is what makes composition preserve relative lower supports.

Proposition 11.17.6 (Monotone convex composition in any dimension)**.** *Let $\psi = (\psi_1, \dots, \psi_m) : T \to [0, M]^m$ satisfy the hypotheses of Proposition 11.17.4. Let H be a finite convex function on a convex open neighbourhood of $[0, M]^m$, nondecreasing in each coordinate there. For every $\lambda \in \Gamma_{>0}$ there are finite maxima Q_j of affine functions with slopes in $\mathbf{Q}^m_{\ge 0}$ and constants in $\Gamma_{\mathbf{Q}}$ such that*

$$Q_{j+1} \le Q_j, \qquad 0 \le Q_j - H \le 4\lambda 4^{-j} \quad \text{on } [0, M]^m. \tag{11.17.8}$$

Consequently $Q_j \circ \psi$ are coefficient-compatible relative PL functions decreasing uniformly to $H \circ \psi$ on the same set T. The analytic PL, classical psh and nef-residue conclusions of Proposition 11.17.4 apply to their pullbacks.

Proof. Put $D = [0, M]^m$ and $\gamma_j = \lambda 4^{-j}$. As in Proposition 11.12.9, the supporting-hyperplane theorem supplies an affine minorant $\ell_a(t) = p_a \cdot t + b_a$ touching H at each $a \in D$. All coordinates of p_a are nonnegative: if $(p_a)_i < 0$,

the support inequality at $a-he_i$ for small $h>0$ would contradict the assumed monotonicity of H. This uses the open neighbourhood of D, including at its boundary.

The sets where $H-\ell_a<\gamma_j$ cover D. By continuity and compactness, finitely many of these supports suffice. For each selected pair (p,b), choose $r\in\mathbf{Q}^m_{\geq 0}$ with $r_i\leq p_i$ and $\|p-r\|_1\leq \gamma_j/(M+1)$, and set $c=\gamma_j\lfloor b/\gamma_j\rfloor\in\Gamma_{\mathbf{Q}}$. On D,

$$0\leq (p\cdot t+b)-(r\cdot t+c)\leq M\|p-r\|_1+(b-c)<2\gamma_j.$$

Let R_j be the maximum of the finitely many resulting affine functions. Then $0\leq H-R_j\leq 3\gamma_j$ on D. Set $Q_j=R_j+4\gamma_j$. Since $4\gamma_j-4\gamma_{j+1}=3\gamma_j$,

$$Q_j-Q_{j+1}\geq -3\gamma_j+4\gamma_j-4\gamma_{j+1}=0,\qquad \gamma_j\leq Q_j-H\leq 4\gamma_j.$$

This proves (11.17.8). At a point of T, choose local affine lower supports L_i for the ψ_i and an active affine branch $\sum_i r_it_i+c$ of Q_j at ψ. The function $\sum_i r_iL_i+c$ is a local affine lower support for $Q_j\circ\psi$, because every $r_i\geq 0$. Finite selections and the coefficient conditions are preserved just as in Proposition 11.17.4. □

The same construction is compatible with a decreasing tuple. More precisely, suppose each $\psi_{i,j}:T\to[0,M]$ has the stated coefficient-compatible PL and support presentations and $\psi_{i,j+1}\leq\psi_{i,j}\to\psi_i$. The functions Q_j depend only on H,D,Γ, not on these presentations. Hence

$$Q_{j+1}(\psi_{1,j+1},\dots,\psi_{m,j+1})\leq Q_j(\psi_{1,j+1},\dots,\psi_{m,j+1})\leq Q_j(\psi_{1,j},\dots,\psi_{m,j}). \tag{11.17.9}$$

The error estimate and continuity of H identify the pointwise limit as $H(\psi_1,\dots,\psi_m)$. Each term is still a relative rational PL function on the same T. Uniform convergence of the tuple gives uniform convergence of these composites by uniform continuity on D; pointwise convergence alone is not upgraded to uniform convergence. This is a finite-tuple closure statement for the displayed generator class, not closure under arbitrary successive approximation stages.

Corollary 11.17.7 (Original smooth approximation of these composites)**.** *Let X be integral projective of positive dimension over a complete nontrivially valued K. Suppose an ordinary open $W \subset X^{\mathrm{an}}$ carries a unit chart and a bounded tuple ψ as in Proposition 11.17.6. Then, near each point of W, the pullback of $H \circ \psi$ is a decreasing limit of original smooth psh functions on one ordinary neighbourhood, uniformly on compact subsets.*

Proof. The proposition supplies a decreasing sequence of actual Q-PL functions with relative supports on all of W. Apply Corollary 11.11.7. Its neighbourhood is fixed before the individual extension and smoothing steps. The endpoint is continuous, so the compact-uniform conclusion follows from the last assertion of that corollary. □

These results combine the supporting-hyperplane argument of Section 11.12.4, sign-preserving rational approximation and the previously established relative-support bridge. They are recorded as reconstructions and consequences of those inputs, not as a claim of a new general regularization theorem. In particular, we have not shown that every smooth facewise-convex function admits such a finite monotone-convex composition presentation. Section 11.22 supplies coefficient-controlled density on any fixed compact image for continuous relatively supported inputs, without requiring such a composition presentation. The remaining reverse-comparison step concerns arbitrary families with different charts and their common analytic domain. The missing full text of the mixed-calculus Theorem 6.13 is not an input to any argument in this section.

11.18 Relative coefficient approximation on a finite polytopal complex

Section 11.12.7 already proves coefficient-compatible decreasing approximation for homogeneous relative convex functions on a fixed rational fan. Its proof approximates the function and all supporting covectors simultaneously inside a rational system. The obstruction following it shows why an arbitrary nonhomogeneous complex cannot be substituted without checking its affine constants. Here we extend that argument to finite polytopal complexes whose vertices share one value-group scale.

The source convention matters. In [APW26, Section 2.1], a rational half-space has an integral normal but an arbitrary real intercept; Section 2.1.6 assumes rational simplicial polyhedral structures in that sense. This does not impose the value-group condition on the vertices used below. The t-wo qualifications must therefore be checked separately when importing PL approximants.

11.18.1 The affine support system

Theorem 11.18.1 (Rational approximation with one geometric scale)**.** *Let $\Gamma \subset \mathbf{R}$ be a nonzero additive subgroup, and fix $\lambda \in \Gamma_{>0}$. Let Π be a finite polytopal complex in $\mathbf{R}^r$ such that every vertex belongs to $\lambda\mathbf{Q}^r$. Let $F : |\Pi| \to \mathbf{R}$ be continuous, affine on each cell, and admit a relative affine lower support near every point. Then there are continuous functions F_j on $|\Pi|$, affine on the same cells with rational slopes and constants in $\lambda\mathbf{Q} \subset \Gamma_{\mathbf{Q}}$, admitting relative affine lower supports and satisfying*

$$F_{j+1} \le F_j, \qquad 0 \le F_j - F \le 3\lambda 8^{-j} \quad \text{on } |\Pi|. \tag{11.18.1}$$

Here polytopal means that the cells are compact polytopes. In particular, the hypothesis on the vertices holds for a finite $(\mathbf{Z}, \Gamma)$-polytopal complex when $\Gamma = \lambda\mathbf{Z}$. The conclusion does not require Γ itself to be discrete if the displayed one-scale hypothesis holds.

Proof. First write $t = \lambda z$ and $\widehat{F}(z) = F(\lambda z)/\lambda$. This is a change of real coordinates and function units for the calculation; the absolute value on any analytic base field remains unchanged. The scaled complex has rational vertices. An affine function with rational coefficients in z pulls back, after multiplication by λ, to a function with rational slopes and $\lambda\mathbf{Q}$-constants in t. It therefore suffices to approximate the function and its supports with rational coefficients in the scaled coordinates.

Write the affine expression on a cell σ as $p_\sigma(z) + c_\sigma$. For every face ρ, introduce an affine support variable $a_\rho(z) + b_\rho$. The conditions are

$$\begin{aligned} &p_\sigma(v) + c_\sigma = p_\tau(v) + c_\tau \quad (v \text{ a vertex of } \sigma \cap \tau),\\ &a_\rho(v) + b_\rho = p_\rho(v) + c_\rho \quad (v \text{ a vertex of } \rho),\\ &a_\rho(v) + b_\rho \le p_\sigma(v) + c_\sigma \quad (\sigma \supseteq \rho,\ v \text{ a vertex of } \sigma). \end{aligned} \tag{11.18.2}$$

The first line is continuity. To justify the support conditions, choose $x \in \operatorname{relint}(\rho)$. A touching affine lower support agrees with $\widehat{F}$ on the affine span

of ρ, since their affine difference has a local minimum of zero there. For $y \in \sigma \supseteq \rho$, apply the support inequality at $x + s(y - x)$ for small $s > 0$. Linearity on σ and equality at x extend the inequality to all of σ. Testing it on vertices is sufficient. Conversely, the last two lines give a lower support on every cell containing ρ. A small neighbourhood of x avoids the other cells: the complex is finite, and a cell containing a point in $\operatorname{relint}(\rho)$ contains ρ. Thus the system is equivalent to the required local relative supports.

All vertices are rational in the scaled coordinates, so (11.18.2) is a finite homogeneous rational linear system in the combined tuple $(p_\sigma, c_\sigma, a_\rho, b_\rho)$. Use the rational-density argument already given in Section 11.12.7: impose the active inequalities as equalities, choose a rational basis of the resulting nullspace, and approximate the tuple within that subspace. The inactive inequalities remain strict for sufficiently close approximations. This preserves both continuity and the existence of supports, rather than merely approximating the cellwise formulas.

Return to the t-coordinates and choose the approximation G_j so that $|G_j - F| \le \epsilon_j := \lambda 8^{-j}$ on $|\Pi|$. This is possible by controlling the error at the finitely many vertices: on each cell the difference is affine, and its absolute value is bounded by the largest vertex error. Put $F_j = G_j + 2\epsilon_j$. Constants still belong to $\lambda\mathbf{Q}$, and

$$F + \epsilon_j \le F_j \le F + 3\epsilon_j, \qquad F_{j+1} \le F + \tfrac{3}{8}\epsilon_j \le F_j.$$

Shifting the supports by $2\epsilon_j$ preserves contact. This proves (11.18.1). For the discrete value-group case, a vertex of a $(\mathbf{Z}, \lambda\mathbf{Z})$-polytope is the solution of a full-rank rational linear system with right-hand side in $\lambda\mathbf{Z}$, so its coordinates lie in $\lambda\mathbf{Q}$. □

On an ordinary unit chart whose image is contained in $|\Pi|$, the pullbacks of F_j are actual Q-PL functions by Proposition 11.11.5, provided the analytic space is good, strictly analytic and pure-dimensional over nontrivially valued K, and $\Gamma = -\log|K^\times|$. Their supports give nef residues by Theorem 11.11.6. In the integral projective setting, Corollary 11.11.7 gives an original smooth decreasing approximation near every point on one ordinary neighbourhood. These are consequences of the stated analytic inputs, not a new current-positivity criterion.

The compactness in the theorem serves to control the error by finitely many vertex values. It need not restrict the original tropical image to a

compact polytope. A single truncation gives the following local version; the translation also clarifies which affine constants are allowed.

Corollary 11.18.2 (Translation and one fixed truncation)**.** *Let Π be a finite polyhedral complex in $\mathbf{R}^r$. Suppose that there are $a \in \Gamma_{\mathbf{Q}}^r$ and $\lambda \in \Gamma_{>0}$ such that every cell is given by rational affine inequalities in the coordinates $z = (t-a)/\lambda$; both normals and intercepts are rational in these coordinates. Let F be continuous, affine on each cell, and have local relative affine lower supports. For every compact $S \subset |\Pi|$, there is an integer $N > 0$ such that*

$$S \subset U := |\Pi| \cap \big(a + (-\lambda N, \lambda N)^r\big),$$

and $F|_U$ admits a decreasing uniform approximation with the bound in (11.18.1), by continuous functions affine on each $\sigma \cap U$, with rational slopes, $\Gamma_{\mathbf{Q}}$-constants and local relative affine lower supports. The set U is fixed for the whole sequence.

Proof. Choose N so that S lies in the interior of the displayed cube. Intersect the scaled complex with $[-N, N]^r$ and include all faces. This is a finite rational polytopal complex. Apply Theorem 11.18.1 to $\widehat{F}(z) = F(a + \lambda z)/\lambda$, with subgroup $\mathbf{Z}$ and scale 1. Its hypotheses concern real polyhedra, so this use of $\mathbf{Z}$ makes no change to an analytic value group. A rational affine branch $p \cdot z + c$ gives, after multiplying by λ and returning to t,

$$p \cdot t + \lambda c - p \cdot a.$$

Its slope is rational and its constant belongs to $\Gamma_{\mathbf{Q}}$. The error is multiplied by λ; order and relative supports are preserved. Restrict to U. Every original cell meets the closed cube in a cell of the truncated complex, so the approximants remain affine on $\sigma \cap U$. No extension beyond this fixed truncation is asserted. □

In particular, this applies to finite $(\mathbf{Z}, \Gamma_{\mathbf{Q}})$-polyhedral complexes over a discrete nontrivial value group $\Gamma = \lambda\mathbf{Z}$, even when the cells are unbounded or have lineality. On a unit chart the inverse image of the open cube is an ordinary open set, so the truncation does not force the analytic domain to vary with the approximation index.

11.18.2 Two distinct restrictions on a fixed subdivision

Example 11.18.3 (Irrational directions)**.** In $\mathbf{R}^2$, put $L(x,y) = x+\sqrt{2}y$, and subdivide the plane into $L \geq 0$, $L \leq 0$ and their common face. The convex function $F = \max\{0, L\}$ is linear on these cones. Every continuous function G which is affine with rational slopes on these same two cones is globally affine. Thus

$$\max_{a\in\{(-1,0),(0,0),(1,0)\}} |G(a) - F(a)| \geq \frac{1}{4}. \tag{11.18.3}$$

Proof. Continuity makes the branch difference aL for some real a. Both coordinates of the slope difference are rational, so a and $a\sqrt{2}$ are rational; hence $a = 0$. The midpoint defect of F at the three displayed points has absolute value $1/2$, whereas an affine function has zero defect. Three errors bounded by e change this defect by at most $2e$, giving $e \geq 1/4$. □

This subdivision is not rational in the APW convention, so the example does not contradict a statement under that convention. The next example has rational directions and isolates the separate intercept condition.

Example 11.18.4 (An intercept outside the value-group hull)**.** Suppose $\Gamma_{\mathbf{Q}} \neq \mathbf{R}$, and choose $b \notin \Gamma_{\mathbf{Q}}$. On $\mathbf{R}^r$, put $F(t) = \max\{0, t_1 - b\}$. Every continuous function that is affine with rational slopes and $\Gamma_{\mathbf{Q}}$-constants on each of the same halfspaces $t_1 \leq b$ and $t_1 \geq b$ is globally affine. For $h > 0$, its uniform error from F at $(b-h, 0), (b, 0), (b+h, 0)$ is at least $h/4$.

Proof. The branch difference vanishes on $t_1 = b$, so it is $a(t_1 - b)$ with $a \in \mathbf{Q}$. If $a \neq 0$, its constant $-ab \in \Gamma_{\mathbf{Q}}$ would force $b \in \Gamma_{\mathbf{Q}}$. Thus the branches agree. The midpoint defect of F has absolute value $h/2$, giving the error bound as before. □

Both examples allow rational approximation when the subdivision varies: approximate $\sqrt{2}$ by rational numbers in the first, and b by elements of $\Gamma_{\mathbf{Q}}$ in the second. On bounded sets the errors tend uniformly to zero; the error shifts of Proposition 11.12.9 enforce decrease. Neither example refutes unrestricted rational approximation. They exclude only freezing the displayed cells while rounding their coefficients. The earlier nonhomogeneous-complex obstruction in Section 11.12.7 addresses a further issue: vertices may belong to a higher-rank $\Gamma_{\mathbf{Q}}$ without sharing the one scale required in Theorem 11.18.1.

The remaining source-comparison task is therefore precise. The real PL output must be replaced by coefficient-compatible relative PL functions on

one fixed domain, while allowing controlled changes of subdivision if its vertices or normals fail these conditions. Theorems for an already rational fan or for a one-scale complex do not establish that replacement for arbitrary smooth inputs. No full Wu Theorem A.3, CLD25 Proposition 8.1.11 or mixed-calculus comparison is inferred here.

11.19 Coefficient closure and relative supports

The one-scale hypothesis of Section 11.18 makes the entire support system rational. With several independent value-group scales, continuity alone may already prevent rational approximation. We first describe that obstruction exactly. We then separate it from the additional requirement of relative convexity, and give a finite strict-support condition under which no further obstruction occurs. All approximations use the same compact support; each statement specifies whether its subdivision is fixed or allowed to vary.

11.19.1 Continuity and the coefficient space

Fix a nonzero additive subgroup $\Gamma \subset \mathbf{R}$ and put $\Gamma_{\mathbf{Q}} = \Gamma \otimes_{\mathbf{Z}} \mathbf{Q} \subset \mathbf{R}$. Let Π be a nonempty finite complex of compact polytopes in $\mathbf{R}^r$, with support $S = |\Pi|$. Write $V(\Pi)$ for the real vector space of continuous functions on S that are affine on each cell. Its uniform norm is the maximum of the absolute values at the finitely many vertices. Define the rational vector space

$$A_\Gamma(\Pi) = \{F \in V(\Pi) : F|_\sigma = p_\sigma \cdot t + c_\sigma,\ p_\sigma \in \mathbf{Q}^r,\ c_\sigma \in \Gamma_{\mathbf{Q}}\}.$$

The formula is required for every cell, but its ambient extension need not be unique. Let $V_\Gamma(\Pi)$ denote the uniform closure of $A_\Gamma(\Pi)$ in $V(\Pi)$. This is a real vector subspace: rational scalar multiplication extends to real scalar multiplication by taking limits. The following description makes this closure explicit when the vertices have value-group coordinates.

Theorem 11.19.1 (Continuity closure)**.** *Suppose that S is connected and every vertex of Π lies in $\Gamma_{\mathbf{Q}}^r$. There is a rational linear subspace $L \subset \mathbf{R}^{r\#\Pi}$ and a real linear map $T : L \times \mathbf{R} \to V(\Pi)$ such that*

$$A_\Gamma(\Pi) = T\big((L \cap \mathbf{Q}^{r\#\Pi}) \times \Gamma_{\mathbf{Q}}\big), \qquad V_\Gamma(\Pi) = T(L \times \mathbf{R}). \tag{11.19.1}$$

The construction below uses only the vertex coordinates and continuity equations. For disconnected S, apply it to each connected component with a separate constant.

Proof. Let E be the rational span of all vertex coordinates, enlarged by one nonzero element of Γ if necessary. Choose a rational basis $\gamma_1, \dots, \gamma_s$ of E. Introduce one ambient slope variable $p_\sigma \in \mathbf{R}^r$ for each nonempty cell. The graph joining two cells with nonempty intersection is connected: otherwise the corresponding finite compact unions would disconnect S. Choose a root cell σ_0, a spanning tree and, for each edge $\sigma\tau$, a vertex $v_{\sigma\tau}$ of the intersection. Along the rooted tree define

$$d_{\sigma_0}(p) = 0, \qquad d_\tau(p) = d_\sigma(p) + (p_\sigma - p_\tau) \cdot v_{\sigma\tau}.$$

Continuity at the remaining intersection vertices requires

$$(p_\sigma - p_\tau) \cdot v + d_\sigma(p) - d_\tau(p) = \sum_{a=1}^{s} \gamma_a A_{\sigma,\tau,v,a}(p) = 0. \tag{11.19.2}$$

Each $A_{\sigma,\tau,v,a}$ is a rational linear form. Let L be their common kernel, and set

$$T(p,c)|_\sigma(t) = p_\sigma \cdot t + d_\sigma(p) + c.$$

For $p \in L$, the expressions agree at every intersection vertex, hence on every intersection; thus T is well-defined. If p is rational and $c \in \Gamma_{\mathbf{Q}}$, all $d_\sigma(p)$ belong to $E \subset \Gamma_{\mathbf{Q}}$. This proves one inclusion in the first equality of (11.19.1). Conversely, choose rational ambient slopes and value-group constants for $F \in A_\Gamma(\Pi)$. Continuity along the tree forces $c_\sigma = c_{\sigma_0} + d_\sigma(p)$. In (11.19.2), all $A_{\sigma,\tau,v,a}(p)$ are rational. The rational independence of the γ_a therefore forces each to vanish, so $p \in L$.

The rational points of L are dense in L, and $\Gamma_{\mathbf{Q}}$ is dense in $\mathbf{R}$, since it contains $\rho\mathbf{Q}$ for some $\rho > 0$. The image of their product is therefore dense in $T(L \times \mathbf{R})$. This image space is finite-dimensional and hence closed in $V(\Pi)$. These facts prove the second equality. For disconnected S, there are only finitely many components, and their function spaces form a finite product. □

Remark 11.19.2. The criterion for a target F is the existence of a representation $F = T(p,c)$ with $p \in L$. On a lower-dimensional cell it is not legitimate to test an arbitrary ambient extension and declare failure. Nor does the construction claim an effective procedure for identifying rational relations among arbitrary unspecified real numbers. When all vertices lie in $\lambda\mathbf{Q}^r$, one can take $s = 1$, and the split equations are exactly the original continuity equations. Then $V_\Gamma(\Pi) = V(\Pi)$, consistently with Theorem 11.18.1. With several scales the equality can fail, as the rectangle calculation in Section 11.12.7 shows.

11.19.2 The relative-support cone

Let $C(\Pi) \subset V(\Pi)$ consist of functions admitting an affine lower support in the relative topology of S near every point, with equality at that point. The support is allowed to depend on the point. This is the local support condition of Section 11.18; it is not ambient convexity on a neighbourhood of S.

Lemma 11.19.3 (Finite support inequalities)**.** *The set $C(\Pi)$ is a closed convex polyhedral cone in $V(\Pi)$, and contains all constants with either sign. Its defining real coefficients need not be rational. More precisely, $F \in C(\Pi)$ if and only if for every nonempty face ρ there is an affine function a_ρ satisfying*

$$\begin{aligned} a_\rho(v) &= F(v) \quad (v \text{ a vertex of } \rho), \\ a_\rho(v) &\le F(v) \quad (v \text{ a vertex of } \sigma,\ \sigma \supseteq \rho). \end{aligned} \tag{11.19.3}$$

Proof. Choose $x \in \operatorname{relint}(\rho)$. A lower support at x agrees with F on $\operatorname{aff}(\rho)$, since their affine difference has a local minimum there. For any $y \in \sigma \supseteq \rho$, apply the local inequality at $x+t(y-x)$ and divide by $t > 0$; this extends it to y. Conversely, (11.19.3) gives a support on each coface of ρ, and a sufficiently small neighbourhood of x meets no other cells. This proves the equivalence.

The system is finite and homogeneous linear in the vertex values and support coefficients. Eliminating the latter preserves polyhedrality. For completeness, to eliminate a variable y, write each inequality as $a_i y + b_i(z) \ge 0$. Keep the inequalities with $a_i = 0$, and require each lower bound from $a_i > 0$ to be no larger than each upper bound from $a_i < 0$. If one type is absent there is no additional restriction. Equalities may be written as pairs of inequalities. Repeating this elimination gives finitely many weak linear inequalities in the vertex values alone, proving closedness as well. Addition, nonnegative scaling and constant shifts preserve the support condition directly. □

Closedness alone does not justify intersecting with a dense rational class. For instance, the ray $\{(x, \sqrt{2}x) : x \ge 0\}$ is a closed real polyhedral cone whose intersection with $\mathbf{Q}^2$ is only the origin. This is a warning about a linear-algebra inference, not a tropical or analytic counterexample. The next statement gives the exact replacement.

Theorem 11.19.4 (Closure of admissible rational functions)**.** *For any finite compact polytopal complex, set*

$$W_\Gamma(\Pi) = \operatorname{span}_{\mathbf{R}}\big(A_\Gamma(\Pi) \cap C(\Pi)\big).$$

Then

$$\overline{A_\Gamma(\Pi) \cap C(\Pi)}^{\,\|\cdot\|_\infty} = W_\Gamma(\Pi) \cap C(\Pi). \tag{11.19.4}$$

If F belongs to this set and $\rho \in \Gamma_{>0}$, there are $F_j \in A_\Gamma(\Pi) \cap C(\Pi)$ such that

$$F_{j+1} \le F_j, \qquad 0 \le F_j - F \le 3\rho 8^{-j} \quad \text{on } S. \tag{11.19.5}$$

Proof. Both $W_\Gamma(\Pi)$ and $C(\Pi)$ are closed, so the left side of (11.19.4) is contained in the right side. To prove density, choose a real basis $g_1, \dots, g_m$ of $W_\Gamma(\Pi)$ from $A_\Gamma(\Pi) \cap C(\Pi)$. Its positive cone lies in $C(\Pi)$ and has nonempty interior in $W_\Gamma(\Pi)$. In particular, $H = \sum_i g_i$ has a ball $B_W(H, a)$ contained in $C(\Pi)$. For $F \in W_\Gamma(\Pi) \cap C(\Pi)$ and $\epsilon > 0$,

$$B_W(F + \epsilon H, \epsilon a) = F + \epsilon B_W(H, a) \subset C(\Pi).$$

Approximate the coordinates of $F + \epsilon H$ in the basis g_i by rational numbers so closely that the resulting function lies in this ball. It belongs to $A_\Gamma(\Pi)$, since that is a rational vector space, and it belongs to $C(\Pi)$ by the ball inclusion. Letting ϵ and the coordinate errors tend to zero proves density. The argument does not require the rational coordinates to be nonnegative.

Choose G_j in this intersection with $\|G_j - F\|_\infty \le \eta_j := \rho 8^{-j}$ and put $F_j = G_j + 2\eta_j$. The added constant belongs to $\Gamma_{\mathbf{Q}}$, and shifting every support by it preserves contact. Thus $F + \eta_j \le F_j \le F + 3\eta_j$ and $F_{j+1} \le F + 3\eta_j/8 \le F_j$. This proves (11.19.5). □

This is an exact closure formula, but not by itself an effective description of $W_\Gamma(\Pi)$. A useful finite certificate is a list of admissible rational functions g_i whose real span contains the target F. The finite support system checks admissibility, and the theorem then gives decreasing approximation. In general we have only $W_\Gamma(\Pi) \subseteq V_\Gamma(\Pi)$; equality requires an argument. The proof is an elementary combination of finite-dimensional convexity and rational approximation, and no priority claim is made for that combination.

11.19.3 A strict-support certificate

We now give a geometric sufficient condition for this equality. Say that $H \in V(\Pi)$ has strict face supports if, for every nonempty face ρ, there is an affine a_ρ agreeing with H on ρ and satisfying

$$H(v) > a_\rho(v) \quad (\sigma \supseteq \rho,\ v \text{ a vertex of } \sigma,\ v \notin \rho). \tag{11.19.6}$$

This is a finite condition on a specified complex. It is not a definition of strict CLD plurisubharmonicity. For a maximal face the inequalities are vacuous.

Proposition 11.19.5 (Strict supports remove the positivity obstruction)**.** *If $H \in A_\Gamma(\Pi)$ has strict face supports, then H is an interior point of $C(\Pi)$ in $V(\Pi)$, and*

$$W_\Gamma(\Pi) = V_\Gamma(\Pi), \qquad \overline{A_\Gamma(\Pi) \cap C(\Pi)} = V_\Gamma(\Pi) \cap C(\Pi).$$

Consequently every target in the latter intersection has the decreasing approximants of (11.19.5)*.*

Proof. For each ρ, choose a linear extension map E_ρ from affine functions on $\mathrm{aff}(\rho)$ to affine functions on $\mathbf{R}^r$. Such a map exists by extending affine coordinates, and all the relevant spaces are finite-dimensional. If $\delta \in V(\Pi)$, replace a_ρ by $a_\rho + E_\rho(\delta|_\rho)$. It agrees with $H + \delta$ on ρ. At the finitely many other incident vertices the change in the support gap is

$$\delta(v) - E_\rho(\delta|_\rho)(v).$$

The absolute values of these expressions are bounded by $M\|\delta\|_\infty$ for a fixed finite M. The finitely many strict gaps in (11.19.6) have a positive minimum. For sufficiently small $\|\delta\|_\infty$, all remain positive, so Lemma 11.19.3 gives $H + \delta \in C(\Pi)$. If there are no strict gaps to test, the same conclusion holds for every δ. This proves interiority.

For $F \in C(\Pi) \cap V_\Gamma(\Pi)$, the points $F + \epsilon H$ are interior to $C(\Pi)$ relative to $V_\Gamma(\Pi)$. The density of $A_\Gamma(\Pi)$ in $V_\Gamma(\Pi)$ gives admissible rational approximants tending to F, exactly as in Theorem 11.19.4. Moreover, a sufficiently small ball about H in $V_\Gamma(\Pi)$ is contained in the closure of the admissible class, hence in $W_\Gamma(\Pi)$. A linear subspace containing such a ball is all of $V_\Gamma(\Pi)$. This proves both equalities and the decreasing conclusion. □

The hypothesis has an explicit source of examples, valid in arbitrary dimension and without a one-scale restriction. The subdivision must be the one specified below; adding arbitrary faces can destroy the strict certificate.

Proposition 11.19.6 (Subdivisions of a finite rational maximum)**.** *Let $Q \subset \mathbf{R}^r$ be a full-dimensional compact polytope given by $h_b \geq 0$, where the h_b have rational slopes and constants in $\Gamma_{\mathbf{Q}}$. Let*

$$H = \max_{1 \leq i \leq N} \ell_i, \qquad \ell_i(t) = p_i \cdot t + c_i, \quad p_i \in \mathbf{Q}^r, \ c_i \in \Gamma_{\mathbf{Q}}.$$

Form the complex of all nonempty regions $R_i = \{t \in Q : H(t) = \ell_i(t)\}$ and all their faces. On any nonempty subcomplex Π of this complex, $H|_{|\Pi|}$ belongs to $A_\Gamma(\Pi)$ and has strict face supports. Every vertex of Π lies in $\Gamma_{\mathbf{Q}}^r$. Thus the continuity criterion of Theorem 11.19.1, component by component, is necessary and sufficient for a relatively supported $F \in V(\Pi)$ to admit the approximants (11.19.5).

Proof. The regions form a polyhedral complex: $R_i \cap R_j$ is the equality face for $\ell_i - \ell_j \geq 0$ on R_i, and similarly on R_j. All defining normals are rational and constants belong to $\Gamma_{\mathbf{Q}}$. At a vertex, a full-rank set of active equations has a rational inverse matrix, so the coordinates belong to $\Gamma_{\mathbf{Q}}$. This argument also applies to lower-dimensional regions.

Fix a face ρ, choose $x \in \operatorname{relint}(\rho)$, and set $I = \{i : \ell_i(x) = H(x)\}$ and $B = \{b : h_b(x) = 0\}$. Define

$$a_\rho = \frac{1}{|I|} \sum_{i \in I} \ell_i - \sum_{b \in B} h_b. \tag{11.19.7}$$

On Q, both $H - |I|^{-1} \sum_{i \in I} \ell_i$ and every h_b in this sum are nonnegative. Their sum vanishes precisely where all the indicated maximum branches and boundary constraints are active. This set is exactly ρ: inside any R_{i_0} with $i_0 \in I$, these are all the constraints active at x, hence they define the minimal face containing x. Consequently $a_\rho = H$ on ρ and $a_\rho < H$ at every point of $Q \setminus \rho$. Formula (11.19.7) supplies the required strict face supports, even after restricting to a subcomplex. The function H is rational affine on each cell. Proposition 11.19.5 and Theorem 11.19.1 finish the proof. □

11.19.4 One fixed refinement

A complex need not itself have strict face supports. Refinement gives a precise remedy when its cells have rational normals and value-group constants. The continuity obstruction, however, survives every fixed refinement of the kind considered here.

Lemma 11.19.7 (Continuity closure under a fixed refinement)**.** *Let Π' be a finite compact polytopal refinement of Π with the same support: each refined cell lies in a coarse cell, and each coarse cell is a union of refined cells. Suppose that every cell of Π has rational tangent space, and all vertices of both complexes belong to $\Gamma^r_{\mathbf{Q}}$. Under the natural inclusion $V(\Pi) \subseteq V(\Pi')$,*

$$V_\Gamma(\Pi') \cap V(\Pi) = V_\Gamma(\Pi). \tag{11.19.8}$$

Proof. Restriction to the refined cells gives $A_\Gamma(\Pi) \subseteq A_\Gamma(\Pi')$, hence one inclusion. For the reverse inclusion, use Theorem 11.19.1 on Π', component by component, to represent $F \in V_\Gamma(\Pi') \cap V(\Pi)$ by slope parameters $p \in L$ and root constants. For each coarse cell σ, take all refined cells $\tau \subseteq \sigma$ of dimension $\dim \sigma$. They cover σ: their finite closed union contains the relative interior densely and therefore contains its closure. Choose a rational basis $u_{\sigma,k}$ for the tangent space of σ. Being affine on σ implies

$$(p_\tau - p_{\tau_0}) \cdot u_{\sigma,k} = 0 \quad (\tau, \tau_0 \subseteq \sigma,\ \dim \tau = \dim \tau_0 = \dim \sigma). \tag{11.19.9}$$

Conversely, these equations, together with continuity, force affinity on σ. Subtract an ambient affine function with slope p_{τ_0}; the remainder is constant on every full-dimensional refined cell. The intersection graph of these cells is connected because they cover the connected compact polytope σ, so their constants agree.

Equations (11.19.9) are rational linear equations in the slope parameters. Their intersection with L is therefore a rational linear subspace containing the chosen representative of F. Approximate within that subspace by rational parameters, and approximate each root constant in $\Gamma_{\mathbf{Q}}$. The resulting functions converge uniformly to F, belong to $A_\Gamma(\Pi')$, and are affine on every coarse cell. On a coarse cell their expression agrees with that of any full-dimensional refined cell, so it has rational ambient slope and constant in $\Gamma_{\mathbf{Q}}$. They thus belong to $A_\Gamma(\Pi)$, proving (11.19.8). Zero-dimensional coarse cells require no slope equation. □

Theorem 11.19.8 (Relative approximation after one fixed refinement)**.** *Suppose each cell of the finite compact complex Π is given by finitely many affine weak inequalities with rational slopes and constants in $\Gamma_{\mathbf{Q}}$. There is a finite compact refinement Π^*, depending only on Π, such that for every $F \in C(\Pi)$ the following are equivalent:*

1. *$F \in V_\Gamma(\Pi)$.*

2. *F is a uniform limit of functions in $A_\Gamma(\Pi^*) \cap C(\Pi^*)$.*

3. *F has approximants in that class satisfying (11.19.5).*

Proof. List all affine functions $h_1, \dots, h_N$ occurring in inequalities for all cells, writing equalities as two inequalities. Choose a cube $Q = [-M\rho, M\rho]^r$ containing S in its interior, with $\rho \in \Gamma_{>0}$ and M a sufficiently large integer. On this cube set

$$H = \sum_{l=1}^{N} |h_l| = \max_{\epsilon \in \{-1,1\}^N} \sum_{l=1}^{N} \epsilon_l h_l. \tag{11.19.10}$$

The regions of this maximum are exactly the nonempty sign regions $\epsilon_l h_l \geq 0$ in Q: equality in the maximum requires equality in each nonnegative term $|h_l| - \epsilon_l h_l$. Include their faces. Each original cell is a union of these arrangement cells, since its defining inequalities occur in the list. Take the subcomplex consisting of arrangement cells contained in S; call it Π^*. It refines every original cell and has the same support. All its vertices lie in $\Gamma_{\mathbf{Q}}^r$, and Proposition 11.19.6 supplies strict face supports for H.

The relative-support condition depends on the function and the support S, not on a chosen subdivision. Thus $F \in C(\Pi)$ also belongs to $C(\Pi^*)$. If (1) holds, restriction gives $F \in V_\Gamma(\Pi^*)$, and Proposition 11.19.5 proves (3). Clearly (3) implies (2). Finally (2) gives $F \in V_\Gamma(\Pi^*) \cap V(\Pi)$, which is $V_\Gamma(\Pi)$ by Lemma 11.19.7. Its rational-tangent hypothesis holds because the original defining normals are rational. □

In particular, a fixed rational value-group refinement can remove the extra support obstruction, but cannot remove the continuity obstruction for a target affine on the old cells. This does not exclude progressively changing subdivisions: their arithmetic continuity equations also change. The following construction shows that the distinction is substantive, even when the target is already PL.

Example 11.19.9 (Moving subdivisions resolve the rectangle test)**.** Use the rectangle and function F of Example 11.12.17, with $a = \sqrt{2}$ and $\Gamma = \mathbf{Z} + a\mathbf{Z}$. Thus $F(x, 0) = ax$, $F(0, y) = y$, and $F = a$ on the top and right edges. Put

$$\delta_j = 8^{-j}, \qquad q_j = \delta_j \lfloor a/\delta_j \rfloor, \qquad q_j^+ = q_j + \delta_j, \qquad b_j = \frac{q_j^+ - a}{\delta_j} \in (0,1) \cap \Gamma_{\mathbf{Q}}.$$

Define $G_j = F$ on the left, top and right edges, and on the bottom edge set

$$G_j(x, 0) = \max\{q_j x,\ a - q_j^+(1 - x)\}, \qquad F_j = G_j + 2\delta_j. \tag{11.19.11}$$

The two branches meet at $x = b_j$. Both have rational slopes and constants in $\Gamma_{\mathbf{Q}}$. Their maximum equals 0 at $x = 0$ and a at $x = 1$, so G_j is continuous around the rectangle. On the bottom edge it is convex. Near each corner its affine branch and the branch on the adjacent edge extend to one ambient affine function, because the edge directions are independent. It therefore has relative affine lower supports everywhere. The same holds for F_j.

Since $q_j < a < q_j^+$, both branches lie below ax, and

$$0 \le ax - G_j(x, 0) = \min\{(a - q_j)x, (q_j^+ - a)(1 - x)\} \le \delta_j.$$

Consequently $F + \delta_j \le F_j \le F + 2\delta_j$ on all of S, and $F_{j+1} \le F + \delta_j/4 \le F_j$. This is a coefficient-compatible decreasing uniform approximation, obtained by adding just one moving breakpoint. It cannot be carried by any single finite refinement with vertices in $\Gamma_{\mathbf{Q}}^2$: Lemma 11.19.7 would then imply $F \in V_\Gamma(\Pi)$, contrary to (11.12.5). Pullback to $S \times [0,1]^{d-1}$ gives the same construction in every positive dimension. These remain abstract polyhedral examples; no balancing or analytic realization is asserted. The changing arithmetic equations can be seen directly. Write $N = 8^j$ and $m = \lfloor Na \rfloor$, so $b_j = m + 1 - Na$. For rational slopes p_1, p_2 on the two bottom segments, p_t on the top and q_l, q_r on the vertical edges, continuity splits as

$$(m + 1)p_1 - mp_2 - p_t = 0, \qquad N(p_2 - p_1) + q_r - q_l = 0.$$

The approximant has $(p_1, p_2, p_t, q_l, q_r) = (m/N, (m + 1)/N, 0, 1, 0)$ and satisfies both equations. The target would have $(a, a, 0, 1, 0)$ and satisfies neither. There is no contradiction with convergence: the equation coefficients change with N. This also explains why a failure of fixed-complex density cannot decide moving-subdivision density.

11.19.5 Propagation of a common scale

There is another useful route back to the one-scale theorem. Rational simplices of dimension at least two have a stronger arithmetic property than general polytopes with rational facet normals. The property propagates through a shared edge.

Proposition 11.19.10 (Scale propagation through edges)**.** *Let* Π *be a finite compact simplicial complex in* $\mathbf{R}^r$*. Suppose every maximal simplex has dimension at least two, every edge has rational direction, and the graph joining maximal simplices that share an edge is connected. If all vertices belong to* $\Gamma^r_{\mathbf{Q}}$*, then for some* $a \in \Gamma^r_{\mathbf{Q}}$ *and* $\lambda \in \Gamma_{>0}$ *every vertex lies in* $a+\lambda\mathbf{Q}^r$*. Consequently every* $F \in C(\Pi)$ *has decreasing approximants in* $A_\Gamma(\Pi) \cap C(\Pi)$*, uniformly on* $|\Pi|$*.*

Proof. First take one simplex with vertices $v_0, \ldots, v_d$, $d \geq 2$. Write $v_i - v_0 = \lambda_i e_i$, with nonzero $e_i \in \mathbf{Q}^r$. The e_i are linearly independent. The edge $v_j - v_i = \lambda_j e_j - \lambda_i e_i$ is a real multiple of a rational vector. A rational left inverse to the matrix with columns e_i shows that the ratio λ_i/λ_j is rational: its two nonzero coordinates are real multiples of two nonzero rational numbers. Thus all vertex differences of this simplex belong to $\lambda_\sigma \mathbf{Q}^r$ for one positive real λ_σ.

If two simplices share an edge, its nonzero vector belongs to both $\lambda_\sigma \mathbf{Q}^r$ and $\lambda_\tau \mathbf{Q}^r$, forcing $\lambda_\sigma/\lambda_\tau \in \mathbf{Q}$. Propagate this relation along the connected graph. Starting from one vertex a, shared vertices then show that every vertex lies in $a + \lambda_0 \mathbf{Q}^r$ for a common positive λ_0. A nonzero coordinate of an edge shows $\lambda_0 \in \Gamma_{\mathbf{Q}}$ after the initial choice of scale; multiplying by a positive integer makes it an element λ of Γ without changing $\lambda_0 \mathbf{Q}^r$. Translate by a and apply Theorem 11.18.1. Translation preserves rational slopes and changes a rational affine constant by $p \cdot a \in \Gamma_{\mathbf{Q}}$, so the approximants retain the required coefficients. □

Example 11.19.11 (Rational normals do not ensure rational triangulations)**.** The rectangle $Q = [0,1] \times [0,\sqrt{2}]$ has integral facet normals and vertices in Γ^2 for $\Gamma = \mathbf{Z} + \sqrt{2}\mathbf{Z}$. Nevertheless it has no finite triangulation all of whose edges have rational direction, even allowing extra vertices. Indeed, the triangles in any finite triangulation of a rectangle are connected through shared edges. To see this, join interior points of two triangles by a polygonal path in the interior, chosen to avoid the finitely many vertices and to cross edges transversely. The proof of Proposition 11.19.10, up to the value-group assertion, puts all vertices of such a rational-direction triangulation in $a+\lambda\mathbf{Q}^2$ for some real $\lambda > 0$. The two boundary vectors $(1,0)$ and $(0,\sqrt{2})$ would then force both $1/\lambda$ and $\sqrt{2}/\lambda$ to be rational, a contradiction.

This example concerns a compact polytope, not a balanced tropical variety and not a counterexample to the conditional theory in APW26. Section 2.1.6

of that source explicitly assumes the existence of a rational simplicial structure. The example explains why this assumption cannot be replaced silently by rationality of facet normals. Nor may Proposition 11.19.10 be applied to simplices meeting only at vertices or to one-dimensional complexes: the shared nonzero edge is the mechanism that compares their scales.

11.19.6 Convex extension on individual cells

The moving-subdivision problem has a further elementary part that can be completed without a global comparison theorem: matching convex boundary data on one polytope. Recording it separately identifies the remaining issue at joins between cells.

Lemma 11.19.12 (Coefficient-compatible convex boundary extension)**.** *Let Q be a positive-dimensional compact polytope defined by rational-slope, $\Gamma_{\mathbf{Q}}$-constant affine inequalities. Suppose g is continuous on its relative boundary, and on each facet is the maximum of finitely many ambient affine functions with those coefficients. Then g extends to a function G on Q which is itself a finite maximum of such affine functions. If an affine real function A satisfies $0 \le A - g \le e$ on the boundary, then for every $\epsilon > 0$ one can arrange*

$$G|_{\partial Q} = g, \qquad 0 \le A - G \le e + \epsilon \quad \text{on } Q. \tag{11.19.12}$$

Proof. For each facet ρ, choose a rational-slope, value-group-constant affine function $h_\rho \ge 0$ on Q whose zero set on Q is exactly ρ. Take one affine branch ℓ of $g|_\rho$. We claim that $\ell - Ch_\rho \le g$ on the whole boundary for some nonnegative rational C. Refine the boundary into finitely many compact polytopes on which g is affine. At a vertex v with $h_\rho(v) = 0$, we already have $\ell(v) \le g(v)$. At all other vertices impose

$$C \ge \frac{\ell(v) - g(v)}{h_\rho(v)}.$$

There are only finitely many bounds, so a nonnegative rational C exists. Vertex inequalities imply the inequalities on every boundary cell. The modified branch still restricts to ℓ on ρ and retains rational slopes and constants in $\Gamma_{\mathbf{Q}}$. Take the maximum of all modified branches, over all facets and all their branches. It agrees with g on each facet and is convex on Q.

If $g \le A$, every modified affine branch is at most A at all vertices of Q, hence at most A throughout Q. To obtain the lower bound, choose a rational-slope, $\Gamma_{\mathbf{Q}}$-constant affine function ℓ_0 with

$$A - e - \epsilon \le \ell_0 \le A - e \quad \text{on } Q.$$

Such a choice follows by approximating the slope and constant of $A-e-\epsilon/2$, using compactness and the density of $\Gamma_{\mathbf{Q}}$. On the boundary, $\ell_0 \le g$. Adding ℓ_0 to the finite maximum therefore preserves the boundary values and gives (11.19.12). □

Proposition 11.19.13 (Decreasing cellwise-convex approximation)**.** *Let* Π *have the coefficient-compatible defining inequalities of Theorem 11.19.8, and let* $F \in V(\Pi)$*. There are continuous rational-slope,* $\Gamma_{\mathbf{Q}}$*-constant PL functions* F_j *on* S*, carried by subdivisions allowed to depend on* j*, such that each* F_j *is convex on every original cell and*

$$F_{j+1} \le F_j, \qquad 0 \le F_j - F \le 2\rho 8^{-j} \quad (\rho \in \Gamma_{>0} \text{ fixed}).$$

No relative affine support across different original cells is asserted.

Proof. Put $d = \dim \Pi$ and fix $\epsilon > 0$. At each vertex choose a value in $\Gamma_{\mathbf{Q}}$ between $F(v)-\epsilon$ and $F(v)$. Induct on cell dimension. Once a continuous function has been defined on the $(k-1)$-skeleton, its restriction to the boundary of each k-cell is a finite maximum of coefficient-compatible affine functions on each facet, by the preceding induction steps. Apply Lemma 11.19.12 with $A = F|_\sigma$ and additional error ϵ. The extensions agree on common faces because those values have already been fixed. After dimension d, the resulting continuous function G satisfies $0 \le F - G \le (d+1)\epsilon$ on S and is convex on every original cell. A common refinement by the finitely many branch comparisons gives a finite coefficient-compatible PL presentation.

For the jth construction choose $\epsilon = \rho 8^{-j}/(d+1)$, write $\eta_j = \rho 8^{-j}$, and set $F_j = G_j + 2\eta_j$. Then $F + \eta_j \le F_j \le F + 2\eta_j$, and $F_{j+1} \le F + \eta_j/4 \le F_j$. The shift belongs to $\Gamma_{\mathbf{Q}}$ and preserves convexity on each cell. □

The qualification is necessary. On $[-1,1]$ with cells $[-1,0]$ and $[0,1]$, the functions $\epsilon(1-|x|)$ are affine on each cell and converge uniformly to the relatively convex function 0, but have no affine lower support at 0: its slope would have to be both at least ϵ and at most $-\epsilon$. Small uniform errors therefore do

not control the support condition at a join. In the boundary-extension proof, the constants C provide domination but no bound on the normal slopes. A relative-convex version needs such control, or another argument preserving supports across the old faces. Section 11.20 supplies that slope control for compatible polyhedral geometry. The value-only cellwise result by itself is not used to infer nef residues or original smooth psh approximation.

11.19.7 Analytic use and remaining limits

To use these results on a fixed ordinary unit chart, retain the hypotheses of Proposition 11.11.5 and Theorem 11.11.6: the ground field is nontrivially valued, $\Gamma = -\log|K^\times|$, and the analytic space is good, strictly analytic and pure-dimensional. Suppose the chart image is contained in S, and $F \in C(\Pi) \cap W_\Gamma(\Pi)$. Theorem 11.19.4 gives a decreasing sequence on the whole same S, with uniform error. Pullback therefore gives actual Q-PL functions on the whole same ordinary chart domain, with nef residues by the cited inputs. If the integral-projective hypotheses of Corollary 11.11.7 also hold, that corollary supplies an original smooth decreasing sequence near each point on one fixed ordinary neighbourhood.

The same analytic conclusion follows from Theorem 11.19.8 when the original complex has rational normals and value-group constants and $F \in C(\Pi) \cap V_\Gamma(\Pi)$. Its refinement changes neither the tropical support nor the ordinary inverse-image domain. In contrast, Example 11.19.9 has not been realized as an analytic tropical chart, so no analytic conclusion is inferred from that example alone.

This application uses the existing model and residue comparisons; a formal real PL expression is not substituted for actual Q-PL membership. In particular, [GRPSH, Theorem 4.3] assumes an actual R-PL function and nontrivial valuation before identifying classical psh with semipositivity. The real coordinate calculations above do not rescale the absolute value, extend the ground field, replace original CLD tests, or enlarge their actual supports.

The distinction from [APW26, Proposition 5.21] remains essential. That proposition produces locally uniform real PL approximants on one neighbourhood; its proof passes through a projective compactification and functions $(\varphi_i + h)/C$. Neither a common finite subdivision for all approximants nor the coefficient and strict-support conditions of this section are supplied by that statement. Our criteria can be applied to an individual output only after

those data are checked, or to a sequence after the domains and error choices are made compatible. We have not proved that every relevant analytic tropical image admits the certificate of Proposition 11.19.6. Thus the general relative coefficient replacement, Wu Theorem A.3 and the stronger CLD25 closure comparison remain open here. The primary full text of the separate mixed-calculus Theorem 6.13 has still not been obtained; it is not an input to any proof in this section.

References. Finite-dimensional reconstruction in this manuscript, extending the continuity-only note of v0.21. Source scope checked against APW26 Sections 2.1, 5.2 and Proposition 5.21 (v1), and GRPSH Theorem 4.3 (v2). The strict-support certificate and closure proofs are given above; no general source theorem is declared corrected or disproved by them.

11.20 Controlled slopes and moving relative approximation

The value-only extension in Section 11.19.6 leaves the normal slopes uncontrolled. Here the fixed geometry of the original cells supplies the missing estimate. We first strengthen the extension lemma, then show that small cellwise Lipschitz errors preserve strict relative supports. A fixed strict barrier reduces a general relatively supported PL target to that case. The resulting approximants may change subdivision, but never change the compact support.

11.20.1 A boundary extension with controlled slopes

Use the standard Euclidean norm in the fixed ambient space. For a function on a compact convex cell σ, write Lip_σ for its Lipschitz seminorm, and set this seminorm to zero on a point. All constants below depend on the original finite polytopes, not on subdivisions of their boundary data.

Lemma 11.20.1 (Controlled convex extension)**.**

Let Q be as in Lemma 11.19.12. There is $M_Q \geq 1$ with the following property. Let A be real affine, let $\epsilon > 0$, and let g be continuous on ∂Q, a finite maximum of rational-slope, $\Gamma_{\mathbf{Q}}$-constant affine functions on each facet. Suppose

$$0 \leq A - g \leq \epsilon \quad \text{on } \partial Q, \qquad \mathrm{Lip}_\tau(g - A) \leq \epsilon \quad \text{for every facet } \tau.$$

Then g has a finite-max extension G with the same coefficient conditions and

$$G|_{\partial Q} = g, \qquad 0 \le A - G \le M_Q\epsilon, \qquad \mathrm{Lip}_Q(G - A) \le M_Q\epsilon. \tag{11.20.1}$$

Proof. Fix a facet ρ and a compatible affine barrier $h_\rho \ge 0$ vanishing exactly on ρ. In a finite-max representation of $g|_\rho$, discard branches never active on a relatively open subset. This does not change the function, including at its boundary, by continuity. For a remaining branch ℓ, the tangential part of $\nabla\ell - \nabla A$ on ρ has norm at most ϵ. The tangent space of ρ is rational. Its orthogonal complement therefore has dense rational points. Add to ℓ a function $q \cdot (t - v_\rho)$, where $v_\rho \in \Gamma^r_{\mathbf{Q}}$ is a vertex and q is rational, orthogonal to the tangent space, chosen so that the new branch $\widetilde{\ell}$ satisfies

$$\widetilde{\ell}|_\rho = \ell|_\rho, \qquad \|\nabla\widetilde{\ell} - \nabla A\| \le 2\epsilon.$$

The added constant is in $\Gamma_{\mathbf{Q}}$. For a zero-dimensional facet the same construction uses its single boundary value and no tangential constraint. At a point where the original branch is active, $|\widetilde{\ell} - A| \le \epsilon$. If $D = \mathrm{diam}(Q)$, it follows that $|\widetilde{\ell} - A| \le (1 + 2D)\epsilon$ on Q.

We next bound the penalty needed to dominate every other facet. If $\tau \cap \rho \ne \varnothing$, then for every $y \in \tau$ one can choose $z \in \tau \cap \rho$ such that

$$\|y - z\| \le K_{\rho\tau} h_\rho(y), \tag{11.20.2}$$

with a fixed finite constant. Indeed, write y as a convex combination of the vertices of τ, keep the vertices in ρ, and replace all other vertices by one fixed vertex $z_0 \in \tau \cap \rho$. One may take the largest ratio $\|v - z_0\|/h_\rho(v)$ over vertices outside ρ. Since $\widetilde{\ell}(z) \le g(z)$ and the function $\widetilde{\ell} - g$ is 3ϵ-Lipschitz on τ,

$$\widetilde{\ell}(y) - g(y) \le 3K_{\rho\tau}\epsilon h_\rho(y).$$

If the two facets are disjoint, h_ρ has a positive minimum on τ, and the uniform bounds above give the same inequality with another fixed constant. Thus, independently of the branch and the boundary subdivision, there is $K_\rho \ge 0$ such that every modified branch is dominated on the boundary after subtracting $C_\rho h_\rho$, where C_ρ is any rational number in $[K_\rho\epsilon, (K_\rho + 1)\epsilon]$.

Take the maximum of these penalized branches over all facets. It extends g exactly and retains the coefficient conditions. Every branch has gradient within $M_1\epsilon$ of ∇A, where M_1 depends only on the chosen facet barriers. Hence $G - A$, a maximum of these affine differences, is $M_1\epsilon$-Lipschitz on

Q. Each branch is at most A at the vertices, so $G \le A$ throughout Q. Comparing with any boundary point gives $A - G \le (1 + M_1 D)\epsilon$. Taking $M_Q = \max\{1, M_1, 1 + M_1 D\}$ proves the lemma. □

Corollary 11.20.2 (Cellwise Lipschitz approximation)**.** *For a compatible finite compact complex Π, every $F \in V(\Pi)$ and every $\delta > 0$ admit a continuous coefficient-compatible PL function G, convex on each original cell, such that*

$$0 \le F - G \le \delta, \qquad \max_{\sigma \in \Pi} \operatorname{Lip}_\sigma(G - F) \le \delta. \tag{11.20.3}$$

Its subdivision may depend on δ.

Proof. Repeat the skeleton induction of Proposition 11.19.13, now using Lemma 11.20.1. Let $M \ge 1$ bound its constants for all positive-dimensional cells and let $d = \dim \Pi$. Choose the vertex errors at most δ/M^d. At step k, both the uniform error and the facetwise Lipschitz error are at most $M^{k-1}\delta/M^d$, and the lemma bounds both errors on the new cell by $M^k\delta/M^d$. The functions agree on shared faces by construction. The finite-max presentations give rational slopes and value-group constants on a finite refinement. If $d = 0$, the vertex choices alone suffice. □

11.20.2 Preserving the relative supports

Lemma 11.20.3 (Stability near a strict PL function)**.** *Let $F_0 \in V(\Pi)$ have strict face supports. There is $\delta_0 > 0$ such that every continuous G, a finite maximum of real affine functions on each cell, has local relative affine lower supports whenever*

$$\max_{\sigma \in \Pi} \operatorname{Lip}_\sigma(G - F_0) \le \delta_0$$

holds. More quantitatively, if $F_0 = F + tH$ with $F \in C(\Pi)$ and fixed H having strict face supports, one may take $\delta_0 = t/(4K_H)$ for a constant $K_H \ge 1$ independent of F and $t > 0$.

Proof. Choose the strict support a_ρ for a face ρ. On any coface σ, put $D_{\rho\sigma} = F_0 - a_\rho$. It is affine and nonnegative, with zero set exactly ρ. The vertex replacement argument used in (11.20.2) gives, for every $y \in \sigma$, a point $z \in \rho$ with $\|y - z\| \le K D_{\rho\sigma}(y)$, for one finite $K \ge 1$ over all proper incidences.

Take $x \in \operatorname{relint}(\rho)$ and an active affine branch supporting $G|_\rho$ at x. Its tangential gradient differs from that of $F_0|_\rho$ by a vector e of norm at most

$\delta := \max_\sigma \operatorname{Lip}_\sigma(G - F_0)$. This follows by testing the supporting inequality in both tangent directions near x. Regard e as a tangent vector in the ambient Euclidean space and set

$$b(y) = a_\rho(y) + e \cdot (y - x) + G(x) - F_0(x).$$

Then $b \le G$ on ρ and $b(x) = G(x)$. For $y \in \sigma \supseteq \rho$, use the point z above and the Lipschitz estimate within σ to obtain

$$\begin{aligned} G(y) - b(y) &= (G - F_0)(y) - (G - F_0)(z) + G(z) - b(z) \\ &\quad + D_{\rho\sigma}(y) - e \cdot (y - z) \\ &\ge D_{\rho\sigma}(y) - 2\delta\|y - z\| \\ &\ge (1 - 2\delta K) D_{\rho\sigma}(y). \end{aligned}$$

Thus $\delta \le 1/(4K)$ suffices. A small neighbourhood of x meets only cofaces of ρ, so b is the required relative support. If ρ is maximal, the active branch alone suffices.

For the quantitative statement, add a support of F to t times a strict support of H. The gap on each coface is at least t times the corresponding gap for H. The same vertex replacement construction therefore gives the distance bound with constant K_H/t, which may itself be less than one. This proves the stated bound without using a uniform bound on the values of $G - F_0$. □

Theorem 11.20.4 (Relative rational approximation with moving subdivisions)**.** *Let $\Gamma \ne 0$ and let Π be a finite compact polytopal complex whose cells are defined by rational-slope, $\Gamma_{\mathbf{Q}}$-constant affine inequalities. Every continuous cellwise-affine F with local relative affine lower supports has continuous coefficient-compatible PL approximants F_j on the same support $S = |\Pi|$, themselves admitting such supports, with*

$$F_{j+1} \le F_j, \qquad 0 \le F_j - F \le 3\rho 8^{-j} \quad (\rho \in \Gamma_{>0} \text{ fixed}). \tag{11.20.4}$$

The subdivisions of the approximants may vary with j. No continuity-closure condition $F \in V_\Gamma(\Pi)$ is required.

Proof. Use the fixed arrangement refinement Π^* and strict function H from Theorem 11.19.8. The function F is still affine on each refined cell and has relative supports. Set $B = \max\{1, \|H\|_\infty\}$ and $\eta_j = \rho 8^{-j}$. Choose

$$t_j = \frac{\eta_j}{2B}, \qquad 0 < \delta_j \le \min\left\{ \frac{\eta_j}{2}, \frac{t_j}{4K_H} \right\}.$$

Apply Corollary 11.20.2 to the real PL target $F+t_jH$ with error δ_j, obtaining G_j. Lemma 11.20.3 gives relative affine supports for G_j, and

$$\|G_j - F\|_\infty \le \delta_j + t_j\|H\|_\infty \le \eta_j.$$

Put $F_j = G_j + 2\eta_j$. This is an allowed value-group shift, preserving the supports and coefficients. The inequalities $F + \eta_j \le F_j \le F + 3\eta_j$ imply (11.20.4). Every construction takes place on the same S. The arrangement refinement is fixed once; only the further finite-max subdivisions used to build G_j vary. □

The same proof applies termwise to a decreasing sequence of such real PL targets u_j on the same compact support, even if their compatible complexes differ. Choose supported rational G_j with $|G_j - u_j| \le \eta_j$ and add $2\eta_j$. Then $G_{j+1} + 2\eta_{j+1} \le u_{j+1} + 3\eta_{j+1} \le u_j + \eta_j \le G_j + 2\eta_j$. The limit is the original pointwise limit of u_j; uniform convergence is claimed only when the targets themselves converge uniformly.

11.20.3 The source interface still to be checked

On an actual ordinary unit chart, the analytic consequences in Section 11.19.7 now apply to every relatively supported real PL target whose finite compact complex has the stated coefficient-compatible geometry. Actual Q-PL identification, nontrivial valuation, nef-residue comparison and integral-projective original-smooth transfer remain separate cited inputs. The arithmetic normalization and original tests are unchanged.

This does not yet cover the arbitrary real intercepts allowed by APW26, nor an arbitrary smooth target without a suitable PL approximation on a common domain. The argument also does not extend the current calculus to arbitrary positive backgrounds or prove first-stage closure. These distinctions are part of the theorem: the new result removes the fixed-subdivision obstruction by controlled moving subdivisions within the compatible-polyhedral class. It is an elementary reconstruction assembled from the explicit extension and support estimates above, not a claim that an unverified source theorem has been proved in its full generality.

Section 11.21 goes further for rational-direction PL germs and for families with an explicit common transport domain. Section 11.21.4 isolates an earlier issue in the literal APW26 PL convention; the gap is not resolved merely by replacing real intercepts with value-group constants.

11.21 Real intercepts, moving germs and common domains

A real intercept can be an obstruction to keeping a subdivision fixed without being an obstruction to approximating the function. There is a local way to remove it: move the centre of a PL star inside the stratum of the original support. The support remains fixed, whereas the break loci move. We prove this germ statement first. We then give a separate transport criterion that works on a prescribed compact set, and explain why the germ statement alone cannot be applied termwise to a source approximation on one neighbourhood.

Section 11.22 gives a subsequent replacement for the fixed-domain approximation question: continuous relative supports suffice, without rational directions or a one-scale input subdivision. The present constructions retain their explicit geometric interpretation and illustrate why termwise germ arguments alone do not control a common domain.

11.21.1 Moving a centre within its stratum

Throughout this section $\Gamma \subset \mathbf{R}$ is a nonzero additive subgroup. A compatible polyhedron is one defined by rational-slope affine inequalities with constants in $\Gamma_{\mathbf{Q}}$. A rational polyhedron, without the word compatible, may have arbitrary real constants. All complexes used locally are finite after restriction to a sufficiently small neighbourhood. Relative supports always mean ambient affine lower supports in the relative topology of the specified support.

Lemma 11.21.1 (Compatible points in a face)**.** *If τ is a nonempty compatible polyhedron, then* $\operatorname{relint}(\tau) \cap \Gamma_{\mathbf{Q}}^r$ *is dense in* $\operatorname{relint}(\tau)$*. Its tangent space $L = T\tau$ is rational.*

Proof. The affine hull is specified by those rational equations that hold identically on τ. Gaussian elimination gives a particular solution $a \in \Gamma_{\mathbf{Q}}^r$ and a basis $e_1, \dots, e_d \in \mathbf{Q}^r$ for the homogeneous solutions. Thus $\operatorname{aff}(\tau) = a + \sum_k \mathbf{R} e_k$. Since $\Gamma_{\mathbf{Q}}$ contains $\rho\mathbf{Q}$ for some $\rho > 0$, points $a + \sum_k b_k e_k$ with $b_k \in \Gamma_{\mathbf{Q}}$ are dense in this affine hull and have compatible coordinates. Intersect with the relatively open set $\operatorname{relint}(\tau)$. The case $d = 0$ is included. □

Lemma 11.21.2 (A fixed support near a stratum)**.** *Let S be the support of a locally finite compatible complex, and let $x \in \operatorname{relint}(\tau)$. There are $R > 0$ and a finite union C of rational polyhedral cones such that*

$$S \cap B(x,R) = (x + C) \cap B(x,R), \qquad C + L = C, \quad L = T\tau.$$

In particular, $b + C = x + C$ for every $b \in \operatorname{aff}(\tau)$. If F is continuous and affine on a finite rational polyhedral subdivision near x, then, after decreasing R,

$$F(x+v) = F(x) + f(v), \qquad v \in C, \quad \|v\| < R, \tag{11.21.1}$$

where f is continuous and homogeneous of degree one, affine on a finite rational conical subdivision of C. If F has relative supports near x, then f has relative supports on all of C.

Proof. Only cells containing x meet a sufficiently small ball. In each such cell, inequalities that are strict at x remain strict in a smaller ball; the active inequalities become homogeneous after translating x to the origin. Their normals are rational, and vanish on L, because they vanish on the face τ. Their tangent cones form the asserted union C, invariant under both signs of every vector in L. This also proves the equality after changing the centre to b.

Apply the same active-constraint construction to the finite rational subdivision of F, together with the original complex. On a cell containing x, its affine expression becomes $F(x) + p_\sigma \cdot v$. The expressions agree on overlaps, first near zero and then everywhere by homogeneity. They define f on C. To transfer a relative support to a point $v \in C$, choose $s > 0$ so small that sv and a neighbourhood of it lie in the ball where (11.21.1) holds. A support a for f near sv gives the support $s^{-1}a(s\,\cdot)$ near v. At zero use the given support directly. □

Theorem 11.21.3 (Rational approximation of a real-intercept PL germ)**.** *Let S be locally the support of a compatible complex and let F be continuous, affine on a locally finite rational polyhedral subdivision of S, with local relative affine lower supports. For every $x \in S$ there is a compact neighbourhood K of x in S and continuous coefficient-compatible PL functions F_j on K, themselves having relative supports, such that*

$$F + \eta_j \le F_j \le F + 3\eta_j, \qquad F_{j+1} \le F_j, \qquad \eta_j = \rho 8^{-j}, \quad \rho \in \Gamma_{>0}. \tag{11.21.2}$$

The intercepts of the subdivision of F need not belong to $\Gamma_{\mathbf{Q}}$. The neighbourhood K depends on the germ F; the theorem makes no uniform-domain assertion for a family of different germs.

Proof. Use Lemma 11.21.2. Choose a closed box B with coordinate endpoints in $\Gamma_{\mathbf{Q}}$ such that $x \in \operatorname{int}(B)$ and $B \subset B(x, R/4)$, and put $K = S \cap B = (x + C) \cap B$. For $b \in \operatorname{relint}(\tau) \cap \Gamma_{\mathbf{Q}}^r$ close to x, define

$$H_b(t) = F(x) + f(t - b), \qquad t \in K.$$

The equality $b + C = x + C$ makes this a function on exactly the same K. It has relative supports, and its cells are intersections of B with translates by b of rational cones. These cells are compatible. Thus H_b satisfies the hypotheses of Theorem 11.20.4, although its function coefficients can still be real.

There is a constant $M < \infty$ with $|H_b - F| \le M\|b - x\|$ on K. Indeed, the segment from $t - x$ to $t - b$ lies in C, since its direction belongs to L. Partition that segment at the finitely many conical cells of f. On each piece its slope is bounded by the norm of a chosen ambient slope of that cell; take M to be the largest of these norms. Lemma 11.21.1 now gives b_j so close to x that this error is at most $\eta_j/2$.

Theorem 11.20.4 supplies a supported coefficient-compatible G_j on K with $\|G_j - H_{b_j}\|_\infty \le \eta_j/2$. Hence $\|G_j - F\|_\infty \le \eta_j$. Set $F_j = G_j + 2\eta_j$. The shift is in $\Gamma_{\mathbf{Q}}$ and gives (11.21.2), including decrease because $3\eta_{j+1} < \eta_j$. No affine map of an analytic space has been introduced: only the real function used in a fixed set of coordinates has changed. □

11.21.2 Transport on a prescribed compact set

For a single germ the preceding construction is enough. For a sequence, a useful alternative is to move an entire one-scale presentation on a domain fixed in advance. The scale below is a real geometric parameter; it is not the normalization of a field valuation.

Theorem 11.21.4 (One-scale transport)**.** *Let C be a finite union of rational cones, let $L \subset \mathbf{R}^r$ be a rational subspace with $C + L = C$, and let $a \in \Gamma_{\mathbf{Q}}^r$. Let U be relatively open in $S = a + C$. Let B be a closed compatible box with nonempty $K = S \cap B \subset U$. Suppose F is continuous, has relative supports*

on U, and is affine on a finite polyhedral structure in a neighbourhood of K, whose cells are defined by inequalities

$$q \cdot (t - a - w) \geq \lambda c, \qquad q \in \mathbf{Q}^r, \quad c \in \mathbf{Q}, \quad w \in L, \quad \lambda > 0. \tag{11.21.3}$$

Here the same w and λ are used for all cells, and equalities are allowed. Then $F|_K$ has decreasing coefficient-compatible relative PL approximants satisfying (11.21.2) on this prescribed K. Neither w nor λ is required to have value-group coordinates.

Proof. Choose $w_j \in L \cap \Gamma_{\mathbf{Q}}^r$ and $\lambda_j \in \Gamma_{\mathbf{Q},>0}$ tending to w and λ, respectively, and define

$$T_j(t) = a + w_j + \frac{\lambda_j}{\lambda}(t - a - w).$$

The maps T_j preserve S bijectively and converge to the identity, as do their inverses, uniformly on compact sets. Choose a relatively open neighbourhood N of K with compact closure in U, contained in the neighbourhood where the stated finite presentation is valid. Sufficiently close choices ensure $T_j^{-1}(K) \subset N$. On K put $H_j = F \circ T_j^{-1}$. Supports are transported by this invertible affine map in the real coordinate space. The transformed cell inequalities are

$$q \cdot (t - a - w_j) \geq \lambda_j c.$$

They are compatible, since only rational numbers multiply elements of $\Gamma_{\mathbf{Q}}$. Their intersections with B form a compatible compact complex with support exactly K. Uniform continuity of F on $\overline{N}$ allows the choices to satisfy $\|H_j - F\|_K \leq \eta_j/2$. Apply Theorem 11.20.4 with error $\eta_j/2$, and add $2\eta_j$ as in the previous proof. □

Corollary 11.21.5 (An edge-connected simplicial certificate)**.** *In Theorem 11.21.4, the one-scale presentation condition follows if F is affine on a finite compact simplicial complex covering a neighbourhood of K in S, every maximal simplex has dimension at least two, every edge has rational direction, the maximal simplices are connected by shared edges, and one vertex p belongs to $a + L$. The vertices need not have value-group coordinates.*

Proof. Use the geometric part of the proof of Proposition 11.19.10, before its value-group step. It gives one positive real λ such that all vertices lie in $p + \lambda\mathbf{Q}^r$. Indeed, within a simplex the rational directions of its edges force the edge scales to have rational ratios; a shared nonzero edge propagates the

ratio between simplices. Each simplex is therefore $p + \lambda$ times a polytope with rational vertices, allowing lower-dimensional affine-hull equations. Its defining inequalities have the form (11.21.3), with $w = p - a \in L$. Apply Theorem 11.21.4. This is a consequence of the earlier scale-propagation proof and the new transport argument, rather than a separate density theorem. □

This certificate is deliberately checkable. It does not apply merely because a source uses the word simplicial: unbounded polyhedra, pieces joined only at vertices, one-dimensional cells, and a missing lineality anchor require additional arguments. In an ambient vector space one may take $L = \mathbf{R}^r$, so the anchor condition is automatic; on a branching support it is a real restriction.

Corollary 11.21.6 (Families with a common transport domain)**.** *Let u_j decrease pointwise on the same $U \subset S$. Suppose each u_j satisfies the hypotheses of Theorem 11.21.4 near the same K, with its own finite structure, scale and lineality translation. Then there are supported coefficient-compatible PL functions v_j on K with*

$$u_j + \eta_j \le v_j \le u_j + 3\eta_j, \qquad v_{j+1} \le v_j.$$

The pointwise limit is unchanged. If instead u_j converges uniformly to a finite function u on K, without monotonicity, a subsequence and the same error-shift construction give decreasing supported coefficient-compatible PL approximants to u on K.

Proof. For the first assertion choose G_j with $\|G_j - u_j\|_K \le \eta_j$ and set $v_j = G_j + 2\eta_j$. Then $v_{j+1} \le u_{j+1} + 3\eta_{j+1} \le u_j + \eta_j \le v_j$. For the second, select u_{k_j} with $\|u_{k_j} - u\|_K \le \eta_j/2$ and approximate that term within $\eta_j/2$. The resulting G_j lies within η_j of u, so the same shift proves the claim. Parameters may be chosen separately for each target; no uniform slope bound or common subdivision is used. □

11.21.3 Why the germ quantifiers matter

Example 11.21.7 (Shrinking stars despite uniform control)**.** Take $\Gamma = \mathbf{Z}$ and let p_j be the secant interpolation of t^2 at the dyadic nodes of $[-1, 1]$, with mesh $h_j = 2^{-j}$. On $[kh_j, (k+1)h_j]$ one has

$$p_j(t) - t^2 = (t - kh_j)((k+1)h_j - t), \qquad 0 \le p_j(t) - t^2 \le h_j^2/4.$$

The slopes $(2k+1)h_j$ increase with k, so p_j is convex. Refining each interval inserts a point of the convex graph below its old chord, hence $p_{j+1} \le p_j$. Also $\mathrm{Lip}(p_j) \le 2$. Nevertheless, the maximal centred interval on which $p_j(t) = p_j(0) + f_j(t)$ for a degree-one homogeneous PL function f_j is $[-h_j, h_j]$: there $p_j = h_j|t|$, and the slope changes at $\pm h_j$. Thus no positive-radius star works for all j. Pullback by the first-coordinate projection gives the same phenomenon in every positive dimension.

This is not a failure of approximation: the functions p_j are already coefficient-compatible, convex and decreasing uniformly to a smooth convex target. It disproves only the inference from termwise PL stars, uniform convergence and a uniform Lipschitz bound to a common star radius. The domain in Theorem 11.21.3 therefore cannot simply be intersected over the approximants furnished by a source theorem.

Corollary 11.21.8 (Original smooth approximation of a rational-direction PL germ)**.** *Let K be complete and nontrivially valued, let X be an integral projective K-variety of positive dimension, and let $W \subset X^{\mathrm{an}}$ be ordinary open. On W fix analytic units $g_1, \dots, g_r$ and the map $\tau = (-\log|g_1|, \dots, -\log|g_r|)$. Let $x \in W$. Suppose near $\tau(x)$ that the actual image is contained in a support S of a locally finite $\Gamma_{\mathbf{Q}}$-compatible complex, where $\Gamma = -\log|K^\times|$, and*

$$u = F \circ \tau,$$

where F is finite continuous, affine on a locally finite rational subdivision of S, and has local relative affine lower supports. Then some ordinary open neighbourhood $V \subset W$ of x carries original smooth psh functions $b_j \downarrow u|_V$. Convergence is uniform on compact subsets of V.

Open proof obligation (RC2). Prove the analytic germ bridge using the required curvewise-to-model implication; GRPSH Theorem 4.3.

For a family on a prescribed compatible box, Corollary 11.21.6 can replace the germ theorem in this proof, after its geometric presentation conditions are checked term by term. Neither route changes the chosen valuation, extends the ground field, enlarges the support of an actual analytic test, or replaces original CLD tests. The auxiliary real support S must be specified in the hypothesis; no unsupported extension of F from the actual image is being asserted. These are consequences of the displayed real constructions and the named model and smoothing inputs, not a full comparison with the generated source classes.

11.21.4 The PL convention in the source regularization

There is a logically earlier issue than the value-group constants. In the fixed version [APW26, Section 1.6; Sections 2.1, 2.1.4–2.1.6], affine functions have real coefficients, rational half-spaces have integral normals and arbitrary real intercepts, and polyhedral structures are declared rational and simplicial unless specified otherwise. The last step of Proposition 5.12, printed page 38, invokes a continuous finite-selection argument to conclude PL membership. The following two examples explain why that step needs an explicit convention or an additional geometric argument. They do not disprove the existence statements of Propositions 5.12 or 5.21.

The examples below concern finite or locally finite subdivisions, as explicitly stated. The displayed definition in the source Section 2.1.6 does not itself list local finiteness. This convention also needs clarification when that definition is used: we do not infer a statement about arbitrary non-locally-finite decompositions from the finite-boundary argument below.

Example 11.21.9 (A maximum with an irrational break direction)**.** On $\mathbf{R}^2$, the affine functions $\ell_1(x, y) = x$ and $\ell_2(x, y) = \sqrt{2}\, y$ are affine on every rational triangulation. Their maximum

$$m(x, y) = \max\{x, \sqrt{2}\, y\}$$

is not affine on the cells of any locally finite rational polyhedral structure near the origin. Indeed, each full-dimensional affine cell must lie on one side of the break line $D = \{x = \sqrt{2}\, y\}$. Hence an open segment of D would lie in the finite union of rational supporting lines of the cells near the origin. One such line would have to contain an open segment of D, and therefore equal D. But D has no nonzero rational tangent vector. This is a contradiction. The function m is nevertheless an ordinary real-polyhedral PL function and is ambient convex. Products with $\mathbf{R}^{r-2}$ give the same distinction in every dimension $r \geq 2$.

Proposition 11.21.10 (Uniform density does not give Lipschitz-small errors)**.** *Let $m(x, y) = \max\{x, \sqrt{2}\, y\}$ and let B be a closed ball of positive radius centred at the origin. If G is continuous and affine on a locally finite rational polyhedral subdivision of a neighbourhood of B, then*

$$\mathrm{Lip}_B(G - m) \geq \frac{\sqrt{3}}{2}.$$

This holds even when the affine coefficients of G are real. Nevertheless, rational numbers $q_j \to \sqrt{2}$ give coefficient-compatible convex functions $\max\{x, q_j y\}$ converging uniformly to m on B.

Proof. A nontrivial segment of the irrational line $D = \{x = \sqrt{2}\, y\}$ inside B cannot be covered by the finitely many rational boundary lines of the subdivision. Choose a point z on that segment in the interior of a full-dimensional cell. Near z, the function G is one affine function with gradient p. On the two open sides of D, the gradients of $G - m$ are respectively $p - (1, 0)$ and $p - (0, \sqrt{2})$. Its Lipschitz seminorm on B bounds both norms, by evaluating differences in small line segments lying in the corresponding open side. The triangle inequality gives

$$\sqrt{3} = \|(1,0) - (0,\sqrt{2})\| \le \|p - (1,0)\| + \|p - (0,\sqrt{2})\| \le 2\,\mathrm{Lip}_B(G - m).$$

For uniform approximation, the maximum operation is 1-Lipschitz in the sup norm of its two entries, so the error is at most $|q_j - \sqrt{2}| \max_B |y|$. The same obstruction holds in every ambient dimension at least two by restricting to the rational plane where the other coordinates vanish. The lower bound is sharp among rational-subdivision PL functions with real affine coefficients: the single affine function $G = (x + \sqrt{2}\, y)/2$ has $G - m = -|x - \sqrt{2}\, y|/2$ and Lipschitz error $\sqrt{3}/2$. Among coefficient-compatible functions the infimum is the same, by approximating this affine gradient by rational vectors; an allowed constant can be chosen independently because the Lipschitz seminorm ignores constants. □

Thus the cellwise Lipschitz estimates used in Section 11.20 cannot simply be demanded for arbitrary real-polyhedral targets on a preassigned rational complex. A rational subdivision of the input was a substantive hypothesis of the germ construction. This obstruction concerns the stronger error norm only; it neither obstructs uniform rational approximation nor disproves the source regularization statement.

The same proof applies to the maximum of two affine functions with gradients p_1, p_2 whenever their break hyperplane cannot be a facet in the permitted subdivision class; the bound is $\|p_1 - p_2\|/2$. This also distinguishes value-group intercepts from rational directions. For example, with $\Gamma = \mathbf{Z}$, on $[-2, 2]$ every coefficient-compatible PL approximation G to $|t - \sqrt{2}|$ has $\mathrm{Lip}(G - |t - \sqrt{2}|) \ge 1$: its finitely many rational breakpoints miss $\sqrt{2}$, so G is affine there. Yet $|t - a_j|$ converges uniformly when $a_j \in \mathbf{Q}$ tends to

$\sqrt{2}$. Thus moving the centre in Theorem 11.21.3 is compatible with uniform error control, without implying Lipschitz convergence. Products give this intercept phenomenon in every positive dimension.

Example 11.21.11 (Compatible coefficients without rational triangulability)**.** Let $\Gamma = \mathbf{Z} + \sqrt{2}\mathbf{Z}$. Each of the four functions $|x|$, $|x-1|$, $|y|$ and $|y-\sqrt{2}|$ admits a rational simplicial conical subdivision of $\mathbf{R}^2$, translated to a point on its break line. Their sum

$$P(x,y) = |x| + |x-1| + |y| + |y-\sqrt{2}|$$

has rational slopes and $\Gamma_{\mathbf{Q}}$ constants on a compatible polyhedral complex. But it admits no locally finite rational simplicial subdivision on a neighbourhood of $[0,1] \times [0,\sqrt{2}]$. Its gradient changes across each of the four bounding lines. No full-dimensional cell of an affine presentation can cross any of those lines. Thus the cells inside the rectangle would give a finite rational-direction triangulation of it, contradicting Example 11.19.11. All the cells contained in the bounded rectangle are compact simplices, irrespective of how simplicial unbounded polyhedra are defined outside it.

The first example concerns rational directions, independently of any valuation. The second retains rational directions and compatible coefficients, but concerns the extra simplicial requirement. Neither example is a counterexample to approximation by supported rational functions: the first is ambient convex and admits such approximations by Section 11.12.4, while the second is already coefficient-compatible. They identify the category in which a finite-selection proof is valid.

Lemma 11.21.12 (Coefficient-compatible finite selections)**.** *On the support of a finite compatible complex, a continuous selection from finitely many affine functions $q_i \cdot t + c_i$, with $q_i \in \mathbf{Q}^r$ and $c_i \in \Gamma_{\mathbf{Q}}$, is coefficient-compatible PL on a finite compatible refinement. The same conclusion holds for a continuous selection from finitely many coefficient-compatible PL functions. No simplicial refinement is asserted.*

Proof. Cut each cell by all equations $(q_i - q_k) \cdot t = c_k - c_i$. These are compatible hyperplanes. On the relative interior of a resulting cell, two branches are either identical or everywhere distinct, with a fixed strict order. A continuous selection must choose one of these distinct branches on each connected relative interior: locally it cannot change to a different branch across a positive value gap. Continuity extends that affine expression to the closed cell.

For PL inputs first take their finite common compatible refinement and apply the same argument to their affine branches there. □

With arbitrary real affine coefficients the same proof produces a real-polyhedral refinement, whose face directions may be irrational. Thus there are two honest ways to use the last step of the source Proposition 5.12: either specify the unrestricted real-polyhedral PL category and separately prove the later rational/model interface, or establish coefficient and subdivision control during the construction. Merely citing a finite-selection theorem does not supply that control. The germ theorem above assumes a rational subdivision of the input and therefore does not silently cover the first alternative. Section 11.22 supplies the second route for continuous relative-support inputs on a fixed compact set. The companion mixed-calculus Theorem 6.13 remains unverified here because its primary full text has not been obtained.

The primary-file check also confirms the localization correction already proved in Lemma 11.11.11. On the compact neighbourhood used in APW Proposition 5.10, the intersection of the closures of its shrinking sets is $\{g - \ell_g = 0\} = \{x\}$, not the printed zero set of $f - \ell_f$. This is a recheck of that earlier repair, not an additional result of the present section.

11.22 Rational affine bumps on a fixed compact support

The coefficient problem can be treated before the PL subdivision is produced. The local modification and finite-cover argument of [APW26, Propositions 5.10 and 5.12] suggest the following replacement: approximate the raised affine supports by coefficient-compatible affine functions, then perform the gluing. A small quadratic term supplies the strict separation needed to absorb that perturbation. The proof below is given in full, including the common domain and the last finite-selection step. It is an elementary reconstruction of that argument with coefficient control, not a claim of a new regularization principle.

11.22.1 Local modifications with controlled coefficients

Fix a nonzero additive subgroup $\Gamma \subset \mathbf{R}$, and set $\Gamma_{\mathbf{Q}} = \Gamma \otimes_{\mathbf{Z}} \mathbf{Q} \subset \mathbf{R}$. An allowed affine function is $a(t) = q \cdot t + c$ with $q \in \mathbf{Q}^r$ and $c \in \Gamma_{\mathbf{Q}}$. Let $S \subset \mathbf{R}^r$

be nonempty and compact. A continuous $F : S \to \mathbf{R}$ has relative affine lower supports if, for every $x \in S$, some ambient affine ℓ_x satisfies $\ell_x(x) = F(x)$ and $\ell_x \le F$ on a relative neighbourhood of x. No polyhedrality of S is needed until the polyhedral conclusion is stated.

Lemma 11.22.1 (A rational affine local bump)**.** *Suppose that $g : S \to \mathbf{R}$ is continuous and has relative affine lower supports. Fix $x \in S$, $r, t > 0$ and an ambient affine ℓ such that*

$$\ell(x) = g(x), \qquad g(y) - \ell(y) \ge t\|y - x\|^2 \quad (y \in S \cap \overline{B}(x, 2r)).$$

For every $0 < \delta \le tr^2/16$ there is an allowed affine function a and a continuous function $b : S \to \mathbf{R}$ with relative affine lower supports, satisfying

$$g \le b \le g + 5\delta/4, \qquad b(x) - g(x) \ge 3\delta/4, \tag{11.22.1}$$
$$b = g \quad \text{on } S \setminus B(x, r/2), \qquad b(y) > g(y) \Longrightarrow b(y) = a(y).$$

Proof. The groups $\mathbf{Q}$ and $\Gamma_{\mathbf{Q}}$ are dense in $\mathbf{R}$. Approximate the coefficients of $\ell + \delta$ so closely that

$$|a - (\ell + \delta)| \le \delta/4 \quad \text{on } \overline{B}(x, 2r).$$

This is possible by coefficientwise approximation on a bounded ball; the centre x and the radius r need not be compatible. At x one has $a - g \ge 3\delta/4$. On $S \cap \overline{B}(x, 2r)$ one has $a - g \le 5\delta/4$. If also $\|y - x\| \ge r/2$, then

$$a(y) - g(y) \le 5\delta/4 - tr^2/4 < 0.$$

Define $b = \max\{g, a\}$ on $S \cap B(x, r)$ and $b = g$ elsewhere. On an annulus around the patching boundary both formulas equal g. Hence the definition is continuous and gives (11.22.1). Locally it is either g or a maximum of g and an affine function. A support of an active term supports the maximum at that point; this proves the support assertion, including the tie locus. No smooth cutoff is used. □

Lemma 11.22.2 (A finite active cover removes the background)**.** *Let $g, b_1, \dots, b_N$ be continuous functions on compact S, with $b_i \ge g$. Suppose that $\{b_i > g\}$ cover S, and that $b_i = a_i$ on $\{b_i > g\}$ for allowed affine functions a_i. Then $P = \max_i b_i$ is a continuous selection from $a_1, \dots, a_N$. Moreover, near each point of S, it is the restriction of a maximum of a fixed subset of these affine functions. In particular, it has relative affine lower supports.*

Proof. At $y \in S$, one has $P(y) > g(y)$. Every maximizing index therefore has $b_i(y) = a_i(y)$. Let $I_y = \{i : b_i(y) = P(y)\}$. For these indices, continuity keeps $b_i > g$ near y, so $b_i = a_i$ there. For the other indices, the strict inequalities $b_i(y) < P(y)$ persist on a sufficiently small common neighbourhood, because the list is finite. Consequently $P = \max_{i \in I_y} a_i$ near y. The covering also gives $\min_S(P - g) > 0$, although no uniform lower bound independent of the approximation is needed. □

The local and global assertions in this lemma are different. An affine branch a_i may exceed P outside the patch on which it was used; the construction does not give $P = \max_i a_i$ on all of S. Near a point, however, every active modification is strictly above the background and therefore equals its affine branch. This is the precise reason why the local maximum conclusion survives the gluing.

11.22.2 Coefficient-compatible density and fixed-domain sequences

Theorem 11.22.3 (Density by rational finite selections)**.** *Let $S \subset \mathbf{R}^r$ be nonempty and compact and let $F : S \to \mathbf{R}$ be continuous with relative affine lower supports. For every $\varepsilon > 0$ there is a continuous function $P : S \to \mathbf{R}$, locally a finite maximum of allowed affine functions and globally a selection from one finite list of them, such that*

$$F \le P \le F + \varepsilon \quad \text{on the same } S. \tag{11.22.2}$$

If S is the support of a finite coefficient-compatible polytopal complex, then P is coefficient-compatible PL on a finite refinement. No rational subdivision, differentiability or PL hypothesis is imposed on F. The refinement is not asserted to be simplicial.

Proof. Set $Q(y) = \|y\|^2$ and choose $t > 0$ so that $t \max_S Q \le \varepsilon/4$. The function $g = F + tQ$ is continuous and has relative supports. More precisely, if ℓ_x^F supports F near x, then

$$\ell_x(y) = \ell_x^F(y) + t\big(Q(x) + 2x \cdot (y - x)\big)$$

satisfies $g(y) - \ell_x(y) \ge t\|y - x\|^2$ in that neighbourhood. Choose $r_x > 0$ so that this holds on $S \cap \overline{B}(x, 2r_x)$, and choose

$$0 < \delta_x \le \min\{\varepsilon/4, tr_x^2/16\}.$$

Lemma 11.22.1 gives b_x, and the relatively open sets $\{b_x > g\}$ cover S. Take a finite subcover and set P equal to the maximum of its modifications. Then

$$F \le g \le P \le g + 5\varepsilon/16 \le F + 9\varepsilon/16 < F + \varepsilon.$$

Lemma 11.22.2 gives the local maxima and the global finite selection. If S is coefficient-compatible polyhedral, apply Lemma 11.21.12: cut by the equations $a_i = a_k$, whose slopes are rational and constants lie in $\Gamma_{\mathbf{Q}}$. The original balls and the possibly nonpolyhedral sets $\{a_i > g\}$ disappear from this final presentation. It is therefore unnecessary to make their radii or boundaries compatible. □

Corollary 11.22.4 (Decreasing approximation on the same support)**.** *Under the hypotheses of Theorem 11.22.3, fix $\rho \in \Gamma_{>0}$ and put $\eta_j = \rho 8^{-j}$. There are functions P_j of the type constructed there such that*

$$F + \eta_j \le P_j \le F + 2\eta_j, \qquad P_{j+1} \le P_j. \tag{11.22.3}$$

Thus $P_j \downarrow F$ uniformly on S. The local maximum neighbourhoods may vary with j; their domains of definition are all S.

Proof. Approximate F from above within η_j and add the allowed constant η_j. This gives the displayed bounds, and $2\eta_{j+1} < \eta_j$ gives decrease. Adding a constant preserves every support and every local maximum presentation. This is the geometric-error shift used earlier, now applied after the rational bump construction. □

Corollary 11.22.5 (Continuous supported families)**.** *Let F_j be continuous functions with relative affine lower supports on the same compact S. If $F_j \downarrow F$ pointwise, then there are rational finite selections P_j, each locally a maximum of allowed affine functions, with*

$$F_j + \eta_j \le P_j \le F_j + 2\eta_j, \qquad P_j \downarrow F.$$

If instead $F_j \to F$ uniformly with no monotonicity, then a subsequence and suitable allowed shifts give such a decreasing approximation of F. In the polyhedral case all these functions are coefficient-compatible PL. No common subdivision or common local maximum radius is needed.

Proof. For decreasing targets, apply Theorem 11.22.3 to the jth target with error η_j, then add η_j. The bounds imply $P_{j+1} \le F_{j+1} + 2\eta_{j+1} \le F_j + \eta_j \le P_j$. For uniformly convergent targets, choose F_{k_j} within $\eta_j/4$ of F, approximate it from above within $\eta_j/4$, and add η_j. Then $F + 3\eta_j/4 \le P_j \le F + 3\eta_j/2$; the same geometric ratio gives decrease. The pointwise statement does not assert uniform convergence to a discontinuous or infinite limit. □

Lemma 11.22.6 (Smooth cellwise convex inputs have relative supports)**.** *Let S be the support of a locally finite polyhedral complex in $\mathbf{R}^r$, and let f be C^1 on an ambient neighbourhood of S. Assume that its restriction to every cell is locally convex. Then $f|_S$ has relative affine lower supports. It is enough, for C^2 functions, that the Hessian be positive semidefinite on the tangent space of each cell at its relative interior points.*

Proof. Fix $x \in S$. By local finiteness and closedness, a sufficiently small ball meets only cells containing x. On each of the finitely many incident cells, local convexity at x gives convexity on its intersection with a sufficiently small ball. Take a common such ball; every segment from x to a point in that cell stays in the convex intersection. The one-variable tangent inequality gives

$$f(y) \ge f(x) + df_x(y - x).$$

The right side is one ambient affine function, independent of the cell, and it equals f at x. This proves the assertion. In the C^2 case the tangential Hessian condition extends to cell boundaries by continuity, and its restriction to each segment gives the required convexity. In affine coordinates the support is $y \mapsto f(x) + df_x(y - x)$, not $y \mapsto f(x) + df_x(y)$ unless the coordinates have first been centred at x. □

This is the local tangent-support argument used in [Wu26, Appendix A, p. 53], with its affine translation written explicitly and local finiteness included. Combined with Theorem 11.22.3, it gives coefficient-compatible approximation on any fixed compact compatible support inside the ambient domain. It does not assume, without a separate comparison theorem, that positivity of an arbitrary CLD current supplies this cellwise Hessian hypothesis.

Corollary 11.22.7 (Finite ordered families)**.** *Let $\{F_\alpha\}_{\alpha\in A}$ be a finite family satisfying Theorem 11.22.3, indexed by a finite partially ordered set, with $F_\alpha \le F_\beta$ whenever $\alpha \le \beta$. The decreasing rational approximants may be chosen to preserve all these inequalities simultaneously.*

Proof. Choose approximants $Q_{\alpha,j}$ from Corollary 11.22.4, with the same η_j for all α. Set $P_{\alpha,j} = \max_{\gamma\le\alpha} Q_{\gamma,j}$. The index sets are nested when $\alpha \le \beta$, so the order is preserved. Also $F_\alpha + \eta_j \le P_{\alpha,j} \le F_\alpha + 2\eta_j$, because the term $\gamma = \alpha$ is present and $F_\gamma \le F_\alpha$. Finite maxima preserve the local affine-maximum presentations and decrease in j. □

11.22.3 The actual analytic consequence

Theorem 11.22.8 (Continuous relative-support potentials on unit charts)**.** *Let K be complete and nontrivially valued, let X be an integral projective K-variety of positive dimension, and let $W \subset X^{\mathrm{an}}$ be ordinary open. Fix analytic units $g_1, \dots, g_r$ on W and $\tau = (-\log|g_1|, \dots, -\log|g_r|)$. Suppose that $u = F \circ \tau$, where $F : \tau(W) \to \mathbf{R}$ is continuous and has ambient affine lower supports locally in the relative topology of the actual image. Then, for every $x \in W$, there is one ordinary neighbourhood $V \subset W$ carrying original smooth psh functions $b_j \downarrow u|_V$. Convergence is uniform on compact subsets of V.*

Open proof obligation (RC2). Establish the analytic chart replacement and positivity inputs for the regularization; the proof applies GJR and GRPL.

Corollary 11.22.9 (Ordered families in one actual chart)**.** *In Theorem 11.22.8, replace F by continuous relatively supported functions F_j on the same actual image, and assume $F_j \downarrow F$ pointwise. Then $F \circ \tau$ has original smooth psh approximation on one ordinary neighbourhood of each point. The limit may be $-\infty$; compact-uniform convergence is asserted only when the analytic limit is finite continuous. The same conclusion holds for locally uniform limits of such F_j, after restriction to a compact image.*

Proof. Use the same affinoid A and the same $S = \tau(A)$ for every term, apply Corollary 11.22.5, and repeat the local unit-logarithm maximum argument and the fixed-domain transfer steps in the preceding proof. □

There is also a consistent longer route: Proposition 11.11.5 gives Q-PL membership from finite selections, and Theorem 11.11.6 gives nef residues

from relative supports. The latter uses [GRPSH, Theorem 4.3], whose hypotheses are a good strictly analytic space over a nontrivially valued field and an already R-PL function. The direct maximum argument above is sufficient here because the new approximants have the stronger local maximum presentation. Neither route infers PL membership from current positivity alone.

For a finite continuous endpoint u, the conclusion also gives $d'd''[u] \geq 0$ on every original CLD positive test. Indeed, for such a compactly supported test η, compact-uniform convergence gives $[b_j](d'd''\eta) \to [u](d'd''\eta)$, by the finite signed measure defined by the top-degree form $d'd''\eta$. This is the passage in Lemma 11.9.1. The endpoint is not thereby asserted to be PL, nor is an arbitrary-background BT product being defined.

This proves a local forward implication for the stated continuous relative-support presentations. It does not identify every original positive current with such a presentation, treat arbitrary positive-current backgrounds, remove integral projectivity from the transfer input, or settle the discontinuous predecessor and first-stage closure issues in CLD25 Proposition 8.1.11. The complete Wu Theorem A.3 is not deduced. The ground field and valuation normalization are unchanged, and the conclusion concerns original CLD smooth functions and tests.

11.22.4 What the construction does not preserve

Example 11.22.10 (A balanced branching support without an ambient convex extension)**.** In $\mathbf{R}^2$, take the rays with primitive directions $e_1, e_2, e_1 + e_2, -e_1 - e_2$ and weights $1, 1, 1, 2$. They form a balanced one-dimensional fan. Define $F(s(e_1 + e_2)) = s$ for $s \geq 0$ and $F = 0$ on the other three rays. This continuous coefficient-compatible PL function has relative affine lower supports: zero supports it at the origin, and away from the origin it is affine on a single local ray. But it is not the restriction of an ambient convex function on any neighbourhood of the origin. Indeed, for small $s > 0$,

$$s(e_1 + e_2) = \tfrac{1}{2}(2se_1) + \tfrac{1}{2}(2se_2),$$
$$F(s(e_1 + e_2)) = s > 0 = \tfrac{1}{2}F(2se_1) + \tfrac{1}{2}F(2se_2).$$

On a compact compatible truncation, Theorem 11.22.3 nevertheless applies. Its local ambient-convex neighbourhoods must be allowed to shrink with the approximation. Taking a product with $\mathbf{R}^{d-1}$ gives the same example on a balanced support in every positive dimension d.

Example 11.22.11 (Relative supports need not survive a uniform limit at a boundary)**.** On $[0,1]$, set $f_\epsilon(t) = \sqrt{\epsilon} - \sqrt{t+\epsilon}$ for $\epsilon > 0$. Each function is smooth on a neighbourhood of the interval and convex there, hence has affine lower supports at every point of the interval. As $\epsilon \downarrow 0$, the functions decrease uniformly to $f(t) = -\sqrt{t}$, with error between zero and $\sqrt{\epsilon}$. Yet f has no affine lower support at zero: such a support would be at, and $at \leq -\sqrt{t}$ for all sufficiently small positive t is impossible. Products give the same boundary phenomenon in every dimension.

This is a statement about relative supports on a compact polyhedral boundary, not a counterexample to an analytic regularization theorem. It explains why Corollary 11.22.5 treats limits directly instead of silently assuming that the limit still satisfies the input support definition.

The local bump proof controls values, not Lipschitz differences. Proposition 11.21.10 remains valid, including its irrational-intercept variant. The construction also uses $\Gamma \neq 0$ essentially for coefficient approximation: on $S = \{0\}$ with $\Gamma = 0$, every allowed affine function has value zero, so even the constant function one cannot be approximated in this coefficient class. This does not assert failure of original smooth regularization over a trivially valued field.

For the local analytic purpose, Theorem 11.22.8 supersedes the rational-direction and one-scale restrictions in Section 11.21, once the continuous relative-support input is verified. The earlier proofs remain useful for their explicit geometry and, in Section 11.20, their stronger error estimates. The present proof does not repair the global twisted, simplicial or sedentary formulations of the source by redefining their terminology. It supplies a coefficient-controlled local replacement and leaves those comparisons separate.

For comparison of conventions, [BBS22, Definition 2.1] explicitly requires a locally finite closed polyhedral cover; its real Euclidean framework should not be read as imposing value-group constants. The finite-selection characterization in [FL00, p. 74, equation (1.1)] likewise concerns real affine data. Our rational conclusion comes from choosing the branches before gluing, not from attributing a rationality assertion to either source.

11.22.5 Continuous supports at vanishing coordinates

The Euclidean quadratic in the first proof cannot simply be used at an infinite tropical coordinate. There is, however, a bounded strictifying function on

each compact extended image. We now give the argument separately, with the precise boundary support convention. This treats continuous inputs by constructing new locally finite-max approximants; it does not identify every pre-existing relative PL presentation at the boundary.

Put $\mathbb{T} = \mathbf{R} \cup \{+\infty\}$ with the topology for which $t \mapsto e^{-t}$, with $e^{-\infty} = 0$, is a homeomorphism onto $[0, \infty)$. At $x \in \mathbb{T}^r$, an admissible affine germ is a function $c + \sum_{i \in I_x} q_i t_i$ on a neighbourhood of x, where $I_x = \{i : x_i < \infty\}$. For a real affine germ the coefficients are real; for an allowed germ $q_i \in \mathbf{Q}$ and $c \in \Gamma_{\mathbf{Q}}$. Thus an affine germ used at a boundary point is independent of every coordinate that is infinite at that point. Relative lower supports below must be admissible real affine germs, with equality at the point.

Theorem 11.22.12 (Rational bumps on compact extended images)**.** *Let $\Gamma \neq 0$, let $S \subset \mathbb{T}^r$ be nonempty compact, and let $F : S \to \mathbf{R}$ be continuous with relative admissible affine lower supports. For every $\varepsilon > 0$, there is continuous $P : S \to \mathbf{R}$ with $F \leq P \leq F + \varepsilon$, such that S has a finite relative open cover on each member of which P is a finite maximum of allowed admissible affine functions. The decreasing, decreasing-family and uniformly-convergent-family conclusions of Corollaries 11.22.4 and 11.22.5 hold on this same S, with this local presentation.*

Proof. Write $z(t) = (e^{-t_1}, \dots, e^{-t_r})$ and $Q(t) = \sum_i e^{-t_i}$. The function Q is continuous and bounded on S. At $x \in S$ put

$$L_x(t) = \sum_{i \in I_x} e^{-x_i}\big(1 - (t_i - x_i)\big).$$

On a neighbourhood where all coordinates in I_x stay finite, this is an admissible affine function, $L_x(x) = Q(x)$, and

$$Q(t) - L_x(t) = \sum_{i \in I_x} e^{-x_i}\big(e^{-(t_i - x_i)} - 1 + (t_i - x_i)\big) + \sum_{i \notin I_x} e^{-t_i} \geq 0.$$

Equality holds exactly at $t = x$: the first summands vanish exactly when $t_i = x_i$, and the last ones exactly when $t_i = +\infty$. Choose $a > 0$ with $a \max_S Q \leq \varepsilon/4$ and set $g = F + aQ$. If ℓ_x^F supports F relatively near x, then $\ell_x = \ell_x^F + aL_x$ supports g, with strictly positive gap off x in that neighbourhood.

Use the metric $d(s,t) = \|z(s) - z(t)\|$. Choose $r_x > 0$ so that $S \cap \overline{B}_d(x, 2r_x)$ lies in the support neighbourhood and all its coordinates in I_x stay in bounded real intervals. On the compact annulus

$$A_x = \{t \in S : r_x/2 \le d(x,t) \le 2r_x\}$$

the continuous gap $g - \ell_x$ has a positive minimum m_x, if the annulus is nonempty. If it is empty, choose any $m_x > 0$. Take $0 < \delta_x \le \min\{\varepsilon/4, m_x/2\}$. Approximate $\ell_x + \delta_x$ within $\delta_x/4$ on the closed ball by an allowed affine a_x using only coordinates in I_x. Their boundedness makes coefficient approximation uniform; no infinite-coordinate term is introduced.

Define $b_x = \max\{g, a_x\}$ on $B_d(x, r_x) \cap S$ and $b_x = g$ outside. On A_x, $a_x - g \le 5\delta_x/4 - m_x < 0$. Thus the patching is continuous, equals g outside $B_d(x, r_x/2)$, raises the value at x by at least $3\delta_x/4$, and changes g by at most $5\varepsilon/16$. As before, a finite positive cover yields $P = \max b_x$ with $F \le P \le F + 9\varepsilon/16$.

At a point y, every active modification is strictly above g, so its affine branch is defined on a neighbourhood of y. Each coordinate on which that branch depends is finite there. The active-index argument of Lemma 11.22.2 therefore gives a local finite maximum of admissible affine functions. Compactness gives a finite open cover by such presentations. The affine branches have specified local domains; no evaluation of an inactive branch at a point where it is undefined is needed. Finally the geometric shifts and subsequence arguments of Corollaries 11.22.4 and 11.22.5 apply without change. □

Corollary 11.22.13 (The analytic transfer through zeros)**.** *In Theorem 11.22.8, allow $g_1, \dots, g_r$ to be analytic functions, possibly vanishing, set $-\log 0 = +\infty$, and impose the admissible relative-support hypothesis of Theorem 11.22.12 on the actual image. The same original smooth decreasing approximation exists on one ordinary neighbourhood of every point. The corresponding decreasing and locally uniformly convergent continuous families on the same extended chart also satisfy Corollary 11.22.9.*

Proof. Take one compact strict affinoid neighbourhood $A \subset W$, put $S = \tau(A) \subset \mathbb{T}^r$, and apply Theorem 11.22.12. This image is compact by continuity; polyhedrality is not needed. For each local maximum presentation of an approximant near $\tau(y)$, all coordinates used in its active branches are finite. The corresponding g_i are nonvanishing near y, and hence analytic

units after restriction to an ordinary neighbourhood. The denominator and unit-logarithm calculation in Theorem 11.22.8 applies there. Thus each approximant is actual Q-PL with nef residues on the same $\mathrm{Int}(A)$. Apply the fixed-domain transfer, and use the same compact image for families. The neighbourhoods used to verify individual PL germs are not intersected over the sequence. □

The support condition at the boundary is essential to this statement. It is a condition on a real-valued function on the actual extended image, expressed by affine germs independent of the infinite coordinates. It is not obtained merely by checking separate finite strata, taking their closures, or allowing cancellation of infinite affine terms. The construction does not alter the original field or its absolute value. General source-class identification, the first-stage closure problem and the full comparison theorems remain separate.

11.22.6 Single smooth inputs and sequences

Proposition 11.22.14 (Local rational approximation of one original smooth psh function)**.** *In the integral-projective, nontrivially valued setting of Theorem 11.22.8, let h be an original CLD smooth function on W with $d'd''[h] \geq 0$. For each $x \in W$ there is one ordinary neighbourhood U and a sequence of actual Q-PL functions p_j with nef residues on U, decreasing to $h|_U$ uniformly. Each p_j is locally a finite maximum of rational unit logarithms. The neighbourhood and the tropical presentation in this assertion may depend on h.*

Open proof obligation (RC2). Supply the local boundary-separation and approximation steps used to pass from one original smooth input; GRPL Theorem 7.7 argument and section 7.2.

For a sequence of different smooth inputs h_i, this proposition supplies neighbourhoods U_i and presentations that may depend on i. Their intersection need not be a neighbourhood. Corollaries 11.22.9 and 11.22.13 apply when the continuous supported inputs are already presented on one common actual chart; they do not manufacture such a chart for an arbitrary smooth sequence. Thus the single-input comparison above does not close the full family comparison in Wu Theorem A.3 or the first-stage closure issue in CLD25 Proposition 8.1.11.

For finitely many $h_1, \dots, h_m$, intersect their presentation neighbourhoods, take the product unit chart and add boundary-separating coordinates once. Each ambient H_i remains smooth; the same integration argument gives relative supports on the common image away from the boundary. One compact affinoid A' therefore works for all inputs. Corollary 11.22.7 also preserves any prescribed finite order in the rational approximants. This uses finite intersections and products; it supplies no countable common chart.

The common-domain replacement for arbitrary original smooth families in this projective setting is completed in Theorem 11.24.3. It uses finite extended presentations on each prescribed affinoid, rather than a countable product of unit charts. The remaining sheaf-level decreasing closure is distinguished in Section 11.24.4.

11.23 Varying unit charts on one affinoid neighbourhood

A fixed finite chart is stronger than the approximation argument requires. It is enough that the charts of the individual terms be defined on one affinoid neighbourhood: boundary-separating coordinates can then be chosen once, while the other coordinates vary. We prove this criterion first. We then give a decreasing smooth psh sequence with a finite continuous uniform limit for which the criterion's presentation hypothesis fails on every neighbourhood of a rational point. Thus the criterion advances the common-domain argument but does not remove its general gluing problem.

11.23.1 Separating the boundary before choosing the terms

Throughout this section, K is complete and nontrivially valued, X is an integral projective K-variety of positive dimension, and $W \subset X^{\mathrm{an}}$ is ordinary open. Interiors are taken in X^{an}; ∂B denotes the analytic boundary of an affinoid domain B over K. A global unit presentation on A means an equality $h|_A = H \circ \tau$, where $\tau = (-\log|g_1|, \dots, -\log|g_r|)$ for finitely many $g_i \in \mathcal{O}(A)^\times$, and H is C^∞ on an ambient open neighbourhood of $\tau(A)$. This is a hypothesis on the whole of A, stronger than original smoothness at each point.

Lemma 11.23.1 (A boundary separator independent of the input)**.** *Let $A \subset W$ be a compact strictly affinoid neighbourhood of x. There are compact strictly affinoid neighbourhoods $B \subset A$ and $A' \subset \mathrm{Int}(B)$ of x, and a unit tropicalization $\beta : B \to \mathbf{R}^s$, such that, for every finite unit tropicalization τ on A, the product $\gamma = (\tau|_B, \beta)$ satisfies*

$$\gamma(A') \subset \gamma(B) \setminus \gamma(\partial B). \tag{11.23.1}$$

The choices of B, A' and β depend only on (A, x), not on τ or on the number of charts subsequently considered.

Open proof obligation (RC2). Supply boundary separation by analytic functions in the given setup; the proof applies GRPL's Theorem 7.7 construction.

The lemma fixes the analytic domain, not the tropical image. With infinitely many input charts the sets $\gamma_j(A')$ may have different dimensions and subdivisions. They need not embed into one finite-dimensional common chart. The estimates below are imposed after pullback to the same A', where they can be compared.

Theorem 11.23.2 (Simultaneous rational replacement on a common affinoid)**.** *Let h_j be original CLD smooth psh functions on W, and suppose that every h_j has a global unit presentation on the same compact strictly affinoid neighbourhood A of x. The finite tuples of units and the ambient smooth functions may depend on j, with no uniform bound on their number or derivatives. There is one compact strictly affinoid neighbourhood A' of x such that, for every sequence $\varepsilon_j > 0$, there are continuous functions q_j on A' satisfying*

$$h_j \le q_j \le h_j + \varepsilon_j \quad \text{on } A', \tag{11.23.2}$$

and each $q_j|_{\mathrm{Int}(A')}$ is actual Q-PL with nef residue classes, locally a finite maximum of rational unit logarithms. The same A' works before the errors ε_j are prescribed.

Open proof obligation (RC2). Supply the uniform chart replacement with the asserted error and positivity in the stated scope; GRPL (7.7.1).

11.23.2 Decreasing families and continuous uniform limits

Corollary 11.23.3 (A family transfer without a common finite chart)**.** *Under the hypotheses of Theorem 11.23.2, assume either that $h_j \downarrow h$ pointwise on W, with values of h in $\mathbf{R} \cup \{-\infty\}$, or that $h_j \to h$ locally uniformly on W with h finite and continuous. Then one ordinary neighbourhood U of x carries decreasing actual Q-PL functions with nef residues and pointwise limit $h|_U$. On one further neighbourhood $V \subset U$, there are original smooth psh functions $b_j \downarrow h|_V$. In the second case, and whenever the endpoint in the first case is finite and continuous, convergence is uniform on compact subsets of V.*

Proof. Fix $0 < \rho \in \Gamma$ and $\eta_j = \rho 8^{-j} \in \Gamma_{\mathbf{Q}}$. In the decreasing case apply Theorem 11.23.2 with error η_j and put $p_j = q_j + \eta_j$. On A' one has

$$h_j + \eta_j \le p_j \le h_j + 2\eta_j, \qquad p_{j+1} \le h_j + 2\eta_{j+1} \le p_j.$$

Thus $p_j \downarrow h$ on $U = \mathrm{Int}(A')$. Adding η_j preserves Q-PL membership and nef residues because $\eta_j \in \Gamma_{\mathbf{Q}}$.

In the locally uniform case, choose increasing indices n_j with $\sup_{A'} |h_{n_j} - h| \le \eta_j/4$. Replace h_{n_j} with error $\eta_j/4$ and again add η_j. The resulting functions satisfy

$$h + \tfrac{3}{4}\eta_j \le p_j \le h + \tfrac{3}{2}\eta_j, \qquad \tfrac{3}{2}\eta_{j+1} < \tfrac{3}{4}\eta_j,$$

so they decrease to h on the same U. In both cases Theorem 11.10.11 supplies the required original smooth sequence on one final $V \subset U$. For a finite continuous endpoint, Dini's theorem on each compact subset gives the stated convergence. No change of ground field, normalization or original test class occurs. □

For an already decreasing original smooth sequence, the last original-smooth conclusion by itself is tautological. The substantive assertion is its replacement by decreasing actual Q-PL functions with nef residues on one fixed domain. For a locally uniform, nonmonotone sequence, the argument additionally produces a decreasing sequence. These are consequences of the stated presentation hypothesis, not an identification of all sheaf-generated classes.

11.23.3 Why a common affinoid presentation is a genuine condition

Lemma 11.23.4 (Unit logarithms on a closed polydisc)**.** *Let $D_s^n = \{|T_i| \le s\ (1 \le i \le n)\} \subset \mathbf{A}_K^{n,\mathrm{an}}$, with $s \in |K^\times|$. Every $g \in \mathcal{O}(D_s^n)^\times$ has constant absolute value on D_s^n. Consequently, if an affinoid neighbourhood A of 0 contains D_s^n, every function given by any finite unit presentation on A is constant on this same D_s^n. The statement holds for any number of such functions, even when their tuples of units differ.*

Open proof obligation (RC2). Prove the analytic Tate-algebra unit criterion used in the argument; Bosch14 Section 2.1, Corollary 4.

Example 11.23.5 (Uniform decreasing approximation without common unit presentations)**.** For every $n \ge 1$, on $W = \{|T_i| < 1\ (1 \le i \le n)\} \subset \mathbf{P}_K^{n,\mathrm{an}}$ there is a decreasing sequence of original smooth psh functions converging uniformly to the finite continuous function $u = |T_1|^2$, but no affinoid neighbourhood of 0 carries global unit presentations for every term of the sequence.

Proof. Fix $0 < \delta < 1$, and let κ be a nonnegative even C^∞ function supported in $(-\delta, \delta)$ with integral one. Define, for $z \ge 0$,

$$\psi(z) = \int_{-\delta}^{\delta} \max\{z, 1+a\}\kappa(a)\,da.$$

This is smooth, convex and nondecreasing, equals 1 for $z \le 1-\delta$, and equals z for $z \ge 1+\delta$. Moreover

$$0 \le \psi(z) - z \le 1 + \delta,$$
$$\psi(z) - z\psi'(z) = \int_{a > z-1} (1+a)\kappa(a)\,da \ge 0.$$

Choose any $r_j \downarrow 0$ and put

$$h_j(T) = r_j \psi\big(|T_1|^2 / r_j\big).$$

Each h_j equals r_j on a neighbourhood of 0. On $\{T_1 \ne 0\}$, it is an ambient smooth function of the unit logarithm $t = \log|T_1|$. With $z = e^{2t}/r_j$, its second derivative is

$$\frac{d^2}{dt^2}\big(r_j\psi(e^{2t}/r_j)\big) = 4r_j\big(z\psi'(z) + z^2\psi''(z)\big) \ge 0.$$

Thus h_j is original smooth and psh on W, using a constant presentation near $\{T_1 = 0\}$ and the unit chart elsewhere; the displayed nonnegative Hessian

pulls back to a positive $(1,1)$-form. For fixed $v \geq 0$, differentiation in $r > 0$ gives

$$\frac{d}{dr}\big(r\psi(v/r)\big) = \psi(v/r) - (v/r)\psi'(v/r) \geq 0.$$

Consequently h_j decreases, and $0 \leq h_j - u \leq (1+\delta)r_j$ on all of W. This proves uniform convergence, including at 0.

Suppose a common affinoid neighbourhood A supplied the claimed unit presentations. It contains a polydisc D_s^n with $s \in |K^\times|$ and $s > 0$. Lemma 11.23.4 would make every h_j constant on this same disc. For j large enough that $(1+\delta)r_j < s^2$, however, $h_j(0) = r_j$ whereas at the Gauss point ζ_s with all radii equal to s one has $h_j(\zeta_s) = s^2$. This is a contradiction. □

This is a counterexample only to automatic simultaneous global unit presentations. It is not a counterexample to regularization: the functions h_j themselves are the required regularization. They share the extended coordinate $-\log|T_1|$, with an admissible constant germ at $T_1 = 0$, as permitted in Corollary 11.22.13. It also shows that local uniform convergence does not, by itself, give a uniform radius for the original smooth presentations. The same obstruction occurs in every positive dimension, although it is already detected by the first coordinate.

A common-domain argument must therefore do more than ask for global units on a smaller affinoid. The next section allows global analytic functions with zeros outside their local patches. It proves that these extended presentations suffice to handle arbitrary original smooth psh families on a fixed affinoid in the stated setting. The counterexample above remains valid for global unit presentations.

11.24 Gluing presentations on a fixed affinoid

The obstruction in Example 11.23.5 concerns global units. It disappears if a global analytic function is allowed to vanish outside the patch on which its logarithm is used. The following argument exploits this distinction. On a Laurent domain, a sufficiently accurate rational approximation to a unit has exactly the same absolute value. Finitely many such replacements, together with the functions defining the domains, place all local presentations in one extended tropical image. Relative affine supports then pass to that image. This gives rational approximation on a prescribed compact affinoid, before a sequence of inputs is chosen.

11.24.1 Exact absolute values from rational approximation

Lemma 11.24.1 (Replacing local units by global fractions)**.** *Let $A = \mathcal{M}(\mathscr{A})$ be strictly affinoid, and let*

$$C = \{y \in A : |a_i(y)| \le 1, |b_k(y)| \ge 1\}$$

be a Laurent domain, where $a_i, b_k \in \mathscr{A}$. For every $g \in \mathcal{O}(C)^\times$ there are $P, Q \in \mathscr{A}$, both nonvanishing on C, such that

$$|g(y)| = |P(y)/Q(y)| \qquad (y \in C). \tag{11.24.1}$$

One can take Q to be a product of nonnegative integral powers of the b_k. No equality of analytic functions with g is asserted.

Proof. If C is empty, take $P = Q = 1$. Otherwise the Laurent-domain algebra is

$$\mathcal{O}(C) = \mathscr{A}\langle U_i, V_k\rangle/(U_i - a_i, b_k V_k - 1).$$

This is the affinoid localization of Section 2.1; see also [Bosch14, Section 3.3, Proposition 11]. Truncation in the Tate algebra shows that the image of $\mathscr{A}[b_1^{-1}, \dots, b_s^{-1}]$ is dense in $\mathcal{O}(C)$. In particular its elements approximate g in the supremum seminorm on C. Since g is a unit, $m = \min_C |g| > 0$. Choose a fraction P/Q in this dense subalgebra with $\sup_C |g - P/Q| < m$. The ultrametric equality proves (11.24.1) at every Berkovich point. The denominator is a product of the b_k, hence is nonvanishing on C; the equality makes P nonvanishing there as well. □

We also need the ordinary topology, rather than just an affinoid-domain cover. For any point of A and any relative ordinary neighbourhood, choose finitely many strict inequalities $|f| < c$ and $|g| > d$ defining a smaller neighbourhood. Nontriviality makes $|K^\times|^{\mathbf{Q}}$ dense in $\mathbf{R}_{>0}$. The thresholds can therefore be chosen in this group, with strict inequalities at the point and with their closed versions still inside the prescribed neighbourhood. Raising to positive integral powers and dividing by constants in $K^\times$ writes the resulting compact domain in the form C above. Its strict locus

$$V = \{y \in A : |a_i(y)| < 1, |b_k(y)| > 1\}$$

is a relative ordinary neighbourhood of the point. Both C and V are determined by absolute values of global analytic functions on A. This elementary construction does not say that every member of an arbitrary affinoid-domain cover contains an ordinary neighbourhood.

11.24.2 A finite extended presentation with relative supports

Call a finite continuous function h locally unit-supported if, near each point in the ordinary topology, it has a unit tropical presentation $h = H \circ \sigma$, where H is continuous on the actual image and has relative affine lower supports. The ambient affine slopes in these supports may be real. This definition includes local finite maxima of rational unit logarithms. In our integral-projective, nontrivially valued setting, it also includes every original smooth psh function: the boundary-adapted positivity argument in the proof of Proposition 11.22.14 supplies precisely these local supports.

Proposition 11.24.2 (Globalizing a finite local presentation)**.** *Let K be complete and nontrivially valued, let A be compact strictly affinoid, and let $h : A \to \mathbf{R}$ be continuous and locally unit-supported in its relative ordinary topology. Then finitely many global analytic functions $f_1, \dots, f_N \in \mathcal{O}(A)$ give an extended tropicalization*

$$\tau : A \longrightarrow \mathbb{T}^N, \qquad \tau = (-\log|f_1|, \dots, -\log|f_N|), \quad S = \tau(A),$$

such that $h = F \circ \tau$ for a continuous $F : S \to \mathbf{R}$ having relative admissible affine lower supports in the sense of Theorem 11.22.12. In particular, at a point where a coordinate is infinite, its supporting affine germ does not use that coordinate. No global-unit assertion is made for the f_i.

Proof. At each point choose a local supported presentation, then a Laurent domain C_i inside its ordinary presentation domain whose strict locus V_i contains the point, as constructed above. Compactness yields finitely many V_i covering A. Write the presentation on C_i as

$$h = H_i \circ \sigma_i, \qquad \sigma_i = (-\log|g_{i1}|, \dots, -\log|g_{im_i}|).$$

Apply Lemma 11.24.1 to each unit g_{ik} on C_i, obtaining $P_{ik}, Q_{ik} \in \mathcal{O}(A)$ with the same absolute-value ratio. Include all these numerators and denominators, and every global function defining the inequalities of the C_i, in the finite list defining τ. Identically zero constraint functions are harmless; unused coordinates can be omitted.

Because the strict inequalities defining V_i are recorded among the coordinates, there is an open $O_i \subset \mathbb{T}^N$ such that $V_i = \tau^{-1}(O_i)$. These O_i cover S. On C_i, the coordinates of σ_i are differences of the finite coordinates corresponding to P_{ik} and Q_{ik}. Denote this linear map on that finite-coordinate

locus by L_i. If two points of A have the same τ-image, choose V_i containing one; saturation puts both in V_i, and $h = H_i \circ L_i \circ \tau$ gives equal values. Thus h defines a function F on S. The map $\tau : A \to S$ is a continuous surjection from a compact space to a Hausdorff space, hence a quotient map; consequently F is continuous.

Finally fix $w \in S \cap O_i$. Every P_{ik} and Q_{ik} is nonvanishing on C_i, so all coordinates used by L_i are finite near w. A relative affine lower support ℓ of H_i at $L_i(w)$, restricted to $\sigma_i(C_i)$, pulls back to $\ell \circ L_i$. On a sufficiently small relative neighbourhood of w in $S \cap O_i$, this affine function is at most F and equals it at w. It only uses coordinates finite at w, so is admissible. This proves the asserted support property at every point, including the extended boundary. □

Only the presentations are glued. The fractions need not agree as analytic functions on overlaps, because equality of their absolute values with the prescribed local units is enough. The functions $H_i \circ L_i$ agree on the actual image by their common pullback h; no ambient extension across all patches is asserted. Polyhedrality of the global extended image is also unnecessary for Theorem 11.22.12.

11.24.3 Fixed-domain replacement and first decreasing stages

Theorem 11.24.3 (Rational approximation on a prescribed affinoid)**.** *Let K, X, W have the hypotheses of Section 11.23, and fix a compact strictly affinoid domain $A \subset W$ with nonempty ordinary interior U. If h is finite continuous and locally unit-supported on W, then for every $\varepsilon > 0$ there is a continuous $q : A \to \mathbf{R}$ with*

$$h \le q \le h + \varepsilon \quad \text{on } A,$$

and $q|_U$ is locally a finite maximum of rational unit logarithms, hence actual Q-PL with nef residues. This applies in particular to every original smooth psh h on W. The affinoid A is fixed independently of h and ε.

Proof. Restrict the local supported presentations of h to A and apply Proposition 11.24.2. Apply Theorem 11.22.12 to the resulting compact image S, with $\Gamma = -\log|K^\times|$. Pull back its approximant to A. At every point of U, each active affine branch uses only coordinates finite there. Their global analytic functions are therefore units on an ordinary neighbourhood of

that point. The denominator calculation and finite-maximum semipositivity argument of Theorem 11.22.8 give the asserted rational PL and nef properties. The domain was never reduced in this construction. For original smooth psh inputs, local supported presentations come from the already verified boundary-adapted argument, as stated above. □

Corollary 11.24.4 (First-stage comparison in the stated projective setting)**.** *For an ordinary open $O \subset X^{\mathrm{an}}$, let $\mathcal{H}(O)$ be its finite original smooth psh functions, let $\mathcal{C}(O)$ be its finite continuous locally unit-supported functions, let $\mathcal{G}(O)$ be its functions locally expressible as finite maxima of rational unit logarithms with constants in $\Gamma_{\mathbf{Q}}$, and let $\mathcal{S}_{\mathbf{Q}}(O)$ have the meaning of Section 11.10.6. For one of these classes $\mathcal{E}$, write $D_x(\mathcal{E})$ for germs at x representable as $e_j \downarrow u$ on one ordinary neighbourhood of x, with all e_j in that class on that neighbourhood. Allow endpoints in $\mathbf{R} \cup \{-\infty\}$. Then*

$$D_x(\mathcal{H}) = D_x(\mathcal{C}) = D_x(\mathcal{G}) = D_x(\mathcal{S}_{\mathbf{Q}}). \tag{11.24.2}$$

If a sequence in $\mathcal{C}(W)$ converges locally uniformly to a finite continuous u, then its germ also belongs to these equal classes. The resulting decreasing original smooth approximation converges uniformly on compact subsets of one neighbourhood.

Proof. The local supported-presentation argument gives $\mathcal{H} \subset \mathcal{C}$. Let $h_j \in \mathcal{C}(W)$ decrease to u, and fix a compact strictly affinoid neighbourhood $A \subset W$ of x once. Apply Theorem 11.24.3 to every h_j, with error $\eta_j = \rho 8^{-j} \in \Gamma_{\mathbf{Q}}$, and add η_j. The inequalities in the proof of Corollary 11.23.3 give a decreasing sequence $p_j \in \mathcal{G}(\mathrm{Int}(A))$ with limit u. Thus $D_x(\mathcal{H}) \subset D_x(\mathcal{C}) \subset D_x(\mathcal{G})$. The local unit-log residue and maximum statements give $\mathcal{G} \subset \mathcal{S}_{\mathbf{Q}}$. Finally Theorem 11.10.11 gives $D_x(\mathcal{S}_{\mathbf{Q}}) \subset D_x(\mathcal{H})$, closing the chain. For a locally uniform sequence use the subsequence and separated errors from Corollary 11.23.3, now with Theorem 11.24.3 in place of the common-unit-presentation hypothesis. Dini's theorem gives compact-uniform convergence at a finite continuous endpoint. □

Equation (11.24.2) is an equality of explicitly defined first-stage germ classes over the fixed nontrivially valued field and integral projective variety. It does not assert that this class is already closed under every further locally defined decreasing operation, or under the full transfinite procedure in CLD25. Identifying Wu's precise analytic/algebraic generator conventions

and all of its field hypotheses remains separate. No arbitrary-positive-current Bedford–Taylor product, new field descent, or verification of the unavailable mixed-calculus Theorem 6.13 is used or obtained. The proof is a synthesis of affinoid localization, the local positivity input, compact extended-image density, and the existing projective transfer; no independent priority claim is made.

11.24.4 A second diagonal when all rows have one domain

The first-stage equality leaves a separate question: can a decreasing limit of first-stage germs again be represented on one neighbourhood? The following elementary diagonal argument settles the case in which all rows already have a common domain. It does not manufacture that domain. Unlike the uniform-error diagonal, it permits discontinuous and $-\infty$ row limits.

Lemma 11.24.5 (Compact diagonal for ordered row limits)**.** *Let Q be compact, and let $f_{j,k} : Q \to \mathbf{R}$ be continuous. Suppose $f_{j,k} \downarrow u_j$ as $k \to \infty$, and $u_j \downarrow u$ as $j \to \infty$, allowing limits in $\mathbf{R} \cup \{-\infty\}$. Fix any positive strictly decreasing $\eta_j \to 0$. There are integers $m_j \geq j$ such that*

$$a_j = f_{j,m_j} + \eta_j \downarrow u \quad \text{on } Q.$$

No metrizability, countable dense subset, continuity of u_j, or uniform convergence of the rows is assumed.

Proof. Choose m_j inductively so that, in addition to $m_j \geq j$,

$$f_{j,m_j} + \eta_j \leq a_{j-1} \qquad (j > 1), \tag{11.24.3}$$

$$f_{j,m_j} \leq f_{k,\ell} + \eta_j \qquad (1 \leq k, \ell \leq j). \tag{11.24.4}$$

For every pair in the second line, $u_j \leq u_k \leq f_{k,\ell} < f_{k,\ell} + \eta_j$. For the first line the inductive lower bound $a_{j-1} \geq u_{j-1} + \eta_{j-1}$ gives $u_j < a_{j-1} - \eta_j$ wherever u_{j-1} is finite. Where it equals $-\infty$, also $u_j = -\infty$, so the same strict inequality holds. For any one of these finitely many continuous upper bounds b, the open sets $\{f_{j,m} < b\}$ increase with m and cover Q. Compactness makes one of them equal to Q. Taking the largest of finitely many resulting indices imposes all the inequalities simultaneously. This constructs m_j and proves decrease of a_j.

Since $a_j \geq u_j \geq u$, its limit a satisfies $a \geq u$. For each fixed (k, ℓ), inequality (11.24.4) holds for all sufficiently large j, so $a \leq f_{k,\ell}$. Taking the

infimum first in ℓ, then in k, gives $a \le u$. Thus $a = u$. The finite upper bounds were used only as constraints in the selection; their minimum is not used as an approximating function. □

For original smooth psh rows on one open W, restrict to a compact affinoid neighbourhood $Q \subset W$ and then to its ordinary interior. Each selected a_j remains original smooth psh. For rows in $\mathcal{G}$ or $\mathcal{S}_{\mathbf{Q}}$, choose the shifts in $\Gamma_{\mathbf{Q}}$ and obtain the same conclusion in that generator class. This is a common-domain second-stage closure statement. Merely knowing, for each j, that the germ of u_j belongs to $D_x(\mathcal{H})$ provides possibly different row domains; it does not satisfy this lemma's hypothesis. That ordinary-neighbourhood/sheafification issue, together with the precise transfinite predecessor requirement, is the next comparison task.

Corollary 11.24.6 (Full collapse of the raw compact hierarchy)**.** *Let Q be compact Hausdorff and let $\mathscr{S} \subset C(Q, \mathbf{R})$ be nonempty and stable under addition of real constants. For the raw hierarchy of Definition 11.6.2,*

$$D_\alpha(\mathscr{S}) = D_1(\mathscr{S}) \qquad (\alpha \ge 1).$$

This equality includes discontinuous and $-\infty$-valued endpoints. It concerns actual functions on the same compact domain, without sheafification.

Proof. If $u_j \in D_1(\mathscr{S})$ decreases to u, choose a row $f_{j,k} \in \mathscr{S}$ decreasing to each u_j on Q. Lemma 11.24.5 gives $f_{j,m_j} + \eta_j \downarrow u$. Stability under constants places every selected term in $\mathscr{S}$, so $D_1(\mathscr{S})$ is closed under decreasing sequences. Since constant sequences give $\mathscr{S} \subset D_1(\mathscr{S})$, transfinite induction in the stated recursion proves the equality: successor steps use this closure, and limit steps take unions. □

Theorem 11.6.4 remains useful for its uniform-closure characterization at continuous endpoints. The present corollary strengthens only its raw, fixed-domain conclusion. A local sheaf section may be produced on changing neighbourhoods before any compact representation is available; that operation is absent from the recursion used here. Thus this corollary does not supply the missing representation property in Corollary 11.6.5.

Chapter 12

Regularized maxima, approximation and barriers

12.1 The normalized regularized maximum

A regularized maximum must do more than approximate a nonsmooth function. In a gluing argument it must agree exactly with a branch which dominates the others. For logarithmic branches it must also ignore a branch near its zero set. We prove these properties with one normalization.

Definition 12.1.1. Fix an even nonnegative function $\theta \in C_c^\infty((-1,1))$ with integral one. For $m \geq 1$ and $\varepsilon > 0$ set

$$M_{\varepsilon,m}(s) = \int_{\mathbb{R}^m} \max_{1\leq i\leq m} (s_i + \varepsilon t_i) \prod_{i=1}^m \theta(t_i)\, dt_1 \cdots dt_m.$$

The subscript m is retained when branches are removed. For $m = 1$, the zero mean gives $M_{\varepsilon,1}(s) = s$.

Theorem 12.1.2. *The function $M = M_{\varepsilon,m}$ is smooth, convex, symmetric in its coordinates, and nondecreasing in each coordinate. It satisfies*

$$M(s + a\mathbf{1}) = M(s) + a, \tag{12.1.1}$$

$$\max_i s_i \leq M(s) \leq \max_i s_i + \varepsilon, \tag{12.1.2}$$

$$|M(s) - M(r)| \leq \max_i |s_i - r_i|. \tag{12.1.3}$$

If $s_i < \max_{j\neq i} s_j - 2\varepsilon$, then on a neighbourhood of s the function is independent of s_i and equals the regularized maximum of the remaining coordinates. Furthermore,

$$\partial_i M \geq 0, \qquad \sum_i \partial_i M = 1, \qquad (\partial_i \partial_j M)_{ij} \geq 0, \qquad D^2 M\, \mathbf{1} = 0.$$

Proof. Put $F(s) = \max_i s_i$. After changing variables, M is the convolution of the locally integrable function F with the smooth compactly supported probability kernel $\varepsilon^{-m} \prod_i \theta(t_i/\varepsilon)$. Differentiating that kernel proves smoothness. For $0 \leq \lambda \leq 1$, apply the convexity inequality for F to $\lambda(s+\varepsilon t)+(1-\lambda)(r+\varepsilon t)$ and integrate. This proves convexity. Monotonicity and permutation symmetry follow directly from the integral.

Translation by $a\mathbf{1}$ adds a inside the maximum, and the kernel has mass one. Jensen's inequality gives the lower bound in (12.1.2), since $\int t_i \theta(t_i) dt_i = 0$. The upper bound follows from $|t_i| < 1$. The pointwise estimate $|F(s+\varepsilon t) - F(r+\varepsilon t)| \leq \|s-r\|_\infty$ proves (12.1.3).

For the deletion assertion choose $j \neq i$ realizing the other maximum. Then $s_i + \varepsilon t_i < s_j + \varepsilon t_j$ throughout the kernel support. The maximum does not use the ith coordinate; integrating out t_i contributes the factor one. The strict gap persists near s. Finally, monotonicity and convexity give the derivative inequalities. Differentiate (12.1.1) once in a, then once in s, to obtain the two remaining identities. □

The factor ε^{-m} is essential if the kernel is written as $\prod_i \theta(t_i/\varepsilon)$. Without it the mass is ε^m, and translation adds $a\varepsilon^m$. This is the normalization error in the displayed regularization in CLD25, Proposition 16.9.1, p. 466. Formula (12.1.1) supplies a direct check independent of any positivity argument.

12.2 Composition and gluing

The smoothness just proved is an ordinary real statement. To use it in CLD geometry, all functions involved must first be represented on a common refinement of their charts. The following chain rule then descends to actual forms.

Proposition 12.2.1. *Let $u_1, \dots, u_m$ be original smooth functions on an analytic open set, and let F be smooth near their joint image. Then*

$$\mathrm{d}'\mathrm{d}''F(u) = \sum_i F_i(u)\mathrm{d}'\mathrm{d}''u_i + \sum_{i,j} F_{ij}(u)d'u_i \wedge d''u_j.$$

If every $\mathrm{d}'\mathrm{d}''u_i$ is positive, F is convex, and $F_i \geq 0$, then $\mathrm{d}'\mathrm{d}''F(u)$ is positive. In particular $M_{\varepsilon,m}(u)$ is smooth psh.

Proof. Apply d' to $d''F(u) = \sum_i F_i(u)d''u_i$ and use the product rule. At a point, factor the positive semidefinite Hessian as $D^2F = B^{\mathsf{t}}B$. The second sum becomes $\sum_k \alpha_k \wedge J\alpha_k$, where $\alpha_k = \sum_i B_{ki}d'u_i$. This is a positive $(1,1)$-form. The first sum is positive because its scalar coefficients are nonnegative. These identities commute with the pullbacks defining the sheaf. □

Corollary 12.2.2 (Gluing with a collar)**.** *Let $V \subset U$ be open, let u be smooth psh on U, and let v be smooth psh on V. Suppose an open neighbourhood O of the relative boundary of V in U satisfies $u > v + 2\varepsilon$ on $O \cap V$. Then*

$$w = \begin{cases} M_{\varepsilon,2}(u,v) & \text{on } V, \\ u & \text{on } U \setminus V \end{cases}$$

is smooth psh on U, and equals v wherever $v > u + 2\varepsilon$.

Proof. Deletion and $M_{\varepsilon,1}(u) = u$ show equality on the collar, so the definitions agree on an open neighbourhood of every interface point. Composition proves positivity on V, and the old function supplies it elsewhere. The last assertion follows by deleting the other branch. □

A partition of unity does not replace this argument. Although it glues unrestricted smooth functions, derivatives of its factors introduce mixed terms of no fixed sign. The collar inequality is the geometric input that makes positive gluing possible.

12.3 Logarithmic branches and finite barriers

Proposition 12.3.1. *Let $f_1, \dots, f_m$ be analytic functions on an analytic open set U, let a_i and c be real constants, and put*

$$q = \max\{c, \log|f_1| + a_1, \dots, \log|f_m| + a_m\}.$$

Interpret $\log 0 = -\infty$. Applying M_ε to the finite branches and deleting the branches equal to $-\infty$ defines a global original smooth psh function q_ε satisfying $q \le q_\varepsilon \le q + \varepsilon$.

Proof. Fix x. If $f_i(x) = 0$, continuity of $|f_i|$ gives a neighbourhood on which $\log|f_i| + a_i < c - 2\varepsilon$. Delete that branch throughout this neighbourhood. For every remaining branch, shrink the neighbourhood so that f_i is a unit. Its logarithmic absolute value is then an original smooth function with zero $\mathrm{d}'\mathrm{d}''$. The constant branch remains finite everywhere. Proposition 12.2.1 therefore proves smoothness and positivity locally. On overlaps the deletion identity shows that the local formulas coincide. The estimates follow from (12.1.2). □

Theorem 12.3.2 (A barrier from nested rational domains)**.** *Let V be an affinoid domain in a good Hausdorff analytic space X. Let $f_1, \dots, f_m, g \in \mathcal{O}(V)$ have no common zero and let $\lambda_i > 0$. For real $a < b$, define*

$$W_t = \{y \in V : |f_i(y)| \le e^t \lambda_i |g(y)| \text{ for all } i\}.$$

Suppose $W_b \subset \mathrm{Int}_X(V)$, and let x satisfy all the inequalities for W_a strictly. There exist an open neighbourhood U of W_a, an original smooth psh function ρ on U, and an open neighbourhood U_0 of x such that

$$\rho < 0 \text{ on } U_0, \qquad \{y \in U : \rho(y) \le 0\} \text{ is compact in } U.$$

Proof. The no-common-zero assumption implies $g \neq 0$ on every W_t: otherwise every f_i would vanish there. These sets are compact rational domains. Choose $c < a$ and set

$$q = \max\{c, \log|f_i/g| - \log\lambda_i : 1 \le i \le m\}$$

on the locus $D(g)$ in V. It is finite and continuous. Choose c so that $q(x) < a$, and put $U = \{q < b\} \subset D(g)$. This subset is contained in W_b, hence in the ordinary interior of V; it is therefore open in X. It contains W_a. For $c \le t < b$, its sublevel $\{q \le t\}$ is exactly W_t.

Choose $\varepsilon > 0$ and a real s with $q(x)+\varepsilon < s < a$. The preceding proposition applies to the analytic functions f_i/g on U. Put $\rho = q_\varepsilon - s$. Then $\rho(x) < 0$. Its nonpositive sublevel is closed in U and contained in $W_s \subset W_a$. As a closed subset of the compact set W_a, it is compact. Continuity gives the required U_0. □

The theorem is deliberately stated with the nested-domain input visible. Establishing such rational data near every point uses the local domain theorems of analytic geometry. The analytic construction above proves what follows once those data have been supplied. It also distinguishes the larger domain U from the smaller set on which ρ is negative; these two sets must not be identified.

12.4 Uniform approximation and decreasing sequences

The domain of approximation is part of the assertion. The useful elementary conversion below does not change that domain with the accuracy parameter.

Lemma 12.4.1. *Let u be continuous on U and let $\mathcal{F}(U) \subset C^0(U)$ be stable under addition of real constants. Suppose u is a locally uniform limit of functions in $\mathcal{F}(U)$. For every relatively compact $V \Subset U$, there is a sequence in $\mathcal{F}(U)|_V$ decreasing to $u|_V$.*

Proof. Choose $v_j \in \mathcal{F}(U)$ with $\sup_{\overline{V}} |v_j - u| \le 4^{-j}$. Set $w_j = v_j + 2 \cdot 4^{-j}$. Then

$$u + 4^{-j} \le w_j \le u + 3 \cdot 4^{-j}, \qquad w_{j+1} \le u + \tfrac{3}{4}\, 4^{-j} < w_j.$$

The squeeze proves the limit. Every function remains defined on the common set V. □

Conversely, decreasing continuous functions with a continuous limit converge uniformly on every compact subset, by Dini's theorem. Neither direction proves that a continuous section at an arbitrary transfinite decreasing-limit stage admits continuous approximants in the preceding stage. That is the missing replacement step discussed in the source concordance, not a consequence of Dini's theorem.

Chapter 13

Continuous Bedford–Taylor products

13.1 Continuous products and the role of the potential

The mixed product of Theorem 11.1.5 can be understood through two operations: multiplication of a positive current by a continuous function, and distributional differentiation. Their domains explain why the notation $d'd''u \wedge T$ is not an unrestricted wedge product of currents.

13.1.1 A recursive construction with stated regularity

Proposition 13.1.1 (Multiplication and weak convergence)**.** *Let T_j be ordinary positive currents of one fixed bidegree converging weakly to an ordinary positive current T. Let u_j be continuous functions converging locally uniformly to a continuous function u. Then*

$$u_j T_j \longrightarrow uT$$

as currents. The products are defined through the Radon measures obtained by testing the positive currents against smooth complementary forms.

Open proof obligation (RC2). Prove uniform approximation near the test support in the required category; the proof explicitly identifies CLD12 Corollaries 5.4.7–5.4.8 as the density input.

Definition 13.1.2 (One continuous potential acting on a closed current)**.** Let T be an ordinary positive current with $d'T = d''T = 0$, and let $u \in \mathcal{C}_{25}$. Define

$$d'd''u \wedge T := d'd''(uT). \tag{13.1.1}$$

At this point the right side, and therefore the construction, uses the specified function u. Dependence only on its curvature current is a separate question.

Proposition 13.1.3 (Positivity and closedness)**.** *The current in* (13.1.1) *is ordinary positive and is killed by both d' and d''. If u is smooth, it is the usual product of the smooth form $d'd''u$ with T.*

Open proof obligation (RC2). The argument uses multiplier convergence, whose native density premise is open.

This argument uses the continuous generated sheaf directly. It does not require the disputed all-stage equality with $C^0 \cap \mathcal{P}_{25}$.

Proposition 13.1.4 (Iteration and local uniform stability)**.** *Let T have bidegree (p,p) and satisfy the preceding positivity and closedness assumptions. For $u_1, \dots, u_q \in \mathcal{C}_{25}$, with $p+q \le n$, the recursive expression*

$$d'd''\big(u_1 d'd''(u_2 \cdots d'd''(u_q T)\cdots)\big) \tag{13.1.2}$$

is a positive closed current of bidegree $(p+q, p+q)$. If $T_j \to T$ weakly through positive closed currents and each $u_{i,j} \to u_i$ locally uniformly through $\mathcal{C}_{25}$, the corresponding recursive products converge weakly to (13.1.2).

Open proof obligation (RC2). The induction uses one-step positivity and multiplier convergence. The cited CLD25 recursion is explanatory here and its citation does not discharge either open native premise.

Open proof obligation (RC3). For $u_i \in \mathcal{U}$, the symmetry and independence of smooth approximations in Theorem 11.1.5 remain pending. For smooth approximants on a common smaller neighbourhood, the smooth products are symmetric. Passing this identity to the recursive products requires the local uniform stability in Proposition 13.1.4, which in turn depends on the open multiplier-convergence assertion in Proposition 13.1.1. Strongly positive forms need not span all symmetric forms.

13.1.2 Independence of the potential is another assertion

Definition 13.1.5 (The continuous Bedford–Taylor property)**.** A positive closed current T has the continuous Bedford–Taylor property on a specified class of continuous potentials if, locally,

$$d'd''[u] = d'd''[v] \quad \Longrightarrow \quad d'd''(uT) = d'd''(vT) \tag{13.1.3}$$

for all potentials u, v in that class. This is the continuous interpretation of the property singled out in [CLD25, Definition 8.6.2].

Remark 13.1.6 (A proved case and a missing inference). The integration current $[1]$ has this property: $u[1] = [u]$, so (13.1.3) is exactly its hypothesis. For an arbitrary positive closed T, the implication also holds if $u - v$ is smooth with $d'd''(u - v) = 0$, by the Leibniz calculation above. If $u - v$ is merely continuous and its current Hessian vanishes, the same argument is unavailable. This is why the potential-independence condition is stated separately in [CLD25, §8.6.1].

Similarly, a positive continuous scalar multiplier preserves positivity of a current but need not preserve either closedness or the Bedford–Taylor property. Example A.1.3 disproves the unrestricted multiplication assertion printed in Corollary 8.6.7 of that text.

13.1.3 Metrics, measures, and actual norms

For a line bundle with continuous local weights in $\mathcal{U}$, the products of its curvature currents are defined locally by the preceding construction and glue: changes of frame add the smooth pluriharmonic function $-\log|f|$ for an analytic unit f. This is a legitimate use of the smooth case in Remark 13.1.6. The argument concerns currents and their products; it does not forget the normalization of the original metric.

Remark 13.1.7 (Constants are invisible to curvature). If $\|\cdot\|' = e^{-c}\|\cdot\|$, then its weights satisfy $\varphi'_e = \varphi_e + c$, and

$$c_1(L, \|\cdot\|') = c_1(L, \|\cdot\|).$$

The two norms are nevertheless different for $c \neq 0$. Equality of currents or of top-degree measures therefore cannot identify actual metrics. This distinction persists when currents are used in Deligne pairings or in adelic intersection theory.

The passage to unbounded potentials requires an integrability domain for products such as uT. The continuous recursion above supplies no automatic product when that multiplication is undefined. In particular, it supplies neither a general non-pluripolar product nor a finite-energy theory by notation alone. Those constructions require their own definitions and comparison theorems.

13.2 The estimates and identities behind continuous products

13.2.1 What a Chern–Levine–Nirenberg estimate controls

Fix an ordinary positive closed current T, a relatively compact test region, and a compactly supported smooth complementary form ω. A local CLN estimate bounds expressions of the shape

$$\left|\left\langle u_0\, \mathrm{d}'\mathrm{d}''u_1 \wedge \cdots \wedge \mathrm{d}'\mathrm{d}''u_q \wedge T, \omega \right\rangle\right| \leq C \prod_{i=0}^{q} \|u_i\|_K,$$

for smooth psh functions on a fixed larger neighbourhood, with the usual choice of nested compact sets and cutoffs. The constant depends on the region, the fixed tests and cutoffs, and the relevant local mass of T. It is not uniform over an unbounded family of background currents.

The estimate is striking because the left side contains second derivatives of every u_i, while the right side contains only their values. Positivity is the reason derivatives can be removed without losing control. It also explains why an estimate for arbitrary signed currents would need a different argument.

13.2.2 The induction mechanism

Choose nested compact neighbourhoods of the support of ω and cutoffs equal to one on the smaller neighbourhoods. Positive-cone domination bounds the coefficients of the test by finitely many positive tests. For one curvature factor, integration by parts moves $\mathrm{d}'\mathrm{d}''$ from u_1 to a cutoff times the test. The derivatives now fall on fixed smooth data. Their coefficients are bounded on the larger compact set, so ordinary positivity bounds the resulting pairing by a fixed local mass of T, times $\|u_1\|_K$.

Repeat this step for the remaining factors, enlarging the compact region at each stage. The closedness of the inner mixed current removes derivatives of that current. Positivity allows the same coefficient estimate at the next step. The scalar u_0 is controlled directly by its sup norm. This is the structure of the CLN argument in CLD25 Chapter 8; the technical formulation keeps track of the degrees and support sets.

There is a printed degree slip in Proposition 8.4.1: the symbol p must be read as the bidimension of the initial current for the displayed pairing to have the stated complementary degree. The estimate is used here with explicitly complementary test degrees, as explained in Appendix A.

13.2.3 Symmetry on the continuous generated class

The recursive expression in the preceding section is written in an order. Its symmetry is a theorem. For smooth potentials, it follows from the wedge algebra. For two continuous generated potentials u, v, first fix a smooth u and pass to the generated continuous class in v: the identity

$$\mathrm{d}'\mathrm{d}''(u\,\mathrm{d}'\mathrm{d}''(vT)) = \mathrm{d}'\mathrm{d}''(v\,\mathrm{d}'\mathrm{d}''(uT))$$

is local and stable under the uniform limits allowed in the definition, by the multiplier-convergence theorem. Now fix such a v and make the same argument in u. This proves the two-factor symmetry on the continuous class. Adjacent transpositions give symmetry for any finite number of factors.

This argument does not require replacing every generated function by one globally chosen sequence of original smooth psh functions. It uses the defining closure property of the continuous sheaf. It also does not establish the separate kernel property: if two different potentials have the same ambient Hessian, their actions on an arbitrary background current still need comparison.

13.2.4 Differences of psh functions

CLD25 calls the real vector sheaf generated by its continuous psh functions the dpsh class. Locally one can write a member as a difference of continuous psh functions. Products extend multilinearly by expanding such differences. Independence of the chosen positive decomposition follows from linearity of the scalar multiplication and distributional differentiation in the recursive construction, together with the established symmetry.

For example, a smooth function on a relatively compact real chart can be written as

$$u = (u + C|x|^2) - C|x|^2$$

with both terms convex when C dominates the negative part of the Hessian. Pulling back this local argument explains why original smooth functions belong to the dpsh class. PL and G-smooth functions require the corresponding piecewise convex decomposition and the source's local chart hypotheses.

The word dpsh does not assert positivity. A dpsh curvature is generally a difference of positive currents. It is a useful domain for multilinear calculus, just as differences of semipositive metrics are useful in arithmetic intersection theory.

13.2.5 Projection formulas and the direction of functoriality

For a proper morphism in the category where smooth pullback and current pushforward are defined, the elementary identity

$$f_*(f^*u\, S) = u\, f_*S$$

comes from evaluating both sides on the same test and using scalar multiplication. Commuting f_* with the distributional derivatives then gives

$$f_*\big(\mathrm{d}'\mathrm{d}''(f^*u) \wedge S\big) = \mathrm{d}'\mathrm{d}''u \wedge f_*S$$

for continuous potentials in the admitted class and positive closed S. Iteration gives the mixed version. The support/properness hypothesis ensures that pulled-back tests have the support needed to evaluate S.

This is a pushforward theorem with pulled-back potentials. It is not a general pullback construction for arbitrary currents. The latter would require defining the action on tests in the opposite direction and controlling singular intersections. This distinction is important when interpreting the functorial psh condition of GK and the analytic curve tests of Gubler–Rabinoff.

13.3 Polar sets, integrability, and the boundary of the BT theorem

13.3.1 The precise input in the polar-negligibility theorem

Theorem 8.12.1 assumes the local existence of suitable smooth psh sublevel neighbourhoods. Its mixed measure is constructed from continuous locally approachable potentials $u_1, \dots, u_n$. The auxiliary function V is itself locally approachable psh and locally integrable. The conclusion is that V is locally integrable for the mixed measure.

The analytic version is Theorem 16.9.2. For the good boundaryless analytic setting used in these notes, the preceding sublevel construction provides the needed local geometry. A formula error in the printed smoothing kernel is corrected in Appendix A; the normalization and domain of the regularized maximum are made explicit there.

These hypotheses are narrower than a theorem about every generated psh function and every positive closed background current. In particular, the theorem is not a full statement of capacity theory for arbitrary discontinuous bounded potentials.

13.3.2 Why integrability implies no polar mass

Suppose $V \le 0$ near a compact set and $\int (-V)\, d\mu < \infty$. For $M > 0$,

$$M\, \mu\{V < -M\} \le \int_{\{V<-M\}} (-V)\, d\mu \le \int (-V)\, d\mu.$$

The sets $\{V < -M\}$ decrease to $\{V = -\infty\}$ as $M \to \infty$. Hence the latter has zero μ-mass. This elementary argument is the final step; the mathematical work lies in proving the integrability bound for the mixed measure.

The source obtains that bound by combining CLN estimates with decreasing smooth approximants of V and uniform smooth approximants of the continuous u_i. A compactly supported psh barrier controls the localization. Positivity and monotone convergence then pass the estimate to V. Each approximation has a specific role; a general weak limit of measures would not by itself preserve the desired integrability bound.

13.3.3 Polar sets and small tropical dimension are different criteria

CLD25 Chapter 16 also proves a support theorem for positive closed currents. Under the nontrivial valuation and boundaryless hypotheses, Theorem 16.8.5 says that a positive closed current of bidimension (p, p) does not charge, after pairing with a smooth (p, p)-form, a closed set whose image in every relevant tropical chart has zero p-dimensional Hausdorff measure.

This criterion is geometric and dimension-theoretic. Polar-negligibility is formulated through a potential taking the value $-\infty$. One should not identify the two notions without another theorem. A set may be small in one chosen tropical image while being visible in another; the support theorem requires all the prescribed charts.

13.3.4 What a complete bounded theory would add

Continuous scalar multiplication is canonical because positive currents have measure coefficients. If a bounded potential is discontinuous, multiplication remains meaningful once a measurable representative is fixed, but the continuity theorem needed for iteration changes. Locally uniform convergence is no longer available. One must control decreasing limits, exceptional sets, and the dependence on the representative.

In complex pluripotential theory, capacities, quasi-continuity, and monotone convergence make these operations compatible. A corresponding non-Archimedean theory must specify its psh class and its test space before claiming the analogy. The toric theory of BGJK and global variational theories provide important established cases, treated in Chapter 21; they should be compared on their actual domains. The continuous arguments above are a foundation for that comparison, not a proof of every bounded or finite-energy statement.

References. [CLD25, §§8.4–8.12, 16.8–16.9].

13.4 Local mass estimates

The first product theorem proves positivity by approximation. Once positivity is known, integration by parts gives a mass estimate depending on the

potential itself, not on derivatives of its approximants. This is the estimate needed for repeated products.

Lemma 13.4.1. *For a smooth real function χ, the coefficient matrix of $\mathrm{d}'\mathrm{d}''\chi \wedge \beta_{q-1}$ satisfies*

$$\left\| H_{\mathrm{d}'\mathrm{d}''\chi\wedge\beta_{q-1}} \right\|_{\mathrm{op}} \le q \left\| D^2\chi \right\|_{\mathrm{op}}.$$

Proof. At the point in question, diagonalize the real symmetric Hessian by an orthogonal basis change, with eigenvalues $\lambda_1, \dots, \lambda_n$. Such a change preserves ω and all the exterior Euclidean norms. The coefficient matrix of the wedge is diagonal; its entry indexed by a q-element set I is $\sum_{i\in I} \lambda_i$. This follows by expanding β_{q-1} and observing that only indices different from the Hessian index survive. The absolute value of each entry is at most $q \max_i |\lambda_i|$. □

Proposition 13.4.2 (One-step mass bound)**.** *Let $T, u, R = \mathrm{d}'\mathrm{d}''(uT)$ satisfy Theorem 9.8.1. Choose $0 \le \chi \in C_c^\infty(\Omega)$ equal one near K, and a compact L containing* $\operatorname{supp}\chi$*. For any $c \in \mathbb{R}$,*

$$\mu_R(K) \le q \left\| D^2\chi \right\|_{\infty,\mathrm{op}} \|u - c\|_{L,\infty}\, \mu_T(L). \tag{13.4.1}$$

Proof. Since R is positive and $\chi = 1$ on K,

$$\mu_R(K) \le R(\chi\beta_{q-1}) = T(u\,\mathrm{d}'\mathrm{d}''\chi \wedge \beta_{q-1}).$$

Here β_{q-1} is constant and (9.7.1) is valid on the continuous product uT. Closedness of T gives $T(\mathrm{d}'\mathrm{d}''(\chi\beta_{q-1})) = 0$, so u may be replaced by $u - c$. Now apply (9.3.5) and Lemma 13.4.1; all coefficients of the test are supported in L. □

The constant in this bound depends on the cutoff and on the bidimension, not on the rate of approximation of u. Subtracting a constant is legitimate because the background is closed. In applications one may take $c = \inf_L u$, replacing the potential norm by its oscillation on L.

13.5 Mixed products in the approximation class

We now prove iteration and symmetry for the explicit approximation class used above. This result is deliberately stated with that class in its hypotheses; it makes no identification with a larger generated or current-psh class.

Definition 13.5.1. Write $\mathcal{U}(\Omega)$ for the continuous functions that locally admit smooth approximants with positive Hessian, converging locally uniformly on one fixed neighbourhood.

Theorem 13.5.2. *Let T be positive and closed of bidimension (q,q), and let $u_1,\dots,u_k \in \mathcal{U}(\Omega)$ with $k \le q$. The recursively defined current*

$$R(u_1,\dots,u_k;T) = \mathrm{d}'\mathrm{d}''\big(u_1\,\mathrm{d}'\mathrm{d}''(u_2\cdots \mathrm{d}'\mathrm{d}''(u_kT)\cdots)\big)$$

is positive and closed of bidimension $(q-k,q-k)$, local in the potentials and background, and symmetric in the u_i. If $T_j \rightharpoonup T$ are positive closed currents and $u_{i,j}\to u_i$ locally uniformly with all $u_{i,j},u_i\in\mathcal{U}(\Omega)$, then

$$R(u_{1,j},\dots,u_{k,j};T_j) \rightharpoonup R(u_1,\dots,u_k;T).$$

Proof. Existence, positivity and closedness follow inductively from Theorem 9.8.1. Each multiplication and derivative commutes with restriction, giving locality.

We prove continuity before symmetry. For $k=0$ it is the assumed convergence of T_j. Suppose it is proved for $k-1$. Let $S_j = R(u_{2,j},\dots,u_{k,j};T_j)$ and $S = R(u_2,\dots,u_k;T)$. By induction $S_j \rightharpoonup S$, and these currents are positive. Theorem 9.6.3 gives $u_{1,j}S_j \rightharpoonup u_1 S$. Differentiation gives the claimed convergence for k. This argument does not require derivative bounds on $u_{i,j}$.

Symmetry is local. Around any point, intersect the finitely many approximation neighbourhoods and choose simultaneous smooth approximants $u_{i,j}$ there. For smooth potentials, Proposition 9.7.2 identifies the recursion with

$$\mathrm{d}'\mathrm{d}''u_{1,j}\wedge\cdots\wedge\mathrm{d}'\mathrm{d}''u_{k,j}\wedge T.$$

All the smooth factors have even total degree, so they commute. Apply the continuity just proved to this expression in either order. The two limits agree on each neighbourhood and hence globally. □

Corollary 13.5.3 (Mixed local mass bound)**.** *Let $K \Subset \operatorname{int} L$ with $L \Subset \Omega$ compact. For $k \le q$ there is a constant C depending on K, L, k, q, n and chosen intermediate cutoffs, such that*

$$\mu_{R(u_1,\dots,u_k;T)}(K) \le C\,\mu_T(L)\prod_{i=1}^{k}\operatorname{osc}_L(u_i).$$

Proof. Choose nested compact sets $K = K_0 \subset \operatorname{int} K_1 \subset \cdots \subset \operatorname{int} K_k \subset L$, with smooth cutoffs equal one on K_{i-1} and supported in K_i. Apply Proposition 13.4.2 first to the outermost product and then to its background, continuing inward. At each step subtract $\inf_L u_i$. The bidimension factors and the Hessian bounds of the finitely many cutoffs multiply into C. □

The same proof shows that the mixed product is separately additive when the potentials involved and their sums belong to $\mathcal{U}$, and is homogeneous for nonnegative scalars. Positivity remains attached to this admitted class. A proof that a different psh definition belongs to $\mathcal{U}$ is a comparison theorem, not part of the formal recursion.

13.6 Intrinsic limits and admitted potentials

Theorem 13.6.1. *In the algebraic scope of Section 8.8, let $T_j \rightharpoonup T$ be ordinary positive currents of fixed bidimension on W, and let $f_j \to f$ locally uniformly with all f_j, f continuous. Then $f_j T_j \rightharpoonup fT$.*

Proof. Fix an actual symmetric compactly supported test η and choose an actual positive control θ and M such that $M\theta \pm \eta \geq 0$. For the corresponding signed measures,

$$|\mu_{T_j,\eta}| \leq M \mu_{T_j,\theta}, \qquad \mu_{T_j,\theta}(W) = T_j(\theta) \longrightarrow T(\theta).$$

Hence their total variations are uniformly bounded and their supports lie in the fixed compact $\operatorname{supp} \eta$. For smooth scalar ϕ, weak convergence of currents gives

$$\int \phi \, d\mu_{T_j,\eta} = T_j(\phi\eta) \longrightarrow T(\phi\eta).$$

One may insert a cutoff equal one near $\operatorname{supp} \eta$ if ϕ is not compactly supported. Scalar density on that compact set and the uniform variation bound extend this convergence to every continuous scalar function on a neighbourhood of the compact set. Finally split

$$(f_j T_j - fT)(\eta) = \int (f_j - f) \, d\mu_{T_j,\eta} + \int f \, d(\mu_{T_j,\eta} - \mu_{T,\eta}).$$

The first term tends to zero by uniform convergence and the variation bound; the second by the continuous-scalar convergence just proved. Symmetrization treats arbitrary tests. □

Definition 13.6.2. Let $\mathcal{U}_X(W)$ denote the continuous functions locally uniformly approximable, on one fixed neighbourhood of each point, by original smooth functions u_j such that the actual smooth form $\mathrm{d}'\mathrm{d}''u_j$ is ordinary positive.

This is an explicitly admitted class. No equivalence with all continuous functions whose curvature current is positive is asserted. In particular, Proposition 8.12.2 does not prove membership in this class.

Theorem 13.6.3. *Let T be an ordinary positive closed current of bidimension (q,q) on W. For $u_1,\dots,u_k \in \mathcal{U}_X(W)$, $k \le q$, the recursive product*

$$R_X(u_1,\dots,u_k;T) = \mathrm{d}'\mathrm{d}''\big(u_1\,\mathrm{d}'\mathrm{d}''(u_2\cdots\mathrm{d}'\mathrm{d}''(u_kT)\cdots)\big)$$

is an intrinsic ordinary positive closed current. It is symmetric in the potentials and commutes with locally uniform convergence of admitted potentials and simultaneous weak convergence of positive closed backgrounds of fixed degree.

Proof. The first product is defined by Theorem 8.14.1 and distributional differentiation. If u_j are smooth positive-curvature approximants on a fixed neighbourhood, Theorem 13.6.1 gives $u_jT \rightharpoonup uT$. Smooth Leibniz on actual tests and closedness of T give

$$\mathrm{d}'\mathrm{d}''(u_jT) = \mathrm{d}'\mathrm{d}''u_j \wedge T \rightharpoonup \mathrm{d}'\mathrm{d}''(uT).$$

Wedge positivity is checked after a common refinement of the smooth form and the test. It follows from Proposition 7.5.2 face by face. Thus the approximating currents are positive, and their limit is positive. It is closed by the differential identities. This argument works locally; a nonnegative partition of unity gives the global assertion.

Iterate this construction. To prove convergence, induct on the number of potentials. The inner products converge by induction and are positive. Apply Theorem 13.6.1 to their multiplication by the outer potentials, then differentiate. For symmetry, choose simultaneous smooth approximants locally for the finitely many potentials. The smooth product is the wedge of their curvature forms with T, hence unchanged by permutation. The convergence just proved identifies its limits in any order. All operators commute with restriction, so the local equalities glue. □

This theorem supplies the intrinsic version of the Euclidean mixed-product argument in Section 13.5 for $\mathcal{U}_X$. The continuous-scalar relations are handled by Theorem 8.14.1. Equality of two nonsmooth potentials modulo the curvature kernel is still a different question, and has not been used in the proof.

13.7 Quantitative continuity of mixed products

Weak convergence of currents is convergence of their values on each fixed smooth compactly supported test. It does not measure the distance between their coefficient measures in total variation. A quantitative version of the local continuity theorem makes this distinction precise and explains which derivatives must be controlled.

We first work on an open real domain $\Omega \subset \mathbb{R}^n$, with the conventions and trace measures of Chapter 9. Let T be ordinary positive and closed of bidimension (q,q). All potentials belong to $\mathcal{U}(\Omega)$, and $R(\mathbf{u};T)$ denotes the product of Theorem 13.5.2. The Euclidean estimate below is not asserted on an arbitrary analytic space by simply choosing one tropical chart: an analytic test can involve further refinements. Its intrinsic qualitative counterpart is Theorem 13.6.3.

Proposition 13.7.1 (A telescoping identity)**.** *For two k-tuples $\mathbf{u},\mathbf{v}$, where $1 \le k \le q$, put*

$$S_i = R(v_1,\dots,v_{i-1},u_{i+1},\dots,u_k;T).$$

Then, as currents,

$$R(\mathbf{u};T) - R(\mathbf{v};T) = \sum_{i=1}^{k} \mathrm{d}'\mathrm{d}''\big((u_i - v_i)S_i\big). \tag{13.7.1}$$

Proof. Replace u_1 by v_1, then u_2 by v_2, and continue. At the ith replacement, symmetry moves the changing factor to the outside of the recursion. Linearity of continuous scalar multiplication and of $\mathrm{d}'\mathrm{d}''$ identifies the difference with $\mathrm{d}'\mathrm{d}''((u_i-v_i)S_i)$. Adding these differences cancels every intermediate product. This proof uses the chosen potentials; it does not replace their differences by abstract curvature classes. □

Theorem 13.7.2 (A bound on each fixed test)**.** *Fix a smooth compactly supported test η of bidegree $(q-k, q-k)$ and a compact $L \Subset \Omega$ whose interior contains its support. There is a constant $C_{\eta,L}$, depending also on the chosen intermediate cutoffs, such that*

$$\begin{aligned}&|\langle R(\mathbf{u};T) - R(\mathbf{v};T), \eta\rangle| \\ &\quad \le C_{\eta,L}\mu_T(L) \sum_{i=1}^{k} \|u_i - v_i\|_L \prod_{j<i} \operatorname{osc}_L(v_j) \prod_{j>i} \operatorname{osc}_L(u_j).\end{aligned}$$

An empty product is one. Only the fixed test and cutoffs, not the varying potentials, are differentiated in this estimate.

Proof. We may symmetrize η, since all currents being paired are symmetric. Choose a compact neighbourhood K of $\operatorname{supp}\eta$ in $\operatorname{int} L$. The telescoping identity gives

$$\langle R(\mathbf{u};T) - R(\mathbf{v};T), \eta\rangle = \sum_i S_i\big((u_i - v_i)\mathrm{d}'\mathrm{d}''\eta\big).$$

The form $\mathrm{d}'\mathrm{d}''\eta$ has bidegree $(q-k+1, q-k+1)$, exactly the bidimension of S_i. Its symmetric coefficient matrix has a finite operator-norm bound A_η. The trace-measure estimate of Chapter 9 therefore gives

$$|S_i((u_i - v_i)\mathrm{d}'\mathrm{d}''\eta)| \le A_\eta \|u_i - v_i\|_L \, \mu_{S_i}(K).$$

Apply Corollary 13.5.3 to the $k-1$ potentials defining S_i, using intermediate compact sets between K and L. There are finitely many positions i, so a common maximum of the cutoff constants suffices. This proves the formula. For $k = 1$, there are no inner potentials and the argument is simply the trace estimate for T. □

Corollary 13.7.3. *If all the oscillations in the preceding theorem are bounded by M and $\mu_T(L) \le B$, then*

$$|\langle R(\mathbf{u};T) - R(\mathbf{v};T), \eta\rangle| \le C_{\eta,L} B\, M^{k-1} \sum_i \|u_i - v_i\|_L.$$

For $k = 1$ the factor M^{k-1} is one, also when $M = 0$. Thus on a bounded family the product is Lipschitz on each fixed smooth test, with respect to uniform distance of the potentials.

The constant depends on the second derivatives of η and on the localization. It is not uniform over all continuous tests of sup norm one. Consequently this corollary gives no total-variation convergence. Nor does weak convergence $T_j \to T$ alone give a numerical rate for the background error. The simultaneous-background theorem is obtained by continuity of scalar multiplication and induction, rather than by assigning a norm distance to arbitrary currents.

13.8 Locality, maxima and contact sets

The maximum operation connects regularization to local comparison arguments. It is useful to record exactly what follows from locality before invoking any comparison principle. Work either in the real setting above or in the intrinsic algebraic setting of Theorem 13.6.3.

Proposition 13.8.1. *Let u, v belong to the admitted local uniform approximation class and put $w = \max(u, v)$. Then w belongs to that class. For fixed additional admitted potentials* $\mathbf{a}$*,*

$$R(w, \mathbf{a}; T)|_{\{u>v\}} = R(u, \mathbf{a}; T)|_{\{u>v\}},$$
$$R(w, \mathbf{a}; T)|_{\{v>u\}} = R(v, \mathbf{a}; T)|_{\{v>u\}}.$$

Proof. On a common approximation neighbourhood choose smooth psh approximants u_j, v_j. Apply the normalized regularized maximum of Chapter 12, with smoothing parameter tending to zero. Convexity and coordinatewise monotonicity preserve positivity of the Hessian. Its uniform error from the ordinary maximum tends to zero, while

$$|\max(u_j, v_j) - \max(u, v)| \le \max(|u_j - u|, |v_j - v|).$$

This proves membership in the admitted class. On the open set $\{u > v\}$ the functions w and u agree, so locality of scalar multiplication and distributional differentiation identifies the products. The second equality is the same argument. □

Example 13.8.2 (The contact contribution)**.** On the real line take $u(t) = t$ and $v(t) = -t$. Their Hessians vanish, but

$$\mathrm{d}'\mathrm{d}'' \max(u, v) = \mathrm{d}'\mathrm{d}''|t| = 2\delta_0.$$

Thus the two restrictions in Proposition 13.8.1 do not determine the product on $\{u = v\}$. For a test χ, integration by parts separately on the two half-lines gives $\int |t|\chi''(t)\, dt = 2\chi(0)$, which proves the formula and its normalization. In a tropical annulus coordinate the same local calculation applies with the relevant lattice calibration.

For a truncation $u_c = \max(u, c)$, locality gives $\mathrm{d}'\mathrm{d}''u_c = \mathrm{d}'\mathrm{d}''u$ on $\{u > c\}$ and $\mathrm{d}'\mathrm{d}''u_c = 0$ on $\{u < c\}$. The level set $\{u = c\}$ can carry mass. Removing that mass by restricting to a strict sublevel complement is an additional operation; it is not the original product. This distinction underlies the separate definitions of truncated and non-pluripolar products in theories where the latter are available.

References. [CLD12, Section 5.6]; [CLD25, Sections 8.4–8.7]. The quantitative estimate combines the local mass bound with the telescoping identity; the maximum statements use regularized maxima and locality. These are consequences in the stated admitted classes, not extensions to arbitrary current-psh potentials.

13.9 Symmetry without choosing simultaneous smooth approximants

The continuous generated sheaf $\mathcal{C}_{25}$ was defined by local uniform closure from smooth psh functions. Its sections need not come with one common one-step approximation. Nevertheless the recursive products are symmetric. The proof uses the defining closure property separately in the two variables of an adjacent interchange.

For the concrete analytic assertions below, use the algebraic scope of Section 8.8: an algebraically closed complete nontrivially valued field and paracompact open subsets of analytifications of separated integral algebraic varieties. The background currents are ordinary positive and closed, and Theorem 13.6.1 supplies multiplication and joint continuity on actual original smooth tests. The closure arguments also apply abstractly in any calculus where these same inputs have been established. For $u \in \mathcal{C}_{25}$ write

$A_u T = \mathrm{d}'\mathrm{d}''(uT)$. The preceding construction gives a positive closed current, and

$$u_j \to u \text{ locally uniformly}, \quad T_j \rightharpoonup T \quad \Longrightarrow \quad A_{u_j} T_j \rightharpoonup A_u T.$$

All currents in this implication have the same degree and are positive closed.

Lemma 13.9.1. *Suppose a property of continuous functions is local, holds for smooth psh functions, and is preserved under locally uniform limits. Then the property holds for every section of $\mathcal{C}_{25}$, provided it is imposed on restrictions to every open subset.*

Proof. The functions satisfying the property form a subsheaf of continuous functions, contain the starting sheaf of smooth psh functions, and are closed under the prescribed limit operation. The minimality in the definition of the generated sheaf gives the conclusion. No countable-stage or fixed-neighbourhood representation of an arbitrary section is asserted. □

Theorem 13.9.2. *For $u, v \in \mathcal{C}_{25}$ and an ordinary positive closed current T of the appropriate degree,*

$$A_u A_v T = A_v A_u T.$$

Consequently every iterated product $A_{u_1} \cdots A_{u_q} T$ is symmetric in the potentials.

Proof. First fix a smooth psh function v. For smooth psh u, the identity is the commutativity of the two smooth curvature factors. Consider the property that the identity holds for every positive closed T on every open restriction. It is local. If $u_j \to u$ locally uniformly, joint continuity gives $A_{u_j} A_v T \to A_u A_v T$ and $A_v A_{u_j} T \to A_v A_u T$. Hence the property is closed under these limits. Lemma 13.9.1 proves it for all $u \in \mathcal{C}_{25}$ and smooth v.

Now consider the property of v that the identity holds for every $u \in \mathcal{C}_{25}$ and every such T, on every open restriction. The first step gives it for smooth psh v. If $v_j \to v$ locally uniformly, then $A_{v_j} T \to A_v T$; applying joint continuity with the fixed continuous multiplier u gives $A_u A_{v_j} T \to A_u A_v T$. The other side converges to $A_v A_u T$ by the same theorem. A second application of the closure principle gives the desired identity for all $v \in \mathcal{C}_{25}$.

In an iterated product the factors inside an adjacent pair produce another positive closed current. The two-factor identity therefore allows every adjacent interchange. Such interchanges generate all permutations. □

This proof distinguishes two types of independence. The product is independent of an approximation used to construct it and of the order of its potential factors. It has not thereby been shown to depend only on the ambient curvature currents $\mathrm{d}'\mathrm{d}''[u_i]$. That latter statement is the kernel property of Chapter 14.

13.10 Chern operators with continuous generated weights

Proposition 13.10.1. *Let a metrized line bundle $\overline{L}$ have local weights in $\mathcal{C}_{25}$. For every ordinary positive closed current T, the local currents $\mathrm{d}'\mathrm{d}''(\varphi_e T)$ glue to a global positive closed current, denoted $c_1(\overline{L}) \wedge T$. These operators commute for different such metrized line bundles.*

Proof. For frames $e' = ge$, one has $\varphi_{e'} - \varphi_e = -\log|g|$. This difference is original smooth and has zero $\mathrm{d}'\mathrm{d}''$ because g is an analytic unit. The two closedness conditions on T give

$$\mathrm{d}'\mathrm{d}''\big((\varphi_{e'} - \varphi_e)T\big) = \mathrm{d}'\mathrm{d}''(-\log|g|) \wedge T = 0.$$

Thus the local currents agree on overlaps. Positivity and closedness are local and were proved for $A_u T$. Commutativity is Theorem 13.9.2 on a common trivializing neighbourhood. □

The gluing uses a very specific kernel: logarithms of analytic units. It therefore remains valid even when an arbitrary-current theorem for all current-harmonic continuous functions is unavailable. This is why line-bundle intersection calculus can be established on a useful class before the general harmonic-kernel problem is settled.

13.11 Proper pushforward and projection

Let $f : Y \to X$ be a proper map between the chosen analytic opens induced by an algebraic morphism over the same field, in the scope just specified. Both spaces are pure dimensional. We index the next statement by bidimension, which is preserved by current pushforward. If T has bidimension (q, q) on Y, then $f_* T$ acts on (q, q)-tests on X.

Lemma 13.11.1. *For an ordinary positive current T and a continuous function u on X,*

$$f_*((u \circ f)T) = u\, f_*T.$$

Proof. For smooth u, evaluate both sides on a compactly supported test η and use $f^*(u\eta) = (u \circ f)f^*\eta$. For continuous u, uniformly approximate it by smooth functions near the compact support of η. Properness makes the inverse image of that support compact. The order-zero estimates on both spaces pass the smooth identity to the limit. This also proves the identity when f is merely proper over the support involved. □

Theorem 13.11.2. *If T is ordinary positive closed and $u_1, \dots, u_s \in \mathcal{C}_{25}(X)$ with $s \le q$, then*

$$f_*\big(A_{u_1 \circ f} \cdots A_{u_s \circ f} T\big) = A_{u_1} \cdots A_{u_s}(f_*T).$$

Proof. Pullback preserves smooth psh functions and locally uniform convergence; the closure principle therefore gives $u_i \circ f \in \mathcal{C}_{25}(Y)$. Pullback preserves ordinary positive test forms, so f_*T is ordinary positive. It is closed because the differentials commute with pullback on tests. For one factor, commutation of current pushforward with distributional differentiation and Lemma 13.11.1 give

$$f_*\mathrm{d}'\mathrm{d}''((u \circ f)T) = \mathrm{d}'\mathrm{d}'' f_*((u \circ f)T) = \mathrm{d}'\mathrm{d}''(u f_*T).$$

Apply this successively to the positive closed inner products to obtain the formula for s factors. □

For a finite flat same-field map of degree d, apply the theorem with $T = [Y]$ and the calibrated identity $f_*[Y] = d[X]$. It gives the degree-d projection formula for these mixed products. If instead one integrates a scalar extension over its new native ground field, Chapter 23 gives $p_*[X_L]_L = [X]_K$ and the corresponding formula has no extra degree factor. The recursive argument uses the supplied calibration identity and does not choose that normalization itself.

This theorem is a statement about currents. It does not say that proper pushforward of an actual delta-form is an actual delta-form, or that an alteration preserves an exact support complement. Those are the distinct representative questions discussed in Chapter 23.

13.12 A monotone consequence and its limit

Proposition 13.12.1. *Suppose $u_{i,j} \in \mathcal{C}_{25}$ decrease pointwise to finite continuous functions u_i, for $1 \le i \le s$. On each compact subset, convergence is uniform. The limits belong locally to $\mathcal{C}_{25}$, and the mixed products converge weakly, also with a weakly convergent sequence of ordinary positive closed background currents.*

Proof. For a compact set K and $\epsilon > 0$, the open sets $\{u_{i,j} - u_i < \epsilon\}$ increase with j and cover K. A finite subcover is contained in one of them. This is Dini's argument and proves uniform convergence. Local compactness and the generated-sheaf closure give the membership assertion. The product convergence is then the locally uniform stability theorem already proved. □

Continuity of the limit is essential to this argument. Without it, the sets used in the compactness proof need not be open. A general bounded-potential monotone theorem requires more than relabelling pointwise convergence as uniform convergence.

Example 13.12.2 (Pointwise multipliers do not suffice)**.** On $\mathbb{R}$, let $T_j = \delta_{1/j}$ and $T = \delta_0$. Choose continuous bumps u_j of height one at $1/j$, supported in intervals of radius $1/(10j^2)$ around $1/j$. Then $u_j \to 0$ pointwise, whereas $u_j T_j = T_j \rightharpoonup \delta_0$. Thus multiplication does not converge to $0T$. The bumps are not psh; this is a counterexample to the bare pointwise multiplier argument, not to a separately proved psh monotone theorem.

References. The generated-class construction is compared with [CLD25, Sections 8.5–8.7]. The proofs above use the ordinary-positive estimates and current operations established in Chapters 8–9; no full strong-cone spanning statement is used.

13.13 Comparison on a continuous global domain

A comparison principle can be separated from the harder problem of defining products for all bounded psh functions. We prove the continuous statement from locality, positivity, and the total mass formula. This identifies a useful consequence of an established calculus without claiming the full bounded-potential theory.

Let X be compact and let $\mathcal{P} \subset C(X)$ be a specified class of continuous potentials in one fixed background class θ. Assume that $\mathcal{P}$ is stable under addition of constants and finite maxima. Suppose its Monge–Ampère operator assigns positive Radon measures of the same finite mass V, is unchanged by adding constants to a potential, and is local on open sets: if two potentials agree on an open set, their measures agree there. These hypotheses are part of the statement. They must be checked in the chosen analytic theory.

Theorem 13.13.1. *For $u, v \in \mathcal{P}$,*

$$\int_{\{u<v\}} \mathrm{MA}(v) \leq \int_{\{u<v\}} \mathrm{MA}(u).$$

Proof. Fix $\varepsilon > 0$ and put $w = \max(u, v - \varepsilon)$. On the disjoint open sets $A_\varepsilon = \{u < v - \varepsilon\}$ and $B_\varepsilon = \{u > v - \varepsilon\}$, locality gives $\mathrm{MA}(w) = \mathrm{MA}(v)$ and $\mathrm{MA}(w) = \mathrm{MA}(u)$, respectively. Positivity and equality of total masses therefore imply

$$\begin{aligned} \mathrm{MA}(v)(A_\varepsilon) &= \mathrm{MA}(w)(A_\varepsilon) \leq V - \mathrm{MA}(w)(B_\varepsilon) \\ &= V - \mathrm{MA}(u)(B_\varepsilon) = \mathrm{MA}(u)(\{u \leq v - \varepsilon\}). \end{aligned}$$

As ε decreases to zero through a sequence, both sets in the final inequality increase to $\{u < v\}$. Continuity from below for finite positive measures proves the result. No assertion about the measure of the equality set $\{u = v\}$ is needed. □

Remark 13.13.2. This proof applies, for example, after the three stated properties have been established for a continuous semipositive metric class on a proper variety. It does not derive those properties from the single condition $\mathrm{d}'\mathrm{d}''[u] + \theta \geq 0$ on original smooth tests. In particular, the maximum must belong to the admitted domain and its measure must be local there. The theorem should not be cited as a regularization theorem or as an extension of the domain of MA.

13.13.1 Why finite maxima preserve a generated class

For the continuous generated class used in Chapter 13, maximum stability follows by the same two-variable closure argument as symmetry. The start-

ing step uses a smooth convex regularized maximum, nondecreasing in each argument. Its Hessian chain rule is

$$\mathrm{d}'\mathrm{d}''F(u,v) = F_1\mathrm{d}'\mathrm{d}''u + F_2\mathrm{d}'\mathrm{d}''v + \sum_{i,j=1}^{2} F_{ij}\, d'u_i \wedge d''u_j, \quad (u_1,u_2) = (u,v).$$

For smooth psh inputs all three contributions are positive: $F_i \geq 0$ and the Hessian matrix (F_{ij}) is positive semidefinite. The regularized maxima converge uniformly to the maximum. Keeping one input fixed and passing through local uniform closure, then doing the same for the other input, proves maximum stability on the generated class. The estimate

$$\|\max(u,v) - \max(\widetilde{u},\widetilde{v})\|_\infty \leq \max(\|u-\widetilde{u}\|_\infty, \|v-\widetilde{v}\|_\infty).$$

For metrics on a fixed line bundle, carry out this argument on local weights. Changing a frame adds the same pluriharmonic function to both weights; maximum commutes with that common addition, so the resulting local metric weights glue. One must still use the global degree theorem to identify their total masses.

13.14 A capacity estimate on the same domain

Assume additionally that $0 \in \mathcal{P}$, that convex combinations remain in $\mathcal{P}$, and that symmetric multilinear positive mixed measures are defined, with the usual expansion for $\theta + \mathrm{d}'\mathrm{d}''(tv)$. These assumptions hold only on the specified domain. For a Borel set $E \subset X$, define

$$\mathrm{Cap}_{\mathcal{P}}(E) = \sup_{v\in\mathcal{P},\, -1\leq v\leq 0} \int_E \mathrm{MA}(v).$$

This definition records its testing class in the subscript. Different classes may give different capacities, even when their operators agree on the intersection.

Proposition 13.14.1. *For $u \in \mathcal{P}$ with $u \leq 0$, $s \geq 0$ and $0 < t \leq 1$,*

$$t^n\,\mathrm{Cap}_{\mathcal{P}}(\{u < -s-t\}) \leq \int_{\{u<-s\}} \mathrm{MA}(u),$$

where n is the number of factors in the Monge–Ampère operator.

Proof. For an admitted v with $-1 \le v \le 0$, put $w = tv - s$. Since $0 < t \le 1$, convexity and constant stability make w admitted. Its curvature is $t(\theta + \mathrm{d}'\mathrm{d}''v) + (1-t)\theta$. The mixed expansion and positivity give $\mathrm{MA}(w) \ge t^n \mathrm{MA}(v)$. Moreover,

$$\{u < -s - t\} \subseteq \{u < w\} \subseteq \{u < -s\}.$$

The comparison theorem therefore yields

$$t^n \int_{\{u<-s-t\}} \mathrm{MA}(v) \le \int_{\{u<w\}} \mathrm{MA}(w) \le \int_{\{u<w\}} \mathrm{MA}(u) \le \int_{\{u<-s\}} \mathrm{MA}(u).$$

Take the supremum over the admitted test potentials v. □

Monotonicity, countable subadditivity and continuity along increasing Borel sets follow directly from the definition: each test supplies a positive measure, and increasing suprema commute. Also $\mathrm{Cap}_{\mathcal{P}}(X) = V$, since $v = 0$ is an allowed test and all test measures have mass V. Continuity along decreasing compact sets does not follow from this argument.

Example 13.14.2. On $[0,1]$, the set function $c(E) = \sup_{k\ge1} \delta_{1/k}(E)$ is a supremum of positive measures of mass one. Yet for $K_j = [0, 1/j]$ one has $c(K_j) = 1$ for every j, while $c(\bigcap_j K_j) = c(\{0\}) = 0$. Thus the measure properties of individual tests do not by themselves make their supremum a Choquet capacity. This example concerns the abstract inference, not a counterexample to a capacity theorem proved for a particular psh class.

The continuous comparison and sublevel estimate are therefore available consequences once their explicit inputs have been verified. Quasicontinuity, pluripolar-set characterizations, monotone convergence for discontinuous bounded potentials, and a Dirichlet existence theorem require further results. The elementary arguments above do not remove those obligations.

References. The proofs combine maximum locality and the fixed-mass identity, followed by the positive mixed expansion. Their analytic inputs are discussed in Chapters 13, 19 and 20; compare the model-based pluripotential framework of [BFJ15].

Chapter 14

Harmonic kernels and potential independence

14.1 Skeletal currents and harmonic charts

14.1.1 The quantifier in the definition

Let T have bidimension (p, p). A calibrated skeletal subset P describes T relative to a chart f if

$$\langle T, \rho\omega \rangle = \int_P \rho\, \omega|_P$$

for every smooth form ω tropicalized by f and every compactly supported smooth scalar ρ. In Definition 7.5.1, the current is skeletal when such a local description exists relative to *every* chart. The describing subset and calibration may depend on the chart.

This last quantifier is the essential point. Being supported on one skeleton only says where a current vanishes. Being skeletal specifies how it acts and how that action transforms when more tropical coordinates are introduced. The integration current of the ambient tropical space is skeletal because every chart has a canonically calibrated locus describing its top-dimensional integration.

Skeletal currents are symmetric and of order zero in the source construction. Positive describing calibrations give a correspondingly strong positivity statement. These conclusions come from their geometric description, inde-

pendently of the false general implication from weak positivity to order zero discussed in Appendix A.

14.1.2 A simple example and a misleading near-example

On a weighted graph, integration over a collection of edges with constant compatible weights is a skeletal current. Multiplying by a nonconstant smooth scalar gives a current with the same support, but its edge density is no longer locally constant. It need not remain skeletal in this particular sense, and it need not remain closed. Thus support, order zero, skeletal representability, and closedness are four different properties.

Similarly, a Radon measure supported on a compact skeleton may have a singular continuous density or an arbitrary atomic distribution. It is a perfectly legitimate measure. It does not follow that it is a skeletal current of the required calibrated type. A theorem about the latter cannot be applied solely from the support of the former.

14.1.3 What harmonicity asks to vanish

The harmonic charts of CLD25 are built from the source's PL-generated harmonic functions and current conditions. Their coordinates have vanishing appropriate Hessian currents, but their ordinary CLD smoothness is not assumed. Harmonicity is intended to preserve the integration and balancing needed for differential operations when the coordinate class is enlarged.

For a skeletal current, a coordinate h interacts with the calibration through $\mathrm{d}'\mathrm{d}''(hT)$. If this vanishes, the weighted slopes of h cancel in the relevant skeletal description. This is a stronger compatibility with T than merely $\mathrm{d}'\mathrm{d}''[h] = 0$ on the ambient integration current. That difference is exactly why the Bedford–Taylor property later appears in the harmonic product theorem.

14.1.4 The mechanism of the harmonic product formula

Proposition 8.11.1 assumes that T is skeletal and has the Bedford–Taylor property. For a G-form α locally obtained from harmonic coordinates, it proves the expected differential identities

$$d'(T \wedge \alpha) = T \wedge d'\alpha, \qquad d''(T \wedge \alpha) = T \wedge d''\alpha.$$

The proof applies Stokes to the skeletal pieces. The internal boundary terms involve the outgoing derivatives of the harmonic coordinate functions. The Bedford–Taylor property turns their ambient Hessian vanishing into $\mathrm{d}'\mathrm{d}''(hT) = 0$, which forces the required weighted cancellation. Thus the hypothesis has a concrete job: it makes the new coordinates harmonic for the current being multiplied.

Corollary 8.11.2 then identifies certain recursively defined BT products with products by the corresponding G-forms. Its skeletal and BT hypotheses are retained here. The subsequent Corollary 8.11.3 is stated for an arbitrary positive closed current but cites this restricted comparison. Section 14.2 gives an explicit counterexample in the abstract tropical category. The restricted theorem cannot be extended by simply dropping those hypotheses.

14.2 The kernel problem: a complete tropical calculation

14.2.1 The equality one wants to descend

For an admitted continuous potential u and a positive closed current T, the expression $\mathrm{d}'\mathrm{d}''(uT)$ is defined. To say that it depends only on $\mathrm{d}'\mathrm{d}''[u]$ asks for

$$\mathrm{d}'\mathrm{d}''[h] = 0 \quad \Longrightarrow \quad \mathrm{d}'\mathrm{d}''(hT) = 0 \tag{14.2.1}$$

for differences h of admitted potentials. This is a statement about the kernel of the ambient Hessian acting on T. It does not follow from approximation independence, which says that two approximating sequences for the *same function* give the same product.

For smooth h, closedness of T gives $\mathrm{d}'\mathrm{d}''(hT) = \mathrm{d}'\mathrm{d}''h \wedge T$, so the implication holds. For a smooth analytic-unit logarithm, this proves exactly the gluing needed for line-bundle weights. For a nonsmooth current-harmonic h, the same Leibniz proof is not available.

14.2.2 An admissible reducible tropical space

Let $H = \mathbb{R} \times \{0\}$, $V = \{0\} \times \mathbb{R}$, and $X = H \cup V$. Give each of the four rays weight one. Use the refined affine structure of CLD25 Example 6.1.4: affine functions are locally restrictions of ambient affine functions on $\mathbb{R}^2$. In particular, not every PL function is declared affine.

Here are the category checks, since the example is meant to test an abstract tropical theorem. The underlying polyhedral structure has the G-tropical and skeletal structures of Examples 5.1.4 and 5.2.3. It is pure of dimension one and has the required richness. Ambient coordinates separate points and differential directions. An affine germ on a polyhedral domain extends from its ambient affine expression, and such domains are locally Boolean combinations of affine inequalities; these give the two decency conditions in Definition 6.3.1. The calibration balances at the vertex because

$$e_1 - e_1 + e_2 - e_2 = 0.$$

Every local affine chart preserves this equation. Boundaries of local polyhedral domains have zero one-dimensional measure. Thus Definition 6.6.2 permits the empty boundary. The larger boundary allowed as a general choice in Example 6.6.3 does not force the vertex to be a boundary point in this balanced example.

Smooth tests for this structure are locally restrictions of ambient smooth superforms. In particular, their restrictions to each whole line are smooth through the origin, and scalar tests have one common value at the vertex.

14.2.3 Two potentials with the same ambient curvature

Put $u = |x||_X$ and $v = |y||_X$. The restrictions of

$$\sqrt{x^2 + \varepsilon^2}, \qquad \sqrt{y^2 + \varepsilon^2}$$

are smooth psh functions decreasing uniformly to u, v as $\varepsilon \downarrow 0$. Thus both belong to the one-stage continuous approximation class. For every smooth compactly supported scalar test ϕ,

$$\langle \mathrm{d}'\mathrm{d}''[u], \phi\rangle = \int_{\mathbb{R}} |x| \frac{d^2}{dx^2}\phi(x,0)\, dx = 2\phi(0,0),$$
$$\langle \mathrm{d}'\mathrm{d}''[v], \phi\rangle = \int_{\mathbb{R}} |y| \frac{d^2}{dy^2}\phi(0,y)\, dy = 2\phi(0,0).$$

Hence $\mathrm{d}'\mathrm{d}''[u] = \mathrm{d}'\mathrm{d}''[v] = 2\delta_0$ on the same actual test space.

Now let $T = \delta_H$, integration of $(1,1)$-tests along the horizontal line. This is a positive current, and Stokes on the whole line proves $d'T = d''T = 0$. Multiplication by the continuous functions gives

$$\mathrm{d}'\mathrm{d}''(uT) = 2\delta_0, \qquad \mathrm{d}'\mathrm{d}''(vT) = 0. \tag{14.2.2}$$

The current T is therefore positive closed but fails the BT kernel property. All identities use the same tests; no tensor-product-only testing or change of category is hidden in the calculation.

14.2.4 The consequence for harmonic pullback

The function $h = u - v$ is PL and satisfies $\mathrm{d}'\mathrm{d}''[h] = 0$. It is harmonic by CLD25 Definition 7.8.2. Take the harmonic map $f = -h : X \to \mathbb{R}$ and the convex function $w(t) = t$. Then $f^*w = -h$ is a difference of the admitted psh potentials, but

$$\mathrm{d}'\mathrm{d}''\big((f^*w)T\big) = -2\delta_0.$$

This is not positive. Thus Corollary 8.11.3 is false in its unrestricted abstract-tropical formulation. The missing skeletal/BT hypothesis in its cited argument is not merely cosmetic. This calculation strengthens the hypothesis-transfer observation in the preceding audit section to a counterexample in the explicitly verified category.

The example does not satisfy the hypotheses of a smooth Berkovich local-potential problem. It does not refute a separate analytic theorem based on residual geometry. Its conclusion is confined to the stated abstract tropical generality.

14.2.5 The graph-support statement also detects the problem

On $X \times \mathbb{R}$, with coordinate t in the second factor, put $S = \delta_{H\times\mathbb{R}}$ and use $f = h$. The graph-intersection construction of §9.5 is

$$S \cap \Gamma_f = \mathrm{d}'\mathrm{d}''\big(\max\{h, t\}\, S\big).$$

On the open set

$$W = \big(X \cap \{|x|, |y| < 1/4\}\big) \times (-5/4, -3/4)$$

we have $h > t$. Hence W is disjoint from Γ_f, but the restriction of the displayed current to W is

$$2\delta_0 \boxtimes [1]_{\mathbb{R}},$$

which is nonzero. Indeed, take a test $\eta = \rho(x, y)\chi(t)\, d't \wedge d''t$ supported in W, with $\rho(0) = 1$ and $\chi \geq 0$ nonzero. Its pairing is $2\int \chi(t)\, dt > 0$.

This disproves the unrestricted harmonic graph-support assertion in Lemma 9.5.4. Its printed proof uses a BT hypothesis on S that is not in the

statement. For an affine f, the relevant potential is smooth affine off the graph and the closed-current Leibniz calculation does prove the support assertion. A version with additional BT hypotheses must verify the corresponding stability and potential class.

14.2.6 What remains a valid research question

The cross shows that arbitrary-current kernel descent cannot be proved from the abstract tropical axioms alone. A positive answer on a smooth analytic space must use extra analytic geometry or a more restrictive harmonic kernel. Possible inputs include residual numerical triviality, approximation preserving the relevant kernel, or a representation theorem for the allowed currents. Finding such an input is a substantive theorem, not a change of notation.

References. [CLD25, Examples 5.1.4, 5.2.3, 6.1.4; Definitions 6.3.1, 6.6.2, 7.8.2; Corollary 8.11.3 and Lemma 9.5.4].

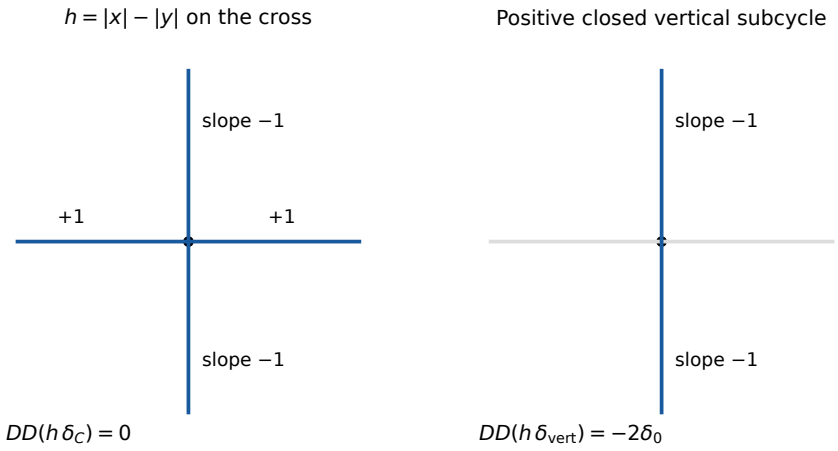


Figure 14.1: The outgoing slopes of h cancel on the whole balanced cross, but not on the positive closed vertical subcycle. This is the kernel obstruction of Section 14.2.

14.3 A local classification of the graph kernel property

The tropical cross gives a counterexample, but its mechanism is more general. On a balanced graph, the ambient Laplacian and the Laplacian weighted by another current are two linear functionals on the vector of outgoing slopes. Potential independence asks for inclusion of their kernels. This elementary linear-algebra problem can be solved completely.

Let a neighbourhood of a vertex v be a union of k rational rays with primitive outgoing vectors $u_1, \dots, u_k \in N$, parameterized by lattice length $t \geq 0$. Give them positive weights m_i satisfying $\sum_i m_i u_i = 0$. Smooth tests are restrictions of ambient smooth superforms. Let T_λ be integration along the same rays with nonnegative constant weights λ_i. We work with compactly supported tests in this neighbourhood, so artificial outer endpoints contribute no boundary terms.

Lemma 14.3.1. *The current T_λ is closed under both d' and d'' if and only if $\sum_i \lambda_i u_i = 0$. If this balancing condition holds and a continuous function h is affine on each ray near v, with outgoing slopes s_i, then its vertex contribution is*

$$d'd''(hT_\lambda) = \left(\sum_i \lambda_i s_i \right) \delta_v \quad \text{near } v.$$

Proof. Stokes on each ray expresses the boundary of its integration current as evaluation at v against the outgoing tangent. Summing with weights gives contraction with $\sum_i \lambda_i u_i$. Thus balancing is sufficient. Conversely, choose an ambient covector a and a smooth cutoff equal to one near v. The associated one-form test detects $\langle a, \sum_i \lambda_i u_i \rangle$. Vanishing for every a forces balancing. The two differentials give the same condition in their respective exterior factors.

For the second assertion, write $h_i(t) = h(v) + s_i t$. Twice integrating by parts against a scalar test ψ gives the vertex terms $\lambda_i s_i \psi(v) - \lambda_i h(v) \partial_{u_i} \psi(v)$. The terms containing $h(v)$ sum to zero by balancing. The remaining sum is the displayed atom. There is no interior term because each h_i is affine near the vertex. □

Theorem 14.3.2. *Assume T_λ is closed. The following conditions are equivalent:*

1. *For every continuous, raywise affine germ h at v, ambient harmonicity $\sum_i m_i s_i = 0$ implies $d'd''(hT_\lambda) = 0$ near v.*
2. *There is a constant $c \geq 0$ such that $\lambda_i = cm_i$ for every i.*

Thus, in this class of currents and PL germs, the full ambient harmonic kernel acts trivially precisely when the current has a common density relative to the ambient calibration.

Proof. Any vector $s = (s_1, \ldots, s_k) \in \mathbb{R}^k$ is realized by a continuous raywise affine germ with a prescribed common value at v. By Lemma 14.3.1, condition (1) is therefore

$$\ker(m_1, \ldots, m_k) \subseteq \ker(\lambda_1, \ldots, \lambda_k).$$

Fix an index j. Since $m_j > 0$, every s decomposes as

$$s = \left(s - \frac{m \cdot s}{m_j} e_j\right) + \frac{m \cdot s}{m_j} e_j,$$

with the first term in $\ker m$. Kernel inclusion gives $\lambda \cdot s = (\lambda_j/m_j)(m \cdot s)$ for all s, so $\lambda = cm$ with $c = \lambda_j/m_j \geq 0$. Conversely, proportionality gives $\lambda \cdot s = c(m \cdot s)$, proving (1). It also automatically gives balancing from that of m. □

The theorem concerns all continuous raywise affine germs. If a particular analytic theory admits a smaller harmonic class, its kernel condition can be weaker. In particular, the theorem does not assert that every such germ on every graph is the restriction of an analytic harmonic function. That realization problem involves the analytic or residual geometry absent from this linear-algebra calculation.

Corollary 14.3.3. *Let a connected finite balanced graph have positive ambient edge weights, and let a closed polyhedral integration current have constant nonnegative weights on its edges. If the local property of Theorem 14.3.2 holds at every vertex, then the current is one constant nonnegative multiple of the ambient integration current.*

Proof. At a vertex, the theorem says that the ratios of current weight to ambient weight are the same on all incident edges. An edge has one ratio

at both ends because both of its weights are constant. Connectedness propagates the ratio through the graph. The statement is componentwise if the graph is disconnected. □

14.3.1 Why ordinary closedness is weaker

An ambient affine function has slopes $s_i = \langle a, u_i \rangle$. Closedness gives $\lambda \cdot s = \langle a, \sum_i \lambda_i u_i \rangle = 0$. Thus closedness already annihilates all slopes coming from ambient affine functions. The harmonic kernel usually contains more slopes: it is the hyperplane $m \cdot s = 0$, whereas the affine slopes form the image of $N^*_{\mathbb{R}} \to \mathbb{R}^k$. Equality of these two spaces requires an additional rank condition.

Example 14.3.4. For the standard trivalent tropical line, $u_1 = (1, 0)$, $u_2 = (0, 1)$ and $u_3 = (-1, -1)$. Balancing forces $\lambda_1 = \lambda_2 = \lambda_3$. With ambient weights one, every closed current in this constant-edge class therefore satisfies the PL kernel property.

For the cross, the directions are $e_1, -e_1, e_2, -e_2$. Balancing only forces weights of the form (a, a, b, b). Theorem 14.3.2 requires $a = b$. The slopes $(1, 1, -1, -1)$ are ambient harmonic and give weighted Laplacian $2(a - b)\delta_v$. Taking $a = 1$, $b = 0$ recovers the earlier counterexample. The failure is measured by the difference between the densities on the two crossing lines.

This calculation explains the role of an extra BT property in a harmonic-coordinate theorem. Such a property is a compatibility requirement between the allowed harmonic functions and the current's calibration. It is not another name for positivity and closedness. In a Berkovich theorem, proving that compatibility may require residual geometry or a representation theorem; it cannot be replaced by the smooth Leibniz rule for coordinates which are not smooth in the original sheaf.

14.4 The obstruction to gluing local potentials

A local potential theorem and a global potential theorem are different assertions even on a smooth real affine manifold. Local potentials differ by elements of the Hessian kernel. Their differences form a cocycle, and this cocycle is the exact obstruction to choosing one global potential. This elementary description is useful when comparing Green currents, metric weights and cohomological vanishing.

Let M be a paracompact real affine manifold, with affine transition maps, and let Aff_M be the sheaf of locally affine real functions. On convex coordinate charts, Theorem 11.5.3 gives convex potentials for any ordinary positive closed $(1,1)$ current. Affine coordinate changes preserve Hessians and convexity. Choose a cover (U_i) by such charts and potentials u_i.

Proposition 14.4.1. *On every overlap, $a_{ij} = u_j - u_i$ is locally affine, and*

$$a_{ij} + a_{jk} = a_{ik}.$$

The resulting class in $\check{H}^1(M, \mathrm{Aff}_M)$ is independent of the local potentials and cover. It vanishes if and only if the current is the Hessian of a global finite locally convex function.

Proof. The difference has zero Hessian, hence is affine on each connected overlap component by Proposition 11.4.1. The cocycle identity is subtraction of three functions. If u_i is replaced by another potential, its difference from the old one is a locally affine function b_i, and the new cocycle differs by $b_j - b_i$. Refinement restricts the same cocycle. Thus the class is intrinsic.

If the class vanishes, after refinement there are locally affine b_i with $a_{ij} = b_j - b_i$. Then $u_i - b_i$ agree on overlaps and glue to a finite continuous function u. Locally it is convex and has the prescribed Hessian. Conversely, a global potential gives $b_i = u_i - u$ and makes the cocycle a coboundary. □

The coefficient sheaf is Aff_M, not the constant sheaf $\mathbb{R}$. On a singular tropical space, the kernel of the Hessian current can differ from the restrictions of affine functions. Replacing that kernel by an ambient affine sheaf would change the obstruction group. The tropical cross in Chapter 14 gives an explicit reason for retaining the actual kernel.

Example 14.4.2 (A positive curvature class on a real torus)**.** Let $M = \mathbb{R}^n/\Lambda$, and let H be a positive definite symmetric matrix. The form $\theta = \sum H_{ij} d'x_i \wedge d''x_j$ descends to M. Locally it is DDq for $q(x) = \frac{1}{2} x^{\mathsf{t}} H x$. On overlaps of lifted charts differing by $\lambda \in \Lambda$,

$$q(x+\lambda) - q(x) = x^{\mathsf{t}} H \lambda + \tfrac{1}{2} \lambda^{\mathsf{t}} H \lambda.$$

This is an affine cocycle. It cannot be a coboundary of local affine functions, because a global distributional potential u would give

$$0 = \langle DD[u], \theta^{n-1} \rangle = \int_M \theta^n > 0,$$

a contradiction. The first equality is distributional integration by parts and $DD(\theta^{n-1}) = 0$. Thus a perfectly regular positive curvature form can have local convex potentials and no global scalar potential.

14.4.1 Metric weights solve a different gluing problem

For a line bundle with local frames e_i and transitions $e_j = g_{ij}e_i$, additive metric weights satisfy

$$u_j - u_i = -\log|g_{ij}|.$$

The transition functions are prescribed by the line bundle. Their logarithms are smooth pluriharmonic where g_{ij} are analytic units, so the local currents $DD[u_i]$ glue even when the weights do not glue as scalar functions. A metrized line bundle is therefore naturally described by local potentials with a specified cocycle. The global scalar-potential problem asks for that cocycle to be trivial in the relevant kernel sheaf.

This also explains a limitation of curvature comparison. Equal local curvatures imply that the difference of weights lies in a Hessian kernel. They do not identify the metric constants or the transition cocycle without further information. In the real affine case the kernel is fully described above; in a non-Archimedean form theory its description depends on the chosen tests and regularity class.

References. The local real potential theorem is [Lag12, Proposition 1.13]. The cocycle argument is the standard gluing construction applied to its affine kernel; the torus automorphy formula is compatible with the metric setting of [BGJK25, Sections 7–8].

Part IV

Delta and harmonic calculi

Chapter 15

Polyhedral intersection calculus

15.1 Tropical preforms and their representations

Let N be a lattice, let $\widetilde{\Omega} \subset N_{\mathbb{R}}$ be open, and let C_i be tropical cycles with smooth real weights as in Chapter 5. Their currents δ_{C_i} have bidegree equal to their codimension.

Definition 15.1.1 (Tropical delta-preform)**.** A delta-preform of bidegree (p, q) on $\widetilde{\Omega}$ is a supercurrent admitting an expression

$$\alpha = \sum_{i=1}^{m} \alpha_i \wedge \delta_{C_i}, \qquad \alpha_i \in A^{p-\ell_i, q-\ell_i}(\widetilde{\Omega}), \qquad \operatorname{codim} C_i = \ell_i.$$

The vector space of these currents is denoted $P^{p,q}(\widetilde{\Omega})$. If every $\ell_i = \ell$, the expression has pure codimension ℓ; the coefficient bidegree is then $(p-\ell, q-\ell)$.

The defining data are representations of a current, rather than additional labels attached to it. The product on P combines the wedge product of the coefficients with stable intersection of tropical cycles. Its independence of the representation is a theorem [GK17, Definition 2.9 and Proposition 2.12]. It is not a product defined on arbitrary supercurrents.

Remark 15.1.2 (Codimension zero does not mean smooth). With smooth weights allowed, a codimension-zero tropical cycle on $N_{\mathbf{R}}$ is a piecewise smooth function on that space. Consequently, a codimension-zero delta-preform can have piecewise smooth coefficients [GK17, Remark 1.7 and Example 2.10]. For example, $|t|[\mathbf{R}]$ has codimension zero but is not represented by a smooth function near 0. Thus codimension and regularity are different attributes.

15.2 Recovering coefficients from a polyhedral current

The product of delta-preforms is a product of currents, not of chosen lists of generators. Its construction therefore needs a detection statement before a multiplication formula. We give the argument in fixed polyhedral codimension, as in GK17, Sections 2.11–2.12.

Lemma 15.2.1. *Let $\mathscr{C}$ be a locally finite polyhedral complex in $N_{\mathbb{R}}$, and let*

$$S = \sum_{\sigma \in \mathscr{C}_d} \alpha_\sigma \wedge \delta_\sigma,$$

where each α_σ is a smooth tangential superform on σ, extending smoothly to a neighbourhood in its affine span. If $S = 0$, then each $\alpha_\sigma = 0$. Consequently two such currents on a common complex are equal exactly when their tangential coefficients agree.

Proof. Fix a point in the relative interior of one d-cell. A sufficiently small neighbourhood avoids the other d-cells, since their intersections with this cell are proper faces and the complex is locally finite. On that neighbourhood, testing S is testing α_σ by integration on the affine span of σ.

Choose affine coordinates on this span. For any coefficient of α_σ, wedge with the complementary primed and double-primed basis elements, choosing their order to fix the sign, and multiply by an arbitrary compactly supported scalar test. Such a test extends to the ambient space by a linear projection and a normal cutoff. The vanishing of S says that the integral of the chosen coefficient against every scalar test is zero. The fundamental lemma for distributions gives that coefficient zero. All coefficients vanish on the relative interior and, by continuity, on its closure. Repeat for each cell. □

A representative $\sum_i \alpha_i \wedge \delta_{C_i}$ has, after a common subdivision, the coefficient $\sum_i m_{i,\sigma}\alpha_i|_\sigma$ on a cell σ. This coefficient is intrinsic to the current on that complex. It is a tangential form: adding an ambient form which restricts to zero on σ changes nothing. This last point is essential when the support has positive codimension.

15.3 The geometric input for stable intersection

We use the stable-intersection theorem for tropical cycles with smooth weights in GK17, Section 1.4. After common subdivision, a generic displacement computes the coefficient of a codimension $\ell+\ell'$ intersection face τ by summing over contributing pairs of faces σ, σ', of codimensions ℓ, ℓ'. A pair contributes only if it survives the displacement test and $N_\sigma + N_{\sigma'}$ has full rank. Its factor is

$$[N : N_\sigma + N_{\sigma'}]\, m_\sigma m_{\sigma'}.$$

The cycle theorem asserts that the resulting cycle is independent of the permitted sufficiently small generic displacement and of subdivision, and that the cycle product is associative and commutative. We use those assertions as geometric inputs. They are not consequences of the coefficient-detection lemma.

Theorem 15.3.1 (Representation independence)**.** *Assume the stated cycle-intersection theorem. For delta-preforms of fixed polyhedral codimensions, the formula*

$$\left(\sum_i \alpha_i \wedge \delta_{C_i}\right) \wedge \left(\sum_j \beta_j \wedge \delta_{D_j}\right) = \sum_{i,j} (\alpha_i \wedge \beta_j) \wedge \delta_{C_i \cdot D_j}$$

is independent of the chosen representations. It is bilinear, associative and graded commutative for total superform degree, and agrees with multiplication by a smooth superform.

Proof. Choose a common complex for all cycles in two proposed representations of each factor, refining further so that the displacement formula applies. Let a_σ and $b_{\sigma'}$ be the intrinsic tangential coefficients of the two currents on this complex. On an intersection face τ the proposed product has coefficient

$$\sum_{\sigma,\sigma'} [N : N_\sigma + N_{\sigma'}](a_\sigma|_\tau) \wedge (b_{\sigma'}|_\tau),$$

with the same contributing-pair rule as the cycle formula. This expression follows by expanding the finite sums and collecting the cycle weights. Lemma 15.2.1 makes $a_\sigma, b_{\sigma'}$ independent of the initial lists of generators. Restriction to τ is well defined for tangential forms, because $T\tau$ lies in both tangent spaces. Thus the coefficient of every intersection face is independent of those lists. Detection proves equality of the resulting currents.

Bilinearity is immediate. Once representation independence is established, expand a triple product using the original smooth coefficients and cycles. Associativity of wedge and of the cycle product identifies the two expansions. Swapping the factors changes the smooth coefficients by the usual total-degree sign; twice the polyhedral codimension is even and adds no sign. Finally, a smooth form is represented using the unit cycle $N_{\mathbb{R}}$, so the formula reduces to its usual multiplication with the current. □

This proof separates two independence questions. The tropical cycle theorem controls displacement and subdivision. The coefficient argument controls representations of an already defined current. Conflating them hides the main point of GK17, Proposition 2.12. Neither independence statement defines a multiplication on arbitrary distributional currents.

15.4 Transverse lines and the determinant

Proposition 15.4.1. *In $\mathbb{R}^2$ with lattice $\mathbb{Z}^2$, let L_a, L_b be distinct lines through the origin with primitive integral direction vectors a, b. With weight one on each line,*

$$\delta_{L_a} \wedge \delta_{L_b} = |\det(a, b)|\delta_0.$$

If the two lines coincide, their stable intersection is zero.

Proof. Distinct lines meet transversely. After any sufficiently small generic translation they meet at one point tending to the origin. The cycle multiplicity is $[\mathbb{Z}^2 : \mathbb{Z}a + \mathbb{Z}b] = |\det(a, b)|$. For coincident lines choose a translation transverse to the line. The displaced lines are disjoint, so no face contributes. Displacement independence makes this the stable product. □

For $u = \max(0, x)$ and $v = \max(0, x + 2y)$, the two divisor directions may be chosen as $(0, 1)$ and $(2, -1)$. Their determinant has absolute value two, so $\mathrm{d}'\mathrm{d}''[u] \wedge \mathrm{d}'\mathrm{d}''[v] = 2\delta_0$ in this calculus. In contrast, the stable square

of the straight divisor $\{x = 0\}$ is zero. These identities express geometric intersection rules; they are not prescriptions for squaring Dirac distributions in ordinary distribution theory.

15.5 Balancing as cancellation of boundary currents

The corner calculation in Chapter 10 has a useful converse. For constant weights, balancing is precisely what removes the boundary term in Stokes' formula. This gives a direct way to prove that a PL corner divisor is again balanced. Smoothly varying weights require an additional distinction, because their derivatives need not vanish.

Proposition 15.5.1. *Let C be a locally finite, pure d-dimensional rational polyhedral complex in $N_{\mathbb{R}}$ with constant real weights m_σ. Tests have compact support away from any designated boundary. The following conditions are equivalent: C is balanced; $d'\delta_C = 0$; $d''\delta_C = 0$.*

Proof. Fix a codimension-one face τ. Write $n_{\sigma,\tau}$ for an outgoing primitive normal lift and put

$$b_\tau = \sum_{\sigma \supset \tau} m_\sigma n_{\sigma,\tau} \pmod{N_{\tau,\mathbb{R}}}.$$

Apply ordinary Stokes on each maximal polyhedron to the coefficients of a superform. The interior derivatives of the weights vanish. Along τ, all tangential coordinates and the induced lattice measure agree, so the sum of the boundary contractions depends only on b_τ. A tangent change of normal has zero contraction in the normal quotient. Hence $b_\tau = 0$ makes the boundary contribution vanish. Summing over the finitely many cells meeting a test support proves both closedness assertions.

Conversely, take a point in the relative interior of τ and a small neighbourhood meeting no other codimension-one face. Choose a compactly supported scalar bump on τ whose lattice integral is one. Given any real linear functional ℓ on $N_{\mathbb{R}}/N_{\tau,\mathbb{R}}$, extend the bump to an ambient test and choose its remaining differential factors to be a tangential volume element and the normal covector ℓ. The same Stokes calculation pairs the boundary current with this test as $\ell(b_\tau)$, up to one fixed nonzero sign. If $d'\delta_C = 0$, every such pairing

vanishes, so $b_\tau = 0$. The argument for d'' is identical with the two copies of the cotangent space exchanged. □

Theorem 15.5.2 (A PL corner is balanced)**.** *Let C have constant real weights and be balanced. If u is continuous and affine on the cells of a subdivision of C, the corner weights of Proposition 10.5.1 define a balanced cycle $u \cdot C$, and*

$$\mathrm{d}'\mathrm{d}''(u\delta_C) = \delta_{u\cdot C}.$$

Proof. On a maximal cell, the second derivatives of u vanish. Integrate twice by parts in the pairing of $u\delta_C$ with $\mathrm{d}'\mathrm{d}''$ of a test. The terms containing the value of u times a normal derivative of the test cancel along each facet by continuity and balancing. The remaining coefficient on a facet τ is

$$\sum_{\sigma \supset \tau} m_\sigma \ell_\sigma(n_{\sigma,\tau}) - \ell_\tau\left(\sum_{\sigma \supset \tau} m_\sigma n_{\sigma,\tau}\right),$$

where ℓ_σ and ℓ_τ are the linear parts of u. This proves the current identity, including the tangent correction. It also proves that the result is a polyhedral current with constant facet weights. Now

$$d'\delta_{u\cdot C} = d'\mathrm{d}'\mathrm{d}''(u\delta_C) = 0, \qquad d''\delta_{u\cdot C} = 0.$$

Proposition 15.5.1 therefore proves balancing of the corner cycle. This last step uses only the differential identities and the preceding Stokes characterization. □

The proof also completes subdivision independence in this constant-weight PL setting. The distribution $\mathrm{d}'\mathrm{d}''(u\delta_C)$ is unchanged by subdivision. On a common refinement, coefficient detection identifies its facet weights. Thus two descriptions give the same weighted cycle; the new facets inside an affine piece already have zero weight by Proposition 10.5.1.

15.5.1 Smooth weights and the cellwise differential

Balancing of smooth weights is a pointwise condition along a facet. It cancels boundary contributions but does not kill derivatives inside cells. Already on $N_\mathbb{R} = \mathbb{R}$, every smooth weight $m(t)$ on the full line satisfies the codimension-zero balancing condition, whereas

$$d'(m\delta_\mathbb{R}) = m'(t)d't \wedge \delta_\mathbb{R}.$$

For $m(t) = e^t$, the right side is nonzero. Thus "balanced with smooth weights" and "closed integration current" are different assertions. The operators d'_P, d''_P record the cellwise terms. The difference between a distributional differential and its cellwise counterpart is the boundary operator. This distinction is used when selecting the universally closed generators in the GK sheaf.

15.6 Why the fan product is independent of displacement

For a finite complete rational fan Σ in a rank-r lattice N, a codimension-p Minkowski weight assigns a real number to each $(r-p)$-dimensional cone and satisfies balancing at the next lower-dimensional cones. There is a geometric reason for exactly these linear equations.

Let X_Σ be the associated complete toric variety. A cone σ of codimension p has orbit closure $V(\sigma)$ of dimension p. An operational Chow class $c \in A^p(X_\Sigma)$ consequently defines

$$m_c(\sigma) = \deg(c \cap [V(\sigma)]).$$

The dimension convention is important: the codimension of the cone is the dimension of its orbit closure. For a cone τ of codimension $p+1$ and a character $u \in M \cap \tau^\perp$, the divisor of that character on $V(\tau)$ gives the relation

$$\sum_{\substack{\sigma \supset \tau \\ \dim \sigma = \dim \tau + 1}} \langle u, n_{\sigma,\tau} \rangle [V(\sigma)] = 0 \quad \text{in } A_p(X_\Sigma).$$

Applying c and taking degree says that the weighted sum of the normal vectors pairs to zero with every such u. This is precisely balancing modulo N_τ. Thus balancing is the numerical expression of principal-divisor relations.

Theorem 15.6.1 (The toric input)**.** *For complete Σ, the assignment $c \mapsto m_c$ identifies operational Chow cohomology, with real coefficients, with Minkowski weights. Its product is computed by the fan displacement formula: at a cone γ of codimension $p+q$, sum*

$$[N : N_\sigma + N_\tau]\, m_c(\sigma) m_{\widetilde{c}}(\tau),$$

over cones of codimensions p, q containing γ whose generic displacement meets. This is the Fulton–Sturmfels theorem; the identification and the diagonal-degeneration formula are used here as inputs.

Displacement independence now has a precise proof from that input. Two generic vectors give two expressions for the same number

$$\deg\big((c\widetilde{c}) \cap [V(\gamma)]\big).$$

They must agree even though the contributing pairs may differ. Associativity and commutativity are inherited from the Chow product, not from an assertion that the displaced set-theoretic intersections are identical. A common toric subdivision compares different fans. Pullback of operational classes and the projection formula preserve the relevant degrees, which is the geometric source of refinement compatibility.

15.6.1 From constant fans to smooth polyhedral weights

Proposition 15.6.2. *Assume the toric fan product and its refinement compatibility. The displacement construction for rational polyhedral cycles with smooth weights is well defined, balanced pointwise, and compatible with subdivision. Its products are associative and commutative.*

Proof. Fix a point x in a common polyhedral complex. Replace every cell containing x by its local tangent cone at x, and freeze its weight at the value at x. The pointwise balancing equations become exactly the balancing equations for this constant-weight fan. A compatible complete fan is obtained by adding zero-weight cones and refining. For each prospective intersection face, the small-displacement incidence condition on the original cells becomes the displacement condition on these tangent cones. The local coefficient is therefore the constant-fan product coefficient. It is independent of the generic vector and of the chosen complete refinement by Theorem 15.6.1.

On any fixed face, this coefficient is a finite sum of products of restrictions of the original smooth weights, multiplied by constant lattice indices. It is therefore smooth on that face. At a lower-dimensional face, freeze the weights again at a point. The balancing equation for the product is the constant-fan balancing equation there, so it holds. The same procedure shows subdivision compatibility: both subdivisions give the same local fan product at every point.

Finally, the local fan of a product at x is the product of the local fans at x, with their frozen weights. Thus both bracketings of a triple product have the same coefficient by associativity of the constant-fan product. Swapping two

factors is treated in the same way. These are pointwise identities of smooth coefficient functions and hence identities of the weighted cycles. □

This argument uses values of weights, not derivatives of weights. It explains why stable intersection extends to smooth weights even though their integration currents need not be closed. Combining it with Theorem 15.3.1 proves representation independence of the resulting delta-preform algebra. The geometric and distributional independence statements are different steps, now linked by an explicit local construction.

15.6.2 Two displacement calculations

Example 15.6.3 (A tropical line intersects itself)**.** Let $L \subset \mathbb{R}^2$ consist of the rays generated by $e_1, e_2, -e_1 - e_2$, all with weight one. For $v = (a, b)$ with $0 < a < b$, the ray $\mathbb{R}_{\geq 0}e_2$ meets the translated diagonal ray $v + \mathbb{R}_{\geq 0}(-e_1 - e_2)$ at $(0, b - a)$. Checking the three pairs of coordinate and diagonal directions shows that this is the only transverse intersection of L and $L + v$. Its lattice index is $|\det(e_2, -e_1 - e_2)| = 1$. Replacing v by ϵv sends this point to the origin. Hence

$$L \cdot L = [0].$$

A straight tropical line has stable square zero, as in Proposition 15.4.1. The difference comes from the three-ray geometry, not from a formal rule about repeated supports.

Example 15.6.4 (Nonprimitive equations and weights)**.** Let $u = \max(0, ax)$ and $v = \max(0, by)$ for positive integers a, b. The supports of their divisors are the coordinate lines, but their weights are a, b. The determinant of the primitive line directions is one, so

$$\mathrm{d}'\mathrm{d}''[u] \wedge \mathrm{d}'\mathrm{d}''[v] = ab\delta_0.$$

Counting only the geometric intersection point loses both multiplicities. Conversely, using the nonprimitive vectors in the determinant and also retaining the weights would count them twice. Primitive directions and cycle weights must therefore be recorded separately.

References. The Stokes and corner-current framework is [GK17, Sections 2–3]. The toric input is [FS, Theorems 2.1 and 3.2]; the passage to smooth weights follows the local-fan construction of [GK17, Remark 1.4], with its pointwise steps written above.

15.7 The delta-algebra and intersection theory

15.7.1 Polyhedral curvature terms

For a smooth real function, the Hessian is a smooth $(1,1)$-form. For a PL function, the cellwise Hessian vanishes and the distributional Hessian is a corner current. A calculus limited to original smooth forms cannot treat that corner current as an ordinary smooth coefficient. GK enlarges the algebra so that certain polyhedral currents become legitimate differential objects with products.

The enlargement is controlled. A typical preform is a finite sum of smooth superforms multiplied by integration currents on weighted polyhedral cycles. Stable intersection supplies the product of the cycle factors. Differential operators have both cellwise and boundary parts. The functorial closedness conditions ensure that the construction survives refinements and pullbacks in the specified algebraic category.

This is why the detailed definition in the preceding sections is needed. Declaring every polyhedral current to be a form would not automatically make its derivatives remain in the class or make products independent of the presentation.

15.7.2 A local multiplication that ordinary distribution theory cannot supply

On $\mathbb{R}^2$, let $u = \max(0, x)$ and $v = \max(0, y)$. Their curvature currents are integration currents on the coordinate lines, with the corresponding normal differential degrees. The delta product is the point mass at the origin, with multiplicity one. Replacing y by $x+2y$ gives the determinant factor two after the correct lattice normalization.

If two divisors coincide, ordinary multiplication of their distributional densities is not defined by writing a square of a Dirac delta. Tropical stable intersection provides a geometric intersection operation in its admissible cycle class. The product belongs to that calculus because its construction includes a displacement or equivalent intersection-theoretic definition. This does not supply a product for every pair of distributions with the same supports.

15.7.3 The metric application

A model metric has PL local weights. The corner formula produces its curvature as a delta-form in the appropriate GK setting. Products of these Chern delta-forms can then be integrated and compared with model intersection numbers. Green currents and star products lead to local heights, with Poincaré–Lelong providing the divisor term.

The arithmetic importance is that the calculus retains lower-degree curvature, not only a final scalar intersection number. It gives local formulas for metric variation and intersections, while compatibility with the model theory verifies the normalization. The resulting local heights can then be assembled in global arithmetic settings with their own adelic hypotheses.

15.7.4 Mihatsch's analytic construction and the comparison boundary

Mihatsch develops delta-forms intrinsically on analytic spaces using compatible skeletal descriptions and a polyhedral-current calculus. The resulting trigrading records smooth bidegrees and polyhedral codimension. In the notation of that paper, the codimension-zero component is the piecewise smooth calculus, and the realization in CLD currents has an injectivity theorem under the stated analytic hypotheses.

These are useful improvements for actual representatives and higher-codimension Green identities. They should not be retroactively inserted into the original GK definition without a comparison argument. In particular, a property proved for the later analytic delta-sheaf does not resolve an original GK representability question solely because both are denoted by B in informal discussions.

The complete-intersection Green formula is a case where the distinction matters constructively. CLD25 states a current identity on original smooth tests. Mihatsch's theorem supplies an actual delta-form on the complement in its own category and an identity against the appropriate delta-form tests. The stronger representation and testing statement is additional mathematical content.

References. [GK17, Sections 2–6, 9–12]; [Mih24, Propositions 4.7, 4.11 and Theorem 5.5].

15.8 Products and tropical intersection in CLD25 Chapter 9

15.8.1 External products come before internal intersections

If S is a current on X and T a current on Y, their external product belongs on $X \times Y$. Formally it is obtained by integrating a test in one variable against one current and then against the other. To make this rigorous, CLD25 Chapter 9 develops integration with parameters and the continuity of the resulting test-valued operations. The construction is more general than merely writing a product of two scalar measures.

An internal intersection on X can then be related to the diagonal in $X \times X$. But pulling a current back to the diagonal is a singular restriction problem. A Gysin construction supplies such an operation in a controlled class; an arbitrary pair of distributions cannot be intersected just because their external product exists.

15.8.2 Why displacement appears

In a real vector space, translate one tropical cycle by a small generic vector. The transverse intersections have lattice-index multiplicities. Passing to the stable value gives the familiar tropical intersection. This can be interpreted through a family of currents and a diagonal or Gysin operation. The analytic formulation explains why the result is compatible with differential operations and why multiplicities are conserved under small displacements.

For two transverse lines in $\mathbb{R}^2$, the determinant computation in Chapter 5 is the entire local calculation. For nontransverse cycles, the displacement construction resolves an excess-dimensional intersection into weighted pieces of the expected dimension. The support of the result is a subset of the set-theoretic intersection, but the multiplicities require the displacement and lattice data.

15.8.3 Relation to the GK product

GK begins with polyhedral currents and stable intersections to construct a delta-algebra on analytic spaces. CLD25 develops product spaces, parameter-dependent integrals, and Gysin operations inside its abstract tropical frame-

work. These approaches have overlapping local models and related formulas. An identification on a chosen class still requires matching the calibration, the allowable generators, and the current realization. The existence of both products is not a proof that the entire algebras coincide.

This point will matter for the research questions in Chapter 24. Extending a geometric intersection product to arbitrary positive closed currents requires new analytic continuity and representation results. The tropical product explains the finite polyhedral case and provides examples, but it does not automatically supply all bounded-potential or unbounded-current products.

References. [CLD25, §§7.5–7.8, 8.10–8.11, Chapter 9]; [GK17, Sections 2–4].

15.9 Changing conventions without changing curvature

The six operators of Section 16.5 use Mihatsch's boundary convention. Cai's convention is the opposite. The current derivatives and the polyhedral derivatives agree, while

$$\partial'_{\mathrm{C}} = d' - d'_P = -\partial'_{\mathrm{M}}, \qquad \partial''_{\mathrm{C}} = d'' - d''_P = -\partial''_{\mathrm{M}}.$$

Here the subscripts label conventions, not different geometric operations. For a continuous piecewise smooth function f, both conventions give

$$\begin{aligned} d'd''f &= d'_P d''_P f + \partial'_{\mathrm{C}} d''_P f \\ &= d'_P d''_P f - \partial'_{\mathrm{M}} d''_P f. \end{aligned}$$

Thus the corner current is $+\partial'_{\mathrm{C}} d''_P f$ in Cai's notation and $-\partial'_{\mathrm{M}} d''_P f$ in Mihatsch's. For $f(t) = \max(0, t)$ it is δ_0, with positive coefficient in both descriptions. This example is a useful normalization test: changing the sign of the boundary operator must not change the sign of the Monge–Ampère mass.

References. [Mih23, Definition 3.4]; [Cai23, Theorem 4.25 and the conventions preceding Theorem 1.2].

15.10 Graph pullback and two pushforwards

Let $F : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ be an integral affine map. The graph construction defines a pullback of delta-forms even when F is not surjective. If p_1, p_2 are the projections from $N'_{\mathbb{R}} \times N_{\mathbb{R}}$ and Γ_F is the graph, the defining formula is

$$F^*\beta = (p_1)_*\big([\Gamma_F] \wedge ([N'_{\mathbb{R}}] \times \beta)\big).$$

The intersection on the right is the delta-product. It is not restriction of the coefficients of an arbitrary distribution. The projection has the required properness on the graph support, locally over the source. The construction preserves total bidegree and commutes with the wedge product and all six operators. These are theorems of the polyhedral calculus, not consequences of continuity of F.

Example 15.10.1 (A non-surjective pullback)**.** Let $i : \mathbb{R} \to \mathbb{R}^2$ be $t \mapsto (t, 0)$, and let $D_x = d'd'' \max(0, x)$ and $D_y = d'd'' \max(0, y)$. Then

$$i^*D_x = \delta_0, \qquad i^*D_y = 0.$$

Indeed pullback commutes with $d'd''$ and with piecewise linear functions. The pulled-back functions are $\max(0, t)$ and 0. Although $i(\mathbb{R})$ is contained in the support of D_y, its delta-pullback is zero. A set-theoretic inverse image of supports does not determine a pullback multiplicity.

Two operations are denoted by pushforward in the literature. The current pushforward F_* is defined by duality with smooth tests; it requires properness on support, or compact support in the version used in Cai's Theorem 5.13(3). It can integrate coefficient degrees along positive-dimensional fibres. The cycle-type pushforward $\widehat{F}_*$ instead preserves the dimension of the supporting cells. A cell on which dimension drops contributes zero. For an s-dimensional cell σ mapped injectively to an s-dimensional cell it is

$$\widehat{F}_*(\alpha_\sigma[\sigma]) = [N_{F(\sigma)} : dF(N_\sigma)]((F|_\sigma)^{-1})^*\alpha_\sigma\,[F(\sigma)].$$

Cai proves a projection formula for each operation with its own domain. The formula involving $\widehat{F}_*$ on the full affine spaces does not remove the support condition from the current pushforward. When F is injective on every supporting cell the two constructions agree.

Proposition 15.10.2 (The lattice factor in dimension one)**.** *For $F : \mathbb{R} \to \mathbb{R}$, $t \mapsto mt$ with a nonzero integer m, normalized using the standard lattices,*

$$F_*[\mathbb{R}] = |m|[\mathbb{R}], \qquad F_*\delta_0 = \delta_0, \qquad F^*\delta_0 = |m|\delta_0.$$

Proof. For a compactly supported top-degree test $a(y)d'y \wedge d''y$, its pullback is $m^2 a(mt)d't \wedge d''t$. Change of variables in the real integral gives the factor $m^2/|m| = |m|$. The second formula follows by evaluation on a scalar test. For the third, write $\delta_0 = d'd'' \max(0, y)$. The slopes of $\max(0, mt)$ have jump $|m|$ at zero. □

These formulas satisfy the projection formula: pushing $F^*\delta_0 \wedge [\mathbb{R}]$ forward gives $|m|\delta_0$, as does $\delta_0 \wedge F_*[\mathbb{R}]$. They also show why the mass of a point and the weight of a one-dimensional cycle must not be normalized by the same informal "degree rule". Passing from this affine construction to analytic spaces requires compatibility of charts and their test quotients. Cai's algebraic construction, GK's original sheaf and Mihatsch's analytic sheaf must therefore still be distinguished. None of the affine formulas alone supplies a trace of arbitrary actual delta-forms through every proper analytic map.

References. [Mih23, Theorem 4.1 and Proposition/Definition 4.2]; [Cai23, Proposition 5.4, Remark 5.5, Definition 5.6 and Theorem 5.13]. The displayed examples are direct computations.

Chapter 16

Analytic delta-forms and their positivity

An analytic domain, a closed analytic subspace and a scalar extension are different constructions; Chapter 2 supplies each of them. The quotient defining a differential object must respect precisely the maps admitted by its theory. In the original GK construction below, the universal quantifier ranges over algebraic morphisms and compatible charts. The later Mihatsch comparison uses analytic morphisms under its own hypotheses. The existence of the underlying analytic map alone does not prove that substitution respects a presentation quotient.

16.1 Delta-forms and delta-currents

Throughout this section, K is algebraically closed, complete, and nontrivially valued. Let X be an integral separated K-scheme of finite type, let $n = \dim X$, and let $W \subset X^{\mathrm{an}}$ be open. These are the hypotheses of the algebraic tropical-chart construction in [GK17, §4] and of the positivity results in [GK19, §0.1]. No hypothesis on the residue characteristic is imposed.

16.1.1 From tropical preforms to forms on the analytic space

Let N be a lattice of rank r, put $N_{\mathbf{R}} = N \otimes_{\mathbf{Z}} \mathbf{R}$, and let $\widetilde{\Omega} \subset N_{\mathbf{R}}$ be open. A tropical cycle with smooth weights is represented by a finite integral real-

affine polyhedral complex together with smooth weight functions m_σ on its maximal faces. For each codimension-one face τ and $x \in \tau$, balancing means

$$\sum_{\sigma \supset \tau} m_\sigma(x)\nu_{\sigma,\tau} \in N_{\tau,\mathbf{R}},$$

where $\nu_{\sigma,\tau}$ is a lift of the primitive outgoing normal and $N_{\tau,\mathbf{R}}$ is the tangent space of τ. If C has codimension ℓ, its integration current δ_C has bidegree (ℓ,ℓ). The weights and coefficients in the following definition are real.

The tropical preforms were defined in Chapter 15.

A tropical chart of X^{an} consists of a very affine open subvariety $U \subset X$, its canonical torus embedding $\varphi_U : U \hookrightarrow T_U$, and

$$V = \mathrm{trop}_U^{-1}(\Omega), \qquad \Omega = \widetilde{\Omega} \cap \mathrm{Trop}(U).$$

Write N_U for the cocharacter lattice of T_U. If $f : X' \to X$ is an algebraic morphism and $(V', \varphi_{U'})$ is a chart with $f(U') \subset U$ and $f^{\mathrm{an}}(V') \subset V$, the torus maps induce an integral affine map

$$F : N_{U',\mathbf{R}} \longrightarrow N_{U,\mathbf{R}}.$$

An ambient preform $\widetilde{\alpha}$ can be pulled back by F and then restricted to the tropical cycle $\mathrm{Trop}(U')$. Here restriction means the product with $\delta_{\mathrm{Trop}(U')}$.

Definition 16.1.1 (The functorial chart quotient)**.** Let $\mathcal{N}^{p,q}(V, \varphi_U)$ be the subspace of ambient preforms $\widetilde{\alpha} \in P^{p,q}(\widetilde{\Omega})$ such that

$$\left(F^*\widetilde{\alpha} \wedge \delta_{\mathrm{Trop}(U')}\right)\big|_{\mathrm{trop}_{U'}(V')} = 0$$

for every algebraic morphism $f : X' \to X$ and every compatible chart as above. Set

$$P^{p,q}(V, \varphi_U) = P^{p,q}(\widetilde{\Omega})/\mathcal{N}^{p,q}(V, \varphi_U).$$

This quotient is independent of the choice of $\widetilde{\Omega}$. It has pullbacks under compatible algebraic morphisms [GK17, Definition 4.4 and Remark 4.5]. The quantifier over all morphisms is part of the definition. It cannot be replaced by the requirement that the restriction to the one cycle $\mathrm{Trop}(U)$ vanish. Indeed, [GK17, Example 4.22] constructs a nonzero chart class whose restriction to that cycle is zero but whose pullback to a compatible chart is nonzero.

Definition 16.1.2 (Closed generators and delta-forms)**.** An element $\omega \in P^{p,q}(V, \varphi_U)$ belongs to $Z^{p,q}(V, \varphi_U)$ if, after every compatible pullback and restriction as in Definition 16.1.1, the resulting supercurrent is both d'-closed and d''-closed. Let $AZ(V, \varphi_U)$ be the algebra generated by $A^{\bullet,\bullet}(\Omega)$ and $Z^{\bullet,\bullet}(V, \varphi_U)$. Thus locally an element has the form

$$\beta = \sum_{j=1}^{s} \alpha_j \wedge \omega_j, \qquad \alpha_j \in A^{\bullet,\bullet}(\Omega), \quad \omega_j \in Z^{\bullet,\bullet}(V, \varphi_U).$$

Sheafifying the chart algebras P gives the sheaf of generalized delta-forms; sheafifying AZ gives the sheaf $B^{\bullet,\bullet}$ of delta-forms.

The universally closed condition makes

$$d'\beta = \sum_j d'\alpha_j \wedge \omega_j, \qquad d''\beta = \sum_j d''\alpha_j \wedge \omega_j$$

independent of the chosen expression. There are injective morphisms of sheaves

$$A^{\bullet,\bullet} \hookrightarrow B^{\bullet,\bullet} \hookrightarrow P^{\bullet,\bullet},$$

and B is a bigraded differential algebra [GK17, §4.6, Definitions 4.9 and 4.13, Proposition 4.15]. The differentials need not preserve the larger algebra P. This is the precise difference between a generalized delta-form and a delta-form in this construction.

16.1.2 The test topology and its dual

It is useful to spell out the continuity condition in the definition of a delta-current. On a fixed chart, freeze a finite tuple $\omega_J = (\omega_j)_{j\in J}$ of closed generators. Let $AZ(V, \varphi_U, \omega_J)$ consist of the sums $\sum_j \alpha_j \wedge \omega_j$. If $C \subset \Omega$ is compact and $s \geq 0$, write $p_{C,s}(\alpha)$ for the usual smooth coefficient seminorm, controlling derivatives up to order s. The quotient seminorm is

$$p_{C,s,\omega_J}(\beta) = \inf_{\beta = \sum_j \alpha_j \wedge \omega_j} \max_{j\in J} p_{C,s}(\alpha_j).$$

On a fixed chart covering of W, choose such tuples chart by chart and use the induced locally convex topology. Finally fix a compact subset of W containing the supports of the test forms.

Definition 16.1.3 (Delta-current)**.** A delta-current on W is a real linear functional on $B_c^{\bullet,\bullet}(W)$ whose restriction to every fixed-chart, fixed-generator, fixed-compact-support space just described is continuous. A delta-current of bidegree (p,q) acts on $B_c^{n-p,n-q}(W)$; its space is denoted $E^{p,q}(W)$.

This is [GK17, Definition 6.3 and §6.4], with bidegree indexing. The two constructions of currents fit into natural maps

$$P^{p,q}(W) \longrightarrow E^{p,q}(W) \longrightarrow D^{p,q}(W).$$

The first associates a functional by integration,

$$\langle [\alpha]_E, \beta \rangle = \int_W \alpha \wedge \beta, \qquad \beta \in B_c^{n-p,n-q}(W);$$

the second restricts a functional to the smaller space $A_c^{n-p,n-q}(W)$. In particular, enlarging the space of test forms reverses the direction of the comparison map on duals.

Proposition 16.1.4 (Functorial detection of a generalized delta-form)**.** *Let $\alpha, \beta \in P^{p,q}(W)$. Then $\alpha = \beta$ if and only if*

$$[f^*\alpha]_D = [f^*\beta]_D$$

on every open $W' \subset (X')^{\mathrm{an}}$ with $f^{\mathrm{an}}(W') \subset W$, for every algebraic morphism $f : X' \to X$.

Open proof obligation (RC2). Prove detection and its converse on actual tropical pullback tests; GK17 Proposition 6.8 supplies the criterion.

Remark 16.1.5 (Which operations have been constructed). There is a product $B^{a,b} \times E^{p,q} \to E^{a+p,b+q}$, and d', d'' act on E by transposition. Proper algebraic morphisms have pushforward on delta-currents [GK17, §6.4, §6.9 and §6.10]. These statements do not give a product of two arbitrary delta-currents or pullback of an arbitrary delta-current. Nor do the displayed comparison maps by themselves establish injectivity, surjectivity, or extension of every ordinary current to the larger test space.

16.1.3 Positivity and positive representatives

The adjectives strong, positive and weak refer to the three Lagerberg cones. They should be distinguished from the following two additional issues: which chart representatives are positive, and which test space is used by a current.

Definition 16.1.6 (Positive generalized delta-form)**.** A chart class $\beta \in P^{p,p}(V, \varphi_U)$ is positive if, for every compatible algebraic morphism and chart, its pulled-back restriction is a tropical delta-preform whose underlying supercurrent is positive. It is positively representable if it has a positive ambient representative $\widetilde{\beta} \in P^{p,p}(\widetilde{\Omega})$. A generalized delta-form on W has either property if the property holds locally on tropical charts.

There are corresponding strong and weak versions [GK19, Definitions 2.2 and 2.6]. Positive representability implies positivity, since ambient affine pullback and restriction to an effective tropical cycle preserve the relevant cones. The converse is not part of the definition or of this implication.

Definition 16.1.7 (Positive delta-current)**.** A symmetric $T \in E^{p,p}(W)$ is positive if $\langle T, \beta \rangle \geq 0$ for every positive $\beta \in B_c^{n-p,n-p}(W)$. It is weakly positive when tested against strongly positive forms, and strongly positive when tested against weakly positive forms.

Restriction of a positive delta-current is a positive ordinary current, because every positive smooth test form is a positive delta-form [GK19, Definition 2.10 and Remark 2.12]. For a generalized delta-form β, the statements proved in that paper are more specific:

$$\begin{aligned} \beta \text{ positively representable} &\implies [\beta]_E \text{ positive}, \\ \beta \text{ positive} &\iff [f^*\beta]_D \text{ positive for every algebraic } f. \end{aligned}$$

These are [GK19, Corollary 2.11 and Proposition 2.13]. In the second line the inverse image of W is the domain of the current. In the first line, positive representability is an actual hypothesis.

Remark 16.1.8 (The smooth case and the middle-degree caution). For a smooth form, positivity of the form and of either associated current agree [GK19, Remark 2.14]. In middle degree, one must not supplement this with the false assertion that strongly positive real Lagerberg forms span all symmetric forms. The counterexample in [BGJK, Example 2.3.6] also shows that the weak cone can contain a line. Thus [GK19, Remark 1.12], which repeats the older spanning assertion, is not a valid general-purpose nondegeneracy argument. In degree $(1, 1)$ the three cones agree, so this particular obstruction does not affect the curvature comparisons below.

16.2 Piecewise smooth weights and four plurisubharmonicity conditions

Definition 16.2.1 (The four conditions for a continuous weight)**.** Let $u : W \to \mathbf{R}$ be continuous.

1. u is current-psh for the original smooth test theory if $d'd''[u]_D$ is positive.
2. u is delta-psh if $d'd''[u]_E$ is positive.
3. u is functorial current-psh if $u \circ f^{\mathrm{an}}$ is current-psh on $(f^{\mathrm{an}})^{-1}(W)$ for every algebraic morphism $f : X' \to X$.
4. u is functorial delta-psh if the analogous condition holds with delta-psh in place of current-psh.

These are the conditions in [GK19, Definition 3.1 and §3.4]; the explicit label *current-psh* records which of the several uses of psh is intended. For a continuous metric on a line bundle L, the condition is imposed on each weight $-\log \|s\|$, where s is a nowhere vanishing local algebraic frame. The ordinary or delta Poincare–Lelong formula for invertible functions makes the condition independent of the frame [GK19, §3.5].

Proposition 16.2.2 (Implications requiring no comparison theorem)**.** *For continuous weights and for continuous metrics,*

$$\textit{functorial delta-psh} \implies \textit{delta-psh} \implies \textit{current-psh},$$
$$\textit{functorial delta-psh} \implies \textit{functorial current-psh} \implies \textit{current-psh}.$$

Proof. Use the identity morphism to remove the functorial quantifier. For the remaining implication, restrict the positive delta-current to smooth test forms. Apply the same argument after every morphism for the functorial version. □

The proposition states implications, not their strictness. In particular, it is not a counterexample to the converse of any arrow. For smooth metrics the arrows collapse to equivalences:

$$c_1(L, \|\cdot\|) \geq 0 \text{ as a smooth form}$$
$$\iff \text{current-psh} \iff \text{functorial delta-psh}.$$

This is [GK19, Corollary 8.9]; it also includes functorial current-psh, and delta-psh follows by Proposition 16.2.2.

Definition 16.2.3 (Delta-metric)**.** A piecewise smooth metric is a delta-metric if every point admits a tropical frame with weight φ for which the facewise $(1,1)$-form $d'_P d''_P \varphi$ is the restriction of an ambient superform with piecewise smooth coefficients. For such a metric, there is a unique $\gamma \in B^{1,1}(W)$ whose associated delta-current, after every algebraic pullback, is the curvature delta-current of the pulled-back metric. This γ is denoted $c_1(L, \|\cdot\|)$.

The local criterion and its equivalence with the functorial curvature representation are [GK17, Definition 9.9, Proposition 9.11 and Definition 9.12]. The existence of one associated curvature current does not by itself supply this stronger representation. For a delta-metric, the statement in [GK19, Proposition 8.8] is the narrower equivalence

$$c_1(L, \|\cdot\|) \geq 0 \text{ as a delta-form} \iff \text{functorial current-psh.}$$

16.2.1 The Hessian term and the corner term

Let C be a tropical cycle of pure dimension d with positive constant weights. Suppose φ is a continuous piecewise smooth function on an open subset of $|C|$. Choose a subdivision on whose faces it is smooth. The operators d'_P, d''_P differentiate on those faces; they do not include jumps of the first derivatives. The tropical Poincare–Lelong formula is

$$d'd''[\varphi] = [d'_P d''_P \varphi] + \delta_{\varphi \cdot C}. \tag{16.2.1}$$

Here $\varphi \cdot C$ is the corner cycle [GK17, Corollary 3.19].

For precision, at a codimension-one face τ choose primitive outgoing normal lifts $\nu_{\sigma,\tau}$ for the maximal faces σ containing τ. Balancing says that $\nu_\tau = \sum_{\sigma \supset \tau} m_\sigma \nu_{\sigma,\tau}$ is tangent to τ. The corner weight at $x \in \tau$ is

$$m_{\varphi \cdot C, \tau}(x) = \sum_{\sigma \supset \tau} m_\sigma \, d\varphi_\sigma(x)(\nu_{\sigma,\tau}) - d\varphi_\tau(x)(\nu_\tau).$$

Changing a normal lift by a vector tangent to τ changes the two terms by equal amounts. This proves independence of the lifts. The full construction and its invariance under subdivision are given in [GK17, §1.10 and Proposition 1.12].

Theorem 16.2.4 (Piecewise smooth criterion)**.** *Let $\|\cdot\|$ be a piecewise smooth metric on $L|W$. On a tropical frame write*

$$-\log\|s\| = \varphi \circ \mathrm{trop}_U,$$

where the tropical domain meets each face convexly. Then the metric is current-psh if and only if, on every such frame, the following hold:

1. *the restriction of φ to each maximal tropical face is convex;*
2. *the corner cycle $\varphi \cdot \mathrm{Trop}(U)$ is effective.*

It suffices to verify the conditions on a basis of tropical frames. Equivalently, after a subdivision, the first condition can be replaced by smooth convexity on the maximal pieces, while retaining the corner condition.

Open proof obligation (RC2). Prove the piecewise-smooth corner-current criterion, including its precise conditions; GK19 Theorem 8.4 and Remark 8.5(a) are the source inputs.

Example 16.2.5 (A corner invisible to the facewise Hessian)**.** For $\varphi(t) = |t|$ and $\eta \in C_c^\infty(\mathbf{R})$, integration by parts separately on the two half-lines gives

$$\langle d'd''[\varphi], \eta\rangle = \int_{\mathbf{R}} |t|\eta''(t)\,dt = 2\eta(0).$$

Thus $d'd''[|t|] = 2\delta_0$, although the facewise Hessian is zero. The sign is reversed for $-|t|$. This atomic curvature is represented by a delta-preform, but not by a smooth $(1,1)$-form.

Example 16.2.6 (The tropical tripod)**.** For $X = \{x_1 + x_2 = 1\} \subset \mathbf{G}_m^2$, the tropicalization has three rays of weight one with primitive outgoing vectors $(1,0), (0,1), (-1,-1)$. Let φ be conic, with respective outgoing slopes a, b, c, and put $u = \varphi \circ \mathrm{trop}$. Every edge restriction is linear. Nevertheless a nonnegative test function η, equal to one at the vertex, gives

$$\langle d'd''[u], \mathrm{trop}^*\,\eta\rangle = a + b + c.$$

In particular, $(a,b,c) = (-1,0,0)$ disproves the claim that facewise convexity alone implies current-psh. This is [GK19, Example 3.3]. The choice $(1,1,-2)$ has zero corner curvature: nonnegativity of each individual outgoing slope would be an unnecessarily strong test.

Proposition 16.2.7 (The precise graph criterion)**.** *Let φ be continuous and piecewise C^2 on a one-dimensional balanced tropical cycle, viewed as a metric graph with positive constant edge weights m_e. Parametrize each edge by a coordinate t in a primitive integral direction; all outgoing derivatives below use this lattice-normalized parametrization. Against restrictions of smooth ambient test functions supported away from the boundary, its curvature is*

$$\Delta\varphi = \sum_e m_e \varphi_e''(t)\, dt + \sum_v \left(\sum_{e \ni v} m_e \partial_{\nu_e} \varphi(v) \right) \delta_v,$$

where the vertices include the breakpoints of the chosen subdivision. It is positive precisely when the densities and the vertex coefficients in this formula are all nonnegative.

Proof. Integrate by parts twice on every edge. Continuity and the balancing condition cancel the terms involving derivatives of the test function at vertices. The remaining boundary terms are the weighted outgoing derivatives of φ. The density measures and the point masses are mutually singular; their positivity can therefore be tested separately. □

Remark 16.2.8 (A scope correction to a graph formulation). The slope-only wording of [GK19, Example 8.6] is correct for piecewise linear functions. For piecewise smooth functions it must retain convexity on the open edges. Indeed, $\varphi(t) = -t^2$ on an open interval has sum of the two outgoing first derivatives equal to zero at every point, while $\varphi'' = -2$. Proposition 16.2.7 supplies the precise replacement.

16.3 Why the universal quotient is necessary

The definitions at the start of this chapter build functoriality into the null space. We give the arguments which this choice makes possible. They also identify the remaining geometric input: pullback and intersection of ambient tropical preforms must already be constructed, with their projection and closedness properties. The following algebra does not replace those results.

Fix a chart (V, φ_U) in the algebraic category of Section 16.1. For a compatible map f and chart upstairs, write $R_f(\widetilde{\alpha})$ for $F^*\widetilde{\alpha} \wedge \delta_{\mathrm{Trop}(U')}$. Thus the null space consists of preforms annihilated by every R_f. The identity map is among these maps, but it is not the only one.

Proposition 16.3.1. *The universal null spaces are stable under compatible pullback. Consequently pullback of a chart class is independent of its ambient representative and is compatible with composition.*

Proof. Suppose $\widetilde{\alpha}$ is null on V, and pull it back along $f : X' \to X$. To show that $F^*\widetilde{\alpha}$ is null upstairs, test it after an arbitrary $g : X'' \to X'$ and compatible chart. The resulting expression is the restriction of $(F \circ G)^*\widetilde{\alpha}$ to the tropical cycle of that chart. It is zero because the composite $f \circ g$ is one of the maps occurring in the original null condition. Thus the pullback descends. Composition and the identity rule hold before quotienting, and hence hold afterward. □

A condition using only R_{id} would not support this proof: its hypothesis contains no information about the composite map. This is the precise reason for the quantifier over morphisms. The product on the quotient additionally uses the ideal property of tropical restriction, proved by the ambient intersection theory. Vanishing as a distribution on one support is not a license to multiply arbitrary distributions.

Proposition 16.3.2. *Let $\beta = \sum_i \alpha_i \wedge \omega_i$ with ω_i universally closed. Then the classes $\sum_i d'\alpha_i \wedge \omega_i$ and $\sum_i d''\alpha_i \wedge \omega_i$ depend only on β. They give differentials satisfying $d'^2 = d''^2 = 0$ and $d'd'' + d''d' = 0$.*

Proof. It suffices to consider an expression for zero. After every compatible pullback and tropical restriction, it is a zero current. Differentiate that equality. The smooth-current Leibniz rule and closedness of every restricted generator identify its derivative with the restriction of $\sum_i d'\alpha_i \wedge \omega_i$. This vanishes after all maps and hence represents zero in the universal quotient. The same argument applies to d''. The three differential identities now follow by applying the corresponding identities to the smooth coefficients in one expression. □

The proof is a relation argument, not merely a rule for differentiating a preferred representative. It is also why requiring closedness only on the original chart would be insufficient. Universal closedness is preserved by pullback through the same composition argument as Proposition 16.3.1. Closure under products uses GK's tropical product theorem for closed preforms. With that input, the graded Leibniz rule descends and gives the full differential algebra.

16.3.1 The nodal example and what it detects

GK17 Example 4.22 uses an integral nodal cubic over $\mathbb{C}_p$, with $p \neq 2, 3$, and its normalization. The construction supplies a chart whose tropical image near a vertex has four outgoing branches. A PL potential has cancelling total slope at the vertex, so its corner current on that single tropical image is zero. After pullback to a chart of the normalization, the potential is zero on one interval and has slope one on the next; its corner current is a unit atom.

The underlying cubic is $xy + px^3 + py^3 = 0$. Its normalization is parametrized on its affine part by

$$x = -\frac{u}{p(1+u^3)}, \qquad y = -\frac{u^2}{p(1+u^3)}.$$

Substitution verifies the equation: the three numerators after multiplication by $p^2(1+u^3)^3$ are $u^3(1+u^3)$, $-u^3$, and $-u^6$. The points $u = 0, \infty$ map to the node. A regular unit on the nodal curve pulls back to a unit with equal values at these two preimages, which explains why the canonical tropical coordinates downstairs can identify data separated by normalization. The complete skeleton and unit-group computation is the geometric part of the cited example.

The conclusion is precise: restriction to one tropical image does not detect equality of chart classes. It is not, by itself, a counterexample to injectivity of the map from delta-forms to ordinary analytic currents. An ordinary analytic test may use a refined moment map; testing on one tropical image is less information than testing on every actual smooth analytic form. Proposition 16.1.4 avoids this ambiguity by using every algebraic pullback and all ordinary tests there.

16.4 Mihatsch's skeleton definition and its comparison maps

Let X be a pure-dimensional Hausdorff good analytic space without analytic boundary, over a complete valued field. Mihatsch permits trivial valuation; statements involving weakly smooth forms additionally use their nontrivial-valuation hypotheses. The data of an actual delta-form consist of compatible polyhedral currents on all admitted skeleta, locally obtained from one ambient delta-form through toric coordinates. Compatibility under adding coordinates is part of the definition.

There are two levels of degree. A term $\alpha \wedge [\sigma]$ with coefficient bidegree (p, q) and support codimension c has tridegree (p, q, c) and total bidegree $(p + c, q + c)$. Thus a PL function has tridegree $(0, 0, 0)$, while its pure corner curvature has tridegree $(0, 0, 1)$ and total bidegree $(1, 1)$. Confusing coefficient degree and total degree gives incorrect test degrees in Green identities.

Proposition 16.4.1 (The coefficient-codimension-zero comparison)**.** *In Mihatsch's analytic setting, piecewise smooth forms identify with the codimension-zero part: $PS_X^{p,q} \simeq B_X^{p,q,0}$. Original smooth forms embed in it. Under the hypotheses for the weakly smooth theory, weakly smooth forms also embed in this piecewise smooth sheaf.*

Reduction used in the comparison. The local presentation in codimension zero gives surjectivity: its coefficients are piecewise smooth forms in the toric coordinates. For injectivity, the top-degree case is decisive. If a piecewise smooth (n, n)-form represents zero on the calibrated tropical part, its remaining possible support lies in lower-dimensional pieces and in the image of the analytic boundary of a compact chart. Those pieces have dimension at most $n-1$ in the vanishing criterion of Mihatsch Proposition 4.7. An (n, n)-form restricts to zero on every such tangent space. It is therefore zero as a piecewise smooth form. For general (p, q), wedge with every complementary piecewise smooth form and apply the top-degree statement. The nondegeneracy of local wedge detection, proved for the analytic presentation spaces in the cited comparison, forces the original form to vanish. Finally harmonic coordinate functions are locally PL, so pullback of smooth superforms by them is piecewise smooth. The relation and injectivity statements for that inclusion are those of the weakly smooth construction. □

This proof records the reduction and its geometric inputs; it does not replace the analytic wedge-detection theorem by dimension counting alone. On an abstract tropical space with a prescribed sheaf of linear functions, the analogous inclusion into all piecewise smooth forms can be strict. The analytic local-presentation theorem is therefore substantive.

16.4.1 Which differential is preserved

On piecewise smooth forms, the facewise operators are d'_P, d''_P. The full current operators d', d'' also contain boundary terms. Mihatsch uses the convention

$$\partial' = d'_P - d', \qquad \partial'' = d''_P - d''.$$

Thus it is not correct to say that inclusion of all piecewise smooth forms preserves their facewise differential as the full differential. On original smooth and weakly smooth forms the boundary operators vanish, which is precisely what makes those smaller inclusions compatible with the full calculus. For $|t|$ on $\mathbb{R}$, the facewise second derivative is zero, while the full second derivative is $2\delta_0$; the codimension-one term must occur in the delta-algebra.

16.4.2 Pullback is a theorem about relations

Mihatsch Theorem 4.5 proves pullback for every analytic morphism between pure-dimensional boundaryless spaces. Locally it is the expected substitution $t_f^\star\gamma \mapsto t_{f\circ\pi}^\star\gamma$. The substantive assertion is that a zero presentation remains zero after substitution. The proof reduces near Abhyankar points to a map of constant fibre dimension followed by a closed immersion. For the former it uses compatible flat maps of skeleta; for the latter it uses the tropical product description near a Cartier divisor. The regular-immersion geometry in Chapter 22 explains this second mechanism.

This gives actual pullback, including restriction to a closed analytic subspace in the stated category. It does not supply arbitrary actual pushforward. A proper map always suggests a transpose operation on currents, but the resulting current need not have local ambient presentations of the required kind. Mihatsch Example 2.45(3) explicitly notes that even a balanced polyhedral pushforward need not be a delta-form on the target tropical space. Additional representation control is necessary.

In the common algebraic category, Mihatsch also states an embedding of original GK delta-forms into his sheaf, specifying its action on ambient generators; the source states this comparison without a formal proof. We retain that status. The proved piecewise-smooth comparison above does give an inclusion of weakly smooth forms into Mihatsch's algebra. It does not give an inclusion of weakly smooth forms into the original GK algebra. This distinction refines, rather than reverses, the comparison table in Chapter 18.

References. [GK17, Definitions 4.4 and 4.9, Propositions 4.7–4.15, Example 4.22, Proposition 6.8]; [Mih24, Proposition 2.38, Example 2.45, Theorem 4.5, Proposition 4.7 and Section 4.2].

16.5 The six differential operators and their signs

The full derivative of a delta-form has a cellwise part and a boundary part. The distinction is already visible for a PL function, but becomes more important when products have several support codimensions. We use Mihatsch's convention, in which the boundary operator is the cellwise derivative minus the current derivative.

$$\partial' = d'_P - d', \qquad \partial'' = d''_P - d'', \qquad d' = d'_P - \partial', \quad d'' = d''_P - \partial''.$$

For coefficient bidegree (a, b) and support codimension c, the total bidegree is $(a + c, b + c)$. The homogeneous degree changes are

operator	coefficient/codimension change	total change
d'_P	$(1, 0, 0)$	$(1, 0)$
d''_P	$(0, 1, 0)$	$(0, 1)$
∂'	$(0, -1, 1)$	$(1, 0)$
∂''	$(-1, 0, 1)$	$(0, 1)$.

An operator landing in a negative coefficient degree is zero. Thus the first derivatives of a continuous piecewise smooth scalar function are cellwise derivatives. Its second derivative can nevertheless acquire a boundary term.

Proposition 16.5.1. *On the delta-algebra the following identities hold:*

$$(d'_P)^2 = (d''_P)^2 = (\partial')^2 = (\partial'')^2 = 0,$$
$$d'_P\partial' + \partial' d'_P = 0, \qquad d''_P\partial'' + \partial'' d''_P = 0,$$
$$d'_P d''_P + d''_P d'_P = 0, \qquad \partial'\partial'' + \partial''\partial' = 0,$$
$$d'_P\partial'' + \partial'' d'_P + d''_P\partial' + \partial' d''_P = 0.$$

The last identity does not assert that each of its two mixed anticommutators vanishes separately.

Proof. The cellwise identities hold on each smooth coefficient form. Expand $(d')^2 = (d'_P - \partial')^2 = 0$ on a trihomogeneous delta-form. Its three terms have support codimension changes zero, one and two. The direct trigrading makes each codimension part vanish separately, giving $(\partial')^2 = 0$ and the primed mixed identity. Repeat for d''. Expanding $d'd'' + d''d' = 0$ and separating the same three codimension changes gives the remaining identities. The codimension-one part contains both mixed anticommutators, with the same sign; there is no further grading that separates them. □

Example 16.5.2. For $f(t) = |t|$ on $\mathbb{R}$, one has $DDf = 2\delta_0$ and $d'_P d''_P f = 0$. Since $\partial' f = \partial'' f = 0$,

$$(d''_P\partial' + \partial' d''_P)f = -2\delta_0, \qquad (d'_P\partial'' + \partial'' d'_P)f = 2\delta_0.$$

Thus the two anticommutators cancel but are individually nonzero. Treating all six operators as pairwise anticommuting would erase exactly the corner curvature that the delta-algebra was designed to retain.

16.5.1 The curvature of a piecewise smooth function

For any continuous piecewise smooth function f, the degree table gives $d'' f = d''_P f$ and hence

$$DDf = d'_P d''_P f - \partial' d''_P f.$$

The first term has tridegree $(1, 1, 0)$; it records the Hessian on the cells. The second has tridegree $(0, 0, 1)$; it records the slope mismatch along codimension-one faces. For a PL function the first term vanishes, so the entire curvature is the negative of the displayed boundary derivative. This is compatible with the positive corner mass of a convex maximum.

If f is harmonic in an enlarged coordinate system, this conclusion concerns its full curvature. It cannot be read as saying that every cellwise Hessian or every individual normal slope is zero. The compatibility theorem for that coordinate system identifies the cancellations in the full operator. This is another reason to retain the operator notation when comparing original smooth, weakly smooth and piecewise smooth forms.

16.6 Codimension as a filtration of the differential complex

Fix a total first degree p. In the complex $(B^{p,\bullet}, d'')$, put $F^c B^{p,q}$ equal to the sum of components of support codimension at least c. Both terms of $d'' = d''_P - \partial''$ preserve this decreasing filtration: the first keeps codimension fixed and the second raises it by one. The filtration is finite in each degree.

Proposition 16.6.1. *The differential on the associated graded complex is d''_P. For a d''_P-closed homogeneous term α of codimension c, the first obstruction to correcting it to a full d''-closed term is the class of $\partial''\alpha$ in the d''_P-cohomology of codimension $c+1$.*

Proof. Modulo F^{c+1} the boundary derivative disappears, proving the first assertion. To make $\alpha + \beta$ closed modulo codimension $c+2$, where β has codimension $c+1$, one must solve

$$d''_P \beta = \partial'' \alpha.$$

The right side is d''_P-closed by Proposition 16.5.1. Its class is therefore the obstruction to this equation. Replacing α by $\alpha + d''_P \gamma$ changes the right side by $-d''_P \partial'' \gamma$, so the obstruction depends only on the associated-graded cohomology class. □

Further corrections can have further obstructions even though the differential has only these two codimension pieces. The standard spectral sequence of a finite filtered complex organizes those successive equations; its existence is not a statement that they all vanish. For a reference calculation, the explicit first obstruction is often more useful than an unsupported claim that cellwise closedness is sufficient. It explains algebraically how a facewise primitive can fail to be a primitive in the actual delta-calculus.

References. The operator convention and trigrading are [Mih24, Definition 2.41 and Section 4.2]. The identities and first-obstruction calculation are consequences of that trigrading and the current differential identities.

16.7 Polyhedral Monge–Ampère theory beyond the toric case

The toric theory of [BGJK25] uses a global convex model. On a balanced polyhedral space there need not be a single ambient convex function representing every positive potential. Botero–Mazzon–Pille-Schneider therefore start with polyhedral convexity. A function is polyhedrally convex, abbreviated PC, when it satisfies the convexity inequality for the polyhedral convex combinations allowed by the space. For a piecewise affine function this is equivalent to nonnegative intersection with every local positive polyhedral cycle. Positivity against the fundamental cycle alone is a weaker condition. Fix a piecewise affine reference function γ and its growth class. The regularizable class $\mathrm{PC}_{\mathrm{reg}}(X,\gamma)$ consists of uniform decreasing limits of piecewise affine PC functions with growth $\gamma + O(1)$. Their Theorem 4.13 extends the mixed polyhedral Monge–Ampère operator to this class, with symmetry, positivity, fixed total mass and integration by parts. The measure is initially defined on their compactification; positivity of γ at infinity ensures that it is supported on X. This is a construction in a specified polyhedral function class. It does not identify that class with $\mathcal{U}_X$ on an arbitrary Berkovich space.

16.7.1 A continuous obstacle for which orthogonality fails

The following computation makes the obstruction to a general envelope argument explicit. It also corrects the negative-axis branch printed in Example 6.23 of [BMP26]. Equip the union X of the two coordinate axes in $\mathbb{R}^2$ with weight one on its four rays, and put $\gamma = |x| + |y|$ on X.

Proposition 16.7.1. *For $\epsilon > 0$, define the continuous obstacle*

$$\varphi(x,0) = |x| + \epsilon, \qquad \varphi(0,y) = \big||y| - \epsilon\big|.$$

Its PC envelope with growth bounded above by $\gamma + O(1)$ is

$$h(x,0) = |x|, \qquad h(0,y) = \max(0, |y| - \epsilon).$$

Moreover, with the polyhedral Monge–Ampère normalization of the slope formula,

$$\mathrm{MA}_{\mathrm{poly}}(h) = 2\delta_{(0,0)} + \delta_{(0,\epsilon)} + \delta_{(0,-\epsilon)}, \qquad \int_X (\varphi - h)\,\mathrm{MA}_{\mathrm{poly}}(h) = 2\epsilon > 0.$$

Proof. The function h is the restriction of the ambient convex function $\max\{|x|, |y|-\epsilon\}$. It is piecewise affine PC, lies below φ, and differs boundedly from γ. Thus it is an admissible competitor. Conversely, let v be any PC competitor. Its restriction to each entire axis is convex: the usual convex combinations on a line are polyhedral convex combinations after subdivision at the point in question. This first gives the assertion at rational points, and then at all points by continuity of PC functions. Since $v(0,\epsilon)$ and $v(0,-\epsilon)$ are at most zero, convexity gives $v(0,0) \leq 0$ and $v(0,y) \leq 0$ for $|y| \leq \epsilon$. Choose C with $v \leq \gamma + C$. For $0 < t < R$ on either horizontal ray, convexity between 0 and R gives $v(t,0) \leq t + tC/R$; letting $R \to \infty$ gives $v(t,0) \leq t$. On a vertical ray, interpolate instead between ϵ and R. For $t > \epsilon$ this gives

$$v(0,\pm t) \leq \frac{t-\epsilon}{R-\epsilon}(R+C) \longrightarrow t - \epsilon.$$

Hence $v \leq h$, proving the envelope identity. The outgoing slopes of h at the origin are $1, 1, 0, 0$; the slope jumps at $(0, \pm\epsilon)$ are one. There are no other corners. This proves the measure formula. The obstacle gap is ϵ at the origin and zero at the other two atoms, giving the integral. □

In the source notation the envelope is $P_\gamma(\varphi - \gamma) = h$, an unshifted function with growth $\gamma + O(1)$. The outside branch $|y - \epsilon|$ printed in the cited example has left limit 2ϵ at $y = -\epsilon$, whereas the middle branch has value zero. Replacing that outside branch by $|y| - \epsilon$ gives exactly the continuous obstacle above. The failure of orthogonality survives this repair. The total mass is four, equal to the total outgoing slope of γ. Thus positivity, correct mass, regularizability and an explicit envelope do not imply orthogonality. In particular the variational implication in [BMP26, Proposition 6.12] must retain both regularity and orthogonality hypotheses, as well as the prescribed mass of the measure. This example concerns the PC class on the balanced cross; it is not a counterexample to the non-Archimedean envelope theorem in Section 19.19.

16.8 Local geometry in tropical Calabi–Yau problems

Amini–Piquerez–Wu work with compact prekähler tropical varieties and locally convex potentials. Their framework contains a background semipositive form ω, and a Kähler form θ used for approximation. Theorem 5.13

of [APW26] has two different conclusions: continuous θ-psh functions admit uniform PL approximations in the same class; an ω-psh function admits decreasing PL approximations in the enlarged classes $\omega + \alpha_j\theta$, with $\alpha_j \downarrow 0$. When the limit is continuous, the latter convergence is uniform by Dini's argument. It is essential to retain the varying background form in the second statement. Their existence and nonexistence results distinguish local models. A point is regular in codimension one in their terminology if a neighbourhood has a tropical chart in $L \times \mathbb{R}^{d-1}$, with L a generalized tropical line. If every point of the open stratum has this property, Theorem 10.4 gives their tropical Calabi–Yau theorem. This phrase is a local chart condition at every point; it does not mean only that a singular set has codimension at least two. The obstruction in Theorem 11.4 is also precise. At a codimension-two face, let Z be the two-dimensional star fanfold and let E be its linear span, of dimension $r \geq 3$. Condition (G) requires that the image of Z in E/F have dimension two for every $(r-2)$-dimensional subspace F. If one such star satisfies (G), their universal tropical Calabi–Yau assertion fails for every strongly semipositive background form. This is a failure of surjectivity of the Monge–Ampère operator onto all admissible measures, not a claim that no equation has any solution.

16.8.1 What the analytic comparison additionally requires

Theorem 12.11 of the same preprint works over $K = \mathbf{C}((t))$ with a smooth projective variety and starts with an existing bounded tropical solution in its toric embedding setup. It assumes a full-dimensional SNC dual complex, strong faithfulness relative to that complex, and a probability measure on it absolutely continuous with respect to its top-dimensional Lebesgue measure. Under these hypotheses the tropical potential extends to a continuous non-Archimedean psh potential with the prescribed Monge–Ampère measure, and has the paper's comparison property. Ordinary injectivity of one tropical chart is not the strong faithfulness assumption, and the theorem does not solve the tropical existence problem that precedes it. The mixed calculus cited there from the companion work is retained as an external dependency. The results just described are source theorems, not independently reconstructed proofs of the companion theory. They supplement the local CLD theory by global positivity and envelope geometry; they do not replace the test-space comparisons required for actual Green forms.

References. [BMP26, Definitions 3.1–3.3, Theorem 4.13, Proposition 6.12 and Example 6.23]; [APW26, Theorems 5.13, 10.4, 11.4 and 12.11].

Chapter 17

Harmonic coordinates and weakly smooth forms

17.1 Harmonic coordinates: the geometric obstruction to original smoothness

17.1.1 Numerical triviality and principal triviality differ

An original affine coordinate is built from logarithms of analytic units. Its residual line-bundle data is therefore principal in the appropriate local description. A numerical harmonic function only asks the associated residual class to be numerically trivial. A degree-zero line bundle on a residual curve need not be trivial or torsion. This difference can prevent a numerical harmonic function from being original smooth.

On a curve whose residue field is algebraic over a finite field, torsion properties of residual degree-zero classes improve the comparison. The relevant statement in GJR is a curve result. It should not be generalized to every dimension over a finite extension of $\mathbb{Q}_p$ merely because the residue field is finite.

17.1.2 What the enlarged sheaf is intended to repair

If harmonic functions are geometrically natural but unavailable as original smooth coordinates, differential constructions based only on analytic units may be too restrictive. GJR allows harmonic tropicalizations and pulls back

real smooth superforms through them. The resulting weakly smooth sheaf contains the original one and supports a differential calculus and integration theory.

The adjective weakly smooth refers to the enlarged coordinate class; it has no connection with weak positivity. Its continuous dual consists of strong currents in the terminology of that paper; again, this says which tests are allowed, not which positive cone is used. The detailed constructions and the direction of the restriction map follow in the next section.

17.1.3 The curve cohomology calculation

Gubler–Jell–Rabinoff also construct a compatible differential theory on weighted metric graphs and use it to calculate weakly smooth Dolbeault cohomology of compact Berkovich curves. For a connected compact curve without boundary in their setting, the dimensions satisfy

$$h^{0,0} = h^{1,1} = 1, \qquad h^{1,0} = h^{0,1} = b_1(X^{\mathrm{an}}).$$

The integer on the right is the first Betti number of the underlying analytic topological space. It is not in general the algebraic genus. A smooth proper curve with good reduction can have positive algebraic genus and $b_1(X^{\mathrm{an}}) = 0$.

This is an important warning for the proposed passage from tropical Hodge theory to Berkovich Hodge theory: one must specify which cohomology is being compared. Residual algebraic geometry is not automatically encoded in the first Betti number of a skeleton. A statement resembling a Hodge diamond may concern tropical Dolbeault groups rather than algebraic de Rham cohomology.

References. [GJR, Sections 7, 10, 15–17]; [GJRc, Theorems 6.4.3 and 6.4.5].

17.2 Harmonic tropicalizations and weakly smooth forms

Throughout this section, K is complete and nontrivially valued, and X is a good strictly K-analytic space. The version of Gubler–Jell–Rabinoff cited here is arXiv:2111.05741v3, dated 7 March 2025. The adjective *weakly* in *weakly smooth* concerns the allowed coordinate functions. It has no connection with the distinction between weakly, strongly, and ordinarily positive forms.

17.2.1 From analytic units to numerically trivial models

A smooth tropicalization is locally a map

$$F = (-\log|f_1|, \dots, -\log|f_r|)$$

defined by invertible analytic functions. To describe the enlargement of this class of maps, we first specify the model-theoretic meaning of harmonicity.

Definition 17.2.1 (Local model functions and harmonicity)**.** Let $x \in X$. An integral model function near x has the form

$$h = -\log\|1\|_{\mathfrak{L}}$$

on an analytic neighbourhood V, where $\mathfrak{V}$ is a formal K°-model of V and $\mathfrak{L}$ is a model of $\mathscr{O}_V$, with its specified generic-fibre trivialization. An $\mathbb{R}$-model function is locally a finite real linear combination of such functions.

An $\mathbb{R}$-model function h is *semipositive at* x if it can be represented on a neighbourhood of x by an $\mathbb{R}$-model line bundle whose restriction to the special fibre is nef. It is *harmonic at* x if both h and $-h$ are semipositive there. These conditions are imposed at every point to obtain semipositivity or harmonicity on X.

Here nefness means nonnegative degree on every proper integral curve in the special fibre. An $\mathbb{R}$-model line bundle means a finite real linear combination of integral model line bundles, with degrees extended linearly. Proposition 7.1 and Definition 7.2 of [GJR] identify this definition with the corresponding condition on the residue class $L_h(x)$ on the reduction of the analytic germ (X, x). In particular,

$$h \text{ is harmonic at } x \quad \Longleftrightarrow \quad L_h(x) \text{ is numerically trivial.}$$

Numerical triviality is the condition $\deg(L_h(x)|_C) = 0$ on the relevant proper curves. It does not assert that $L_h(x)$ is trivial or torsion.

Proposition 17.2.2 (Elementary direction and functoriality)**.** *If f is an invertible analytic function, then $\log|f|$ is harmonic. Harmonic functions are stable under pullback by analytic morphisms and under extension of the ground field. Every real harmonic function is locally a finite real linear combination of integral harmonic functions.*

Open proof obligation (RC2). Prove the nontrivial harmonic and PL assertions currently assigned to GJR Propositions 7.6 and 7.5.

Remark 17.2.3 (The relevant topology). The PL functions form a sheaf for the analytic G-topology. Harmonicity and semipositivity in Definition 17.2.1 are local in the ordinary analytic topology and do not form sheaves for the G-topology [GJR, Remark 7.3(3)]. A covering by formal domains therefore cannot be substituted without argument for analytic neighbourhoods in this definition.

Definition 17.2.4 (Harmonic tropicalization)**.** A harmonic tropicalization of a compact good strictly analytic space V is a map $h = (h_1, \dots, h_r) : V \to \mathbb{R}^r$ whose coordinates are integral harmonic functions.

This is [GJR, Definition 8.1]. A smooth tropicalization is harmonic, but a harmonic tropicalization need not arise from analytic units. The integrality of the coordinates is part of the definition. The last assertion of Proposition 17.2.2 explains how general real harmonic functions are nevertheless represented in such charts.

17.2.2 The sheaf and its differential operations

Definition 17.2.5 (Weakly smooth preforms and forms)**.** Let $U \subset X$ be open with compact strictly analytic closure $\overline{U}$. A preform of type (p, q) on U is a pair (h, α), where $h : \overline{U} \to \mathbb{R}^r$ is a harmonic tropicalization and α is a smooth Lagerberg (p, q)-form on $h(\overline{U})$. Two pairs (h, α) and (k, β) represent the same preform if

$$\mathrm{pr}_1^*\, \alpha = \mathrm{pr}_2^*\, \beta \quad \text{on } (h, k)(\overline{U}).$$

The sheafification of this presheaf is denoted $\mathcal{A}^{p,q}_{\mathrm{wk}}$. Its sections are *weakly smooth forms*.

This is [GJR, Definitions 10.3 and 10.8]. Passing to a product chart puts any finite collection of preforms on a common tropicalization. On that chart, wedge products and derivatives are those of Lagerberg forms:

$$d'(h, \alpha) = (h, d'\alpha), \qquad d''(h, \alpha) = (h, d''\alpha).$$

For types (p, q) and (p', q'), the commutativity sign is $(-1)^{(p+q)(p'+q')}$, and the Leibniz sign for either differential is $(-1)^{p+q}$. These are total-degree signs.

Theorem 17.2.6 (Comparison of the three sheaves)**.** *There are injective morphisms of sheaves*

$$\mathcal{A}^{\bullet,\bullet}_{\mathrm{sm}} \longrightarrow \mathcal{A}^{\bullet,\bullet}_{\mathrm{wk}} \longrightarrow \mathcal{A}^{\bullet,\bullet}_{\mathrm{ps}}.$$

The first sheaf is the original CLD sheaf. The last consists of piecewise smooth forms and is endowed with the polyhedral differentials d'_P, d''_P. Both morphisms preserve wedge products and the indicated differentials. Weakly smooth forms admit functorial pullback by analytic morphisms.

Open proof obligation (RC2). Supply the exact inclusions and comparison of the smooth, weakly smooth and delta-form categories; GJR Proposition 10.14 and its supporting constructions.

Proposition 17.2.7 (Harmonicity detected by the weak differential)**.** *A real-valued function f on X is harmonic if and only if*

$$f \in \mathcal{A}^{0,0}_{\mathrm{wk}}(X), \qquad d'd''f = 0.$$

Open proof obligation (RC2). Prove injectivity of pullback on the harmonic chart and the converse with the stated face hypotheses; GJR Proposition 10.9 supplies injectivity.

Example 17.2.8 (A harmonic function which is not CLD-smooth)**.** Assume that K is algebraically closed and that its residue field is not algebraic over a finite field. Gubler–Jell–Rabinoff construct a smooth proper genus-three curve with a theta graph as skeleton and an elliptic residual component E at one vertex v. Let z be the origin of E and choose a nontorsion point $x \in E(\widetilde{K})$.

The local construction produces an integral harmonic function h whose residue at v has degree zero but represents

$$[z] - [x] \neq 0 \quad \text{in} \quad \mathrm{Pic}^0(E) \otimes_{\mathbb{Z}} \mathbb{R}.$$

On a smooth analytic curve, the quotient of harmonic germs by CLD-smooth harmonic germs at such a point is identified with this residual Picard space. Thus h is weakly smooth by Proposition 17.2.7 and is not CLD-smooth. The construction and the nonvanishing obstruction are given in [GJR, §17.3 and Proposition 17.4]; the sheaf comparison used here is explained in Remark 15.10.

The obstruction is algebraic: degree zero only detects numerical triviality, while the nonzero Picard class prevents a representation by the local logarithms that give smooth harmonic functions. Hence the first inclusion in Theorem 17.2.6 can be strict even in bidegree $(0,0)$.

Remark 17.2.9 (Cases in which the first inclusion is equality). On a smooth analytic curve potentially locally isomorphic to $\mathbb{P}^{1,\mathrm{an}}$, or over a residue field algebraic over a finite field, every harmonic function is CLD-smooth and $\mathcal{A}_{\mathrm{sm}} = \mathcal{A}_{\mathrm{wk}}$ [GJR, Remark 15.10]. Strictness is therefore not a statement about every analytic curve.

17.2.3 Strong currents and restriction of test spaces

Now assume that X is Hausdorff, boundaryless, and of pure dimension d. Write $\mathcal{A}^{r,s}_{\mathrm{wk},c}(X)$ for compactly supported weakly smooth forms, endowed with the locally convex topology of [GJR, §12.1]. In charts, its defining convergence uses fixed compact support control and uniform convergence of every derivative of the representing Lagerberg forms.

Definition 17.2.10 (Strong currents, in the bidegree convention of these notes)**.** Set

$$\mathcal{D}^{p,q}_{\mathrm{wk}}(X) := \big(\mathcal{A}^{d-p,d-q}_{\mathrm{wk},c}(X)\big)',$$

where the prime denotes the continuous real dual. Its elements are the *strong currents* of bidegree (p,q).

In [GJR, §12.2], the notation is $\mathcal{D}_{r,s} = (\mathcal{A}^{r,s}_c)'$: the indices record the degrees of the test forms. Thus their $\mathcal{D}_{r,s}$ is our $\mathcal{D}^{d-r,d-s}_{\mathrm{wk}}$. The explicit dual formula is the reliable way to translate conventions.

Proposition 17.2.11 (The direction of the comparison map)**.** *Restriction of functionals gives a canonical map*

$$\rho : \mathcal{D}^{p,q}_{\mathrm{wk}}(X) \longrightarrow \mathcal{D}^{p,q}_{\mathrm{sm}}(X).$$

Using the assertion in [GJR, §12.1] that the original CLD test-space topology is the induced subspace topology, this map is surjective. Its kernel consists precisely of the functionals vanishing on all original smooth test forms. An extension of a smooth-test current is unique if those forms are dense in the weak test space.

Proof. Let $E = \mathcal{A}^{d-p,d-q}_{\mathrm{sm},c}(X)$ and $F = \mathcal{A}^{d-p,d-q}_{\mathrm{wk},c}(X)$. Continuity of $E \hookrightarrow F$ gives restriction $F' \to E'$. If $T \in E'$, the induced subspace topology supplies a continuous seminorm P on F such that $|T(e)| \le P(e)$ for every $e \in E$. The real Hahn–Banach theorem extends T to a linear functional on F dominated by P, hence to an element of F'. This proves surjectivity. The

kernel description is immediate; a continuous functional vanishing on a dense subspace vanishes everywhere. □

Remark 17.2.12 (What this proposition does not identify). The Hahn–Banach extension is not canonical and does not, by this argument, preserve positivity. Density is a separate question; it does not follow from the inclusion of sheaves. In particular, enlarging the test forms does not by itself give an inclusion of the corresponding spaces of currents, or an isomorphism of those spaces.

The differential of a strong current is defined by duality with the same total-degree signs as for smooth-test currents. Restriction commutes with d' and d''. Proper direct images and multiplication by weakly smooth forms are available [GJR, §12.2]. These constructions do not give arbitrary currents a wedge product or a pullback under every analytic morphism.

Theorem 17.2.13 (Poincare–Lelong against weak tests)**.** *Let f be a regular meromorphic function on X, meaning locally a nonzero quotient of analytic functions that are non-zero-divisors on the local rings. Then*

$$d'd''[\log|f|] = \delta_{\mathrm{div}(f)} \quad \textit{in} \quad \mathcal{D}^{1,1}_{\mathrm{wk}}(X).$$

Open proof obligation (RC2). Prove the Poincare–Lelong equality on the full weakly smooth test class of the statement; GJR Theorem 12.7 is supplied.

Example 17.2.14 (Polyhedral differentiation misses a corner mass)**.** On $X = \mathbb{G}_m^{\mathrm{an}}$, put $t = -\log|T|$ and $u = |t|$. This is a piecewise smooth function. On the two cells $t \geq 0$ and $t \leq 0$ it is affine, so

$$d'_P d''_P u = 0.$$

Its current derivative is different. For a smooth compactly supported test function φ on the real skeleton,

$$\int_{\mathbb{R}} |t|\varphi''(t)\,dt = 2\varphi(0).$$

Consequently $d'd''[u]$ is twice the Dirac measure at the radius-one Gauss point. This explicitly distinguishes polyhedral differentiation from differentiation of currents. On this genus-zero curve weakly smooth forms agree with CLD-smooth forms; u is not weakly smooth at that point.

For weakly smooth forms α, Stokes gives $d'[\alpha] = [d'\alpha]$ and $d''[\alpha] = [d''\alpha]$. For arbitrary piecewise smooth forms, these identities require the missing face terms; they cannot be inferred from the injection into currents in [GJR, §12.4].

17.2.4 Positivity, regularization, and the scope of comparison

Theorem 17.2.15 (Model positivity implies original current positivity)**.** *Let X be good, separated, and strictly analytic. If h is a semipositive $\mathbb{R}$-PL function, then the original CLD current $d'd''[h]$ is positive on $X \setminus \partial X$. The same conclusion holds locally for uniform limits of semipositive PL metric weights.*

Open proof obligation (RC2). Prove metric/current positivity and the converse with their exact hypotheses; GJR Theorem 7.14 and Remark 7.16 are supplied.

Theorem 17.2.16 (Regularization on smooth analytic curves)**.** *Let X be a rig-smooth boundaryless analytic curve over K, and let f be subharmonic in Thuillier's sense. For every relatively compact open subset $U \subset X$, there is a decreasing net of weakly smooth psh functions on U converging pointwise to $f|_U$.*

Open proof obligation (RC2). Prove regularization on curves in the stated net and weakly smooth scope; GJR Proposition 15.11 is supplied.

Remark 17.2.17 (Canonical metrics and an explicitly corrected argument). The canonical tropicalization in Raynaud uniformization is harmonic [GJR, Proposition 16.2]. It need not be CLD-smooth [GJR, Corollary 17.5]. Remark 16.4 explicitly explains that Gubler–Künnemann's earlier argument in the 2017 paper, Examples 8.15 and 9.17, used the incorrect smoothness assertion to conclude that canonical metrics are δ-metrics. GJR leaves the latter conclusion open in the original δ-metric theory. This corrects the cited argument; it does not prove that the metrics are not δ-metrics.

The version concordance is recorded in Section 18.3.4; the elementary formula corrections are collected in Remark A.1.12.

17.3 Ordinary forms, G-forms, and harmonic forms in CLD25

The change of psh definition in CLD25 does not change its ordinary smooth test forms. Three sheaves must nevertheless be distinguished. We retain the good Hausdorff hypotheses on X; whenever currents enter the comparison, X is purely n-dimensional and boundaryless.

Definition 17.3.1 (The analytic affine and PL structures)**.** An affine function in the sense of CLD25 is, locally for the ordinary topology, of the form

$$a \log |f| + c, \qquad a \in \mathbb{Q}, \quad c \in \mathbb{R}, \quad f \in \mathcal{O}_X^{\times}.$$

A PL function is a continuous function which has this description G-locally, on analytic domains. Finite sums are included in this description by clearing rational denominators and multiplying units. These are the structures of [CLD25, Proposition 12.4.1 and §12.4.6]. They use rational slopes and real constants; this notation does not silently mean arbitrary $\mathbb{R}$-PL functions.

Here a G-cover consists of analytic domains such that, at each point, finitely many members containing that point have union a neighbourhood. An ordinary open cover is a G-cover, but a G-cover need not provide a single member which is a neighbourhood of a point on an interface.

Definition 17.3.2 (The two smooth sheaves)**.** For a polyhedral subset P of a real affine space E, let $A^{p,q}(P)$ denote smooth ambient superforms restricted to P, modulo forms whose restrictions to the tangent spaces of every cell in P vanish. More precisely, this is the quotient sheaf of [CLD25, §1.9.1–§1.9.3]; it is independent of subdivision.

For an analytic domain V, form the direct limit

$$\mathcal{P}^{p,q}(V) = \varinjlim_{(h:V \longrightarrow E,\, P)} A^{p,q}(P),$$

where h has PL coordinates, P contains $h(V)$, and transition maps are affine maps between ambient spaces. Products of charts give common refinements. The associated sheaf on the analytic G-site is denoted $A^{p,q}_{G,25}$ here. Its sections are the smooth G-forms of [CLD25, §3.3.3–Definition 3.3.4].

Write $\pi : X_G \to X$ for the map to the ordinary topological site. The ordinary smooth sheaf $A^{p,q}_{25} \subseteq \pi_* A^{p,q}_{G,25}$ consists of forms represented, on an ordinary neighbourhood of each point, by an *affine* chart and a smooth superform. This is [CLD25, Definition 6.2.1].

Proposition 17.3.3 (The ordinary sheaf is unchanged)**.** *There is a canonical identification*

$$A^{p,q}_{\mathrm{sm}} = A^{p,q}_{25}.$$

It respects wedge product and the two differentials. The operators called J correspond by the sign conversion already specified above.

Open proof obligation (RC2). Prove both local moment-map presentation and matching nullity of representatives on the common product chart; CLD25 Proposition 15.1.1(b), CLD12 Lemma 3.2.2 and CLD25 Proposition 3.4.4.

The distinction between the two sheaves is the covering condition, not the replacement of analytic units by arbitrary continuous coordinates. Indeed, [CLD25, Proposition 15.1.1(a)] still presents every G-form by analytic moment maps, but on a G-cover. The ordinary smooth forms remain the test objects in [CLD25, §7.1.1].

Definition 17.3.4 (The harmonic subsheaf)**.** A PL function h is harmonic in [CLD25, Definition 7.8.2] if

$$d'd''[h] = 0.$$

More general harmonic functions are defined to be local finite real linear combinations of such PL harmonic functions. The sheaf $A^{p,q}_{\mathrm{harm},25}$ consists of G-forms locally represented by $h^*\alpha$, where $h = (h_1, \dots, h_r)$ has PL harmonic coordinates and α is a smooth superform on a neighbourhood of a polyhedron containing $h(U)$; see [CLD25, §7.8.7].

Affine functions are harmonic, so these definitions give inclusions

$$A^{p,q}_{\mathrm{sm}} = A^{p,q}_{25} \;\subseteq\; A^{p,q}_{\mathrm{harm},25} \;\subseteq\; \pi_* A^{p,q}_{G,25}.$$

All three are differential algebras, but integration into currents has different differential properties. Denote the operators on G-forms by d'_G, d''_G. Every PL function satisfies

$$d'_G d''_G h = 0,$$

because it is affine on a G-cover. This does not imply $d'd''[h] = 0$: distributional differentiation can detect the interfaces. Already on the real line, $h(t) = |t|$ has zero piecewise Hessian, whereas its distributional Hessian is $2\delta_0$. The harmonic condition is therefore an additional condition on a PL function.

For a harmonic form ω, the extra boundary terms vanish:

$$d'[\omega] = [d'_G\omega], \qquad d''[\omega] = [d''_G\omega].$$

This is [CLD25, Theorem 7.8.5 and Corollary 7.8.8]. The resulting map to currents is injective in the present boundaryless setting by [CLD25, Corollary 7.3.9 and Theorem 12.11.1]. It does not redefine currents as duals of all G-forms.

Remark 17.3.5 (Comparison with GJR requires a numerical theorem). CLD25 starts with a vanishing *original curvature current* for a PL function. GJR defines harmonicity through numerical triviality of the residual model class, with its own integral and real PL classes; see Definition 17.2.1. An identification of the two harmonic sheaves requires a PL curvature comparison with matching hypotheses, and then a comparison of their chart presentations. The degree-one vanishing assertion of [CLD25, Theorem 18.8.4] addresses the numerical/current step for its integral PL metrics; its scope is discussed in Section 19.3. No unrestricted identification with the GJR weakly smooth sheaf is assumed here. Also, CLD25 does not define every continuous solution of $d'd''[h] = 0$ to be harmonic: [CLD25, Remark 7.8.3] explicitly distinguishes that question from the PL-generated definition.

17.4 Graph forms and harmonic cohomology

17.4.1 The first-order equations on a graph

Let Σ be a finite connected metric graph without boundary, and choose orientations of its edges. On an open edge with coordinate t, forms have the familiar expressions $f(t)$, $a(t)d't$, $b(t)d''t$, and $c(t)d't \wedge d''t$. At vertices, the smoothness conditions are those of the harmonic-coordinate graph calculus. In particular, the first-order compatibility is tied to a weighted conservation law.

For constant coefficients on oriented edges, closed $(1,0)$-forms correspond to flows satisfying Kirchhoff's law: the weighted outgoing coefficients sum to zero at each vertex. In the unweighted case, the incidence matrix has rank $|V| - 1$ for a connected graph. Its kernel therefore has dimension

$$|E| - |V| + 1 = b_1(\Sigma).$$

This linear algebra explains the occurrence of the graph genus in $H^{1,0}$. Subdividing an edge adds one vertex and one edge and leaves the dimension unchanged.

17.4.2 Why integration detects the top cohomology

For a compact graph without boundary, Stokes gives $\int_\Sigma d''\alpha = 0$ for a $(1,0)$-form α. Thus integration descends to a functional on $H^{1,1}$. Conversely, a top-degree form of total integral zero can be integrated along edges to produce a primitive; constants of integration are chosen so that the vertex conditions hold. The obstruction is precisely the total integral.

This gives $H^{1,1} \simeq \mathbb{R}$ in the connected compact case. If the graph has boundary, flux may escape through boundary vertices, changing the obstruction and the cohomology. The source gives the exact boundary-dependent dimension formulas. The boundary must therefore be specified before applying a duality statement.

17.4.3 The circle as a complete example

For a circle of length ℓ, take a periodic coordinate on its universal cover. A closed $(1,0)$-form has a constant coefficient after passing to its cohomology class, and its period is the invariant. A $(0,1)$-form $b(t)d''t$ is exact precisely when its integral over one period is zero. A $(1,1)$-form has a primitive of the required periodic type precisely when its total integral is zero. Hence the four cohomology groups in bidegrees $(0,0), (1,0), (0,1), (1,1)$ all have dimension one.

The pairing of $a, d't$ with $b, d''t$ is $ab\ell$ in the simplest constant representatives. It is nondegenerate. This gives a concrete model for the Poincaré duality proved in the curve theory, while retaining the metric length in the pairing.

17.4.4 The good-reduction curve as a contrasting example

A smooth proper curve with good reduction can have a one-point skeleton and first Betti number zero. Its weakly smooth Dolbeault groups in bidegrees $(1,0)$ and $(0,1)$ are therefore zero in the cited theorem, while $H^{1,1}$ remains one-dimensional by integration. If the algebraic genus is positive, algebraic differential forms still exist; they belong to a different cohomology theory.

This example prevents a misleading inference from the graph picture. The weakly smooth calculus is designed to have robust tropical analytic properties and duality. It is not claimed to reproduce every algebraic Hodge number. To compare it with algebraic de Rham theory, one would need an additional construction accounting for the residual geometry.

17.4.5 From graph forms to analytic forms

The passage to Berkovich curves uses skeleta adapted to the forms and harmonic maps between graph presentations. It is not a claim that every form factors through one minimal skeleton. Enlarging a skeleton can add trees and local coordinates while preserving the appropriate cohomological information. Compatibility under these changes is what makes the analytic sheaf and its cohomology intrinsic.

This gives a successful example of the general strategy discussed in Chapter 24: first understand a finite tropical complex, then prove compatibility under refinement, then identify the resulting analytic cohomology. The proof contains more than a topological deformation retraction, because the differential and vertex-smoothness conditions also have to be preserved.

References. [GJRc, Sections 2–3 and Theorems 6.4.3, 6.4.5].

17.5 The differential complex on a compact graph

We give the graph calculation underlying the curve comparison. It is useful both as a complete example of a differential theory and as a warning about what a skeleton retains. Let Γ be a finite connected metric graph with at least one edge. We first take all edge weights to be one and no designated boundary. Multiple edges and loops are allowed; a loop contributes two outgoing germs at its vertex.

Choose an orientation of every edge e and a parameter $t_e \in [0, \ell_e]$. A smooth function is continuous on Γ, smooth up to each endpoint on each edge, and satisfies the following local conditions. At a vertex of valency one it is constant nearby. At valency two it is an ordinary smooth function across the vertex. At higher valency its outgoing first derivatives have sum zero.

A $(1,0)$-form has edgewise expressions $a_e(t_e)d't_e$. Reversing orientation reverses the coefficient. At a leaf it vanishes nearby, at a two-valent vertex

the one-form extends smoothly across, and at higher valency its outgoing coefficients have sum zero. The definition of $(0,1)$-forms is identical with $d''t_e$. A $(1,1)$-form has expressions $c_e(t_e)d't_e\wedge d''t_e$, unchanged by orientation reversal. It vanishes near leaves, is smooth across two-valent vertices, and has no extra matching condition at higher-valent vertices. These are the unweighted GJR graph conventions.

The edgewise formulas are

$$d''f = f_e'\,d''t_e, \qquad d''(a_e\,d't_e) = -a_e'\,d't_e\wedge d''t_e, \qquad \int_\Gamma c = \sum_e \int_0^{\ell_e} c_e(t)\,dt.$$

The vertex conditions ensure that these derivatives lie in the indicated form spaces. They also imply Stokes: the endpoint terms of $\int_\Gamma d''a$ sum to the outgoing coefficient sums, hence vanish.

17.5.1 The incidence matrix

Let V, E denote the vertices and the chosen oriented edges. Define $B:\mathbb{R}^E\to\mathbb{R}^V$ by putting $+1$ at an edge's tail and -1 at its head; a loop column is zero. Connectedness implies

$$\operatorname{im} B = \{z\in\mathbb{R}^V : \textstyle\sum_v z_v = 0\}, \qquad \dim\ker B = |E| - |V| + 1.$$

Indeed the columns have coordinate sum zero. Along a path, signed sums of columns produce the difference of the two endpoint basis vectors. Since any vertex can be joined to a fixed one, these differences span the zero-sum hyperplane. We write $b = |E| - |V| + 1 = b_1(\Gamma)$.

Proposition 17.5.1. *The d''-closed $(1,0)$-forms are exactly the edgewise constant forms whose coefficient vector belongs to $\ker B$. Consequently $H^{1,0}(\Gamma)\simeq\ker B$ and $h^{1,0}=b$.*

Proof. The displayed formula for d'' forces $a_e'=0$ on each edge. The vertex conditions then reduce to $Ba=0$. At a leaf this equation forces the incident constant to be zero, so the required vanishing on a neighbourhood follows as well. At valency two it makes the constants match with the orientation signs. Conversely these conditions give a smooth closed form. There are no forms of type $(1,-1)$, hence no boundaries in this group. □

Theorem 17.5.2 (Periods and exactness)**.** *Integration around cycles induces an isomorphism*

$$H^{0,1}(\Gamma) \simeq \mathrm{Hom}(H_1(\Gamma, \mathbb{R}), \mathbb{R}).$$

Proof. For a $(0,1)$-form $\beta = b_e(t)d''t_e$, set $I_e = \int_0^{\ell_e} b_e(t)dt$. The period on a cycle $z \in \ker B$ is $\sum_e z_e I_e$. If $\beta = d''f$, these integrals are endpoint differences and all periods vanish.

Conversely suppose all periods vanish. Fix a basepoint and define $f(x)$ by integrating β along a path from it to x. Any two paths differ by cycles, so f is well defined. It is continuous and has derivative b_e on each edge. Near a leaf, $b_e = 0$, so f is constant. At a two-valent vertex, smooth matching of the one-form gives smooth matching of its primitive. At a higher-valent vertex, the outgoing derivatives of f are the outgoing coefficients of β, whose sum is zero. Thus f belongs to $A^{0,0}(\Gamma)$ and $\beta = d''f$.

To prove surjectivity, represent a functional on $\ker B$ by some vector $I \in \mathbb{R}^E$. On each edge choose a smooth bump supported strictly inside the edge and of integral I_e. These coefficients define a $(0,1)$-form because they vanish near every vertex. Its periods are the prescribed functional. □

This proof is stronger than a dimension count: it supplies an exactness criterion and a primitive. It also shows why smoothness at vertices cannot be ignored. Edgewise primitives always exist, but their values and derivatives must satisfy the graph's matching conditions.

Theorem 17.5.3 (Top-degree exactness)**.** *The image of $d'' : A^{1,0}(\Gamma) \to A^{1,1}(\Gamma)$ is precisely the kernel of integration. Hence $H^{1,1}(\Gamma) \simeq \mathbb{R}$.*

Proof. Stokes gives one inclusion. For the other, let $c_e(t)d't_e \wedge d''t_e$ have total integral zero and put $J_e = \int_0^{\ell_e} c_e(t)dt$. An edgewise solution is

$$a_e(t) = A_e - \int_0^t c_e(s)ds.$$

At a tail the outgoing coefficient is A_e; at a head it is $-A_e + J_e$. Therefore the vertex equations are

$$(BA)_v = -\sum_{e:\, e^+ = v} J_e.$$

The right side has total sum $-\sum_e J_e = 0$. By the description of $\operatorname{im} B$, constants A_e solving these equations exist. They give the value matching at

two-valent vertices and the coefficient sums at higher-valent ones. Derivative matching at a two-valent vertex follows by differentiating $a'_e = -c_e$ and using the smooth matching of c. At a leaf, c vanishes nearby and the vertex equation sets a to zero there. Thus a is an admitted $(1,0)$-form and $d''a = c$.

An interior bump on any one edge gives a $(1,1)$-form of nonzero integral, proving that integration is surjective on the quotient. □

Corollary 17.5.4 (Graph duality)**.** *For this compact connected graph,*

$$h^{0,0} = h^{1,1} = 1, \qquad h^{1,0} = h^{0,1} = b.$$

Integration of wedge products gives perfect pairings between complementary bidegrees.

Proof. A function with zero edge derivatives is constant by connectedness, giving $h^{0,0} = 1$. The other dimensions were proved above. For $(1,0)$ and $(0,1)$, write the first class as a constant flow $a \in \ker B$ and the second by its edge integrals I. Their pairing is $\sum_e a_e I_e$. It is the natural pairing of $\ker B$ with its dual from Theorem 17.5.2, hence perfect. Constants pair perfectly with the integral coordinate of $H^{1,1}$. Stokes ensures independence of representatives. □

17.5.2 What changes when a boundary is designated

Let $\partial\Gamma \subset V$ be nonempty and impose no vertex matching condition there, apart from continuity of functions. Write B_{int} for the incidence matrix with only the rows indexed by $V \setminus \partial\Gamma$. These rows are independent. Indeed a linear relation, extended by zero at boundary vertices, gives a vector in $\ker B^t$, hence a constant vector; because it is zero at one boundary vertex, it is zero everywhere.

Proposition 17.5.5. *For a connected graph with nonempty designated boundary,*

$$h^{0,0} = 1, \quad h^{0,1} = b, \quad h^{1,0} = b + |\partial\Gamma| - 1, \quad h^{1,1} = 0.$$

Proof. Closed $(1,0)$-forms are the constant flows in $\ker B_{\mathrm{int}}$, whose dimension is $|E| - |V| + |\partial\Gamma|$. The period proof for $H^{0,1}$ is unchanged: path integration still gives a continuous primitive and there are fewer endpoint conditions to check. The proof of top-degree exactness now solves the vertex equations only

at interior vertices. Since B_{int} is onto, these equations always have a solution, with no total-integral constraint. Thus every $(1,1)$-form is exact. □

For an interval with both endpoints designated as boundary, $h^{1,0} = 1$ and $h^{1,1} = 0$. For the same interval with no designated boundary, the leaf conditions give $h^{1,0} = 0$ and $h^{1,1} = 1$. The topological interval alone does not determine this differential complex: the boundary convention is part of its data.

17.5.3 Weights and the curve comparison

If an edge has positive weight w_e, put $t_{0,e} = t_e/w_e$, so its new length is ℓ_e/w_e. Pullback replaces a one-form coefficient a_e by $w_e a_e$ and a two-form coefficient c_e by $w_e^2 c_e$. The weighted vertex conditions become the preceding unweighted ones. Since $d/dt_{0,e} = w_e d/dt_e$, this is an isomorphism of differential algebras. Its integration rule agrees with $\sum_e w_e \int c_e(t_e)dt_e$.

The passage from these graph groups to weakly smooth forms on a Berkovich curve is a separate theorem. In the setting of GJR's curve paper, a suitable strictly semistable model gives a pullback from graph forms inducing the cohomology isomorphism. Finite extension and Galois descent handle a curve without such a model over the initial field. We use that comparison, not a claim that every continuous retraction induces an isomorphism of differential complexes.

In particular, a skeleton consisting of one isolated point has no graph $(1,1)$-forms under the definitions above. It cannot by itself compute the top-degree cohomology of a proper analytic curve. The comparison theorem uses an appropriate model; one may blow up to avoid smooth connected components of the special fibre and obtain a nontrivial skeleton. Attaching the resulting tree does not change b_1, but it supplies the graph differential data needed for top-degree integration.

17.6 Residual genus and topological genus

The discrepancy between algebraic genus and graph genus can be calculated without differential forms. Let C_0 be a connected proper nodal curve over an algebraically closed field. Let $\widetilde{C}_v$ be its normalized irreducible components

and let Γ be its dual graph. There is one vertex per component and one edge per node, including loop edges for self-nodes.

Proposition 17.6.1. *The arithmetic genus satisfies*

$$p_a(C_0) = b_1(\Gamma) + \sum_v g(\widetilde{C}_v).$$

Proof. Normalization gives an exact sequence

$$0 \longrightarrow \mathscr{O}_{C_0} \longrightarrow \bigoplus_v \nu_* \mathscr{O}_{\widetilde{C}_v} \longrightarrow \bigoplus_{e \in E} k_e \longrightarrow 0.$$

At a node, the last map takes the difference of the two branch values. Its exactness follows from the local description $k[[x, y]]/(xy)$: a pair of power series in x and y comes from this ring exactly when its constant terms agree. Taking Euler characteristics yields

$$\chi(\mathscr{O}_{C_0}) = \sum_v (1 - g(\widetilde{C}_v)) - |E|.$$

Since C_0 is connected and proper, $h^0(\mathscr{O}_{C_0}) = 1$. Thus $p_a = 1 - \chi = |E| - |V| + 1 + \sum_v g(\widetilde{C}_v)$, as asserted. □

For a proper semistable model of a smooth curve, constancy of Euler characteristic in the flat family identifies this arithmetic genus with the genus of the generic fibre. The skeleton retains the graph contribution $b_1(\Gamma)$; the remaining summands are genera of residual components. Good reduction gives a one-vertex dual graph and all the genus is residual. Totally degenerate reduction has genus-zero residual components and all the genus is graph-theoretic.

Consequently the weakly smooth cohomology formula $h^{1,0} = b_1$ cannot be identified with the dimension g of algebraic differentials without an additional hypothesis. For a good-reduction elliptic curve these dimensions are respectively zero and one. This does not contradict either theory: their complexes retain different information. In a proposed tropical-to-Berkovich comparison, the relevant sheaf and the treatment of residual components must be specified before drawing a Hodge-theoretic conclusion.

References. Graph definitions and the unweighting are [GJRc, Section 2.2]; the graph calculation above is an explicit incidence-matrix proof of the cohomology and duality statements. The analytic comparison is [GJRc, Theorems 6.4.2–6.4.5].

17.7 Residual divisors and harmonic coordinates

A harmonic function on a curve satisfies a conservation law for outgoing slopes. An original smooth harmonic function satisfies an additional local algebraic condition. The difference is visible on the residual curve of a type II point. This section calculates that difference and then revisits the canonical tropicalization example of Gubler–Jell–Rabinoff with consistent coordinates.

Work over a complete algebraically closed nontrivially valued field K. At a type II point v of a smooth analytic curve, the residual function field determines a smooth projective curve C_v over $\widetilde{K}$. Its closed points correspond to tangent directions. We use the slope formula: for an analytic meromorphic function whose normalized reduction at v is $\widetilde{f}$, outgoing slopes of $-\log|f|$ are the orders of $\widetilde{f}$, with the sign reversed for $\log|f|$. The identification of tangent directions and this slope formula are geometric inputs; they are not consequences of the abstract graph calculation.

The sign can be checked in the open disc direction at the Gauss point: with outward distance $t = -\log|T|$, the reduction of T has order one at 0, while $\log|T| = -t$ has outgoing slope -1.

Proposition 17.7.1. *Let h be a harmonic germ represented by a PL function on a finite star at v. If s_p is its outgoing slope in the direction $p \in C_v$, then*

$$D_h = \sum_p s_p[p], \qquad \deg D_h = \sum_p s_p = 0.$$

If h is a real linear combination of logarithms of local analytic units, then D_h has zero class in $\mathrm{Pic}^0(C_v) \otimes_{\mathbb{Z}} \mathbb{R}$.

Proof. Only finitely many directions have a nonzero slope, so the divisor is defined. Harmonicity is precisely the vanishing of the outgoing slope sum; the residual field is algebraically closed, so all residual points have degree one. For $h = \log|f|$, the slope formula identifies D_h with $-\operatorname{div}(\widetilde{f})$, hence with a principal divisor. Multiplying by a real scalar and adding preserves the zero divisor class. Constants have all slopes zero. □

The converse is not asserted from this calculation alone: lifting a residual principal relation to analytic units is a local analytic theorem. Thuillier's comparison identifies the relevant quotient of harmonic germs by smooth harmonic germs with $\mathrm{Pic}^0(C_v) \otimes \mathbb{R}$ in the curve setting used below. GJR

records this input in Remark 15.10 and the proof of Proposition 17.4. It also records that original smooth harmonic germs are the sheaf generated by logarithms of analytic units. We use these two comparison results explicitly.

Corollary 17.7.2. *In this setting, a harmonic germ with nonzero class $[D_h]$ cannot be an original smooth function. A degree-zero residual divisor is therefore not enough to guarantee original smoothness.*

Proof. If the germ were original smooth, it would be a smooth harmonic germ, hence a real linear combination of logarithms of analytic units. Proposition 17.7.1 would make its residual class zero. □

Example 17.7.3 (Why the residue field matters)**.** If a smooth projective curve is defined over an algebraic closure of a finite field, every point of its Jacobian is torsion. Indeed the variety and any chosen point descend to a finite field; the group of points over that finite field is finite. Thus $\mathrm{Pic}^0(C_v) \otimes \mathbb{R} = 0$. Over other algebraically closed fields, an elliptic residual curve can have nontorsion points, and their degree-zero divisor classes survive tensoring with $\mathbb{R}$. For an abelian group A, the kernel of $A \to A \otimes \mathbb{Q}$ is its torsion subgroup, and extension from $\mathbb{Q}$ to $\mathbb{R}$ is faithful. This justifies the last assertion. The vanishing of the residual obstruction is consistent with the equality of the two harmonic sheaves in the finite-residue-field case; the analytic equality itself uses the cited comparison theorem.

17.8 A consistent theta-graph calculation

Let a, b, c be three edges from P to Q, all oriented in that direction, with lengths $A, B, C > 0$. Choose the cycles

$$\gamma_1 = c - a, \qquad \gamma_2 = c - b.$$

The notation means traverse c from P to Q and return along the indicated other edge. Use the basis dual to (γ_1, γ_2) on $H^1(\Theta, \mathbb{R})$. The edge-length pairing has matrix

$$G = \begin{pmatrix} A + C & C \\ C & B + C \end{pmatrix}, \qquad \det G = AB + AC + BC > 0.$$

Thus $G\mathbb{Z}^2$ is a full lattice in $\mathbb{R}^2$, and the graph Jacobian is $\mathbb{R}^2/G\mathbb{Z}^2$. The positivity also follows directly from the positive edge-length pairing on the cycle space.

Proposition 17.8.1. *A lift of the Abel–Jacobi map based at P, on a star around P, has the following formulas, where t is distance from P:*

$$a:\ (-t,0), \qquad b:\ (0,-t), \qquad c:\ (t,t).$$

Modulo $G\mathbb{Z}^2$, these formulas agree at Q and define a continuous map on the whole graph. Each coordinate is harmonic near P.

Proof. Integrating the two harmonic one-forms corresponding to γ_1, γ_2 gives coefficient vectors $(-1,0)$ on a, $(0,-1)$ on b, and $(1,1)$ on c. Their sums at P vanish. At Q, the three lifted values are $(-A,0)$, $(0,-B)$, and (C,C). Their differences from the third are the negatives of the first and second columns of G, so they represent the same point of the quotient. The period of each cycle is the corresponding column of G, as required by the Abel–Jacobi definition. □

Attach an elliptic curve E at P, with the directions a, b, c marked by distinct points x, y, z, where z is the identity and x, y are nontorsion. Attach a projective line at Q, with three distinct markings. The lifting theorem for metrized complexes, used in GJR Section 17.3, realizes this data by a smooth proper analytic curve after choosing the lengths in the value group. This realization and compatibility of analytic and tropical Abel–Jacobi maps are the two geometric inputs in the example.

Theorem 17.8.2 (The residual obstruction in the example)**.** *The two lifted coordinate functions are harmonic but not original smooth near P. Their outgoing-slope divisors are respectively*

$$D_1 = [z] - [x], \qquad D_2 = [z] - [y].$$

Consequently the canonical tropicalization of the Jacobian of the realized curve need not be an original smooth tropicalization.

Proof. Proposition 17.8.1 gives the slope lists $(-1,0,1)$ and $(0,-1,1)$. Each has sum zero, giving harmonicity. Matching the directions with the three residual markings gives the displayed divisors. The standard identification $E(\widetilde{K}) \simeq \mathrm{Pic}^0(E)$ sends x to $[x] - [z]$. Nontorsion of x, y and Example 17.7.3 show that both classes remain nonzero after tensoring with $\mathbb{R}$. Corollary 17.7.2 proves nonsmoothness. The analytic Abel–Jacobi map identifies these local functions with pullbacks of local coordinates of

the canonical tropicalization of the Jacobian. If that tropicalization were original smooth, pullback of original smooth functions by the analytic map would make these functions original smooth, a contradiction. The realized curve has genus $2 + 1 + 0 = 3$, using the graph and residual genus formula of Proposition 17.6.1; hence this produces the abelian threefold of GJR. □

Remark 17.8.3 (Two coordinate slips in the cited version). In GJR v3, Section 17.2 names the first cycle as $c - b$ and the second as $c - a$, but its Gram matrix and Abel–Jacobi formulas use the opposite ordering. Taking $(c - a, c - b)$ as above makes the displayed formulas consistent. In the proof of Proposition 17.4, the printed first-coordinate slope list $(1, -1, 0)$ along (a, b, c) is incompatible with both those formulas and the subsequently stated divisor $[z] - [x]$. The list $(-1, 0, 1)$ gives the stated divisor. These are indexing and slope corrections; they preserve the non-smoothness example. In particular, one should not use the printed list to replace $[z] - [x]$ by $[x] - [y]$: distinct nontorsion x, y do not by themselves imply that $x - y$ is nontorsion.

17.9 Consequences for the choice of differential sheaf

A harmonic tropicalization is built from local numerical harmonic functions, including the functions above. Pullback of a real smooth function along such a map is weakly smooth by definition. Original smoothness, on the other hand, requires presentation through logarithms of analytic units. The obstruction is therefore a change of coordinate class, not a failure of the elementary real chain rule.

For a smooth real function F and harmonic coordinates $h_1, \dots, h_r$, the weakly smooth calculus has

$$\mathrm{d}'\mathrm{d}''F(h) = \sum_{i,j} F_{ij}(h)\, d'h_i \wedge d''h_j.$$

The first-derivative terms vanish in that calculus because $\mathrm{d}'\mathrm{d}''h_i = 0$ there. This is not a proof that h_i are original smooth functions, and it cannot be inserted into the original smooth Leibniz rule against an arbitrary original current without a comparison theorem. Chapter 14 isolates exactly that background-current kernel requirement.

The example also explains the limitation of an earlier argument about canonical metrics. GK17 used smoothness of canonical tropicalizations to claim that all canonical metrics on abelian varieties are original delta-metrics. GJR Remark 16.4 explicitly withdraws that argument: its conclusion requires a further proof, while weakly smooth coordinates are available. The non-smooth coordinate example does not by itself prove that a particular canonical metric fails to be an original delta-metric; a metric might admit another presentation. This distinction between a failed presentation argument and a false existence statement is essential when citing the correction.

References. [GJR, Remark 15.10, Proposition 15.11, Proposition 16.2, Remark 16.4, Sections 17.1–17.4]; the graph calculation above fixes all orientations before applying the cited geometric comparisons.

Chapter 18

Comparing the differential theories

18.1 Forms, positivity, and intersection theory

Line of work	Main geometric problem	Relation to CLD
Tropicalization and canonical measures	Describe analytic images, multiplicities, degeneration and equidistribution	Supplies geometry used by the calculus; important parts predate CLD12.
Gubler's forms and currents	Present original analytic forms using very affine algebraic charts	An equivalent presentation under the stated hypotheses, not an enlarged sheaf.
Gubler–Künnemann	Multiply polyhedral curvature and construct local intersections	Enlarges the differential algebra to delta-forms while retaining compatibility with original tests.
Gubler–Jell–Rabinoff	Use numerical harmonic functions as coordinates	Enlarges the smooth coordinate class and hence the test space; produces weakly smooth forms and strong currents.
Gubler–Rabinoff	Detect PL positivity and define classical psh analytically	Compares model geometry and curve tests; supplies independent routes to specified positivity equivalences.

Line of work	Main geometric problem	Relation to CLD
Model, toric, and variational theories	Build semipositive metrics, envelopes, measures and solutions	Global or symmetry-restricted theories requiring explicit comparison with local currents.

The purpose of the table is to prevent two opposite mistakes. These papers are not all reformulations of CLD, but neither are they unrelated theories that merely use the same words. Their common local calculations involve slopes, Hessians, weighted integration and metric variation. Their differences concern the allowable representatives, the tests, the category of spaces, and the mode of passage to limits.

The following comparison keeps the objects and their test spaces explicit. Chapters 19–21 then use these distinctions in the study of model metrics, canonical measures and local heights.

18.2 Objects, morphisms, and comparison hypotheses

Every comparison has four entries: the underlying space and field, the object being compared, the operation or property, and the tests used to detect equality. A statement about one entry cannot silently change the others. For example, equality of a Chern measure is a top-degree comparison; equality of first Chern currents tests lower-degree forms; equality of metrized line bundles also retains constants and transition data.

There are three especially useful kinds of theorem in this subject. An equivalence of presentations, such as Gubler's algebraic presentation of original CLD forms, changes how the same object is described. An enlargement, such as the weakly smooth sheaf, changes the allowable coordinates and hence the tests. A comparison criterion, such as a PL model-positivity theorem, identifies properties on a restricted class of objects. None of these three kinds should be substituted for another.

The tables and proofs below can be used as a checklist while reading a new paper. They also explain why the same notation dd^c, $B^{p,q}$, or psh can have different domains in different sources. A correct formula includes its domain even when that domain is suppressed typographically.

18.3 Comparison of forms, potentials, and metrics

The preceding definitions give two different kinds of comparison: maps between spaces of forms and currents, and implications between positivity conditions on a fixed function or metric. Keeping these two comparisons separate prevents several misleading identifications.

18.3.1 Forms and their dual test spaces

Construction	Defining data	Comparison justified here
Original CLD forms	Analytic-unit tropicalizations, superforms, common refinements	Gubler's algebraic presentation gives the same sheaf under Theorem 8.1.4.
GJR weakly smooth forms	Numerically harmonic coordinate functions	Original forms inject into weakly smooth forms; strictness can occur.
Piecewise smooth forms	Smooth formulas on polyhedral pieces	The derivatives are d'_P, d''_P; current derivatives also see corners.
GK delta-forms B	Smooth coefficients and universally closed polyhedral generators	$A_{\mathrm{sm}} \hookrightarrow B \hookrightarrow P$, with P the generalized class.
Original currents D_{sm}	Continuous dual of $A_{\mathrm{sm},c}$	Other test theories restrict to these tests.
Strong currents D_{wk}	Continuous dual of $A_{\mathrm{wk},c}$	Restriction $D_{\mathrm{wk}} \to D_{\mathrm{sm}}$; see Proposition 17.2.11 for its topological qualifications.
Delta-currents E	Continuous dual of B_c	Restriction $E \to D_{\mathrm{sm}}$; functorial detection is a separate theorem.

The table gives no general inclusion between weakly smooth forms and delta-forms. A larger test space reverses the natural arrow on continuous duals. Equality on original tests is consequently a weaker assertion than equality on all tests of an enlarged theory unless an additional detection theorem applies.

18.3.2 Regularity determines the available theorem

Input	What is justified	Further conclusion needing another input
Original smooth u	Positive Hessian form and positive current agree; pullback preserves psh.	No automatic global model approximation on a prescribed line.
$u \in \mathcal{U}$	Current positivity, first-stage decreasing approximation, and the continuous mixed product.	No automatic comparison with every continuous current-psh function.
$u \in \mathcal{P}_{25} \cap L^1_{\mathrm{loc}}$	Original current positivity and generated-class functoriality.	No automatic one-step smooth approximation or equality of all continuous generated stages.
Real PL u under Proposition 19.2.11	Model, generated, and classical curvewise positivity are equivalent under the stated good Hausdorff boundaryless strict hypotheses.	The original-current converse is the additional source comparison in Remark 19.3.1.
Arbitrary continuous metric	A well-defined first Chern current.	Local or global positive approximation must be established.
Unbounded potential	A current when locally integrable.	Multiplication by another current, a non-pluripolar product, and energy finiteness are separate requirements.

18.3.3 Relation with adelic metrics and Deligne pairings

The same local distinction remains relevant in adelic geometry. A model line bundle, an actual collection of local norms, and the associated collection of curvature currents carry different data. An adelic completion also specifies compatibility of models and a mode of convergence. In particular, the equality of two local currents does not supply the lost constants of the metrics, nor does a local positivity statement assert a global numerical nef condition.

For example, let L be the trivial line on a smooth projective analytic curve and compare the constant norms $\|1\| = 1$ and $\|1\|' = e^{-c}$. Both have zero curvature. Any construction retaining the norm of the distinguished section still distinguishes them when $c \neq 0$. This elementary observation already explains why a current comparison is not an isometry theorem.

The present book establishes the vocabulary and local comparison results needed before such applications. It does not use any companion research manuscript as a source for the definitions or for the corrections in this volume. The earlier internal research summaries are not used as additional theorems of this series.

18.3.4 A terminology concordance

The following translations retain the mathematical distinctions needed when reading the French sources. A shared English word is not an additional comparison theorem.

Source term	Meaning used here	Precise reference in this book
forme lisse	An original smooth form obtained from analytic-unit tropicalizations	Definition 8.1.1
courant de bidimension (p, q)	A functional on tests of bidegree (p, q)	Definition 9.1.2
support propre in CLD12	Compact support disjoint from the analytic boundary	Definition 9.1.1
positif, *faiblement positif*, *fortement positif*	Three different cones, with complementary dual cones for currents	Definitions 7.2.2 and 9.1.5
localement psh-approchable in CLD12	Local uniform smooth psh approximation on one fixed neighbourhood	Definition 11.2.1 and remark 11.2.6
localement approchable in CLD12	A difference of two functions in that positive approximation class	Remark 11.2.6
localement approchable in CLD25	The first decreasing smooth psh stage $\mathcal{P}_{25,1}$	Definition 11.2.2 and proposition 11.2.3

Source term	Meaning used here	Precise reference in this book
psh continue in the two CLD versions	Current positivity in CLD12; a continuous generated sheaf in CLD25	Definitions 11.1.1 and 11.2.2
Weakly smooth, in GJR	Smooth superforms pulled back along harmonic tropicalizations	Definition 17.2.5
Strong current, in GJR	A continuous functional on weakly smooth test forms	Definition 17.2.10
Strongly positive current	A symmetric current nonnegative on weakly positive tests	Definition 9.1.5
Functorially psh, in GK	Current positivity after all morphisms in the stated algebraic category	Definition 16.2.1
Classically psh, in GR	Subharmonicity after analytic extension and every smooth analytic curve test	Definitions 19.2.1 and 19.2.2

For the GJR source, the earlier arXiv version and the version cited here have different numbering. In v2, model semipositivity implies current psh in Theorem 7.15, with harmonic consequence Corollary 7.16 and metric Remark 7.17. The corresponding v3 references are Theorem 7.14, Corollary 7.15, and Remark 7.16. The curve regularization result in the body of v3 is *Proposition* 15.11. The bibliography pins v3, and Theorem 17.2.16 preserves its decreasing-net formulation.

The two meanings of the word “strong” can be read directly from their definitions:

$$\underbrace{D^{p,p}_{\mathrm{wk}} = (A^{n-p,n-p}_{\mathrm{wk},c})'}_{\text{choice of test space}} \qquad \text{and} \qquad \underbrace{T(\alpha) \geq 0 \text{ for weakly positive } \alpha}_{\text{choice of positive cone}}.$$

Neither choice determines the other. Likewise, weak smoothness has no relation by definition to weak positivity.

18.4 Extensions of the original calculus

The original construction gives a local differential language, integration and currents, logarithmic divisor identities, a positive approximation calculus, and the comparison with model measures. Its scope is already substantially broader than a measure theory on proper algebraic varieties.

Three obstacles explain much of the subsequent literature. First, singular but geometric weights such as PL model functions are not automatically original smooth functions; an algebra that can multiply their curvature terms is desirable. This leads toward GK delta-forms. Second, model-harmonic functions can be larger than the original smooth pluriharmonic class; using them as coordinates leads toward GJR weakly smooth forms. Third, positivity of a distribution does not itself exhibit positive approximants or a local potential for an arbitrary closed current. This leads toward regularization, curve criteria, and further pluripotential theory.

There are also literal statements that need repair. The strong-positive cone does not span all symmetric forms in middle degree; a continuous facewise-convex function can have negative corner mass; and certain printed signs disagree with the total-degree algebra. These are treated in Appendix A, together with the usable replacements. The existence of such corrections does not erase the geometric construction of the theory, but it does require care when a later proof cites the affected lemma.

The standard model metric on $\mathcal{O}_{\mathbb{P}^1}(1)$ gives a common instance of these constructions: its local weight has a slope jump, its Chern measure is the unit mass at the Gauss point, and its Green function satisfies Poincaré–Lelong. The metric and height calculations are given together in Examples 20.3.1 and 22.2.1. A constant rescaling leaves the curvature fixed but changes the Green function; the local-height recursion records that change.

18.5 What a comparison theorem preserves

The constructions in the preceding chapters can be compared at several levels: functions, metrics, forms, currents, measures and cohomology classes. Moving from one level to another usually forgets information. The following elementary calculations explain the resulting limitations. They are useful when a theorem is applied outside the setting in which it was first stated.

18.5.1 A top-degree measure does not determine every curvature factor

On $\mathbb{R}^2$, consider the convex quadratic functions

$$u(x) = \tfrac{1}{2}x_1^2 + x_2^2, \qquad v(x) = x_1^2 + \tfrac{1}{2}x_2^2.$$

Their Hessian matrices are $\operatorname{diag}(1,2)$ and $\operatorname{diag}(2,1)$. Thus $DDu \neq DDv$, while

$$(DDu)^2 = (DDv)^2 = 4\, d'x_1 \wedge d''x_1 \wedge d'x_2 \wedge d''x_2.$$

This equality is simply equality of the determinants. It neither identifies the matrices nor identifies their mixed products with an independently chosen form. For instance, wedging with $d'x_1 \wedge d''x_1$ detects the coefficient in the second direction and gives different results.

There is no conflict with a uniqueness theorem for a Monge–Ampère equation in a fixed class. Such a theorem supplies additional information: a fixed reference curvature, a boundary or asymptotic condition, and usually a normalization of the potential. The two quadratic functions above do not satisfy the same prescribed asymptotic condition. On a real torus their constant Hessians also define different cohomology classes.

Consequently, the identification of a Chambert–Loir measure with a top-degree analytic product is an important comparison theorem, but it is not by itself an identification of the underlying differential sheaves. The latter requires comparison in every bidegree and compatibility with restriction, multiplication and differentiation. The separate constructions in Chapters 8, 16 and 17 provide precisely the additional objects that must be compared.

18.5.2 A tropical pushforward can lose an analytic measure

Example 18.5.1. Let K be complete and nontrivially valued. In $\mathbb{G}_m^{\mathrm{an}}$, let x be the rational point $T = 1$ and let y be the Gauss point with $|T(y)| = 1$. Then $x \neq y$, but both have tropical coordinate $-\log|T| = 0$. Hence

$$\operatorname{trop}_*\, \delta_x = \operatorname{trop}_*\, \delta_y = \delta_0.$$

The bounded continuous function $|T-1|/(1+|T-1|)$ has value zero at x and value $1/2$ at y, so the two analytic measures are distinct.

For measures supported on a prescribed section $s : N_{\mathbb{R}} \to X^{\mathrm{an}}$ of tropicalization, the situation changes. If $\mu = s_*\nu$, then $\mathrm{trop}_* \mu = \nu$, and therefore

$$\mu = s_* \,\mathrm{trop}_* \mu.$$

The section makes recovery possible. In the toric Monge–Ampère comparison, invariance and the canonical section are substantive hypotheses: they identify a class on which tropical pushforward retains all the required information. Removing those hypotheses would produce the ambiguity of Example 18.5.1.

The same reasoning applies to a finite family of moment coordinates. A larger family can distinguish more points, and Payne's inverse-limit theorem uses a sufficiently rich system of them all. A result on one selected tropicalization must not be read as a result on that inverse limit without checking compatibility under refinement. Chapter 6 proves the affine reconstruction and identifies exactly where the complete coordinate family enters.

18.6 Changing the test space of a current

Let $E_0 \subset E$ be real locally convex Hausdorff vector spaces, where the inclusion is continuous but the topology on E_0 need not be the topology induced by E. Continuous duals have a restriction map

$$r : E' \longrightarrow E_0', \qquad r(T) = T|_{E_0}.$$

The following facts separate detection of a current from extension of a current. They are elementary functional analysis, but both issues occur when original smooth tests are compared with weakly smooth or delta-form tests.

Proposition 18.6.1. *The map r is injective if and only if E_0 is dense in E. A functional $T_0 \in E_0'$ belongs to its image if and only if there is a continuous seminorm p on E such that*

$$|T_0(v)| \le p(v) \qquad (v \in E_0).$$

Proof. A continuous functional vanishing on a dense subspace is zero. Conversely, if the closure of E_0 is a proper closed subspace, the Hausdorff locally convex quotient by that closure has a nonzero continuous functional, by Hahn–Banach separation. Its pullback is a nonzero element of $\ker r$.

If T_0 extends continuously, continuity of the extension bounds it by a constant times the maximum of finitely many continuous seminorms; absorb

that constant into p. Conversely, the stated domination lets the real Hahn–Banach theorem extend T_0 to a linear functional on E dominated by p. That extension is continuous. □

Example 18.6.2 (Density does not imply existence of extension)**.** Take $E = C([0,1])$ with the uniform norm and $E_0 = C^\infty([0,1])$ with its usual smooth topology. The inclusion is continuous and dense, so restriction is injective. The functional $T_0(f) = f'(0)$ is continuous on E_0, but has no continuous extension to E. Indeed,

$$f_j(t) = j^{-1}\sin(jt), \qquad \|f_j\|_\infty \le j^{-1}, \qquad T_0(f_j) = 1.$$

The required uniform-norm bound is impossible.

For actual differential tests, this means that a statement about the direction of an arrow between sheaves does not by itself determine injectivity or surjectivity of the dual arrow. One needs density for uniqueness and a continuity estimate for existence. Moreover, the Hahn–Banach extension in Proposition 18.6.1 does not automatically preserve positivity, closedness, wedge products or geometric representatives. Each of those requirements adds another condition.

An equality first checked on decomposable exterior-product tests has the same issue. It extends to the full test space if their span is dense in the relevant test topology and both sides are continuous there. An algebraic spanning statement on one chart is not a substitute for this topological density on the actual quotient. In the Green-form constructions of Chapter 22, the diagonal and cutoff arguments are used to reach all actual compactly supported tests, not just one chosen family.

18.7 A common chain of comparison and its hypotheses

Consider a line bundle with a continuous metric. To pass from its geometric construction to an intersection measure, one often uses the following chain of results. Each arrow has a distinct mathematical content.

First, choose local frames and form their weights $u_e = -\log\|e\|$. On overlaps, $u_{e'} - u_e = -\log|g_{ee'}|$. Analytic transition functions are units, so

their logarithms are original smooth and have zero Hessian. This proves that the local currents $DD[u_e]$ glue. It says nothing yet about their positivity.

Second, a model or curvewise criterion may establish positivity in its specified class. If the available theorem produces a locally uniform positive approximation on one fixed neighbourhood, then the weights lie in the admitted class of Chapter 11. This is stronger information than a bare inequality against current tests. For real PL metrics, Chapter 19 identifies the hypotheses under which the geometric criterion supplies that approximation.

Third, the continuous product theorem defines local mixed currents. Gluing these products uses the special transition functions of the line bundle. Their smooth zero-Hessian property gives the required kernel action on closed backgrounds; there is no need to assert that every continuous function with zero ambient curvature has the same property. The distinction is essential in view of the cross example.

Finally, comparison with the model measure is proved on the geometric approximants and passed to the limit in the theorem's topology. At this stage one obtains a top-degree measure. Total mass is controlled by the algebraic intersection number on a proper space. Neither properness nor that mass identity proves local regularization of arbitrary current-psh weights. They enter after the potential class and its products have been established.

The construction therefore combines different contributions without identifying their domains by notation: CLD supplies the intrinsic differential language; Gubler's algebraic presentation makes the original charts accessible in algebraic geometry; GK retains singular polyhedral curvature in an intersection algebra; GJR and GR supply harmonic and curvewise geometric comparisons; model and toric theories identify the resulting measures and heights in their respective classes. The proofs used at one stage should be cited at that stage, with their field, regularity and test-space assumptions retained.

References. The examples and functional-analytic criteria above are elementary. The geometric comparisons are the separately stated theorems of [CLD12, Gubler, GK17, GK19, GJR, GRPL, GRPSH, BGJK25]; the proof locations in this book are Chapters 8, 13, 16, 19 and 20.

Part V

Metrics and measures

Chapter 19

Model metrics and semipositivity

The construction of a formal metric, including frame independence and invariance under an admissible blow-up, is Proposition 3.11.1. We now study the additional numerical positivity conditions. A residual line bundle is represented on models of a germ reduction as explained in Section 3.10; its degree conditions retain information beyond the real dimension of a chosen skeleton.

For the distinction between formal and algebraic domination of models, and for cofinal models carrying ample extensions, see Section 11.15.1. The separate reduced-fibre hypothesis and its effect on section norms are treated in Sections 11.15.2–11.15.3.

19.1 Model semipositivity and the role of curve tests

19.1.1 A definition using formal models

Let K be algebraically closed, complete and nontrivially valued, let X be a strictly K-analytic space, and let L be an analytic line bundle. A metric is piecewise linear if there is a covering in the analytic Grothendieck topology on which L has frames of norm identically one. On a compact reduced strictly analytic space this is equivalent to being induced by a formal model [GK19,

§5.9 and §6.1]. The topology in this statement is material: a frame on an analytic domain need not extend to an ordinary open neighborhood.

Definition 19.1.1 (Local model semipositivity)**.** Suppose first that X is reduced. A piecewise linear metric on L is semipositive at $x \in X$ if there are a compact strictly analytic neighborhood V of x and a formal model $(\mathfrak{V}, \mathfrak{L})$ inducing the metric such that

$$\deg(\mathfrak{L}|_C) \geq 0$$

for every integral closed curve $C \subset \mathfrak{V}_s$ that is proper over the residue field. For nonreduced X, impose the condition on X_{red}. The metric is semipositive on an open set if it is semipositive at each of its points.

This is [GK19, Definition 6.2]. Independence from the inducing model uses common dominating models and the projection formula [GK19, Lemma 6.3]. For a proper algebraic model, this recovers numerical nonnegativity of its line bundle on every special-fibre curve. It is a local non-Archimedean condition; it does not by itself specify an adelic metric at all places.

Theorem 19.1.2 (The algebraic piecewise linear comparison)**.** *Retain the field hypotheses above. Let X be an algebraic variety over K, let L be a line bundle, and let $W \subset X^{\mathrm{an}}$ be open. For a piecewise linear metric on $L|W$, the following conditions are equivalent:*

1. *the metric is semipositive on W in Definition 19.1.1;*
2. *after every algebraic morphism $f : X' \to X$, the pulled-back metric is delta-psh on $(f^{\mathrm{an}})^{-1}(W)$;*
3. *after every such morphism, the pulled-back metric is current-psh for the original smooth test theory;*
4. *its first Chern delta-form is positive on W;*
5. *for every closed integral algebraic curve $C \subset X$, the restriction to $W \cap C^{\mathrm{an}}$ is current-psh.*

Open proof obligation (RC2). Prove the model equivalence in the stated scope, including curve closure and the detection of each model degree; the text uses GK19 Theorems 6.10, 0.1 and 4.1 and Lemma 6.8.

Proof retained in the RC2 evidence archive. Prove the model equivalence in the stated scope, including curve closure and the detection of each

model degree; the text uses GK19 Theorems 6.10, 0.1 and 4.1 and Lemma 6.8.

Remark 19.1.3 (What the theorem does not identify). The condition $d'd''[-\log\|s\|]_D \geq 0$ on W alone is not one of the five equivalent conditions as stated in [GK19, Theorem 6.10]. Nor does the theorem concern arbitrary continuous metrics in place of piecewise linear ones. These are changes of hypotheses that require separate results.

19.1.2 Analytic curve tests in the 2025 comparison

The analytic extension in [GRPL] is best stated for functions. Here k is any complete nontrivially valued field, and X is a good strictly k-analytic space.

Definition 19.1.4 (Real piecewise linear function)**.** A function $h : X \to \mathbf{R}$ is piecewise linear if, locally at each point, finitely many strictly analytic domains V_j cover a neighborhood and there are $g_j \in \mathcal{O}_X(V_j)^\times$ such that $h|_{V_j} = \log|g_j|$. It is real piecewise linear if it is locally a finite real linear combination of such functions.

For a real piecewise linear h, the metric $\|1\| = e^{-h}$ is locally induced by a formal model of the trivial line bundle with real coefficients. Semipositivity at a point means that this model can be chosen with nonnegative degree on each proper curve in the special fibre. Equivalently, the residue class

$$\mathcal{L}_h(x) \in \mathrm{Pic}(\mathrm{red}(X,x)) \otimes_{\mathbf{Z}} \mathbf{R}$$

is nef. Here $\mathrm{red}(X,x)$ is Temkin's reduction of the germ; the equivalence with the formal-model definition is part of the comparison recalled in [GRPL, §5].

Theorem 19.1.5 (Curve tests after analytic extension)**.** *Let $h : X \to \mathbf{R}$ be real piecewise linear. The following conditions are equivalent:*

1. *h is semipositive at every point of X;*
2. *there exists an algebraically closed analytic extension F/k such that, for every smooth analytic curve C/F and every morphism $\varphi : C \to X_F$, the function $h_F \circ \varphi$ is semipositive on C.*

When these conditions hold, the corresponding curvewise assertion holds over every analytic field extension. On smooth analytic curves, semipositivity of a real piecewise linear function is equivalent to subharmonicity in Thuillier's sense.

Open proof obligation (RC2). Supply the exact curve and scalar tests and their relationship with model positivity; the text invokes GRPL Proposition 5.1 and Corollary 5.4.

Theorem 19.1.6 (Closure and local approximation in the real PL class)**.** *Let X be good and strictly k-analytic.*

1. *If h_j are semipositive real piecewise linear functions converging pointwise to a real piecewise linear function h, then h is semipositive.*
2. *If X is boundaryless and h is real piecewise linear, then h is semipositive at x if and only if on some strictly analytic neighborhood of x it is a uniform limit of locally Fubini–Study functions.*

Open proof obligation (RC2). Prove the stated uniform closure and approximation claims with their precise hypotheses; GRPL Theorems 1.4 and 1.5 are supplied.

Remark 19.1.7 (Current positivity and the dated status statement)*.* The curvewise condition in Theorem 19.1.5 is not defined by testing a single current on X. The March 2025 version of [GRPL, Remark 1.8] explicitly records as unresolved whether a real piecewise linear function that is current-psh for the original CLD test forms must be semipositive, and relates this to pullback stability. This is a status statement about that version. Later claims must be compared with it using their definitions and hypotheses; the analytic curve theorem alone does not settle this additional comparison.

19.1.3 Uniform limits of model metrics

For a line bundle on a proper K-variety, a semipositive approximable metric in Zhang's sense is a uniform limit of semipositive rational-formal metrics. Uniform convergence means uniform convergence of the logarithm of the ratio to a fixed continuous reference metric.

Proposition 19.1.8 (Recognizing a formal metric in the closure)**.** *Assume the ground field hypotheses of Theorem 19.1.2, and let L be a line bundle on a proper K-variety. A formal metric on L is semipositive if and only if it is a uniform limit of semipositive rational-formal metrics.*

Open proof obligation (RC2). Prove the formal metric closure and current comparison in the stated scope; GK19 Proposition 7.2 supplies the comparison.

The proposition recognizes semipositivity when the limiting metric is already formal. An arbitrary continuous limit need not be induced by one formal model. Similarly, a topology controlling adelic boundary behavior requires additional data beyond the uniform convergence of metrics on a proper analytic space.

19.2 Curvewise positivity and the 2025 comparison results

The curvewise definition of Gubler–Rabinoff in [GRPSH] is relevant to the comparison with both versions of CLD: it tests pullbacks of functions to curves and does not begin by pulling back a current. In this section, "classically psh" always has the meaning of [GRPSH, Definition 4.1].

19.2.1 The quantifiers in the curvewise definition

Let K be any complete non-Archimedean field, possibly trivially valued, and let X be a K-analytic space. For an analytic extension K'/K, write

$$\pi_{K'/K} : X_{K'} = X\widehat{\otimes}_K K' \longrightarrow X.$$

Definition 19.2.1 (Classically plurisubharmonic functions)**.** An upper semicontinuous function $u : X \to \mathbb{R}\cup\{-\infty\}$ is *classically psh* if, for every analytic extension K'/K, every K'-analytic curve C, and every K'-analytic morphism $f : C \to X_{K'}$, the function

$$u \circ \pi_{K'/K} \circ f$$

is subharmonic on C in the sense of [GRPSH, Definition 3.4].

Here an analytic curve is separated and of pure dimension one. We spell out the one-dimensional condition, including the convention at $-\infty$, to make the definition usable without an implicit change of category.

Definition 19.2.2 (Harmonic comparison on curves)**.** First let F be nontrivially valued and let C be a smooth strict F-analytic curve. On a strictly affinoid domain $A \subset C$, a harmonic function is, locally after finite separable base extension, the pullback from a semistable skeleton of a real piecewise affine function whose weighted outgoing slopes sum to zero at every interior vertex. No balancing condition is imposed at boundary vertices.

An upper semicontinuous function $v : C \to \mathbb{R} \cup \{-\infty\}$ is subharmonic in Thuillier's sense if it is not identically $-\infty$ on any connected component and if, for every strictly affinoid domain $A \subset C$ and every harmonic $a : A \to \mathbb{R}$,

$$v \leq a \text{ on } \partial A \quad \Longrightarrow \quad v \leq a \text{ on } A.$$

For a possibly singular or non-strict curve C over an arbitrary complete field F, Gubler–Rabinoff require that, after every *nontrivially valued* analytic extension F'/F and every morphism from a connected smooth strict F'-analytic curve into $C_{F'}$, the pullback is either Thuillier-subharmonic or identically $-\infty$.

The harmonic comparison and the general-curve extension are [GRPSH, §§3.1–3.3 and Definition 3.4]. Thus on a connected smooth strict curve over a nontrivially valued field, the general definition is Thuillier's class with the constant function $-\infty$ adjoined. This explicit enlargement is necessary for decreasing limits. The source's §4.1.3 shows that in Definition 19.2.1 it is enough to test connected smooth strict curves after analytic base extension; combining this with the curve definition permits nontrivially valued extensions throughout. The quantifier over extension fields must not be dropped.

Proposition 19.2.3 (Functoriality and the maximality property)**.** *Classically psh functions form a sheaf, and they are stable under analytic pullback and base change. Suppose $\mathcal{F}$ is any assignment of upper semicontinuous functions to analytic spaces which is stable under these operations and whose functions on analytic curves are subharmonic. Then every member of $\mathcal{F}(X)$ is classically psh.*

Open proof obligation (RC2). Prove the common-extension step in the converse direction; GRPSH Proposition 4.2(5).

Thus the curve-test definition gives a *maximal* compatible class, not a definition by taking the smallest class generated from smooth functions. This logical distinction matters when comparing it with the generation process in CLD25. The word "smallest" in the introductory description in [GRPSH, §1.2] should not be used as a replacement for its operative Definition 4.1; the implication in Proposition 19.2.3 follows directly from that definition.

19.2.2 What the PL comparison actually proves

Theorem 19.2.4 (Curvewise positivity and nef models)**.** *Assume that the valuation of K is nontrivial. Let X be good and strictly K-analytic, and let $h : X \to \mathbb{R}$ be an $\mathbb{R}$-PL function. Then the following conditions are equivalent:*

1. *h is semipositive in the local model sense;*
2. *the residue class $L_h(x)$ is nef at every $x \in X$;*
3. *h is classically psh.*

Open proof obligation (RC2). Prove all equivalences between the model, residual and curvewise classes; the text supplies GJR Proposition 7.1 and GRPSH Theorem 4.3.

Corollary 19.2.5 (Two harmonicity definitions on the PL class)**.** *Under the hypotheses of Theorem 19.2.4, an $\mathbb{R}$-PL function h is harmonic in the GJR model sense if and only if both h and $-h$ are classically psh.*

Proof. Apply Theorem 19.2.4 to h and to $-h$. The two residue classes are nef exactly when $L_h(x)$ has degree zero on all the curves used to test nefness, which is numerical triviality. □

Remark 19.2.6 (A PL hypothesis cannot be discarded). Gubler–Rabinoff call a real function *pluriharmonic* when both it and its negative are classically psh [GRPSH, Definition 7.1]. These functions are continuous. Corollary 19.2.5 identifies them with GJR harmonic functions only after imposing the $\mathbb{R}$-PL condition. The cited paper proves this PL condition for a good quasicompact strict analytic space admitting a strictly semistable formal model [GRPSH, Proposition 7.8]; it does not assert that condition for every analytic space.

19.2.3 The relation with CLD25 psh functions

We now restrict to a good Hausdorff boundaryless strictly analytic space of pure dimension over a nontrivially valued field. We use $\mathcal{P}_{25}$ from Definition 11.2.2 and write $\mathrm{Psh}_{\mathrm{cl}}$ for the class of Definition 19.2.1.

Proposition 19.2.7 (A verified one-way comparison)**.** *Every CLD25 psh function is classically psh:*

$$\mathcal{P}_{25}(X) \subseteq \mathrm{Psh}_{\mathrm{cl}}(X).$$

Consequently, a CLD25 psh function which is also $\mathbb{R}$-PL is semipositive in the model sense.

Open proof obligation (RC2). Prove the curvewise comparison in the precise source category; the argument appeals to GRPSH section 1.2.

This argument uses pullbacks of functions at each stage of the construction. It does not supply a pullback of an arbitrary current. In particular, it cannot start with only the assumption $d'd''[u] \geq 0$ against the original CLD test forms unless an additional comparison has already placed u in the CLD25 class.

Remark 19.2.8 (A distinction between forward implications). For an $\mathbb{R}$-PL function on the common boundaryless setting, the source-supported implications include

$$\mathcal{P}_{25} \implies \mathrm{Psh}_{\mathrm{cl}} \iff \text{model semipositivity} \implies d'd''[u] \geq 0.$$

The middle equivalence is restricted to PL functions; the final implication is [GJR, Theorem 7.14]. The inclusion at the left is not an assertion that the two full sheaves of possibly singular functions are equal. A separate claim in CLD25 about a converse current criterion must be assessed at its own stated hypotheses and proof, rather than inferred from the curvewise theorem.

Proposition 19.2.9 (Pointwise stability in the curvewise class)**.** *Let (u_j) be a net of classically psh functions on a K-analytic space. If its pointwise limit $u : X \to \mathbb{R} \cup \{-\infty\}$ is upper semicontinuous, then u is classically psh.*

Open proof obligation (RC2). Prove the curvewise psh limit property and its common-extension step in the exact scope, without using GRPSH Theorem 6.5 as proof.

19.2.4 An independent identification on the PL class

The following comparison combines the local model approximation in GJR with the curvewise criterion of Gubler–Rabinoff. Its proof does not use the equality of the two continuous generated sheaves claimed in CLD25 Proposition 8.1.11, or the converse current criterion claimed in its Theorem 18.8.4.

Lemma 19.2.10 (Locally Fubini–Study functions belong to the continuous generated class)**.** *Let X be good and strictly analytic over a nontrivially valued complete field K. If an $\mathbb{R}$-PL function h is locally Fubini–Study, then $h \in \mathcal{C}_{25}(X)$.*

Open proof obligation (RC2). Establish the finite nonnegative combination of maxima of analytic-unit logarithms on an ordinary neighbourhood in the stated scope; GJR section 7.9 supplies this presentation.

Proposition 19.2.11 (Four equivalent positivity conditions for PL functions)**.** *Let K be complete and nontrivially valued, and let X be a good Hausdorff boundaryless strictly K-analytic space of pure dimension n. For an $\mathbb{R}$-PL function $h : X \to \mathbb{R}$, the following conditions are equivalent:*

1. *h is semipositive in the local model sense;*
2. *$h \in \mathcal{C}_{25}(X)$;*
3. *$h \in \mathcal{P}_{25}(X)$;*
4. *h is classically psh.*

Each of these conditions implies positivity of the original current $d'd''[h]$.

Open proof obligation (RC2). Prove the fixed-neighbourhood Fubini–Study approximation and the reverse curvewise model implication; GJR Proposition 7.10 and GRPSH Theorem 4.3.

Remark 19.2.12 (What the PL identification leaves separate). This proposition does not identify $\mathcal{C}_{25}$ with $C^0 \cap \mathcal{P}_{25}$ for arbitrary continuous functions. It also does not prove that every semipositive PL function lies in the one-step smooth approximation class $\mathcal{U}$: the local charts used to regularize different LFS approximants may vary. Most importantly, the proof does not supply a converse starting only with $d'd''[h] \geq 0$. The four-condition equivalence is obtained through model approximation and curvewise tests, independently of that additional current criterion.

19.3 The PL curvature comparison stated in CLD25

There is a substantive additional claim in the July 2025 text. It must be stated accurately even though the elementary implications in the preceding sections do not prove it.

Remark 19.3.1 (The residual criterion stated in the source). Let X be a Hausdorff boundaryless analytic space and let L carry an integral PL metric in the terminology of CLD25. At a point x, write $\widetilde{L}_x$ for the residual line bundle on the reduction of the analytic germ. The degree-one case of [CLD25, Theorem 18.8.4] states

$$\begin{aligned} &c_1(L, \|\cdot\|) \geq 0 \text{ on some neighbourhood of } x \\ \iff\quad &\widetilde{L}_x \text{ is numerically nef.} \end{aligned} \tag{19.3.1}$$

The full theorem treats mixed products of integral PL curvature currents and residual intersection classes. It is not a theorem about arbitrary finite continuous metrics.

The residual numerical condition is an intersection-theoretic condition: on a suitable proper premodel of the reduction, the residual line has nonnegative degree on the relevant proper integral curves. It is neither an effective-section condition nor the definition of positivity of a test current. The forward direction in (19.3.1) is consequently a substantial comparison.

The source proof reduces near a germ to residual dimension of definition, computes the curvature along fibres when the relevant dimension equals the number of factors, and uses a negligibility argument for the remaining locus. That proof relies on the earlier construction of residual bundles, tropical calibration, and fibre integration. The elementary counterexamples in Section A.1 do not refute this theorem. Conversely, those earlier constructions are not replaced by the positivity definition or by the scalar estimate in Proposition 11.1.2.

Remark 19.3.2 (The precise boundary of verification). Theorem 19.18.12 reconstructs the current criterion in the pure-dimensional category, including mixed products. It supplies the Laurent realization and noncharging argument while citing the deeper reduction, lifting and fibre-integration theorems. Those inputs have not all been independently re-proved here. Theorem 19.18.15 now proves first-stage approximation for integral PL metrics

with semiample residue, and Corollary 19.18.18 gives the generated-psh comparison for strictly integral metrics over a nontrivially valued field. The general integral comparison is now proved by Corollary 19.18.23, using prescribed graded realization. A converse for arbitrary continuous metrics remains separate. The comparison with the earlier status statement in [GRPL, Remark 1.8] is chronological: that statement dates from March 2025, whereas CLD25 dates from July 2025.

There are local issues which can be identified precisely. The numerical zero-cycle test on p. 539 must use subvarieties of dimension equal to the number of Chern factors. The proof of Corollary 18.8.8 also invokes a strictly integral PL result under a more general integral PL hypothesis; the required reduction is not provided there. Theorem 19.18.22 supplies a graded proof in the integral category without that strictness reduction. These points are described in Remark A.1.7. They do not justify a blanket rejection of the residual criterion.

Remark 19.3.3 (What an extension to continuous metrics would require). Suppose a finite continuous metric satisfies $c_1(L, \|\cdot\|) \geq 0$. To apply a theorem about PL metrics to it, one would first need an appropriate family of PL approximants whose semipositivity is controlled on a fixed neighbourhood or on a fixed global line. Density of arbitrary model functions among continuous functions gives no such positivity. Likewise, positivity of the weak limit of currents gives no sign information about every approximating current. Thus even accepting (19.3.1) does not produce a converse for all continuous metrics.

19.4 Model semipositivity and global Monge–Ampère theory

19.4.1 Gubler–Martin: a common model language

Gubler–Martin studies semipositive metrics over general complete non-Archimedean fields, removing a recurring division between discretely valued fields and algebraically closed fields. Algebraic, formal, and piecewise linear descriptions are compared in their stated proper or strictly analytic settings. Density of rational PL metrics gives a flexible supply of approximating metrics.

Density alone is not a positivity theorem. Approximating a continuous weight by arbitrary PL functions need not preserve convexity or model nefness. A separate result says that pointwise limits of semipositive model metrics, when the limiting metric is itself a model metric in the theorem's class, remain semipositive. The endpoint hypothesis matters: it is not a claim that every arbitrary pointwise limit is a finite continuous model metric.

The proof uses model geometry and numerical tests. It is not a corollary of weak closure of the positive-current cone unless a converse comparison between current positivity and model nefness has already been proved. This illustrates a recurrent relation to CLD: model theorems can supply an independent route to positivity and then be compared with analytic currents.

19.4.2 Two objects both called a closed $(1,1)$-form

In model-based variational pluripotential theory, a closed $(1,1)$-form may mean an element of a direct limit

$$\mathcal{Z}^{1,1}(X) = \varinjlim_{\mathcal{X}} N^1(\mathcal{X}/K^\circ)_{\mathbb{R}},$$

with transition maps given by pullback to dominating models. A model function has a curvature class in this space. This is an algebraic numerical object, not the original CLD sheaf of smooth $(1,1)$-forms.

Given θ in this model space, a model function ϕ is θ-psh when $\theta + dd^c\phi$ is represented by a nef class. The notation resembles a differential positivity condition because the intended analogy is strong. The definition still requires a comparison theorem before it can be replaced by positivity against all CLD tests.

For metrics in classes where the measure comparison is established, the resulting top-degree Monge–Ampère measure agrees with the analytic or model measure. That common endpoint does not identify all lower-degree objects or all singular potential classes.

19.4.3 Envelopes and the variational method

For a continuous obstacle f, the θ-psh envelope is, schematically, the upper regularization of the supremum of admitted θ-psh functions below f. The difficult question is whether the envelope remains continuous and has the

required approximation properties. This is where algebraic positivity, multiplier or test ideals, and global generation enter.

The energy functional is built from mixed intersections of a reference metric and the variable metric. In an admissible continuous class, its first variation is integration against the Monge–Ampère measure. Maximizing energy minus the integral against a prescribed measure leads to the equation

$$\mathrm{MA}(\phi) = \mu.$$

Turning this variational principle into a theorem requires compactness or coercivity, envelope regularity, and a differentiability theorem. The formal Euler–Lagrange equation alone is not a solution method.

This global programme concerns a fixed ample line bundle on a proper variety and a normalized measure of the correct total mass. It is different from proving a local potential theorem for every positive closed current on every analytic domain. Both problems are important, and neither formulation automatically implies the other.

19.4.4 Residue characteristic and the scope of the results

The foundational results of Boucksom–Favre–Jonsson assume residue characteristic zero in their initial forms. Gubler–Jell–Künnemann–Martin develops envelope continuity using test ideals in positive characteristic, with the geometric and resolution assumptions stated in that paper. Its surface case uses available resolution results in the relevant dimension.

Fang–Gubler–Künnemann treats mixed characteristic using global test-ideal methods. In the final version used here, Theorem 1.1 gives envelope continuity under the specified regular-model hypothesis, and more generally under resolution of singularities for the projective models. Theorem 1.2 solves the Monge–Ampère equation for positive Radon measures supported on the skeleton of an SNC-model, assuming both the stated resolution and embedded-resolution hypotheses. The solution is a continuous semipositive metric, unique up to scaling.

These assumptions must remain in the theorem. It would be incorrect to cite the result as an unconditional general mixed-characteristic local BT theory. Conversely, the conditional nature of the resolution input does not make the analytic and algebraic methods trivial: the test ideals supply the regularity needed for the variational argument.

References. [GM, BFJ15, GJKM, FGK].

19.5 Maximum principles from analytic curves

The curvewise definition has a useful consequence which does not depend on smooth regularization: maxima can be detected geometrically. Two inputs of Gubler–Rabinoff are responsible. First, a connected boundaryless analytic space is connected by overconvergent analytic curves after suitable analytic field extensions. Second, the boundary of a pure-dimensional affinoid has a dimension-decreasing stratification whose minimal strata are its Shilov points. We state where these geometric results enter and prove their potential-theoretic consequences.

Theorem 19.5.1 (Local maximum principle for the curvewise class)**.** *Let u be classically psh on a K-analytic space X. If u attains a local maximum at $x \in \mathrm{Int}(X)$, then u is constant on a neighbourhood of x.*

Open proof obligation (RC2). Supply the overconvergent curve-connectivity bridge and the curve maximum principle used on the lifted fibres; GRPSH Theorem 5.4 and Proposition 3.5(6).

The compact-fibre argument is needed: local constancy upstairs at individual points of a fibre does not alone give local constancy on its image. Compactness permits all those neighbourhoods to be used simultaneously. The analytic extension fields are likewise part of the curve definition; they cannot be omitted merely by restricting to curves defined over the original field.

Theorem 19.5.2 (The Shilov maximum principle)**.** *If u is classically psh on an affinoid space X with Shilov boundary S, then*

$$\max_X u = \max_S u.$$

Open proof obligation (RC2). Prove the boundary stratification down to Shilov points; GRPSH Lemma 6.3 is used in addition to the open local maximum principle.

References. These are [GRPSH, Theorems 6.1 and 6.4], with the geometric inputs in Theorem 5.4 and Lemma 6.3 separated from their consequences. In the proof of Theorem 6.4 in the consulted v2, the phrase asserting continuity of the psh function should be replaced by upper semicontinuity; that weaker property already

makes its maximum set closed. This is a repair of the wording of the proof, not a failure of the maximum theorem.

19.6 Why pointwise limits suffice in the curvewise class

Theorem 19.6.1. *Let (u_j) be a net of classically psh functions converging pointwise to $u : X \to \mathbb{R} \cup \{-\infty\}$. If u is upper semicontinuous, then it is classically psh. No monotonicity assumption is needed.*

Open proof obligation (RC2). Supply the native curve comparison maximum principle used after pointwise convergence on the finite boundary. The present route names the now-open Shilov maximum principle as an alternative.

The finiteness of the testing boundary is the decisive feature. It converts separate pointwise inequalities into one simultaneous eventual inequality. There is no need to dominate a moving family of current coefficients, and no product of currents occurs. This is why the result is a pointwise-closure theorem for a function class, rather than a Bedford–Taylor continuity theorem.

References. [GRPSH, Theorem 6.5]. The distinction from convergence of mixed products is explained in Chapters 9 and 13.

19.7 Finite-dimensional spaces of pluriharmonic functions

Write $\mathrm{PH}(X)$ for the real functions h for which both h and $-h$ are classically psh. Such a function is continuous, since it is both upper and lower semicontinuous. The class is a real vector space by the convex-cone property of classically psh functions.

Theorem 19.7.1. *Let a quasicompact analytic space X be covered by finitely many affinoid domains $U_1, \dots, U_s$, and let S be the union of their Shilov boundaries. Then evaluation induces an injective linear map*

$$\mathrm{PH}(X) \longrightarrow \mathbb{R}^S, \qquad h \longmapsto (h(s))_{s \in S},$$

and in fact $\|h\|_{C^0(X)} = \max_{s \in S} |h(s)|$. In particular $\dim \mathrm{PH}(X) \leq |S|$.

Open proof obligation (RC2). The Shilov maximum principle for both signs is open in this projection; prove it or give an independent norm-detection route.

Corollary 19.7.2. *On a quasicompact X as above, a net in* $\mathrm{PH}(X)$ *which converges pointwise to a finite real function converges uniformly, and its limit is pluriharmonic.*

Open proof obligation (RC2). The finite-boundary norm and evaluation theorem used for the finite-dimensional argument remains open.

The corollary is the finite-dimensional norm argument behind the locally uniform convergence result of Gubler–Rabinoff. On this quasicompact domain, the argument works for nets and itself establishes continuity of the limit. It is a direct consequence of the finite-boundary evaluation theorem, not a new regularization theorem. On a connected compact boundaryless space, the local maximum principle further gives $\mathrm{PH}(X) = \mathbb{R}$.

The finite set S is a determining set for this particular pluriharmonic space. It does not determine arbitrary continuous functions, arbitrary currents, or every cohomology class of a differential sheaf. Likewise, identifying this kernel with the kernel of an original CLD Hessian on a larger function class requires a comparison theorem. The word pluriharmonic does not erase that requirement.

References. [GRPSH, Proposition 7.3 and Theorems 7.4, 7.6]. The net corollary above is the elementary finite-dimensional consequence of their evaluation map, written with its quasicompact-domain hypothesis.

19.8 Numerical nefness under proper pullback

The model comparison uses two kinds of independence. Numerical nefness is unchanged by a proper surjective pullback, while a model metric is unchanged by replacing the model with a dominating one. We first prove the numerical statement over an ordinary field, then explain how it enters the model calculation.

Proposition 19.8.1. *Let $f : Y \to Z$ be a proper surjective morphism of proper varieties over a field k, and let M be a line bundle on Z. Then M is nef if and only if f^*M is nef. The statement also holds for real linear combinations of line bundles.*

Proof. Let $C \subset Y$ be an integral proper curve. If $f(C)$ is a point, $f^*M|_C$ is trivial and has degree zero. Otherwise its image is an integral curve D, and the cycle projection formula gives

$$\deg_k(f^*M|_C) = [k(C) : k(D)] \deg_k(M|_D).$$

Thus nefness of M implies nefness of f^*M. Conversely, fix an integral curve $D \subset Z$. The fibre over its generic point is a nonempty finite-type scheme over $k(D)$. A closed point of that fibre has residue field finite over $k(D)$. Its closure in Y, with reduced structure, is an integral proper curve C mapping onto D. Applying the displayed equality and dividing by the positive field degree shows $\deg_k(M|_D) \geq 0$. This proves the converse. All degree expressions are linear in the line-bundle class, giving the real-coefficient version. □

The proof includes inseparable morphisms: the field degree is the full algebraic degree. Replacing it by the number of distinct geometric points of a fibre would invalidate the argument in positive characteristic. It also explains why nefness is tested on curves, even when the model has much larger dimension.

19.9 Equal model metrics on a common normal model

For the next two results we restrict to a complete discretely valued field K, with valuation ring R, uniformizer π and $\lambda = -\log|\pi|$. Let X be an integral projective K-variety. All models are normal integral projective flat R-models with generic fibre identified with X. The normal and discretely valued hypotheses permit a direct Cartier-divisor calculation; this proof is not a substitute for the general formal-model comparison over arbitrary valuation rings.

Lemma 19.9.1. *Let $\mathcal{X}_s = \sum_Y m_Y Y$, and let $D = \sum_Y a_Y Y$ be a vertical Cartier divisor. Equip the generically trivial bundle $\mathcal{O}(D)$ with its model metric and use its canonical rational section s_D as the generic trivialization. At the divisorial point ξ_Y,*

$$-\log \|s_D(\xi_Y)\| = \frac{a_Y}{m_Y}\lambda.$$

In particular, a vertical model metric with identically zero weight has $D = 0$.

Proof. At the generic point of Y, the normal model has a discrete valuation ring. Its valuation ord_Y satisfies $\mathrm{ord}_Y(\pi) = m_Y$. The analytic valuation extending the fixed one on K is therefore $v_{\xi_Y} = (\lambda/m_Y)\,\mathrm{ord}_Y$. If a local equation of D is g, the canonical section is g times the local model frame, which has norm one. Hence its weight is $v_{\xi_Y}(g) = a_Y\lambda/m_Y$. If all weights vanish, every Weil coefficient is zero. On a normal integral scheme the map from Cartier divisors to Weil divisors is injective, so $D = 0$. □

Proposition 19.9.2. *Suppose $\mathcal{L}$ and $\mathcal{L}'$ on two such models extend the same line bundle L with specified generic-fibre identifications. Assume a common normal projective model $\mathcal{Y}$ dominates both. If their induced metrics on L^{an} agree, the pullbacks of $\mathcal{L}$ and $\mathcal{L}'$ to $\mathcal{Y}$ are isomorphic compatibly with the generic identifications. Consequently their special-fibre nefness conditions agree.*

Proof. On $\mathcal{Y}$, the quotient of the two pulled-back line bundles is generically trivial. The specified generic trivialization determines a rational section whose Cartier divisor is vertical. The quotient metric has weight zero because the original metrics agree and model metrics pull back along the generic identity. Lemma 19.9.1 forces its divisor to vanish. The rational section is therefore a global nonvanishing regular section and gives the required isomorphism.

Each morphism on special fibres is proper and surjective. Apply Proposition 19.8.1 to their reduced components, or directly to all integral curves as in its proof. Nefness on the source is equivalent to nefness of the original line bundle. The two pulled-back bundles are isomorphic, so the two original nefness conditions agree. For rational model metrics, first raise both to a common positive tensor power. □

The existence of common dominating normal models is a geometric input. In this projective DVR setting it is obtained from the closure of the generic graph and normalization; finiteness of normalization uses the usual excellence hypotheses, satisfied by a complete DVR. In the general analytic setting the corresponding work is carried out by admissible formal modifications. The elementary divisor argument above begins after this input, rather than assuming that arbitrary formal models can be compared by a set-theoretic common refinement.

19.10 Why the atomic model measure survives a modification

We retain the normal projective DVR setting and use the ordinary projection formula for algebraic intersection cycles. Let $f : \mathcal{Y} \to \mathcal{X}$ be a proper birational model morphism inducing the identity on X. Put $\mathcal{Y}_s = \sum_E m_E E$ and let $\mathcal{L}_1, \dots, \mathcal{L}_n$ be model line bundles on $\mathcal{X}$, where $n = \dim X$.

Proposition 19.10.1. *The atomic sums defining the model measure on $\mathcal{X}$ and on $\mathcal{Y}$ with pulled-back line bundles are equal as measures on X^{an}. Exceptional components whose images have dimension less than n contribute zero.*

Proof. If $\dim f(E) < n$, the proper pushforward of the n-cycle $[E]$ is zero. The projection formula therefore gives

$$\deg_E(f^*\mathcal{L}_1 \cdots f^*\mathcal{L}_n) = 0.$$

If E dominates a component Y of $\mathcal{X}_s$, the local ring at the generic point of E dominates the discrete valuation ring at the generic point of Y. Both have the same fraction field, because f is birational. An overring of a discrete valuation ring in its fraction field is either that ring or the field; domination excludes the latter. Thus the two discrete valuation rings are equal. Properness and separatedness give a unique centre of this valuation on $\mathcal{Y}$, so there is exactly one such component over Y. It has the same residue function field, the same multiplicity of π, and the same normalized analytic valuation. Hence

$$m_E = m_Y, \qquad \xi_E = \xi_Y, \qquad \deg_E(f^*\mathcal{L}_1 \cdots f^*\mathcal{L}_n) = \deg_Y(\mathcal{L}_1 \cdots \mathcal{L}_n).$$

For completeness, the existence of that component is the codimension-one part of the proper birational model comparison: extending the generic inverse

over the discrete valuation ring gives its centre, whose closure dominates Y. Equivalently, a proper birational morphism to a normal variety is an isomorphism over the generic points of prime divisors. Summing the preceding equalities, with the zero contributions from exceptional components, proves equality of the two atomic measures. □

This proof isolates the two numerical mechanisms. Pullback does not create mass on exceptional components contracted to smaller-dimensional sets, and the old divisorial valuations retain their original masses. It does not say that every modification of a metric leaves the measure unchanged: replacing the pulled-back bundle by a twist involving an exceptional divisor changes the metric and can redistribute the masses. The blow-up example in Section 20.6 is of this latter kind.

References. [GK19, Lemma 6.3 and Theorem 6.10]; [GK17, Theorem 10.5]; [CL06]. The proofs here treat the normal projective DVR case; the more general formal-model statements retain the hypotheses of those sources.

19.11 From curves in a special fibre to analytic curve tests

The converse direction of a semipositivity criterion is geometric. A current inequality on an analytic curve can detect a negative model degree only if that curve meets the relevant residual component. The lifting theorem provides this incidence. We explain the subsequent degree argument before treating the pointwise-limit theorem, whose proof uses a different finite-support mechanism.

Throughout this section the ground field is complete and nontrivially valued, and analytic spaces are good and strict. A real PL function h is represented locally by a formal model $\mathfrak{L}$ of the trivial line bundle, with real coefficients. Its semipositivity means nonnegativity of the degrees of $\mathfrak{L}_s$ on proper integral curves in a suitable local formal model.

Lemma 19.11.1 (The degree detected by a lifted curve)**.** *Let Y be a proper integral curve in the special fibre of a formal model $\mathfrak{W}$. Suppose an analytic curve C in a neighbourhood U has a formal model $\mathfrak{C}$ mapping to $\mathfrak{W}$, and has*

a proper integral special-fibre component Y' dominating Y. If the restriction of the model function to C is semipositive, then

$$\deg(\mathfrak{L}|_Y) \geq 0.$$

Proof. The nonconstant morphism of proper integral curves $Y' \to Y$ is finite and surjective. Write its degree as $e = [\kappa(Y') : \kappa(Y)] > 0$, including inseparable degree. Pullback of the model metric to C is represented by the restricted pullback of $\mathfrak{L}$. Semipositivity on C and the model-degree criterion give

$$0 \leq \deg(\pi^*\mathfrak{L}|_{Y'}) = e \deg(\mathfrak{L}|_Y).$$

The equality is the projection formula on proper curves. Division by the positive number e proves the assertion. Real coefficients cause no problem because both sides depend linearly on the line-bundle class. □

The construction in [GRPL, Theorem 4.1 and proof of Theorem 5.3] supplies the situation of this lemma. Starting from the generic point η of Y, it produces an analytic point z reducing to η and a one-dimensional Zariski-closed subspace through z in a sufficiently small analytic neighbourhood. After choosing a formal model of the inclusion, the closure of the reduction of z is the required Y'. There are two dimension and boundary checks in this argument.

First, the formal closure of the analytic curve has a one-dimensional special fibre. A point in that fibre mapping to η cannot have zero-dimensional closure, because its image contains the generic point of a curve. Its closure is therefore a component. Second, the closure of the reduction of an interior analytic point is proper over the residue field. Conversely, properness of Y makes z an interior point of the original formal generic fibre. Choosing a neighbourhood of z keeps it interior in the smaller domain, so the corresponding Y' is proper as well. This is the interior-point criterion recalled in [GRPL, Section 5.2].

Thus the proof needs a local lift with controlled reduction, not a global algebraic curve through an arbitrary analytic point. The affine lifting theorem and the local analytic lifting theorem have different scopes. A Zariski-closed curve in an affinoid domain need not extend to a Zariski-closed subspace of the whole analytic space.

19.11.1 Why normalization and field extension occur

Once all Zariski-closed analytic curves detect the model degrees, one can replace them by smooth curve tests. Over an algebraically closed analytic extension, first pass to the reduction of a curve and then normalize it. The normalization is a rig-smooth curve because the ground field is perfect. The normalization morphism is finite and surjective. Semipositivity descends under this morphism by the same positive-degree argument used above. Boundary points impose no proper residual-curve condition in dimension one; equivalently, one can use smooth boundaryless subdomains to test interior points.

This explains the quantifiers of Theorem 19.1.5. The test maps are allowed after analytic field extension and are not restricted to curves already embedded globally in X. Descent under field extension is a theorem about the residue line-bundle classes. It is not obtained by defining a pullback of every CLD current.

19.12 Why pointwise convergence suffices for a PL limit

For general distributions, pointwise convergence of functions does not imply convergence of their derivatives as currents. The PL closure theorem avoids this invalid inference. It transfers the derivative to a specially chosen test whose own curvature has finite support, turning the limit into a finite sum of point evaluations.

Lemma 19.12.1 (Finite-support testing)**.** *Let C be a proper analytic curve on which PL functions have finite signed curvature measures DDh satisfying integration by parts. Let $U \subset C$ be open and let h_j, h be PL on U, with $h_j \to h$ pointwise. Suppose they admit PL extensions to C. If a nonnegative PL function ϕ has compact support in U, then*

$$\int_C \phi \, DD\widetilde{h}_j \longrightarrow \int_C \phi \, DD\widetilde{h},$$

where tildes denote arbitrary such extensions. If DDh_j is positive on U, the limit is nonnegative.

Proof. By locality, $DD\phi$ is supported in the compact support of ϕ, hence in U. Write $DD\phi = \sum_{a=1}^{N} c_a \delta_{x_a}$. Integration by parts gives

$$\int_C \phi \, DD\widetilde{h}_j = \int_C \widetilde{h}_j \, DD\phi = \sum_{a=1}^{N} c_a h_j(x_a) \longrightarrow \sum_{a=1}^{N} c_a h(x_a) = \int_C \phi \, DD\widetilde{h}.$$

There is no interchange of an infinite integral with a pointwise limit: the middle expression is a fixed finite sum. Nonnegativity follows from $\phi \geq 0$, its support in U, and positivity of DDh_j there. The chosen extensions do not affect the integrals. □

Proposition 19.12.2 (The curvewise pointwise-limit argument)**.** *On a smooth strict analytic curve, a pointwise limit of semipositive real PL functions is semipositive if the limit is again real PL.*

Open proof obligation (RC2). Prove the curvewise limiting comparison and nontrivial model/geometric step imported from GRPL's Theorem 5.6 argument.

Corollary 19.12.3. *On a good strict analytic space, a pointwise limit h of semipositive real PL functions h_j is semipositive whenever h is real PL.*

Proof. Pull the functions back along each smooth curve test after analytic field extension. Pointwise convergence survives this operation and each pulled-back function remains real PL. Proposition 19.12.2 gives semipositivity on that curve. Theorem 19.1.5 then gives semipositivity of h. □

The real PL hypothesis on the limit is used twice: it gives a formal model with finitely many curvature atoms on a curve, and it makes the residue-model criterion applicable. The proof does not assert a general pointwise-closedness theorem for arbitrary continuous current-psh functions. Nor does it establish weak convergence of DDh_j against every continuous test from pointwise convergence alone. It proves exactly the numerical inequalities needed for a PL endpoint.

References. [GRPL, Theorems 4.1, 5.3, 5.6 and Corollary 5.4]; the curve integration-by-parts formula is obtained there from formal intersection theory and local heights.

19.13 Why the approximation neighbourhood can be fixed

A pointwise perturbation of a nef residue class is not yet a local approximation theorem. If the neighbourhood on which the perturbation is positive shrinks with its coefficient, there may be no common domain on which to take a limit. Gubler–Jell–Rabinoff's construction resolves this by realizing all perturbations on one formal model and using one reduction tube.

Work over a complete nontrivially valued field on a good strictly analytic space. Let x be an interior point, and let h be a semipositive real PL function at x. The residue-model construction used below is [GJR, Proposition 7.10]: after shrinking once and taking common dominating models, there are a compact analytic neighbourhood V of x, a formal model $\mathfrak{V}$, and a projective residue model P with the following data.

The closure Q of $\mathrm{red}(x)$ in $\mathfrak{V}_s$ maps properly onto P. A nef real line-bundle class L on P represents the residue of h. Choose an ample line bundle A on P. Its residue class is realized by a PL function a on V. The model line bundles representing h and a restrict to the pullbacks of L and A on Q. These simultaneous equalities, not merely equality of numerical classes at x, are arranged by the common-model construction.

Proposition 19.13.1. *With these data, the single open set* $W = \mathrm{Int}_X(V) \cap \mathrm{red}^{-1}(Q)$ *contains* x*, and*

$$h_\epsilon = h + \epsilon a \quad (\epsilon \in \mathbb{Q}_{>0})$$

is locally Fubini–Study on W *for every* $\epsilon > 0$*. Moreover* $h_\epsilon \to h$ *uniformly on* W*.*

Proof. Anticontinuity of reduction makes the tube open in V; intersecting with $\mathrm{Int}_X(V)$ makes W open in X and still containing x. The numerical theorem that nef plus a positive multiple of ample is ample shows that $L+\epsilon A$ is ample on P, hence semiample. Its pullback to Q remains semiample. For every $y \in W$, the closure of $\mathrm{red}(y)$ lies in Q; the residue bundle of h_ϵ is represented there by the restriction of that pullback, hence is semiample. This is exactly the locally Fubini–Study condition at y.

The function a is continuous on the fixed compact set V, so $\|a\|_V < \infty$. Therefore $\|h_\epsilon - h\|_W \le \epsilon \|a\|_V \to 0$. Neither Q nor W depends on ϵ. □

The difficult geometric input is the simultaneous realization on a formal model. Once it is available, the fixed-domain and uniform-error assertions follow from the displayed estimate. This identifies the role of the model construction more precisely than a pointwise appeal to openness of the ample cone.

19.13.1 Which CLD approximation class follows

Proposition 19.13.2. *On a good Hausdorff boundaryless strictly analytic space over a complete nontrivially valued field, a semipositive real PL function belongs locally to the continuous generated sheaf $\mathcal{C}_{25}$ of Definition 11.2.2.*

Open proof obligation (RC2). Supply the finite maxima presentation used for local Fubini–Study functions; GJR section 7.9.

This is a two-stage argument. It does not choose one common smooth-approximation neighbourhood for all h_ϵ. Consequently the conclusion stated here is membership in $\mathcal{C}_{25}$, not an unproved identification with the one-step class $\mathcal{U}$. This distinction is exactly why the generated-sheaf definition is useful.

Corollary 19.13.3. *For a real PL function h in this setting, the following conditions are equivalent: model semipositivity, membership in $\mathcal{C}_{25}$, membership in $\mathcal{P}_{25}$, and classical psh in the sense of Gubler–Rabinoff.*

Proof. The preceding proposition gives the first implication. Proposition 11.2.4 gives $\mathcal{C}_{25} \subseteq \mathcal{P}_{25}$. Proposition 19.2.7 gives classical psh. The PL curvewise comparison, Theorem 19.2.4, returns model semipositivity. Every implication is used on the common stated category. □

Single-current positivity is not included among these equivalent conditions. Obtaining it from any of them is already established. Inferring the others from that single-current condition is the separate residual-current comparison discussed earlier. This corollary is a synthesis of the cited theorems, not a resolution of that additional implication.

The complementary pointwise-closure theorem is Corollary 19.12.3. Its proof was given in Section 19.12 by finite-support testing on curves. The two results use different topologies: the fixed-tube construction gives a uniform approximation to a semipositive PL function, whereas pointwise closure starts with semipositive PL functions and requires a PL limit. Together they explain

the two directions of the local PL comparison without identifying pointwise convergence with weak convergence of currents.

References. [GJR, Proposition 7.10]; [GRPL, Theorem 5.6]. The generated-sheaf corollary combines these constructions with the explicitly stated comparisons of Chapters 11 and 19.

19.14 The analytic step in a residual positivity criterion

The residual criterion of CLD25 Theorem 18.8.4 compares two different kinds of positivity: degrees of residual intersections and positivity against analytic test forms. Its proof first treats points where the residual construction has dimension at most the number of Chern factors, then removes an exceptional set by a support theorem. We isolate the latter inference so that its hypotheses can be checked independently of the residual geometry.

Lemma 19.14.1 (Positivity across a negligible set)**.** *Let T be a symmetric current on X whose scalar pairings with smooth tests are signed Radon measures. Let $E \subset X$ be closed. Assume that $T|_{X\setminus E}$ is positive and that $|\mu_{T,\eta}|(E) = 0$ for every compactly supported smooth test η of complementary degree. Then T is positive on X.*

Proof. Fix a positive test η. The scalar measure $\mu_{T,\eta}$ restricts to a positive measure on $X \setminus E$, because testing it against a nonnegative smooth scalar is testing T against a positive form there. Smooth scalar density extends this positivity to the associated Radon measure. Since its total variation gives zero mass to E, it is positive on all of X. In particular $T(\eta) = \mu_{T,\eta}(X) \geq 0$. This holds for every positive test, proving the claim. □

It is not enough to say that E has empty interior, or that it is small for one chosen tropicalization. The noncharging hypothesis concerns every actual test measure of T. If T is not yet known positive, one also cannot apply a theorem about positive currents directly to T. A finite signed decomposition supplies a legitimate route.

Proposition 19.14.2. *Suppose locally $T = \sum_{j=1}^{N} c_j S_j$, where $c_j \in \mathbb{R}$ and each S_j is positive and closed of the same degree. If every S_j gives zero mass to E after pairing with any complementary smooth form, then the noncharging hypothesis of Lemma 19.14.1 holds for T.*

Proof. For a fixed test η, linearity gives $\mu_{T,\eta} = \sum_j c_j \mu_{S_j,\eta}$. Total variations satisfy

$$|\mu_{T,\eta}| \le \sum_j |c_j|\, |\mu_{S_j,\eta}|.$$

The right side vanishes on E, so the left side does too. A finite partition subordinate to local decompositions proves the assertion on the compact support of any global test. □

For a product of Chern currents with local weights $h_i = a_i - b_i$ in an admitted continuous psh class, such a decomposition is explicit:

$$\prod_{i=1}^{d} \mathrm{d}'\mathrm{d}'' h_i = \sum_{I \subseteq \{1,\dots,d\}} (-1)^{d-|I|} \bigwedge_{i \in I} \mathrm{d}'\mathrm{d}'' a_i \wedge \bigwedge_{i \notin I} \mathrm{d}'\mathrm{d}'' b_i.$$

Every summand without its sign is positive and closed by the continuous-product theorem. Thus a support theorem for positive closed currents can be applied to those summands before positivity of the original signed product has been established. This is the correct order of the argument.

19.14.1 Dimension and the exceptional set

Suppose X is pure of dimension n and T has bidegree (d,d), hence bidimension $(n-d, n-d)$. CLD25 Theorem 16.8.5 uses negligible sets in tropical Hausdorff dimension $n-d$. Its hypotheses include a nontrivial valuation, absence of analytic boundary, closedness and positivity. To use it for the preceding decomposition, verify these conditions for each positive summand and show that the exceptional set has negligible image in every relevant tropical chart.

The geometric part of CLD25 Theorem 18.8.4 produces such a set from strata of a residual model, using Proposition 11.10.4 and Example 16.8.4. The numerical input is tested on integral residual subvarieties W of dimension d:

$$\deg\big(c_1(M_1) \cdots c_1(M_d) \cap [W]\big) \ge 0.$$

The dimension is forced: each Chern class lowers cycle dimension by one. A zero-cycle degree therefore requires $\dim W = d$, not an unspecified codimension. The index slip in the introductory explanation on p. 539 of CLD25 is corrected in this form.

Near a point where the residual construction has dimension d, the product is a scalar multiple $\lambda\delta_F$ of a calibrated fibre current. Positivity is then exactly $\lambda \geq 0$, and the comparison theorem identifies the sign of λ with the displayed residual degree. When the residual dimension is smaller than d, the product vanishes for degree reasons. The remaining strata form the exceptional locus. Once its noncharging property has been verified, Lemma 19.14.1 extends positivity to the entire neighbourhood.

This separates the proof into verifiable components: the residual-model realization, the calibrated-fibre formula, the dimension bound for the exceptional locus, and the measure argument above. The measure and signed-decomposition steps have just been proved. Sections 19.18.1–19.18.7 supply the remaining assembly, with named geometric inputs and a precise pure-dimensional conclusion.

Example 19.14.3 (Why the noncharging condition matters)**.** On the real line, the signed measure $T = -\delta_0$ is zero, hence positive, on $\mathbb{R} \setminus \{0\}$. It is not positive on $\mathbb{R}$. The closed exceptional set has empty interior, but $|T|(\{0\}) = 1$. Thus positivity off a closed set cannot be extended without an actual noncharging assertion.

References. [CLD25, Theorem 16.8.5, Example 16.8.4 and Theorem 18.8.4]. The measure-theoretic transfer is proved above; the residual geometric hypotheses remain explicit.

19.15 Envelope calculus with explicit analytic inputs

The global variational method has an elementary core once its analytic inputs are available. It is useful to prove that core separately: one can then see exactly where envelope regularity, orthogonality, and compactness enter. In this section X is compact and θ denotes the background class in a fixed global pluripotential theory, not an unspecified original smooth CLD form.

Assume the admitted θ-psh class is stable under addition of constants and convex combinations. For a continuous obstacle f, assume its envelope $P(f)$

exists as the largest admitted function below f. These are assumptions on the chosen theory, including any upper-semicontinuous regularization necessary to make the supremum admitted.

Proposition 19.15.1. *The envelope satisfies monotonicity, translation invariance, concavity, and the sup-norm estimate*

$$f \le g \implies P(f) \le P(g), \qquad P(f+c) = P(f) + c,$$
$$P((1-t)f + tg) \ge (1-t)P(f) + tP(g) \quad (0 \le t \le 1),$$
$$\|P(f) - P(g)\|_\infty \le \|f - g\|_\infty.$$

If u is an admitted obstacle, then $P(u) = u$.

Proof. Every competitor below f is below g when $f \le g$, proving monotonicity. Adding c gives a bijection between the two competitor sets for translation invariance. The convex combination $(1-t)P(f)+tP(g)$ is admitted and lies below $(1-t)f+tg$, proving concavity. If $\varepsilon = \|f-g\|_\infty$, then $g-\varepsilon \le f \le g+\varepsilon$; monotonicity and translation invariance give the norm estimate. Finally u is its own largest competitor. □

Assume next that admitted continuous functions have positive Monge–Ampère measures $\mathrm{MA}(u)$ of fixed mass V, an increasing concave energy E, and the first-variation and endpoint inequalities

$$\int_X (u - v)\,\mathrm{MA}(u) \le E(u) - E(v) \le \int_X (u - v)\,\mathrm{MA}(v).$$

The inequalities follow from concavity and the first variation along the segment from v to u; they are also proved by the energy calculation in Chapter 20 on its approximation domain. In the model-based theory their validity is a theorem of that theory, with its own domain.

Definition 19.15.2. The envelope has the orthogonality property if

$$\int_X (f - P(f))\,\mathrm{MA}(P(f)) = 0.$$

If f and $P(f)$ are continuous, this is equivalent to the measure being supported on the contact locus $\{P(f) = f\}$.

Contact-locus interpretation. The integrand is nonnegative. Each compact subset where $f - P(f) \ge \varepsilon > 0$ has measure zero if the integral vanishes.

Taking the union over positive reciprocal integers gives zero measure on the noncontact locus. Conversely a measure concentrated on the contact locus makes the integrand zero almost everywhere. Continuity makes that locus closed, giving the support statement. □

Theorem 19.15.3 (Differentiation through a continuous envelope)**.** *Assume every continuous obstacle has a continuous admitted envelope, orthogonality holds, and* MA *is weakly continuous under uniform convergence in this class. Then*

$$\frac{d}{dt}\Big|_{t=0} E(P(f+th)) = \int_X h \ \mathrm{MA}(P(f)), \qquad f, h \in C(X).$$

Proof. Put $u = P(f)$ and $u_t = P(f + th)$. The norm estimate gives $\|u_t - u\|_\infty \le |t|\|h\|_\infty$. On the contact locus for u, one has $u = f$ and $u_t \le f + th$, so $u_t - u \le th$ almost everywhere for $\mathrm{MA}(u)$. On the contact locus for u_t, one has $u_t = f + th$ and $u \le f$, so $u_t - u \ge th$ almost everywhere for $\mathrm{MA}(u_t)$. The energy inequalities yield

$$t \int h \ \mathrm{MA}(u_t) \le E(u_t) - E(u) \le t \int h \ \mathrm{MA}(u).$$

For positive t, divide directly; for negative t, reverse the inequalities. Uniform convergence of u_t and weak continuity of MA make the two endpoint integrals tend to the same value. This proves both one-sided derivatives and their equality. □

Remark 19.15.4. This short proof deliberately assumes continuity of every envelope and uniform-continuity of the Monge–Ampère operator on the admitted class. It is a synthesis of the energy inequalities and contact-locus identity, not a new proof of those analytic inputs. BFJ's differentiability theorem is formulated in a broader finite-energy framework and includes work needed when these stronger continuous hypotheses are unavailable. The argument above must not be used to remove their hypotheses or their envelope construction.

Proposition 19.15.5 (Differentiability forces orthogonality)**.** *Conversely, assume the derivative formula of Theorem 19.15.3 holds for all continuous directions and that $P(f)$ is continuous. Then orthogonality follows.*

Proof. Set $h = P(f) - f \le 0$. For $0 \le t \le 1$, the obstacle $f + th$ lies between $P(f)$ and f. Hence monotonicity and $P(P(f)) = P(f)$ give $P(f+th) = P(f)$ exactly. The derivative formula gives $\int (P(f) - f)\,\mathrm{MA}(P(f)) = 0$. □

19.16 The Euler–Lagrange step and what it does not prove

Let μ be a positive measure of mass V, and define $F_\mu(u) = E(u) - \int u\,d\mu$ on the admitted class. Since $E(u+c) = E(u) + cV$, the functional is unchanged by constants. A normalization is required for compactness and uniqueness.

Theorem 19.16.1. *Under the continuous-envelope hypotheses above, if an admitted continuous function u maximizes F_μ, then $\mathrm{MA}(u) = \mu$. Conversely, a solution of this equation maximizes F_μ among the admitted continuous functions.*

Proof. For any $h \in C(X)$ put $u_t = P(u + th)$. Since $u_t \le u + th$ and $\mu \ge 0$, maximality gives

$$E(u_t) - E(u) \le \int (u_t - u)\,d\mu \le t \int h\,d\mu.$$

Use positive and negative t and the envelope derivative to deduce $\int h\,\mathrm{MA}(u) = \int h\,d\mu$. Equality against every continuous test identifies the measures. Conversely, concavity gives

$$E(v) - E(u) \le \int (v - u)\,\mathrm{MA}(u) = \int (v - u)\,d\mu,$$

hence $F_\mu(v) \le F_\mu(u)$. □

The theorem does not produce a maximizer. In BFJ's method one enlarges the domain to finite-energy potentials to gain compactness, proves the appropriate upper semicontinuity and integrability, obtains a maximizer there, and then proves continuity under the measure hypotheses. None of these steps follows from the displayed Euler–Lagrange equation. Likewise, uniqueness modulo constants uses a global Hodge-index or energy rigidity theorem; concavity by itself need not be strict.

19.16.1 The precise BFJ existence theorem

In the version cited here, BFJ Theorem A assumes a complete discretely valued field of residue characteristic zero, a smooth projective variety which descends to a one-variable function field having this completion, and an ample line bundle. The prescribed positive Radon measure has the degree as its mass and is supported on the dual complex of an SNC model. Under these hypotheses it is the Monge–Ampère measure of a continuous semipositive metric, unique up to a constant factor. Here "algebraizable" is the source's extra descent condition, not merely the assertion that the analytic space is the analytification of a projective variety over K. The latter is already assumed. The difference matters when applying the theorem to a variety over a local field. Later envelope theorems may weaken particular inputs in specified settings, but their conclusions must be cited with their own geometric and characteristic hypotheses.

References. [BFJ15, Proposition 2.15, Theorem 7.2, Remark 7.4, Theorems A and 8.1]; [GJKM, FGK].

19.17 A corrected negligibility statement

For a product of d curvature factors on a pure n-dimensional space, the relevant tropical Hausdorff dimension is $n-d$. It is the dimension of the tests, or equivalently the bidimension of the current. It is not the number d of curvature factors.

Proposition 19.17.1 (Noncharging for admitted differences)**.** *Work in the algebraic scope of Theorem 13.6.3, over a nontrivially valued complete field. Let X be pure of dimension n, Hausdorff and boundaryless. Locally let $h_i = a_i - b_i$ with $a_i, b_i \in \mathcal{U}_X$, for $1 \le i \le d \le n$. If E is closed and tropically $\mathcal{H}^{n-d}$-negligible, then for every actual compactly supported smooth test η of bidegree $(n-d, n-d)$, the following signed measure has total variation zero on E:*

$$\mathrm{d}'\mathrm{d}''h_1 \wedge \cdots \wedge \mathrm{d}'\mathrm{d}''h_d \wedge \eta.$$

Proof. Expand the product by multilinearity as in Proposition 19.14.2. Each unsigned term is an ordinary positive closed current of bidimension $(n-d, n-d)$ by Theorem 13.6.3. Apply CLD25 Theorem 16.8.5 with its parameter $p = n - d$ to each term. The finite signed sum is noncharging by the total-

variation inequality in Proposition 19.14.2. A finite partition on the support of η handles local decompositions. □

Example 19.17.2 (The printed exponent in CLD25 Corollary 16.8.6)**.** Take $X = \mathbf{G}^{\mathrm{an}}_{m,K}$ over an algebraically closed nontrivially valued field, and put $u = \max(0, \log|T|)$. This is a finite continuous member of $\mathcal{U}_X$: regularize the maximum of the two smooth pluriharmonic branches. The standard formal metric on $\mathcal{O}(1)$, restricted to this chart, or the slope formula, gives

$$\mathrm{d}'\mathrm{d}''[u] = \delta_\zeta,$$

where ζ is the unit-radius Gauss point. In the logarithmic coordinate the slopes jump from 0 to 1; all directions off the skeleton contribute zero. The singleton $E = \{\zeta\}$ is tropically $\mathcal{H}^1$-negligible, since its image under every moment map has at most one point. A nonnegative smooth compactly supported scalar test equal to one near ζ gives mass one on E. Thus the printed $\mathcal{H}^p$ hypothesis in Corollary 16.8.6, with $n = p = 1$, is false as written. The corrected exponent is $n - p$; E is not $\mathcal{H}^0$-negligible.

Proposition 19.17.1 proves the corrected assertion in its explicit nontrivial-valuation and admitted-potential scope. For PL metrics, Proposition 19.18.8 below supplies the additional Shilov-section argument over a trivially valued field. Correcting the exponent alone does not enlarge the full potential class.

19.18 The logical structure of the residual criterion

The following formulation separates the analytic inference from its geometric inputs. Let T be a symmetric current on a neighbourhood U of x, with signed Radon scalar pairings. Suppose the proposed residual criterion assigns a number a_W to each relevant proper integral residual subvariety W of dimension d.

Proposition 19.18.1 (Conditional residual test)**.** *Assume the following two properties.*

1. *If all $a_W \geq 0$, there is a closed set $E \subset U$ such that T is positive on $U \setminus E$ and every scalar total variation of T gives zero mass to E.*
2. *For each W and every sufficiently small neighbourhood V of x, there is a nonempty open subset $V_W \subset V$ on which $T = c_W a_W \delta_{F_W}$, with $c_W > 0$ and δ_{F_W} a nonzero positive integration current.*

Then T is positive near x if and only if all $a_W \geq 0$.

Proof. The forward direction follows by restricting a positive current to V_W. If $a_W < 0$, a positive test on which δ_{F_W} is nonzero gives a negative value. Conversely, use the first assumption and Lemma 19.14.1. □

For CLD25 Theorem 18.8.4, a_W is $\deg(c_1(M_1)\cdots c_1(M_d) \cap [W])$. The conditional proposition is not a replacement for the geometric assertions in the following list. It identifies exactly what they must supply.

1. Residual realization and change of premodel: Chapter 17, especially the residual/analytic dimension comparison in Proposition 17.6.4. These remain geometric inputs.
2. Calibrated fibre identity: Proposition 18.6.1 and paragraph 18.8.3. Their flatness, equidimensionality and native completed-residue-field normalizations must be retained.
3. Dimension bound: applying Proposition 11.10.4 with threshold $d+1$ gives dimension at most $n-d-1$ for strata with residual dimension greater than d. Example 16.8.4 converts this into tropical $\mathcal{H}^{n-d}$-negligibility.
4. The complement of the union of positive neighbourhoods need only be contained in that bad locus. Equality with the whole bad locus is unnecessary. Noncharging of every scalar pairing, not smallness alone, completes the argument.
5. Passage from numerical nefness to a generated psh metric: Proposition 17.8.4 is stated for strictly integral PL metrics. Corollary 18.8.8 is written for integral PL metrics. An extension of the former statement is a separate requirement.

Theorem 19.18.12 below reconstructs the residual current criterion in the Hausdorff boundaryless pure-dimensional category, with the geometric inputs named explicitly. It does not strengthen the potential-class assertion of Theorem 13.6.3. The independent PL equivalence in Corollary 19.13.3 retains its good, strict, nontrivially valued setting and reaches the generated class $\mathcal{C}_{25}$, not the one-stage class $\mathcal{U}$.

References. CLD25, fixed version of 25 July 2025, pp. 464–465, 497–498, 532–533, 538–541. The following sections distinguish the supplied deductions from the imported geometric theorems.

19.18.1 The coefficient belongs to the base

The scalar in the fibre description can be determined before any positivity argument. This also fixes a notational error in CLD25, paragraph 18.8.3: the bundles M_i live on Y, so their residual intersection and the accompanying multiplicity and defect must be evaluated at y, not at x. We give the deduction on actual smooth tests.

Proposition 19.18.2 (Fibre coefficient, flat case)**.** *Let $f : V \to W$ be a flat morphism of Hausdorff boundaryless analytic spaces, pure of dimensions n and d, respectively. Let $M_1, \dots, M_d$ be integral PL metrized line bundles on W. Suppose their top Chern measure is supported on a closed discrete set $S \subset W$, with*

$$\mu = \bigwedge_{i=1}^{d} c_1(M_i) = \sum_{z \in S} a_z \delta_z.$$

Assume the relative integration formula of CLD25 Proposition 18.3.5 and that every nonempty fibre over S is pure of dimension $n-d$. If $L_i^{\otimes q} \simeq f^* M_i$ isometrically for a common positive integer q, then

$$\bigwedge_{i=1}^{d} c_1(L_i) = q^{-d} \sum_{z \in S} a_z \delta_{V_z}. \tag{19.18.1}$$

Here V_z is an $\mathscr{H}(z)$-analytic fibre with its native integration current, including its cycle multiplicities; an empty fibre contributes zero.

Proof. Let η be any actual compactly supported smooth form of bidegree $(n-d, n-d)$ on V. Its support K has compact image $f(K)$. Choose a compactly supported smooth scalar cutoff α on W, equal to one near $f(K)$. The support of η is proper over W, so Proposition 18.3.5 applies to $\eta \wedge f^*\alpha$. With

$$u_\eta(z) = \int_{V_z} \eta$$

on fibres of the prescribed dimension, with value zero on the others. Tensor homogeneity and relative integration give

$$q^d \left\langle \bigwedge_i c_1(L_i), \eta \right\rangle = \int_W \alpha u_\eta \, d\mu = \sum_{z \in S \cap f(K)} a_z \int_{V_z} \eta.$$

This sum is finite because S is closed and discrete and $f(K)$ is compact. The formula holds for every such η, proving the current identity. No degree factor

is introduced by regarding V_z over $\mathscr{H}(z)$: this is already the normalization in the relative integration theorem. □

CLD25 Proposition 18.4.3 supplies the discrete top measure, while Theorem 18.4.7, not Proposition 18.4.3 alone, supplies its residual coefficient. In the notation $f : (X, x) \to (Y, y)$, $\dim Y = d$, shrinking away the other atoms gives the correctly typed formula

$$T = q^{-d} a_y \delta_{f^{-1}(y)}, \qquad a_y = m_Y(y)\,\mathrm{def}_k(y) \int_{P_{\widetilde{\mathscr{H}(y)}/\widetilde{k}}} \prod_{i=1}^{d} c_1(\widetilde{M}_{i,y}). \quad (19.18.2)$$

The graded intersection integral, $m_Y(y)$ and $\mathrm{def}_k(y)$ retain the definitions of that theorem. The positive prefactor affects the mass, although not its sign. A reduction to an ungraded premodel may introduce an additional graded-field degree; it must not be suppressed. Theorem 18.4.7 and the relative integration theorem remain imported inputs, not conclusions of the short proof above.

19.18.2 How residual dimension produces a flat presentation

Write r_x for simultaneous residual definition dimension. CLD25 Proposition 17.6.4 is the lifting input: on an almost good germ with trivializable underlying bundles, a graded field of definition $F \subset \widetilde{\mathscr{H}(x)}$ of transcendence degree r_x and with $\dim_F \Omega_{F/\widetilde{k}} = r_x$ gives a flat boundaryless presentation through a germ of dimension r_x. If $\widetilde{k}$ is perfect, this applies without tensor powers. For a finite family of line bundles in general, it applies after a common power q of the residue characteristic exponent.

Its construction has three distinct inputs. First, choose homogeneous generators with independent differentials and lift them to analytic units. The associated torus moment sends x to a Gauss point. A finite separable extension of the graded residue field is realized by a tame extension of the completed residue field, hence by an etale morphism of germs. Second, the given embedding into $\widetilde{\mathscr{H}(x)}$ selects a section after base change; the germ-reduction correspondence restricts the target to the required quasi-compact open. The resulting map is boundaryless. Flatness at the Abhyankar target is supplied by the analytic flatness theorem cited there, not by boundarylessness alone. Third, the residual-bundle equivalence of Theorem 17.4.9 lifts the bundles and identifies their pullbacks isometrically. In the inseparable case,

Lemma 10.6.14 descends the powered line bundles to the separable graded subfield before this construction.

Thus the power q repairs inseparability in a field of definition. It is not a strictification of the metric. Nor does a premodel of dimension at most d imply that the minimal definition dimension is exactly d: it may be smaller, in which case the d-fold curvature product vanishes. If the minimum is d, formula (19.18.2) applies with its full coefficient. These two cases must be kept separate in testing a residual subvariety of dimension d.

19.18.3 Reduction strata and the exceptional set

Two elementary consequences of the reduction theorems explain the geometry of the final positivity argument. They also specify which reduction topology is being used.

Lemma 19.18.3 (Specialization reaches every neighbourhood)**.** *Let V be an affinoid space with graded reduction map $\rho : V \to \widetilde{V}$. Let $x \in V$ and let w be a specialization of $\rho(x)$. Then every neighbourhood of x in V meets $\rho^{-1}(w)$.*

Proof. CLD25 Lemma 11.7.2 identifies the closure of $\rho^{-1}(w)$ with the union of fibres over the generizations of w. The point $\rho(x)$ is one such generization. Hence x lies in that closure, which is precisely the asserted neighbourhood property. For clarity, the nontrivial input in that lemma is the closed-image statement for reduction of an affinoid neighbourhood at an interior point, together with surjectivity of reduction. Anticontinuity alone proves only one inclusion in the closure formula. □

Lemma 19.18.4 (The dimension threshold)**.** *Let $0 \le d \le n$ and let V be an affinoid of dimension at most n. The locus where the closure of the reduction point has dimension greater than d is weakly analytic of dimension at most $n-d-1$. Its image under every local moment map has zero $(n-d)$-dimensional Hausdorff measure.*

Proof. Choose finitely many homogeneous generators of the graded reduction, and lift them to analytic functions g_j of spectral radii r_j. A reduction point has dimension at least $d+1$ exactly when some $d+1$ of these generators have algebraically independent residues there. For the corresponding subtuple I, this says that its moment has value the Gauss point η_{r_I}. The locus is therefore a finite union of fibres $g_I^{-1}(\eta_{r_I})$. The analytic dimension inequality bounds

their dimension over $\mathscr{H}(\eta_{r_I})$ by $n-(d+1)$. This is the proof mechanism of Proposition 11.10.4 with threshold $d+1$, not d. Example 16.8.4 supplies the stated tropical Hausdorff negligibility. If $d \geq n$, the locus is empty. □

To assemble the criterion on a reduction neighbourhood, one needs a finite analytic-domain cover whose graded reductions glue to a space S, together with line bundles $\mathcal{L}_i$ realizing the residual bundles at every point in the interior. Put $P = \overline{\{\rho(x)\}}$. Anticontinuity makes $\rho^{-1}(P)$ open. Intersect with the interior of the analytic neighbourhood before applying a boundaryless positivity statement. If all degree-d residual intersections on P are nonnegative, the cases of minimal definition dimension at most d give positive neighbourhoods around the corresponding points. The closed complement of their union is contained in the higher-dimensional reduction locus. Lemma 19.18.4 and a noncharging theorem then remove that complement. Only containment is required; equality with the entire higher-dimensional locus is generally unwarranted.

Conversely, let $W \subset P$ be an integral residual subvariety of dimension d, with generic point w. The closure property of Lemma 19.18.3, when available on the glued reduction neighbourhood, places a point over w in every positive neighbourhood of x. The residual realization at that point identifies the restrictions $\mathcal{L}_i|_W$ with its premodel data. The lower-definition-dimensional case gives zero intersection; the dimension-d case detects its sign by (19.18.2). This proves the numerical implication once the realization and the premodel intersection comparison are supplied. The Laurent gluing and simultaneous realization are supplied in Propositions 19.18.9 and 19.18.10; Lemma 19.18.11 explains the numerical comparison. Under the boundaryless hypothesis, goodness already follows from CLD25 paragraph 11.4.1. Thus an affinoid neighbourhood is available in this argument.

There is a further propagation of the exponent correction: CLD25 Proposition 18.3.7 repeats the $\mathcal{H}^p$ hypothesis of Corollary 16.8.6 for a product of p curvature factors. Example 19.17.2, viewed as the metric $\|1\| = e^{-u}$ on the trivial line bundle, refutes this printed metric version as well. In the admitted-potential scope, Proposition 19.17.1 supplies the required $\mathcal{H}^{n-p}$ version. Proposition 19.18.8 extends the verified replacement to PL metrics over arbitrary complete base fields, including the trivial valuation.

19.18.4 Why a tensor power does not remove strictness

CLD25 Definition 17.2.3 distinguishes integral PL metrics from strictly integral PL metrics. The latter require a strict analytic space and local orthonormal frames on strict analytic domains. The common power in Proposition 17.6.4 addresses inseparability, whereas the following elementary example rules out a general strictification by tensor powers, even after finite extension.

Example 19.18.5 (An irrational constant norm)**.** Take $K = \mathbf{Q}_p$ with $|p| = p^{-1}$, $X = \mathcal{M}(K)$ and the trivial line $L = Ke$ with $\|e\| = c = p^{-\sqrt{2}}$. This is an integral PL metric because constant norms are allowed. For every positive integer q, an orthonormal basis of $L^{\otimes q}$ would have the form $ae^{\otimes q}$ with

$$|a|c^q = 1, \qquad |a| = p^{q\sqrt{2}}.$$

But $|K^\times| = p^{\mathbf{Z}}$. After a finite extension K'/K of ramification index e, the value group is $p^{\frac{1}{e}\mathbf{Z}}$, which still cannot contain $p^{q\sqrt{2}}$. On the one-point space the local orthonormal-frame condition is the same condition. Hence no such tensor power becomes strictly integral, even after finite extension. Nevertheless its weight is constant, hence smooth and psh with zero curvature. This refutes the proposed reduction method, not Corollary 18.8.8 itself.

A constant change of weight preserves both curvature and the psh property, so a metric already known to be a constant rescaling of a strictly integral metric can be treated in that restricted situation. The general integral PL case needs a compatible treatment of the constants on all chart overlaps, or an extension theorem in graded geometry. Enlarging the value group by a transcendental valued extension merely moves the problem: one must then descend psh membership to the original domain. Neither tensor homogeneity nor Proposition 17.6.4 provides that descent. Theorem 19.18.15 below handles arbitrary constants for semiample residues. Theorem 19.18.17 supplies the fixed-neighbourhood argument in the strict, nontrivially valued setting. For general nef integral PL data, Proposition 19.18.20 supplies the required compatibility, and Theorem 19.18.22 proves the generated-class implication.

References. CLD25, fixed arXiv v2: Definition 17.2.3, Remark 17.2.4, Lemma 17.5.1, Proposition 17.6.4; Propositions 18.3.5 and 18.4.3, Theorem 18.4.7, Proposition 18.6.1 and paragraphs 18.8.2–18.8.3; Lemma 11.7.2 and Proposition 11.10.4. The fibre deduction, threshold argument and strictification counterexample are written out above; deep lifting and intersection inputs are attributed explicitly.

19.18.5 PL weights and the noncharging theorem

The general approximation question in Section 11.6 is unnecessary for PL weights. Their local polyhedral description gives an explicit first-stage approximation of each convex summand. We first make this observation precise, then deal with the trivially valued case of the support argument.

Proposition 19.18.6 (PL weights are admitted differences)**.** *Let X be Hausdorff and almost good, and let $h_1, \dots, h_s$ be finitely many real-valued PL functions. Every point has one open neighbourhood U on which*

$$h_i = a_i - b_i, \qquad a_i, b_i \in \mathcal{U}_X(U).$$

The functions a_i, b_i can be chosen as pullbacks of finite maxima of affine functions under one analytic moment. They have decreasing smooth psh approximants on this same U, with an explicit uniform error bound.

Proof. We use the local chart and polyhedral decomposition results of CLD25 Proposition 12.4.7, Proposition 6.3.4 and Proposition 8.8.2(b). Almost goodness makes the associated affine tropical space decent. After taking a compact domain neighbourhood and concatenating the finitely many moments, these results write each h_i as the difference of two convex PL functions on the convex hull of the tropical image. A convex PL function on a finite polytope is a finite maximum of affine functions there. Extend those affine functions to the ambient real vector space. Their maximum is defined on that whole vector space and has the required restriction. Shrinking to the interior of the chosen domain gives one open U for all the functions.

For $v = \max_{1 \le j \le N} \ell_j$, put

$$v_\epsilon(t) = \epsilon \log \left(\sum_{j=1}^{N} e^{\ell_j(t)/\epsilon} \right), \qquad \epsilon > 0.$$

Then $v \le v_\epsilon \le v + \epsilon \log N$. If $p_j = e^{\ell_j/\epsilon} / \sum_k e^{\ell_k/\epsilon}$ and b_j is the gradient of ℓ_j, then

$$\nabla^2 v_\epsilon = \epsilon^{-1} \left(\sum_j p_j b_j b_j^t - \Big(\sum_j p_j b_j \Big) \Big(\sum_j p_j b_j \Big)^t \right) \ge 0,$$

$$\frac{\partial v_\epsilon}{\partial \epsilon} = - \sum_j p_j \log p_j \ge 0.$$

Thus $v_\epsilon \downarrow v$ as $\epsilon \downarrow 0$. Pullback by the fixed moment produces original smooth psh functions on U, with the same uniform error bound. Apply this construction to both summands of every h_i. No assertion about later ordinal stages is used. □

The chart theorem is an external geometric input; the displayed regularization proves the first-stage assertion directly. In particular, this is stronger information than merely recording that a PL weight belongs to the generated difference class. For arbitrary analytic spaces below, we use the CLD continuous-product construction and its locally uniform continuity in Proposition 8.7.2 on these explicitly approximated functions. This is a named input in that category; it does not enlarge the algebraic scope of Theorem 13.6.3.

Lemma 19.18.7 (Tropical negligibility along a Gauss section)**.** *Suppose k is trivially valued, fix $0 < r < 1$, and let $k_r = k((t))$ with $|t| = r$. Let $\pi : X_r \to X$ be scalar extension and $\sigma : X \to X_r$ its continuous Gauss section. If a closed subset $E \subset X$ is tropically $\mathcal{H}^a$-negligible, then $\sigma(E)$ is closed and tropically $\mathcal{H}^a$-negligible in X_r. Here X is Hausdorff and good and $a \geq 0$.*

Proof. The image of σ is the fixed-point set of the continuous retraction $\sigma\pi$, hence is closed in the Hausdorff space X_r. The section is a homeomorphism onto its image, so $\sigma(E)$ is closed. We explain the finite local description underlying CLD25 Lemma 16.8.7. Work over an affinoid $V = \mathcal{M}(A) \subset X$. On a compact analytic domain $W \subset V_r$, let f be an analytic unit. A finite rational refinement of W, followed by approximation in the rational-domain algebra, expresses its absolute value as $|\phi|/|\psi|$, where $\phi, \psi \in A\widehat{\otimes}_k k_r$ are nonzero on the relevant compact domain. The approximation can preserve absolute values: choose its error smaller than the positive minimum of the absolute value being approximated.

Approximate ϕ, ψ in turn by finite Laurent sums with coefficients in A, still preserving their absolute values there. The Gauss-section formula is

$$\left|\sum_\nu a_\nu t^\nu\right|(\sigma(z)) = \max_\nu |a_\nu(z)| r^\nu.$$

Cover $\sigma^{-1}(W)$ by the finitely many compact sets on which specified numerator and denominator terms attain these maxima. On each nonempty piece

K, the selected coefficients are nonzero and their ratio g is an analytic unit on a neighbourhood of K in V. For an integer e,

$$\log|f|\circ\sigma = \log|g| + e\log r \quad \text{on } K.$$

For a finite tuple of units, refine these pieces simultaneously. Its tropicalization on each piece is thus the tropicalization of a moment on X, followed by translation. Cover the pieces by finitely many compact analytic domains on which those units are defined. The hypothesis on E and translation invariance of Hausdorff measure now give zero $\mathcal{H}^a$ measure for the image of $\sigma(E)\cap W$. The assertion is G-local, so these affinoid computations suffice. □

Proposition 19.18.8 (Noncharging for PL curvature products)**.** *Let k be any complete non-Archimedean field, including a trivially valued field. Let X be Hausdorff, boundaryless and pure of dimension n, and let $L_1,\dots,L_d$ have PL metrics, with $1\le d\le n$. Put $T=\bigwedge_i c_1(L_i)$, using the CLD continuous-product construction. If $E\subset X$ is closed and tropically $\mathcal{H}^{n-d}$-negligible, then*

$$|\mu_{T,\eta}|(E)=0$$

for every actual compactly supported smooth test η of bidegree $(n-d,n-d)$. No positivity of T is assumed.

Proof. Boundarylessness implies goodness (CLD25 paragraph 11.4.1), so local frames and Proposition 19.18.6 apply. Expand the product of the local weight differences into a finite signed sum of ordinary positive closed products S. Their existence and locally uniform approximation are supplied by the continuous-product theorem just specified. If the valuation is nontrivial, apply CLD25 Theorem 16.8.5 to each S, with its bidimension parameter equal to $n-d$. Proposition 19.14.2 finishes this case.

Suppose the valuation is trivial and use the extension and section of Lemma 19.18.7. Write S_r for the product of the pulled-back convex summands. The source inputs CLD25 Proposition 16.8.2 and Corollary 18.3.3(a), restricted here to the explicit smooth approximation sequences, give

$$\operatorname{supp}(S_r\wedge\pi^*\eta)\subseteq\sigma(X),\qquad \pi_*(S_r\wedge\pi^*\eta)=S\wedge\eta.$$

These are the Shilov-support and scalar-extension integration inputs. They follow first for smooth products from the canonical tropical calibrations and then pass to the locally uniform limits; no equality of the higher generated

classes is needed here. By the lemma, $\sigma(E)$ is tropically $\mathcal{H}^{n-d}$-negligible. The nontrivially valued theorem applied on X_r says that $S_r \wedge \pi^*\eta$ has total variation zero there. Its support lies in $\sigma(X)$, and $\pi^{-1}(E) \cap \sigma(X) = \sigma(E)$. Consequently,

$$|\mu_{S,\eta}|(E) \le \pi_*|\mu_{S_r,\pi^*\eta}|(E) = 0.$$

Finite signed summation and a finite partition of unity on the test support finish the proof. Scalar extension uses the native calibration of Corollary 18.3.3(a), with factor one. This is distinct from a finite analytic covering of calibrated degree e, for which the corresponding factor is e. □

This supplies the corrected PL-metric scope needed in the residual argument. It does not prove noncharging for every function in the full transfinite dpsh class, nor does it construct pullback of an arbitrary current under scalar extension. The argument only pulls back the functions defining the specified products.

19.18.6 Laurent gluing with arbitrary radii

We now fill in the gluing step of the residual argument. The boundaryless hypothesis already guarantees an affinoid neighbourhood of the point. Thus the non-good-germ concern recorded in version 0.8 is not an additional obstruction under the hypotheses of CLD25 Theorem 18.8.4. What needs proof is that the particular Laurent cover is a formal cover in the graded sense.

Proposition 19.18.9 (The reduction of a Laurent cover)**.** *Let $V = \mathcal{M}(A)$ be affinoid, $x \in V$, and $f_1, \dots, f_m \in A^\times$. Set $r_i = |f_i(x)|$ and, for $\epsilon \in \{\pm 1\}^m$, put*

$$V_\epsilon = \{z \in V : |f_i(z)|^{\epsilon_i} \le r_i^{\epsilon_i} \text{ for all } i\}, \qquad A_\epsilon = \mathcal{O}(V_\epsilon).$$

The graded schemes $S_\epsilon = \operatorname{Spec}(\widetilde{A}_\epsilon)$ glue along the reductions of the intersections to a separated graded $\widetilde{k}$-scheme S of finite type. Their reduction maps glue to a surjective anticontinuous map $\rho : V \to S$, with $\rho^{-1}(S_\epsilon) = V_\epsilon$. If $s \in S$, then

$$\overline{\rho^{-1}(s)} = \bigcup_{t \text{ a generization of } s} \rho^{-1}(t).$$

Open proof obligation (RC2). Establish the residual-to-analytic Laurent-domain identification used for the coordinate bounds; TemLocal Proposition 3.1(ii).

Proposition 19.18.10 (Simultaneous residual realization)**.** *Let x be a point of a Hausdorff boundaryless analytic space X, and let $L_1, \dots, L_d$ have integral PL metrics. One can choose an affinoid neighbourhood V of x, a Laurent cover as above and graded line bundles $\mathcal{L}_i$ on its glued reduction S such that, for every $y \in \mathrm{Int}_X(V)$,*

$$P_y = \overline{\{\rho(y)\}}_{\mathrm{red}}, \qquad (P_y, \mathcal{L}_1|_{P_y}, \dots, \mathcal{L}_d|_{P_y})$$

is a simultaneous proper premodel of the residual bundles on the germ (X, y).

Open proof obligation (RC2). Prove the finite homogeneous refinement and lift required for simultaneous realization; TemLocal Lemma 2.4.

19.18.7 Numerical tests and the residual criterion

A change of premodel should preserve the numerical test, not an unnormalized top intersection number. The following projection-formula argument explains the distinction. We state its ungraded form; CLD25 paragraphs 10.4.6–10.4.11 and 10.7.14 give its graded version by a splitting extension.

Lemma 19.18.11 (Numerical positivity under proper domination)**.** *Let $g : Z \to Y$ be a proper surjective morphism of proper integral varieties over a field, and let $M_1, \dots, M_d$ be line bundles on Y. Their Chern product has nonnegative degree on every integral d-dimensional subvariety of Y if and only if the pulled-back product has that property on Z. The same equivalence holds with all degrees required to vanish.*

Proof. For an integral d-dimensional $A \subset Z$, the projection formula expresses its intersection as that of $g_*[A]$ on Y. This pushforward is zero if the image has smaller dimension; otherwise it is $[k(A) : k(g(A))][g(A)]$, with positive coefficient. This proves one direction. Conversely, given an integral d-dimensional $B \subset Y$, choose a closed point of the nonempty generic fibre over B. Its residue field is finite over $k(B)$. Its reduced closure $A \subset g^{-1}(B)$ is proper and generically finite over B, of some degree $e > 0$, and $\dim A = d$. Hence

$$\deg\left(\prod_i c_1(g^* M_i) \cap [A]\right) = e \, \deg\left(\prod_i c_1(M_i) \cap [B]\right).$$

Division by the positive integer e gives the converse, for either nonnegativity or vanishing. □

Finite families of residual bundles admit common proper dominating premodels by CLD25 paragraphs 10.7.5–10.7.7. After such domination their pullbacks agree. The lemma, with the graded intersection conventions, explains the independence stated in paragraph 10.7.14. In particular, a presentation through a premodel of dimension $m < d$ has numerically zero d-fold product. If $m = d$, numerical positivity is detected by its top degree. Positive field-degree and multiplicity factors may change that degree; they preserve its sign.

Theorem 19.18.12 (Residual criterion in pure dimension)**.** *Let X be Hausdorff, boundaryless and pure of dimension n over a complete non-Archimedean field. Let $L_1, \dots, L_d$, $1 \le d \le n$, have integral PL metrics, and put $T = \bigwedge_i c_1(L_i)$. For $x \in X$, the following are equivalent:*

1. *T is positive on an open neighbourhood of x.*
2. *On a proper simultaneous residual premodel at x, the Chern product has nonnegative degree on every integral subvariety of dimension d.*

Likewise, T vanishes near x if and only if all these degrees vanish. The premodel may be the P_x of Proposition 19.18.10.

Proof. We retain three geometric inputs with their original hypotheses: residual-to-analytic lifting after a common power (CLD25 Proposition 17.6.4), the residual top-mass formula (Theorem 18.4.7), and flat relative fibre integration (Proposition 18.6.1). Section 19.18.2 describes the first input and Proposition 19.18.2 checks the normalization of the latter two. They are cited theorems, not being re-proved here.

Step 1: points of small definition dimension. At a point whose simultaneous residual definition dimension is at most d, these inputs give, after a common tensor power q, a flat presentation through a base of dimension $m \le d$. If $m < d$, both the curvature product and its numerical class vanish. If $m = d$, after shrinking the presentation the product equals

$$q^{-d} a_z \, \delta_{f^{-1}(z)}.$$

The fibre current is positive and nonzero near the point. Formula (19.18.2) and premodel independence identify the sign of a_z with the numerical sign. This proves both implications in the small-dimension case.

Step 2: extend positivity across the remaining locus. Use Proposition 19.18.10 and write $P = P_x$. The set

$$U = \mathrm{Int}_X(V) \cap \rho^{-1}(P)$$

is an open neighbourhood of x. If the numerical condition holds on P and $y \in U$ has $\dim P_y \leq d$, it holds on $P_y \subset P$ as well. This premodel bounds the residual definition dimension by d, so Step 1 gives a positive neighbourhood of y. Let $O \subset U$ be the union of all these neighbourhoods and set $E = U \setminus O$. Then E is closed in U and is contained in $\{y : \dim P_y > d\}$. On each Laurent chart the latter locus has weakly analytic dimension at most $n-d-1$, by Lemma 19.18.4. Thus E is tropically $\mathcal{H}^{n-d}$-negligible. Proposition 19.18.8 gives zero total variation on E for every scalar test measure of T. Lemma 19.14.1 proves positivity on U.

Step 3: recover every numerical test. Suppose T is positive on a neighbourhood N of x contained in $\mathrm{Int}_X(V)$. Let $W \subset P$ be integral of dimension d and let w be its generic point. Since w is a specialization of $\rho(x)$, the closure formula of Proposition 19.18.9 gives a point $y \in N \cap \rho^{-1}(w)$. Now $P_y = W$, and the restrictions $\mathcal{L}_i|_W$ realize the residual bundles at y. Step 1 and premodel independence show that their degree on W is nonnegative. This proves the converse.

Finally, replace L_1 by its dual to apply the positivity statement to $-T$. Vanishing of all degrees is equivalent to both numerical signs being nonnegative, hence to both T and $-T$ being positive. Positive test forms span the symmetric tests and T is symmetric, so this is equivalent to $T = 0$ near x. □

This is a source-based reconstruction of the residual current criterion in the stated pure-dimensional category. The Laurent overlaps, simultaneous bundle realization, corrected noncharging exponent and trivial-valuation passage have now been supplied. The conclusion concerns current positivity. It does not identify an arbitrary positive integral PL metric with a member of the generated psh class, and it does not prove smooth regularization for every continuous current-psh metric. The following sections treat semiample integral residues and the strict fixed-neighbourhood argument; Theorem 19.18.22 then supplies the general integral PL bridge by prescribed graded realization. Proposition 19.18.6 expresses a PL weight as a *difference* of admitted functions; positivity of its difference current alone is not a proof that the difference itself is admitted.

References. [Tem04, Proposition 3.1(ii), Lemma 2.4, and Section 4]; [CLD25, paragraph 11.4.1, Propositions 6.3.4, 8.8.2, 12.4.7, 16.8.2, Theorem 16.8.5, Lemma 16.8.7, Corollary 18.3.3 and Theorem 18.8.4]. The proof uses the named analytic product, reduction, lifting and mass theorems; it is not an independent reproof of those inputs.

19.18.8 Normalized maxima and residual sections

The numerical criterion does not by itself give an approximation of a metric. For that purpose, the residual sections must first be lifted with their norms intact. In the integral PL category these norms can be arbitrary positive real numbers. Keeping them in the formulas both repairs a normalization error in CLD25 and gives a direct proof of the semiample case.

Lemma 19.18.13 (The normalized maximum formula)**.** *Let (X, x) be a Hausdorff almost-good analytic germ, let L have an integral PL metric, and let s be a local nonvanishing section of its underlying line bundle. Put $C = \|s(x)\|^{-1}$. Let $a_1, \dots, a_N$ be analytic units near x, and set*

$$r_i = \frac{C}{|a_i(x)|}. \tag{19.18.3}$$

If the homogeneous residual sections $\widetilde{a_i(x)}\widetilde{s(x)}$ are global sections generating $\widetilde{L}_x$, then on one ordinary open neighbourhood of x

$$\|s\|^{-1} = \max_i r_i|a_i|, \qquad h = h(x) + \max_i \log \frac{|a_i|}{|a_i(x)|}, \quad h = \log \|s\|^{-1}. \tag{19.18.4}$$

Conversely, this identity on a neighbourhood implies that the stated residual sections generate.

Proof. We use the correspondence between domains containing x and their reductions as germs, and the residual norm of CLD25 Section 17.4. Let $b = \|s\|^{-1}$. Define the analytic domains

$$W = \{b \geq r_i|a_i| \text{ for every } i\}, \qquad V_i = \{b = r_i|a_i|\}, \qquad V = W \cap \bigcup_i V_i.$$

These are analytic domains locally because the metric is integral PL. Every one contains x, by the choice of r_i. The set V is exactly the locus where $b = \max_i r_i|a_i|$. In the reduction of the germ, its defining conditions say

$$v(\widetilde{s(x)})^{-1} \geq v(\widetilde{a_i(x)}) \quad \text{for every } i, \qquad \text{with equality for at least one } i.$$

The positive real numbers r_i normalize the analytic inequalities at x; they are not required to be absolute values of scalars. Equivalently, the last display is the maximum identity of CLD25 Lemma 10.6.15. That lemma identifies it with generation by the residual sections. Thus generation is equivalent to (V, x) having the full reduction of (X, x). The germ-reduction criterion then says that V is a neighbourhood of x. Restrict to its ordinary interior and take logarithms. This proves both directions. □

Remark 19.18.14 (Corrections to the printed formulas). In the fixed CLD25 v2 PDF, page 484, Lemma 17.4.10 prints $r_i = |a_i(x)|\|s(x)\|^{-1}$ and defines W using $\leq$. In the maximum formula these must be replaced by (19.18.3) and $\geq$, respectively. The last line of that proof must also retain the inverse on the residual norm of s. The same normalization is printed again in Corollaries 17.4.11 and 17.4.12, page 485. Proposition 17.8.2, page 497, drops the constants altogether when applying Corollary 17.4.12. The valid conclusion is the weighted maximum (19.18.4); this is sufficient for tropical convexity.

These are literal formula corrections, not counterexamples to the intended semiample criterion. To check the first correction, take $X = \mathcal{M}(\mathbf{Q}_p)$, $\|s\| = 1$ and $a_1 = p$. Its nonzero homogeneous reduction generates the residual line. The printed right side is $|p|^2$, whereas the left side is 1. With $r_1 = |p|^{-1}$ the equality holds. More generally, replacing a_i by $c_i a_i$, $c_i \in k^\times$, leaves every normalized ratio $|a_i|/|a_i(x)|$ unchanged.

Theorem 19.18.15 (Semiample residues and first-stage approximation)**.** *In the setting of Lemma 19.18.13, suppose that $\widetilde{L}_x^{\otimes q}$ is globally generated for some positive integer q. There is one ordinary open neighbourhood U of x on which $h = \log \|s\|^{-1}$ belongs to the first-stage class $\mathcal{U}_X(U)$. More precisely, there are finitely many analytic units a_i on U such that*

$$h = h(x) + \frac{1}{q} \max_i \ell_i, \qquad \ell_i = \log \frac{|a_i|}{|a_i(x)|}.$$

The functions

$$h_\epsilon = h(x) + \frac{\epsilon}{q} \log \left(\sum_{i=1}^{N} e^{\ell_i/\epsilon} \right), \qquad \epsilon > 0, \tag{19.18.5}$$

are original CLD smooth psh functions on that same U, decrease to h as $\epsilon \downarrow 0$, and satisfy

$$0 \leq h_\epsilon - h \leq \frac{\epsilon \log N}{q}.$$

No strictness or nontriviality of the valuation is required.

Proof. Quasicompactness of the germ reduction provides a finite family of homogeneous generating sections of $\widetilde{L}_x^{\otimes q}$. Relative to $\widetilde{s(x)^{\otimes q}}$, write their nonzero coefficients in $\widetilde{\mathscr{H}(x)}$. The homogeneous-unit lifting used in CLD25 Corollary 17.4.12 lifts these coefficients to analytic units a_i near x. This is a germ statement: after one finite shrinking all units and their inverses are defined on the same neighbourhood. Apply Lemma 19.18.13 to $L^{\otimes q}$ and $s^{\otimes q}$.

The finite tuple (a_i) is an analytic moment. On its real tropical space, each ℓ_i is affine, including its constant term. The log-sum-exp calculation in Proposition 19.18.6 gives a positive semidefinite covariance Hessian and nonnegative ϵ-derivative. Its pullback therefore has positive $d'd''$, and is s-smooth in the original moment-chart sense. The elementary maximum bounds give the stated uniform estimate on all of U. This constructs one sequence on a fixed domain, rather than merely proving membership in a generated closure. □

Changing the frame to $s' = fs$ subtracts the smooth pluriharmonic function $\log|f|$ from every weight in the construction. Consequently the conclusion is intrinsic to the metric. These are local approximating weights; no global approximation of metrics on an arbitrary nontrivial line bundle is asserted. Arbitrary constants in a finite maximum have thus been handled directly. No valued extension, and hence no descent of psh functions, enters this argument.

19.18.9 A fixed reduction tube for nef approximation

For a nef residue, one adds a small ample class and then applies the semiample result. The delicate point is not the inequality on a single residual variety. It is the existence of one open neighbourhood where all the perturbed metrics have semiample residues. The following formulation separates this geometric input from the elementary limiting argument.

Proposition 19.18.16 (Approximation from compatible premodels)**.** *Let V be a compact analytic domain neighbourhood of x in a Hausdorff boundaryless space. Let $\rho : V \to S$ be an anticontinuous reduction map, with S a graded scheme, and put $P = \overline{\{\rho(x)\}}_{\mathrm{red}}$. Assume the following data have been constructed on this one domain:*

1. *integral PL weights h, a and graded line bundles $\mathcal{L}, \mathcal{A}$ on S, whose restrictions to $P_y = \overline{\{\rho(y)\}}_{\mathrm{red}}$ are premodels of their residues for every $y \in \mathrm{Int}_X(V)$;*
2. *a proper graded variety Q, line bundles M, A on Q, and a morphism $g : P \to Q$ with $\mathcal{L}|_P \simeq g^*M$, $\mathcal{A}|_P \simeq g^*A$;*
3. *for all positive integers m, n, the line bundle $M^{\otimes n} \otimes A^{\otimes m}$ is semiample.*

Then on the single ordinary open neighbourhood

$$U = \mathrm{Int}_X(V) \cap \rho^{-1}(P) \tag{19.18.6}$$

every $h + \epsilon a$, $\epsilon \in \mathbf{Q}_{>0}$, is locally in the first-stage class and belongs to $\mathcal{C}_{25}(U)$. Moreover $h \in \mathcal{C}_{25}(U)$.

Proof. Anticontinuity makes $\rho^{-1}(P)$ open; the interior in (19.18.6) ensures that U is an ordinary open neighbourhood of x in the original space. For $y \in U$, closedness of P implies $P_y \subset P$. If $\epsilon = m/n$, the residue of the integral PL weight $nh + ma$ is represented by

$$(\mathcal{L}^{\otimes n} \otimes \mathcal{A}^{\otimes m})|_{P_y} \simeq (g|_{P_y})^*(M^{\otimes n} \otimes A^{\otimes m})$$

on P_y. Pullback and restriction preserve global generation of a power, hence semiampleness. Theorem 19.18.15, followed by division by n, applies near every y. The local neighbourhood may depend on both y and ϵ; the sheaf property nevertheless gives $h + \epsilon a \in \mathcal{C}_{25}(U)$ on this same U.

Since a is continuous on compact V, choose a real constant B with $a + B \geq 0$ on V. For $j \geq 1$ set $h_j = h + (a + B)/j$. Constants preserve $\mathcal{C}_{25}$, so each h_j belongs to $\mathcal{C}_{25}(U)$, and

$$h_j \downarrow h, \qquad \sup_U |h_j - h| \leq \frac{\sup_V |a + B|}{j}.$$

Closure under locally uniform limits, or under this decreasing continuous limit, gives the last assertion. No intersection of a countable family of neighbourhoods is used. Nor have we produced a single smooth sequence on U for the limiting h. □

For an ordinary projective variety Q, condition (3) follows from M nef and A ample: $M^{\otimes n} \otimes A^{\otimes m}$ is ample, hence semiample. In the graded formulation we have stated the required semiampleness explicitly. This avoids hiding an

algebraic descent or ampleness theorem in the analytic step. The proposition is a conditional criterion: it proves the consequence of compatible premodels, not their existence in every graded situation.

Theorem 19.18.17 (The strict nef case on a fixed neighbourhood)**.** *Let k be complete and nontrivially valued, let X be good, Hausdorff and strictly k-analytic, and let $x \in \mathrm{Int}(X)$. If an $\mathbf{R}$-PL function h in the GJR sense is semipositive at x, then there is an ordinary open neighbourhood U of x on which $h \in \mathcal{C}_{25}(U)$. One can choose bounded $a \in \mathrm{PL}$ on a compact neighbourhood and $B \in \mathbf{R}$ so that $h + (a + B)/j$ decreases uniformly to h on U, and each term is locally Fubini–Study up to an additive constant.*

Open proof obligation (RC2). Supply the fixed-local approximation in the strict nef setting; GJR Proposition 7.10.

This is a source-based expansion of GJR's proof, not a new approximation theorem. It supplies the fixed-neighbourhood justification missing from a bare germwise reading of CLD25 Proposition 17.8.4 in the common strict, nontrivially valued setting. The smooth regularization neighbourhoods for the different perturbations need not agree: only the domain on which their generated-class membership is asserted must agree. The conclusion is deliberately $\mathcal{C}_{25}$; no diagonalization to $\mathcal{U}_X$ has been established.

19.18.10 The strict comparison and the graded compatibility problem

Corollary 19.18.18 (Current positivity for strictly integral PL metrics)**.** *Let X be Hausdorff, boundaryless, strictly analytic and pure of dimension $n \geq 1$ over a complete nontrivially valued field. Let L have a strictly integral PL metric. At a point x, the following conditions are equivalent:*

1. *$c_1(L)$ is a positive current on a neighbourhood of x, against the original CLD tests;*
2. *the residual line bundle $\widetilde{L}_x$ is nef;*
3. *a local weight of L belongs to $\mathcal{C}_{25}$ on a neighbourhood of x.*

The same conclusion holds after a constant rescaling of such a metric.

Proof. Boundarylessness implies goodness. With $d = 1$, Theorem 19.18.12 identifies (1) and (2). In this strict category the residual line of a strictly

integral metric agrees with the residue of its GJR PL local weight, through the formal-model description; this is the comparison of CLD25 paragraph 17.7.10 and GJR Proposition 7.1. Thus Theorem 19.18.17 gives (2)⇒(3). Finally, $\mathcal{C}_{25}$ is generated locally from smooth psh functions by continuous locally uniform limits. Positivity of $d'd''$ is local and passes through these limits: for each compactly supported smooth test, uniform convergence of the weights gives convergence of their distributional derivatives. This proves (3)⇒(1). An additive constant in the weight preserves both curvature and generated-class membership, proving the last assertion. □

For an integral PL metric with semiample residue, Theorem 19.18.15 already works over every complete field. The strict nef proof uses the prescribed-model compatibility of GJR Lemma 6.14. For the general integral case, one must supply a graded analogue with the two bundle identifications required in Proposition 19.18.16; simultaneous realization of given metrics alone does not assert this additional domination. Proposition 19.18.20 below constructs precisely that analogue.

One could instead enlarge the valued field to accommodate the radii. That route must prove descent of generated psh membership to the original open set. Descent of numerical nefness, or the scalar-extension formula for Chern currents, only returns numerical information or current positivity. Using the desired converse to descend would be circular. Similarly, a continuous Gauss section is not an analytic morphism and does not automatically pull arbitrary smooth psh functions back to original smooth psh functions. The Gauss-section noncharging argument of Lemma 19.18.7 is a different statement.

Setting	Conclusion established here
Integral PL; semiample residue	First-stage smooth approximation on one neighbourhood; arbitrary complete base field.
GJR **R**-PL; nef residue; strict, nontrivial valuation	Fixed-tube locally Fubini–Study approximation and $\mathcal{C}_{25}$ membership.
Strictly integral PL; pure dimension; boundaryless	Original-current positivity, residual nefness and local $\mathcal{C}_{25}$ membership are equivalent over a nontrivially valued field.
General integral PL; nef residue	The prescribed graded construction below proves local $\mathcal{C}_{25}$ membership over every complete field; see Theorem 19.18.22.
General continuous current-positive weight	No approximation or generated-class converse follows from the PL results alone.

References. CLD25 v2, Lemma 10.6.15, Theorem 17.4.9, Lemma 17.4.10, Corollaries 17.4.11–17.4.12, and Propositions 17.8.2–17.8.4; GJR arXiv v3, Corollary 5.9, Lemmas 6.11 and 6.14, Section 7.9 and Proposition 7.10. The normalization corrections were checked against the printed PDF. The fixed-tube proof is an expansion of GJR's argument with its model-theoretic inputs retained.

19.18.11 Realizing a prescribed graded premodel

The conditional criterion in Proposition 19.18.16 requires more than separate realizations of two residual bundles. Their analytic realizations must be compared with the same prescribed projective premodel. We now construct that comparison. The construction takes place over the original valued field; arbitrary radii occur as homogeneous degrees in the reduction.

Write $F = \widetilde{\mathscr{H}(x)}$ and $R = P_{F/\widetilde{k}}$. A proper premodel Q comes with an embedding of its graded function field into F and a centre morphism $\pi_Q : R \to Q$. It need not have function field equal to F. This distinction matters when the residue is defined over a smaller graded field.

Lemma 19.18.19 (A finite cover adapted to a premodel)**.** *Let X be Hausdorff and boundaryless over a complete non-Archimedean field k, and let $x \in X$. Let Q be a proper graded premodel of $F/\widetilde{k}$, with a finite affine cover $Q_i = \operatorname{Spec} B_i$. Choose homogeneous nonzero generators a_{ij} of the finitely generated graded $\widetilde{k}$-algebra $B_i \subset F$. There exist an affinoid neighbourhood V of x, analytic units $f_{ij}, g_1, \dots, g_t$ on V, and an assignment $\epsilon \mapsto i(\epsilon)$, for $\epsilon \in \{\pm 1\}^t$, with the following properties:*

1. $\widetilde{f_{ij}(x)} = a_{ij}$;
2. *the Laurent domains*

$$V_\epsilon = \{z \in V : |g_l(z)|^{\epsilon_l} \le |g_l(x)|^{\epsilon_l} \text{ for all } l\}$$

cover V, contain x, and satisfy

$$|f_{i(\epsilon)j}(z)| \le |f_{i(\epsilon)j}(x)| \quad (z \in V_\epsilon, \text{ all } j); \tag{19.18.7}$$

3. *the reduction of the germ (V_ϵ, x) is contained in $\pi_Q^{-1}(Q_{i(\epsilon)})$.*

Open proof obligation (RC2). Construct the Laurent refinement and the simultaneous bounds in the residual setting; TemLocal Lemma 2.4 and Theorem 4.5 Step 1.

Historical proof-local display withheld. (19.18.8)

The finite refinement is essential. Merely choosing lifts of coordinates on each Q_i does not show that their boundedness domains cover an analytic neighbourhood. The reflection of germ inclusions supplies exactly that missing implication.

Proposition 19.18.20 (Prescribed simultaneous realization)**.**

Under the hypotheses of Lemma 19.18.19, let $L_1, \dots, L_s$ be integral PL metrized line bundles near x. Let $M_1, \dots, M_s$ be graded line bundles on Q, and fix residual isomorphisms

$$\pi_Q^* M_l \simeq \widetilde{L}_{l,x} \qquad (1 \le l \le s).$$

One can choose V and its Laurent cover so that the separated glued graded reduction $\rho : V \to S$ carries line bundles $\mathcal{L}_l$ realizing all the L_l. With

$$P = \overline{\{\rho(x)\}}_{\mathrm{red}},$$

there is a proper surjective morphism of premodels $g : P \to Q$ and compatible isomorphisms

$$\mathcal{L}_l|_P \simeq g^* M_l. \tag{19.18.9}$$

For every $y \in \mathrm{Int}_X(V)$, the closure $P_y = \overline{\{\rho(y)\}}_{\mathrm{red}}$ is a proper premodel of the reduction at y, and $\mathcal{L}_l|_{P_y}$ represents $\widetilde{L}_{l,y}$. All these data are chosen for the finite family before any tensor powers or positive rational perturbations are selected.

Proof. Choose the affine cover of Q so that every M_l has a homogeneous frame t_{li} on every Q_i. Apply Lemma 19.18.19. We first construct the map of premodels and then arrange the line bundles on the same cover.

Put $A_\epsilon = \mathcal{O}(V_\epsilon)$ and $S_\epsilon = \operatorname{Spec} \widetilde{A}_\epsilon$. Proposition 19.18.9 glues these charts to S. All contain the point $q = \rho(x)$. The coordinate ring of $P \cap S_\epsilon$ is

$$C_\epsilon = \operatorname{im}\Big(\widetilde{A}_\epsilon \longrightarrow F\Big). \tag{19.18.10}$$

For $i = i(\epsilon)$, each f_{ij} is bounded on V_ϵ by its absolute value at x. Consequently its homogeneous reduction is defined in degree $|f_{ij}(x)|$ and maps to a_{ij} in F. Thus $B_i \subset C_\epsilon$. This inclusion gives a morphism $P \cap S_\epsilon \to Q_i$. The morphisms agree on overlaps: they induce the same embedding of graded function fields into F, their source is integral, and Q is separated. They therefore glue to a dominant $g : P \to Q$. The premodel P is proper by Proposition 19.18.10, or directly by the full-germ valuative criterion used in its proof. As Q is separated, g is proper. Dominance and properness make it surjective.

Notice that we have constructed a map from P, not necessarily from all of S. Polynomial relations among the a_{ij} need only hold in the image ring (19.18.10). They need not hold among the chosen analytic lifts f_{ij}. This is why coordinate lifting suffices.

We now treat the metrics. Trivialize each underlying L_l near x by a frame e_l. Under the prescribed residual isomorphism, the generic value of t_{li} is of the form $\beta_{li}\widetilde{e_l(x)}$, with $\beta_{li} \in F$ homogeneous and nonzero. Lift these finitely many β_{li} to analytic units b_{li} and put $e_{li} = b_{li}e_l$. For $i = i(\epsilon)$, the residual frame of e_{li} generates $\widetilde{L}_{l,x}$ on the reduction of (V_ϵ, x). CLD25 Corollary 17.4.5, in rank one, implies that e_{li} has constant norm $c_{li} = \|e_{li}(x)\|$ on this domain after shrinking at x. Apply it to all finitely many pairs (l, ϵ) and shrink V once more. The coordinate bounds and the Laurent refinement remain valid under this shrinking.

Reduce the normed free modules generated by these frames. On an overlap, the ratio of two frames is an analytic unit of constant absolute value, so its reduction is an invertible homogeneous transition function. These transitions glue to $\mathcal{L}_l$ on S, exactly as in Proposition 19.18.10. Their degrees retain the constants c_{li}; no normalization into $|k^\times|$ is made. On $P \cap S_\epsilon$, identify the reduced frame of e_{li} with g^*t_{li}. On overlaps, the transition ratios agree in F by the prescribed residual isomorphism. Both are regular homogeneous

units on an integral chart overlap, hence agree there. This proves (19.18.9), including compatibility with the original residual isomorphisms.

All the objects S, P, g in the conclusion are understood to be reconstructed from the final shrunken V by the displayed image-ring construction; the retained coordinate bounds justify the same maps. Finally, the assertion at every interior y is the germ-reduction calculation in Proposition 19.18.10, applied to these same constant-norm frames. It identifies both the premodel and the residual bundles. Since only a finite cover, finitely many coordinates and finitely many frames were used, the construction does not depend on any later perturbation parameter. □

19.18.12 Nef perturbations in the graded category

The remaining algebraic input is semiampleness of positive rational perturbations. It can be proved after a graded splitting extension and then descended. This algebraic extension is distinct from extending the valued field of the analytic space.

Lemma 19.18.21 (Graded nef-plus-ample semiampleness)**.** *Let Q be a proper graded variety over a graded field K. Suppose M is nef and A is very ample. Then $M^{\otimes n} \otimes A^{\otimes m}$ is semiample for every pair of positive integers m, n.*

Proof. Choose a finite polyradius r such that the graded field extension $K' = K(T/r)$ splits Q, M, A and the degrees of a projective embedding defined by A; this is CLD25 paragraph 10.3.4. Taking degree-one parts gives an ordinary projective variety Q'^1 over K'^1, an ordinary nef line bundle M'^1, and a very ample line bundle A'^1. Here nefness uses the graded base-change theorem of paragraph 10.4.11, while the very ample assertion follows from paragraphs 10.3.2–10.3.3. These are the named algebraic inputs to this reduction.

The classical nef-plus-ample theorem makes $(M'^1)^{\otimes n} \otimes (A'^1)^{\otimes m}$ ample. Choose a positive power q which is globally generated. The equivalence between strict graded bundles and their degree-one parts shows that the corresponding graded bundle on $Q_{K'}$ is globally generated by homogeneous sections. Global generation descends to Q by CLD25 paragraph 10.2.4. For completeness, the descent uses

$$H^0(Q, D) \otimes_K K' \simeq H^0(Q_{K'}, D_{K'}), \qquad D = (M^{\otimes n} \otimes A^{\otimes m})^{\otimes q}.$$

This flat base-change identity follows by computing sections on a finite affine cover and its intersections. Express a finite set of homogeneous generators

on the right as sums of tensors. The finitely many homogeneous components of their coefficients in $H^0(Q, D)$ generate after extension to K'. Faithful flatness makes the cokernel of their evaluation map zero already on Q. Thus D is globally generated, as required. The exponent q may depend on m, n; no uniform exponent is needed. □

The proof uses the classical nef-plus-ample theorem in the same way as the strict argument of GJR Proposition 7.10. The additional work is the graded splitting and descent of homogeneous generation. It supplies the precise semiampleness needed in Proposition 19.18.16, without claiming that a purely numerical descent proves analytic psh descent.

Theorem 19.18.22 (Integral PL nefness gives generated psh)**.** *Let k be any complete non-Archimedean field, allowing trivial valuation. Let X be Hausdorff and boundaryless, and let L have an integral PL metric near $x \in X$. If $\widetilde{L}_x$ is nef, then a local weight h of L belongs to $\mathcal{C}_{25}(U)$ on an ordinary open neighbourhood U of x. More precisely, one can choose an integral PL weight a on a compact neighbourhood V, a real constant B, and a single $U \subset V$ such that*

$$h_j = h + \frac{a+B}{j} \downarrow h, \qquad \sup_U |h_j - h| \leq \frac{\sup_V |a+B|}{j}, \tag{19.18.11}$$

and each h_j is locally in $\mathcal{U}_X$ on U. Neither strictness nor a nontrivial valuation is required.

Proof. Represent $\widetilde{L}_x$ by a nef graded line bundle M on a proper premodel. Graded Chow's lemma, CLD25 Lemma 10.7.3, gives a proper birational quasi-projective dominating premodel Q. Since the original premodel is proper, Q is proper and projective in the graded sense. Pullback preserves nefness. Replace M by its pullback and choose a very ample graded line bundle A on Q. These operations concern only the two fixed residual bundles.

The essential surjectivity of CLD25 Theorem 17.4.9(b) lifts $\pi_Q^* A$ to an integral PL metric on an analytically trivial line bundle near x. Denote its weight by a, and fix a residual isomorphism. Apply Proposition 19.18.20 simultaneously to L and this auxiliary metrized line, with their prescribed M, A on Q. It gives $V, S, \rho, P, g, \mathcal{L}, \mathcal{A}$ satisfying the first two hypotheses of Proposition 19.18.16. Lemma 19.18.21 gives its third hypothesis.

Set $U = \mathrm{Int}_X(V) \cap \rho^{-1}(P)$. That proposition proves that $h + \epsilon a$ is locally first-stage and belongs to $\mathcal{C}_{25}(U)$ for every positive rational ϵ. Choose B with

$a+B \geq 0$ on compact V. Its final argument yields (19.18.11) and $h \in \mathcal{C}_{25}(U)$. Every analytic unit and every open neighbourhood in this argument was constructed over k. The splitting extension in Lemma 19.18.21 was used only to establish a statement about graded line bundles; its global generators were descended before the analytic regularization was applied. □

This fills the prescribed-premodel step isolated in version 0.10. It reconstructs the integral PL implication intended in CLD25 Proposition 17.8.4 by making its domain and bundle compatibility explicit. It is a synthesis of the cited reduction and algebraic results, not a new regularization theorem for arbitrary continuous weights. In particular, no assertion that $h \in \mathcal{U}_X(U)$ has been made: the neighbourhoods supporting smooth approximations of the individual h_j may depend on j.

19.18.13 The completed integral PL comparison

Corollary 19.18.23 (Integral PL current, residue and generated class)**.** *Let X be Hausdorff, boundaryless and pure of dimension $n \geq 1$ over any complete non-Archimedean field. Let L have an integral PL metric. At a point x, the following conditions are equivalent:*

1. *$c_1(L)$ is positive near x against all original CLD compactly supported smooth tests;*
2. *$\widetilde{L}_x$ is nef;*
3. *a local weight of L belongs to $\mathcal{C}_{25}$ near x.*

Proof. For $d = 1$, Theorem 19.18.12 gives (1)⇔(2), with its named product, noncharging, lifting and mass inputs. Theorem 19.18.22 gives (2)⇒(3). To prove (3)⇒(1), note that current positivity is local, holds for smooth psh weights, and is preserved by locally uniform limits of continuous weights. Indeed, for a fixed compactly supported smooth test η, uniform convergence on its support gives $\langle d'd''[h_j], \eta\rangle \to \langle d'd''[h], \eta\rangle$. The class of continuous current-positive weights is therefore closed under all the operations generating $\mathcal{C}_{25}$. □

Remark 19.18.24 (Scope of the completed implication). The semiample case still has the stronger conclusion of Theorem 19.18.15: one decreasing smooth sequence on one neighbourhood. The general integral nef case has the generated-class conclusion of Theorem 19.18.22. Neither result settles the continuous predecessor replacement in CLD25 paragraph 8.1.11 or the passage from an arbitrary continuous positive current potential to a locally smooth-approximable potential. The general analytic Bedford–Taylor, capacity and Green-form questions retain their separate hypotheses and unresolved steps.

For comparison with GJR, Theorem 19.18.17 allows real PL coefficients in the strict, nontrivially valued category and uses formal models. Theorem 19.18.22 treats integral PL metrics with arbitrary homogeneous degrees over any complete field, using graded reductions. Their common mechanism is a fixed reduction tube, an ample residual perturbation, local regularization of finite maxima, and a continuous limit. The categories and the strengths of the approximation conclusions should remain visible when either theorem is cited.

References. Temkin [Tem04, Lemma 2.4, Proposition 3.1(ii), Lemma 4.4 and Theorem 4.5]; CLD25 v2, paragraphs 10.2.4, 10.3.2–10.3.4, 10.4.11 and 10.7.1, Lemma 10.7.3, Lemma 17.3.2, Corollary 17.4.5, Theorem 17.4.9 and Proposition 17.8.4. The proofs above supply the prescribed-cover construction, map on the reduced closure, compatible bundle identifications, and fixed-domain deduction. The named germ correspondence, graded nef base change, Chow lemma and residual lifting remain imported theorems.

19.19 Relative volumes and the orthogonality mechanism

A local positive current does not by itself define a global envelope theory. The latter requires a global class of semipositive metrics and an approximation theorem in that class. Relative volumes provide a separate algebraic route to the derivative needed by the variational method. This route explains both the strength and the scope of the results of Burgos Gil–Gubler–Jell–Künnemann–Martin and Boucksom–Gubler–Martin. Use additive metric weights. Let X be a projective geometrically integral K-variety of dimension n, let L be ample, and let ϕ be a continuous semipositive metric. The relative volume $\mathrm{vol}(L, \phi_1, \phi_0)$ is the asymptotic relative determinant norm of the sup norms

on $H^0(X, mL)$, with the normalization of [BGM22]. Their Theorem A, valid over an arbitrary complete non-Archimedean field, asserts

$$\frac{d}{dt}\Big|_{t=0} \operatorname{vol}(L, \phi + tf, \phi) = \int_{X^{\mathrm{an}}} f \ \mathrm{MA}(\phi), \qquad f \in C^0(X^{\mathrm{an}}). \tag{19.19.1}$$

Here $\phi + tf$ need not remain semipositive. Differentiation transverse to the semipositive cone is part of the theorem. Over a nontrivially valued field semipositivity is the uniform closure of nef model metrics; the arbitrary-field statement uses the compatible Fubini–Study formulation. It must not be read as a differentiability theorem for every local CLD-psh function.

Proposition 19.19.1 (From the volume derivative to orthogonality)**.** *Assume continuity of semipositive envelopes for (X, L). For a continuous metric ψ, let $P\psi$ be its semipositive envelope. Then*

$$\int_{X^{\mathrm{an}}} (\psi - P\psi)\, \mathrm{MA}(P\psi) = 0.$$

Open proof obligation (RC2). Establish the required volume–energy identity for continuous envelopes; BGM22 Equation (0.2).

Since the integrand is continuous and nonnegative, the measure is supported on the contact set $\{P\psi = \psi\}$. Indeed, on any compact subset where the gap is at least $\epsilon > 0$, its mass is at most ϵ^{-1} times the displayed integral. This explains the precise place where orthogonality enters the variational solution of a Monge–Ampère equation. The earlier discretely valued result [BGJKM20, Theorem C], in residue characteristic zero on a smooth projective variety, removes the additional algebraicity condition in the corresponding BFJ orthogonality result. The arbitrary-field volume theorem is later and more general as a differentiability statement. Its orthogonality consequence still assumes continuity of envelopes. "Arbitrary field" in the former theorem does not erase the latter hypothesis. This also identifies why Proposition 16.7.1 is compatible with these theorems. Its polyhedral PC class has no automatic relative-volume identity with the stated derivative. The formal envelope inequalities alone do not force contact-locus support. The algebraic volume input is doing substantive work.

References. [BGJKM20, Theorems B–D]; [BGM22, Theorems A–B and Equation (0.2)].

19.19.1 Finite-energy measures and their topology

Boucksom–Jonsson formulate a global theory from a space $\mathcal{D}$ of test functions, a positive class ω, an energy E_ω and an admissible cone $\mathcal{D}_\omega$. With probability-measure normalization, the dual energy is

$$J_\omega(\mu) = \sup_{\phi \in \mathcal{D}_\omega} \left(E_\omega(\phi) - \int \phi \, d\mu \right), \qquad \mathcal{M}^1_\omega = \{\mu : J_\omega(\mu) < \infty\}.$$

The strong topology is the coarsest refinement of weak convergence making J_ω continuous. It contains strictly more information than weak convergence alone. Positivity and total mass one do not imply finite energy. Their Theorem A constructs a complete quasi-metric on this space, compatible with the Dirichlet functional on Monge–Ampère images, under their synthetic orthogonality property. The density of those images is one conclusion; an actual local CLD potential for every measure is not. Their Definition 2.15 expresses orthogonality through a directed limit of admitted test potentials below an obstacle. It is not an assumption that every envelope has a continuous representative. The paper verifies its synthetic hypotheses for projective Berkovich spaces over arbitrary non-Archimedean fields. Thus it would be incorrect to carry the continuity-of-envelopes hypothesis from Proposition 19.19.1 into every finite-energy measure theorem, or conversely to infer that all continuous envelopes have been constructed from the synthetic result. The two formulations must be compared at the level of their definitions.

References. [BJ-S, Definitions 2.1, 2.15 and Theorems A–B].

Chapter 20

Chern measures and intersection degrees

20.1 Residual bundles and the recovery of model measures

20.1.1 What a formal metric contributes

Let $\mathfrak{X}$ be a formal model of a proper analytic space and let $\mathfrak{L}$ be a line bundle on it. A local formal frame has norm one on the generic fibre. This rule determines the model metric on $L = \mathfrak{L}|_X$. A change of formal frame is multiplication by a formal unit, whose norm is compatible with the gluing. Tensor powers allow rational model metrics.

The resulting weights are piecewise linear in the appropriate analytic sense. Their curvature is generally singular. A basic example is the standard metric on $\mathcal{O}_{\mathbb{P}^1}(1)$, whose curvature is concentrated at the Gauss point. Calling this metric formal does not mean its original CLD weight is a smooth function.

20.1.2 Why residual geometry enters positivity

At an analytic point, a PL metric determines residual line-bundle data on a suitable reduction of the germ. Positivity of that data is tested by degrees on residual curves. This can see geometry invisible in one real coordinate picture. If the residual variety contains a curve on which the line bundle

has negative degree, a numerical semipositivity assertion fails even when a chosen local polyhedral presentation looks harmless.

CLD12 develops the residual construction and uses residual positivity to obtain approximation results and identify model Monge–Ampère measures. It does not thereby prove every converse between continuous current positivity and model semipositivity. Chapter 19 develops the later model and curve tests; Chapter 19 states the stronger residual comparison claimed in CLD25 with its precise regularity scope.

20.1.3 The special-fibre formula

For orientation, assume a normal proper flat model over a discrete valuation ring, with special fibre $\sum_Y m_Y Y$, and integral model line bundles $\mathfrak{L}_1, \dots, \mathfrak{L}_n$. In the usual model-measure normalization,

$$c_1(\overline{L}_1) \cdots c_1(\overline{L}_n) = \sum_Y m_Y \deg_Y(\mathfrak{L}_1 \cdots \mathfrak{L}_n)\, \delta_{\xi_Y}.$$

Here ξ_Y is the divisorial point associated with the component Y. Rational model metrics introduce the corresponding tensor-power denominators. The conventions for ξ_Y and the component multiplicities must be retained together; omitting m_Y while keeping the same normalized valuation changes the formula.

The mass is the generic-fibre intersection degree. In particular, it does not acquire an extra factor $-\log|\pi|$ merely because the underlying absolute value has that scale. That logarithmic scale enters the values of vertical model functions and local heights; the Chern measure itself is normalized to have algebraic degree as mass. A detailed blow-up calculation in Section 20.6 checks both facts simultaneously.

20.1.4 How the analytic and model constructions meet

The comparison is a theorem, with two constructions to compare. One side comes from approximating local metric weights and taking limits of differential products. The other side is a finite sum of algebraic intersection degrees. The proof identifies the analytic mass near a divisorial point with the corresponding residual intersection number and shows that no other mass remains. Locality, Poincaré–Lelong, and tropical degree calculations are all used in this passage.

Once this comparison is established for suitable model metrics and the appropriate continuity theorem is available, uniform limits extend the agreement to a larger class of semipositive metrics. It remains necessary to say which approximation is being used: local smooth approximation, global approximation on one line bundle, or approximation by semipositive models. Their equivalence is not part of the notation $c_1(\overline{L})$.

References. [CLD12, §§6.1–6.9]; [CL06].

20.2 Metrics and residual curvature in CLD25 Chapters 17–18

20.2.1 Vector bundles first, curvature products for line bundles

CLD25 Chapter 17 develops metrics on vector bundles, their PL forms, and residual bundles. The statements about curvature products in CLD25 Chapter 18 principally use line bundles and first Chern currents. The presence of vector-bundle metrics in the foundations should not be read as a complete higher-Chern-form or Bott–Chern theory for every such metric.

For a line bundle, a metric is locally a weight, but the transition rule is constrained by analytic units. Residual reduction attaches geometric data to PL weights and makes numerical positivity available. Formal metrics and strictly integral PL metrics are compared in the appropriate nontrivially valued strict analytic category. Extending a conclusion from strictly integral PL to integral or real PL data requires the relevant reduction, tensor, or descent argument.

20.2.2 Relative Monge–Ampère and residual intersections

The model-measure formula of Chapter 20 is a top-degree case. CLD25 develops curvature products of PL metrics as structured currents and calculates them using residual intersection data. Relative formulas allow products to be evaluated along fibres. This is the mechanism that feeds the complete-intersection and abelian applications.

The central distinction is between a numerical residual class and an actual metric. Residual numerical triviality can force a curvature current to vanish in a specified theorem, but it does not make the metric canonically trivial.

Constants and algebraically nontrivial degree-zero data remain relevant to heights and pairings.

20.2.3 The canonical current of a curve is not a probability measure

Section 18.2 treats a different geometric metric: the semi-metric on the canonical bundle induced from Temkin's Kähler seminorm. Over an algebraically closed nontrivially valued field, the basic example on the affine line is

$$\|dT\|(\eta_{a,r}) = r.$$

At a rigid point the seminorm vanishes. Hence this is not a finite continuous metric of the sort used in the elementary BT recursion. Its logarithm is nevertheless locally integrable in the source construction, and its restriction to suitable skeletal pieces is PL.

Theorem 18.2.16 identifies its curvature on a smooth analytic curve with a canonical skeletal current, evaluated using a skeleton adapted to the test. The adapted skeleton may change when the test changes. On a compact curve, the degree of the canonical bundle is $2g - 2$, so this object is not by definition a positive probability measure. In particular, it should not be confused with the normalized admissible measure used in Arakelov theory on a metrized graph or with the canonical probability measure of a polarized dynamical system.

This example is important for the learning programme: a natural geometric norm can be singular and fail to be continuous at a dense set, while still admitting a meaningful curvature current. The required integrability and support arguments are part of the theorem. They cannot be replaced by an informal appeal to the continuous theory.

References. [CLD25, Chapters 10–18, especially §§12.10–12.11, §14.7, Proposition 17.7.5, and Theorem 18.2.16].

20.3 Metrics, curvature, and the first substantial applications

20.3.1 A metric is locally a function, but globally more than curvature

Let L be a line bundle with a continuous metric. For a nonvanishing local frame e, set

$$h_e = -\log \|e\|.$$

If $e' = fe$ with f invertible, then

$$h_{e'} = h_e + v(f), \qquad v(f) = -\log |f|. \tag{20.3.1}$$

The unit logarithm $v(f)$ is original smooth, being a moment-map coordinate, and has $\mathrm{d}'\mathrm{d}''[v(f)] = 0$. Hence the local currents $\mathrm{d}'\mathrm{d}''[h_e]$ glue to the first Chern current $c_1(L, \|\cdot\|)$. This gluing uses a smooth harmonic change of weight and does not require a kernel theorem for arbitrary continuous harmonic multipliers. The trivialization rule is why a local Hessian becomes a global curvature.

For a nonzero rational or meromorphic section s, the Poincaré–Lelong formula in our normalization is

$$\mathrm{d}'\mathrm{d}''[-\log \|s\|] + \delta_{\mathrm{div}(s)} = c_1(L, \|\cdot\|), \tag{20.3.2}$$

interpreted where the locally integrable current is defined. Equivalently, $\mathrm{d}'\mathrm{d}''[\log |f|] = \delta_{\mathrm{div}(f)}$ for a meromorphic function in the trivial-metric case. This is consistent with the concavity of $v(f)$ on a torus Gauss section: the positive divisor appears with a minus sign in $\mathrm{d}'\mathrm{d}''[v(f)]$.

Changing the metric to $e^{-c}\|\cdot\|$, with $c \in \mathbb{R}$, adds c to every local weight and leaves curvature unchanged. It does change the metrized line bundle and can change a local height. Thus curvature is an invariant of metric data, not a complete replacement for those data.

20.3.2 The standard metric on the projective line

Example 20.3.1. On $\mathbb{P}^1_K$ with homogeneous coordinates $[Z_0 : Z_1]$, the standard model metric on $\mathcal{O}(1)$ satisfies, on the chart $z = Z_1/Z_0$,

$$\|Z_0\|(x) = \frac{1}{\max(1, |z(x)|)}.$$

On the torus Gauss skeleton, with $t = v(z)$, its weight is

$$h(t) = \log \max(1, e^{-t}) = \max(0, -t).$$

Its left slope is -1 and its right slope is 0. The one-dimensional corner formula gives one unit of curvature at $t = 0$. Analytically this is

$$c_1\big(\mathcal{O}(1), \|\cdot\|_{\mathrm{std}}\big) = \delta_{\xi_G}, \tag{20.3.3}$$

where ξ_G is the Gauss point of the closed unit disc. The identity on all of $\mathbb{P}^{1,\mathrm{an}}$ also follows from the model formula in Section 20.6; the skeleton computation identifies the slope and its sign.

The section Z_0 vanishes at infinity. Therefore

$$\mathrm{d}'\mathrm{d}''[-\log \|Z_0\|] + \delta_\infty = \delta_{\xi_G}.$$

Both sides have total mass one after the divisor term is included. Integrating the displayed Hessian alone gives zero, as Stokes requires on the proper curve. The pole of the Green function at infinity is part of the global calculation and must not be lost when using the real line chart.

20.3.3 What an original GK delta-metric requires

A piecewise smooth metric has local weights that can be represented by piecewise smooth functions on tropical charts. The original GK definition of a *delta-metric* imposes an additional condition: the piecewise Hessian on the tropical image must come from a piecewise smooth ambient $(1,1)$-superform in the prescribed way. This is [GK17, Definition 9.9]. Its purpose is to ensure that the complete curvature, including the corner part, is a delta-form with the required pullback behavior.

The characterization in [GK17, Proposition 9.11] is functorial: a first Chern delta-form must realize the Chern delta-current after every allowed pullback, with the equality taking place in GK's delta-current space, not only after restriction to original smooth tests. The resulting curvature is d'- and d''-closed. Piecewise linear metrics satisfy the original condition because

their interior Hessian is zero. Formal/model metrics therefore provide the fundamental examples. It is inaccurate to replace this statement by "every piecewise smooth metric is an original GK delta-metric." The later Mihatsch category incorporates piecewise smooth functions more systematically, as explained in the previous section.

20.3.4 Positivity has a geometric meaning

For a convex smooth weight on a real chart, positivity is simply positivity of its Hessian. For a piecewise smooth weight, the facewise Hessians and the corner contributions both enter. On a line, convexity means nonnegative second derivatives on the pieces and nonnegative slope jumps. This local calculation is the prototype of the metric positivity criteria.

For piecewise linear metrics in the hypotheses of [GK19, Theorem 6.10], model semipositivity is equivalent to the stated functorial positivity conditions, to positivity of the Chern delta-form, and to the relevant curve tests. The adjective *functorial* is essential in that theorem. Its list of equivalences should not be shortened to arbitrary positivity of a current on one fixed space. The later work of Gubler–Rabinoff and CLD develops additional analytic and residual criteria; these have their own scope, discussed in Section 20.6.

20.3.5 Comparison with model intersection numbers

On a proper n-dimensional variety with formally metrized line bundles $\overline{L}_1, \dots, \overline{L}_n$, GK identifies the wedge of the Chern delta-forms with the Chambert–Loir measure obtained from intersection numbers on a model:

$$c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_n) = \mathrm{MA}(\overline{L}_1, \dots, \overline{L}_n). \tag{20.3.4}$$

This is [GK17, Theorem 10.5]. It does not require the model line bundles to be nef for the equality to make sense: in general the measure is signed. Semipositivity is what makes the coefficients nonnegative.

This comparison is a major reason to learn the calculus. A wedge product defined by tropical local data computes a measure whose atoms and masses come from algebraic intersection theory on the special fibre. Section 20.6 explains the model formula; Section 22.2 uses it in local heights.

Remark 20.3.2 (Constant rescaling). Let $\overline{L}' = (L, e^{-c}\|\cdot\|)$ for a real constant c. The weight changes from φ_e to $\varphi_e + c$, and $\mathrm{d}'\mathrm{d}''[c] = 0$, so $c_1(\overline{L}') = c_1(\overline{L})$. Whenever the local-height recursion and degree formula apply on a proper n-fold, rescaling one metric by e^{-c} changes the local height by

$$c \deg_X(L_1 \cdots L_n),$$

where the displayed line bundles are the other n entries. This follows by taking $\rho = c$ in (22.2.4); it does not follow from the curvature alone.

20.4 Fubini–Study weights and mixed Chern measures

20.4.1 Sections produce a metric

Let L be a line bundle and let $s_0, \dots, s_N$ be sections of $L^{\otimes m}$ with no common zero. Choose real weights a_i. On an open set with frame e of L, write $s_i = f_i e^{\otimes m}$ and put

$$\phi_e = \frac{1}{m} \max_i \bigl(\log|f_i| + a_i\bigr), \qquad \|e\| = e^{-\phi_e}. \tag{20.4.1}$$

Use $\log 0 = -\infty$. The maximum is finite because the sections have no common zero. Under $e' = ge$, the coefficients become $f_i' = f_i g^{-m}$, so $\phi_{e'} = \phi_e - \log|g|$. Hence the local formulas define a metric.

Near a point, any section vanishing at that point is dominated in the maximum by a nonvanishing section after shrinking. The relevant nonvanishing coefficients are analytic units. Thus (20.4.1) is locally a maximum of affine functions of unit logarithms. This explains its piecewise affine character and positive corner curvature. Arbitrary real a_i give real weighted FS data; to regard the metric as a formal or rational model metric over the original field, one must impose the corresponding value-group and denominator conditions.

20.4.2 Smoothing the maximum without losing the chart conditions

On a real chart, $\max(\ell_0, \dots, \ell_N)$ can be approximated by smooth convex regularized maxima. For instance, log-sum-exp gives a convenient local estimate

$$\max_i \ell_i \le \varepsilon \log \sum_i e^{\ell_i/\varepsilon} \le \max_i \ell_i + \varepsilon \log(N+1).$$

This formula illustrates uniform approximation and convexity on the chart. Near a zero of a section, it should not be used to pretend that $\log|f_i|$ is a smooth finite coordinate. A compatible regularized-maximum construction discards dominated terms and agrees with the surviving affine term where it is separated. CLD's FS regularization provides the analytic construction under its stated hypotheses.

On $\mathbb{P}^1$, the standard weight $\max(0,-t)$ is particularly instructive. Smooth it only in a small interval around $t=0$, leaving it equal to its affine branches outside. Its second derivative is a nonnegative measure of total mass one, since the endpoint slopes remain -1 and 0. As the interval shrinks, these measures converge weakly to δ_0. In analytic terms, the curvature tends to the Gauss-point mass. The need to keep the affine branches is also what permits smoothness in the original sense near the rigid endpoints.

20.4.3 Three legitimate meanings of a mixed Chern expression

The notation

$$\mu = c_1(L_1, \|\cdot\|_1) \wedge \cdots \wedge c_1(L_n, \|\cdot\|_n)$$

does not define itself. There are several constructions, with comparisons on their common domains.

For *smooth semipositive metrics*, the local positive $(1,1)$-forms $\mathrm{d}'\mathrm{d}''\phi_{i,e}$ glue to $c_1(\overline{L}_i)$. Their ordinary wedge is a top-degree form and hence a measure.

For *continuous metrics whose local weights lie in* $\mathcal{C}_{25}$, use the recursion of Proposition 13.1.4 with background $[1]_X$. Smooth unit-logarithm frame changes have zero Hessian, so the local products glue. In top degree the resulting positive current is a Radon measure. Metrics given as controlled differences of admissible positive metrics can be treated by the corresponding multilinear signed theory once its independence statements have been established.

For *formal/model metrics*, the special-fibre formula gives an atomic measure, while GK constructs it as a product of Chern delta-forms. The equality of these constructions is a comparison theorem. The model definition does not rely on pretending the original curvature current was smooth.

20.4.4 What uniform approximation does to the measure

Suppose globally defined smooth semipositive metrics $\|\cdot\|_{i,j}$ on the fixed L_i converge uniformly in their weights to $\|\cdot\|_i$. On every trivializing neighbourhood their weights satisfy the hypotheses of the continuous multiplier and BT theorems. Consequently

$$\bigwedge_{i=1}^{n} c_1(L_i, \|\cdot\|_{i,j}) \rightharpoonup \bigwedge_{i=1}^{n} c_1(L_i, \|\cdot\|_i).$$

Independence of approximating sequences follows from the intrinsic recursive construction; it is not inferred just from the existence of some weakly convergent subsequence.

For uniform limits of semipositive FS or model metrics, the model-theoretic measure is likewise defined using its continuity theorem. To compare it with the CLD product, compare first on the admissible common class and then pass to the uniform limit using continuity on both sides. Thus the proof has two inputs: equality at the approximating level and convergence of each construction. A raw current inequality alone does not supply either input.

20.4.5 Why the total mass is the intersection degree

Let Y/K be smooth projective of dimension n. In the smooth intersection comparison, one has

$$\int_{Y^{\mathrm{an}}} c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_n) = \deg_K(L_1 \cdots L_n).$$

For a globally smoothly approximable semipositive tuple as above, every approximating measure has this mass. Properness makes Y^{an} compact, so the constant function 1 is an admissible test for the top-degree current. Weak convergence therefore preserves the mass.

This explains the statement studied in the earlier discussion: global approximation plus the smooth degree formula gives the total mass formula. The word *global* cannot be dropped in this short argument. On a noncompact open space, vague convergence can lose mass at infinity; for example $\delta_j \to 0$ on compact tests in $\mathbb{R}$, although each measure has mass one.

References. [CLD12, §§6.3–6.9]; [CLD25, §§8.7, 17.8, 18.8]; [GK17, Theorem 10.5].

20.5 Total variation and the curve estimate

20.5.1 The norm of a signed measure

For a finite signed Radon measure ν, write its Jordan decomposition as $\nu = \nu^+ - \nu^-$, with mutually singular positive measures. Its total variation measure is $|\nu| = \nu^+ + \nu^-$, and

$$\|\nu\|_{\mathrm{TV}} = |\nu|(X) = \nu^+(X) + \nu^-(X).$$

Equivalently, it is the supremum of $\sum_i |\nu(E_i)|$ over finite measurable partitions, or the supremum of $|\int f\, d\nu|$ over bounded measurable f with $|f| \le 1$. For $\nu = \sum a_i \delta_{x_i}$ at distinct points, the norm is $\sum |a_i|$. Its signed mass, by contrast, is $\sum a_i$.

If μ_0, μ_1 are positive finite measures, then

$$|\mu_1 - \mu_0| \le \mu_1 + \mu_0, \qquad \|\mu_1 - \mu_0\|_{\mathrm{TV}} \le \mu_1(X) + \mu_0(X).$$

Cancellation can make the inequality strict. For example $\mu_0 = \mu_1 \neq 0$ gives zero on the left.

20.5.2 The bound $2 \deg L$ and its hypotheses

On a smooth projective analytic curve, let two admissible semipositive metrics on the same line bundle L have curvature measures μ_0, μ_1. Their local weight difference $g = \phi_1 - \phi_0$ is a global continuous function, and

$$\mathrm{d}'\mathrm{d}''[g] = \mu_1 - \mu_0.$$

Each positive measure has mass $\deg_K L$. Therefore

$$\|\mathrm{d}'\mathrm{d}''[g]\|_{\mathrm{TV}} \le 2 \deg_K L. \tag{20.5.1}$$

The signed mass of $\mathrm{d}'\mathrm{d}''[g]$ is zero, but its total variation need not be zero. Inequality (20.5.1) measures how much positive and negative curvature can move while preserving the degree.

The bound does not hold for an arbitrary continuous function g without the hypothesis that its Hessian is the difference of these two positive measures of controlled mass. It is a measure estimate after the curvature construction and degree comparison have been established, not a separate general regularization theorem.

20.5.3 What the estimate buys in an integral

For a continuous test coefficient a,

$$\left| \int a \, \mathrm{d}'\mathrm{d}''[g] \right| \leq \|a\|_\infty \|\mathrm{d}'\mathrm{d}''[g]\|_{\mathrm{TV}} \leq 2 \deg_K L \, \|a\|_\infty.$$

Thus a uniform error in a potential or coefficient produces a controlled error against this curvature difference. For smooth functions on a proper curve, integration by parts gives

$$\int a \, \mathrm{d}'\mathrm{d}''g = \int g \, \mathrm{d}'\mathrm{d}''a.$$

Extensions to nonsmooth functions require the admissible approximation or multiplier theorem; the formal symmetry of the display alone does not justify them. This is the analytic use of total variation in curve energy and approximation arguments.

20.6 Model measures with the original absolute value

20.6.1 The special-fibre formula

Let K now be discretely valued, $R = K^\circ$, and $\lambda = -\log|\pi| > 0$ for a uniformizer. Let $\mathcal{X}$ be a normal projective flat model of a projective n-fold X, with special fibre $\sum_Y m_Y Y$. Each component determines a divisorial point ξ_Y, whose valuation extends the original valuation of K, so $v_{\xi_Y}(\pi) = \lambda$.

If $\mathcal{L}_i$ extends $L_i^{\otimes e_i}$, the model measure is

$$\mu = \frac{1}{e_1 \cdots e_n} \sum_Y m_Y \deg_{\widetilde{K}}\big(c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_n) \cap [Y]\big) \delta_{\xi_Y}. \tag{20.6.1}$$

The residual degree includes residue-field degrees. There is no extra factor λ multiplying this Chern measure in the normalization used here: its mass is the algebraic intersection degree. The scale λ does appear in values of logarithmic model functions and in lengths of skeleton edges. Keeping both is what makes the slope formula agree with (20.6.1).

Relative nefness makes the coefficients nonnegative. Without nefness, the same formula defines a signed measure. The equality with the analytic CLD/GK construction on the common class is the content of the measure

comparison, and compatibility with base change is needed when using the algebraically closed version of the GK theorem over a smaller ground field.

20.6.2 A blow-up with the scale still visible

Blow up $\mathbb{P}^1_R$ at a rational smooth point of the special fibre. The resulting special fibre is $F + E$, where E is exceptional, and

$$F \cdot E = 1, \qquad F^2 = E^2 = -1.$$

The generic fibre is still $\mathbb{P}^1_K$. The model line $\mathcal{O}(E)$, with its canonical generic trivialization, defines a function $g = -\log \|1\|$ with

$$g(\xi_F) = 0, \qquad g(\xi_E) = \lambda.$$

The intervening skeletal edge has length λ. Its slope from F to E is therefore $+1$, not $+\lambda$. The outgoing slope at the other end is -1, and

$$\mathrm{d}'\mathrm{d}''[g] = \delta_{\xi_F} - \delta_{\xi_E}.$$

This agrees with $E \cdot F = 1$ and $E^2 = -1$. Normalizing $\lambda = 1$ hides the distinction between value and slope, but is unnecessary.

More generally, $D = aF + bE$ gives values $a\lambda, b\lambda$ and curvature

$$(b - a)\delta_{\xi_F} + (a - b)\delta_{\xi_E}.$$

Its mass is zero. If the curvature is positive, then $a = b$, and $D = a\,\mathrm{div}(\pi)$ gives a constant model function. Thus even this simple example separates a metric constant from its vanishing curvature.

References. [CL06, §2]; [GK17, Theorem 10.5 and Remark 12.7].

20.7 Hessians, corner measures, and mixed mass

20.7.1 The smooth calculation fixes the factorial

On $\mathbb{R}^n$, write

$$\mathrm{d}'\mathrm{d}''u = \sum_{i,j} u_{ij}\, d'x_i \wedge d''x_j, \qquad \Omega = \bigwedge_{i=1}^{n} (d'x_i \wedge d''x_i).$$

Expanding the nth power and using the exterior signs gives

$$(\mathrm{d}'\mathrm{d}''u)^n = n!\det(D^2u)\,\Omega.$$

There are $n!$ ways to choose the row indices, and the signed sum over column indices is the determinant. In this paired-order normalization, Ω is positive. If all d' factors are moved to the left, the factor $(-1)^{n(n-1)/2}$ must be inserted.

For $u(x) = \frac{1}{2}x^{\mathsf{T}}Ax$ with A positive semidefinite, the measure has constant density $n!\det A$. If A has rank less than n, the top-degree measure vanishes although $\mathrm{d}'\mathrm{d}''u$ can be nonzero. This returns to the warning in Chapter 7: lower-degree curvature is not determined by its self-top-degree mass.

20.7.2 The PL analogue uses subgradients

For a finite convex function u, its subgradient at x consists of covectors a such that

$$u(y) \geq u(x) + \langle a, y - x\rangle \quad \text{for all } y$$

in the relevant convex domain. For a maximum of affine functions, the subgradient at a point is the convex hull of the slopes of the active affine branches. The real Alexandrov Monge–Ampère measure measures the volume swept out by these subgradients. In the superform normalization above, the corresponding top-degree measure has the additional factor $n!$.

For $u(x, y) = \max(0, x, y)$, the subgradient at the origin is the triangle with vertices $(0, 0), (1, 0), (0, 1)$, of area $1/2$. Away from the origin, the subgradient is a segment or a point and has area zero. Consequently

$$(\mathrm{d}'\mathrm{d}''u)^2 = \delta_0.$$

The corner set is one-dimensional, but the top-degree product is concentrated where two independent slope changes meet. This is the convex-geometric prototype of the complete-intersection mass in the logarithmic maximum formula.

20.7.3 Mixed products are polarized intersections

Let $u = \max(0, x)$ and $v = \max(0, y)$. The subgradient of $su + tv$ at the origin is the rectangle $[0, s] \times [0, t]$, so its normalized top-degree mass is $2st$. On the other hand,

$$(\mathrm{d}'\mathrm{d}''(su + tv))^2 = s^2(\mathrm{d}'\mathrm{d}''u)^2 + 2st\,\mathrm{d}'\mathrm{d}''u \wedge \mathrm{d}'\mathrm{d}''v + t^2(\mathrm{d}'\mathrm{d}''v)^2.$$

Each self-square vanishes because only one real direction is involved. Comparing coefficients gives $\mathrm{d}'\mathrm{d}''u \wedge \mathrm{d}'\mathrm{d}''v = \delta_0$. Replacing the two slopes by primitive vectors produces the lattice determinant from tropical stable intersection.

This calculation explains why the tropical intersection and continuous convex-potential constructions agree on their common elementary examples. A theorem identifying the constructions globally must still handle charts, multiplicities, and limits. Agreement of one formula is a normalization check, not the full comparison theorem.

20.7.4 Why smoothing is local and why truncation is different

On a convex real domain, convolution with a nonnegative normalized kernel preserves convexity, but it is defined only on a smaller domain unless the function has been extended. This is the origin of the fixed-neighbourhood requirement in local approximation. A sequence whose domains shrink to the point does not give a uniform approximation theorem on an open neighbourhood.

Truncating a singular psh function by $\max(u, -M)$ is a different operation. It produces bounded functions, but their curvature measures can carry mass on the truncation level. As $M \to \infty$, deciding what mass remains and what mass is excluded from a non-pluripolar product is part of the definition and theorem. It cannot be settled by treating the truncations as uniformly convergent on a neighbourhood of the singular set.

20.7.5 The analytic logarithmic maximum

For $u = \log \max_i |f_i|$, the real maximum calculation explains the local convex structure on the complement of the common zero set. The analytic theorem must also establish local integrability, extension across the zero set, and the multiplicity of the resulting cycle. A regular sequence supplies the expected codimension, and the calibrated fibre comparison supplies the intersection multiplicity. The real determinant and subgradient calculations explain the shape of the answer; the analytic geometry proves that it is the correct answer for the given equations.

References. [CLD12, §5.7]; [CLD25, §§1.7, 8.9 and 18.7]; [BGJK25, Section 4].

20.8 A global smooth approximation of a fixed Fubini–Study metric

The local regularized maximum of Chapter 12 gives a global construction when the local expressions come from one finite system of sections. The transition rule for a line-bundle frame is exactly the translation rule required by the regularization. This is a useful case in which local smooth approximation can be upgraded to global approximation without an additional gluing theorem.

Let X be a good analytic space in the original smooth-form category, let L be an analytic line bundle, and let $s_0, \dots, s_N$ be global sections of $L^{\otimes m}$ without a common zero, with $m \geq 1$. Fix real constants a_i. In a frame e, write $s_i = f_i e^{\otimes m}$ and put

$$\phi_e = \frac{1}{m} \max_{0 \leq i \leq N} \bigl(\log |f_i| + a_i\bigr).$$

Use the normalized regularized maximum $M_{\varepsilon,k}$ constructed in Chapter 12, with the same even one-variable kernel in every coordinate. We will use its four properties: convexity and coordinatewise monotonicity; translation covariance by the diagonal vector; the bound $\max \leq M \leq \max + \varepsilon$; and exact deletion of a coordinate lying more than 2ε below another coordinate.

Theorem 20.8.1. *For every $\varepsilon > 0$, the Fubini–Study metric above admits a global original smooth semipositive metric with local weights $\phi_{e,\varepsilon}$ satisfying*

$$0 \leq \phi_{e,\varepsilon} - \phi_e \leq \frac{\varepsilon}{m} \qquad \text{on every trivializing open set.}$$

The difference is independent of the frame. The metrics therefore converge uniformly on the whole space in the metric-ratio distance, even when X is not compact.

Proof. Fix $x \in X$ and a frame near x. Let $I_x = \{i : f_i(x) \neq 0\}$. This set is nonempty. Shrink the neighbourhood so that every f_i, $i \in I_x$, is a unit. Choose $i_0 \in I_x$. Since $|f_j| \to 0$ at x for $j \notin I_x$, a further shrinking gives

$$\log |f_j| + a_j < \log |f_{i_0}| + a_{i_0} - 2\varepsilon \qquad (j \notin I_x),$$

with the usual interpretation $\log 0 = -\infty$. On this neighbourhood define

$$\phi_{e,\varepsilon} = \frac{1}{m} M_{\varepsilon,|I_x|}\bigl((\log |f_i| + a_i)_{i \in I_x}\bigr).$$

This is original smooth: the finite logarithms are coordinates of a moment map by units, and M is a smooth real function. If two such neighbourhoods overlap, enlarge the coordinate list to the union of their surviving indices near a point of overlap. Every extra coordinate is either finite and inactive, or vanishes and remains inactive after shrinking. Exact deletion shows that both formulas give the same value. Thus the local constructions agree, including at zeros of sections.

If $e' = ge$, then each finite input changes by $-m \log |g|$. Diagonal translation covariance gives $\phi_{e',\varepsilon} = \phi_{e,\varepsilon} - \log |g|$. Hence these weights define a metric on L. Each input has zero $d'd''$, and the Hessian of M is positive semidefinite. The convex-composition formula of Chapter 12 proves $d'd''\phi_{e,\varepsilon} \geq 0$. This proves semipositivity. Finally, the maximum bound for M gives the stated estimate on each neighbourhood. Since the transition term is the same for the smoothed and original weights, their difference is global, and the estimate is uniform on all of X. □

This proof uses a common finite section system. It does not imply that every metric with locally smoothly approximable weights has a globally approximating sequence of metrics. General local-to-global approximation must also preserve the line bundle's transition data. Here that preservation is supplied by the diagonal translation identity.

Proposition 20.8.2. *Pullback preserves finite real-weighted Fubini–Study metrics. Tensor product also preserves this class, after passing to a common tensor exponent. The corresponding smooth approximations have positive curvature, and the uniform errors add under tensor product.*

Proof. Pullback takes the section system to its pullback, which still has no common zero: the original evaluation map from a trivial bundle onto $L^{\otimes m}$ is surjective, and pullback preserves this surjection. The local weight formula is simply composed with the analytic map.

For tensor product, first make both exponents equal by taking powers of the individual sections and multiplying their constants accordingly. If the resulting local lists are $A_i = \log |f_i| + a_i$ and $B_j = \log |g_j| + b_j$, the product sections have list $A_i + B_j$. The identity

$$\max_{i,j}(A_i + B_j) = \max_i A_i + \max_j B_j$$

identifies the metric with the tensor product. Curvatures of smooth metric weights add, so semipositivity is preserved. The triangle inequality gives

addition of uniform errors. The tensor product of the chosen smooth approximations need not equal the particular regularized maximum of the product section system; both are legitimate approximations of the same limiting metric. □

20.9 Metric variation, closedness and conservation of total mass

We next isolate the formal calculation used in degree comparisons. Work on a proper boundaryless pure n-dimensional analytic space where smooth integration and Stokes' theorem are available. All metrics in the next proposition are original smooth.

Proposition 20.9.1. *If two smooth metrics on L have curvature forms θ and θ', then $\theta' - \theta = d'd''u$ for a global smooth function u. For every smooth d'- and d''-closed form S of bidegree $(n-1, n-1)$,*

$$\int_X (\theta' - \theta) \wedge S = 0.$$

Consequently the total mixed Chern integral of a fixed tuple of line bundles is independent of the smooth metrics.

Proof. The differences of local weights agree on overlaps, because the two frame-change terms cancel. They define u, and differentiating gives $\theta' - \theta = d'd''u$. Closedness of S implies $d'd''(uS) = d'd''u \wedge S$. Stokes' theorem gives zero after integration. To compare two tuples, replace the metrics one at a time. The wedge of all unchanged Chern forms is closed under both differentials, so the preceding identity applies at every replacement. □

This argument proves metric independence, not the value of the integral. To identify that value with $\deg_K(L_1 \cdots L_n)$ still requires an intersection comparison or a Poincaré–Lelong induction with admissible integrability. These are supplied by the source theorems cited earlier. Once that normalization is known, every smooth semipositive metric tuple has the same algebraic mass.

Lemma 20.9.2 (From smooth tests to continuous tests)**.** *Let X be compact Hausdorff, let $\mathscr{S} \subset C(X,\mathbb{R})$ be uniformly dense, and let μ_j, μ be finite signed Radon measures with*

$$\sup_j \|\mu_j\|_{\mathrm{TV}} + \|\mu\|_{\mathrm{TV}} < \infty.$$

If $\int a\, d\mu_j \to \int a\, d\mu$ for every $a \in \mathscr{S}$, the convergence holds for every $a \in C(X,\mathbb{R})$.

Proof. Choose $b \in \mathscr{S}$ with $\|a-b\|_\infty < \eta$. If the displayed bound is M, then

$$\left|\int a\, d(\mu_j - \mu)\right| \le M\eta + \left|\int b\, d(\mu_j - \mu)\right|.$$

Take the upper limit in j and then let η tend to zero. □

In the algebraic CLD setting of Chapter 8, smooth functions are uniformly dense on compact spaces by the cutoff and partition argument there. Positive measures of a fixed degree have uniformly bounded total variation because total variation equals mass. Thus weak convergence as top-degree currents upgrades to weak convergence of Radon measures. This is often used implicitly when a current formula is stated as a measure formula.

Corollary 20.9.3. *Assume the proper algebraic CLD setting and the mixed-product construction for local uniform smooth psh limits established in Chapter 13. For a tuple of finite Fubini–Study metrics, their smoothed mixed Chern measures converge weakly to the intrinsic mixed measure of the original metrics. The limit is independent of the chosen regularizing kernels. Its mass is $\deg_K(L_1 \cdots L_n)$ whenever the smooth degree comparison is normalized as above.*

Proof. Theorem 20.8.1 gives uniformly convergent smooth semipositive metrics on the fixed line bundles. The local mixed-product theorem identifies their current limit and proves independence of the approximating sequences. The local limits glue because frame changes are smooth unit logarithms with zero Hessian. Positivity and the degree formula bound all masses. Lemma 20.9.2 upgrades convergence to continuous tests. Finally, test against the constant function one to recover the mass. □

There is no assertion of total-variation convergence in this result. Even on $\mathbb{R}$, positive smooth probability densities concentrating at the origin converge

weakly to δ_0 while their difference from δ_0 has total variation two. Likewise, smoothing a model metric can replace an atom by a diffuse measure without making the total-variation distance small. Weak convergence is the appropriate conclusion.

References. [CLD12, Sections 6.3–6.9]; [CLD25, Sections 17.8 and 18.8]; [GK17, Theorem 10.5]. The global approximation and the measure-theoretic passage are proved above; the algebraic degree comparison remains an explicitly cited input.

20.10 Integration by parts for metric differences

A global difference of metric weights is more flexible than a global psh function: its curvature is measured relative to a fixed line bundle. This leads to energy identities which connect the local differential calculus with arithmetic metric variation. We prove the identities in an explicit approximation domain. No general regularization theorem is hidden in the choice of that domain.

Let X be proper, pure n-dimensional and boundaryless, in the algebraic CLD setting where the preceding integration, positive-current and continuous-product theorems hold. Fix an original smooth semipositive metric on a line bundle L, with curvature θ. Write

$$\theta_u = \theta + \mathrm{d}'\mathrm{d}''u, \qquad V = \int_X \theta^n.$$

An admitted function in this section is a continuous global function u which is a uniform limit of global original smooth functions u_j with $\theta + \mathrm{d}'\mathrm{d}''u_j \geq 0$. Thus the metric with weights $\varphi_e + u$ is a uniform limit of smooth semipositive metrics on the same L. Locally its weights belong to $\mathcal{U}_X$. This global requirement is stronger than merely having local smooth approximants, and stronger than positivity of a curvature current.

Lemma 20.10.1. *The admitted domain contains constants and is convex. All mixed products of its curvature currents are well-defined, positive and closed, and*

$$\int_X \theta_{u_1} \wedge \cdots \wedge \theta_{u_n} = V.$$

Proof. Constants do not change curvature. If u_j, v_j are admissible smooth approximants, $(1-t)u_j + tv_j$ has curvature $(1-t)\theta_{u_j} + t\theta_{v_j} \geq 0$ for $0 \leq$

$t \le 1$, and approximates $(1-t)u + tv$ uniformly. Local continuous products give existence and compatibility with uniform limits. Changes of frame are smooth pluriharmonic functions, so these local products glue. For smooth approximants, Proposition 20.9.1 gives total mass V. Weak convergence of the positive measures and testing against one give the same formula in the limit. □

Proposition 20.10.2 (Integration by parts)**.** *Let $u, v, a_1, \dots, a_{n-1}$ be admitted, and put $S = \theta_{a_1} \wedge \cdots \wedge \theta_{a_{n-1}}$. Then*

$$\int_X u\,(\theta_v - \theta) \wedge S = \int_X v\,(\theta_u - \theta) \wedge S. \tag{20.10.1}$$

The equality also holds for differences of admitted functions in the two scalar positions, interpreting the curvature factors multilinearly.

Proof. First assume that all functions are smooth. Closedness of S and Stokes give

$$\int_X u\,\mathrm{d}'\mathrm{d}''v \wedge S = -\int_X d'u \wedge d''v \wedge S = \int_X v\,\mathrm{d}'\mathrm{d}''u \wedge S.$$

The second equality follows by the other integration by parts, or by applying J to the middle integrand and using symmetry of S. There are no boundary terms because X is proper and boundaryless.

Choose simultaneous global smooth approximants for the finitely many functions. Every expression such as $\theta_{v_j} \wedge S_j$ is a positive mixed measure of mass V and converges weakly to the corresponding continuous mixed measure. Uniform convergence of u_j permits multiplication: the error from $u_j - u$ is bounded by $V\|u_j - u\|_\infty$, and the remaining error tends to zero by weak convergence against the continuous function u. Treat the term with θ in the same way. Passing to the limit proves the displayed equality. Subtracting equalities proves its bilinear extension. □

This proof does not define a wedge of distributional first derivatives of arbitrary continuous functions. The smooth middle expression is used to establish an identity whose limiting terms are already defined mixed measures. That distinction is useful whenever a formal Dirichlet expression occurs in a non-Archimedean argument.

20.11 Energy and the variation of Monge–Ampère measures

Definition 20.11.1. For an admitted u, define the unnormalized energy

$$E_\theta(u) = \frac{1}{n+1}\sum_{j=0}^{n}\int_X u\,\theta_u^j \wedge \theta^{n-j}.$$

In particular $E_\theta(0) = 0$. We do not divide by V, so the definition is meaningful even when $V = 0$.

Theorem 20.11.2 (First variation and the difference formula)**.** *For admitted u, v put $h = u - v$ and $u_t = v + th$, $0 \le t \le 1$. Then*

$$\frac{d}{dt}E_\theta(u_t) = \int_X h\,\theta_{u_t}^n, \tag{20.11.1}$$

$$E_\theta(u) - E_\theta(v) = \frac{1}{n+1}\sum_{j=0}^{n}\int_X h\,\theta_u^j \wedge \theta_v^{n-j}. \tag{20.11.2}$$

Proof. All expressions along this segment are polynomials in t, by multilinearity of the continuous mixed products. Differentiation is therefore algebraic. Put $\Theta = \theta_{u_t}$. Differentiating the jth summand of the energy gives

$$\int_X h\,\Theta^j\theta^{n-j} + j\int_X u_t\,\mathrm{d}'\mathrm{d}''h \wedge \Theta^{j-1}\theta^{n-j}.$$

By Proposition 20.10.2, the second integral equals $j\int_X h(\Theta - \theta)\Theta^{j-1}\theta^{n-j}$. Thus the summand becomes

$$(j+1)\int_X h\,\Theta^j\theta^{n-j} - j\int_X h\,\Theta^{j-1}\theta^{n-j+1}.$$

Summing over j cancels consecutive terms and leaves $(n+1)\int_X h\Theta^n$. This proves the derivative formula. Integrate from zero to one and expand $\theta_{u_t} = (1-t)\theta_v + t\theta_u$. The coefficient of each mixed term is

$$\binom{n}{j}\int_0^1 t^j(1-t)^{n-j}\,dt = \frac{1}{n+1}.$$

The integral identity follows by j integrations by parts, or by the factorial formula $j!(n-j)!/(n+1)!$. This proves the difference formula. □

Corollary 20.11.3. *Energy is increasing on the admitted domain and satisfies*

$$|E_\theta(u) - E_\theta(v)| \le V\|u - v\|_\infty, \qquad E_\theta(u + c) = E_\theta(u) + cV.$$

Proof. All $n+1$ mixed measures in (20.11.2) are positive of mass V. If $u \ge v$, each summand is nonnegative. Bounding the absolute value of h gives the Lipschitz estimate. For a constant difference c, every curvature equals θ_u and every mass equals V, giving the translation formula. □

Proposition 20.11.4 (Concavity)**.** *The energy is concave on the admitted convex domain. If all functions along the segment are smooth, its second derivative is*

$$\frac{d^2}{dt^2} E_\theta(u_t) = -n \int_X d'h \wedge d''h \wedge \theta_{u_t}^{n-1} \le 0.$$

Proof. Differentiate (20.11.1) in the smooth case to get $n \int h \mathrm{d}'\mathrm{d}''h \wedge \theta_{u_t}^{n-1}$. Integration by parts gives the displayed expression. The form $d'h \wedge d''h$ is strongly positive of bidegree $(1,1)$, so its product with the semipositive curvature power has nonnegative integral. For continuous admitted endpoints, choose smooth approximants for both endpoints and use their affine segments. The Lipschitz estimate gives convergence of their energies at each of the three arguments in the concavity inequality. Passing to the limit proves concavity without assigning a separate meaning to the nonsmooth first-derivative integral. □

Combining concavity with the first variation gives the useful pair of bounds

$$\int_X (u - v)\theta_u^n \le E_\theta(u) - E_\theta(v) \le \int_X (u - v)\theta_v^n.$$

Indeed, the derivative of a differentiable concave function on $[0,1]$ is nonincreasing, so its integral lies between its two endpoint values. These inequalities concern energy, not pointwise comparison of the two measures.

Remark 20.11.5 (Relation to local heights and variational theories). The sum of mixed integrals in (20.11.2) is the differential-calculus expression that occurs when all factors of an intersection pairing are changed from one metric to another. Identifying it with an arithmetic or Deligne pairing requires that pairing's normalization and comparison theorem. The identities proved here use neither an adelic summation nor a claim that every semipositive model metric has a global original smooth approximation.

Global variational theories often define energy first on model or Fubini–Study metrics and extend it by order or completion. Their domains can be larger and their topology can be different. The present formulas provide a common calculation on the overlap of domains; they do not identify those completions. Likewise, energy concavity alone supplies neither a comparison principle for arbitrary bounded psh functions nor a solution of a Monge–Ampère equation.

References. [CLD12, Sections 6.3–6.9]; [CLD25, Sections 8.7 and 17.8]; [GK17, Sections 10–12]. The proofs combine continuous mixed-product convergence, smooth Stokes and elementary polynomial identities. Chapter 21 discusses the larger variational frameworks separately.

20.12 The graph Laplacian as an intersection matrix

The slope formula, model measure, and negative intersection pairing become the same calculation on a semistable curve. Keeping the length scale visible also resolves several normalization ambiguities in local heights. We first work on a finite connected compact metric graph Γ. Subdivide loops if necessary so that every edge has two distinct endpoints. Denote its edge lengths by $\ell_e > 0$.

For a continuous function which is C^2 on edges and has one-sided derivatives at vertices, set

$$\Delta f = -f''\,dt - \sum_v \left(\sum_{e \ni v} \partial_{e,v} f \right) \delta_v.$$

This is the nonnegative-operator convention: it is the negative of the curvature convention $\mathrm{d}'\mathrm{d}''f$ used for graph potentials in this book. The graph has no designated boundary here; leaf vertices contribute to the displayed sum. For a graph serving as an analytic domain with boundary, one must instead specify which tests vanish at the boundary.

Proposition 20.12.1. *For continuous edgewise C^2 functions f, g, with integrable second derivatives and endpoint derivative limits,*

$$\int_\Gamma f\, d\Delta g = \sum_e \int_e f' g'\, dt = \int_\Gamma g\, d\Delta f.$$

In particular Δf has total mass zero. If $\Delta f = 0$, then f is constant.

Proof. On an oriented edge $[0,\ell]$, integration by parts gives $-\int fg'' = \int f'g' - [fg']_0^\ell$. At the initial endpoint the outgoing derivative is $g'(0)$, and at the final endpoint it is $-g'(\ell)$. Thus the two atomic contributions in Δg cancel precisely the bracket term. Sum over all edges. Setting $f = 1$ proves zero total mass; setting $f = g$ when $\Delta f = 0$ proves $f' = 0$ on each edge. Continuity and connectedness make the edgewise constants equal. □

For a function affine on every edge, its vertex values $h = (h_v)$ determine it. The coefficient vector of Δh is Lh, where

$$(Lh)_v = \sum_{e=vw} \frac{h_v - h_w}{\ell_e}, \qquad h^{\mathsf{T}} L h = \sum_{e=vw} \frac{(h_v - h_w)^2}{\ell_e}.$$

Parallel edges are counted separately. In terms of an oriented incidence matrix B and $C = \operatorname{diag}(\ell_e^{-1})$, one has $L = BCB^{\mathsf{T}}$. Reversing an edge changes one column of B by a sign and leaves L unchanged. The kernel is the space of constant vectors, and the image is the hyperplane of vectors with coordinate sum zero.

Proposition 20.12.2 (The vertical divisor calculation)**.** *Let $\mathcal{X}$ be a regular proper split strictly semistable curve over a complete DVR R, with reduced special fibre $\sum_v C_v$. Assume distinct branches at each node belong to distinct components, after subdivision if needed. Put $\lambda = -\log|\pi|$ for a uniformizer. Every node gives an edge of length λ. For vertical divisors $D = \sum_v a_v C_v$ and $E = \sum_v b_v C_v$, their model functions have values $h_D(v) = \lambda a_v$ and $h_E(v) = \lambda b_v$, and*

$$\mathrm{d}'\mathrm{d}''h_D = \sum_v \deg(\mathcal{O}(D)|_{C_v})\,\delta_v,$$

$$\int_\Gamma h_D\,\mathrm{d}'\mathrm{d}''h_E = \lambda(D\cdot E) = -\sum_{e=vw} \frac{(h_D(v)-h_D(w))(h_E(v)-h_E(w))}{\ell_e}.$$

Proof. The off-diagonal intersection number $C_v \cdot C_w$ is the number of their nodes, since the model is regular and split semistable. The whole special fibre is the principal divisor of π, so its intersection with each component is zero. Hence C_v^2 is minus the number of edges incident to v. It follows that

$$\deg(\mathcal{O}(D)|_{C_v}) = \sum_{e=vw} (a_w - a_v).$$

The model-weight formula of Lemma 19.9.1 gives $h_D(v) = \lambda a_v$. The slope along $e = vw$ away from v is $(h_D(w) - h_D(v))/\lambda = a_w - a_v$. Summing

gives the first formula. Multiplying its version for E by h_D and summing over vertices gives $\lambda \sum_v a_v (E \cdot C_v) = \lambda (D \cdot E)$. Grouping the two endpoint contributions of each edge gives the last expression. □

Corollary 20.12.3. *One has $D^2 \leq 0$, with equality exactly when all coefficients a_v are equal. Thus the vertical intersection pairing is negative definite modulo the whole special fibre.*

This is the graph form of the vertical Hodge index statement. It is proved here under the split regular reduced-fibre hypotheses. With component multiplicities or nonsplit residue fields, the intersection matrix and graph weights must be adjusted; the unweighted formula should not be transplanted unchanged. In a model with a node of thickness m, the corresponding skeleton edge has length $m\lambda$; resolving that node replaces it by a chain of length-λ edges. Linear interpolation on the chain gives the same energy.

20.13 Solving the graph Green equation

Theorem 20.13.1. *For a finite signed atomic measure $\nu = \sum_v d_v \delta_v$ with $\sum_v d_v = 0$, there exists a continuous PL function g_ν such that*

$$\Delta g_\nu = \nu, \qquad \textit{equivalently} \qquad \mathrm{d}'\mathrm{d}'' g_\nu = -\nu.$$

It is unique modulo constants. Given a probability measure μ, the normalization $\int g_\nu \, d\mu = 0$ determines it uniquely.

Proof. Choose a vertex set containing the support of ν. The symmetric matrix L has kernel the constants, so its image is their orthogonal complement. The sum-zero condition gives $d \in \operatorname{im} L$. Solve $Lh = d$ and interpolate affinely on every edge. Its Laplacian is exactly ν. Any two solutions differ by a harmonic function, hence by a constant by Proposition 20.12.1. Subtracting the μ-integral imposes the normalization, since μ has mass one. □

The Green pairing of two degree-zero atomic measures is

$$\langle \nu, \eta \rangle_\Gamma = \int_\Gamma g_\nu \, d\eta.$$

It is independent of the additive constant and of the normalizing probability measure, since η has mass zero. Integration by parts gives symmetry and positivity:

$$\langle \nu, \eta \rangle_\Gamma = \sum_e \int_e g'_\nu g'_\eta \, dt, \qquad \langle \nu, \nu \rangle_\Gamma \geq 0,$$

with equality in the second expression only for $\nu = 0$. This positive Green pairing corresponds to the negative vertical intersection form with its sign reversed.

Example 20.13.2 (Two terminals and effective resistance)**.** For vertices $p \neq q$, put $\nu = \delta_p - \delta_q$ and choose $g_\nu(q) = 0$. The number

$$r(p, q) = g_\nu(p) - g_\nu(q) = \langle \nu, \nu \rangle_\Gamma > 0$$

is the effective resistance when an edge of length ℓ is a resistor of resistance ℓ. On a segment it is the distance. For two parallel paths of lengths a, b, a potential difference V produces total current $V/a + V/b$. Setting that current to one gives $r = ab/(a + b)$. These statements follow directly from $Lh = d$; no probabilistic interpretation is required.

Proposition 20.13.3 (Subdivision does not change the solution)**.** *Inserting a vertex with no charge into an edge leaves the PL solution, Green pairing and effective resistances unchanged.*

Proof. If the two new lengths are a, b, the zero-charge equation at the inserted vertex is $(h - h_0)/a + (h - h_1)/b = 0$. Hence $h = (bh_0 + ah_1)/(a + b)$, exactly linear interpolation. The endpoint currents are $(h_0 - h_1)/(a + b)$ and its negative, as before. The edge energy is

$$\frac{(h - h_0)^2}{a} + \frac{(h_1 - h)^2}{b} = \frac{(h_1 - h_0)^2}{a + b}.$$

Uniqueness modulo constants now proves the assertion for the full graph. □

20.14 Canonical graph measures and residual genus

Let R_e be the effective resistance between the endpoints of e after deleting its interior. If deletion disconnects those endpoints, set $R_e = +\infty$ and $(\ell_e + R_e)^{-1} = 0$. The resistance in the full graph is the parallel combination

$$r_e = \frac{\ell_e R_e}{\ell_e + R_e},$$

interpreted as $r_e = \ell_e$ for a bridge. Write $b = |E| - |V| + 1$.

Lemma 20.14.1 (Foster's identity)**.**

$$\sum_e \frac{r_e}{\ell_e} = |V| - 1, \qquad \sum_e \frac{\ell_e}{\ell_e + R_e} = b.$$

Proof. Let L^+ be the inverse of L on the sum-zero hyperplane, extended by zero on constants. If b_e is the incidence column for e, the defining Green equation gives $r_e = b_e^{\mathsf{T}} L^+ b_e$. Therefore

$$\sum_e \ell_e^{-1} b_e^{\mathsf{T}} L^+ b_e = \operatorname{tr}(L^+ B C B^{\mathsf{T}}) = \operatorname{tr}(L^+ L) = |V| - 1.$$

The last operator is the identity on a space of dimension $|V| - 1$ and zero on constants. Since $r_e/\ell_e + \ell_e/(\ell_e + R_e) = 1$, subtracting the first equality from $|E|$ gives the second. □

The canonical measure of a metric graph is the signed measure

$$\mu_{\mathrm{can}} = \sum_v \left(1 - \frac{\mathrm{val}(v)}{2}\right) \delta_v + \sum_e \frac{dt}{\ell_e + R_e}.$$

This explicit formula and its resistance-kernel characterization are classical; see Baker–Rumely, Section 14. Its mass is one: the vertex part has mass $|V| - |E| = 1 - b$, and Foster's identity makes the edge part have mass b. It is generally not positive. A trivalent vertex has coefficient $-1/2$.

A semistable curve carries more data than the unweighted graph: attach the residual genus $q(v) \geq 0$ to each vertex. Put $g = b + \sum_v q(v)$ and assume $g \geq 1$. The polarized canonical divisor and admissible probability measure are

$$K_\Gamma = \sum_v (2q(v) + \mathrm{val}(v) - 2)[v],$$

$$\mu_{\mathrm{ad}} = \frac{\delta_{K_\Gamma} + 2\mu_{\mathrm{can}}}{2g} = \frac{1}{g}\left(\sum_v q(v)\delta_v + \sum_e \frac{dt}{\ell_e + R_e}\right).$$

The identification of this measure with Zhang's admissible measure is the graph-theoretic input from admissible pairing theory. Its positivity and normalization are immediate from the formula, and can be checked without that identification: all displayed coefficients are nonnegative and the total mass is $(\sum q(v) + b)/g = 1$. Also $\deg K_\Gamma = 2g - 2$.

Example 20.14.2. For good reduction, the graph is a point with $q(v) = g$, so $\mu_{\mathrm{ad}} = \delta_v$. For a totally degenerate genus-one curve, the graph is a circle of length ℓ and all $q(v)$ vanish. The formula gives $\mu_{\mathrm{ad}} = dt/\ell$. For a tree with positive residual genera, every edge is a bridge, and the measure is $g^{-1}\sum_v q(v)\delta_v$. Thus neither total graph length nor the number of components alone determines the measure.

Three measures must now be distinguished: the generally signed μ_{can} of the bare graph, the positive probability measure μ_{ad} of the polarized graph, and the Chern measure of a specified metrized line bundle. An admissibly metrized dualizing line has Chern measure $(2g-2)\mu_{\mathrm{ad}}$, not a probability measure unless that degree is one. A general semipositive model line of degree d instead gives its own positive atomic measure of mass d. Equality with an admissible measure is a theorem about the chosen metric, not a consequence of the word canonical.

References. [Zh93, Appendix, especially Lemma a.6 and Theorem a.7]; [BRg, Sections 1, 4 and 14]. The finite-dimensional Green, energy, subdivision and Foster calculations above are included to make all signs and scale factors explicit. The full analytic admissible metric construction is cited, not reconstructed from a skeleton alone.

20.15 Green kernels for finite measures on a compact graph

The finite matrix calculation extends to measures which need not be atomic. This extension is useful for admissible metrics: the target curvature may have a density along the edges, whereas a model metric has atomic curvature. We give the functional-analytic construction on a finite connected compact metric graph Γ with positive edge lengths. It does not involve probability theory.

Fix $z \in \Gamma$, inserting it as a vertex if necessary. Let H_z be the real vector space of continuous functions u on Γ whose restriction to each edge is in H^1 and for which $u(z) = 0$. Put

$$\mathcal{E}(u,v) = \sum_e \int_e u'v'\,dt, \qquad \|u\|_H^2 = \mathcal{E}(u,u).$$

The derivatives here are weak derivatives on real intervals. Continuity at vertices is imposed separately; the construction does not impose a Kirchhoff condition on an arbitrary member of H_z.

Lemma 20.15.1. *The space H_z is a Hilbert space. With d the path metric and $D = \operatorname{diam}\Gamma$,*

$$|u(x) - u(y)|^2 \le d(x,y)\mathcal{E}(u,u), \qquad \|u\|_\infty \le \sqrt{D}\,\|u\|_H.$$

Proof. Integrate u' along a shortest path from y to x. Cauchy–Schwarz bounds the square of the integral by the path length times its derivative energy, which is at most the total energy. Taking $y = z$ gives the second estimate. A Cauchy sequence in the energy norm is therefore uniformly Cauchy, and its edge derivatives converge in L^2. On each edge, the fundamental theorem for H^1 functions identifies the uniform limit with the function obtained by integrating the limiting derivative and the limiting endpoint value. The limit has matching vertex values and vanishes at z. It belongs to H_z and the convergence is in energy norm. Definiteness follows because zero energy makes the function constant on a connected graph, and its value at z is zero. □

Theorem 20.15.2. *For every finite signed Radon measure ν on Γ with $\nu(\Gamma) = 0$, there is a unique $g_\nu \in H_z$ such that*

$$\mathcal{E}(g_\nu, v) = \int_\Gamma v\,d\nu \quad (v \in H_z).$$

Equivalently, $\Delta g_\nu = \nu$ in the weak graph sense. The same equation holds against all continuous edgewise H^1 tests, without a condition at z. Moreover,

$$\|g_\nu\|_H \le \sqrt{D}\,\|\nu\|_{\mathrm{TV}}, \qquad \|g_\nu\|_\infty \le D\,\|\nu\|_{\mathrm{TV}}.$$

Proof. Lemma 20.15.1 bounds the linear functional $v \mapsto \int v\,d\nu$ by $\sqrt{D}\|\nu\|_{\mathrm{TV}}\|v\|_H$. The Hilbert-space representation theorem gives its unique representing vector and the first bound; the second follows from the sup-norm estimate. Subtracting $v(z)$ from a general test leaves its derivative unchanged and does not change its integral against ν, because the latter has mass zero. This proves the assertion about all tests. For a smooth edgewise function, integration by parts recovers precisely the convention $\Delta = -\mathrm{d}'\mathrm{d}''$ of Section 20.12. □

The necessary mass condition can also be read directly from the equation: the constant test has zero derivative, so $\int 1\,d\nu = 0$. On a single-point graph the condition forces $\nu = 0$ and the normalized solution is zero. The nontrivial case is the graph with edges considered below.

20.15.1 Construction and symmetry of the kernel

For $y \in \Gamma$, apply the theorem to $\delta_y - \delta_z$ and denote the solution by k_y. Define

$$j_z(x, y) = k_y(x).$$

Proposition 20.15.3. *The function j_z is continuous on $\Gamma \times \Gamma$, is symmetric, vanishes when either argument is z, and satisfies*

$$\Delta_x j_z(x, y) = \delta_y - \delta_z, \qquad |j_z(x, y)| \le \sqrt{d(x, z) d(y, z)}.$$

Proof. The representing-vector identity gives $k_y(x) = \mathcal{E}(k_x, k_y) = k_x(y)$. The evaluation functional at x has norm at most $\sqrt{d(x, z)}$, so $\|k_x\|_H \le \sqrt{d(x, z)}$ and Cauchy–Schwarz gives the stated bound. Likewise, the functional $v \mapsto v(x) - v(x')$ has norm at most $\sqrt{d(x, x')}$; hence

$$\|k_x - k_{x'}\|_H \le \sqrt{d(x, x')}.$$

This proves continuity of the representing vectors in H_z and therefore joint continuity of their inner products. The differential equation and normalization are the defining properties of k_y. □

Corollary 20.15.4. *The normalized potential and energy pairing are*

$$g_\nu(x) = \int_\Gamma j_z(x, y)\, d\nu(y),$$
$$\mathcal{E}(g_\nu, g_\omega) = \iint_{\Gamma^2} j_z(x, y)\, d\nu(x)\, d\omega(y)$$

for finite signed measures of mass zero. The second expression is independent of z and positive definite when $\omega = \nu$.

Proof. The continuous bounded map $y \mapsto k_y \in H_z$ can be integrated against a finite signed measure. Pairing that integral with $v \in H_z$ gives $\int v\, d\nu$, so uniqueness identifies it with g_ν. Evaluation is continuous on H_z, giving the first formula. The second follows by pairing and Fubini; the kernel is bounded and both measures have finite total variation. Changing the normalization point adds only a constant to each solution, which leaves energy unchanged. Finally, zero energy implies $g_\nu = 0$, and hence ν vanishes on all continuous edgewise H^1 functions. These include the PL functions on arbitrary subdivisions, which are uniformly dense in $C(\Gamma)$. Thus $\nu = 0$. □

20.15.2 Normalization by a probability measure

Let μ be any positive Radon measure of mass one. The word probability specifies only this normalization; no stochastic construction is used. Set

$$a(x) = \int j_z(x,t)\, d\mu(t), \qquad c = \iint j_z(s,t)\, d\mu(s) d\mu(t),$$
$$g_\mu(x,y) = j_z(x,y) - a(x) - a(y) + c.$$

Theorem 20.15.5. *This is the unique continuous symmetric Green kernel with energy-class sections satisfying*

$$\Delta_x g_\mu(x,y) = \delta_y - \mu, \qquad \int g_\mu(x,y)\, d\mu(x) = 0.$$

It is independent of the auxiliary point z. Its sections are in general neither PL nor smooth.

Proof. Integration of the defining equation for j_z gives $\Delta a = \mu - \delta_z$. Subtract to obtain the equation for g_μ. Symmetry is inherited from j_z, and integration in x cancels all four terms because $\int a\, d\mu = c$. If another normalized kernel has the same equation, the difference of its sections has zero energy and is constant. The normalization makes this constant zero. This also proves independence of z. A measure with a nonzero density on an edge requires a nonzero distributional second derivative there, so its potential need not be PL. □

Proposition 20.15.6 (Weak convergence of measures and uniform convergence of potentials)**.** *Suppose ν_j and ν have mass zero, $\nu_j \rightharpoonup \nu$ against $C(\Gamma)$, and their total variations are bounded uniformly. Then $g_{\nu_j} \to g_\nu$ uniformly, and their self-energies converge.*

Proof. Pointwise convergence follows by testing against $j_z(x,\cdot)$. Uniform continuity of j_z on the compact product and the total-variation bound make the potentials equicontinuous. A finite net in Γ upgrades pointwise convergence to uniform convergence: control the finitely many values on the net, then the oscillation on each neighboring ball. Finally split

$$\int g_{\nu_j}\, d\nu_j - \int g_\nu\, d\nu = \int (g_{\nu_j} - g_\nu)\, d\nu_j + \int g_\nu\, d(\nu_j - \nu).$$

The first term tends to zero by uniform convergence and bounded total variation, and the second by weak convergence. □

The scope of this theorem is a fixed compact metric graph. A classical point of a Berkovich curve can lie at infinite distance along a ray outside such a graph, and its logarithmic Green function need not be continuous. To pass from the graph construction to an admissible metric on the analytic curve, one uses the curve-theoretic comparison between the Laplacian, the retraction, and the Chern measure. That comparison, and the treatment of divisor singularities, are additional analytic theorems. The Hilbert-space proof does not establish a higher-dimensional non-Archimedean Green theorem.

References. [BRg, Zh93]. The argument above combines the elementary graph energy estimate with Hilbert-space representation and bounded-kernel integration; it supplies the measure extension of the preceding finite-matrix construction.

Chapter 21

Canonical and toric metrics

21.1 Abelian varieties: residual geometry plus a real quadratic form

21.1.1 The uniformization picture

After the appropriate hypotheses and extensions in the uniformization theorem, an abelian analytic space is described using an extension

$$0 \longrightarrow T \longrightarrow E \longrightarrow B \longrightarrow 0$$

and a discrete lattice M with $A^{\mathrm{an}} \simeq E^{\mathrm{an}}/M$. The torus T supplies real tropical directions; B has good reduction and supplies residual algebraic geometry. The skeleton of A is a real torus of dimension equal to the toric rank, which can be smaller than $\dim A$.

This is a useful test for every theory in the series. In good reduction, the skeleton can be a point but an ample line bundle still has positive top-degree mass. In total degeneration, the real torus carries the relevant continuous distribution of mass. A general abelian variety combines both effects.

21.1.2 The canonical metric is not simply a quadratic function

For a line bundle with a cubist canonical metric, the pullback to E decomposes into a formal metric coming from B and a real quadratic contribution in the toric directions. Schematically, with the notation of CLD25 Chapter 19,

$$c_1(p^*L) = c_1(q^*M_B) + \tau^*\mathrm{d}'\mathrm{d}''\left(\tfrac{1}{2}q_L\right). \tag{21.1.1}$$

The affine part of the weight does not contribute to curvature. The quadratic form q_L encodes the polarization in the tropical directions; the residual line bundle M_B supplies the good-reduction contribution.

The map τ is naturally harmonic in the enlarged framework. It need not be original CLD-smooth. This is precisely the issue behind the correction of the canonical delta-metric examples in GK, acknowledged by GJR. A formula using harmonic coordinates should not be relabelled as an original smooth formula without proving that coordinate comparison.

21.1.3 Positivity and canonical measure

For an ample line bundle, the canonical metric is semipositive in the established geometric theories. In the CLD25 description, the positivity separates into the real quadratic and residual numerical conditions. The stated general comparison in Proposition 19.3.1 depends on the residual curvature comparisons from CLD25 Chapter 18; that dependency should remain visible when auditing the argument.

The canonical top-degree measure on A is the Haar measure on its canonical skeleton, normalized to have total mass $\deg_L(A)$. In the good-reduction case this is an atom. In the totally degenerate case it is a translation-invariant density on a real torus. The intermediate case has a lower-dimensional real support with a coefficient carrying the residual intersection degree.

Example 21.1.1 (A Tate elliptic curve)**.** For a Tate parameter q with $0 < |q| < 1$, the skeleton is a circle of length $\ell = -\log|q|$. If L has degree $d > 0$, its canonical measure is $d\,dt/\ell$ on that circle. The normalization is forced by the total degree: integrating the density gives d. Rescaling the absolute value changes both dt and ℓ, so their ratio has the required invariant total mass.

21.1.4 Relation to Gubler's earlier work

Canonical measures on abelian varieties and their tropical distributions were a major part of Gubler's work before and alongside the differential-form theories. CLD25 places these phenomena inside its current and harmonic-coordinate calculus. This is a synthesis and an additional analytic interpretation, not evidence that the underlying canonical measures first appeared in 2025.

The distinction between the ambient abelian variety and a subvariety is also important. The ambient canonical measure is translation invariant on its skeleton. A subvariety has its own tropical support, multiplicities, and possible degeneration dimensions; one cannot replace its measure by ambient Haar measure restricted set-theoretically. This leads to the tropical canonical-measure and equidistribution programme discussed next.

References. [CLD25, §18.7, Chapter 19, Appendix B]; [Mih24, Theorem 5.5]; [GJR, Remark 16.4 and Section 17]; [Gub10].

21.2 The toric pluripotential theory of Burgos Gil–Gubler–Jell–Künnemann

21.2.1 Why toric symmetry is analytically powerful

On the dense torus, an invariant metric can be encoded by a real function on $N_{\mathbb{R}}$. With the weight convention used here, positivity is expressed through convexity after choosing the corresponding trivialization; other toric references use a concave metric function with the opposite sign. The fan controls the behaviour at infinity, so the correct domain is the extended tropical toric space N_{Σ}, not just $N_{\mathbb{R}}$.

The boundary strata are essential. A function may be convex on the open real vector space but fail the extension conditions required by a metric on the complete toric variety. Likewise, mass can concentrate near or on strata at infinity. The extended topology and the sheaf of forms must be part of the definition.

21.2.2 The complex comparison and the positivity cones

The comparison paper relates tropical superforms and currents with suitable invariant complex toric objects. This makes complex analytic estimates available in the tropical setting, with carefully matched positivity and symmetry conventions. It also exposes an important difference from a naive analogy: real strong positivity in middle degree does not span all symmetric forms. The Plücker example in Chapter 7 is a concrete manifestation.

One should therefore transfer a theorem through the actual comparison functor and its hypotheses. The statement that tropical (p, q)-forms resemble complex forms does not identify every positive cone or every distributional

property. This is why the ordinary-positive coefficient estimate is the safe input for the continuous multiplier argument.

21.2.3 A genuine background-current BT theorem

Theorem 4.1.2 of the toric pluripotential paper constructs products

$$\mathrm{d}'\mathrm{d}''u_1 \wedge \cdots \wedge \mathrm{d}'\mathrm{d}''u_q \wedge T$$

for the stated locally bounded tropical psh functions and a positive closed tropical current T. It includes positivity, locality, symmetry, and the relevant continuity properties. This is a substantial arbitrary-background theorem in its tropical toric category.

For these tropical psh functions, local boundedness has particularly strong regularity consequences: finite psh functions in the local real convex picture are continuous, and the extended-boundary condition is part of the theory. Thus this theorem should not be described as merely the unresolved discontinuous bounded case of an arbitrary Berkovich psh class. Its domain is a specific, well-developed convex and toric setting.

The paper also treats unbounded products under additional hypotheses on singular loci. The conditions preventing excess intersection of singular sets are part of those theorems. They cannot be dropped to define a wedge of arbitrary singular currents.

21.2.4 What comparison with Berkovich geometry yields

Tropicalization relates the toric psh functions and measures to invariant non-Archimedean metrics. In the continuous classes where the comparison theorem applies, the mixed products agree after the appropriate pullback or pushforward interpretation. This gives explicit solutions and formulas in toric and related abelian settings.

The remaining step for a general analytic space is not simply to choose a toric chart once. A function or current may require refinements, different local presentations, and compatibility with all tests. A tropical background current representing one chart must descend through the relations between charts. The representation-independence problem in Chapter 24 is exactly where the special toric theorem meets the general analytic programme.

References. [BGJK]; [BGJK25, Theorems 4.1.2, 4.2.2 and Sections 5–7].

21.3 Canonical measures, equidistribution, and local heights

21.3.1 Gubler's canonical-measure programme

For a subvariety Z of an abelian variety, the canonical mixed measure is not generally a restriction of ambient Haar measure. Gubler describes its tropical pushforward using weighted real polyhedral pieces and the residual geometry of the Raynaud extension. If $d = \dim Z$ and the good-reduction abelian part has dimension b, the dimensions appearing in the general description can range from $d - b$ to d in the source theorem, subject to nonnegative dimensions.

In total degeneration, $b = 0$, so the full d-dimensional tropical picture is available. In mixed reduction, some mass can occur in smaller real dimensions. This is why arguments based on dimension in a real torus need additional care outside the totally degenerate case.

The paper explicitly corrects an earlier claim that the tropical image is always pure-dimensional in this general abelian setting. The corrected theorem retains its polyhedral description and measure formula. This author-acknowledged correction is a useful example of how to distinguish a lost auxiliary claim from the surviving main result.

21.3.2 Why tropical equidistribution can prove arithmetic statements

A small generic family of algebraic points has Galois-orbit measures converging to the normalized canonical measure under the relevant equidistribution hypotheses. Tropicalization pushes this convergence to a real torus, where the limiting measure has an explicit polyhedral density. The arithmetic assumption of small height thereby becomes a geometric assertion about the distribution of mass.

In Gubler's totally degenerate Bogomolov argument, difference maps on powers of a subvariety interact with this distribution. After the appropriate reduction by stabilizers, a generically finite difference map must transport canonical mass compatibly with degree. On suitable tropical pieces, the map's rank behaviour conflicts with the distribution forced by a hypothetical dense family of small points. The contradiction excludes that family.

This is a proof mechanism, not a substitute for the full argument with its stabilizer and genericity hypotheses. Its relevance here is that tropical multiplicities and canonical measures have arithmetic force before one introduces a full differential-form algebra. CLD and GK later provide local curvature and intersection languages compatible with these measures.

21.3.3 Toric heights and convex duality

For a semipositive toric metric, the real metric function has a Legendre–Fenchel dual on the moment polytope, often called its roof function. Local toric height formulas express intersection heights as integrals of these concave roof functions, with lattice-normalized volume and the standard factorial conventions. Gubler–Hertel extends the local-height and induction formulas to arbitrary rank-one non-Archimedean absolute values in the stated toric setting.

The elementary reason convex duality enters is that the slopes of the metric function encode characters, while its values record the norm. The moment polytope organizes those slopes. Mixed intersection numbers then correspond to mixed convex-geometric data. The numerical output is a height, whereas GK's local calculus also retains intermediate Chern and Green objects.

21.3.4 The constants that curvature forgets

If one replaces a metric by e^{-c} times that metric, the local weight increases by c and the curvature does not change. The local height against a fixed positive mixed measure changes by

$$c \deg(L_1 \cdots L_n).$$

This is already visible in the projective-line example above. A theory of actual norms or adelic line bundles therefore carries more information than the collection of curvature currents. A comparison between currents cannot by itself identify a Deligne pairing with its norm.

21.4 A reading route through the Gubler programme

For the local differential problem, read the algebraic-chart presentation first, then GK's definitions of preforms, relations and closed generators. Work through the corner formula before reading the local-height construction. The positivity paper should be read with the ordinary/strong/weak cone distinction already in place.

For harmonic coordinates, start with model functions and residual numerical triviality, then read the definition of a harmonic tropicalization and the sheaf it generates. Only after that should one compare strong currents with original currents. The curve cohomology paper provides concrete calculations and a warning about confusing topological and algebraic genus.

For positivity, read the Gubler–Rabinoff PL lifting and curve criterion together with its quantifiers under analytic field extension. Compare the resulting theorem on the PL class with CLD25's generated classes and residual criterion. A theorem on arbitrary continuous functions needs an additional approximation or curve-detection input.

For global applications, retain the model definition of semipositivity and the proper ample setting of envelopes and variational methods. In the toric setting, current products, convexity, boundary strata, and global equations admit an explicit comparison. Chapter 24 explains how to extract further questions from these results without silently changing the class of objects.

References. [Gub10, Theorem 1.1 and its erratum]; [Gub07, GH]; [GK17, GK19, GJR, GRPL, GRPSH].

21.5 Convex weights and the normalization of Monge–Ampère

Toric formulas become easier to compare once the two sign conventions are fixed. In this section $u \in N_{\mathbb{R}}$ is the tropical coordinate $-\log|\cdot|$. We use a convex local weight f. The concave metric function in Gubler–Hertel is $\psi = -f$. If its moment polytope is $\Delta \subset M_{\mathbb{R}}$, the slopes of f lie in $P = -\Delta$. Lebesgue measure on $M_{\mathbb{R}}$ is normalized so that M has covolume one.

Proposition 21.5.1. *For a smooth function f in lattice coordinates on $N_{\mathbb{R}} \simeq \mathbb{R}^n$,*

$$(\mathrm{d}'\mathrm{d}''f)^n = n!\det(D^2 f)\, d'u_1 \wedge d''u_1 \wedge \cdots \wedge d'u_n \wedge d''u_n.$$

Proof. Expand $\mathrm{d}'\mathrm{d}''f = \sum_{i,j} f_{ij} d'u_i \wedge d''u_j$. A nonzero term in the nth power uses each primed and each double-primed index once. Choosing the order of the primed indices contributes $n!$ equal terms. The remaining permutation of the double-primed indices contributes its sign, so their sum is the determinant. The interleaved order displayed above is the positive-volume convention of Chapter 7. □

Thus the differential-form Monge–Ampère measure is $n!$ times the real Alexandrov measure. Omitting this factorial would already give the wrong degree for the standard simplex. The following calculation proves the atomic formula for finite polyhedral weights and fixes this normalization without using an analytic measure comparison.

21.6 Polyhedral Monge–Ampère by a gradient calculation

Let

$$f(u) = \max_{1 \le i \le s} \{\langle a_i, u\rangle + b_i\}, \qquad P = \mathrm{conv}(a_1, \dots, a_s),$$

where $a_i \in M_{\mathbb{R}}$ and P has dimension n. No rationality assumption is needed for the real calculation. At $x \in N_{\mathbb{R}}$ define the subdifferential

$$\partial f(x) = \{p : \ f(u) \ge f(x) + \langle p, u - x\rangle \text{ for every } u\}.$$

Lemma 21.6.1. *The subdifferential is the convex hull of the slopes active at x:*

$$\partial f(x) = \mathrm{conv}\{a_i : f(x) = \langle a_i, x\rangle + b_i\}.$$

Proof. Each active affine function supports f at x, so every convex combination of its slopes belongs to $\partial f(x)$. If p lies outside that convex hull, strict separation gives a vector v such that $\langle p, v\rangle$ exceeds every active $\langle a_i, v\rangle$. For sufficiently small $t > 0$, inactive affine functions remain below the maximum at $x + tv$. Hence

$$\frac{f(x+tv) - f(x)}{t} = \max_{i \text{ active}} \langle a_i, v\rangle < \langle p, v\rangle,$$

contradicting the subgradient inequality. □

Use the smooth convex functions

$$f_\epsilon(u) = \epsilon \log \left(\sum_{i=1}^{s} e^{(\langle a_i, u\rangle + b_i)/\epsilon} \right), \qquad 0 \le f_\epsilon - f \le \epsilon \log s.$$

The estimate is global on $N_{\mathbb{R}}$. It follows by factoring out the largest exponential and bounding the remaining sum between one and s.

Lemma 21.6.2 (The gradient parametrizes the slope polytope)**.** *For every $\epsilon > 0$, $\nabla f_\epsilon : N_{\mathbb{R}} \to \operatorname{int} P$ is a diffeomorphism.*

Proof. Let w_i be the corresponding positive exponential weights, normalized to sum to one. Differentiation gives

$$\nabla f_\epsilon = \sum_i w_i a_i, \quad D^2 f_\epsilon = \epsilon^{-1} \left(\sum_i w_i a_i a_i^t - \overline{a}\,\overline{a}^t \right), \qquad \overline{a} = \sum_i w_i a_i.$$

For a nonzero vector v, its quadratic value is ϵ^{-1} times the weighted variance of the numbers $\langle a_i, v\rangle$. This variance is positive because the a_i affinely span $M_{\mathbb{R}}$. The Hessian is therefore positive definite, and the gradient is locally invertible and globally injective by strict convexity. Its value lies in $\operatorname{int} P$ since all weights are positive.

Given $p \in \operatorname{int} P$, some Euclidean ball of radius $c > 0$ about p lies in P. The support function $h_P(u) = \max_i \langle a_i, u\rangle$ satisfies $h_P(u) - \langle p, u\rangle \ge c|u|$. Also $f_\epsilon \ge f \ge h_P + \min_i b_i$. Thus $f_\epsilon(u) - \langle p, u\rangle$ is coercive and has a minimizer. At that point its derivative vanishes, giving $\nabla f_\epsilon = p$. This proves surjectivity, and the inverse function theorem supplies a smooth inverse. □

Theorem 21.6.3 (The atomic formula)**.** *Let $\mathcal{V}$ be the finite set of vertices of the domains-of-linearity complex of f. Then, weakly as measures on $N_{\mathbb{R}}$,*

$$(\mathrm{d}'\mathrm{d}'' f_\epsilon)^n \longrightarrow n! \sum_{x \in \mathcal{V}} \operatorname{vol}_M(\partial f(x))\, \delta_x.$$

The total mass is $n!\operatorname{vol}_M(P)$.

Proof. For a continuous compactly supported test η, the change of variables $p = \nabla f_\epsilon(u)$ gives

$$\int \eta(u)(\mathrm{d}'\mathrm{d}'' f_\epsilon)^n = n! \int_{\operatorname{int} P} \eta(x_\epsilon(p))\, dp, \qquad x_\epsilon = (\nabla f_\epsilon)^{-1}.$$

For almost every $p \in P$, the function $f(u) - \langle p, u \rangle$ has a unique minimizer $x(p)$, which is a vertex. To see the exceptional set directly, use the finite polyhedral decomposition of f. If a positive-dimensional face consists of minimizers, then p belongs to its subdifferential, which lies in an affine space of dimension at most $n-1$: every tangent vector to that face has a prescribed pairing with p. There are finitely many such faces. Outside their subdifferentials and the boundary of P, the coercive function has a unique vertex minimizer.

For each such p, the coercive bound in Lemma 21.6.2 bounds all $x_\epsilon(p)$ for $0 < \epsilon \leq 1$. Uniform convergence $f_\epsilon \to f$ makes every cluster point a minimizer of $f - \langle p, \cdot \rangle$. Uniqueness gives $x_\epsilon(p) \to x(p)$. Dominated convergence applies because P has finite volume and η is bounded. The set of p with minimizer x is $\partial f(x)$ up to its lower-dimensional boundary. This gives the atomic formula. Taking the constant integrand one in the change-of-variables formula, and using that these subgradient polytopes partition P up to null sets, gives the total mass. □

The continuous convex-product theorem identifies this weak limit with $(\mathrm{d}'\mathrm{d}''f)^n$. Thus the atom is controlled by a dual polytope, rather than by the number of affine pieces meeting at the vertex. The same number of pieces can produce different masses when their slopes have different lattice geometry.

21.6.1 Mixed products

For finite convex functions, the subdifferential of a sum is the Minkowski sum of the subdifferentials. In the polyhedral case this follows from directional derivatives: the directional derivative at x is the support function of the active-slope polytope. Directional derivatives add, and support functions determine compact convex sets. Hence

$$\partial \left(\sum_i t_i f_i \right)(x) = \sum_i t_i \partial f_i(x) \qquad (t_i > 0).$$

On a common subdivision, apply Theorem 21.6.3 to $\sum_i t_i f_i$ and compare the coefficient of $t_1 \cdots t_n$. With mixed volume normalized by $V(P, \ldots, P) = \mathrm{vol}_M(P)$, the result is

$$\mathrm{d}'\mathrm{d}''f_1 \wedge \cdots \wedge \mathrm{d}'\mathrm{d}''f_n = n! \sum_x V(\partial f_1(x), \ldots, \partial f_n(x)) \delta_x.$$

Here the polynomiality of volume under Minkowski sums is the usual mixed-volume theorem. Polynomiality on the current side follows from multilinearity of the continuous products. The factorial is checked by setting all $f_i = f$. If the sum of the slope spans is lower dimensional, both sides vanish; otherwise the preceding full-dimensional calculation applies to their sum.

Example 21.6.4 (Simplex and box)**.** For $f(u) = \max(0, u_1, \dots, u_n)$, the only full-dimensional subdifferential is the standard simplex at the origin, of lattice volume $1/n!$. Hence $(\mathrm{d}'\mathrm{d}''f)^n = \delta_0$. For $g(u) = \sum_{i=1}^n a_i \max(0, u_i)$ with $a_i > 0$, the subdifferential at the origin is the box $\prod_i [0, a_i]$, so

$$(\mathrm{d}'\mathrm{d}''g)^n = n! \left(\prod_i a_i \right) \delta_0.$$

The first formula corresponds to the degree-one polarization on projective space; the second has the multinomial factor of a product polarization on $(\mathbb{P}^1)^n$. These comparisons use the toric measure theorem below.

21.7 From the real measure to the analytic toric measure

Let K be algebraically closed, complete and nontrivially valued, let Σ be a complete rational fan, and let L be a toric line bundle on the normal toric variety X_Σ. Write Ψ for the concave support function associated to a toric rational section. Gubler–Hertel's classification identifies continuous semipositive toric metrics with concave functions ψ such that $\psi - \Psi$ is bounded. The classification includes the extension to the toric boundary; it is not an assertion only about the dense torus.

Their measure comparison says, in our convex sign convention,

$$\mathrm{trop}_* \, c_1(L, \|\cdot\|)^n = (\mathrm{d}'\mathrm{d}''f)^n, \qquad c_1(L, \|\cdot\|)^n = \rho_*(\mathrm{d}'\mathrm{d}''f)^n, \qquad f = -\psi,$$

where ρ is the toric Gauss section, and the real measure is extended by zero on the boundary of the extended tropicalization. This is an imported analytic comparison theorem. The preceding gradient proof computes its right side in the polyhedral case; it does not by itself identify the analytic measure on tests that do not factor through tropicalization. The second equality is the part that locates the analytic measure on the Gauss section.

Remark 21.7.1 (Smooth on the real vector space is not smooth at the analytic boundary). The functions f_ϵ used above are smooth on $N_{\mathbb{R}}$. They need not give original CLD-smooth weights at boundary points of X_Σ^{an}. On the affine line, the expression

$$\epsilon \log(1 + |T|^{1/\epsilon})$$

extends continuously to $T = 0$, but is not locally constant there. Original smooth functions are locally constant at a rigid point in this setting, as explained in Chapter 8. Thus this convenient real regularization is not the global original-smooth approximation constructed in Chapter 20. The latter uses exact deletion in a regularized maximum to handle vanishing sections.

21.8 Roof functions and the information lost by curvature

The convex conjugate of f is $f^*(p) = \sup_u(\langle p, u\rangle - f(u))$. In the concave toric convention the roof is

$$\vartheta(m) = \inf_u(\langle m, u\rangle - \psi(u)) = -f^*(-m), \qquad m \in \Delta.$$

It is concave because it is an infimum of affine functions of m. For a PL weight, its graph is the upper concave envelope of the coefficient data in this sign convention. The roof contains values as well as slopes.

Proposition 21.8.1. *Replacing the metric by $e^{-c}\|\cdot\|$ replaces f by $f + c$, ψ by $\psi - c$, and ϑ by $\vartheta + c$. Its curvature measure is unchanged.*

Proof. The metric weight is minus the logarithm of the norm, so it increases by c. Taking the infimum in the roof formula adds the same constant. The differential $\mathrm{d}'\mathrm{d}''$ kills constants. □

Gubler–Hertel's diagonal toric local-height formula is

$$\lambda_{\overline{L}}^{\mathrm{tor}}(X_\Sigma) = (n+1)! \int_\Delta \vartheta(m)\, d\,\mathrm{vol}_M(m).$$

If all $n + 1$ metric factors are scaled by e^{-c}, the change is $(n + 1)c \deg_L(X)$, since $\deg_L(X) = n!\,\mathrm{vol}_M(\Delta)$. If only one factor of a mixed local height is scaled, the change is c times the degree of the remaining n line bundles. These are consistent formulas for different variations; they must not be confused.

Example 21.8.2. On $\mathbb{P}^1$ take the usual toric section of $\mathscr{O}(1)$ with $\Delta = [0,1]$, $\Psi(u) = \min(0,u)$, and $f = -\Psi = \max(0,-u)$. Its roof is zero and its curvature is the mass-one atom at the Gauss point. After the constant metric change above, the roof is the constant c, the curvature remains the same atom, and the diagonal toric local height changes by $2c$. This single example checks the sign of the roof, the factorial in dimension one, and the number of metric factors in a height.

References. The toric classification, measure comparison and height formula are [GH, Theorems 2.5.8, 2.5.10 and 2.6.6]. The real gradient and atomic calculations are proved here; the broader tropical current framework is [BGJK25, Sections 3–4].

21.9 Tate's limiting construction for metrics

The canonical metric is obtained by a contraction, before one chooses a differential-form theory. This separates its existence from the later question whether its curvature has an original smooth, weakly smooth, or delta-form representative.

Let X be a proper K-variety, let $F : X \to X$ be a morphism, and fix an isomorphism $\Phi : F^*L \simeq L^{\otimes q}$ with $q > 1$. The isomorphism is part of the data. For continuous metrics m_1, m_2 on L, define

$$d(m_1,m_2) = \sup_{x \in X^{\mathrm{an}}} \left| \log \frac{\|s(x)\|_{m_1}}{\|s(x)\|_{m_2}} \right|.$$

The ratio is independent of a nonzero local frame s. Properness makes the supremum finite. Identifying every metric with its weight difference from one fixed metric identifies this metric space with the complete Banach space $C(X^{\mathrm{an}},\mathbb{R})$.

Theorem 21.9.1. *There is a unique continuous metric m_{can} for which Φ is an isometry. Starting from any continuous metric m_0, form m_{j+1} by taking the qth root of the metric on $L^{\otimes q}$ transported from F^*m_j through Φ. Then m_j converges uniformly to m_{can}, and*

$$d(m_j,m_{\mathrm{can}}) \le \frac{q^{-j}}{1-q^{-1}} d(m_1,m_0).$$

Proof. Let $\mathcal{T}$ be the described operator. A weight difference u becomes $q^{-1}u \circ F$, so $d(\mathcal{T}m,\mathcal{T}m') \le q^{-1}d(m,m')$. Consequently

$$d(m_{j+1},m_j) \le q^{-j} d(m_1,m_0).$$

Summing the geometric series proves that the sequence is Cauchy and gives the error bound. Completeness gives a limit, and continuity of $\mathcal{T}$ makes it fixed. If two metrics were fixed, their distance would be at most q^{-1} times itself, hence zero. The fixed-point equation is exactly the assertion that Φ is an isometry. □

Corollary 21.9.2. *Suppose a class of continuous semipositive metrics is closed under pullback, positive tensor roots, and uniform limits. If it contains a starting metric on L, then m_{can} belongs to that class.*

Proof. Inductively every iterate belongs to the class. Theorem 21.9.1 supplies the required uniform convergence. □

The model definition of semipositivity has these stability properties in its proper setting. For ample L, a suitable positive tensor power has a globally generated model, providing a semipositive starting metric. This constructs canonical semipositivity by model approximation. It does not establish original smoothness of the limit: a uniform limit of original smooth functions need not be original smooth. Even a limit of smooth semipositive metrics would only establish the corresponding approximation property.

Proposition 21.9.3 (Dependence on the polarization isomorphism)**.** *Replace Φ by $a\Phi$, where $a \in K^{\times}$. With the transport convention above, the new canonical metric is $|a|^{-1/(q-1)} m_{\mathrm{can}}$. Its curvature is unchanged.*

Proof. A metric scaled by $c > 0$ on L scales by c on F^*L and by c^q on $L^{\otimes q}$. Since Φ was an isometry, the new isometry condition for $a\Phi$ is $|a|c^q = c$. Solving gives $c = |a|^{-1/(q-1)}$. A constant change of weight has zero curvature. □

For definiteness in the rest of this book, Φ denotes the map $F^*L \to L^{\otimes q}$, so the factor is $|a|^{-1/(q-1)}$. A rigidification removes this scalar ambiguity for the canonical constructions on abelian varieties. The invariant content is that changing the polarization isomorphism changes a constant of the metric; curvature alone cannot recover that constant.

21.10 Canonical measures and their normalization

Now assume X has dimension n, L is ample, and F is finite surjective. Let $\mu = c_1(L, m_{\mathrm{can}})^n$ be the measure constructed by the continuous semipositive model-metric theory. This construction, its continuity, and its projection formula are inputs independent of original smooth representability.

Proposition 21.10.1. *Writing $V = \deg_L(X) > 0$, one has $\deg F = q^n$, $\mu(X^{\mathrm{an}}) = V$, and $F_*\mu = \mu$.*

Proof. The algebraic projection formula gives $(\deg F)V = \deg((F^*L)^n) = q^nV$, hence the degree formula. The total-mass theorem gives $\mu(X^{\mathrm{an}}) = V$. The fixed metric identity gives $c_1(F^*\overline{L})^n = q^n\mu$ on the source. Pushing forward and using the measure projection formula gives

$$q^n F_*\mu = F_* c_1(F^*\overline{L})^n = (\deg F)\mu = q^n\mu.$$

No pullback of an arbitrary current is used: the pulled-back objects here are actual metrized line bundles and their defined Chern measures. □

Invariance alone does not characterize this measure. For $t \mapsto mt$ on $\mathbb{R}/\mathbb{Z}$, both Haar probability measure and the atom at zero are invariant. Hence an argument claiming Haar measure solely from invariance under multiplication is incomplete. Gubler's theorem supplies the additional geometry: for an abelian variety, the canonical measure is supported on its canonical skeleton and corresponds there to Haar measure, with mass equal to the intersection degree. The skeletal support and translation invariance are substantial parts of that theorem.

Lemma 21.10.2. *A finite Borel measure on a real torus invariant under a dense subgroup of translations is invariant under every translation. If its mass is V, it is V times Haar probability measure.*

Proof. For a continuous function f, translations by $a_j \to a$ give uniform convergence $f(\cdot + a_j) \to f(\cdot + a)$ on the compact torus. Taking integrals transfers invariance to a. To prove uniqueness, let m be Haar probability measure. Translation invariance and Fubini give

$$\int f\, d\mu = \int\!\!\int f(x + a)\, d\mu(x)\, dm(a) = V \int f\, dm.$$

□

21.10.1 The circle formula

On a circle of length ℓ, the normalized Green function for Haar measure is, for $0 \le t \le \ell$,

$$g(t) = \frac{t^2}{2\ell} - \frac{t}{2} + \frac{\ell}{12}.$$

The endpoint values agree. On the open edge $g'' = 1/\ell$; at the identified endpoint the derivative jumps from $1/2$ to $-1/2$, contributing $-\delta_0$ to the curvature. Therefore

$$\mathrm{d}'\mathrm{d}''g = \frac{dt}{\ell} - \delta_0, \qquad \int_0^\ell g(t)\frac{dt}{\ell} = 0.$$

The integral follows from $1/6 - 1/4 + 1/12 = 0$. This is a graph Green equation for a point of the circle. On a Tate elliptic curve, the canonical Chern measure of a degree-d line is $d\,dt/\ell$ on the skeleton. The formula for g does not by itself provide the logarithmic singularity at a chosen classical point of the analytic curve; constructing that analytic Green function additionally uses the attached analytic branches.

21.11 The same limiting argument for heights

Over a global field, choose a Weil height h_L satisfying $|h_L(FP) - q h_L(P)| \le C$. This bounded-error functoriality is a standard input of height theory. Then

$$\widehat{h}_L(P) = \lim_{j\to\infty} q^{-j} h_L(F^j P)$$

exists, satisfies $\widehat{h}_L(FP) = q\widehat{h}_L(P)$, and differs from h_L by at most $C/(q-1)$.

Proof. Consecutive terms differ by at most Cq^{-j-1}; sum the geometric series. Shifting the index gives the functional equation. If two functions satisfy this equation and differ by a bounded function, iterate and divide by q^j to show they agree. Thus the limit is independent of the initial height within its bounded equivalence class. If h_L is bounded below, the limit is nonnegative. □

For a symmetric ample line on an abelian variety, take $F = [m]$ and $q = m^2$. The theorem of the square gives the parallelogram identity for a Weil height

up to a bounded error. Applying it to $[m]^j P, [m]^j Q$, dividing by m^{2j}, and taking limits removes that error:

$$\widehat{h}(P+Q) + \widehat{h}(P-Q) = 2\widehat{h}(P) + 2\widehat{h}(Q).$$

This produces the quadratic Néron–Tate height. Over a number field, its zero points are torsion: if $\widehat{h}(P) = 0$, the orbit $[m]^j P$ has bounded Weil height and bounded degree over the ground field. Northcott finiteness makes two orbit points equal, so $(m^i - m^j)P = 0$. Conversely torsion gives zero by the scaling identity. Over a function field, this use of Northcott is unavailable in this form, and constant-field trace phenomena must be retained; one cannot carry over the zero-height characterization unchanged.

These metric and height constructions precede CLD. Gubler's canonical-measure programme links them to tropical distributions; CLD supplies currents and local curvature identities; GK adds products and local intersections; GJR enlarges the coordinates needed for canonical tropicalizations. The constructions address successive mathematical questions, and their chronology reflects a genuine dependence of ideas rather than four alternative names for one theory.

References. [Gub10, Section 3, Example 7.2 and Corollary 7.3]; [Gub07]; [GJR, Section 16 and Remark 16.4].

21.12 Canonical measures and two kinds of dimension

The real dimension of a canonical tropicalization of an abelian variety is its toric rank. The algebraic dimension is the toric rank plus the dimension of its good-reduction abelian part. Gubler's canonical-measure formula keeps track of both. Forgetting the second part would give an incorrect measure formula even for an abelian variety with good reduction.

In the setting of [Gub10], start with a discretely valued field and pass to the completed algebraic closure with the extended absolute value. Let A be an abelian variety of dimension g, with Raynaud uniformization

$$0 \longrightarrow T^{\mathrm{an}} \longrightarrow E \longrightarrow B^{\mathrm{an}} \longrightarrow 0, \qquad A^{\mathrm{an}} = E/M.$$

Here T is a split torus of rank r, B has good reduction and dimension b, and $g = r + b$. The tropicalization of the period group M is a full lattice $\Lambda \subset N_{\mathbb{R}} \simeq \mathbb{R}^r$, giving

$$\tau : A^{\mathrm{an}} \longrightarrow N_{\mathbb{R}}/\Lambda.$$

This construction uses the torus torsor and its canonical metric data. It is not a claim that A^{an} admits one global moment map by analytic units. On a proper connected abelian variety, global analytic functions are constant.

Theorem 21.12.1 (The polyhedral description of canonical measures)**.** *Let $X \subset A$ be an irreducible closed subvariety of dimension d in the stated source setting. There are finitely many rational simplices Δ_j in $N_{\mathbb{R}}/\Lambda$ whose union is $\tau(X^{\mathrm{an}})$, with*

$$\max(0, d - b) \leq \dim \Delta_j \leq \min(d, r).$$

For canonically metrized line bundles $\overline{L}_1, \dots, \overline{L}_d$ on A, one has

$$\tau_*\big(c_1(\overline{L}_1|_X) \cdots c_1(\overline{L}_d|_X)\big) = \sum_j c_j \lambda_{\Delta_j}.$$

Here λ_{Δ_j} is relative Lebesgue measure on the simplex, not an ambient r-dimensional measure unless its dimension is r. If all L_i are ample, the coefficients can be chosen strictly positive.

Open proof obligation (RC2). Prove the canonical measure and dimension conclusions in the stated setting; Gubler Theorem 1.1 is supplied.

21.12.1 Nondegenerate simplices in an alteration

Choose a semistable alteration $f : Y \to X^{\mathrm{an}}$ of generic degree e, in the formal-model setting of the source. A stratum S of the special fibre corresponds to a canonical simplex Δ_S in the skeleton, with

$$\dim S + \dim \Delta_S = d.$$

The map to A induces an integral affine map on Δ_S and a residual map from S to the good-reduction abelian part. The simplex is nondegenerate for f when both maps preserve the respective dimensions:

$$\dim f_{\mathrm{aff}}(\Delta_S) = \dim \Delta_S, \qquad \dim q_s \Phi_s(S) = \dim S.$$

The residual lift Φ_s is determined up to period translation, which does not change either dimension. This is the definition in [Gub10, Section 6.3].

Proposition 21.12.2. *A nondegenerate simplex of dimension s satisfies $d - b \le s \le \min(d, r)$.*

Proof. The residual image has dimension $\dim S = d - s$ and lies in a variety of dimension b. Thus $d - s \le b$. The injective affine map of the simplex into $N_{\mathbb{R}}$ gives $s \le r$, and $s \le d$ is automatic from the dimension formula for strata. □

For a single ample line bundle L with canonical metric, the source gives a density formula on each nondegenerate simplex. In its simplex coordinates, let $\ell_S : \mathbb{R}^s \to N_{\mathbb{R}}$ be the injective linear part of f_{aff}, let b_L be the positive definite bilinear form from the polarization, and set

$$\Lambda_S = \ell_S^{-1}(\Lambda), \qquad \Lambda_S^L = \{b_L(\ell_S(\,\cdot\,), \lambda) : \lambda \in \Lambda\} \subset (\mathbb{R}^s)^*.$$

These are full lattices in the source setting. For a measurable subset Ω of the relative interior of the simplex, [Gub10, Theorem 6.7] gives

$$c_1(f^*\overline{L})^d(\Omega) = \frac{d!}{(d-s)!}\, \deg_H(S)\, \frac{\operatorname{covol}(\Lambda_S^L)}{\operatorname{covol}(\Lambda_S)}\, \operatorname{vol}(\Omega).$$

The symbol $\deg_H(S)$ denotes the residual intersection degree defined from the proper residual cycle in the source's Section 5.17, with H the induced ample line bundle on the good-reduction part. It is not an undefined degree of an arbitrary open stratum. For ample L and a nondegenerate simplex it is positive. The density is the product of a residual algebraic degree and a real lattice-volume factor.

The formula is compatible with keeping the original absolute value. If all logarithmic values are rescaled by a positive constant c, then the period lattice and simplex coordinates scale by c, while b_L scales by c^{-1}. Thus Λ_S has covolume multiplied by c^s, Λ_S^L is unchanged in the corresponding dual coordinates, and $\operatorname{vol}(\Omega)$ is multiplied by c^s. These factors cancel. This is a change-of-coordinate check, not permission to renormalize a finite field extension independently.

21.12.2 Projection and the support of the measure

Over one fixed ground field, the projection formula gives

$$f_* c_1(f^*\overline{L}_1) \cdots c_1(f^*\overline{L}_d) = e\, c_1(\overline{L}_1|_X) \cdots c_1(\overline{L}_d|_X).$$

The measure downstairs is therefore computed by pushing the skeletal measure upstairs and dividing by the alteration degree. This is a computation of a measure, for which proper pushforward is available. It is not a theorem that arbitrary actual differential representatives descend through the same alteration. The ground-field distinction in Chapter 23 still applies if an extension was used before constructing the alteration.

Corollary 21.12.3. *For ample L_i, the support of the tropical pushforward of their canonical mixed measure is all of $\tau(X^{\mathrm{an}})$.*

Proof. Every simplex in Theorem 21.12.1 has positive coefficient. An open neighbourhood of any point of a simplex meets its relative interior in positive relative Lebesgue measure; for a zero-dimensional simplex this means a positive atom. Hence every point in the union belongs to the measure support. Conversely, the measure is supported on this finite union, which is closed. □

Remark 21.12.4. A measure containing pieces of different dimensions does not by itself show that its support is not pure-dimensional. For example, Lebesgue measure on $[0,1]$ plus a positive atom at zero has pure one-dimensional support but a zero-dimensional measure component. Likewise, the dimension interval in the canonical-measure theorem does not refute purity of the tropical image. The author's erratum records a gap in an earlier purity proof; it does not supply a counterexample. These are distinct conclusions.

21.12.3 The ambient abelian variety and Haar measure

When $X = A$, one has $d = g = r + b$, so the dimension inequalities force every nondegenerate simplex to have dimension r. The canonical measure is supported on the canonical skeleton, and its tropical pushforward is

$$\tau_* c_1(\overline{L})^g = \deg_L(A)\, m_{N_{\mathbb{R}}/\Lambda},$$

where $m_{N_{\mathbb{R}}/\Lambda}$ is Haar probability measure. This is the ambient calculation in [Gub10, Example 7.2]. The total degree fixes the scalar.

The real uniqueness argument is short. A finite measure invariant under the dense subgroup of rational-period translations is invariant under every translation, by uniform continuity of continuous tests. If ν is such a measure

and m is Haar probability, then for a continuous test a, Fubini and translation invariance give

$$\int a\,d\nu = \int\int a(x+y)\,d\nu(x)\,dm(y) = \nu(N_{\mathbb{R}}/\Lambda)\int a\,dm.$$

The geometric input is the translation invariance of the canonical measure, or equivalently the constant density obtained from the canonical-metric formula. Invariance under the single map $[m]$ alone would not suffice: a Dirac mass at zero is also fixed by its pushforward.

At good reduction, $r = 0$, the real torus is one point and this formula becomes a degree-weighted Dirac measure. At total degeneration, $b = 0$, the support has full real dimension g. Mixed reduction lies between these extremes. These examples explain why a top-degree measure comparison can be correct while a claim that one tropicalization retains every lower-degree differential object is false.

References. [Gub10, Theorem 1.1 and its erratum, Sections 5.17 and 6.3–6.8, Theorem 6.12, Example 7.2]. The density theorem is cited; its dimension, normalization and support consequences are proved here.

21.13 The variational passage to equidistribution

The local measures of this book become arithmetic tools when metric variation detects the distribution of small points. The final variational argument is elementary; obtaining its lower bound is the arithmetic theorem. We separate these two steps.

Lemma 21.13.1. *Let M be compact Hausdorff, μ_j probability measures, and $a_j \to a$. Let $f \in C(M)$ and suppose that for all sufficiently small positive and negative t, a function $H_f(t)$ satisfies*

$$\liminf_j \left(a_j + t\int f\,d\mu_j\right) \geq H_f(t), \qquad H_f(0) = a,$$

and is differentiable at zero. Then $\int f\,d\mu_j \to H_f'(0)$. The same proof applies to a net.

Proof. For $t > 0$, subtract a and divide by t to obtain

$$\liminf_j \int f\,d\mu_j \geq \frac{H_f(t) - H_f(0)}{t}.$$

For $t < 0$, division reverses the inequality and exchanges the lower and upper limits, giving the same quotient as an upper bound for $\limsup_j \int f\,d\mu_j$. Let t tend to zero from its two sides. The quantities are bounded by $\|f\|_\infty$, so these limits are finite, and differentiability makes the bounds equal. □

In the arithmetic application, a_j is the original point height and μ_j is the normalized Galois-orbit measure at one place. Changing the local weight by tf, with the place factor absorbed into t, changes the point height exactly by $t\int f\,d\mu_j$. Arithmetic bigness and the fundamental inequality provide the function H_f. Its derivative is the integral of f against the normalized Chern measure. These are the substantive results in the variational equidistribution method of Szpiro–Ullmo–Zhang and Yuan, and in Gubler's function-field tropical setting.

One cannot simply vary an arbitrary semipositive metric in both directions by an arbitrary continuous f and assume semipositivity persists. The arithmetic lower bound must apply to the permitted perturbations, or be obtained through a volume or envelope argument. The squeeze lemma proves equidistribution after that input; it does not prove the input.

Lemma 21.13.2. *If probability measures μ_j, μ on a compact space satisfy convergence against a uniformly dense vector subspace $D \subset C(M)$, then $\mu_j \rightharpoonup \mu$ against all continuous functions.*

Proof. For $f \in C(M)$ choose $g \in D$ with $\|f - g\|_\infty < \varepsilon$. Then

$$\left|\int f\,d\mu_j - \int f\,d\mu\right| \le 2\varepsilon + \left|\int g\,d\mu_j - \int g\,d\mu\right|.$$

First take the limit in j, then let ε decrease to zero. □

The uniform mass bound is essential in this density argument. It is the measure analogue of the coefficient estimates used earlier for currents. Neither density nor weak convergence alone permits products of singular currents; here only scalar tests of probability measures are involved.

21.14 Difference maps and the stabilizer

Let X be an integral closed subvariety of an abelian variety A over an algebraically closed field. Denote its stabilizer by G. After quotienting by the

positive-dimensional stabilizer, the relevant case has finite stabilizer. The next argument explains the generic finiteness needed in the Bogomolov proof, including in positive characteristic; separability is not required.

Proposition 21.14.1. *If G is finite, then for some integer N, the difference map*

$$\alpha : X^N \longrightarrow A^{N-1}, \qquad (x_1, \dots, x_N) \longmapsto (x_2 - x_1, \dots, x_N - x_{N-1})$$

is generically finite onto its image.

Proof. The set-theoretic intersection $\bigcap_{x \in X}(X - x)$ equals the stabilizer: membership means $X + t \subseteq X$, which is equality because both are irreducible of the same dimension. The noetherian topology gives a finite subcollection $x_1, \dots, x_N$ whose intersection has dimension zero. Indeed a descending chain of closed intersections stabilizes; if its current intersection is larger than the full one, choose one further translate that makes it smaller. The fibre of α over $\alpha(x_1, \dots, x_N)$ is identified with $\bigcap_i (X - x_i)$ by sending a common translation t to $(x_1 + t, \dots, x_N + t)$. Thus this fibre is zero-dimensional, possibly nonreduced. Upper semicontinuity of fibre dimension for the proper map α shows that its generic fibre is zero-dimensional. Since X^N is integral, this is generic finiteness onto the image. □

Notice what this argument does not say: the map need not be finite on every fibre. The diagonal is contracted to zero. The coexistence of generic finiteness and this contracted diagonal is precisely what the measure argument exploits.

21.15 The tropical measure contradiction

We state the inputs in the totally degenerate setting of Gubler's theorem. Let K be the function field of a projective integral variety over an algebraically closed constant field, regular in codimension one, with the product formula determined by an ample class. Suppose A/K is totally degenerate at a place v. Pass to the completed algebraic closure at v for analytic geometry. We use:

1. The tropicalization of a d-dimensional subvariety is a pure d-dimensional finite polytopal set in the tropical real torus.

2. The normalized canonical measure pushes forward to a measure with strictly positive Lebesgue densities on its top-dimensional cells, and no mass on lower-dimensional polyhedral sets.

3. Generic small nets of points equidistribute to this measure, and the same theorem applies to products and images under homomorphisms.

These are the tropical geometry and arithmetic theorems proved in Gubler's work. The following proof is the argument after these inputs.

Theorem 21.15.1. *A positive-dimensional integral subvariety $X \subset A$ with finite stabilizer cannot have Zariski-dense points of arbitrarily small Néron–Tate height.*

Proof. Suppose otherwise and put $d = \dim X > 0$. Choose N as in Proposition 21.14.1 and let $Y = \alpha(X^N)$. Then $\dim Y = Nd$. The external tensor product polarization on X^N has height equal to the sum of the heights of its coordinates. Dense small points on X therefore give dense small points on X^N. More explicitly, the product of a Zariski-dense set with itself is dense; directing finite avoidance conditions and height bounds gives a generic net $P_j \in X^N$ with height tending to zero. Its image is generic in Y, since inverse images of proper closed subsets of Y are proper closed subsets of X^N. Quadratic height estimates for the difference homomorphism make the image heights tend to zero. Let μ and ν be the normalized tropical canonical measures on $\mathrm{Trop}(X)^N$ and $\mathrm{Trop}(Y)$. Equidistribution and compatibility of normalized finite Galois-orbit measures with the map give

$$\alpha_{\mathrm{trop},*}\mu = \nu, \qquad \alpha_{\mathrm{trop}}(u_1, \dots, u_N) = (u_2 - u_1, \dots, u_N - u_{N-1}).$$

For completeness, a map of finite transitive Galois sets has fibres of equal cardinality, so the pushforward of uniform orbit measure is uniform orbit measure. Pushforward of weak convergence by a continuous map follows by testing against pullbacks of continuous functions. Choose a small compact d-dimensional polytope σ in the relative interior of a top cell of $\mathrm{Trop}(X)$, contained in one affine lift of the real torus. The product σ^N has dimension Nd and strictly positive μ-mass. If V is the d-dimensional tangent space of σ, the differential of α_{trop} on V^N kills the diagonal copy of V. Hence

$$\dim \alpha_{\mathrm{trop}}(\sigma^N) \leq (N-1)d < Nd.$$

The compact image $\tau = \alpha_{\mathrm{trop}}(\sigma^N)$ is a lower-dimensional polyhedral subset of $\mathrm{Trop}(Y)$, so $\nu(\tau) = 0$. But the measure identity implies

$$0 < \mu(\sigma^N) \leq \mu(\alpha_{\mathrm{trop}}^{-1}(\tau)) = \nu(\tau) = 0,$$

a contradiction. This last inequality uses a Borel set directly; no choice of a continuous cutoff equal only at a single point would suffice in its place. □

The complete theorem also treats stabilizers and zero-dimensional images. Gubler proves that total degeneration passes to the required abelian quotients and forces the constant-field trace to vanish. The trace theorem then identifies zero-height points with torsion in this setting. After quotienting by the stabilizer, the theorem above forces the remaining image to be a torsion point. Pulling back gives a torsion translate of an abelian subvariety. These trace and quotient assertions are cited geometric inputs; the number-field Northcott argument is not a substitute for them over function fields.

The proof also explains the limitation of the argument in mixed reduction. The canonical measure of a d-dimensional subvariety may have pieces of several tropical dimensions. A lower-dimensional image need not have measure zero. One must then compare the dimensions actually carrying canonical mass, not merely the algebraic dimensions of source and target. Gubler's later canonical-measure theorem supplies exactly the refined information needed to formulate this problem.

References. [Gub07, Theorem 1.1, tropical equidistribution theorem, and Section 6]; [Gub10, Theorem 1.1]; [Yua08, Section 3].

21.16 Tropical boundary conditions and logarithmic curvature

The real vector space $N_{\mathbb{R}}$ describes the dense torus. Passing to N_{Σ} adds strata, changes the test sheaf, and allows curvature supported on the boundary. A calculation made only in $N_{\mathbb{R}}$ can therefore lose a divisor term. We begin with a complete one-dimensional calculation.

For the fan consisting of the zero cone and the positive ray, write $N_{\Sigma} = \mathbb{R} \cup \{+\infty\}$. On the analytic affine line this coordinate is $t = -\log|z|$, and $z = 0$ maps to $+\infty$. A smooth tropical function on a neighbourhood of $+\infty$ is constant on a sufficiently long tail. A positive-degree form vanishes there: it must come from the zero-dimensional boundary stratum.

Proposition 21.16.1 (The boundary slope formula)**.** *Let $U = (a, +\infty]$. Let f be a finite convex nonincreasing function on (a, ∞), and extend it to $+\infty$ by its limit in $\mathbb{R} \cup \{-\infty\}$. Write $\mu = f''$ for its distributional second derivative on the open interval. The increasing one-sided derivatives have a limit $\ell \leq 0$ at $+\infty$. On U,*

$$d'd''[f] = \mu - \ell\, \delta_{+\infty}.$$

Both terms on the right are positive measures. If $f(+\infty)$ is finite, then $\ell = 0$.

Proof. A test $\eta \in A_c^{0,0}(U)$ vanishes near a and is equal to a constant c for large t. Its Hessian has ordinary compact support in (a, ∞), so $[f](d'd''\eta) = \int f(t)\eta''(t)\, dt$ is well-defined even when $f(+\infty) = -\infty$. Distributional integration by parts for a convex function gives, for R beyond the nonconstant part of η and chosen away from derivative jumps,

$$\int_a^R f\eta''\, dt = \int_{(a,R)} \eta\, d\mu - cf'(R).$$

On every tail $[b, \infty)$ the positive measure μ has finite mass, bounded by $-f'_+(b)$, since the derivative is increasing and at most zero. Letting $R \to \infty$ proves the formula. If $\ell < 0$, choose $\ell < \ell/2 < 0$; for all sufficiently large t, the derivative is at most $\ell/2$, so $f(t) \to -\infty$. Thus a finite boundary value forces $\ell = 0$. □

Example 21.16.2. The function $f(t) = -t$ has zero interior Hessian but $d'd''[f] = \delta_{+\infty}$. Under $t = -\log|z|$, it is $\log|z|$. This is the boundary contribution in Poincaré–Lelong. By contrast, $f(t) = e^{-2t}$ has boundary value zero, interior Hessian $4e^{-2t}dt$, and no boundary atom. Both functions are psh on U, but neither is tropical smooth near the boundary.

The second example is also a useful complex comparison. Its complex pullback is $|z|^2$, a smooth function at $z = 0$. Thus complex smoothness of an invariant pullback does not force the source function to be smooth for the tropical boundary sheaf. Boundary compatibility is a genuine condition, not merely continuity.

21.16.1 Psh functions on a general tropical toric variety

For a rational fan Σ, the BGJK psh class on $U \subset N_\Sigma$ consists of upper semicontinuous functions with values in $\mathbb{R} \cup \{-\infty\}$, convex on $U \cap N_\mathbb{R}$, and satisfying

$$u(p + \infty v) \leq u(p) \quad \text{whenever } [p, p + \infty v] \subset U,\ v \in |\Sigma|.$$

Here $p + \infty v$ is the limiting point in the stratum attached to the cone containing v in its relative interior. The theorem in [BGJK25, Theorem 3.1.4] identifies this class with continuous extended-real functions whose restriction to the dense stratum is convex. Continuity is taken in $\mathbb{R} \cup \{-\infty\}$, identified with $[0, \infty)$ by the exponential. In particular, a finite psh function is continuous and locally bounded. A function equal to $-\infty$ is not a bounded potential.

The proof of this boundary continuity is not a formal consequence of convexity on the dense stratum. It uses the compactified-ray inequalities and basic polyhedral neighbourhoods. In dimension one, convexity and a finite upper boundary limit force monotone decrease: a positive secant slope would force growth to $+\infty$. The higher-dimensional proof applies this observation along the cone directions and controls nearby transverse directions by convexity. We use the full theorem as an input when passing to the BT construction.

21.17 Invariant analytic smoothness can use additional coordinates

Proposition 21.17.1. *Let K be a complete non-Archimedean field and $t = -\log|T|$ on $\mathbb{G}_m^{\mathrm{an}}$. The function*

$$h = \min(0, t)^2$$

is original CLD smooth and invariant under the maximal affinoid torus action, but is not the pullback of a smooth real function by this one-coordinate tropicalization.

Proof. On the open analytic set $T \neq 1$, the two functions T and $T - 1$ are units. The ultrametric inequality gives

$$\min(0, -\log|T|)^2 = (-\log|T|)(-\log|T - 1|).$$

Indeed, for $|T| < 1$ the second factor is zero; for $|T| > 1$ the two factors agree; for $|T| = 1$ the first is zero. The right side is a smooth polynomial in the two tropical coordinates. Near $T = 1$, on $|T - 1| < 1$, we have $|T| = 1$ and $h = 0$. Hence the local smooth presentations glue across the omitted point.

The function factors through tropicalization, hence is invariant under the stated action. Tropicalization is surjective onto $\mathbb{R}$, so any function in that one-coordinate chart representing h would have to be $s \mapsto \min(0, s)^2$. Its second derivative jumps at zero; it is not smooth on $\mathbb{R}$. □

This is the example of [BGJK25, Example 2.3.5], with the local domains made explicit. In the consulted preprint the displayed map $T \mapsto (T, T-1)$ is written as a torus-valued map on all of $\mathbb{G}_m$. It is torus-valued only after deleting $T = 1$; the zero presentation above supplies the missing neighbourhood. The example itself remains valid. It disproves surjectivity of the fixed-chart smooth pullback onto all invariant analytic smooth functions, not injectivity of that pullback.

21.18 Transporting the BT construction through the complex torus

Suppose first that Σ is smooth. Write $\tau : X_\Sigma(\mathbb{C}) \to N_\Sigma$ for complex tropicalization, $U \subset N_\Sigma$ open, and $V = \tau^{-1}(U)$. Normalize d^c so that $dd^c = (i/\pi)\partial\bar{\partial}$. The source constructs a pullback with

$$\tau^* d' = \pi^{-1/2}\partial\tau^*, \qquad \tau^* d'' = \pi^{-1/2} i\bar{\partial}\tau^*, \qquad \tau^* d' d'' = dd^c \tau^*.$$

It preserves integration of top-degree forms. At boundary strata its image can be smaller than the invariant complex smooth forms, as the preceding $|z|^2$ example shows. The decisive theorem is instead a statement about the closed positive current cones.

Throughout this comparison, complex positivity means the middle cone used in [BGJK25, Section 2.1]: a form is a sum of $i^{p^2} a\, \alpha \wedge \overline{\alpha}$ with $a \geq 0$ and arbitrary $(p, 0)$-forms α, and a current is nonnegative on this cone in complementary degree. This convention matches ordinary Lagerberg positivity. It should not be read as changing silently to a different convention for positive currents in a complex reference.

Theorem 21.18.1 (The complex current comparison input)**.** *The pushforward τ_* is an isomorphism from the cone of closed positive complex currents on V invariant under the maximal compact torus and the real-structure involution F of [BGJK], to the cone of closed ordinary positive Lagerberg currents on U.*

Open proof obligation (RC2). Prove the positive closed current lifting/comparison with the prescribed support and invariance conditions; BGJK Theorem 7.1.5 is supplied.

Theorem 21.18.2 (Toric continuous BT products)**.** *For a closed positive Lagerberg current T of bidegree (p,p) on U, and finite psh functions $u_1,\dots,u_q$ with $p+q\le n$, there are local products*

$$u_1\,DDu_2\wedge\cdots\wedge DDu_q\wedge T,\qquad DDu_1\wedge\cdots\wedge DDu_q\wedge T,\qquad DD=d'd''.$$

The second is closed and positive. The products agree with the smooth formulas and are continuous when all potentials converge locally uniformly and the positive closed background currents converge weakly. These properties determine the products uniquely.

Open proof obligation (RC2). Prove the toric/complex function comparison and local decreasing regularization on the stated neighbourhoods; BGJK25 Proposition 3.2.1 and Theorem 3.3.4.

21.18.1 Unbounded potentials need a separate condition

In the unbounded theorem of [BGJK25, Theorem 4.2.2], the sets D_j containing $\{u_j=-\infty\}$ are closed unions of strata of positive codimension. If B is a closed union of strata containing $\operatorname{supp}T$, the hypothesis is

$$\dim\left(B\cap\bigcap_{j\in I}D_j\right)\le n-p-|I|\quad\text{for every } I\subseteq\{1,\dots,q\}.$$

The dimension includes the empty-index case. Under this proper-intersection condition the products exist and are continuous for decreasing sequences of the potentials with the background current fixed. The proof transfers the corresponding complex integrability theorem, using that a tropical stratum of real dimension d corresponds to a complex torus orbit of complex dimension

d. One must not replace the fixed background in this unbounded assertion by an arbitrary weakly converging sequence without another theorem.

For a nonsmooth fan, a smooth refinement gives a proper tropical toric map. Products of the scalar potentials can then be defined upstairs and pushed down, with independence proved by a common refinement and the projection formula. This is a specific toric descent theorem. It does not provide a general pullback of an arbitrary singular current to a resolution.

21.19 Truncation can discard divisor mass

For $u(t) = -t$ on a neighbourhood of $+\infty$, let $u_j = \max(u, -j)$. The boundary slope formula and the interior corner calculation give

$$DDu_j = \delta_j \ \rightharpoonup \ \delta_{+\infty} = DD[u].$$

However,

$$\mathbf{1}_{\{u>-j\}} DDu_j = \mathbf{1}_{\{t<j\}} \delta_j = 0.$$

Thus an extension defined by retaining the full decreasing-limit current and one defined by discarding the truncation-level mass give different answers. The latter is the mechanism underlying non-pluripolar products: it deliberately omits mass on the pole set. Calling both objects the same Monge–Ampère operator would conceal this choice.

Proposition 21.19.1 (The elementary monotonicity behind truncation)**.** *Suppose a measure-valued operator M is defined on bounded truncations $u_j = \max(u, -j)$, and is local on the sets $E_j = \{u > -j\}$: for $k \geq j$, $\mathbf{1}_{E_j} M(u_k) = \mathbf{1}_{E_j} M(u_j)$. Assume these sets are measurable and the measures positive. Then*

$$\nu_j := \mathbf{1}_{E_j} M(u_j) \leq \nu_k \quad (k \geq j).$$

Their increasing limit is a measure. It is locally finite if the masses $\nu_j(K)$ are uniformly bounded on each compact neighbourhood. Without that bound, the construction need not give a current with locally finite coefficients.

Proof. Since $E_j \subseteq E_k$, restriction of the positive measure ν_k to E_j equals ν_j by locality. Therefore $\nu_k - \nu_j$ is positive and supported outside E_j. The pointwise increasing limit on measurable sets is countably additive by monotone convergence for nonnegative series. Uniform local mass bounds give local finiteness. □

The proposition is measure theory once the stated locality is known. It does not prove that locality for every analytic psh class, independence of all choices of truncation, or a full-mass theorem. In the toric class the extended continuity makes E_j open, which simplifies this step. In a general unbounded analytic theory, the corresponding locality and exceptional-set statements are substantial additional inputs.

References. [BGJK25, Sections 2–4], in the consulted arXiv:2102.07392v1; [BGJK, Theorem 7.1.5].

21.20 Boundary mass and extension of toric currents

A tropical toric boundary changes the space of tests. Near a coordinate with value $+\infty$, a smooth function is eventually independent of that coordinate, and a positive-degree differential involving that coordinate vanishes. Consequently, a current may be well defined on all tropical tests although its invariant complex lift would have infinite mass near the corresponding divisor. The additional mass condition in the toric correspondence theorem is designed precisely for this phenomenon.

21.20.1 Weighted co-coefficients

Work in a smooth toric chart, with coordinates $u_1, \dots, u_n$ in a product of copies of $\mathbb{R} \cup \{+\infty\}$ and $\mathbb{R}$. For a current of bidegree (p,p), put $q = n - p$. If $|I| = |J| = q$, its co-coefficient T^{IJ} is the measure obtained by testing against a scalar multiple of the basis form indexed by I, J, with the usual sign normalization. It is defined on the locus where all coordinates occurring in $I \cup J$ are finite. Write

$$E^{I\cup J} = \bigcup_{i\in I\cup J} \{u_i = +\infty\}, \qquad w_{IJ}(u) = \exp\left(-\sum_{i\in I} u_i - \sum_{j\in J} u_j\right).$$

Definition 21.20.1 (**C**-finite local mass)**.** A positive tropical current has **C**-finite local mass if, in every toric coordinate chart, the image measures of $w_{IJ}|T^{IJ}|$ under the open inclusions from the above finite-coordinate loci are locally finite on the whole chart. For a current defined off a union E of stratum closures, the condition is required across E as well.

The letter **C** refers to the complex realization, not to an additional arbitrary norm on the tropical current. Indeed, on the dense torus $u_i = -\log|z_i|$, logarithmic forms contain dz_i/z_i. Conversion between logarithmic co-coefficients and ordinary complex co-coefficients therefore introduces exactly the factors $|z_i| = e^{-u_i}$ above. Fixed powers of 2 and π depend on the normalization of the complex differential operators and do not affect local finiteness.

Example 21.20.2 (Tropical continuity without finite complex mass)**.** On $U = (0, +\infty] \subset \mathbb{R} \cup \{+\infty\}$, define the positive $(0,0)$ current

$$T(g\, d'u \wedge d''u) = \int_0^\infty g(t) e^{2t}\, dt.$$

Every compactly supported tropical $(1,1)$ test vanishes near $+\infty$. On each fixed test-presentation space the integral is therefore bounded by a constant times the coefficient supremum. This defines a continuous current. Its weighted co-coefficient, however, is

$$e^{-2t} e^{2t}\, dt = dt,$$

which has infinite mass on every neighbourhood $(R, +\infty]$ of the boundary. It fails the **C**-finite condition. A positive complex lift would restrict on the punctured disc to a positive multiple of the function $|z|^{-2}$, whose mass near zero contains $\int_0^\varepsilon r^{-1}\, dr = +\infty$. Thus no positive complex lift exists. This current is not closed; its failure of closedness is essential.

Theorem 21.20.3 (Toric lifting and closedness)**.** *Let Σ be a smooth fan, $U \subset N_\Sigma$ open, and $V = \mathrm{trop}_\infty^{-1}(U)$ in the complex toric manifold. A positive Lagerberg current on U admits a positive invariant complex lift precisely when it has* **C***-finite local mass. Every positive closed Lagerberg current on U has this local mass property. On closed positive currents, the lift is unique after imposing the compact-torus invariance and the real involution used in the correspondence theorem.*

Open proof obligation (RC2). Prove lifting and locally finite mass at the boundary strata where the supplied BGJK propositions/theorem enter.

21.20.2 The toric extension theorem

Theorem 21.20.4 (Toric Skoda–El Mir)**.** *In the preceding smooth toric setting, let $E \subset U$ be the intersection with a union of stratum closures. Let T be a positive closed Lagerberg current on $U \setminus E$ with $\mathbf{C}$-finite local mass across E. Its coefficientwise extension by zero is a positive closed current on U.*

Proof. On the locus where the indices of a co-coefficient are finite, w_{IJ} is positive with reciprocal locally bounded. The weighted local mass assumption thus implies the unweighted local finiteness needed to extend that co-coefficient by zero on its own domain. These extensions define a positive tropical current $\widetilde{T}$, by the coordinate transformation rule and positivity under scalar cutoffs.

It remains to prove closedness. Put $D = \mathrm{trop}_{\infty}^{-1}(E) \subset V$. Because E is a union of stratum closures, D is an analytic subset of the complex toric manifold. Lift T on $V \setminus D$ by the closed positive correspondence theorem. The weighted mass condition is exactly finite complex local mass across D. The classical complex Skoda–El Mir theorem extends that lift by zero to a positive closed current $\widetilde{S}$ on V. The coefficient transformation identifies its tropical pushforward with $\widetilde{T}$. Since pushforward commutes with both differentials, $\widetilde{T}$ is closed. □

This is a proof of the tropical theorem conditional only on the stated complex extension theorem and the toric correspondence. It is not a proof of Skoda–El Mir from elementary real calculus. Its geometric hypothesis on E is also indispensable to this argument: an arbitrary closed subset of a tropical support need not lift to a complex analytic subset. Compare the direct Hausdorff-size criterion of Section 9.16, whose hypotheses and proof are different.

References. [BGJK, Theorem 7.2.4]. The complex Skoda–El Mir theorem is the explicit external extension input in that proof.

21.20.3 Decomposition by strata

Order-zero co-coefficients can be restricted to the Borel strata. For a positive tropical current, the restrictions are compatible with all toric coordinates and give a canonical decomposition

$$T = \sum_{\sigma \in \Sigma} T_\sigma.$$

The component T_σ gives no coefficient mass to the complement of the stratum $N(\sigma)$. Its topological support can nevertheless meet the boundary of that stratum. This distinction between concentration on a measurable stratum and support in its closure is the same distinction that occurs for an ordinary measure on an open interval.

Corollary 21.20.5. *If T is positive and closed, every component T_σ in this canonical decomposition is positive and closed.*

Proof. Positivity follows from the coefficient restriction construction. The dense-stratum component is the extension by zero of the restriction of T to that stratum. Theorem 21.20.3 supplies the weighted mass condition, so Theorem 21.20.4 makes that component closed. Subtract it and repeat on the strata of the next dimension, working in their toric neighbourhoods. At each stage the remaining current is a sum of positive stratum components, so it is positive; it is closed because the earlier components are closed. The same extension theorem applies. Induction over the finitely many strata proves the assertion. □

References. The canonical decomposition is [BGJK, Theorem 6.1.6]; its compatibility with closedness is Corollary 7.2.5 there. The co-coefficient transformation at boundary faces is an essential part of that construction.

The theorem explains why boundary pieces can be separated in toric current arguments. It does not justify separating an arbitrary positive closed current according to arbitrary measurable subsets: the derivative of a restriction usually has boundary terms. The special strata, weighted mass estimate, and analytic extension theorem account for the stronger conclusion here.

21.21 The toric projection formula with every curvature factor visible

Proper pushforward is particularly useful for changing a fan. Its compatibility with a BT product follows from the same operation on smooth tests and a fixed-neighbourhood limit. Writing all factors prevents a degree error when one potential is left undifferentiated.

Proposition 21.21.1. *Let $E : N'_{\Sigma'} \to N_\Sigma$ be a proper equivariant map between tropical toric varieties of dimensions n' and n, with both fans smooth. Let $U \subset N_\Sigma$ be open and $U' = E^{-1}(U)$. Let T be a closed ordinary positive current of bidegree (p,p) on U'. For finite psh functions $u_1, \dots, u_q$ on U, with $p+q \le n'$, one has*

$$E_*\big(DDE^*u_1 \wedge \cdots \wedge DDE^*u_q \wedge T\big) = DDu_1 \wedge \cdots \wedge DDu_q \wedge E_*T,$$
$$E_*\big(E^*u_1\, DDE^*u_2 \wedge \cdots \wedge DDE^*u_q \wedge T\big) = u_1\, DDu_2 \wedge \cdots \wedge DDu_q \wedge E_*T.$$

The bidegrees on the target are $(p+q+n-n', p+q+n-n')$ and $(p+q-1+n-n', p+q-1+n-n')$, respectively. If these indices lie outside the possible range, the corresponding current space is zero.

Proof. For smooth potentials, evaluate on a compactly supported test η on U. Properness makes $E^*\eta$ compactly supported, and by definition of pushforward the left side is T evaluated on the pullback of the full target product. Pullback commutes with DD on functions and with wedge product. This is the right side. The total number of curvature factors gives the displayed degrees.

In the continuous case choose local smooth psh approximations on fixed neighbourhoods of the target support and use a finite subdivision of the test by cutoffs. Properness ensures that the source tests remain in a common compact set. The pulled-back potentials converge uniformly on this compact set, and remain psh by the toric psh pullback theorem. Theorem 21.18.2 passes both sides to the limit. The argument proves a projection formula for a current already defined on the source; it does not construct a pullback of E_*T. □

In the consulted arXiv:2102.07392v1, Proposition 4.3.2 abbreviates the displayed product in a way that omits some DD operators. Remark 4.3.4 also prints the same undifferentiated leading factor on the right side for products

assigned different bidegrees. The formulas above are the degree-consistent versions. For a smooth refinement E and no background factor, the differentiated product downstairs must be

$$DDu_1 \wedge \cdots \wedge DDu_q = E_*\big(DDE^*u_1 \wedge \cdots \wedge DDE^*u_q\big).$$

The companion product has E^*u_1 undifferentiated on the right and u_1 undifferentiated on the left. These are local formula repairs, not failures of the smooth projection identity or of the limiting construction.

There is one further elementary qualification to positivity wording. A product $u_1 DDu_2 \wedge \cdots \wedge DDu_q \wedge T$ need not be positive if u_1 can be negative. For $u_1 = -1$ and a nonzero positive background it is negative. The differentiated product is positive without a sign condition on the potentials, whereas positivity of the product with an undifferentiated factor requires that factor to be nonnegative. The main product statement above imposes positivity only on the differentiated product.

References. [BGJK25, Proposition 4.3.2 and Remark 4.3.4], with version fixed to arXiv:2102.07392v1.

21.22 Quasiperiodic convex functions and real torus metrics

A totally degenerate abelian variety has a real torus as its canonical skeleton. The relevant potential on its universal cover is not periodic: its failure of periodicity is an affine function prescribed by the polarization. Separating the quadratic part from a periodic correction makes both positivity and normalization explicit. The following results are real statements in arbitrary dimension. Their analytic interpretation uses the uniformization and metric-comparison theorems cited at the end.

Let $V = \mathbb{R}^n$, let $\Lambda \subset V$ be a full lattice, and choose a positive-definite symmetric bilinear form b, with matrix H in fixed coordinates. Put

$$q(x) = \tfrac{1}{2} b(x, x), \qquad M = V/\Lambda, \qquad \theta = DDq.$$

A continuous periodic function u defines $f = q + u$ with automorphy rule

$$f(x + \lambda) - f(x) = b(x, \lambda) + q(\lambda), \qquad \lambda \in \Lambda.$$

Conversely this rule makes $f - q$ periodic. Since the right side is affine, the local Hessians DDf glue to a closed form or current on M. For smooth u,

the inequality $\theta + DDu \geq 0$ means that f is convex. For continuous u, we use this convexity as the definition in this real torus setting.

Lemma 21.22.1. *The affine functions $c_\lambda(x) = b(x,\lambda) + q(\lambda)$ satisfy*

$$c_{\lambda+\mu}(x) = c_\lambda(x+\mu) + c_\mu(x).$$

Adding a periodic function to q preserves these transition functions. Adding an affine function $\ell(x) + a$ changes c_λ by the constant $\ell(\lambda)$, while adding a constant alone leaves them unchanged.

Proof. Expand $q(\lambda+\mu) = q(\lambda) + q(\mu) + b(\lambda,\mu)$ and use symmetry of b. The statements about additions follow by subtracting the values at $x+\lambda$ and x. □

The cocycle is the real analogue of transition functions for a line bundle. In the non-Archimedean uniformization, integral slopes and the corresponding Raynaud data are additional requirements for it to come from a specified algebraic polarization. The real calculation does not manufacture that algebraic line bundle.

21.23 Global regularization on a real torus

Theorem 21.23.1. *Let u be continuous and Λ-periodic with $q+u$ convex. There are smooth periodic functions $u_j \to u$ uniformly such that $\theta + DDu_j$ is positive definite. Consequently continuous convex torus potentials belong to a single global uniform smooth-approximation domain.*

Proof. Choose nonnegative smooth compactly supported even kernels ρ_ε on V with integral one. Convolution $u_\varepsilon = u * \rho_\varepsilon$ is smooth and periodic and converges uniformly to u, since u is uniformly continuous on the compact torus. If $f = q + u$, then

$$f * \rho_\varepsilon = q + u_\varepsilon + a_\varepsilon, \qquad a_\varepsilon = \int q(y)\rho_\varepsilon(y)\,dy.$$

The first moments vanish by evenness. Convolution preserves convexity, so $q + u_\varepsilon$ is convex. For $0 < t < 1$, put $u_{\varepsilon,t} = (1-t)u_\varepsilon$. Then

$$DD(q + u_{\varepsilon,t}) = (1-t)DD(q+u_\varepsilon) + t\,DDq \geq t\theta > 0.$$

Take $\varepsilon \to 0$ and $t \to 0$. Boundedness of u gives uniform convergence. Every approximation is defined on the same torus and has the same automorphy rule after adding q. □

This argument is global because convolution respects translations by Λ. It does not extend unchanged to arbitrary tropical charts with changing tangent spaces, nor does it prove original-smooth regularization for every Berkovich current-psh function. Its geometric strength and its limitation are both visible in the translation symmetry.

21.24 The gradient map and total Monge–Ampère mass

Proposition 21.24.1. *If u is smooth periodic and $D^2(q+u) > 0$, then the gradient of $f = q + u$ is a diffeomorphism $V \to V^*$ and satisfies*

$$\nabla f(x+\lambda) = \nabla f(x) + H\lambda.$$

It descends to a diffeomorphism $V/\Lambda \to V^*/H\Lambda$.

Proof. The derivative identity follows by differentiating the automorphy rule. The positive Hessian is periodic, so compactness gives a uniform positive lower bound for its smallest eigenvalue. Hence f is strictly convex. For each $p \in V^*$, the function $f(x) - \langle p, x\rangle$ is coercive, because q is positive definite and u bounded. It has a unique minimizer, which satisfies $\nabla f(x) = p$. Thus the gradient is bijective. Its derivative is invertible everywhere, so the inverse function theorem gives a smooth inverse. Equivariance then gives the quotient diffeomorphism. □

Theorem 21.24.2. *Let dx be the coordinate Lebesgue measure and let* $\operatorname{covol}(\Lambda)$ *use this measure. For every continuous periodic u with $q+u$ convex, the continuous BT measure on M has total mass*

$$\int_M (\theta + DDu)^n = n! \det(H) \operatorname{covol}(\Lambda).$$

For $u = 0$ the measure is $n!\det(H)dx$, and its normalization to mass one is Haar probability measure on M.

Proof. For smooth strictly positive potentials, the local Hessian formula gives $(DDf)^n = n!\det(D^2 f)dx$. Proposition 21.24.1 and change of variables on the two tori give

$$\int_M \det(D^2 f)dx = \operatorname{covol}(H\Lambda) = \det(H)\operatorname{covol}(\Lambda).$$

For continuous u, use Theorem 21.23.1 and the continuous mixed-product convergence proved in Chapter 13. Compactness permits testing against the constant function one, so total mass passes to the limit. The last assertion follows because $D^2 q = H$ is constant. □

The equality of total masses is necessary for solving a Monge–Ampère equation, but is not an existence proof. The gradient construction above starts with a potential and computes its measure. Solving for a potential with a prescribed possibly singular measure is the additional nonlinear theorem in the real and non-Archimedean applications.

21.25 Uniqueness in the smooth strictly positive domain

Theorem 21.25.1. *Let u, v be smooth periodic potentials such that $\alpha = \theta + DDu$ and $\beta = \theta + DDv$ are positive definite. If $\alpha^n = \beta^n$, then $u - v$ is constant.*

Proof. Put $w = u - v$ and $S = \sum_{j=0}^{n-1} \alpha^j \wedge \beta^{n-1-j}$. The factorization of powers gives $\alpha^n - \beta^n = DDw \wedge S$. Since S is closed and M has no boundary, integration by parts yields

$$0 = \int_M w(\alpha^n - \beta^n) = -\int_M d'w \wedge d''w \wedge S.$$

Each summand in the last integrand is nonnegative. In particular the summand with α^{n-1} vanishes in integral. In coordinates diagonalizing the positive matrix of α at a point, this density is a positive constant times a positive-definite quadratic form in the first derivatives of w. For $n = 1$ it is simply the square of the derivative. A nonnegative continuous density with zero integral is zero everywhere. Thus $dw = 0$, and connectedness of the torus makes w constant. □

For continuous semipositive potentials, the same formal identity alone does not prove strictness: the mixed measure may degenerate where the gradient is nonzero, and a classical gradient may not exist. The general uniqueness theorem requires its own convex or pluripotential argument. The smooth result above should be cited with its strict positivity hypothesis, rather than extended by an unproved passage to equality in a limit.

21.25.1 Relation with abelian uniformization

For a totally degenerate polarized abelian variety, the polarization supplies the positive bilinear form and the automorphy data. Canonical weights have the quadratic form just described, up to the permitted affine normalization. Periodic continuous convex perturbations give the corresponding invariant semipositive metrics. The comparison of their Chern measures with these real Monge–Ampère measures is a geometric theorem; it includes the lattice and degree normalization. In mixed reduction, the Raynaud abelian part supplies additional residual intersection factors. A quadratic potential on the tropical torus alone does not encode them.

References. [BGJK25, Sections 7–8]; [Gub10]. The regularization, gradient normalization and smooth uniqueness proofs in this section are real calculations; the cited sources supply their analytic realization and the broader equation-solving theorems.

21.26 Convex duality, roof functions and toric heights

Curvature records slopes and their variation. A height also records the additive constants of a metric. Convex duality packages both kinds of information in a function on a bounded polytope. We derive the real-variable formulas first and then specify the comparison with Gubler–Hertel's toric local height. This separates the factorial normalization from the sign convention for the metric weight.

Let $M \simeq \mathbb{Z}^n$ be a lattice, let $N = \mathrm{Hom}(M, \mathbb{Z})$, and let $P \subset M_{\mathbb{R}}$ be a full-dimensional compact convex polytope. Lebesgue measure dp is normalized

by M. Its support function is $h_P(x) = \max_{p\in P}\langle p, x\rangle$ on $N_{\mathbb{R}}$. Consider finite convex functions f with $f - h_P$ bounded. Define

$$f^*(p) = \sup_{x\in N_{\mathbb{R}}} \big(\langle p, x\rangle - f(x)\big), \qquad \vartheta_f(p) = -f^*(p) \quad (p \in P).$$

We call ϑ_f the roof in this convex convention. It is concave and upper semi-continuous. Its domain is the polytope of slopes of f.

Proposition 21.26.1. *If $|f-h_P| \le C$ on $N_{\mathbb{R}}$, then f^* is finite on P, satisfies $|f^*| \le C$ there, and equals $+\infty$ outside P. Moreover*

$$f(x) = \sup_{p\in P}\big(\langle p, x\rangle + \vartheta_f(p)\big).$$

For two such functions f, g with the same P, one has

$$\|\vartheta_f - \vartheta_g\|_{P,\infty} = \|f - g\|_{N_{\mathbb{R}},\infty}.$$

Proof. The convex conjugate of h_P is zero on P and $+\infty$ outside P. On P, the inequality $\langle p, x\rangle \le h_P(x)$ gives an upper bound zero, attained at $x = 0$. If $p \notin P$, separation of p from the closed convex set P gives x with $\langle p, x\rangle > h_P(x)$; positive multiples of x make the supremum infinite. Reversing the inequalities $h_P - C \le f \le h_P + C$ under conjugation proves the domain and bounds.

At each x, a finite convex function on all of $N_{\mathbb{R}}$ has a supporting affine function. If its slope is p, then $f(y) \ge f(x) + \langle p, y - x\rangle$ for all y. The bounded asymptotics force $p \in P$, and this inequality gives $f^*(p) = \langle p, x\rangle - f(x)$. The reverse inequality for every p is the defining Fenchel inequality. Together they prove the reconstruction formula.

If $|f - g| \le \varepsilon$, conjugation gives $|f^* - g^*| \le \varepsilon$ on P. Applying the reconstruction formula in the other direction gives the reverse norm inequality. Hence equality holds. □

Thus uniform approximation of a metric weight is exactly uniform approximation of its roof in this fixed-polytope class. The roof does not discard constants: $\vartheta_{f+c} = \vartheta_f + c$. This is the information missing from a curvature measure.

21.26.1 The gradient map and total Monge–Ampère mass

Proposition 21.26.2. *Suppose f is smooth, has positive definite Hessian everywhere, and $f - h_P$ is bounded. Then its gradient is a diffeomorphism*

$$\nabla f : N_{\mathbb{R}} \xrightarrow{\sim} \operatorname{int} P.$$

Consequently, in the superform normalization,

$$\int_{N_{\mathbb{R}}} (DDf)^n = n! \operatorname{vol}_M(P).$$

Proof. Every tangent affine function to f has slope in P, by the asymptotic bound as in Proposition 21.26.1. The Hessian is invertible, so the gradient has open image; this image lies in $\operatorname{int} P$. For any p in that interior, there is $\delta > 0$ such that $h_P(x) - \langle p, x\rangle \geq \delta|x|$. Thus $f(x) - \langle p, x\rangle$ is coercive and has a minimum. Strict convexity makes its minimizer unique, and its derivative vanishes exactly when $\nabla f(x) = p$. This proves bijectivity. The inverse function theorem gives a smooth inverse.

The Jacobian determinant of the gradient is $\det D^2 f$. Change of variables gives $\int \det D^2 f \, dx = \operatorname{vol}_M(P)$. The factor $n!$ comes from expanding the nth wedge of the Hessian form, as in Section 21.5. □

For example, if $v_1, \dots, v_s$ are the vertices of P, the function $\log \sum_a e^{\langle v_a, x\rangle}$ has bounded difference from h_P and positive definite Hessian. Its Hessian is the covariance matrix of the vectors v_a with strictly positive weights. Full dimension ensures that no nonzero direction has zero variance.

The mass identity extends to all finite convex f in the stated asymptotic class by the Alexandrov definition. The union of its subdifferentials contains $\operatorname{int} P$, because the coercive minimization argument still applies, and is contained in P by the slope bound. The boundary of P has zero n-dimensional volume. Hence the total Alexandrov mass is $\operatorname{vol}_M(P)$, and the superform measure is $n!$ times that mass. This uses the classical Alexandrov measure theorem, including its agreement with the Hessian determinant in the smooth case.

21.26.2 Energy as the integral of a roof

Define the normalized convex energy by

$$\mathcal{E}_P(f) = n! \int_P \vartheta_f(p) \, dp.$$

It vanishes for h_P, is increasing in f, and satisfies $\mathcal{E}_P(f+c) = \mathcal{E}_P(f) + cD$, where $D = n!\,\mathrm{vol}_M(P)$. Proposition 21.26.1 also gives

$$|\mathcal{E}_P(f) - \mathcal{E}_P(g)| \le D\|f-g\|_\infty.$$

Proposition 21.26.3. *Let f_t be a smooth family of smooth functions with positive definite Hessians, the same polytope P, and bounded asymptotics. Assume $|\dot f_t|$ is uniformly bounded for t near zero. Then*

$$\frac{d}{dt}\Big|_0 \mathcal{E}_P(f_t) = \int_{N_\mathbb{R}} \dot f_0\,(DDf_0)^n.$$

Proof. For $p \in \operatorname{int} P$, let $x_t(p)$ be the unique point with $\nabla f_t(x_t) = p$. Differentiating $f_t^*(p) = \langle p, x_t\rangle - f_t(x_t)$ cancels the terms containing $\dot x_t$ and gives

$$\frac{d}{dt}\vartheta_{f_t}(p) = \dot f_t(x_t(p)).$$

The uniform derivative bound also bounds roof difference quotients, by the norm estimate. Dominated convergence on the finite-volume polytope permits differentiation under the integral. The change of variables $p = \nabla f_0(x)$ then proves the formula. The boundary of P has measure zero and does not enter this calculation. □

The positive-definite Hessian hypothesis gives the smooth inverse gradient maps used in this proof. Strict convexity alone would not justify the inverse-function step. An arbitrary path of PL maxima can have nondifferentiable corners moving through the support of its Monge–Ampère measure. One should then use the energy difference formula or uniform approximation, not evaluate an undefined pointwise derivative at a moving atom.

21.26.3 The toric metric convention and the factor $(n+1)!$

In Gubler–Hertel's convention, a toric section has a concave metric function ψ and a concave roof $\psi^\vee$ on the stability polytope Δ. To compare with our positive-Hessian convention, set $f = -\psi$ and $P = -\Delta$. Then

$$\vartheta_f(p) = \psi^\vee(-p).$$

The sign change on the slope lattice preserves lattice volume. It is not permissible to change ψ to $-\psi$ while keeping its slope polytope unchanged without this identification.

Theorem 21.26.4 (Toric local height; geometric comparison)**.** *For a proper normal toric variety of dimension n over a complete non-Archimedean field, a semipositive toric metric on a toric line bundle has toric local height*

$$\lambda_{\overline{L}}^{\mathrm{tor}}(X) = (n+1)! \int_{\Delta} \psi^{\vee}(m)\, d\,\mathrm{vol}_M(m) = (n+1)\mathcal{E}_P(f).$$

Here the toric local height is the difference from the canonical toric metric on the same line bundle, using the same auxiliary meromorphic sections. It is not an unnormalized local height independent of all choices by definition.

Open proof obligation (RC2). Supply the analytic-to-convex comparison, local-height induction and continuity in the prescribed metric class; the text offers GH Definition 2.6.1 and Theorem 2.6.6.

Example 21.26.5 (A one-dimensional normalization check)**.** Let $P = [0, d]$ with $d > 0$, and take $f(x) = \max(0, dx + a)$ for $a \geq 0$. Then

$$\vartheta_f(p) = \frac{a}{d}p, \quad DDf = d\delta_{-a/d}, \quad \mathcal{E}_P(f) = \frac{ad}{2}, \quad \lambda^{\mathrm{tor}}(f) = ad.$$

Indeed, the conjugate is $-ap/d$ on the interval, by maximizing at the corner $x = -a/d$. Integration gives the energy. Independently, with $h_P(x) = \max(0, dx)$, the curve energy formula gives

$$\frac{1}{2}\int_{\mathbb{R}} (f - h_P)(DDf + DDh_P) = \frac{1}{2}(0 \cdot d + a \cdot d) = \frac{ad}{2}.$$

Changing f to $f + c$ keeps curvature fixed, raises the roof by c, raises normalized energy by cd, and raises the self-height by $2cd$. Changing only one of the two height slots instead changes the height by cd.

For several line bundles, polarization replaces the single roof integral by the corresponding mixed integral. Its symmetry and multilinearity reflect those of local intersection theory. Gubler–Hertel extends the formulas to arbitrary rank-one non-Archimedean absolute values, so discreteness of the value group is not part of the toric height theorem. This is a different extension problem from enlarging a local sheaf of differential forms: the objects here are metrized line bundles and their numerical local heights.

References. [GH, Sections 2.4–2.6 and Appendix A]. The convex-domain, norm, gradient and variation calculations are proved above; the local-height identification retains its geometric comparison inputs.

21.27 Constructing toric potentials from a measure

The direct Monge–Ampère calculation starts from a convex potential and measures its subgradients. The inverse problem prescribes that measure. For finitely supported measures, the inverse problem reduces to a finite-dimensional convex minimization. This construction is particularly useful for understanding the existence theorem: the unknowns are the heights of finitely many supporting planes, and the prescribed masses are the volumes of the regions on which those planes dominate.

Let $P \subset M_{\mathbb{R}}$ be a compact convex body with nonempty interior, and let dp be lattice-normalized Lebesgue measure. Its support function is $h_P(x) = \sup_{p \in P} \langle p, x \rangle$. We first use the Alexandrov normalization $\mathrm{MA}_{\mathbb{R}}(f)(E) = \mathrm{vol}(\partial f(E))$; the top supercurrent is $n!\,\mathrm{MA}_{\mathbb{R}}(f)$.

Theorem 21.27.1 (A finite measure with prescribed masses)**.** *Let $x_1, \dots, x_s \in N_{\mathbb{R}}$ be distinct, and let $m_i > 0$ with $\sum_i m_i = \mathrm{vol}(P)$. There is a finite convex function f with $f - h_P$ bounded and*

$$\mathrm{MA}_{\mathbb{R}}(f) = \sum_{i=1}^{s} m_i \delta_{x_i}.$$

It is obtained by minimizing, modulo a common additive constant in $a = (a_i)$, the function

$$\Phi(a) = \sum_i m_i a_i + \int_P \max_i \bigl(\langle p, x_i \rangle - a_i \bigr)\, dp. \tag{21.27.1}$$

The minimizer is unique modulo that common constant.

Proof. Write $g_a(p) = \max_i (\langle p, x_i \rangle - a_i)$ on P. The function Φ is convex and continuous. Because the total masses agree, $\Phi(a + c\mathbf{1}) = \Phi(a)$. Normalize $\min_i a_i = 0$, so all $a_i \geq 0$. Choose C with $|\langle p, x_i \rangle| \leq C$ for all $p \in P$ and all i. Some $a_j = 0$, and hence $g_a(p) \geq \langle p, x_j \rangle \geq -C$. Consequently

$$\Phi(a) \geq \sum_i m_i a_i - C\,\mathrm{vol}(P).$$

Since every m_i is positive, the normalized sublevel sets are bounded and closed. A minimum exists. This coercivity argument does not require any general-position condition on the x_i.

The cells associated with a are

$$C_i(a) = \{ p \in P : \langle p, x_i \rangle - a_i \geq \langle p, x_j \rangle - a_j \text{ for every } j \}.$$

The equality set of two distinct planes lies in a hyperplane and has Lebesgue measure zero. Almost every $p \in P$ therefore has one active index. Differentiating the integral in a coordinate direction is justified by dominated convergence, since the difference quotient of a maximum is bounded by one. It gives

$$\frac{\partial \Phi}{\partial a_i} = m_i - \mathrm{vol}(C_i(a)).$$

At a minimizer, the derivative vanishes in every direction in $\mathbb{R}^s$. One may take an unrestricted representative, since the functional is constant along $\mathbf{1}$. Thus each cell has the prescribed volume.

Extend g_a by $+\infty$ outside P, and put

$$f(x) = \sup_{p \in P} \big(\langle p, x \rangle - g_a(p) \big).$$

The function g_a is continuous on P, convex, and closed after this extension. Hence $f^* = g_a$. Boundedness of g_a on P gives boundedness of $f - h_P$. Since $g_a(p) \geq \langle p, x_i \rangle - a_i$, one has $f(x_i) \leq a_i$. Equality holds on the nonempty cell $C_i(a)$, and therefore

$$f(x_i) = a_i, \qquad \partial f(x_i) = C_i(a).$$

Almost every point in the interior of P belongs to a unique active cell and is a differentiability point of g_a, with gradient x_i. The subgradient duality $p \in \partial f(x) \iff x \in \partial g_a(p)$ then implies that subgradients over points $x \notin \{x_1, \ldots, x_s\}$ occupy only a null subset of P. The boundary of a full-dimensional convex body also has Lebesgue measure zero. Thus the Alexandrov measure of f is exactly the asserted atomic measure.

For uniqueness of the minimizer, let a, b be two minimizers. Convexity implies equality in the integral convexity inequality at their midpoint. The pointwise nonnegative difference between the two sides consequently vanishes almost everywhere. At a point where the active indices of both maxima are unique, equality forces those indices to be the same. On the resulting full-measure set, $g_a - g_b$ belongs to the finite set $\{b_i - a_i\}$. By continuity, this holds everywhere on P. Since P is connected, $g_a - g_b$ is constant. Each $C_i(a)$ has positive volume, so its interior contains a point with the same unique active index for b; hence every $b_i - a_i$ equals that constant. □

This proof establishes uniqueness in the finite family used by the construction. Uniqueness among all convex solutions in the prescribed asymptotic

class is a further comparison theorem; it should not be inferred merely from the uniqueness of this finite-dimensional minimizer. The general uniqueness statement is part of the convex Monge–Ampère theorem cited below.

21.27.1 A compact-support existence theorem

Theorem 21.27.2. *Let μ be a positive compactly supported Radon measure on $N_{\mathbb{R}}$ of mass $\operatorname{vol}(P)$. There is a finite convex function f with $f - h_P$ bounded and $\operatorname{MA}_{\mathbb{R}}(f) = \mu$.*

Proof. Choose finitely supported positive measures μ_j of the same total mass, supported in a fixed compact set K, and converging weakly to μ. Such measures are obtained by partitioning K into Borel sets of diameter tending to zero and assigning each set's mass to one of its points. Ignore cells of zero mass. Apply Theorem 21.27.1 to obtain dual functions g_j on P.

All slopes of these maxima lie in K. For any Euclidean norms on the dual vector spaces, there is therefore one constant R such that every g_j is R-Lipschitz on P. Fix $p_0 \in P$ and add constants to arrange $g_j(p_0) = 0$. The functions are uniformly bounded by $R\operatorname{diam}(P)$. Arzelà–Ascoli gives a subsequence converging uniformly on P to a convex continuous function g.

Let $f_j = g_j^*$ and $f = g^*$, using $+\infty$ extensions off P. The elementary Legendre estimate gives

$$\sup_{x \in N_{\mathbb{R}}} |f_j(x) - f(x)| \le \|g_j - g\|_{C^0(P)}.$$

Thus the convergence is uniform even on all of $N_{\mathbb{R}}$, and $f - h_P$ is bounded. Local uniform convergence of finite convex functions implies weak convergence of their Alexandrov measures. Equivalently, one can use the continuous real Bedford–Taylor theorem together with the equality $(DDf)^n = n!\operatorname{MA}_{\mathbb{R}}(f)$. This standard convex-measure continuity is the one external analytic input in the limiting step. It yields $\operatorname{MA}_{\mathbb{R}}(f) = \mu$. □

Compact support was used to bound all slopes of the dual functions by one constant. For an arbitrary finite measure, atoms may escape every bounded set; the same compactness proof no longer gives uniformly Lipschitz roofs. The broader theorem permits unbounded metric potentials and requires an appropriate full-mass asymptotic condition. It is not the same assertion as existence of a continuous metric on a proper toric variety.

21.27.2 The one-dimensional formula

When $P = [\alpha, \beta]$ and $\sum_i m_i = \beta - \alpha$, the construction has a direct expression:

$$f(x) = \alpha x + \sum_i m_i \max(x - x_i, 0) + c.$$

The slope jumps by m_i at x_i, so $f'' = \sum_i m_i \delta_{x_i}$. The slopes at the two ends are α and β, giving the required bounded difference from h_P. The additive constant is the only freedom. This is the same slope-mass calculation that appears in metric-graph Green functions, but the boundary condition is now specified by a moment interval instead of a mean-zero normalization on a compact graph.

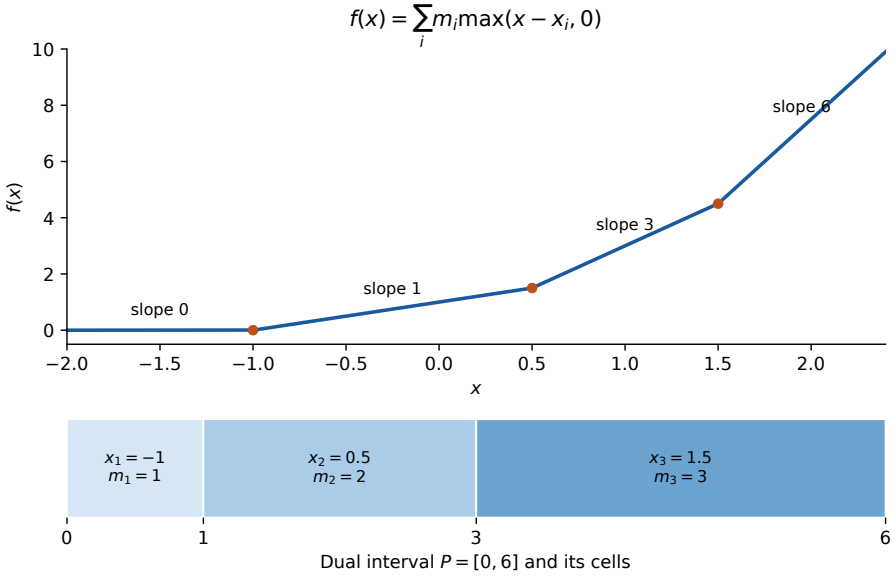


Figure 21.1: Prescribed atoms at $-1, 1/2, 3/2$ with masses $1, 2, 3$. The slope jumps of the primal potential are the lengths of the dual cells.

For a lattice polytope associated with an ample toric line bundle, the bounded convex functions constructed above correspond to continuous semi-positive toric metrics. Their non-Archimedean Chern measures are obtained by the tropical comparison theorem, with the factor $n!$ and the canonical section of tropicalization retained. The inverse construction on $N_{\mathbb{R}}$ does not assert that an arbitrary measure on the entire analytic toric variety is de-

termined by its tropical pushforward. The canonical invariant class and its section are part of the comparison.

References. This is a finite-dimensional proof of the compact-support case of the convex existence theorem used in [BGJK25, Theorem 6.2.13 and Appendix A]. The general theorem, its uniqueness, and its applications to toric metrics are treated there; the convex/metric dictionary is [GH, Theorem 2.5.8]. The minimization argument combines the Legendre correspondence with the volumes of the cells $C_i(a)$, and is included as a proof reconstruction rather than as a new existence result.

21.28 Toric metrics on subvarieties: the local weighted germ

A measure pushed to a tropical image does not by itself remember which analytic branch supplied its mass. This issue occurs both for subvarieties of abelian varieties and for subvarieties of toric varieties. It becomes essential when tropicalization is finite-to-one rather than injective.

21.28.1 The abelian case

In Gubler–Stadlöder's setting over an algebraically closed complete nontrivially valued field, Raynaud uniformization provides a lattice Λ, a real vector space $N_{\mathbb{R}}$, and affine cocycle functions z_λ. Continuous toric metrics on a line bundle L correspond to continuous functions satisfying

$$f(w+\lambda) = f(w) + z_\lambda(w).$$

When L is ample, convexity of f is equivalent to semipositivity of the metric. Their approximation theorem preserves this cocycle; ordinary convolution or an arbitrary polyhedral approximation without the cocycle would not define a metric on the quotient. For a closed subvariety $X \subset A$, the support S_X of the canonical measure is independent of the chosen ample L, and its map to the tropicalization is piecewise linear, surjective and finite-to-one. The measure formula on a polytopal decomposition of S_X has multiplicities depending on X, L and the face, but not on the chosen toric metric. The canonical metric gives a sum of Lebesgue measures on those faces. This statement does not say that all faces have dimension $\dim X$ or that all weights are one. The real Monge–Ampère measure on each face uses the affine space and normalization specified in that formula.

References. [GS23, Theorems 1.1–1.3, Proposition 3.8 and Corollary 1.4].

21.28.2 The toric-subvariety comparison

Lin considers a subvariety X of a toric variety Y over an algebraically closed complete nontrivially valued field. Put $d = \dim X$, and restrict to the torus locus. At a point x of the tropical skeleton, the relevant object is the weighted tropicalization of the analytic germ at x. A sufficiently small neighbourhood $U(x)$ gives a representative C_x of that germ. For a semipositive toric metric with tropical Green function g, Theorem A of [Lin26] gives

$$(\operatorname{trop}|_{U(x)})_*\big(c_1(L|_X, \|\cdot\|)^d|_{U(x)}\big) = \mathrm{MA}_{\mathrm{poly},C_x}(g).$$

On an open face Δ of dimension d, the corresponding formula reduces to $d!m_\Delta \,\mathrm{MA}_{\mathbb{R}}(g|_\Delta)$, transported by the face chart. Here the real measure is normalized by the face lattice and m_Δ is its tropical weight. On lower-dimensional faces the full weighted neighbourhood is generally needed. Restricting g to that face and taking its intrinsic real Hessian can lose transverse intersection mass.

Example 21.28.1 (Why a zero-dimensional restriction loses curvature)**.** On $\mathbb{R}$, let $g(t) = \max(0, t)$. Its restriction to the face $\{0\}$ is the constant zero function. Nevertheless $d'd''g = \delta_0$. The mass is the difference of the slopes on the two adjacent rays; it cannot be recovered from the value of the restricted function. This elementary example explains the need for a weighted germ. It is not a substitute for the higher-dimensional comparison theorem.

Lemma 21.28.2 (Pushforward does not identify branches)**.** *If a map $q : S \to P$ has distinct points s_1, s_2 with $q(s_1) = q(s_2)$, then equality of pushforward measures under q does not imply equality of measures on S.*

Proof. The distinct measures δ_{s_1} and δ_{s_2} have the same pushforward. □

Consequently a tropical equality becomes an analytic equality only after one has supplied a support theorem, branchwise local formulas, or a sufficiently faithful family of maps with compatible measures. The strong faithfulness and absolute continuity assumptions in Section 16.8 are one specific way of obtaining this control. They should not be suppressed when applying a tropical solution to an analytic equation.

References. [Lin26, Theorem A, Corollary B and Remark 9.8]; the two elementary examples above isolate the information lost by restriction or pushforward.

Part VI

Green objects, descent and limits

Chapter 22

Green objects and higher codimension

The geometric ingredients of the local Green construction are now available separately: completed residue fields and analytic fibres in Section 2.6, analytic flatness in Section 2.9, and Abhyankar dimension in Section 3.7. Proposition 2.6.4 gives the degree of a finite locally free fibre as a sum of lengths times residue-field degrees. In the argument below, proving that the required local map is finite and flat remains a geometric step; counting its underlying points would lose the multiplicity.

22.1 Bochner–Martinelli and Levine: what higher codimension changes

22.1.1 A complete intersection supplies a logarithmic maximum

Let X be a pure n-dimensional boundaryless analytic space in the source category. Let $f_1, \dots, f_r$ be a regular sequence defining a closed analytic subspace Y of pure codimension r. On $U = X \setminus Y$, set

$$u = \log \max(|f_1|, \dots, |f_r|).$$

This is finite and continuous on U. Its singularity along Y is exactly what is needed for a Green current. The source constructs the relevant products and extensions and states

$$\mathrm{d}'\mathrm{d}''\big(u(\mathrm{d}'\mathrm{d}''u)^{r-1}\big) = \delta_Y. \tag{22.1.1}$$

This is Theorem 18.7.2 of CLD25, explicitly attributed there to Mihatsch. For $r = 1$ it is Poincaré–Lelong.

The expression on the left has two levels. On U, the continuous-potential product is available. Extending the resulting current to X uses support and dimension results. The formula does not define an unrestricted product $(\mathrm{d}'\mathrm{d}''u)^r$ for every unbounded u on all of X. CLD itself points out this limitation immediately after the theorem.

22.1.2 The truncation argument and the nonformal step

For $\varepsilon > 0$ define

$$u_\varepsilon = \log \max(\varepsilon, |f_1|, \dots, |f_r|) = \max(\log \varepsilon, u).$$

The truncated function is finite and continuous. Its top relative curvature is described by a fibre over a small Gauss point in the parameter space. For a fixed test η of degree $(n-r, n-r)$, the support and dimension lemmas ensure that the differentiated test can be treated away from Y in the extension argument. On the relevant compact support, sufficiently small truncations agree with u.

The remaining step compares the calibrated fibre current with δ_Y. This uses the variation theorem from CLD25 Chapter 14 and the regular-sequence geometry. It is the step that retains the scheme-theoretic multiplicities. The equality is therefore a theorem about the analytic complete intersection, not merely an identity of real functions on a tropical support.

There are several local typographical slips in the printed proof, including the second expression for u_ε and dimension indices. The corrected truncation above is the one used in this exposition. The theorem is also available through Mihatsch's independent delta-form treatment, discussed in Chapter 16; the categories of representatives and tests must still be kept distinct.

22.1.3 The Green sign

With $\varphi = \min_i v(f_i) = -u$, the expression

$$\gamma = (-1)^{r-1}\varphi(\mathrm{d}'\mathrm{d}''\varphi)^{r-1} = -u(\mathrm{d}'\mathrm{d}''u)^{r-1}$$

has bidegree $(r-1, r-1)$ and satisfies $\mathrm{d}'\mathrm{d}''[\gamma] + \delta_Y = 0$ in the appropriate current realization. This computation reconciles the logarithmic-maximum convention in CLD25 with the valuation-minimum convention used in the delta-form literature. It also shows why the sign cannot be guessed from the divisor case without counting the $r-1$ additional factors.

22.1.4 Levine's formula on projective space

Let $X = \mathbb{P}^n$ and let $Y \simeq \mathbb{P}^{n-r}$ be the linear subspace defined by $x_0 = \cdots = x_{r-1} = 0$. On $U = X \setminus Y$, the rational projection

$$f : U \longrightarrow \mathbb{P}^{r-1}, \qquad [x_0 : \cdots : x_n] \longmapsto [x_0 : \cdots : x_{r-1}]$$

is a morphism. Give $\mathcal{O}(1)$ the standard formal metric, and write

$$\alpha = c_1(\mathcal{O}_{\mathbb{P}^n}(1)), \qquad \beta = c_1(f^*\mathcal{O}_{\mathbb{P}^{r-1}}(1)), \qquad h = \log \frac{\max_{0\le i\le n} |x_i|}{\max_{0\le i<r} |x_i|}.$$

The homogeneous ratios make h a well-defined function on U. Set

$$\omega = \sum_{j=0}^{r-1} \alpha^{r-1-j}\beta^j.$$

After the source's extension of $h\omega$ to a current Λ on X, Theorem 18.7.5 gives

$$\mathrm{d}'\mathrm{d}''\Lambda + \delta_Y = \alpha^r. \tag{22.1.2}$$

Away from Y, the algebra is transparent: $\mathrm{d}'\mathrm{d}''h = \alpha - \beta$, so

$$(\alpha - \beta)\sum_{j=0}^{r-1} \alpha^{r-1-j}\beta^j = \alpha^r - \beta^r = \alpha^r,$$

because β comes from a space of dimension $r-1$. The nonformal part identifies the singular contribution along Y. The proof blows up the centre of projection, uses Poincaré–Lelong for the exceptional divisor, and pushes the resulting intersection current down. The exceptional fibres supply degree one, producing δ_Y.

22.1.5 Why this does not solve every Green problem

The local formula uses a regular sequence of global functions on the chosen domain. The projective formula uses a specific linear cycle and a specific curvature term. A general homologically trivial cycle may not be globally a complete intersection, and local representatives may fail to glue in a prescribed differential-form sheaf. Even if a global Green current exists, it need not have an actual delta-form representative on the exact complement of the cycle.

Thus the formulas give substantial explicit cases and a proof mechanism. To reach a general strong Green statement, one must still specify the representative class, the full test class, the support condition, the gluing obstruction, and the descent under field extension. Chapter 24 turns these requirements into a precise comparison problem.

22.2 Green objects, star products, and local heights

22.2.1 The equation behind the construction

For the statements taken directly from original GK in this subsection and the next, work over a complete algebraically closed nontrivially valued field. The model examples can be compared after the base change just described. Mihatsch's statements below have their own analytic hypotheses, which we state separately.

For a codimension-p cycle Z, the integration current δ_Z has bidegree (p,p). A Green object g_Z has bidegree $(p-1,p-1)$ and satisfies an equation

$$\mathrm{d}'\mathrm{d}''g_Z + \delta_Z = \omega_Z, \tag{22.2.1}$$

where ω_Z belongs to a specified class of forms. One must always state that class. In classical complex arithmetic intersection it is a class of smooth forms; in GK it is a class of δ-forms; in the model theory discussed below, "forms" and "currents" are classes assembled from Chow groups of models. Sharing equation (22.2.1) does not make these categories identical.

For a metrized line bundle $\overline{L}$ with a nonzero meromorphic section s, the basic Green potential is $g_D = -\log\|s\|$, where $D = \operatorname{div}(s)$. The Poincaré–Lelong equation reads

$$\mathrm{d}'\mathrm{d}''[g_D] + \delta_D = c_1(\overline{L}).$$

It separates two sources of singularity: the zero or pole of the section, and the curvature of the metric. For a model metric, that curvature may itself be polyhedrally singular. This is precisely why the δ-calculus is useful.

Suppose D meets Z properly and the metric belongs to the GK class. The Green star product is

$$g_D * g_Z = g_D \wedge \delta_Z + c_1(\overline{L}) \wedge g_Z. \tag{22.2.2}$$

The first term means restrict the logarithmic potential to the cycle and push its current forward to X. Proper intersection ensures that this restriction is the permitted one. The second term is the action of an allowed δ-form on a current. These operations are part of the construction; the formula is not an invitation to multiply any two singular distributions. Applying $\mathrm{d}'\mathrm{d}''$ and using the Green equations gives

$$\mathrm{d}'\mathrm{d}''(g_D * g_Z) + \delta_{D\cdot Z} = c_1(\overline{L}) \wedge \omega_Z.$$

For two divisors, changing the order changes the representative by a d'- and d''-boundary. The class in the appropriate quotient is symmetric; integration on a proper space kills these boundaries by Stokes. This is the mechanism behind symmetry of local heights.

References. [GK17, Definition 11.1, §11.2, Propositions 11.4–11.5 and Corollary 11.6].

22.2.2 The local-height recursion and metric variation

Let X be proper of dimension n, and take $n+1$ metrized divisors $\widehat{D}_i$ with properly meeting supports, or the more flexible pseudodivisors of GK with empty total intersection of supports. The integral of their star product defines a local height. Its useful computational form is the recursion

$$\lambda_{\widehat{D}_0,\dots,\widehat{D}_n}(X) = \lambda_{\widehat{D}_0,\dots,\widehat{D}_{n-1}}(\operatorname{div}(s_n)) - \int_{X^{\mathrm{an}}} \log\|s_n\|\, c_1(\overline{L}_0)\cdots c_1(\overline{L}_{n-1}). \tag{22.2.3}$$

Each application replaces one intersection with a lower-dimensional cycle and one analytic integral. For model metrics, this agrees with the local height defined by model intersection theory.

If only the first metric is changed to $\|\cdot\|_0' = e^{-\rho}\|\cdot\|_0$, then

$$\lambda' - \lambda = \int_{X^{\mathrm{an}}} \rho\, c_1(\overline{L}_1)\cdots c_1(\overline{L}_n). \tag{22.2.4}$$

When the remaining metrics are semipositive, the mixed measure is positive and has mass $\deg_X(L_1\cdots L_n)$. Hence

$$|\lambda' - \lambda| \le \|\rho\|_\infty \deg_X(L_1\cdots L_n). \tag{22.2.5}$$

This estimate explains why uniform approximation can extend intersection constructions in a controlled positive class. A limit argument still needs the measures, positivity, and independence of approximation to be established; continuity of functions alone supplies none of those automatically.

References. [GK17, Propositions 12.4 and 12.6, Remark 12.7].

Example 22.2.1 (A height changes while curvature stays fixed)**.** Use the standard model metric on $\mathcal{O}_{\mathbb{P}^1}(1)$ from Example 20.3.1. Write $z = Z_1/Z_0$ and $t = -\log|z|$. The Green functions of the points 0 and ∞ are

$$g_0 = \max(0, t), \qquad g_\infty = \max(0, -t),$$

and both Chern measures are δ_{ξ_G}, where ξ_G is the Gauss point. Formula (22.2.3) gives

$$\lambda(0,\infty) = g_0(\infty) + \int g_\infty\, \delta_{\xi_G} = 0.$$

The closures of the two points in $\mathbb{P}^1_R$ are disjoint, giving the same model intersection. Now multiply the first norm by e^{-c}. Its Green function becomes $g_0 + c$, its curvature is unchanged, and its local height increases by c.

Thus even the simplest local height cannot be recovered from the two curvature measures alone. Later we shall encounter a further loss of information coming from degree-zero line bundles, but scalar changes already establish the point. A theory that retains metrized line bundles contains more arithmetic information than a theory of their Chern currents.

Remark 22.2.2 (The variation bound for a signed measure). Write $\mu = c_1(\overline{L}_1)\cdots c_1(\overline{L}_n)$ in (22.2.4). For a finite signed mixed measure, the general estimate is

$$|\lambda' - \lambda| \le \|\rho\|_\infty \|\mu\|_{\mathrm{TV}}.$$

If μ is positive, its total variation is its mass, and the degree formula gives (22.2.5). For signed μ, the signed mass need not control the total variation. Positivity is therefore essential to that particular degree bound.

22.2.3 Higher codimension: an explicit Green form and a length formula

Mihatsch supplies a particularly concrete higher-codimension construction. Let X be a good Hausdorff pure n-dimensional analytic space without boundary, and suppose $Z = V(f_1, \dots, f_r)$ is a complete intersection of codimension r. On $X \setminus Z$ set

$$\varphi = \min_i v(f_i), \qquad \gamma = (-1)^{r-1}\varphi(\mathrm{d}'\mathrm{d}''\varphi)^{r-1}.$$

Although φ is expressed using a minimum, the product is defined in the specified δ-algebra. The theorem states

$$\mathrm{d}'\mathrm{d}''[\gamma] + \delta_Z = 0, \tag{22.2.6}$$

including the corresponding integration identity against compactly supported δ-form tests. The codimension is visible in the exponent: γ has bidegree $(r-1, r-1)$, as a Green object should. For $r = 1$, it is the familiar logarithmic function $v(f)$.

A separate theorem recovers formal intersection lengths. Assume R is a DVR with $v(\pi) = 1$. Let $\mathfrak{X}$ be separated and flat over $\operatorname{Spf} R$, special, and pure of formal dimension $n+1$. Let $\mathfrak{Z}$ be a local complete intersection of pure codimension r, and let $\mathfrak{Y} \subset \mathfrak{X}$ be closed, Cohen–Macaulay, flat over R, and pure of formal dimension r. If $\mathfrak{Y} \cap \mathfrak{Z}$ is Artinian, the distance function defined by the ideal of $\mathfrak{Z}$ gives the same expression for γ, and

$$\int_{Y \setminus \partial Y} \gamma = \operatorname{length}_R \mathcal{O}_{\mathfrak{Y}\cap\mathfrak{Z}}, \qquad Y = \mathfrak{Y}^{\mathrm{an}}. \tag{22.2.7}$$

The theorem also proves the compact support needed for this integral. Flatness matters: a purely vertical formal subscheme can have empty generic fibre

while retaining a nonzero intersection length. The general local-complete-intersection Green statement appears separately as a conjecture in that paper; the proved length formula must not be confused with a proof of that whole conjecture.

References. [Mih24, Theorem 5.5, Conjecture 5.9, Corollary 5.10, and Theorem 5.11].

Example 22.2.3 (Intersection length and valuation scale)**.** In $\operatorname{Spf} R\langle x\rangle$, the sections $x = 0$ and $x = \pi^m$ have intersection ring

$$R\langle x\rangle/(x, x - \pi^m) \simeq R/(\pi^m).$$

Its R-length is m. The Green function $-\log|x|$ of the first section takes the value $-\log|\pi^m| = m\lambda$ at the generic point of the second, where $\lambda = -\log|\pi|$. Thus the logarithmic value equals the algebraic length under the normalization $v(\pi) = 1$; for the original absolute value it is the length multiplied by λ.

For global calculations, the 2025 BGK preprint adds another useful endpoint. Prescribed piecewise smooth Green data on an arbitrary projective variety can be expressed on a common extended tropicalization, and the corresponding local heights admit tropical formulas. The extended target includes strata at infinity; it is not merely one bounded graph. The result does not assert that every continuous singular metric is captured by one fixed finite tropicalization. It tells us that a finite collection of data in the stated class can be treated together, which is exactly what an intersection calculation requires.

References. [BGK25, Theorem F, Proposition 8.2.2, and Theorem 8.2.3].

22.3 Truncation for a coordinate complete intersection

The complete-intersection theorem combines a differential calculation with a geometric multiplicity calculation. To make both visible, we give the full truncation argument for the coordinate origin in arbitrary dimension. The one external measure input will be the standard projective model-measure formula of Chapter 20. This case explains the signs, the extension of the current and the convergence of the truncated curvature without assuming a general extension theorem.

Let K be complete, algebraically closed and nontrivially valued. Choose $a \in K$ with $0 < |a| < 1$, and put $X = (\mathbb{A}^r_K)^{\mathrm{an}}$, $r \geq 1$. Write $z_1, \dots, z_r$ for the coordinates and o for the K-rational origin. On $U = X \setminus \{o\}$ define

$$u = \log \max_i |z_i|, \qquad u_j = \log \max\bigl(|a|^j, |z_1|, \dots, |z_r|\bigr).$$

Each u_j is a finite Fubini–Study weight on the affine chart: it comes from the sections $a^j Z_0, Z_1, \dots, Z_r$ of $\mathcal{O}_{\mathbb{P}^r}(1)$. Theorem 20.8.1 makes it admissible for the local uniform smooth-potential calculus. The function u has the same property locally on U, because one coordinate is nonzero near each of its points.

Lemma 22.3.1 (Smooth forms near a rational point)**.** *Every original smooth function is locally constant near a K-rational analytic point. Every original smooth form of positive total degree vanishes on some neighbourhood of that point. For forms in one fixed finite local presentation, the vanishing neighbourhood may be chosen independently of the smooth coefficient functions.*

Proof. An original smooth presentation uses finitely many analytic units f_i. At a K-rational point x, the number $f_i(x)$ belongs to $K^\times$. Continuity gives a neighbourhood on which $|f_i - f_i(x)| < |f_i(x)|$. The ultrametric equality then gives $|f_i| = |f_i(x)|$ throughout that neighbourhood. Thus every tropical coordinate $-\log|f_i|$ is constant there. Smooth scalar composites are constant and their positive-degree differential factors vanish. Taking the intersection for the finitely many coordinates proves the uniform statement for a fixed presentation. □

The lemma is particular to the original smooth sheaf. It is not an assertion about all continuous functions near a rational point, nor about every enlarged differential-form sheaf. It explains why some logarithmic extensions are easier on original tests than on stronger test classes.

Lemma 22.3.2. *Let ξ_j be the Gauss point of the closed polydisc $|z_i| \leq |a|^j$. With the standard algebraic-degree normalization,*

$$(d'd''u_j)^r = \delta_{\xi_j} \quad \text{on } X.$$

Moreover $\xi_j \to o$ in X, and therefore these measures converge weakly on compactly supported continuous tests to δ_o.

Proof. The standard formal model of $\mathbb{P}^r$ has irreducible special fibre of multiplicity one, and $\deg_{\widetilde{\mathbb{P}^r_K}} \mathcal{O}(1)^r = 1$. The model-measure comparison therefore identifies its top Chern measure with the single Dirac mass at the standard Gauss point. Apply the projective automorphism

$$[Z_0 : Z_1 : \cdots : Z_r] \longmapsto [a^j Z_0 : Z_1 : \cdots : Z_r].$$

The pullback weight in the frame Z_0 is u_j, and the inverse image of the standard Gauss point is ξ_j. Functoriality for an analytic isomorphism proves the formula. This uses the model comparison, rather than inferring an analytic Dirac mass from a tropical vertex alone.

For a polynomial $f = \sum_\alpha a_\alpha z^\alpha$,

$$|f(\xi_j)| = \max_\alpha |a_\alpha|\,|a|^{j|\alpha|} \longrightarrow |a_0| = |f(o)|.$$

The analytic topology of affine space is generated by these evaluations, so $\xi_j \to o$. Evaluation of a continuous compactly supported test then proves convergence of the Dirac masses. □

Theorem 22.3.3. *The current $u(d'd''u)^{r-1}$ on U has a unique extension G of bidegree $(r-1, r-1)$ to X on original smooth tests. It satisfies*

$$d'd''G = \delta_o.$$

Consequently $\gamma = -G$ is a zero-curvature Green current for the coordinate origin.

Proof. A compactly supported test for G has bidegree $(1,1)$. By Lemma 22.3.1, it vanishes on a neighbourhood of o, so its support is a compact subset of U. Pair it with the current already defined on U. This defines G on all such tests. On a fixed finite presentation the neighbourhood of vanishing is common to all coefficients, so continuity reduces to continuity of the current on U. The inductive test topology then gives continuity on the full test space. If two extensions existed, their difference would be supported at o; every test vanishes near o, so that difference would be zero.

Let η be a smooth compactly supported scalar test. Its form $d'd''\eta$ vanishes near o. On its compact support, $\max_i |z_i|$ has a strictly positive minimum. For all sufficiently large j, the functions u_j and u therefore agree on a neighbourhood of this support. Locality of the admitted mixed product gives

$$\langle G, d'd''\eta\rangle = \left\langle u_j(d'd''u_j)^{r-1}, d'd''\eta\right\rangle = \left\langle (d'd''u_j)^r, \eta\right\rangle.$$

The second equality is the recursive mixed-product identity on the whole affine space, where u_j is finite and admissible. Lemma 22.3.2 identifies the last term with $\eta(\xi_j)$, which tends to $\eta(o)$. The distributional $d'd''$ convention gives $\langle d'd''G, \eta\rangle = \langle G, d'd''\eta\rangle$, proving the theorem. □

22.3.1 The algebraic multiplicity that the coordinate proof does not supply

Replace the coordinate equations by $z_1^{a_1}, \dots, z_r^{a_r}$ with positive integers a_i. Their common zero has the same underlying point, but the coordinate-ring quotient has basis

$$z_1^{b_1} \cdots z_r^{b_r}, \qquad 0 \le b_i < a_i,$$

and length $a_1 \cdots a_r$. The complete-intersection current is therefore $(a_1 \cdots a_r)\delta_o$. This number is not recoverable from the support set $\{o\}$ alone. A proof for the logarithmic maximum of these powered functions must retain the degree of the finite monomial map or carry out an equivalent intersection calculation.

For a general regular sequence, the geometric fibre theorem performs this task, including nonreduced fibres and inseparable multiplicities. The coordinate proof above should therefore be read as a full proof of its stated case and an explanation of the general proof's structure. It does not replace the variation-of-fibres theorem used in CLD25, nor the actual delta-form construction in Mihatsch.

There is a second distinction concerning tests. The extension in Theorem 22.3.3 was possible because original positive-degree tests vanish near a rational point. A theorem on all actual delta-form tests must verify its own extension and support statements; those tests need not satisfy the same vanishing lemma. Equality of the induced original currents does not by itself settle equality on the stronger test space.

References. [CLD25, Theorem 18.7.2]; [Mih24, Theorem 5.5]; [GK17, Theorem 10.5]. The proof above is restricted to the coordinate origin and explicitly uses the standard model-measure comparison.

22.4 Extension across a closed analytic subset

The coordinate proof extends because positive-degree original tests vanish near a rational point. In higher dimension the relevant inequality compares

the test degree with the dimension of the removed analytic subset. It is the test degree, not the current degree, which controls this argument.

Let X be a pure n-dimensional boundaryless analytic space in the original CLD smooth-test category. Let $Z \subset X$ be a closed analytic subset of dimension at most d, and put $U = X \setminus Z$. We use the following geometric input: for $\max(a, b) > d$, restriction induces a topological isomorphism

$$A_c^{a,b}(X) \longrightarrow A_c^{a,b}(U).$$

Its inverse extends tests by zero. In particular every such test on X vanishes on a neighbourhood of Z. This is the smooth-test dimension argument underlying CLD25 Lemma 7.1.13, using its Lemmas 3.3.6 and 6.4.5. We retain the test bidegrees explicitly: the complementary indices in the last sentence of the printed proof of 7.1.13 must not replace the degrees of the tests actually being paired.

Proposition 22.4.1. *If a current T on U has bidegree (p, q) and $\max(n - p, n - q) > d$, it has a unique extension $\widetilde{T}$ to X. It is given by*

$$\langle \widetilde{T}, \alpha \rangle = \langle T, \alpha|_U \rangle.$$

If the same degree inequality holds for a differential of T, then taking that differential commutes with this unique extension.

Proof. Take the continuous dual of the test-space isomorphism above with $(a, b) = (n - p, n - q)$. This proves existence, continuity and uniqueness. For the second assertion, restriction of the differentiated extension agrees with the differential on U. Uniqueness in its new degree therefore identifies it with the extension of that differential. The new-degree inequality is indispensable to this last step. □

For a codimension-r subset, $d = n - r$. A current of bidegree $(r - 1, r - 1)$ extends uniquely, since its tests have degree $(n - r + 1, n - r + 1)$. Its $d'd''$ has bidegree (r, r) and tests of degree $(n - r, n - r)$. The strict inequality has disappeared. A nonzero current supported on Z, such as δ_Z, is now possible.

Example 22.4.2. On the punctured analytic line, $u = \log|z|$ has $d'd''u = 0$. Its original-current extension satisfies $d'd''[u] = \delta_0$. Extending the zero current gives zero, whereas differentiating the extension gives δ_0. Thus an unqualified assertion that extension by zero commutes with differentiation is false when the relevant support condition fails in the differentiated degree.

22.5 The complete-intersection calculation with its geometric inputs

We now separate the analytic limiting argument from the geometric fibre comparison. Let $f_1, \dots, f_r$ define a pure codimension-r analytic complete intersection Z in X, and write

$$u = \log \max_i |f_i| \quad \text{on } U, \qquad u_\varepsilon = \max(\log \varepsilon, u), \quad \varepsilon > 0.$$

The latter expression means $\log \max(\varepsilon, |f_1|, \dots, |f_r|)$ on all of X. Assume the continuous-potential products in this calculation are available with locality and the recursion

$$d'd''\big(u_\varepsilon (d'd''u_\varepsilon)^{r-1}\big) = (d'd''u_\varepsilon)^r.$$

Assume also the following geometric comparison on the full original test space:

$$(d'd''u_\varepsilon)^r \rightharpoonup \delta_Z \quad (\varepsilon \downarrow 0). \tag{22.5.1}$$

This is the substantive input. In CLD25 it is obtained by identifying truncated products with calibrated fibre currents in Corollary 18.6.2 and then applying the fibre comparison in Proposition 18.7.3. It includes the multiplicities of the complete intersection.

Theorem 22.5.1. *Under these hypotheses, the unique original-current extension G of $u(d'd''u)^{r-1}$ satisfies $d'd''G = \delta_Z$.*

Proof. Proposition 22.4.1 gives G. Fix a compactly supported smooth test η of degree $(n-r, n-r)$. The form $d'd''\eta$ has degree $(n-r+1, n-r+1)$ and therefore vanishes near Z. Its support C is a compact subset of U. If C is empty, all pairings below vanish. Otherwise the continuous function u has a finite minimum M on C. Choose $\varepsilon < e^M$. By continuity and compactness, after decreasing ε if necessary, $u_\varepsilon = u$ on a neighbourhood of C. Locality gives

$$\begin{aligned} \langle d'd''G, \eta\rangle &= \langle G, d'd''\eta\rangle \\ &= \big\langle u_\varepsilon (d'd''u_\varepsilon)^{r-1}, d'd''\eta\big\rangle \\ &= \big\langle (d'd''u_\varepsilon)^r, \eta\big\rangle . \end{aligned}$$

Let ε tend to zero and use (22.5.1). The result is $\langle \delta_Z, \eta\rangle$. Every original test of the required degree was allowed, so this is the asserted current identity. □

Only a finite lower bound for u was used; the printed requirement that this bound be strictly positive is unnecessary and generally false. The small parameter is compared with e^M, not with M. This corrects both the truncation expression and the inequality used to make truncations agree on the test support.

22.5.1 What the fibre comparison must verify

CLD25 Theorem 14.7.2 is a theorem about compact analytic X satisfying Serre's condition (S_r), a regular r-dimensional target, and a morphism purely of relative dimension $n - r$. After fixing a moment map, it gives local constancy of the calibrated top-dimensional tropical fibre near a rigid target point. Its hypotheses should be retained when the theorem is used. In particular, the relative dimension in this application is $n - r$, not r.

If a fixed test has a finite chart decomposition to which this theorem applies, equality of calibrated fibres gives equality of its fibre integrals on each chart. Summing the chart contributions gives the desired equality for that test. The neighbourhood in the target may depend on the test and its chart decomposition. No uniform neighbourhood valid for every test is needed for weak convergence.

The reduction from a general regular sequence to those geometric hypotheses is not reproduced by the truncation calculation. In particular, Serre conditions concern localizations, not only the depth at one point. The present proof keeps the calibrated-fibre statement as an explicit input. It proves the complete analytic passage once that input is supplied, without claiming to have independently repaired every geometric reduction in the source.

22.5.2 Why extension uniqueness ends at the critical degree

For $s < r-1$, the same extension argument applies in both degrees (s, s) and $(s+1, s+1)$. Thus locality and uniqueness give

$$d'd''\big(u\widetilde{(d'd''u)^s}\big) = \widetilde{(d'd''u)^{s+1}}.$$

At $s = r-1$ the right-hand extension is no longer uniquely determined by its restriction to U. The geometric fibre limit selects the missing contribution δ_Z. This is the precise place where the analytic complete intersection, rather than only its complement, enters the Green identity.

References. [CLD25, Lemma 7.1.13, Theorem 14.7.2, Corollary 18.6.2, and Section 18.7]. The geometric inputs and the conditional analytic proof are distinguished above.

22.6 The algebra behind finite-flat multiplicities

The finite-flat step in a regular-immersion formula ensures that the same scheme-theoretic multiplicity appears on the zero fibre and on nearby tropical fibres. In the finite case, the relevant local flatness criterion has a short module-theoretic proof. This proof also shows why nilpotents must be retained.

Lemma 22.6.1. *Let $(R, \mathfrak{m})$ be a Noetherian regular local ring of dimension r, with regular parameters $t_1, \dots, t_r$. If a finite R-module M has this sequence as an M-regular sequence, then M is free. Its rank is*

$$\dim_{R/\mathfrak{m}} M/\mathfrak{m}M.$$

Proof. Induct on r. If $r = 0$, the ring is a field. For $r > 0$, put $t = t_1$. The quotient ring R/tR is regular local of dimension $r - 1$, and the remaining parameters are regular on M/tM. By induction, M/tM is free over R/tR, say of rank e. Lift a basis to M. Nakayama's lemma makes the resulting map $R^e \to M$ surjective. Let its kernel be K, which is finite because R is Noetherian.

Since t is a nonzerodivisor on M, one has $\operatorname{Tor}_1^R(M, R/tR) = 0$, as follows from the two-term resolution of R/tR. Tensoring the exact sequence with R/tR gives

$$0 \longrightarrow K/tK \longrightarrow (R/tR)^e \longrightarrow M/tM \longrightarrow 0.$$

The right-hand map is the chosen basis isomorphism, so $K/tK = 0$. Nakayama gives $K = 0$, proving freeness. Reduction modulo $\mathfrak{m}$ identifies the rank. □

Corollary 22.6.2 (Finite miracle flatness)**.** *Let $R \to A$ be a finite local homomorphism of Noetherian local rings, with R regular, A Cohen–Macaulay, and $\dim A = \dim R$. Then A is a finite free R-module.*

Proof. Parameters of R generate an ideal whose quotient in A is finite-dimensional over the residue field of R, hence artinian. Their images therefore

form a system of parameters of A. In a Cohen–Macaulay local ring every system of parameters is regular. Apply Lemma 22.6.1 to the finite R-module A. □

The remaining commutative-algebra input is the system-of-parameters characterization of Cohen–Macaulay rings. In the analytic application, one must separately establish finiteness of the map and the dimension equality at the relevant points. The presence of a regular sequence somewhere in the source does not establish those geometric facts.

Example 22.6.3 (Equal dimensions without flatness)**.** Let $R = k[[t]]$ and $A = R[\epsilon]/(\epsilon^2, t\epsilon)$. This is a finite local R-algebra of dimension one. As an R-module it is $R \oplus k\epsilon$, so it is not flat: t kills the nonzero element ϵ. Its generic fibre has dimension one over $k((t))$, while its special fibre has length two. The missing Cohen–Macaulay condition is detected by depth zero at the closed point.

Example 22.6.4 (Flatness retains nonreduced degree)**.** In contrast, $R[\epsilon]/(\epsilon^2)$ is free of rank two over R, with basis $1, \epsilon$. Every fibre has length two, although its reduced support has one point. Thus finite-flat degree counts scheme length, not the number of distinct points.

For a finite locally free morphism of rank e, tensoring a local free basis with any residue field shows that every fibre algebra has dimension e. This is the precise constancy used in a tropical weight comparison. It survives inseparability and nonreduced fibres. In a complete-intersection example $f_i = x_i^{a_i}$ in a regular coordinate ring, the zero-fibre quotient has basis $x_1^{b_1} \cdots x_r^{b_r}$ with $0 \le b_i < a_i$, hence length $\prod_i a_i$. The corresponding cycle in a Green identity must retain this multiplicity.

References. This is the finite-module proof of the local flatness criterion underlying [Mih24, Proposition 3.13]. Compare Stacks Project, Tags 00R4 and 07DY.

22.7 Regular sequences and the finite-flat comparison

The geometric input in the preceding truncation proof has an independent construction in Mihatsch's Proposition 3.13. The decisive points are Abhyankar points of the complete intersection. At these points the ambient

local ring has dimension exactly equal to the length of the regular sequence. This produces the Cohen–Macaulay hypothesis required for a finite-flat comparison. It does not require a Serre condition on an entire neighbourhood.

Lemma 22.7.1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension r. If $f_1, \dots, f_r \in \mathfrak{m}$ is an A-regular sequence, then A is Cohen–Macaulay. In this situation, if a finite local homomorphism $B \to A$ has regular local source of dimension r and $\dim A = r$, then A is flat over B.*

Proof. The sequence gives $\operatorname{depth} A \geq r$, whereas depth is at most dimension. Thus equality holds. For the second assertion, let $t_1, \dots, t_r$ be a regular system of parameters of B. Finiteness makes $A/(t_1, \dots, t_r)A$ artinian, so their images are a system of parameters of A. Every system of parameters in a Cohen–Macaulay local ring is regular. Corollary 22.6.2 now applies. Equivalently this is the finite case of miracle flatness. The first assertion supplies the Cohen–Macaulay assumption used in the second. □

The dimension hypothesis cannot be replaced by the mere existence of a regular sequence. In particular, a regular sequence of length r at one local ring does not imply that this ring satisfies (S_r) at all its localizations. The following example makes the issue explicit.

Example 22.7.2. For any field k and $r \geq 1$, put

$$A = \bigl(k[x, y, t_1, \dots, t_r]/(x^2, xy)\bigr)_{(x, y, t_1, \dots, t_r)}.$$

The elements $t_1, \dots, t_r$ form a regular sequence, but A does not satisfy (S_1), hence does not satisfy (S_r). Its underlying spectrum is pure of dimension $r + 1$, and the quotient by the sequence is pure of dimension one.

Proof. A polynomial variable is a nonzerodivisor over any coefficient ring, and this remains true after successively quotienting by the previous variables and localizing. Thus the sequence is regular. The nilradical is (x) and the reduced ring is the regular local ring $k[y, t_1, \dots, t_r]_{(y, t_1, \dots, t_r)}$, proving the dimension and purity assertions.

The nonzero class of x has annihilator $\mathfrak{p} = (x, y)$, a nonminimal prime. In $A_{\mathfrak{p}}$ this class remains nonzero and is killed by the maximal ideal. Hence $\operatorname{depth} A_{\mathfrak{p}} = 0$, while $\dim A_{\mathfrak{p}} = 1$. This violates (S_1). Every open neighbourhood of the closed point of $\operatorname{Spec} A$ contains its generization $\mathfrak{p}$, so shrinking around that point does not remove the failure. □

This is a counterexample to the unrestricted commutative-algebra inference, not to the Green identity. It explains why the sentence in the proof of CLD25 Proposition 18.7.3 asserting a neighbourhood with (S_r) needs justification beyond the regular sequence alone. Mihatsch instead makes the Cohen–Macaulay argument only at the relevant Abhyankar points, where the required dimension equality holds.

The openness theorem cited at this point, [Duc09, Theorem 4.4], is a theorem about the locus where the local ring has the specified property; its definition is given in Section 3.4 of that source. Openness does not prove that the complete intersection lies in that locus. In Example 22.7.2, the depth at the closed maximal ideal is at least r, but the local ring fails Serre's condition because one of its localizations has depth zero and positive dimension. This distinction between one depth inequality and the ring condition is the missing commutative-algebra issue.

22.7.1 The weighted product near a regular immersion

Let X be a pure n-dimensional affinoid analytic space and let $Z = V(f_1, \dots, f_r)$ be defined by a regular sequence. Put $m = n - r$,

$$U^{\times} = X \setminus V(f_1 \cdots f_r), \qquad U^{\times}_{\geq C} = \{x \in U^{\times} : v(f_i(x)) \geq C \text{ for all } i\}.$$

Choose a moment map $g : X \to \mathbb{G}_m^s$. Write $T(X, g)$ for its weighted tropicalization in the top dimension of X; lower-dimensional pieces of the set-theoretic image are not included in this notation.

Theorem 22.7.3 (Mihatsch's product theorem)**.** *For all sufficiently large C, one has an equality of weighted top-dimensional polyhedra*

$$T(U^{\times}_{\geq C}, (f, g)) = [C, \infty)^r \times T(Z, g),$$

where the first factor has its standard lattice weight.

Open proof obligation (RC2). Establish the analytic moment-image, boundary dimension and finite-fibre properness inputs in the exact regular immersion scope; the proof takes them from Mihatsch Proposition 3.13.

The proof keeps the regular sequence until the finite-flat step, so it retains nonreduced multiplicities. Replacing Z by its reduced support before this step would generally change e_σ. The product structure is not a product decomposition of the analytic space, and it does not say that every local test

is originally expressed in the coordinates (f, g). The latter issue is the reason for the refinement argument in the next section.

References. [Mih24, Proposition 3.13 and its proof]; [CLD25, Theorem 14.7.2 and Proposition 18.7.3]. The commutative-algebra inputs are the regular-sequence criterion for Cohen–Macaulayness and the local flatness criterion, Stacks Project, Tags 07DY and 00R4; Serre conditions are defined in Tag 031O.

22.8 The complete-intersection identity on delta-form tests

We now use Mihatsch's analytic delta-sheaf $B^{p,q}$, not the original smooth sheaf $A^{p,q}$. The support theorem, integral and Stokes theorem are those of [Mih24, Corollary 4.9, Definition 4.10 and Proposition 4.11]. They are part of the input calculus. In particular, equality will be tested on every actual compactly supported delta-form of the indicated degree.

Let X be pure of dimension n and boundaryless, and let $Z = V(f_1, \dots, f_r)$ be a complete intersection of codimension r. Distinguish the two open sets

$$U = X \setminus Z, \qquad U^{\times} = X \setminus V(f_1 \cdots f_r) \subseteq U.$$

On U, the function $\varphi = \min_i v(f_i)$ is finite and piecewise smooth. Near a point where some f_i vanish, discard those indices: their valuations are larger than the finite minimum on a sufficiently small neighbourhood. The remaining functions are analytic units there and give a tropical presentation of φ. Thus φ is defined on the exact complement U, not only on $U^{\times}$.

Lemma 22.8.1 (The diagonal current)**.** *Let $\phi(x) = \min(x_1, \dots, x_r)$ on $\mathbb{R}^r$, and let Δ be the diagonal with primitive parameter $t \mapsto (t, \dots, t)$. In the polyhedral delta-calculus,*

$$(-1)^{r-1}(\mathrm{d}'\mathrm{d}''\phi)^{r-1} = \delta_{\Delta}, \qquad (-1)^{r-1}\phi(\mathrm{d}'\mathrm{d}''\phi)^{r-1} = t\delta_{\Delta}.$$

Open proof obligation (RC2). Prove the fan computation with the stated diagonal support and coefficients; Mihatsch Lemma 5.7 is supplied.

Define the actual delta-form

$$\gamma = (-1)^{r-1}\varphi(\mathrm{d}'\mathrm{d}''\varphi)^{r-1} \in B^{r-1,r-1}(U).$$

Its support is contained in the locus where all finite $v(f_i)$ agree, hence in $U^{\times}$. On this locus it is the pullback of $t\delta_{\Delta}$. Away from Z, $\mathrm{d}'\mathrm{d}''\gamma = 0$. This

last identity follows from the affine function t on the closed diagonal and the differential rules in the delta-calculus.

22.8.1 Why one tropical presentation is insufficient

For an original smooth test, one can use a finite affinoid covering and inclusion–exclusion: the test is a pullback of a superform, and its high-degree restriction to lower-dimensional boundary images vanishes. For an arbitrary delta-form, its restriction to a skeleton can involve the other charts in its definition. The smooth argument cannot simply be repeated by replacing the letter A with B.

Fix $\eta \in B_c^{n-r,n-r}(X)$. Near $\mathrm{Supp}(\eta|_Z)$ choose finitely many affinoid domains K_i, moment maps g_i and presentations of η in their interiors. The support theorem puts $\eta|_Z$ on the top-dimensional skeleton of Z. Each such skeleton has finite fibres under its tropicalization. Shrinking the affinoids separates the relevant fibre points from the boundary. Choose a smooth partition of unity λ_i with support in these interiors and let $\varepsilon = \sum_i \lambda_i$, equal to one near $\mathrm{Supp}(\eta|_Z)$.

Lemma 22.8.2 (Local product identity with a cutoff)**.** *With these choices, let ν_i be the pushforward of $\lambda_i\eta|_Z$ from its skeleton under g_i. For sufficiently large C, and every polyhedral delta-form α on $\mathbb{R}^r$, the corresponding pushforward upstairs, restricted to $(C,\infty)^r$, is*

$$(f,g_i)_{\mathrm{trop},*}\big(f_{\mathrm{trop}}^{\star}\alpha \wedge \lambda_i\eta\big) = \alpha \times \nu_i.$$

All restrictions to the skeletal domain and to $\{\varphi > C\}$ are understood in this formula. Pushforward is taken only where the support is proper over the target.

Open proof obligation (RC2). Prove the supported cutoff product and its compact gluing step for actual delta tests; Mihatsch Lemma 5.8 is supplied.

Theorem 22.8.3 (Complete-intersection Green identity)**.** *For every actual $\eta \in B_c^{n-r,n-r}(X)$,*

$$\int_{X\setminus Z} \gamma \wedge \mathrm{d}'\mathrm{d}''\eta = -\int_Z \eta|_Z. \tag{22.8.1}$$

In particular $\mathrm{d}'\mathrm{d}''[\gamma] + \delta_Z = 0$ on the original smooth test space as well.

Open proof obligation (RC2). Prove degree vanishing for the actual supported delta tests after the cutoff; the proof applies Mihatsch equation (5.22).

This is the proof of [Mih24, Theorem 5.5], reorganized around the product theorem, the cutoff refinement and the one-dimensional integral. It supplies the complete-intersection identity on the full stated test class. It does not solve a Green problem for an arbitrary higher-codimension cycle, where global regular-sequence equations need not exist. Nor does the proof descend an actual form through an alteration. Those are different geometric problems, discussed in Chapter 23.

22.9 The blow-up proof of Levine's formula

We now prove the projective-space formula from the divisor identity and the degree of a projective-space fibre. This isolates the geometric step suppressed by the telescoping calculation on the complement. Work in the common algebraic setting of original GK, with K algebraically closed, complete and nontrivially valued. Use its delta-metrics, intersection products, and proper pushforward of currents.

Let $Y \subset \mathbb{P}^n$ be the linear subspace $X_0 = \cdots = X_{r-1} = 0$, with $1 \leq r \leq n$. Denote the blow-up by $\pi : \widetilde{X} \to \mathbb{P}^n$ and the resolved projection by $b : \widetilde{X} \to \mathbb{P}^{r-1}$. For $r = 1$, the blow-up in the Cartier divisor is the identity; the following formulas still apply.

Lemma 22.9.1. *The blow-up is the incidence variety in $\mathbb{P}^n \times \mathbb{P}^{r-1}$ defined by $X_i U_j = X_j U_i$ for $0 \leq i, j < r$. Its exceptional Cartier divisor E satisfies*

$$E \simeq Y \times \mathbb{P}^{r-1}, \qquad \mathcal{O}(E) \simeq \pi^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes b^* \mathcal{O}_{\mathbb{P}^{r-1}}(-1).$$

Proof. On an affine chart of $\mathbb{P}^n$, the ideal of Y is generated by the first r affine coordinates. In the blow-up chart $U_i \neq 0$, set $u_j = U_j/U_i$. Its equations are $X_j = X_i u_j$ for $j < r$. These are precisely the usual Rees-algebra charts of the blow-up of this coordinate ideal. They glue to the incidence variety. On Y, all the equations in the U-coordinates disappear, giving the product description of E. The pullback ideal becomes the invertible ideal $\mathcal{O}(-E)$, locally generated by X_i. The sections $X_0, \ldots, X_{r-1}$ of $\pi^*\mathcal{O}(1)$ have common factor cutting out E; after dividing by it, they generate $\pi^*\mathcal{O}(1) \otimes \mathcal{O}(-E)$ and define b. Hence this line is $b^*\mathcal{O}(1)$, which is the asserted line-bundle identity. □

Give both hyperplane lines their standard model metrics and set $a = \pi^*\alpha$, $\beta = b^* c_1(\overline{\mathcal{O}}(1))$. The quotient metric on $\mathcal{O}(E)$ makes the Green function of its canonical section, outside E, equal to

$$H = \log \frac{\max_{0\le i\le n} |X_i|}{\max_{0\le i<r} |X_i|}.$$

This is an equality of norms, not just of curvatures: it follows by evaluating the quotient of the two maximum norms in a chart $U_i \neq 0$. Poincaré–Lelong therefore gives

$$\mathrm{d}'\mathrm{d}''[H] = a - \beta - \delta_E.$$

Theorem 22.9.2. *Put* $\Omega = \sum_{j=0}^{r-1} a^{r-1-j}\beta^j$ *and define the current* $\Lambda = \pi_*([H]\wedge\Omega)$. *Then*

$$\mathrm{d}'\mathrm{d}''\Lambda + \delta_Y = \alpha^r.$$

On $\mathbb{P}^n \setminus Y$, *it is represented by the formula* $h\sum_{j=0}^{r-1}\alpha^{r-1-j}\beta^j$ *of Section 22.1.*

Proof. All factors of Ω are closed delta-forms, so multiplication of the divisor equation and the current Leibniz rule give

$$\mathrm{d}'\mathrm{d}''([H]\wedge\Omega) = a^r - \beta^r - \delta_E\wedge\Omega = a^r - \delta_E\wedge\Omega.$$

Here $\beta^r = 0$ because it is pulled back from the $(r-1)$-dimensional projective space. Proper pushforward commutes with the differential, and $\pi_* a^r = \alpha^r$ by the projection formula and the generic degree one of π. It remains to compute the exceptional term. On $E = Y\times\mathbb{P}^{r-1}$, the form a comes from Y and β from the second factor. Pair $\pi_*(\delta_E\wedge a^{r-1-j}\beta^j)$ with any complementary admitted test. The test also comes from Y after restriction to E. If $j < r-1$, the integrand has insufficient degree in the projective-space fibre, so its fibre integral is zero. For $j = r-1$, the fibre integral of β^{r-1} is $\deg \mathcal{O}_{\mathbb{P}^{r-1}}(1)^{r-1} = 1$. The remaining integral is exactly the test integral over Y. Hence $\pi_*(\delta_E\wedge\Omega) = \delta_Y$. This proves the current identity. Outside Y, the map π is an isomorphism, so the representative is the displayed one. The argument constructs a current globally and a specified form on the exact complement; it does not assert that arbitrary proper pushforward preserves actual delta-forms. □

The product structure of E was essential to this short proof. For a general regular immersion, the exceptional divisor is the projectivized normal bundle, and pushing powers of its tautological class produces Segre classes. The linear-subspace formula cannot simply be copied with a general cycle substituted for Y. Such a generalization requires both the normal-cone intersection formulas and admissible Green representatives.

22.10 Metric variation and completion of local heights

The one-metric variation formula gives a simultaneous continuity theorem without differentiating singular functions. Fix line bundles $L_0, \dots, L_n$ on a proper n-dimensional variety and fixed pseudodivisor data for which the initial local heights are defined. Suppose all metrics under consideration are semipositive, and the model or delta-metric height theory has symmetry and the one-metric variation formula.

Theorem 22.10.1. *For two metric tuples $\boldsymbol{m}, \boldsymbol{m}'$, put $\varepsilon_i = d(m_i, m_i')$ and $V_i = \deg_X \prod_{j \neq i} L_j$. Then*

$$|\lambda(\boldsymbol{m}) - \lambda(\boldsymbol{m}')| \leq \sum_{i=0}^{n} \varepsilon_i V_i.$$

Consequently local height extends uniquely to uniform limits of admitted semipositive metrics, with the fixed pseudodivisor data.

Proof. Replace the metrics one at a time, in the order $0, \dots, n$. The ith difference is the integral of the ith weight difference against a mixed positive measure formed from the other metrics, some old and some new. Its mass is V_i, since the underlying line bundles did not change. Its absolute value is at most $\varepsilon_i V_i$. Sum the $n+1$ inequalities. For uniformly Cauchy metric sequences, the estimate makes the height sequence Cauchy. If two sequences have the same limiting metrics, their mutual distances tend to zero, so the estimate makes their height limits equal. This proves existence and independence of approximation. Uniqueness follows from density of the original metric class in its uniform completion. □

Remark 22.10.2. This is a completion theorem after a height theory has been constructed on a positive class. It is not a construction of arbitrary mixed measures from uniform convergence alone. Symmetry, positivity, the degree formula and the one-metric variation identity are the inputs that make the proof work. For signed integrable metrics, expand into differences of semipositive metrics or use a uniform total-variation bound; the signed total degree alone is insufficient.

22.10.1 From local metric variation to global height variation

For a global field with fixed product-formula weights n_v, changing a metric only at finitely many places gives the sum of the local variations:

$$h' - h = \sum_v n_v \int_{X_v^{\mathrm{an}}} \rho_v \, c_1(\overline{L}_{1,v}) \cdots c_1(\overline{L}_{n,v}).$$

For point heights the mixed measure is replaced by the normalized finite orbit measure. A change by a principal scalar has local weights $-\log|a|_v$; the product formula makes the corresponding global sum zero. An arbitrary independent rescaling at each place does not have this cancellation. This is why local constants matter even though principal global changes leave heights unchanged.

References. [CLD25, Theorem 18.7.5]; [GK17, Sections 7, 11–12]; [GH]. The Levine proof uses only the stated divisor, projection and degree theorems after the explicit blow-up calculation.

22.11 Three meanings of a non-Archimedean Green object

The word Green is used for three different constructions in the recent literature. Their equations share some notation but their unknowns belong to different spaces. Keeping the categories explicit prevents a solution in one theory from being mistaken for a solution in another.

22.11.1 Model-theoretic currents

Dubocs works in the non-Archimedean Arakelov theory of Bloch–Gillet–Soulé. The ground data are an excellent discrete valuation ring and a proper smooth variety over its fraction field. Section 1.1 assumes the existence of regular proper flat models and common dominating regular models. These model hypotheses hold, for example, under the indicated resolution hypothesis; excellence alone is not stated to imply them. Forms and currents are built from bivariant intersection classes and Chow groups of models and their special fibres. They are not defined as continuous linear functionals on CLD test forms. Let F be a metrized vector bundle of rank r with a regular section whose zero scheme is Z. In [Dub26, Theorem 4.1], blow up Z, let s be the

canonical exceptional section, and let Q_F be the tautological quotient with its induced metric. The constructed model current is

$$\Lambda = \pi_*\big(\mathrm{div}_v(s) \wedge f^* c_{r-1}(Q_F)\big), \qquad dd^c\Lambda + \delta_Z = [c_r(F)].$$

The formula is useful because it expresses the higher-codimension Green current through an exceptional divisor and a quotient Chern class. When $F = L^{\oplus r}$, Theorem 6.1 gives the corresponding Levine formula. Neither statement asserts that Λ is an actual Mihatsch delta-form on $X^{\mathrm{an}} \setminus |Z|^{\mathrm{an}}$. A comparison would have to define realization maps, match Chern classes and pushforwards, and verify the identity against the required analytic tests.

22.11.2 Monge–Ampère Green functions

A Green function in Boucksom–Jonsson's pluripotential theory solves a nonlinear measure equation, for example $\mathrm{MA}_\omega(\phi_x) = \delta_x$ with their normalized measure. It is a scalar psh potential. Their Theorem A shows that a divisorial Green function is PL in dimension one but need not be PL in higher dimension. More strongly, Theorem B constructs on $\mathbf{P}^3_{\mathbf{C}((t))}$ a continuous potential whose Monge–Ampère measure has finite support in a dual complex, while the potential does not factor through the retraction of any SNC model. Thus a skeletal support theorem for a measure is not a skeletal factorization theorem for its potential. This result rules out a general shortcut from finitely many measure atoms to a potential on a finite skeleton. It does not contradict the comparison theorem in Section 16.8, where strong faithfulness, a pre-existing tropical solution and absolute continuity are additional hypotheses.

22.11.3 Actual forms for cycle equations

The strong Green problem considered here asks for an actual form of bidegree $(r-1, r-1)$ on the exact cycle complement, satisfying a linear $d'd''$ identity against all admitted actual tests. Mihatsch's complete-intersection result belongs to this third category. It has stronger representative and testing content than a current identity on original smooth tests, but retains its own geometric hypotheses. Dubocs's construction and Monge–Ampère Green functions provide comparisons and possible methods; neither removes those hypotheses. For zero curvature one must also eliminate the curvature term $[c_r(F)]$ or its analogue. Homological triviality of a cycle is not a construction

of a primitive in the actual delta-sheaf. Section 24.15 supplies a relevant vanishing theorem on closed CLD tests; Sections 22.8 and 23.4 explain the additional representative and descent requirements.

References. [Dub26, Section 1.1 and Theorems 4.1, 6.1]; [BJ24, Theorems A–B]; [Mih24, Theorem 5.5].

Chapter 23

Functoriality and descent

The analytic scalar extension X_L is constructed by completed tensor products in Section 2.6. That construction determines its points, functions and projection to X. It does not determine which ground-field calibration is used when integrating upstairs. We therefore specify the calibration before applying a degree or trace formula. The Galois-product example, Example 2.6.3, also shows why a descent argument must use the analytic fibre product rather than the Cartesian product of point sets.

23.1 Field extension, degree, and descent of representatives

23.1.1 Extending the field does not reset the valuation

Fix a complete valued field K and a finite extension L/K, with the absolute value extended from K. If π_K is a uniformizer in a discretely valued example, its value remains $-\log|\pi_K| = \lambda_K$ in L. A uniformizer π_L has value λ_K/e when the ramification index is e. Thus skeleton lengths and logarithmic weights are compared using one fixed real scale.

Normalizing the valuation independently in both fields by assigning value one to each uniformizer changes that scale. A formula that compares local heights or slopes across the extension must then include the corresponding conversion factor. The safest convention for an analytic descent problem is to retain the original absolute value throughout.

23.1.2 The finite flat degree formula and the base field

For a finite flat map $f : Y \to X$ of constant degree d over one fixed field, the calibrated projection formula gives $f_*[Y] = d[X]$ in the applicable algebraic setting. Scalar extension is normalized differently: with X_L integrated over L, the base-change projection satisfies $p_*[X_L]_L = [X]_K$. Section 23.2 explains and proves the distinction from the cited calibration theorem. A degree factor must therefore be attached to an explicit choice of ground-field calibration, not to the notation $X_L \to X$ alone.

23.1.3 What averaging can and cannot descend

If an object upstairs belongs to a class stable under the relevant finite trace, one may hope to define a descended object by $d^{-1}\pi_*$ and verify the desired identity by the projection formula. For currents, proper pushforward is often available. For an actual sheaf of delta-forms or another specified representative class, stability under that trace must be proved.

A pushed-forward current satisfying an equation is not automatically an actual form in the required sheaf. The operation may forget the local presentation or introduce singularities outside the admitted class. A descent theorem must show that the trace of the upstairs representative has local expressions downstairs and that those expressions satisfy the correct relations on overlaps.

23.1.4 Finite descent and alteration descent are different

A proper alteration is not a finite flat cover on the entire space. Exceptional loci, changes of relative dimension, and ramification can affect both support and representative regularity. The existence of a proper pushforward current remains useful, but dividing by a generic degree does not automatically reconstruct an actual form on the original complement.

For the strong Green problem, one must in addition control the support: the upstairs construction may require removing the inverse image of the cycle together with an exceptional divisor. A descended current on the original space does not prove that its representative is defined on the exact original complement. This support issue is independent of the normalization by degree.

23.1.5 Compatibility conditions for descent

To use an extension or a cover in a proof, establish in order: the upstairs equation on the full admitted tests; the trace operation on the actual representative class; commutation with d' and d''; the projection formula with the correct degree; and preservation of the precise support complement. If the goal is only an equality of currents, say so. If the goal is an actual Green form, the representative step cannot be omitted.

This template is particularly relevant to comparisons between CLD currents, GK delta-forms, and Mihatsch's analytic delta-forms. Their operations overlap, but the domains of trace and pushforward need not coincide. The correct comparison carries the object and its testing category through every step.

References. The finite-map and current projection formulas in [CLD25, Chapters 13–16]; the representative-sensitive constructions in [Mih24].

23.2 Two different degree formulas

The field of definition of an integral is part of its normalization. A finite map between spaces over one fixed field and a change of the ground field are different operations. The earlier version of these notes did not separate them clearly enough. We correct that ambiguity here.

Theorem 23.2.1 (Scalar extension; external geometric input)**.** *Let X be an analytic space of level n in the CLD25 framework, let L/K be a complete valued extension with the extended original absolute value, and let $p : X_L \to X$ be the scalar-extension projection. With the canonical calibrations over L and K, respectively,*

$$p_*[X_L]_L = [X]_K.$$

Equivalently, for an admissible compactly supported top-degree smooth form η on X,

$$\int_{X_L,L} p^*\eta = \int_{X,K} \eta.$$

There is no factor $[L : K]$ in this scalar-extension formula, even when the extension is finite.

This is CLD25, Propositions 14.3.6, 15.3.1 and 16.1.1: calibration commutes with scalar extension; the associated measures push forward without

changing mass; testing yields the current identity. The calibration theorem is imported here. Once it is given, the displayed integral and current identities follow directly from the definitions.

For a compact morphism $f : Y \to X$ between analytic spaces over the same field, CLD25 Proposition 16.1.2 instead gives $f_*[Y] = (\deg_c f)[X]$, with calibrated degree. In a finite flat algebraic situation where this is the usual constant degree d, the right side is $d[X]$. If L/K is finite, the underlying space X_L can also be viewed as a K-analytic space. Its K-calibration is then not the L-calibration in Theorem 23.2.1. One cannot retain the L-normalized integral and insert the K-degree factor at the same time.

Example 23.2.2 (Dimension zero)**.** The integration mass of $\operatorname{Spec} L$ over L is one. Viewed as a finite zero-cycle over K, its degree is $[L : K]$. Both statements are correct. The discrepancy is entirely the ground-field normalization. It already rules out a universal instruction to divide every pushed-forward base-extended current by $[L : K]$.

23.3 Pushforward on an actual test space

Proposition 23.3.1. *Suppose $f : Y \to X$ has a continuous pullback f^* on the smooth test spaces of interest, preserves admissible test supports, and commutes with d', d''. For a current T on Y define f_*T by*

$$(f_*T)(\eta) = T(f^*\eta).$$

Then pushforward commutes with d', d''. If pullback preserves the chosen positive cone, it preserves positivity of currents of the corresponding dual type. If a is a smooth form,

$$f_*\big(f^*a \wedge T\big) = a \wedge f_*T,$$

with the same wedge order on both sides.

Proof. Continuity of f^* makes the displayed functional a current. Apply the definition of a distributional derivative and $f^*d' = d'f^*$ to a test; the transpose sign is unchanged because the total bidegree shift under pushforward is even. The proof for d'' is identical. A positive test pulls back to a positive test, proving positivity. Finally, $f^*(a \wedge \eta) = f^*a \wedge f^*\eta$ identifies the two pairings in the projection formula. □

Proper analytic maps supply the required support control in the standard current formalism; more generally one may work with properness on the support of the current, provided a cutoff argument defines the pairing. A general map does not pull compact supports back to compact supports. Thus the same formula cannot be applied to every map without checking its domain.

Corollary 23.3.2. *Under these hypotheses, if $T_j \rightharpoonup T$ and all pushforwards are defined by the displayed test pullback, then $f_*T_j \rightharpoonup f_*T$. A current identity $\mathrm{d}'\mathrm{d}''G + \delta_Z = R$ upstairs pushes forward to*

$$\mathrm{d}'\mathrm{d}''(f_*G) + f_*\delta_Z = f_*R.$$

Proof. For each downstairs test, its pullback is one fixed upstairs test, so weak convergence applies. The second assertion follows by linearity and commutation with derivatives. □

23.4 The extra step for representatives

The preceding corollary produces a current. Suppose the intended conclusion is an actual form in a specified sheaf $B^{p,q}$ on $X \setminus |Z|$. One additionally needs a trace operation

$$\mathrm{Tr}_f : B^{p,q}(Y \setminus f^{-1}|Z|) \longrightarrow B^{p,q}(X \setminus |Z|)$$

with the appropriate degree shift when dimensions differ, compatibility with the current realization, and compatibility with the differentials. This map is additional geometric information. The continuous dual construction of f_*T does not provide it.

Lemma 23.4.1 (Exact support in finite descent)**.** *Let $f : Y \to X$ be a finite surjective map of topological spaces. If a representative upstairs is defined only on $Y \setminus (f^{-1}Z \cup E)$, then the directly available downstairs domain is at most $X \setminus (Z \cup f(E))$. To claim a representative on all of $X \setminus Z$, one must remove the obstruction along $f(E) \setminus Z$ by an extension theorem or a different construction.*

Proof. For $x \notin Z \cup f(E)$ every point of $f^{-1}(x)$ lies in the domain upstairs. For $x \in f(E) \setminus Z$ at least one point of that fibre is excluded, so a trace requiring data on the whole fibre cannot be evaluated from the given representative. A current extension may still exist, but that is a different assertion. □

Averaging over a finite group does not bypass this issue. For continuous functions on a topological quotient $X = Y/G$, the average $|G|^{-1}\sum_{g\in G} g^*u$ is invariant and descends continuously by the quotient topology. For differential representatives, invariance must be related to the downstairs sheaf by a separate descent theorem. The function argument alone neither identifies that sheaf nor proves its trace compatibility.

23.5 Finite harmonic maps of metric graphs

Finite maps of curves have a transparent one-dimensional model. A map of metric graphs may be one-to-one on its underlying points and still carry a nontrivial degree, because an edge is traversed with an integral expansion factor. Keeping this factor leads to the correct pullback of measures, trace of functions and Green-kernel normalization.

Let $\phi : \Gamma' \to \Gamma$ be a surjective map between finite connected compact metric graphs, linear on edges after subdivision, with no contracted edges. For every source edge e' mapping into e, let $m_{e'}$ be the positive integral ratio of target length to source length. At a vertex y above x, assume that

$$\sum_{e'\mapsto e \text{ at } y} m_{e'} = d_y$$

is independent of the outgoing edge e at x. This is the harmonicity condition; d_y is the local degree. At an interior point of an edge, the local degree is its expansion factor. We assume both graphs have at least one edge.

Lemma 23.5.1. *The sum $\sum_{y\in\phi^{-1}(x)} d_y$ is independent of $x \in \Gamma$. Denote it by $d = \deg\phi$.*

Proof. On the interior of a target edge, the inverse image consists of finitely many edge segments, and the sum of their constant expansion factors is constant. As one approaches an endpoint x, group these segments by the source vertex y to which they converge. Harmonicity makes the sum in each group equal to d_y, independently of the chosen target edge. The degree therefore has the same value on every adjacent edge. Connectedness finishes the proof. □

23.5.1 Trace and pullback

For a continuous function v on Γ', define its weighted trace by

$$(\operatorname{Tr}_\phi v)(x) = \sum_{y \in \phi^{-1}(x)} d_y v(y).$$

Proposition 23.5.2. *The trace is continuous, positive, and satisfies*

$$\|\operatorname{Tr}_\phi v\|_\infty \le d\|v\|_\infty, \qquad \operatorname{Tr}_\phi(\phi^* u) = du.$$

There is consequently a pullback of finite signed Radon measures, uniquely determined by

$$\int_{\Gamma'} v \, d(\phi^*\nu) = \int_\Gamma \operatorname{Tr}_\phi(v)\, d\nu.$$

It preserves positivity, multiplies total mass by d, and satisfies $\phi_*\phi^*\nu = d\nu$. In particular $\phi^*\delta_x = \sum_{y \mapsto x} d_y \delta_y$.

Proof. Continuity on edge interiors is immediate. At a vertex x, group the approaching inverse branches by their endpoint y. Continuity of v and harmonicity give the limit $\sum_y d_y v(y)$ from every outgoing direction. The norm estimate and positivity follow from positive local degrees. The representation theorem for continuous linear functionals on $C(\Gamma')$ defines the measure pullback. Applying its definition to 1 and to $u \circ \phi$ gives the mass and push-pull formulas. □

On a single edge of expansion m, the pullback of a measure with density $a(t)\,dt$ has density $m^2 a(ms)\,ds$ in a coordinate where $t = ms$. One factor m is the local degree and the other is the change of variable. Omitting either factor would break the Laplacian identity below.

Proposition 23.5.3 (Energy identities)**.** *For continuous edgewise H^1 functions,*

$$\begin{aligned} \mathcal{E}_{\Gamma'}(\phi^* u, \phi^* w) &= d\,\mathcal{E}_\Gamma(u, w), \\ \mathcal{E}_\Gamma(\operatorname{Tr}_\phi v, u) &= \mathcal{E}_{\Gamma'}(v, \phi^* u), \\ \mathcal{E}_\Gamma(\operatorname{Tr}_\phi v, \operatorname{Tr}_\phi v) &\le d\,\mathcal{E}_{\Gamma'}(v, v). \end{aligned}$$

Proof. On a source edge, $(u\circ\phi)' = m_{e'}u'\circ\phi$ and $dt = m_{e'}ds$. Its contribution to the first energy is therefore $m_{e'}\int_e u'w'\,dt$. Sum over all inverse edges and use the constant-degree lemma. On the interior of a target edge,

$$(\mathrm{Tr}_\phi\, v)'(t) = \sum_{e'\mapsto e} v'_{e'}(s_{e'}(t)).$$

This proves the second identity after change of variable. Weighted Cauchy–Schwarz gives

$$\left|\sum_{e'\mapsto e} v'_{e'}\right|^2 \le \left(\sum_{e'\mapsto e} m_{e'}\right)\left(\sum_{e'\mapsto e}\frac{|v'_{e'}|^2}{m_{e'}}\right).$$

The first factor is d; integrating the second over the target edge recovers the source energies. This proves the inequality and also shows that the continuous trace is edgewise H^1. □

23.5.2 Laplacians and vertex multiplicities

Theorem 23.5.4. *Use the convention $\Delta = -DD$ of Chapter 20. If v has a finite measure Laplacian on Γ', then*

$$\Delta(\mathrm{Tr}_\phi\, v) = \phi_*(\Delta v).$$

If u has a finite measure Laplacian on Γ, then

$$\Delta(\phi^* u) = \phi^*(\Delta u).$$

Functions here are continuous and edgewise H^1, and the equations hold against all continuous edgewise H^1 tests.

Proof. For the trace identity, pair with a test a on Γ and use Proposition 23.5.3:

$$\langle \Delta\,\mathrm{Tr}_\phi\, v, a\rangle = \mathcal{E}_{\Gamma'}(v,\phi^* a) = \int_{\Gamma'}\phi^* a\, d(\Delta v) = \int_\Gamma a\, d\phi_*(\Delta v).$$

For pullback, first let u be PL. Its Laplacian is concentrated at vertices, with coefficient minus the sum of outgoing slopes. At a source vertex y above x, the corresponding sum is

$$\sum_{e' \text{ at } y} m_{e'}\,\partial_e u(x) = d_y \sum_{e \text{ at } x}\partial_e u(x),$$

by harmonicity. This is the asserted weighted pullback of each atom. Subdivision permits the same calculation at any additional break point.

For general u, let $\nu = \Delta u$, which has total mass zero. Approximate ν weakly by finite atomic signed measures ν_j of mass zero and uniformly bounded total variation. The graph Green construction gives normalized PL solutions u_j converging uniformly to $u - u(z)$. In fact they converge in energy:

$$\mathcal{E}(u_j - u + u(z), u_j - u + u(z)) \le \|u_j - u + u(z)\|_\infty \, \|\nu_j - \nu\|_{\mathrm{TV}} \longrightarrow 0.$$

The pullback energy identity preserves this convergence. The bounded continuous trace map makes $\phi^*\nu_j \rightharpoonup \phi^*\nu$. Passing to the PL identity proves the pullback formula on the full energy test class. □

Example 23.5.5 (One skeletal point with degree greater than one)**.** The map $[0, L] \to [0, mL]$, $s \mapsto ms$, has one point in each fibre, but local and global degree m. Its trace is $v \mapsto (t \mapsto mv(t/m))$, and the pullback of an atom is $m\delta_{t/m}$. This is the skeletal normalization of the monomial map $z \mapsto z^m$ between annuli. The algebraic degree is retained by the expansion factor even though the skeletal map is a homeomorphism.

23.5.3 Green-kernel normalization under a finite map

Let μ have mass one on Γ, and set $\mu' = d^{-1}\phi^*\mu$. Both measures have mass one, so the normalized kernels of Theorem 20.15.5 are defined. For $x' \in \Gamma'$ and $y \in \Gamma$,

$$g_\mu(\phi(x'), y) = \sum_{y' \in \phi^{-1}(y)} d_{y'} g_{\mu'}(x', y'). \tag{23.5.1}$$

Proof. Both sides, as functions of x', have Laplacian $\phi^*\delta_y - \phi^*\mu$. Their difference is constant. The right side has zero μ'-mean by normalization. For the left side, the measure pullback formula and $\mathrm{Tr}_\phi(\phi^* a) = da$ give

$$\int_{\Gamma'} g_\mu(\phi(x'), y)\, d\mu'(x') = \int_\Gamma g_\mu(x, y)\, d\mu(x) = 0.$$

Thus the constant vanishes. □

A single lifted point does not generally give the pullback kernel; the weighted sum over its fibre does. These formulas are complete statements in the finite harmonic-graph category. To apply them to a morphism of analytic

curves, one must supply compatible skeleta and the theorem identifying local analytic degrees with the graph expansion factors and residual degrees. The existence of such skeleta is a geometric input, not a consequence of the energy identities.

References. The proofs combine the graph Laplacian and Green kernel of [BRg] with the harmonicity condition expressing local degree. They provide a one-dimensional verification of the normalizations used in the analytic projection formulas. The skeletal and slope geometry is treated in [GRW16]; no existence theorem for compatible skeleta is proved by these elementary graph calculations.

23.6 Boundary completions and arithmetic intersection

Local currents and global arithmetic intersection numbers are related through metrics, but have different domains of definition. Cai–Gubler's abstract divisorial spaces make the completion process itself part of the theory. For an open variety U and a boundary divisor B on a compactification, a sequence of metrized divisors is Cauchy when its differences, after passage to common models, are bounded between $-\epsilon B$ and ϵB, together with the corresponding inequalities for boundary Green functions. Two consequences matter for applications. First, this topology controls growth at the boundary; uniform convergence on each compact subset of U^{an} is not the defining condition. Second, the completion retains an underlying compactified geometric line bundle, not merely its restriction to U. Different boundary data can restrict to the same line bundle on U.

Example 23.6.1 (The open line does not determine its boundary extension)**.** On $U = \mathbf{A}^1 \subset \mathbf{P}^1$, every $\mathcal{O}_{\mathbf{P}^1}(a\infty)$ restricts to the trivial line bundle, but these extensions have degrees a. Thus the restriction to U forgets numerical data at infinity. Any comparison theorem requiring the same compactified object has strictly more input than an isomorphism of the restrictions.

Over a proper adelic curve in their sense, [CG26, Theorem 7.22] extends arithmetic intersection to arithmetically integrable compactified metrized divisors. The pairing is real-valued, symmetric and multilinear, agrees with the proper theory, and has the specified projection, base-change and boundary-continuity properties. The positivity and integrability classes are part of its domain. A positive local CLD current alone is not a membership test for

that domain. The relative-energy extension in Theorem 11.3 has a different range: it can take the value $-\infty$. It requires relatively nef metrics on compactified geometric objects admitting arithmetically nef reference metrics. The reference and the new metric must have the same underlying compactified geometric object, a stronger requirement than having the same line bundle on U. Mixed relative energy measures their change of arithmetic intersection. In the finite-energy part, this has the form

$$I(\mathbf{h}) - I(\mathbf{g}) = E(\mathbf{g}, \mathbf{h}).$$

Writing this as a difference presupposes that the displayed values are finite. The source formulates the extended-valued statement with its admissible energy class, so that no subtraction of two infinities is required. In a proof using the formula, finiteness must be checked rather than inferred from the notation.

Lemma 23.6.2 (Independence of a finite-energy reference)**.** *Suppose an energy E on a class of metrics satisfies the cocycle identity $E(a,c) = E(a,b) + E(b,c)$, all terms below are finite, and the known intersection values satisfy $I(b) - I(a) = E(a,b)$. Then the prescription $I(c) := I(a) + E(a,c)$ is independent of the reference a among those known metrics.*

Proof. Substitute the two assumed identities to get $I(a) + E(a,c) = I(a) + E(a,b) + E(b,c) = I(b) + E(b,c)$. □

The lemma explains the algebra of the extension. It does not prove the analytic construction, finiteness or cocycle property of relative energy; those are substantial inputs from [CG26]. It also separates this global completion theory from the local degree normalization under scalar extension discussed earlier in this chapter.

References. [CG26, Sections 2, 7, 10 and Theorem 11.3].

23.7 Finite changes of the ground field and local values

The two local calculations below use the original absolute values. The ordinary top-integral calculation uses NAG-002503, NAG-002511, NAG-002514 and NAG-002518 in the scalar-extension source `research_v040/f9_scala`

`r_extension_candidate.tex`, which accompanies this reconstruction as a separate article. Equality of archimedean local values remains open here.

Proposition 23.7.1 (Conditional place-weight summation)**.** *Let $F \subset E$ be number fields. Fix unsquared absolute values extending those on $\mathbb{Q}_p$ and the usual real absolute value, and put $c_v(F) = [F_v : \mathbb{Q}_v]/[F : \mathbb{Q}]$. Suppose the completion-degree identity $\sum_{w|v}[E_w : F_v] = [E : F]$ is supplied. Suppose real numbers a_v and a_w satisfy $a_w = a_v$ for $w \mid v$, and their weighted families are finitely supported or absolutely summable. Then*

$$\sum_w c_w(E)a_w = \sum_v c_v(F)a_v.$$

Proof. We first establish the completion-degree identity under the stated number-field hypotheses.

Finite places. Let F be a field with a specified real-valued nonarchimedean absolute value v. Let $H = \widehat{F}_v$, let E/F be a finite extension of degree n, and let $w \mid v$ mean an absolute value on E restricting to this very absolute value on F. Write E_w for the resulting completion. Set $C = H \otimes_F E$. There is a canonical indexing of the Artinian local factors C_w of C by $w \mid v$, with canonical valued residue field E_w. In particular

$$n = \sum_{w|v} \ell_{C_w}(C_w)[E_w : H].$$

For separable E/F all lengths are one.

A basis of E/F gives $\dim_H C = n$. Its ideals are finite-dimensional H-subspaces, so descending chains stabilize. By [StacksAlg, Tag 00JB, Lemma 10.53.6] it is the finite product of its localizations at its maximal ideals. For a maximal ideal P let $D = C/P$, a finite field extension of H. NAG-001450 supplies the unique extension of the absolute value of H, completeness of D, and equivalence with a finite-dimensional coordinate norm, without separability. The map $E \to D$, $a \mapsto 1 \otimes a \bmod P$, is a unital field embedding, so defines an extension w_P of v. The image of E is dense in D: the images of an F-basis e_i of E span D over H, and each $\sum_i h_i e_i$ is a limit of $\sum_i f_i e_i$ with $f_i \in F$. Indeed the error is bounded by $\max_i |h_i - f_i|\,|e_i|_{w_P}$. Thus D identifies isometrically with E_{w_P}, fixing E.

Conversely, for any extending w, the inclusion of F into E_w extends isometrically to H. Multiplication gives a map $C \to E_w$. Its image is a

finite-dimensional H-domain, hence a field: multiplication by each nonzero element is an injective linear endomorphism and therefore surjective. With its induced norm this field is complete by NAG-001450, hence closed in E_w. It contains the dense copy of E and consequently is all of E_w. Its kernel is maximal. Starting from P, the isometry from the previous paragraph agrees with both structure maps on E and on H (on H by density of F); thus the resulting kernel is exactly P. Starting from w, the norm on the residue field is the induced norm by uniqueness, so this construction recovers w. This proves the asserted bijection without counting embeddings. The algebra here is formed over the original, possibly incomplete, F; it is finite-dimensional over H and already complete for a coordinate norm.

The simple modules in a composition series of C_w are its residue field. Each therefore contributes exactly $[E_w : H]$ to the H-dimension. Adding the contributions of the full local factors gives the formula. To check separability, take a finite tower of simple generators for E/F. At each step the minimal polynomial of the new generator over the intermediate field divides a separable polynomial over F. It is coprime to its derivative. Starting with H, if the tensor algebra for the intermediate extension is a product of fields, tensoring with the next simple extension quotients a polynomial ring over each factor by the image of that minimal polynomial. The Bezout identity with its derivative survives in the field. The distinct monic irreducible factors are pairwise coprime, and the Chinese remainder map gives a product of fields again. Induction proves the length-one assertion. In characteristic zero every irreducible nonconstant polynomial is separable, since its derivative is nonzero and of smaller degree.

At a finite place v of a number field F over p, use the absolute value extending the specified one on $\mathbb{Q}_p$. The same result for $F/\mathbb{Q}$ shows that $F_v/\mathbb{Q}_p$ is finite. For $w \mid v$ in a finite extension E/F the identification above uses this $F_v \subset E_w$ isometrically. Since the number fields have characteristic zero, the length-one case gives $\sum_{w|v}[E_w : F_v] = [E : F]$ at each finite place.

Archimedean places. Let $F \subset E$ be number fields, $n = [E : F]$, and v an absolute value on F extending the usual absolute value on $\mathbb{Q}$. Count each absolute value w on E extending this fixed v once. With the usual unsquared moduli on all archimedean completions, one has

$$F_v \otimes_F E \simeq \prod_{w|v} E_w, \qquad \sum_{w|v} [E_w : F_v] = n.$$

If $F_v = \mathbb{R}$ and the factors comprise r real and s complex completions, then $r + 2s = n$. If $F_v = \mathbb{C}$, there are n complex factors, each of degree one over F_v.

Elementary field input. For a nonconstant complex polynomial P, the leading term implies $|P(z)| \to \infty$ as $|z| \to \infty$. A sequence approaching its infimum can therefore be taken in a bounded square. Successive subdivision into four closed subsquares, each time retaining one containing infinitely many sequence terms, gives a convergent subsequence by completeness of the real coordinates. Continuity gives a minimizer z_0. If $P(z_0) \neq 0$, write $P(z_0 + h)/P(z_0) = 1 + ch^k + O(|h|^{k+1})$ with $k \geq 1$ and $c \neq 0$. Choose δ among $1, -1, i, -i$ with $\operatorname{Re}(c\delta) < 0$. It has a unit-modulus k-th root u: writing $k = 2^a b$ with b odd, the b-th power permutes those four units, and successive square roots of a unit v are available since $(1 + v)/|1 + v|$ squares to v for $v \neq -1$, and $i^2 = -1$. For small positive t,

$$|P(z_0 + tu)/P(z_0)|^2 = 1 + 2\operatorname{Re}(c\delta)t^k + O(t^{k+1}) < 1,$$

a contradiction. Division by a linear factor and induction show that every complex polynomial splits. Conjugation of roots then shows that each irreducible real polynomial is linear or quadratic with a pair of nonreal conjugate roots. A quotient by such a quadratic is $\mathbb{C}$ over $\mathbb{R}$, by evaluation at either root: a nonreal root and 1 already span $\mathbb{C}$ over $\mathbb{R}$.

Completion factors. First take any number field D, with base $\mathbb{Q}$ and $H = \mathbb{R}$. More generally let D/B be a finite extension of number fields and fix a completion H of B already identified isometrically with $\mathbb{R}$ or $\mathbb{C}$. Write $A = H \otimes_B D$. Adjoin a finite generating list for D/B one element at a time. At each step the minimal polynomial over the preceding field is separable: in characteristic zero it is coprime to its derivative. Over each field factor reached so far the Bezout identity with its derivative persists. Factoring the resulting polynomial and applying the Chinese remainder identity for coprime factors gives a product of fields. By the preceding paragraph these fields are real or complex over a real factor, and complex over a complex factor. Starting with H yields a product decomposition of A; its H-dimension is $[D : B]$ by a basis. In particular all local lengths here are one.

Each factor carries the usual modulus, independent of the choice between conjugate identifications with $\mathbb{C}$. In fact any multiplicative absolute value ν on $\mathbb{C}$ extending the usual absolute value on $\mathbb{R}$ satisfies $\nu(i) = 1$, hence $\nu(z) \leq |\operatorname{Re} z| + |\operatorname{Im} z| \leq 2|z|$. Apply this inequality to z^j and let $j \to \infty$;

applying it also to z^{-1} gives $\nu(z) = |z|$. Thus each factor restricts to the specified absolute value on H.

The image of D in a factor spans it over H; it is dense because the finitely many coefficients of a linear combination can be approximated in B. The factor is complete, so is the completion of D for the induced w. Conversely, for any extending w, the inclusion of B in D_w extends isometrically to H. Multiplication defines $A \to D_w$. Exactly one primitive factor idempotent maps to 1 in this field, so the image is the corresponding factor. Its norm is its usual one by the preceding uniqueness argument (immediate for a real factor); it is therefore complete and closed. It contains dense D, hence equals D_w. These assignments are inverse: the completion isometry fixing D fixes the prescribed image of H by density of B, hence gives the same kernel in A. This identifies factors with places themselves, including exactly one factor per complex-conjugate pair over a real place.

Apply the argument first to $F/\mathbb{Q}$ and $\mathbb{R}$ to obtain the claimed alternatives for F_v. Apply it again to E/F with $H = F_v$. Taking F_v-dimensions gives precisely the two counts above.

These two calculations give the asserted degree identity for every place in the proposition. At infinity all moduli are unsquared.

The tower formula gives $c_w(E) = c_v(F)[E_w : F_v]/[E : F]$. Sum first over the finite set of w above each v and use the completion-degree identity above. Absolute summability permits regrouping. At the archimedean base place $\mathbb{Q}_v = \mathbb{R}$; a complex completion has degree two and its absolute value remains unsquared. □

For the non-Archimedean smooth top integrals in the native scalar-extension theorem NAG-002503, the local equality is $a_w = a_v$. The extension degree is already in the place weight and must not be inserted a second time into this native local value. The completion-degree identity holds at all places by the preceding proof. Equality of the actual archimedean local height values, the product formula, and summability remain separate inputs.

Finite-place matching of the local value in NAG-002523

The assertion to be applied is at lines 629–632 of the original `research_v040/f9_scalar_e`
Lines 613–614 take equality of the local values and summability as separate hypotheses. Fix a finite extension of number fields E/F, a finite place v and

a place w above it. Put $K = F_v$ and $L = E_w$, with the isometric inclusion for the original absolute values.

The local assertion. Let X be good Hausdorff and pure of analytic dimension d over K, let $\mathcal{F}$ be coherent on X_G, and let α be a compactly supported ordinary smooth real (d,d)-form on X. Take the entire scalar extension $p : X_L \to X$. Use on X_L the pulled-back coefficient and the ordinary form presented on inverse images of the presenting domains by the pulled-back units and the same real differential form. These prescriptions define the upstairs data independently of the presentations. Their native local value satisfies

$$a_w = \int_{X_L/L,p^*\mathcal{F}} p^*\alpha = \int_{X/K,\mathcal{F}} \alpha = a_v.$$

The same conclusion applies to independently specified upstairs data when an L-isomorphism identifies the space, coefficient and form with this triple.

Checking the object for data from a scheme. For example, let Z be of finite type over F with coherent module $\mathcal{M}$, and assume $X = (Z_K)^{\mathrm{an}}$ has the stated analytic hypotheses. The scheme obtained by extending Z_E to E_w is canonically $Z_K \times_K L$, by associativity on affine rings and compatibility with localization. By NAG-001709 its analytification is X_L, compatibly with the projection. On an affine $\mathrm{Spec}(R)$ of Z_K let M represent its coefficient, and let A be an analytic radius chart. For $A_L = A\widehat{\otimes}_K L$ the two coefficient constructions give the same A_L-module:

$$A_L \otimes_A (A \otimes_R M) \simeq A_L \otimes_R M \simeq A_L \otimes_{R\otimes_K L} (M \otimes_K L).$$

Here NAG-001719 gives the analytic chart description and NAG-001686 gives the coherent pullback. The isomorphisms commute with restriction. For the ordinary test the criterion is equality of its germ on every inverse-image chart with the pulled-back presentation. Scalar extension of the scheme and coefficient alone imposes no such condition on a separately chosen test.

Proof of the integral identity. NAG-002503, lines 354–384 of the scalar source, constructs the two pullbacks and proves independence on overlaps. The support of the pulled-back test lies in the inverse image of the original compact support. Cover the latter support by the interiors of finitely many affinoids downstairs. Their full inverse images are compact affinoids, and the

inverse image of the support is closed in their union, hence compact. The local weight calculation in NAG-002514 restricts to the entire simultaneous norm level of the units at an interior top-cell parameter. It uses the same transverse integral projection and lattice index on both sides. Write G_K, G_L for the two complete transverse Gauss fields and Q for the finite coefficient module on the downstairs transverse Gauss fibre. NAG-002511 identifies the upstairs module with $Q \otimes_{G_K} G_L$. Each local Artin factor contributes its module length times its full residue-field degree. Summing over all factors computes the vector-space dimension in the respective ground field, so the numerator of the cell weight is unchanged. NAG-002514 keeps the orientation and lattice measure. The compact chart inclusion–exclusion in NAG-002503, lines 386–410, now gives the claimed equality for the signed test.

Fix $x \in X$ and an affinoid neighbourhood $U = \mathcal{M}_K(A)$ of x. Put $H = \mathcal{H}(x)$. By NAG-001616 the entire inverse image of U has algebra $A_L = A\widehat{\otimes}_K L$. Since L/K is finite isometric and X_L is good Hausdorff, NAG-002518 (scalar source, lines 421–469) gives $(X_L)_{[K]}$ with the same points, domains, structure sheaf and completed residue fields. The bounded K-maps $A \to A_L$ present a morphism $(X_L)_{[K]} \to X$ over K. Apply NAG-001617 to this morphism on the entire inverse image of U. The whole fibre over x has H-affinoid algebra $A_L\widehat{\otimes}_A H$; its points and their completed residue fields are those over x in X_L.

Apply NAG-001689 to $K \to L$ and $K \to A \to H$. Here L is finite K-affinoid by NAG-002518, evaluation at x is bounded, and H is affinoid over its own complete coefficient field. Tensor symmetry and the bounded iterated pushout give an H-algebra isomorphism with bounded inverse

$$(A\widehat{\otimes}_K L)\widehat{\otimes}_A H \simeq H\widehat{\otimes}_K L, \qquad (a \otimes b) \otimes h \longmapsto a(x)h \otimes b.$$

Its inverse sends $h \otimes b$ to $(1 \otimes b) \otimes h$. A finite K-basis of L has bounded coordinate functionals by NAG-001450. Tensoring those functionals and the basis map with H gives inverse bounded coordinate maps between $H \otimes_K L$ with its projective tensor norm and $H^{[L:K]}$ with its maximum norm. Thus $H \otimes_K L$ is already complete. This finite H-algebra splits as $\prod_{y|x} B_y$ by Stacks Tag 00JB: each local residue field is finite over H, has its unique complete extending norm by NAG-001450, and gives exactly one analytic point of the factor. Nilpotents vanish under a seminorm. A composition series of B_y has successive quotients $\mathcal{H}(y)$, whence

$$\sum_{y|x} \operatorname{length}_{B_y}(B_y)[\mathcal{H}(y) : H] = \dim_H(H \otimes_K L) = [L : K].$$

This is the rank of the scalar fibre. The integral uses the native transverse dimension computed above. These local arguments apply to finite isometric extensions of complete nonarchimedean fields, including inseparable extensions, with the full residue degree and possible nilpotent lengths. For a coherent generic cycle its separate multiplicities are those of NAG-002506 and its finite pushforward has the degree of NAG-002509. Restriction of the constants of integration on the same upstairs space has the factor $[L : K]$ by NAG-002520.

Remaining premise for arithmetic descent. The finite-place result discharges the local-value premise for exactly the primitive integral displayed above. For a separately defined local height one still needs a native identification with that primitive on both realizations, or a base-change proof for its own defining operation. At infinity even the real-to-complex case requires a specified primitive and a proof comparing its two normalizations. The reviewed place-weight identity enters only after the local values are matched. The global regrouping in NAG-002523 requires finite support or absolute summability of the weighted families.

Inherited proof dependency. The integral application uses the F6 full-level argument through NAG-002503, NAG-002514, NAG-002405, NAG-002370 and NAG-002365 in that dependency order. The independent F6 receipt passes the specified NAG-002365 repair subject to its native prerequisites. Full support for this application requires the consuming baseline to include the repaired proof and to discharge its prerequisites. In the consuming English baseline, Theorem 5.41.1 contains that scoped repair.

23.7.1 The finite zero-dimensional calculation

For this subsection, let K be any field complete for a specified real-valued nonarchimedean absolute value, which may be trivial. Let $C \neq 0$ be a finite commutative K-algebra, M a finite C-module, and P an invertible C-module. Write $C = \prod_p C_p$ for its Artin local factors, $E_p = C_p/\mathfrak{m}_p$, and

$$w_p(M) = \operatorname{length}_{C_p}(M_p)[E_p : K] = \dim_K M_p.$$

The equality follows by a composition series. The degrees are full field degrees. A metric on P means, at each of these finitely many analytic points, a norm on

the one-dimensional E_p-space $P \otimes_C E_p$ compatible with the unique extending absolute value.

Proposition 23.7.2 (Finite heights, restriction of scalars, and norms)**.** *For a generator s of P, define*

$$h_{C,M}(P,h,s) = -\sum_p w_p(M)\log\|s(p)\|_h. \tag{23.7.1}$$

If $C \to D$ is a map of finite commutative K-algebras, N is a finite D-module, and the metric and section are pulled back, then

$$h_{D,N}(D\otimes_C P, q^*h, 1\otimes s) = h_{C,\mathrm{Res}_C N}(P,h,s).$$

If D is locally free of constant rank n over C and $N = D\otimes_C M$, the right side is $n h_{C,M}(P,h,s)$.

The K-line

$$\mathcal{N}_M(P) = \det_K(P\otimes_C M)\otimes \det_K(M)^\vee$$

has a metric characterized, for every generator e of P, by

$$\|\mathcal{N}_M(e)\| = \prod_p \|e(p)\|^{w_p(M)}.$$

It is well-defined, and $h_{C,M}(P,h,s) = -\log\|\mathcal{N}_M(s)\|$. For $M = C$ it is the determinant norm line $\mathrm{Nm}_{C/K}(P)$. For $h' = e^{-u}h$,

$$h_{C,M}(P,h',s) - h_{C,M}(P,h,s) = \sum_p w_p(M)u(p), \qquad |\Delta h| \le \|u\|_\infty \dim_K M.$$

Moreover (23.7.1) equals the frozen K-analytic integral of the smooth function $-\log\|s\|$ on $\mathcal{M}(C)$ with coefficient module M.

Proof inputs. NAG-001450 (finite extending absolute values); the already accepted Artin decomposition leaf STACKS-00JB; NAG-002362, NAG-002405, NAG-002424, NAG-002425 (zero-dimensional module weights, chart integration and ordinary germs); elementary finite-dimensional determinants and composition series..

Proof. A multiplicative seminorm on C kills its nilpotents. Its kernel selects one local factor and its maximal ideal, and the induced norm on the finite field E_p is unique by NAG-001450. These are precisely the points of $\mathcal{M}(C)$ with

its canonical finite topology. The idempotent factors are open and closed. On each factor a function on this finite point set is constant, hence is an ordinary smooth moment function.

Every invertible C-module has a generator: it is free on each local factor, and the finitely many local generators combine using the idempotents. If $a \in C^{\times}$, a composition series of M_p is stable under multiplication by a. Each simple quotient is E_p with multiplication by its residue $a(p)$. A basis respecting the series gives block triangular matrices and hence

$$|\det_K(m_a \mid M)| = \prod_p |\operatorname{Nm}_{E_p/K}(a(p))|^{\operatorname{length}_{C_p} M_p} = \prod_p |a(p)|^{w_p(M)}. \quad (23.7.2)$$

For completeness, the last equality does not count separable embeddings. If $b \in E_p^{\times}$ has minimal polynomial f, the characteristic polynomial of multiplication by b on E_p is $f^{[E_p:K(b)]}$, by a $K(b)$-basis. A finite splitting extension for f is obtained by adjoining roots successively. Each root has the same absolute value as b: the corresponding embedding of $K(b)$ into that finite extension pulls back an extending absolute value, unique by NAG-001450. Taking the product of all roots *with their multiplicities* gives $|\operatorname{Nm}_{E_p/K} b| = |b|^{[E_p:K]}$, including inseparability.

Replacing a generator e by ae multiplies $\mathcal{N}_M(e)$ by $\det_K(m_a \mid M)$ and multiplies the proposed norm by exactly its absolute value, by (23.7.2). Thus the prescription defines a norm on the K-line and gives the asserted identity for s. If $M = 0$, both determinant lines have their canonical unit and all sums are zero; the same argument applies.

For restriction of scalars, the p-part of N is the direct sum of the D-local parts lying over p. Its K-dimension is the sum of theirs. Uniqueness of extending absolute values makes the pulled-back metric at each such point have exactly the norm of $s(p)$. Grouping the finite sum proves the projection formula. A locally free rank-n D over the Artin semilocal C is free of that rank, by choosing bases on each factor. Hence $D \otimes_C M \simeq M^n$ as C-modules, giving the rank formula. Subtracting the two log norms proves the variation and its bound.

Finally, compute the frozen integral separately on the clopen factors. For the empty moment tuple on C_p the rank-zero tangent lattice has index one, the Gauss field is K, and the full finite algebra is C_p. The module weight of NAG-002362 is therefore $\dim_K M_p$. The global-chain construction NAG-002405 supplies this chart and its interior whole-fibre weight; in dimension zero its boundary image is empty. Rank-zero polyhedral integration evaluates

the constant at the single point. NAG-002425 identifies that chart sum with the ordinary compact integral. This gives exactly (23.7.1). □

Intersection meaning and its boundary. For a specified zero-dimensional intersection represented by a finite K-algebra C, its coherent coefficient M contributes the actual length $\operatorname{length}_{C_p} M_p$ at p, followed by the ordinary residue degree $[E_p : K]$. In particular $C = L[\epsilon]/(\epsilon^r)$ has total weight $r[L : K]$. The determinant on the scalar unit $a \in L^\times$ has absolute value $|a|^{r[L:K]}$. Neither the reduction L nor a separable point count gives this normalization. This identifies the finite base case; it does not construct a positive-dimensional cycle product.

Finite scalar extension of the Artin local height

The zero-dimensional starting value required at lines 674–691 of the F9 descent supplement is defined in NAG-002635, `F9_Local_Height_Interface_v038.tex`, lines 197–214. The projection at lines 215–222 uses a single ground field. The following application compares the same primitive in its two native ground fields.

Let K be complete nonarchimedean, let L/K be finite isometric and let $C \neq 0$ be a finite commutative K-algebra with its canonical finite topology. Take a finite C-module M, an invertible C-module P, a generator s of P, and a metric h as in NAG-002635. Thus at each local factor C_p the metric is a norm on the line over $k_p = C_p/\mathfrak{m}_p$ for its extending absolute value. Write M_p for the corresponding summand and $s(p)$ for the residue of s. The primitive is

$$h_K = -\sum_p \ell_{C_p}(M_p)[k_p : K] \log \|s(p)\|_h.$$

Form $D = C \otimes_K L$, $N = M \otimes_K L$, and the line and generator P_L, s_L obtained by tensoring. All finite tensors carry their canonical complete topologies. Give P_L the pullback metric. Then its height h_L, defined by the same formula over L, equals h_K.

Proof. Each idempotent of the factorization $C = \prod_p C_p$ acts on D. Decomposing its image further into local Artin algebras gives $C_p \otimes_K L = \prod_{q|p} D_q$ (Stacks Tag 00JB). Write $k_q = D_q/\mathfrak{m}_q$. These are all the analytic lifts of p: a bounded multiplicative seminorm kills nilpotents and factors through one k_q, and each k_q gives a point by the unique complete extending absolute

value (NAG-001450). Its restriction to k_p is the original one by the same uniqueness over K. More precisely the analytic point fibre over p has algebra $k_p \otimes_K L = (C_p \otimes_K L)/\mathfrak{m}_p(C_p \otimes_K L)$. The quotient map has nilpotent kernel, so the quotient indexes the same lifts. The coefficient at a lift is computed in D_q on N_q, retaining the original module and the full base-changed algebra.

For a local frame e of the k_p-line the transported norm is $\|ae_L\| = |a|_{k_q}\|e\|_h$ for $a \in k_q$. Isometry on k_p makes this independent of e; in particular $\|s_L(q)\| = \|s(p)\|$. The generator has positive norm at every point. A composition series of N_q has quotients k_q. The decomposition by idempotents and extension of a K-basis of M_p therefore give, for each p,

$$\sum_{q|p} \ell_{D_q}(N_q)[k_q : L] = \sum_{q|p} \dim_L N_q = \dim_L(M_p \otimes_K L) = \dim_K M_p = \ell_{C_p}(M_p)[k_p : K].$$

Multiplying by $-\log\|s(p)\|$ and summing proves the claim. This argument includes zero coefficient summands and inseparable extensions. When the same upstairs data are measured over K, the weight at q uses $[k_q : K] = [k_q : L][L : K]$. Thus the restricted value is $[L : K]h_L$.

Identification of the integral. This integral interpretation uses NAG-002635 through NAG-002405 and NAG-002370 to the same scoped NAG-002365 repair in Theorem 5.41.1. The consuming English baseline contains that repair. Its remaining proof prerequisites still require verification. The equality of finite Artin sums above uses the finite field norms and Artin lengths directly.

NAG-002635, lines 242–258 and 299–308, identifies this specific height with an ordinary compact $(0, 0)$-integral over either field. The function $-\log\|s\|$ is constant on each clopen local factor. Its empty moment tuple has lattice index one and coefficient $\dim_K M_p$ downstairs, respectively $\dim_L N_q$ upstairs. The pullback test on every lift has the same value proved above. Consequently a finite place v and $w \mid v$ in NAG-002523 can take $a_v = h_{F_v}$ and $a_w = h_{E_w}$ for these actual data.

Boundary of the application. NAG-002523, lines 606–634, assumes equality of the real local numbers and leaves archimedean comparison as a separate input. NAG-002635 assumes a nonarchimedean base at lines 57–63. The recovered passages supply neither an archimedean height primitive on the real and complex realizations nor a convention for its comparison over

a real place becoming complex. The all-place degree identity fixes the place weights only. For a higher-dimensional DVR model number, NAG-003211, lines 428–457 of its source, requires a proved downstairs smooth realization. The Artin formula supplies the dimension-zero case only. Finite support or absolute summability of the weighted family is a separate global hypothesis.

Chapter 24

Limits, cohomology and remaining comparisons

24.1 Topological limits, differential limits, and Hodge questions

24.1.1 Payne's inverse-limit theorem

Payne's inverse-limit theorem reconstructs an analytification from a sufficiently rich system of tropicalizations as a topological space. A point is recovered from its compatible values in all those tropicalizations. The system of embeddings and transition maps is part of the theorem.

A sheaf of differential forms is additional data. To recover it as a direct limit of tropical sheaves, one must compare the local presentations, quotient relations, and restriction maps. A current is a continuous dual, so one must also compare topologies. Cohomology introduces another passage to kernels and images. None of these operations is determined solely by a homeomorphism of underlying spaces.

24.1.2 A concrete obstruction to an automatic Hodge transfer

Even if a filtered direct limit of complexes is exact in an algebraic category, the problem at hand may involve topological vector spaces, support conditions, and completed or continuous duals. Images need not be closed, and a primitive available on one tropical presentation may acquire a different sin-

gular support after refinement. A claimed Hodge or Aeppli comparison must therefore specify the category in which the limit is taken.

The curve calculation in Chapter 17 is an effective sanity check. Its weakly smooth Dolbeault dimensions involve the first Betti number of the analytic topology, while the algebraic genus also contains residual contributions. A proposed identification with algebraic Hodge theory must explain those contributions rather than assuming the two genera coincide.

24.1.3 Differential and cohomological comparison requirements

First compare finite-chart smooth complexes on a fixed relatively compact analytic domain. Next prove compatibility under refinement, including the zero relations and test topology. Then identify which compact support condition is preserved. Only after these steps should one compare cohomology or seek representatives with an exact prescribed support complement.

For Green problems, the last support step is essential. A primitive on an alteration, a current after proper pushforward, and an actual differential form on the original complement are different outputs. A limit theorem should state which output it produces and should prove descent of that output, not just convergence of its pairings on a smaller collection of tests.

References. [Pay09]; [GJRc]; the explicit representative constructions in [Mih24].

24.2 Further analytic problems

24.2.1 Regularization with a fixed test theory

The basic comparison asks when a continuous function with positive original CLD Hessian has local smooth positive approximants on one fixed neighbourhood. The forward implication from approximation to current positivity is elementary; the converse is a regularization theorem. A useful result should state the field, analytic category, topology of approximation, and whether the neighbourhood is uniform throughout the approximation.

Such a theorem would make many current inequalities usable in the established continuous BT calculus. It would also clarify comparisons between raw current positivity, generated CLD25 psh, numerical model positivity, and curvewise psh. The PL cases and curve results provide strong tests, but

proving the general continuous statement requires controlling all positive tests, not only a selected finite chart or a family of curves whose detecting power has not been established.

24.2.2 A local potential theorem for closed currents

Given a positive closed $(1,1)$-current T, a local potential theorem seeks u with $\mathrm{d}'\mathrm{d}''[u] = T$ in a specified regularity class. This is logically distinct from regularizing a potential already known to exist. It also requires more than a degree-zero or primitive condition tested on a restricted subspace: the equality must hold on the full test space stated in the theorem.

A proof would need a local exactness statement for the differential complex, an analytic mechanism giving the required regularity, and a way to preserve positivity. The tropical and toric cases suggest techniques, but a primitive on one real chart need not descend through all analytic relations. This is one reason the intrinsic descent problem is central to the long-term programme.

24.2.3 Arbitrary-background action and kernel descent

The continuous recursion already defines an action of suitable potentials on an arbitrary ordinary positive closed background current. To promote this to an action of curvature objects, one needs the kernel condition (14.2.1). The cross example proves that abstract tropical positivity and closedness do not suffice in full generality.

A productive analytic theorem would identify a geometric class of kernel functions that annihilates every allowed background current, or a geometric class of background currents for which the full kernel acts trivially. Smooth pluriharmonic functions give the elementary case; skeletal BT currents give a structured case. The general smooth-Berkovich question must use the additional analytic geometry explicitly.

24.2.4 Bounded and unbounded potentials

Beyond continuous potentials, a complete theory should specify monotone convergence, locality on the appropriate topology, capacities, quasi-continuity, negligible sets, comparison and domination principles, and suitable Dirichlet problems. These are related but separate theorems. A

non-pluripolar product also requires a definition for unbounded potentials and a proof that truncation and exceptional-set choices are compatible.

The toric theory and global energy theories give established frameworks in which many of these issues can be addressed. The goal of a general local analytic theory is to connect such frameworks with the original test calculus while retaining their domains and comparison maps. It is not achieved by relabelling every model-theoretic θ-psh function as a locally integrable CLD potential without proof.

24.2.5 Actual Green forms and exact support

A strong Green statement for a codimension-r cycle Z may ask for an actual form γ of degree $(r-1,r-1)$ on $X^{\mathrm{an}} \setminus |Z|^{\mathrm{an}}$ with

$$\int_{X^{\mathrm{an}}\setminus|Z|^{\mathrm{an}}} \gamma \wedge \mathrm{d}'\mathrm{d}''\eta = -\int_{Z^{\mathrm{an}}} \eta$$

for every actual compactly supported test in the specified calculus. The complete-intersection theorem gives an explicit model of such a statement in its own category. A general cycle requires a global construction and compatibility of local representatives.

The requirements are substantive: the complement is the original complement; the representative lies in the chosen sheaf; the testing is not restricted to external products or a chosen tropical chart; and field extension must descend with the correct degree and the original absolute value. A current identity after arbitrary proper pushforward does not by itself satisfy these requirements. It may be a useful intermediate step if its relation to the desired representative is proved.

24.3 What filtered limits do preserve

A topological inverse-limit description and a differential direct-limit description are different statements. Once a differential complex has actually been identified with a filtered direct limit, its cohomology is easy to compute algebraically. The difficult part is often establishing that identification with the required support and topology. We separate these steps.

Theorem 24.3.1. *Let $(C_i^\bullet, d_i)$ be a filtered system of complexes of real vector spaces, with transition maps commuting with the differentials. Then the canonical map is an isomorphism:*

$$\varinjlim_i H^k(C_i^\bullet) \xrightarrow{\sim} H^k\left(\varinjlim_i C_i^\bullet\right).$$

Injectivity of the transition maps is not required.

Proof. In a filtered colimit of vector spaces, every element has a representative at one stage, and an element represented at a stage is zero precisely when it becomes zero at some later stage. Let z represent a cocycle in the limit. Choose a representative $z_i \in C_i^k$. Its differential is zero in the colimit, hence $d_i z_i$ becomes zero at a later stage j. The image z_j is then a cocycle at that stage, proving surjectivity.

For injectivity, suppose a stage cocycle z_i becomes a boundary in the limit, say $z = dy$. Represent y at another stage and take a common later stage for both representatives. The equality $z = dy$ holds at a further stage by the same zero criterion. Thus the image of the cohomology class of z_i is already zero at that stage, so it is zero in the colimit of cohomology groups. □

Corollary 24.3.2. *For a filtered system of double complexes with commuting transition maps, filtered colimits commute with Bott–Chern and Aeppli cohomology, defined algebraically by*

$$H_{BC}^{p,q} = \frac{\ker d' \cap \ker d''}{\operatorname{im} d'd''}, \qquad H_A^{p,q} = \frac{\ker d'd''}{\operatorname{im} d' + \operatorname{im} d''}.$$

Proof. A numerator condition involves finitely many equations, so each equation that holds in the colimit holds after passage to one common later stage. A denominator relation involves finitely many preimages, which can

also be represented at one common stage. The surjectivity and injectivity argument of the theorem then applies verbatim. This proof concerns algebraic quotients, without any assertion that the images are topologically closed. □

For an analytic application one must first construct a system whose colimit is the desired sheaf of forms or global test complex. A finite collection of local presentations can be refined by concatenating moment maps, but that observation alone does not identify a global complex on a noncompact space. Covers, restrictions, support control and the allowed quotient relations still have to be carried through. The theorem does not supply that geometric step.

Example 24.3.3 (Compact support and pullback)**.** The constant function one on a point has compact support. Its pullback along $\mathbb{R} \to \{*\}$ has support all of $\mathbb{R}$. Thus an arbitrary map does not induce pullback between compactly supported test spaces. Properness over the relevant support, or a specified alternative support construction, is needed before forming a direct system of compactly supported complexes.

24.4 Why inverse limits of spaces are not enough

Example 24.4.1 (A topological test of the inference)**.** Let $X_j = \{0,1\}^j$ with the discrete topology and the maps that forget the last coordinate. Their inverse limit is the compact space $X = \{0,1\}^{\mathbb{N}}$. The pullback colimit of $H^0_{\mathrm{sing}}(X_j, \mathbb{R})$ consists of real functions depending on finitely many coordinates. However $H^0_{\mathrm{sing}}(X, \mathbb{R})$ consists of all real-valued functions on X.

Indeed, a continuous path into X has constant projection to every discrete X_j, hence is constant. Every singular one-simplex is therefore constant, so every zero-cochain is a cocycle and there are no zero-coboundaries. The function $x \mapsto \sum_{j\geq 1} x_j 3^{-j}$ does not depend on finitely many coordinates and gives an explicit class outside the colimit.

This is not a counterexample to Payne's theorem or to a specific Berkovich comparison. It disproves only the general inference that a homeomorphism to an inverse limit automatically identifies singular or differential cohomology with the colimit of the finite-stage groups. Different cohomology theories have different continuity properties. For the intended differential theory, one must prove the required complex-level comparison.

24.5 Duality changes direction under a limit

Proposition 24.5.1. *For a filtered system of real vector spaces, there is a natural algebraic isomorphism*

$$\mathrm{Hom}_{\mathbb{R}}(\varinjlim_i V_i, \mathbb{R}) \simeq \varprojlim_i \mathrm{Hom}_{\mathbb{R}}(V_i, \mathbb{R}).$$

Proof. A functional on the colimit restricts to a compatible family of functionals on the stages. Conversely, a compatible family assigns the same scalar to any two representatives of a colimit element, so it defines a unique linear functional. These constructions are inverse. □

With locally convex topologies, continuous duals require the chosen inductive-limit topology and its continuity criterion. Even when that criterion is available, the direction remains inverse on duals. One cannot replace it by a second direct limit without an additional theorem.

Example 24.5.2 (Perfect finite-stage pairings need not remain perfect)**.** Take $V_j = W_j = \mathbb{R}^j$, with the standard dot product and transition maps adding a zero last coordinate. Each pairing is perfect and the pairings are compatible. Both direct limits are the finite-support sequence space $\mathbb{R}^{(\mathbb{N})}$. Their induced pairing is nondegenerate, but the resulting map

$$\mathbb{R}^{(\mathbb{N})} \longrightarrow \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{(\mathbb{N})}, \mathbb{R}) = \mathbb{R}^{\mathbb{N}}$$

is not surjective. The functional $v \mapsto \sum_i v_i$ is well defined because v has finite support, but it would require the infinite coefficient sequence $(1, 1, \ldots)$ in the second factor. Thus nondegeneracy survives in this example while algebraic perfectness does not.

Lemma 24.5.3 (One sufficient stabilization condition)**.** *If all transition maps of a filtered system of finite-dimensional vector spaces are injective and their dimensions have a uniform finite bound, then the system stabilizes after some stage and its colimit is finite dimensional.*

Proof. The nonempty set of dimensions has a largest value. Choose a stage realizing it. Every later injective map from this stage has a target of the same dimension, hence is an isomorphism. Filteredness makes these later stages cofinal. □

Such stabilization is one possible route for transferring finite-dimensional duality, but it is a hypothesis to prove for the actual transition maps. A common tropical refinement need not induce an injective map on a cohomology group merely because the refinement map is surjective on the underlying support.

24.6 Cohomological annihilation and a current equation

There is another distinction when cohomology is used to construct a Green current. Vanishing on the kernel of a differential gives an algebraic factorization. A continuous current requires a topological estimate, and an actual differential form requires further regularity. The following elementary result makes the first two steps precise.

Theorem 24.6.1 (A continuous lifting criterion)**.** *Let $D : E \to F$ be a continuous linear map of real locally convex spaces and let $L : E \to \mathbb{R}$ be continuous linear. A continuous linear functional G on F satisfying $G \circ D = L$ exists if and only if there is a continuous seminorm p on F such that*

$$|L(e)| \le p(De) \qquad (e \in E).$$

Proof. If G exists, continuity bounds it by a continuous seminorm on F, giving necessity. Conversely, the estimate implies that L vanishes on $\ker D$. Hence $g(De) = L(e)$ is well defined on $D(E)$ and satisfies $|g(y)| \le p(y)$. The real Hahn–Banach theorem extends g to a linear functional G on F with the same bound. Since p is continuous, G is continuous. □

Example 24.6.2 (The kernel condition is not enough)**.** Take $E = F = \ell^2(\mathbb{N})$, $D(x)_n = x_n/n$, and $L(x) = \sum_n x_n/n$. The functional L is continuous by Cauchy–Schwarz, and $\ker D = 0$. If $G \circ D = L$, evaluation on the standard basis forces $G(e_n) = 1$ for every n. No bounded functional on ℓ^2 has these coefficients. Explicitly, take $x_n = n$ for $1 \le n \le N$ and zero otherwise. Then $L(x) = N$ and $\|Dx\|_2 = \sqrt{N}$, so no uniform bound $|L(x)| \le C\|Dx\|_2$ holds.

For a current equation on a pure n-dimensional space, take $E = A_c^{n-r,n-r}$, $F = A_c^{n-r+1,n-r+1}$, $D = d'd''$, and $L(\eta) = -\int_Z \eta$. A functional G supplied by the theorem is a current of bidegree $(r-1, r-1)$ satisfying the

required transposed equation, with the book's fixed differential sign convention. Merely knowing that $\int_Z \eta = 0$ whenever $d'd''\eta = 0$ supplies only the kernel condition. One still needs the continuous estimate.

Even after this estimate, Hahn–Banach gives no actual representative on $X^{\mathrm{an}} \setminus |Z|^{\mathrm{an}}$, no local integrability of a chosen coefficient expression, and no control of additional singular support. Those belong to a regularity and representation theorem. If the target asks for equality on delta-tests, the spaces E, F in the argument must already be the full delta-test spaces with their topology. Solving the equation only on original smooth tests does not perform that enlargement.

The local complete-intersection construction of Chapter 22 addresses these stronger requirements by an explicit geometric representative and a proof on actual tests. The functional argument here serves a different purpose: it identifies which estimates and comparisons a general cohomological existence strategy would have to supply.

24.7 The coefficients carried by a tropical face

A tropical cohomology class remembers tangent directions as well as the topology of the underlying space. The coefficient system is already nonconstant at a vertex where several cells meet. We describe it first for a finite polyhedral complex in a real vector space; strata at infinity are then handled by quotient maps.

Let $\mathscr{C}$ be a polyhedral complex in $N_{\mathbb{R}}$. For a face σ, let L_σ be its parallel linear space. Define

$$F_p(\sigma) = \sum_{\tau \supseteq \sigma} \bigwedge^p L_\tau \ \subseteq \bigwedge^p N_{\mathbb{R}}, \qquad F^p(\sigma) = \mathrm{Hom}_{\mathbb{R}}(F_p(\sigma), \mathbb{R}).$$

The sum is a sum of subspaces, not a direct sum. If $\sigma \subseteq \tau$, then $F_p(\tau) \subseteq F_p(\sigma)$; dualizing gives a restriction $F^p(\sigma) \to F^p(\tau)$. At a boundary stratum of an extended tropicalization, first project to the quotient tangent space of that stratum. These maps define the multi-cotangent sheaf $\mathcal{F}^p$.

Example 24.7.1. Let $Y \subset \mathbb{R}^2$ be the union of the two coordinate axes. At the origin $F_1(0) = \mathbb{R}^2$, whereas on the interior of either axis F_1 is its one-dimensional tangent line. Thus a degree-one coefficient at the vertex is a covector with two components, and restriction retains the component along the chosen axis. For the tropical line with rays $e_1, e_2, -e_1 - e_2$, the three outgoing restrictions are $a, b, -a - b$. Their sum is zero. This relation comes from the common ambient covector; it is not three independent endpoint values.

Example 24.7.2. Let X be the union in $\mathbb{R}^4$ of the planes spanned by e_1, e_2 and by e_3, e_4. At the origin, $F_1(0) = \mathbb{R}^4$, but

$$F_2(0) = \mathbb{R}(e_1 \wedge e_2) \oplus \mathbb{R}(e_3 \wedge e_4), \qquad \dim F_2(0) = 2 \neq 6 = \dim \bigwedge^2 F_1(0).$$

Thus one must take exterior powers before summing the tangent spaces. Replacing F_p by $\bigwedge^p F_1$ changes the theory at singular points.

24.7.1 From local closed forms to a coefficient sheaf

For a polyhedral space X in the sense of Jell–Shaw–Smacka, put

$$\mathcal{L}^p = \ker(d'' : \mathcal{A}^{p,0} \longrightarrow \mathcal{A}^{p,1}).$$

On an ordinary real open set its sections are locally constant primed p-forms. Across a face, only their restrictions to the tangent spaces of adjacent cells are visible. This is why the appropriate coefficient sheaf is $\mathcal{F}^p$.

Theorem 24.7.3 (Polyhedral comparison)**.** *Let X be a paracompact, second countable Hausdorff polyhedral space with the extended-affine charts and boundary compatibility of [JSS]. There is a canonical isomorphism $\mathcal{L}^p \simeq \mathcal{F}^p$, and*

$$H^{p,q}_{d''}(X) \simeq H^q(X, \mathcal{F}^p), \qquad H^{p,q}_{d'',c}(X) \simeq H^q_c(X, \mathcal{F}^p).$$

If X has a face structure, these groups are also its tropical cohomology groups, respectively with and without compact support.

Open proof obligation (RC2). Prove the polyhedral Poincare lemma and the fine/acyclic chart-sheaf route; the proof uses the stated JSS theorem, proposition and lemma.

For $p = 0$ the coefficient sheaf is $\mathbb{R}$, so this row is ordinary real cohomology. For $p > 0$ it is generally not determined by the topology. The two plane example explains the extra data without referring to an analytic space. A claim that Payne's topological inverse limit automatically recovers all rows would have to recover these coefficient systems and their transition maps as well.

24.8 Compact support changes the Poincaré lemma

A radial homotopy usually spreads the support to the star centre and does not solve the compactly supported problem. The following explicit construction provides the required replacement on $\mathbb{R}^n$. It also identifies the obstruction that integration detects.

Lemma 24.8.1. *Write coordinates on $\mathbb{R}^n$ as $(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$, and an ordinary compactly supported q-form as $\omega = \alpha(x,t) + dt \wedge \beta(x,t)$, with no dt in α, β. Choose $\rho \in C_c^\infty(\mathbb{R})$ with integral one and put $\chi(t) = \int_{-\infty}^{t} \rho(s)\, ds$. Define*

$$I\omega = \int_{\mathbb{R}} \beta(x,s)\, ds, \qquad E\eta = \rho(t)\, dt \wedge \eta(x),$$

$$H\omega = \int_{-\infty}^{t} \beta(x,s)\, ds - \chi(t) I\omega.$$

Then $H\omega$ is compactly supported and $dH + Hd = \mathrm{id} - EI$. Moreover $Id = -dI$ and $dE = -Ed$.

Proof. Below the supports of β and ρ both terms in $H\omega$ vanish. Above them they are both $I\omega$ and cancel. The x-support is contained in the compact projection of $\operatorname{supp}\omega$. Thus the support is compact. Direct differentiation gives

$$d\omega = d_x\alpha + dt \wedge (\partial_t \alpha - d_x \beta).$$

Integrating its second coefficient gives $Id\omega = -d_x I\omega$, because α vanishes at both ends. The same calculation gives $dE = -Ed$. Finally the dt-part of $dH\omega$ is $dt \wedge (\beta - \rho I\omega)$, while its x-part cancels the integral terms of $Hd\omega$ and leaves α. Their sum is $\omega - EI\omega$. This also proves the formula in degree zero, where $H\omega = 0$. □

Theorem 24.8.2. *For superforms on $\mathbb{R}^n$,*

$$H^{p,q}_{d'',c}(\mathbb{R}^n) = 0 \quad (q \neq n), \qquad H^{p,n}_{d'',c}(\mathbb{R}^n) \simeq \bigwedge^{p}(\mathbb{R}^n)^*.$$

The second isomorphism integrates each double-primed top-degree coefficient. In particular a compactly supported (p,n)-form is d''-exact with compact support if and only if all these coefficient integrals vanish.

Proof. Keep the primed exterior factor fixed. Up to the fixed degree sign in d'', the remaining complex is the ordinary compactly supported de Rham complex, with coefficients in $\bigwedge^p(\mathbb{R}^n)^*$. Lemma 24.8.1 gives a homotopy equivalence with the same complex on $\mathbb{R}^{n-1}$ shifted by one degree; IE is the identity. Iterate this reduction. On $\mathbb{R}^0$, only degree zero remains and its cohomology is $\mathbb{R}$. The iterated fibre integral is the ordinary top-degree integral up to the orientation sign. This proves the assertions and constructs a primitive whenever the obstruction vanishes. □

Combining this calculation with ordinary Poincaré exactness gives Euclidean duality: $H^{p,0}_{d''}(\mathbb{R}^n)$ pairs perfectly with $H^{n-p,n}_{d'',c}(\mathbb{R}^n)$ by integration. The pairing is the exterior-product pairing of two finite-dimensional coefficient spaces. The compact support in the second factor is essential.

24.9 Where tropical Poincaré duality holds

Theorem 24.9.1 (Jell–Shaw–Smacka)**.** *If X is an n-dimensional tropical manifold, locally modelled on matroidal fans with the boundary charts specified in [JSS, Definition 4.15], integration induces an isomorphism*

$$H^{p,q}_{d''}(X) \longrightarrow \left(H^{n-p,n-q}_{d'',c}(X)\right)^*, \qquad [\alpha] \longmapsto \left([\beta] \longmapsto \int_X \alpha \wedge \beta\right).$$

Open proof obligation (RC2). Prove the local matroidal and global Poincare duality in the stated scope; the text supplies JSS Theorem 4.33.

Proposition 24.9.2. *Give the union Y of the two coordinate axes in $\mathbb{R}^2$ weight one on each axis. Then $H^{0,0}_{d''}(Y) = \mathbb{R}$ and $H^{1,1}_{d'',c}(Y) = \mathbb{R}^2$. The integration pairing is the sum of the two coordinates and is not perfect.*

Proof. A d''-closed smooth function is locally constant and Y is connected, proving the first assertion. Restriction identifies the complexes $A^{1,\bullet}_c(Y)$ with

the direct sum of the corresponding complexes on the two real lines. To check surjectivity, extend an arbitrary compactly supported coefficient $a(x)$ on the horizontal line as $a(x)\rho(y)d'x$ or $a(x)\rho(y)d'x\wedge d''x$, with $\rho = 1$ near zero. This restricts to zero on the vertical line. The analogous vertical extension realizes the other summand. Tangential equality on the two axes is precisely the quotient relation for these bidegrees, proving injectivity. These restrictions commute with d''.

By Theorem 24.8.2 each line contributes one top compactly supported class, detected by its integral. Integration on Y adds the two integrals. The class with integrals $(1,-1)$ is nonzero but pairs to zero with every constant. This proves the failure. □

The cross is balanced. Its defect is that the two one-dimensional branches have independent top classes while the whole space has only one constant class. This supplies a concrete obstruction to deriving duality from balancing, and a separate reason not to transfer a tropical Hodge theorem through a topological inverse limit alone. It is not a counterexample to the theorem for tropical manifolds or to any analytic comparison proved with stronger hypotheses.

References. [JSS, Sections 2–4]; the relation with the analytic limit question is developed in Section 6.5 and Chapter 24.

24.10 The same topological star can carry different cohomology

Let Y be a fan consisting of m distinct rays with nonzero primitive directions $v_1,\dots,v_m$ in a real vector space V. Give the rays positive weights w_i satisfying $\sum_i w_i v_i = 0$. Each ray is parametrized by $t \mapsto tv_i$, $t \geq 0$. Define

$$R : V^* \longrightarrow \mathbb{R}^m, \qquad \lambda \longmapsto (\lambda(v_1),\dots,\lambda(v_m)).$$

Proposition 24.10.1. *For the ambient superform sheaf on Y, edgewise integration induces an isomorphism*

$$H^{1,1}_{d'',c}(Y) \simeq \mathbb{R}^m / \operatorname{im} R.$$

Consequently its dimension is $m - \dim \operatorname{span}\{v_i\}$. Integration on the weighted fan is induced by $(b_i) \mapsto \sum_i w_i b_i$.

Proof. For $\alpha \in A_c^{1,0}(Y)$ write $\alpha|_i = a_i(t)d't$. Since $d''\alpha|_i = -a_i'(t)d't \wedge d''t$, its integral on the ith ray is $a_i(0)$. The values at the vertex come from a common ambient covector, so the vector of integrals lies in $\operatorname{im} R$. Hence the map to the quotient is well-defined. It is surjective, since independent bump top forms supported away from the vertex realize arbitrary edge integrals.

For injectivity, first subtract $d''(\chi\lambda)$, where λ is a constant ambient primed covector and χ a cutoff equal to one near the vertex, to reduce to a form β with zero edge integrals. The local polyhedral Poincaré lemma gives $\beta = d''\alpha_0$ near the vertex. Choose another cutoff supported there. Subtracting $d''(\chi\alpha_0)$ makes β vanish near the vertex, but changes its integrals by the vector of vertex values of α_0. Add back $d''(\chi\lambda_0)$ for an ambient covector λ_0 with those same vertex values. Its derivative vanishes near the vertex. The resulting form $\widetilde{\beta}$ is supported away from the vertex and has zero integral on every ray.

On each ray put $a_i(t) = \int_t^\infty \widetilde{b}_i(s)\,ds$, where $\widetilde{\beta}|_i = \widetilde{b}_i(t)d't \wedge d''t$. The zero integral makes a_i vanish near zero, and compact support makes it vanish for large t. These edgewise primitives extend to ambient forms in disjoint neighbourhoods away from the vertex and glue by zero. Their d'' is $\widetilde{\beta}$. Restoring the subtracted exact forms proves injectivity. Finally balancing says precisely that the functional $\sum_i w_i b_i$ annihilates $\operatorname{im} R$. □

For the rank-two uniform matroid on m elements, the rays are the coordinate vectors in $\mathbb{R}^m/\mathbb{R}\mathbf{1}$. Their span has dimension $m-1$, and their only linear relation is their sum. The proposition gives $H_c^{1,1} = \mathbb{R}$. For the planar cross, $m = 4$ and the span has dimension two, so $H_c^{1,1} = \mathbb{R}^2$. Both underlying spaces are a four-valent topological star when $m = 4$, but their superform coefficient sheaves differ.

This calculation is the local analogue of the harmonic-coordinate issue on analytic curves. Recording only an embedded graph may impose extra relations among its directions. Harmonic tropicalizations use the geometric harmonic functions needed for the intrinsic theory, rather than assuming one fixed low-dimensional embedding records every differential direction. The analytic comparison still requires the model and refinement theorems of GJR; the star calculation explains what those theorems must preserve.

References. For the flag description, [AK]; for the polyhedral Poincaré lemma and matroidal duality, [JSS, Theorems 3.16 and 4.33]. The rank-two balancing and star quotient calculations above are written out in the conventions of this book.

24.11 Hodge and Aeppli calculations on a real torus

A compact real torus provides a complete model in which the two differentials, their cohomology, and the $d'd''$ equation can be computed directly. This model is useful for separating a valid differential statement from a proposed passage through tropical limits. The proof below uses Fourier series on one fixed torus. It does not require a projective embedding, and it does not assert a Hodge theorem for every tropical space.

Let $M = V/\Lambda$ be a real torus of dimension n, with a fixed translation-invariant Euclidean inner product. Complexify the superform spaces during the computation and recover real forms by conjugation. The dual lattice is Λ^*. Every smooth form has a Fourier expansion

$$\alpha(x) = \sum_{k \in \Lambda^*} e^{2\pi i \langle k, x \rangle} \alpha_k, \qquad \alpha_k \in \bigwedge^p V_{\mathbb{C}}^* \otimes \bigwedge^q V_{\mathbb{C}}^*.$$

Its coefficients decay faster than every negative power of $|k|$. Conversely, such decay defines a smooth form, with termwise differentiation. These standard Fourier facts follow by repeated integration by parts on a fundamental domain and the absolute convergence of sufficiently high powers of $(1 + |k|)^{-1}$.

24.11.1 Contraction on each nonzero mode

For $k \neq 0$, write $k' = \sum k_i d'x_i$ and $k'' = \sum k_i d''x_i$ in orthonormal coordinates. On that Fourier mode, the differential operators are exterior multiplication by $2\pi i k'$ and $2\pi i k''$, with the total-degree wedge convention. Let ι'_k and ι''_k be contraction with the corresponding metric dual vectors in the first and second exterior factors. Define

$$h'_k = \frac{\iota'_k}{2\pi i |k|^2}, \qquad h''_k = \frac{\iota''_k}{2\pi i |k|^2}.$$

The second contraction includes the total-exterior-algebra sign. For a primed p-form a' and a double-primed form b'', our convention is

$$\iota''_k(a' \wedge b'') = (-1)^p a' \wedge \iota_k b''.$$

Without this sign, the two cross-factor anticommutation identities below would not hold.

Lemma 24.11.1. *On every nonzero mode,*

$$d'h'_k + h'_k d' = 1, \qquad d''h''_k + h''_k d'' = 1,$$
$$d'h''_k + h''_k d' = 0, \qquad d''h'_k + h'_k d'' = 0.$$

The norms of h'_k, h''_k are bounded by a constant times $|k|^{-1}$. They therefore act on smooth Fourier series, and also on distributions.

Proof. For a covector λ and vector v, exterior multiplication ε_λ and contraction satisfy $\varepsilon_\lambda \iota_v + \iota_v \varepsilon_\lambda = \lambda(v)$. Applying this in the matching factor gives $|k|^2$; applying it across the two disjoint factors gives zero. Division by $2\pi i|k|^2$ proves the identities. The contraction norm is $|k|$, proving the estimate. S-mooth coefficients are rapidly decreasing, while distribution coefficients have at most polynomial growth, so these estimates preserve the corresponding Fourier classes. □

Let P_0 denote the projection to the zero Fourier mode, equivalently the coefficientwise average. Set $h' = h'' = 0$ on that mode and use the preceding formulas on the others. The four identities then hold globally with 1 replaced by $1 - P_0$ in the first two. Thus the only obstruction visible in these complexes is the constant coefficient.

Theorem 24.11.2 (Dolbeault cohomology of a real torus)**.** *Every d''-closed smooth (p, q) form is cohomologous to its unique constant-coefficient average. In particular,*

$$H^{p,q}_{d''}(M) \cong \bigwedge^p V^* \otimes \bigwedge^q V^*, \qquad \dim H^{p,q}_{d''}(M) = \binom{n}{p}\binom{n}{q}.$$

The same conclusion holds with distributional forms, and the inclusion of smooth forms induces the indicated cohomology isomorphism.

Proof. If $d''\alpha = 0$, the homotopy identity gives $\alpha - P_0\alpha = d''h''\alpha$. The primitive is smooth, or distributional, in the same category as α. Conversely, the average of a derivative is zero because its zero Fourier mode vanishes. Hence no nonzero constant form is exact. The construction commutes with conjugation of Fourier series, so it restricts to real forms. □

24.11.2 The $d'd''$ lemma

Theorem 24.11.3. *A smooth form α with $d'\alpha = d''\alpha = 0$ has a smooth $d'd''$ primitive if and only if $P_0\alpha = 0$. One primitive is*

$$\beta = h''h'\alpha, \qquad d'd''\beta = \alpha.$$

The statement also holds for distributions. In particular, a form closed under both differentials which is exact under either one is $d'd''$-exact.

Proof. Necessity follows because a derivative has average zero. For sufficiency, work on a nonzero mode. Since $d''h'\alpha = -h'd''\alpha = 0$,

$$d'd''h''h'\alpha = d'h'\alpha = \alpha.$$

Summing is legitimate by the contraction estimates; the multiplier is of order at most $|k|^{-2}$. If the bidegree has a zero entry, the corresponding negative-degree primitive space is zero, and the same identities force every mean-zero form closed under both operators in that bidegree to vanish. Finally, exactness under either differential implies zero average. □

The two contractions and their anticommutation signs are essential. Solving separately for a d' primitive and a d'' primitive does not by itself provide one common $d'd''$ primitive on an arbitrary double complex. Here the explicit identities make the implication valid.

Definition 24.11.4. For the superform double complex, the Bott–Chern and Aeppli groups are

$$H^{p,q}_{BC} = \frac{\ker d' \cap \ker d'' \subset A^{p,q}}{d'd''A^{p-1,q-1}},$$
$$H^{p,q}_{A} = \frac{\ker(d'd'' : A^{p,q} \to A^{p+1,q+1})}{d'A^{p-1,q} + d''A^{p,q-1}}.$$

Proposition 24.11.5. *On M, both groups are represented uniquely by constant forms, so they have the same dimensions as $H^{p,q}_{d''}(M)$. The assertion holds for smooth forms and for distributions.*

Proof. The Bott–Chern assertion is Theorem 24.11.3. For Aeppli cohomology, take a nonzero mode of α with $d'd''\alpha = 0$. Then $d'\alpha$ is d''-closed, and

$$\begin{aligned}\alpha &= d'h'\alpha + h'd'\alpha \\ &= d'h'\alpha + h'd''h''d'\alpha \\ &= d'h'\alpha - d''h'h''d'\alpha.\end{aligned}$$

This is a sum of an exact term for each differential. The multipliers have polynomially bounded size, so the formulas sum in the desired category. A sum of derivatives has zero average, proving uniqueness of the constant representative. □

24.11.3 Pairings and current equations

Integration gives a perfect pairing

$$H^{p,q}_{d''}(M) \times H^{n-p,n-q}_{d''}(M) \longrightarrow \mathbb{R}, \qquad ([\alpha],[\beta]) \longmapsto \int_M \alpha \wedge \beta.$$

It is well defined by Stokes. To prove nondegeneracy, replace both classes by constant forms. A basis element $d'x_I \wedge d''x_J$ pairs, up to its fixed sign and the positive volume of the torus, with the complementary basis element indexed by I^c, J^c. The pairing matrix is invertible. The same argument identifies the Bott–Chern/Aeppli complementary pairing.

This calculation proves more than finite-dimensionality: it gives explicit continuous solution operators for all nonzero Fourier modes. Hence the obstruction to a closed current being $d'd''$-exact is exactly its constant cohomology component. By contrast, on a proposed direct limit of tropical complexes, finite-stage nondegenerate pairings alone provide neither a limiting solution estimate nor an actual form representative. The Fourier operator has supplied both an algebraic homotopy and a topological bound in this particular model.

24.12 Positive currents and their global torus potentials

Theorem 24.12.1. *Let T be an ordinary positive closed $(1,1)$ supercurrent on M. Its coefficientwise average is a constant positive semidefinite form θ. There is a continuous real function u, unique up to a constant, such that*

$$T = \theta + d'd''[u].$$

On V, the periodic lift of u added to a quadratic potential for θ is finite convex.

Proof. The average of the positive matrix-valued coefficient measure is positive semidefinite, giving θ. The distributional $d'd''$ lemma produces a real

distribution u with $d'd''u = T - \theta$. Locally write $\theta = d'd''q$ for a quadratic function. Then $d'd''(u+q) = T \geq 0$. The real convex-distribution theorem, Corollary 11.5.2, makes $u+q$ a finite convex continuous function locally. Thus u has a continuous representative, and these representatives agree on overlaps because they define the same distribution.

On V, the lifted function $u+q$ has positive distributional Hessian globally, hence is convex. If two potentials give the same current, their difference has zero Hessian and is affine on V. A Λ-periodic affine function is constant. □

This conclusion is a feature of real convexity. Positive closed complex $(1,1)$ currents can have logarithmically singular potentials; a positive real Hessian distribution on an open Euclidean domain instead has a finite convex representative. The real torus theorem must not be transferred to a singular tropical quotient or an arbitrary analytic space merely because the notation for the two differential operators is similar.

Corollary 24.12.2 (Rigidity in a null direction)**.** *Let H be the positive semidefinite matrix of θ and u the preceding potential. The lift of u is constant along every vector in* $\ker H$*. If the image of* $\ker H$ *is dense in M, then u is constant and $T = \theta$.*

Proof. For $e \in \ker H$, the quadratic function $q(x) = \frac{1}{2}x^{\mathsf{t}}Hx$ is constant along $x + \mathbb{R}e$. The convex function $q+u$ restricts to a convex function of one real variable which is bounded, because u is periodic and continuous. A convex function bounded on all of $\mathbb{R}$ is constant: any nonzero secant slope would force unbounded growth on one end. Thus $u(x+te) = u(x)$. Invariance under a dense subgroup and continuity give the final claim. □

Example 24.12.3. On $\mathbb{R}^2/\mathbb{Z}^2$, let $H = \left(\begin{smallmatrix} 1 & a \\ a & a^2 \end{smallmatrix}\right)$ with a irrational. Its kernel is generated by $(-a, 1)$, whose image is dense in the torus. Every positive closed $(1,1)$ current with average H is therefore the constant current H. If $a = 0$, nonconstant potentials depending only on x_1 are possible, subject to $1 + u'' \geq 0$. The closure of the null foliation, rather than the rank of H alone, accounts for the difference.

References. The arguments in this section combine the elementary Fourier contraction of a flat torus with the real convex-potential theorem of [Lag12, Proposition 1.13]. They are included as explicit model calculations. For the broader comparison of superform cohomology with tropical cohomology, see [JSS];

no passage from the flat-torus calculation to a general Berkovich Hodge theorem is used here.

24.13 The Hodge index calculation in bidegree $(1,1)$

The flat-torus computation also gives an explicit index theorem. The statement concerns symmetric real $(1,1)$ classes. Their constant representatives are symmetric matrices, and the primitive condition is a trace condition. This makes the sign and the equality case completely visible.

Theorem 24.13.1. *Assume $n \geq 2$, and let θ be a constant positive definite $(1,1)$ form on M. On the space of symmetric Bott–Chern $(1,1)$ classes, the pairing*

$$Q([\alpha],[\beta]) = \int_M \alpha \wedge \beta \wedge \theta^{n-2}$$

has signature $(1, \frac{n(n+1)}{2} - 1)$. Its restriction to the primitive subspace

$$\left\{ [\alpha] : \int_M \alpha \wedge \theta^{n-1} = 0 \right\}$$

is negative definite. In particular, a smooth symmetric closed form in that primitive subspace has $Q([\alpha],[\alpha]) = 0$ if and only if it is $d'd''$-exact.

Proof. Stokes makes the pairing invariant under adding a $d'd''$-exact form to either factor. By Proposition 24.11.5, replace all classes by constant representatives. Make a real linear coordinate change in which the matrix of θ is the identity; the lattice changes but the torus volume stays positive. Let A be the symmetric matrix of α. Expanding $\det(I + tA)$, or diagonalizing A, gives

$$\alpha \wedge \theta^{n-1} = \frac{\operatorname{tr} A}{n}\, \theta^n,$$
$$\alpha^2 \wedge \theta^{n-2} = \frac{(\operatorname{tr} A)^2 - \operatorname{tr}(A^2)}{n(n-1)}\, \theta^n.$$

Primitivity is precisely $\operatorname{tr} A = 0$. On this subspace the second expression is a negative multiple of $\operatorname{tr}(A^2) = \sum_i \lambda_i^2$, and vanishes only when $A = 0$. The line spanned by the identity is positive and orthogonal to the trace-zero subspace. Since the dimension of the symmetric matrices is $n(n+1)/2$, this gives the

signature. A zero constant representative is equivalent to $d'd''$-exactness by Theorem 24.11.3. □

For a general positive matrix H of θ, the same formulas use $H^{-1/2}AH^{-1/2}$. Thus the invariant primitive condition is $\operatorname{tr}(H^{-1}A) = 0$. The coordinate change is only a device for the calculation; it does not replace the lattice normalization in integration.

24.13.1 A related inequality for metric potentials

Let u, v be continuous periodic functions with $\alpha_u = \theta + DDu \geq 0$ and $\alpha_v = \theta + DDv \geq 0$. The global regularization of Section 21.23 defines their continuous mixed products. Set

$$I_\theta(u, v) = -\int_M (u - v)(\alpha_u^n - \alpha_v^n).$$

Proposition 24.13.2. *One has $I_\theta(u, v) \geq 0$. For smooth u, v, it is the sum of nonnegative Dirichlet terms*

$$I_\theta(u, v) = \sum_{j=0}^{n-1} \int_M d'(u - v) \wedge d''(u - v) \wedge \alpha_u^j \wedge \alpha_v^{n-1-j}.$$

Proof. In the smooth case, factor the difference of powers and integrate by parts against $u - v$. Every remaining factor is a closed positive $(1, 1)$ form, so each integrand is nonnegative. For continuous potentials use common global smooth semipositive approximations. The measures converge weakly, the potentials converge uniformly, and their total masses remain fixed. The moving-test convergence of Chapter 9 therefore passes the displayed nonnegativity to the limit. The integral defining I_θ is meaningful without assigning ordinary first derivatives to the continuous potentials. □

The index theorem is a nondegeneracy statement on a finite-dimensional cohomology space. The inequality for I_θ is a positivity statement on potentials in one fixed class. They use related integration-by-parts signs, but their equality cases require different arguments. In the smooth strictly positive setting, Section 21.25 proves that $I_\theta(u, v) = 0$ forces $u - v$ constant. The full degenerate continuous uniqueness problem belongs to the corresponding convex Monge–Ampère theorem.

References. These are matrix and integration-by-parts consequences of the flat-torus calculations above. Their role in the torus metric setting is compared with [BGJK25, Sections 7–8]; no general tropical Hodge–Riemann theorem is assumed.

24.14 Subject and literature concordance

The following concordance groups the sources by the mathematical construction being used. It supplements the notation index in Section 0.2.3. A source listed in one row may also contribute to another; the grouping does not assert equivalence of its objects with every object in that row.

Construction	Sources	Comparison developed here
Tropical cycles and skeleta	[Gub13, GRW16, GRW17, Pay09, DT26]	Multiplicities, lattice degrees, faithful tropicalization, and the topology of analytic spaces; Sections 5.2, 6.3, 24.1.
Smooth forms and currents	[CLD12, Gubler, CLD25]	Analytic-unit charts, algebraic presentations, test topology, and the revised tropical framework; Chapters 7–9.
Positive cones and current estimates	[CLD12, CLD25, BGJK]	Dual positivity, order zero, and the distinction between ordinary and weak positivity; Section 9.1.1 and Appendix A.
Delta-forms and intersections	[GK17, GK19, Mih24, Mih23, Cai23, BGK25]	Corner terms, products, analytic constructions, and compactification; Sections 16.1, 15.7, 22.2.

Construction	Sources	Comparison developed here
Harmonicity and PL positivity	[GJR, GJRc, GRPL, GRPSH]	Enlarged coordinates and test spaces, model semipositivity, and curvewise criteria; Sections 17.2–19.2.4.
Measures, metrics, and heights	[CL06, GM, BFJ15, Gub10, GS23, Lin26, CG26]	Model measures, global Monge–Ampère theory, canonical measures, and local heights; Sections 19.4, 20.6, 22.2, 21.3.

24.15 Cycle classes: what cohomological vanishing gives

A construction of Green forms must first identify the obstruction represented by the cycle. Liu's tropical cycle class map is directly relevant because its target is built from original CLD forms, rather than from a separately chosen finite polyhedral model. Let X be smooth and separated of finite type over a complete nontrivially valued field. Liu constructs a rational sheaf $\mathscr{T}^p$ whose realification is the sheaf of d''-closed $(p,0)$-forms. The fine resolution by CLD forms relates

$$H^{p,q}_{\mathrm{trop}}(X) = H^q(X^{\mathrm{an}}, \mathscr{T}^p), \qquad H^{p,q}_{\mathrm{trop}}(X) \otimes_{\mathbb{Q}} \mathbb{R} \simeq H^{p,q}_{d''}(X^{\mathrm{an}}).$$

His cycle map $\mathrm{cl}_{\mathrm{trop}} : \mathrm{CH}^p(X) \to H^{p,p}_{\mathrm{trop}}(X)$ is functorial and product-compatible. Its integration formula, after realification, is

$$\int_{X^{\mathrm{an}}} \mathrm{cl}_{\mathrm{trop}}(Z) \wedge \eta = \int_{Z^{\mathrm{an}}} \eta, \qquad d''\eta = 0, quad\eta \in A_c^{n-p,n-p}(X^{\mathrm{an}}). \quad (24.15.1)$$

The left side means the pairing of a representative of the class with the closed test. Independence of that representative follows from Stokes. The theorem proves that this pairing agrees with cycle integration.

Theorem 24.15.1 (Liu's de Rham vanishing consequence)**.** *Let k be a finite extension of $\mathbb{Q}_p$, let X/k be proper and smooth, and let Z have codimension p with $\mathrm{cl}_{\mathrm{dR}}(Z) = 0$. After base change to $\mathbf{C}_p$, integration over Z^{an} vanishes on every d''-closed original CLD test of complementary bidegree.*

Open proof obligation (RC2). Establish the pairing-vanishing result in the stated scope; Liu20 Theorem 1.8 is supplied.

24.16 Finite tropical models and Hodge theory

The fan theorems of Amini–Piquerez provide powerful finite-dimensional models, with hypotheses that are visible combinatorially. For a rational simplicial fan Σ, let $\overline{\Sigma}$ be its canonical compactification. The tropical Feichtner–Yuzvinsky theorem identifies

$$H^{p,p}(\overline{\Sigma}, \mathbb{Q}) \simeq A^p(\Sigma, \mathbb{Q}).$$

For the full cohomology ring to be concentrated on the diagonal and equal to the Chow ring, their Theorem 1.3 additionally assumes that Σ is a tropical homology manifold. Integral coefficients require further unimodularity, saturation and integral-duality hypotheses. Rational and integral statements should be recorded separately. Homological smoothness is itself a local condition. In [AP-D, Theorem 1.1], a fan is homologically smooth precisely when the fan and all its star fans satisfy tropical Poincaré duality. Their local-to-global theorem gives duality between ordinary tropical cohomology and Borel–Moore homology. Compactly supported cohomology enters the associated pairing. Replacing Borel–Moore homology by ordinary homology on a noncompact space would change that theorem.

Example 24.16.1 (The complete one-dimensional fan)**.** For the fan in $\mathbb{R}$ with rays generated by 1 and -1, the Stanley–Reisner relation is $x_+x_- = 0$ and the linear relation is $x_+ - x_- = 0$. Hence

$$A^\bullet(\Sigma, \mathbb{Q}) = \mathbb{Q}[h]/(h^2), \qquad \deg h = 1.$$

The compactification has $H^{0,0} = H^{1,1} = \mathbb{Q}$ and no other cohomology. Multiplication pairs 1 and h nondegenerately. This is a complete finite calculation, using the stated fan comparison. It proves neither a general $d'd''$-lemma on an analytic curve nor an identification of all forms on its Berkovich analytification with forms on one segment.

The remaining transition to Berkovich geometry has three parts. One must identify the relevant complexes and their transition maps, prove that cohomology and the needed support conditions pass through the chosen limit, and control the topology needed for duality or solution operators. Payne's topological inverse-limit theorem addresses none of these three analytic assertions by itself. The limit and closed-range issues in Sections 24.5 and 24.6 remain in force even when every individual tropical model has perfect finite-dimensional duality.

References. [Liu20, Theorems 1.1, 1.7–1.9]; [AP-FY, Theorems 1.1–1.3]; [AP-D, Theorems 1.1–1.3].

24.16.1 From local fans to global tropical Hodge classes

There are two further layers in the Amini–Piquerez programme. The local layer establishes the Kähler package for suitable Chow rings: for example, effective unimodular quasi-projective quasilinear fans are Chow-Kähler in [AP-F, Theorem 1.10]. The global layer constructs Kähler classes and uses a tropical Steenbrink complex to obtain Hard Lefschetz, Hodge–Riemann and weight-monodromy results for smooth projective tropical varieties, with rational coefficients and the definitions of [AP-H, Theorem 1.4]. Neither assertion is a statement about every balanced polyhedral complex. The tropical Clemens–Schmid sequence addresses the representation of cohomology by tropical cycles. Under the additional rational triangulability hypothesis, [AP-CS, Theorem 1.1] identifies the classes generated by codimension-p tropical cycles on a smooth projective tropical variety with

$$\ker\big(N : H^{p,p}_{\mathrm{trop}}(X,\mathbb{Q}) \longrightarrow H^{p-1,p+1}_{\mathrm{trop}}(X,\mathbb{Q})\big).$$

The monodromy-kernel condition is part of the conclusion; one should not replace it by all of $H^{p,p}$. These cycles are tropical cycles. An algebraic lifting statement would require an additional theorem. Likewise, a tropical cycle class and Liu's cycle class of an algebraic cycle become comparable only after the relevant realization map has been constructed. The full proofs of the Steenbrink and Clemens–Schmid constructions are not reproduced here. The finite fan calculation, the cycle-pairing theorem and the limit requirements isolate the parts needed for the analytic programme. Extending this account to a self-contained global tropical Hodge theory would require a separate development of those complexes and their comparison maps.

24.17 The next mathematical comparisons

The preceding additions change the form of the remaining questions. Some unrestricted claims already have counterexamples; others have precise positive theorems in narrower categories. The following programme records the assertions still needed for the general analytic goals of this book. An external theorem with an adequate statement is not a missing proof merely because its proof is not repeated here.

1. *Approximation and residual geometry.* For CLD25 Proposition 8.1.11, prove the predecessor-replacement step or restrict the conclusion to the proved first stage. The pure-dimensional residual current criterion is reconstructed in Theorem 19.18.12. The separate integral PL bridge is now supplied by Theorem 19.18.22 and Corollary 19.18.23. These tasks do not by themselves prove smooth regularization for every continuous current-psh metric.

2. *Larger potential classes and kernels.* Theorem 13.6.3 already permits the stated arbitrary ordinary positive closed background and admitted continuous potentials. Extending the potential class requires new approximation or monotone-convergence results. Passing to curvature classes additionally requires a kernel theorem. The unrestricted abstract kernel statement is false; an analytic replacement must specify functions, backgrounds and actual relations.

3. *Capacity and global envelopes.* Compare the toric, polyhedral, model-metric and synthetic finite-energy theories on their common domains. A general local analytic package of capacity, quasi-continuity, comparison and Dirichlet existence has not been established here. The counterexample in Section 16.7 excludes a universal orthogonality shortcut. The model and field hypotheses of each positive existence theorem remain essential.

4. *Representatives and local potentials.* Polyhedral pullback and current pushforward do not supply a general analytic trace in the actual delta-sheaf. Prove that trace with its domain and support conditions before using it for descent. Likewise, existence of a distributional primitive is insufficient for a local psh potential theorem for arbitrary positive closed $(1,1)$-currents.

5. *Cohomology and actual Green forms.* Combine a precisely stated finite-stage Hodge comparison with control of limits, closed ranges and support. Liu's vanishing theorem supplies a necessary pairing condition, not an actual primitive. The general strong Green problem still requires a representative on the exact complement and the identity on the full actual test space, with compatible finite-field descent.

6. *Energy and arithmetic compatibility.* On an explicitly identified common domain, compare limiting measures and relative energies with CLD/GK currents and arithmetic intersection. Retain the boundary topology and the compactified reference object. In mixed characteristic, list resolution or model hypotheses theorem by theorem; do not replace a conditional global theorem by an unconditional local statement.

The next editorial expansion should develop the global tropical Hodge complexes and the mixed finite-energy comparison in more detail. The next proof repair should address continuous predecessor replacement after the integral PL bridge completed below. These are separate tasks: expanding a literature account is not evidence that an unresolved analytic implication has been proved.

Version 0.9 supplies the Laurent gluing, simultaneous residual realization and PL noncharging argument over arbitrary complete fields, then assembles the pure-dimensional residual criterion. It also corrects the previous manuscript's non-good-germ concern: boundarylessness already implies goodness here. The integral psh comparison requires the additional argument below; it cannot be replaced by merely expressing a PL weight as a difference of admitted functions.

Version 0.11 supplies prescribed graded premodel realization, the simultaneous bundle identifications, and the nef integral PL implication on a fixed tube. P0-A4 is therefore complete in the stated integral PL category, with its named imported inputs. The next approximation task is P0-B: continuous predecessor replacement in CLD25 paragraph 8.1.11. A generated-class conclusion for a nef PL weight is not a first-stage smooth sequence for that weight.

Version 0.12 proves continuous endpoint collapse for a decreasing hierarchy on one fixed compact set. The remaining P0-B task is to justify a common compact representation, or to construct weaker continuous replacements in $\mathcal{C}_{25}$ on one fixed neighbourhood. These are different targets: the first would

give first-stage approximation, while the second is enough for the unindexed equality. The same-ordinal forward inclusion and the locally integrable hierarchy equality are already established in Section 11.6.3; they should not be listed as unresolved.

Version 0.13 adds a projective envelope route. Theorem 11.7.5 proves that bounded first-stage decreasing stability suffices for equality at every continuous ordinal stage; after that input is supplied, ordinal control is no longer a separate obstacle. The next source-verification task is the chain in [Wu26]: Theorem A.3, bounded localization, Theorem 4.17 and Corollary 4.18. Proposition 11.7.3 corrects the barrier normalization and background conversion, and Lemma 11.7.1 supplies sequential continuous obstacles from the local approximation property. Section 11.8 now supplies a fixed-domain diagonal lemma, bounded-family tropical localization and projective barrier construction. Section 11.9 supplies one such common-domain direction for explicitly specified rational PL generators over a nontrivially valued field. The approximation and single-generator tasks are now treated in Section 11.22 under its stated hypotheses. The common-domain first-stage comparison is now supplied by Corollary 11.24.4; remaining tasks are sheaf-level decreasing closure, complete source-class identification and trivial-valuation extension. Outside the projective envelope category, bounded decreasing stability remains the analytic target; no claim over arbitrary complete fields follows from the equicharacteristic-zero envelope theorem.

For an already strict R-PL germ, Section 11.13 now supplies coefficient-compatible decreasing rational approximation on one residue tube, and a projective original-smooth comparison. An arbitrary sequence still needs a common finite presentation for the ordered-family corollary. Identification of arbitrary pre-existing relative real PL presentations remains separate; continuous relative-support approximation is supplied by Section 11.22. Section 11.14 repairs the single-polarization cofinal input after exhibiting the fixed-model failure of GM Lemma 4.12. Finite simultaneous polarizations and the numerical and bounded-correction parts of GM Proposition 4.14 are reconstructed in the stated formal-domination scope. Section 11.15 supplies algebraic model-morphism domination using non-Noetherian formal GAGA, and reduced-special-fibre cofinality under the algebraically closed hypothesis. These particular interfaces are therefore closed; the broader analytic comparisons remain open. Section 11.17 narrows the remaining reverse-comparison problem. The explicit smooth family obstructing ambient convex extension does admit decreasing coefficient-compatible relative PL approximation in

its original chart. More generally, finite coordinatewise-monotone convex compositions of bounded relative rational PL functions have such approximations, including the decreasing-tuple construction (11.17.9). Section 11.22 now supplies the required density directly for continuous relative-support functions on a fixed compact image. Proposition 11.22.14 applies it to one original smooth psh input using a boundary-adapted chart. Theorem 11.24.3 now supplies common-domain replacement for arbitrary original smooth sequences whose charts vary; the further sheaf-level closure is separate. The earlier fixed rational-fan result is extended to one-scale polytopal complexes by Theorem 11.18.1. To apply this result to real PL output from another source, one must verify the vertex condition or prove a replacement allowing the subdivision to vary. Examples 11.18.3 and 11.18.4 rule out freezing arbitrary subdivisions while merely rounding their coefficients.

Section 11.19 now gives the exact closure on a fixed complex: a relatively supported target must lie in the real span of admissible rational functions. Under the strict face-support certificate this reduces to the explicit continuity equations. Edge-connected rational simplices of dimension at least two provide a separate route to one common value-group scale. For this fixed-complex approach, the hypotheses must still be verified on the source output. Section 11.22 instead constructs new affine branches before gluing and so avoids fixing those subdivisions.

Lemma 11.19.12 and Proposition 11.19.13 complete the coefficient-compatible convex extension on individual cells. Their value estimates do not bound normal slopes at the original faces. Section 11.20 now proves the slope-controlled version and preserves relative supports for compatible compact complexes. Section 11.22 handles continuous relative-support inputs with arbitrary real affine supports on one compact domain, including extended images under its admissible boundary convention. The global twisted and simplicial source formulations still require separate identification.

For the current approximation status, Section 11.22 supersedes the earlier density tasks listed above. Continuous relative-support inputs on one compact ordinary or extended image admit coefficient-compatible decreasing approximation; the extended case retains its admissible boundary-support hypothesis. A single original smooth psh input, and a finite collection of such inputs, are covered by Section 11.22.6. Section 11.24 completes the arbitrary original-smooth-family replacement and the stated first-stage germ comparison. The open tasks are further sheaf-level decreasing closure, contin-

uous predecessor replacement, and precise source-class identification. None of the abstract boundary or fixed-chart counterexamples negates those open analytic assertions.

Appendix A

Source corrections and concordance

A.1 Corrections and qualifications of source statements

This section gives the mathematical evidence for the source qualifications used in the main text. It concerns the versions listed in Section 0.2.2, and it does not claim an audit of every page of the 608-page text. The counterexamples below address literal statements; the separate remarks about incomplete arguments do not assert that their conclusions are false. Return to Sections 11.2, 13.1 and 19.3 for the statements used in the lecture.

A.1.1 Weak positivity does not imply order zero

Example A.1.1 (A weakly positive current of order one)**.** On $\mathbb{R}^4$, write a symmetric $(2,2)$-test form in the matrix normalization

$$\alpha = -\sum_{|I|=|J|=2} H_{IJ}(x)\, d'x_I \wedge d''x_J, \qquad H_{IJ} = H_{JI}.$$

Define the coefficient functional

$$\ell(H) = H_{12,34} - H_{13,24} + H_{14,23}.$$

If $H = a\xi\xi^{\mathsf{T}}$ is a strong generator, the two-covector ξ is decomposable. Its Plücker relation is

$$\xi_{12}\xi_{34} - \xi_{13}\xi_{24} + \xi_{14}\xi_{23} = 0.$$

Thus $\ell(H) = 0$ for every strongly positive form, pointwise in x. Define

$$T(\alpha) = \frac{\partial}{\partial x_1} \ell(H_\alpha(x))\Big|_{x=0}, \tag{A.1.1}$$

and let T vanish on antisymmetric forms. This is a continuous distribution of order at most one. It is symmetric, and it vanishes on every strongly positive test. Both T and $-T$ are therefore weakly positive.

Choose a constant symmetric form α^0 with $\ell(H_{\alpha^0}) = 1$, and a smooth compactly supported function χ equal to one near 0. The tests

$$\alpha_j = j^{-1} \sin(jx_1)\chi(x)\alpha^0$$

have one fixed compact support and coefficient suprema tending to zero, whereas $T(\alpha_j) = 1$ for every j. Hence T is not of order zero.

The same phenomenon is recorded in [BGJK, Examples 2.3.6 and 5.2.8]. It directly contradicts the weak-positive assertion in [CLD25, Proposition 7.2.9, p. 162]. The proof there uses the relative-interior statement for the strong cone in Proposition 1.2.13 as if that cone could dominate every symmetric form. Example 7.4.3 shows that the required domination is impossible.

The replacement used in this book is Proposition 9.1.7, for ordinary positive currents. In degrees $(1,1)$, $(n-1,n-1)$, and the extremal degrees, the coincidence of the relevant cones removes this particular middle-degree obstruction. The counterexample must not be cited as a failure of order zero for ordinary positive $(1,1)$-currents.

Return to positivity and continuous multipliers.

A.1.2 A locally integrable distribution does not fix point values

Remark A.1.2 (The literal implication in Proposition 8.3.2). The statement in [CLD25, Proposition 8.3.2, pp. 201–202] quantifies over functions on a convex real open set which are not identically $-\infty$. Its third condition is local integrability and positivity of $d'd''[u]$, and it concludes pointwise finite convexity. The function in Remark 11.2.9 satisfies the third condition and fails the conclusion. The convolution step in the printed proof also invokes uniform convergence to this arbitrary function, which does not follow from local integrability.

For continuous functions, Proposition 11.2.8 gives a complete repair. Theorem 11.3.2 supplies the full L^1_{loc} repair: there is a unique finite convex representative, and convolution converges locally uniformly to that representative. This distinguishes the representative from arbitrary modifications on sets of measure zero. Adding upper semicontinuity alone does not repair the printed statement, as the same upward-point-spike example shows.

Return to the corrected continuous comparison.

A.1.3 Multiplication need not preserve a closed current

Example A.1.3 (Failure of the scalar multiplication assertion)**.** In the tropical line $X = \mathbb{R}$, let $T = [1]$ be the integration current of bidegree $(0, 0)$. It is positive and closed and has the Bedford–Taylor property, as also stated in [CLD25, Example 8.6.4]. Let

$$u(x) = 1 + x^2.$$

This is a smooth positive function, but

$$d'(uT) = [2x\, d'x] \neq 0.$$

Thus uT is not closed and cannot be a Bedford–Taylor current under [CLD25, Definition 8.6.2]. This disproves Corollary 8.6.7, p. 208, literally.

Even deleting closedness from that definition would not fix the claimed preservation of potential independence. Indeed, $a(x) = x$ and $b(x) = 0$ are smooth psh functions with the same zero Hessian. Nevertheless,

$$d'd''(a\,uT) = [6x\, d'x \wedge d''x] \neq 0 = d'd''(b\,uT).$$

The correct smooth multiplication statement is Proposition 8.6.5 of the source: the factor is a positive *closed smooth form*. A general positive continuous scalar does not satisfy the closedness requirement.

Return to the continuous recursive construction.

A.1.4 The successor step in the generated-sheaf comparison

Remark A.1.4 (The missing implication in Proposition 8.1.11). CLD25, Proposition 8.1.11, pp. 197–198, claims

$$\mathcal{C}_{25,\alpha} = C^0 \cap \mathcal{P}_{25,\alpha} \quad \text{for every ordinal } \alpha.$$

At a successor stage, the right side permits a continuous endpoint u which is locally a decreasing limit of $u_j \in \mathcal{P}_{25,\alpha}$. The approximants u_j are not assumed continuous. The printed proof instead compares decreasing and uniform limits of members of $\mathcal{C}_{25,\alpha}$. The induction hypothesis identifies these with the *continuous* members of $\mathcal{P}_{25,\alpha}$; it does not make every u_j continuous.

Dini's theorem for continuous approximants proves the first-stage equality in Proposition 11.2.3. It does not supply the missing continuous approximants at an arbitrary stage. Even a uniform convergence conclusion for the original upper semicontinuous approximants would not place each of them in the continuous stage. An additional replacement or approximation argument is required.

We have no counterexample to the stated equality. The usable conclusions include the first-stage equality, the inclusions in Proposition 11.2.4, and the same-ordinal forward inclusion of Proposition 11.6.6. Theorem 11.6.4 handles a hierarchy on one fixed compact set, but transferring sheaf-generated sections to that hierarchy requires the extra representation in Corollary 11.6.5. Proposition 11.6.7 verifies the neighbouring L^1 comparison independently of the continuous gap. The independent PL synthesis later in the text supplies an equality on that specified class without invoking the disputed successor step. Section 11.7.4 supplies a sufficient bounded-stability criterion, and Corollary 11.7.6 applies the comparison inputs of [Wu26] in the projective envelope category. This is a scoped source-dependent application; it does not repair the original proof on every analytic space.

Remark A.1.5 (The stationarity wording). Lemma 8.1.4 of CLD25, p. 196, says that the ordinal hierarchy is eventually stationary, and its proof invokes a cardinality argument. Remark 8.1.6 on the same page says that stationarity is not known. Read without a qualification on the ordinal, these sentences conflict. A hierarchy of subobjects inside a fixed set of local sections cannot grow strictly through all ordinals; the eventual set-theoretic stabilization and the question whether the *first* stage suffices are different matters. The lecture does not choose an unstated ordinal bound on the authors' behalf. It uses only the minimal-sheaf definition and the explicitly proved first-stage comparison.

Return to the generated-sheaf definition.

Normalized residual sections. The literal corrections to CLD25 Lemma 17.4.10, Corollaries 17.4.11–17.4.12 and Proposition 17.8.2 are recorded, with the corrected proof and a one-point test, in Lemma 19.18.13 and Remark 19.18.14. The semiample conclusion is retained with its weighted maximum.

A.1.5 A fixed neighbourhood is part of an approximation statement

Remark A.1.6 (Proposition 17.8.4 and a repair in a common category). The proof of [CLD25, Proposition 17.8.4, pp. 497–498] perturbs a nef residual class by a rational multiple of an ample class. Applying Proposition 17.8.2 to each perturbation gives psh on a neighbourhood of the chosen point. The argument then takes a uniform limit on one fixed neighbourhood. The printed application alone does not show that the psh neighbourhood can be chosen independently of the perturbation.

The logical point is elementary. On a real interval, let

$$v_\varepsilon(t) = \varepsilon t^2 - t^4.$$

For every $\varepsilon > 0$, this function is convex near 0, since $v_\varepsilon'' = 2\varepsilon - 12t^2$. On a fixed compact interval it converges uniformly to $-t^4$, which is not convex on any neighbourhood of 0. Thus positivity on varying neighbourhoods is insufficient for the claimed limiting step.

In the good strictly analytic, nontrivially valued setting at interior points, [GJR, Proposition 7.10] supplies one fixed neighbourhood and a uniform family of locally Fubini–Study approximants. Theorem 19.18.17 expands its residue-model construction: a single reduction tube works for every perturbation. Proposition 19.18.16 isolates the exact compatibility used. The PL synthesis uses this fixed-domain argument. This is an independent repair under its stated hypotheses, not a counterexample to Proposition 17.8.4. The graded construction of Proposition 19.18.20 and Theorem 19.18.22 now gives the integral PL version over arbitrary complete fields.

A.1.6 Scope and indexing in the residual comparison

Remark A.1.7 (What is and is not established about Theorem 18.8.4). We did not obtain a counterexample to [CLD25, Theorem 18.8.4] or a refutation of its entire proof. Its claim is stated in Remark 19.3.1. The following local problems are identifiable.

1. On p. 539, the residual numerical criterion for a product of d Chern classes is expressed as a zero-cycle degree. The testing subvarieties must therefore have dimension d. The printed reference to codimension m does not give that dimension in general; later in the proof, the subvarieties are correctly taken of dimension d.

2. In the same proof, the rational-domain inequalities formed from $f_i^{\varepsilon_i}$ require the right-hand side $|f_i(x)|^{\varepsilon_i}$. Omitting the exponent on that side fails to make the chosen point satisfy the equality when $\varepsilon_i = -1$ and $|f_i(x)| \neq 1$.

3. Corollary 18.8.8, p. 541, states its equivalence for integral PL metrics, whereas its cited implication from Proposition 17.8.4 has a *strictly* integral PL hypothesis. The distinction is made in the source's definitions; a reduction and descent argument is needed to use that implication in the larger category.

The first two items are incorporated in the reconstruction of Theorem 19.18.12. For the third, Theorem 19.18.15 treats arbitrary integral PL metrics with semiample residue, and Corollary 19.18.18 treats nef strictly integral PL metrics over a nontrivially valued field. Theorem 19.18.22 completes the general integral PL implication using prescribed graded premodels over the original field; Corollary 19.18.23 gives the resulting pure-dimensional comparison. None of them authorizes replacing the theorem by a general statement about arbitrary continuous metrics.

Remark A.1.8 (A bidegree/bidimension slip in the CLN estimate). In [CLD25, Proposition 8.4.1, p. 202], the current is called of bidegree (p, p) while the test has degree $(p - q, p - q)$ after q curvature factors have been inserted. The displayed pairing has the advertised degree when p is the *bidimension* of the initial current: its resulting bidegree is $(n - p + q, n - p + q)$. This is an indexing correction; the lecture uses the explicit complementary test degrees throughout.

Return to the PL source comparison.

A.1.7 Corrections in the other sources

Remark A.1.9 (The 2012 source is not an error-free reference version). The earlier text also needs qualifications. Its commutativity exponents in §1.2.6 and §3.1.4 conflict with the explicit total-degree algebra; Remark 7.1.5 gives the adopted correction. The real strong-cone decomposition in Lemma 5.2.3 is disproved by Example 7.4.3. The proof of the analytic Lemma 5.3.3 relies on that decomposition and requires a separate repair; the real example alone is not a counterexample satisfying every analytic hypothesis. Its continuous facewise-convexity criterion in Lemma 5.5.3 misses the corner mass; [GK19, Remark 3.2 and Example 3.3] supplies the correction. The general pullback sentence following Definition 5.5.1 must be distinguished from the proved pullback result on a specified approximation class.

Remark A.1.10 (Two cautions about the GK positivity text). The incorrect real strong-span statement is repeated in [GK19, Remark 1.12(i)]. The analytic argument of Lemma 2.16 uses the affected decomposition; this identifies a proof obligation, rather than by itself disproving the analytic conclusion. Also, the slope-only criterion in Example 8.6 needs the edgewise Hessian condition for a piecewise smooth function. The smooth function $-t^2$ on an interval has zero sum of its opposite outgoing first derivatives at each interior point and negative second derivative. The full measure formula in Proposition 16.2.7 retains both the edge density and the vertex masses.

Remark A.1.11 (A substantive correction already acknowledged in GJR). GJR, Remark 16.4, explicitly says that the argument in [GK17, Examples 8.15 and 9.17] used an incorrect smoothness assertion for canonical tropicalizations of abelian varieties. Its construction in §17 shows that these maps need not be original CLD-smooth. The argument that canonical metrics are delta-metrics in that original sense consequently does not follow. This corrects the argument; it does not prove that the metrics fail to be delta-metrics. The distinction is important when using canonical metrics as examples of an enlarged calculus.

Remark A.1.12 (Elementary corrections in GJR v3). In GJR §10.6, pp. 63–64, the commutativity exponent should be $(p+q)(p'+q')$, and the last operator in the d' Leibniz rule should be d'. The printed alternatives fail respectively on two independent $(1,0)$-forms and by bidegree counting. In the proof of Proposition 10.17, the affine Taylor polynomial is $g(\nu) + Dg_\nu(\omega - \nu)$; Proposition 17.2.7 uses this corrected expression. These are local corrections to formulas, not counterexamples to the corresponding theorems.

A.2 Additional source checks and explicit repairs

A.2.1 The regularized maximum in Proposition 16.9.1

Status: local formula error, with a repair. The displayed kernel in CLD25, p. 466, integrates

$$\max_i(s_i + t_i) \prod_{i=1}^{m} \theta(t_i/\varepsilon)\, dt_1 \cdots dt_m$$

with $\int \theta = 1$, but omits the factor ε^{-m}. Its total kernel mass is ε^m. Consequently, adding a constant a to every s_i changes the displayed expression by $a\varepsilon^m$, rather than by a. It cannot have the claimed translation behaviour and approximation bounds in that form.

A corrected regularized maximum is

$$M_\varepsilon(s) = \int_{\mathbb{R}^m} \max_i(s_i + \varepsilon t_i) \prod_{i=1}^{m} \theta(t_i)\, dt, \tag{A.2.1}$$

where θ is a smooth nonnegative even kernel supported in $[-1, 1]$ and of integral one. Convolution makes the expression smooth; convexity and coordinatewise monotonicity are preserved. Jensen's inequality and the zero mean give

$$\max_i s_i \le M_\varepsilon(s) \le \max_i s_i + \varepsilon.$$

If one coordinate lies more than 2ε below the maximum, it never attains the perturbed maximum and the expression is independent of that coordinate. This last property is what permits logarithmic branches vanishing identically at some points to be discarded locally.

A.2.2 The domain and sublevel qualifications in the same proof

Status: local domain and statement qualifications. The proof uses logarithms $\log|f_i/g|$. The condition that the f_i and g have no common zero does not prevent all f_i from vanishing at a point where $g \neq 0$. At such a point every logarithmic branch is $-\infty$, so the real smoothing formula is not directly defined. Add a finite constant branch below the desired sublevel threshold.

The maximum is then finite; branches sufficiently far below it can be discarded. On a neighbourhood where a remaining logarithmic branch is active, its analytic function is nonzero and supplies a legitimate unit coordinate.

The intended barrier conclusion also needs two neighbourhoods: a smooth psh function on a larger domain, negative on a smaller neighbourhood of the point, with a compact nonpositive sublevel contained in the larger domain. The literal wording "negative on U and $\{\rho \leq 0\}$ compact" makes the sublevel equal to all of U. The proof itself chooses a shifted function that is negative near the chosen point, not necessarily everywhere on its domain.

These corrections concern the local construction used to apply the polar-negligibility theorem. They do not constitute a counterexample to the existence of suitable local psh barriers. The normalized smoothing and constant branch supply the appropriate mechanism in the good analytic setting of the lecture.

A.2.3 The hypothesis transfer in Corollary 8.11.3

Status: false in the unrestricted abstract tropical category. Proposition 8.11.1 assumes a skeletal current with the Bedford–Taylor property. Corollary 8.11.2 retains a skeletal current locally expressible as a difference of such currents. Corollary 8.11.3 is then stated for an arbitrary positive closed current T and a convex function pulled back by a harmonic chart. Its proof cites the preceding corollary to identify the BT product with multiplication by a G-form.

The cited statement alone does not supply that identification for arbitrary T. One needs an extension theorem, a decomposition preserving the necessary hypotheses, or an independent positivity argument for the harmonic pullback. The skeletal/BT assumptions cannot simply disappear between the two statements. The cross calculation in Section 14.2 gives a counterexample in an explicitly verified abstract tropical category. It does not claim a counterexample on a smooth Berkovich space.

The ordinary continuous-generated-potential construction in Chapter 13 has an independent proof by positive continuous multiplication and closure under uniform limits. It does not need this unrestricted harmonic-coordinate implication. When a theorem uses the latter implication specifically, its additional justification must be supplied.

A.2.4 Local corrections in the higher-codimension formulas

Status: formula and index corrections. In the setup of §18.7.1, the complete intersection Y has dimension $n - r$, so the dimension bound at its points must be compared with $n - r$, not with the codimension r. In the truncation formula, the correct identity is

$$\log \max(\varepsilon, |f_1|, \dots, |f_r|) = \max(\log \varepsilon, u), \qquad u = \log \max_i |f_i|.$$

Writing $\log \max(\varepsilon, u)$ is not meaningful when u is negative and is not equal to the left side. On a compact support away from Y, u has a finite lower bound; it need not be strictly positive. Proposition 18.7.3 likewise uses relative dimension $n - r$ in its proof, rather than the printed r in its introductory statement.

The Levine calculation also provides a simple independent sign check: on the complement of the cycle, $\mathrm{d}'\mathrm{d}''h = \alpha - \beta$, so $\mathrm{d}'\mathrm{d}''(h\omega) = \alpha^r - \beta^r$. The sign of the singular term is then fixed by Poincaré–Lelong. The final identity is the one stated in Theorem 18.7.5 and displayed in Chapter 22. A contradictory intermediate sign must be corrected locally rather than propagated through the calculation.

Sections 22.4–22.7 give the extension and fibre-comparison arguments with their geometric inputs stated. There is also a separate commutative-algebra issue in the proof of Proposition 18.7.3: a regular sequence alone does not put a neighbourhood of its zero set in Serre's (S_r) locus. Example 22.7.2 disproves that unrestricted inference. The relevant finite-flat argument can instead be made at the Abhyankar points, where dimension and depth coincide. This repairs the needed mechanism in the stated category without asserting that the entire original Green identity is false.

A.2.5 A correction acknowledged in the canonical-measure literature

Status: author-acknowledged gap in an earlier auxiliary claim. In the introduction to *Non-archimedean canonical measures on abelian varieties*, Gubler explicitly withdraws an earlier purity claim for the tropical image in the general abelian setting, after a referee identified a gap. The corrected theorem allows polyhedral pieces of different dimensions and retains the canonical-measure formula. The totally degenerate case has the expected pure dimension in that result.

This correction is useful because it shows exactly what changes. One must not use the earlier purity claim in a dimension argument for mixed reduction. One may still use the corrected measure theorem with its actual dimension range. The correction says nothing by itself about the present-day status of every related purity question.

A.2.6 Closedness in the radial mass formula

In the consulted arXiv:1008.2856v1, Proposition 3.4 of Lagerberg states radial trace-mass monotonicity for a weakly positive current without listing closedness. The Stokes calculation in its proof needs closedness. Example 9.13.3 gives a counterexample to the unrestricted wording: a nonnegative compactly supported smooth scalar on $\mathbb{R}$ has a constant ball mass for large radii, so the mass divided by the radius decreases. The corrected result, including its regularized proof, is Theorem 9.13.2.

Corollary 3.5 of the same source does include closedness and ordinary positivity. These assumptions have separate roles. Closedness supplies the ball-growth estimate; ordinary positivity supplies coefficient domination by the trace. The distinction between weak and ordinary positivity is already explicit in the source's Remark 3.3. The correction here is pinned to the consulted preprint wording, without asserting that every later printed version has the same omission.

A.2.7 Local domains and degrees in the toric paper

Three local repairs concern the consulted arXiv:2102.07392v1. In Example 2.3.5, the expression $T \mapsto (T, T-1)$ is used as a map from $\mathbb{G}_m$ to $\mathbb{G}_m^2$. Its second coordinate is not invertible at $T = 1$, so this map is defined only off that point. The invariant function in the example is nevertheless original smooth: use that chart away from $T = 1$, and use the constant zero expression near $T = 1$, where $|T| = 1$. Section 21.17 gives the local calculation. Thus the example's conclusion survives the domain correction.

In Proposition 4.3.2 and Remark 4.3.4, some displayed factors lack the $d'd''$ operators required by their assigned degrees. The differentiated product and the companion product with one scalar factor must be written separately. Proposition 21.21.1 records both formulas and their target bidegrees. Applying $d'd''$ changes the leading scalar factor into a curvature factor; it is not an operation that leaves the bidegree unchanged.

Finally, positivity of the differentiated product does not imply positivity of the companion product when its undifferentiated scalar can be negative. Taking that scalar to be -1 and the remaining positive product nonzero gives a negative current. The companion product is defined without a sign restriction, but its positivity requires an additional nonnegativity assumption on that scalar. These qualifications leave the correspondence and product constructions available with their correct statements.

A.2.8 Upper semicontinuity in the classical maximum proof

The proof of Gubler–Rabinoff, Theorem 6.4 in arXiv:2506.13548v2, invokes continuity to say that the maximum set of a classically psh function is closed. The definition only provides upper semicontinuity. That is sufficient: if $M = \max_X u$, the set $\{u = M\} = \{u \geq M\}$ is closed for an upper semicontinuous function. The rest of the open-and-closed argument is unchanged. The component reduction in the same proof is a finite cover by irreducible components, whose intersections may be nonempty. The reconstructed proof in Theorem 19.5.2 uses precisely these two elementary formulations.

This is a hypothesis slip in an intermediate sentence, with an immediate repair. It has a different mathematical status from the weak-positive order-zero assertion disproved above. Recording the distinction prevents a local wording problem from being presented as a failure of the classical psh theory.

A.3 A compact register of conclusions and dependencies

Source location	Assessment	Usable replacement or remaining obligation
CLD12 §1.2.6, §3.1.4	Sign error	Use total-degree graded commutativity.
CLD12 Lemma 5.2.3; proof of 5.3.3	False real statement; analytic proof obligation	Use ordinary positive forms for coefficient domination; the Plücker example gives the obstruction.
CLD12 Lemma 5.5.3, continuous form	False literally	Retain corner mass; use the smooth version or an explicit Hessian-plus-corner criterion.

Source location	Assessment	Usable replacement or remaining obligation
CLD12 after Definition 5.5.1	General pullback not justified by the definition	Use smooth or specified approximable potentials, or a separate functorial theorem.
CLD25 Proposition 7.2.9, weak-positive part	False literally	Ordinary positivity gives order zero; weak positivity in middle degree need not.
CLD25 Proposition 8.3.2	False pointwise statement	Use continuous functions or distinguish an L^1 class from its convex representative.
CLD25 Corollary 8.6.7	False literally	Positive continuous scalars preserve positivity, but not closedness or the BT kernel property.
CLD25 Proposition 8.1.11	Successor-step gap	Indexed inclusion and fixed-compact barrier theorem proved; sheaf globalization remains. The neighbouring L^1 equality is valid.
CLD25 Lemma 8.1.4 / Remark 8.1.6	Conflicting wording	Separate eventual set-theoretic stabilization from first-stage stabilization.
CLD25 Proposition 8.4.1	Degree slip	Use bidimension p for the displayed test degree.
CLD25 Corollary 8.11.3	False in the stated abstract generality	The cross example disproves the arbitrary-current assertion; retain the proved skeletal/BT hypotheses.
CLD25 Lemma 9.5.4	False in the stated abstract generality	The cross-times-line example has mass off the graph; the affine case remains valid.
CLD25 Proposition 16.9.1	Kernel normalization and local-domain issues	Use (A.2.1), a constant branch, and nested barrier neighbourhoods.
CLD25 Proposition 17.8.4	Fixed-neighbourhood step missing	GJR Proposition 7.10 gives an independent repair in its specified analytic category.

Source location	Assessment	Usable replacement or remaining obligation
CLD25 Theorem 18.8.4 and Corollary 18.8.8	Local corrections; graded PL bridge supplied	Theorem 19.18.22 replaces the missing strictness transfer; arbitrary continuous weights remain separate.
CLD25 §18.7	Local formula/ index/sign corrections	Use the correct truncation and dimensions; retain the extension theorem and regular-sequence hypotheses.
GK positivity Remark 1.12 / proof of Lemma 2.16	Real error; analytic proof obligation	Ordinary-positive replacement.
GK positivity Example 8.6	Incomplete for piecewise smooth functions	Include edgewise Hessian densities as well as vertex slopes.
GK canonical examples; GJR Remark 16.4	Acknowledged invalid smoothness argument	Canonical tropicalizations can fail original smoothness; original delta-metric membership is a separate question.
GJR §10.6, Proposition 10.17 proof	Local algebra/Taylor slips	Correct total degree, differential operator, and constant term.
Gubler canonical-measure introduction	Acknowledged earlier purity gap	Use the corrected mixed-dimensional theorem.
CLD25 Proposition 18.7.3 proof	Regular-sequence inference insufficient	Example 22.7.2; use the local dimension and Cohen–Macaulay inputs in Section 22.7.
GJR v3 §17.2 and Proposition 17.4 proof	Cycle ordering and slope-list slips	The consistent theta-graph calculation and residual obstruction are in Section 17.8.
Lagerberg v1 Proposition 3.4	Closedness missing from literal statement	The counterexample and corrected radial mass proof are in Section 9.13.

Source location	Assessment	Usable replacement or remaining obligation
BGJK25 v1 Example 2.3.5	Torus-chart domain too large	Exclude $T = 1$ from the unit chart and use a constant expression near it; Section A.2.7.
BGJK25 v1 Proposition 4.3.2 and Remark 4.3.4	Differential degrees and positivity of a scalar factor	Retain DD on the curvature factors; a companion product needs a nonnegative scalar to be positive. See Proposition 21.21.1.
GRPSH v2 Theorem 6.4 proof	Continuity invoked unnecessarily	Upper semicontinuity closes the maximum set; Section A.2.8.

The register is an audit of identified statements in fixed versions, not an assertion that every page of every source has been independently verified. The most consequential repairs have complete elementary evidence in this volume. The missing-step entries identify precise obligations for a proof reconstruction and do not assert the negation of the theorem.

A.3.1 Which arguments remain usable after a correction

The false strong-span lemma affects any proof that bounds all coefficients by a strongly positive control form in middle degree. Replacing that step by ordinary-positive domination repairs the continuous multiplier argument used here. It does not automatically repair every theorem involving a different positive cone.

The scalar-BT counterexample prevents treating the class of BT currents as a module under arbitrary positive continuous functions. It does not obstruct the definition of uT as an order-zero current or the construction of $\mathrm{d}'\mathrm{d}''(uT)$ for an admitted psh potential and a positive closed background.

The successor-stage gap prevents using the claimed equality of all continuous generated stages as a shortcut to smooth approximation. It does not obstruct direct arguments using the continuous generated sheaf, nor the independent PL comparison proved through numerical model approximation and curve tests.

The fixed-neighbourhood gap affects a particular limiting argument. A separate theorem producing approximants on one common neighbourhood

repairs the implication in its own category. This is why GJR's model construction is mathematically useful even if the intended CLD conclusion is correct.

A.4 Current estimates and curvature comparisons

A.4.1 Why "replace smooth by psh" is not a definition

A differential algebra has products, two first-order differentials, pullbacks, and relations between local representatives. A psh function is generally not differentiable in the ordinary sense. Even when its Hessian is a positive current, its first derivatives and products with singular forms require definitions. The class is also not closed under negation, so it cannot simply replace the coefficient ring of a differential algebra.

The BT construction supplies one carefully defined operation: an admitted potential acts on a positive closed current by $\mathrm{d}'\mathrm{d}''(uT)$. It does not immediately supply every product $d'u \wedge \alpha$, every pullback of a current, or an algebra of arbitrary singular forms. Extending GK by psh coefficients therefore requires a choice of domain for all operations and a proof that the relations between representatives are respected.

A.4.2 One coefficient estimate, written out

On a fixed real chart, identify a symmetric test of degree (q, q) with its coefficient matrix $H(x)$ in the chosen sign normalization. For ordinary positivity, choose a positive definite control matrix I on the finite coefficient space and a cutoff equal to one near the support. If $\|H(x)\|_{\mathrm{op}} \leq M$, then

$$MI + H(x) \geq 0, \qquad MI - H(x) \geq 0.$$

Pairing with an ordinary positive current gives a bound on the coefficient functional by its value on the fixed positive control form. Localizing and summing over finitely many charts gives the corresponding bound for the actual test.

The matrix argument uses the ordinary cone, which has full-dimensional interior in the symmetric space. For the strong cone in middle degree, the Plücker functional in the audit vanishes on every generator, so no strong

control form can dominate every symmetric direction. This is the exact point where the superficially similar proof fails.

A.4.3 How to pass from smooth scalars to continuous scalars

Fix a positive test form β. The functional $a \mapsto T(a\beta)$ on smooth compactly supported scalar functions is positive and bounded by a sup norm on each fixed compact support. It therefore extends to a Radon measure. For a general symmetric test, write it as a difference of ordinary positive tests and use the resulting signed measures. Compatibility on overlaps makes uT well-defined for continuous u.

This construction does not identify the current with one global matrix of measures in a fixed tropical coordinate system. It builds compatible scalar measures for actual tests. That distinction is what lets the definition survive chart refinement.

A.4.4 The two-error convergence estimate

Suppose $T_j \rightharpoonup T$ through positive currents and $u_j \to u$ locally uniformly. For a fixed test η, first split

$$u_j T_j - uT = (u_j - u)T_j + u(T_j - T).$$

The first term is small by the uniform scalar bound and local boundedness of the positive masses. For the second, approximate u uniformly by a smooth scalar a. The term $a(T_j - T)$ tends to zero on the fixed test by weak convergence, while the two remaining errors are bounded by $\|u - a\|$ times the same mass control. This proves convergence without requiring derivatives of u_j or a common chart for all their approximants.

A.4.5 Total mixed mass and properness

For smooth semipositive metrics on fixed line bundles over a proper n-fold, the mixed degree formula gives the total mass. For globally uniform limits, weak convergence preserves this mass because the constant test 1 is allowed on the compact space. Local uniform convergence on a noncompact domain only controls compactly supported tests. The sequence δ_j on $\mathbb{R}$ has mass one but converges to zero on those tests.

Thus the global mass formula is a consequence of a global comparison and compactness. It is not evidence for an additional local regularization theorem. This distinction answers the recurring question whether a mass or total-variation lemma is a direct consequence of established results: once the measure construction and degree comparison are available, the elementary estimate often is direct; establishing those inputs is the substantial part.

A.4.6 Restriction to a subspace and ambient extension

If a function belongs to a functorial positive approximation class, restriction to an analytic subspace can be obtained by restricting the approximants, provided the morphism and test theory are in the admitted category. The converse problem is different: a positive function or metric on a subspace need not extend positively to an ambient neighbourhood by ordinary extension of continuous functions.

An ambient extension theorem must control curvature in transverse directions and maintain the line-bundle transition data. If it is obtained from global sections or an FS construction, the generation and positivity hypotheses do the work. A proof that only extends the underlying continuous function has not proved a positive extension theorem. Thus a positive extension theorem requires both an extension of the data and control of its curvature.

A.5 Further version-specific corrections

Remark A.5.1 (CLD25 Corollary 16.8.6). The printed tropical Hausdorff exponent is the number of curvature factors. The bidimension of their product instead gives the complementary exponent. Example 19.17.2 disproves the printed statement on $\mathbf{G}_m^{\mathrm{an}}$ using one Gauss point; Proposition 19.17.1 proves a replacement for admitted differences over a nontrivially valued field. For PL metric weights, Proposition 19.18.8 additionally treats trivial valuation. The full transfinite source potential class is not identified here.

Remark A.5.2 (BMP26 Example 6.23). The negative-axis branch of the printed obstacle is discontinuous at $-\epsilon$. Proposition 16.7.1 supplies a continuous correction, proves its envelope formula and computes all atoms of its measure. The example still disproves unrestricted orthogonality in that polyhedral category.

Remark A.5.3 (Cai and Mihatsch: a convention difference). Cai writes $d' = d'_P + \partial'$, while Mihatsch writes $d' = d'_P - \partial'$. Section 15.9 gives the translation. This is not a false curvature formula in either source. The distinction is recorded separately from mathematical errors.

For the remaining CLD25 residual argument, Section 19.18 separates non-charging and the formal positivity inference from residual realization and the calibrated fibre formula. Theorem 19.18.12 completes this assembly in pure dimension, using explicitly cited lifting, integration and graded-reduction theorems; those inputs are not independently re-proved. Similarly, Lemma 11.6.1 sharpens the diagnosis of Proposition 8.1.11: upper semicontinuity suffices for uniform convergence on compact sets, but does not replace discontinuous predecessors by continuous ones.

Remark A.5.4 (The inherited negligibility exponent). CLD25 Proposition 18.3.7 repeats the exponent of Corollary 16.8.6. The metric form of the Gauss-point counterexample refutes this printed assertion too; see Section 19.18.3. The corrected exponent is $\mathcal{H}^{n-p}$; Proposition 19.18.8 verifies the PL-metric scope over every complete base field.

Remark A.5.5 (Base-point normalization). In CLD25 paragraph 18.8.3, the residual bundles $\widetilde{M}_i$ belong to the base germ (Y, y). Formula (19.18.2) places the coefficient there, with its multiplicity and defect, and retains the tensor factor q^{-d}. This is a correction of the displayed notation using Proposition 18.6.1, not a new fibre-integration theorem.

The prescribed graded comparison is proved in Proposition 19.18.20. Together with graded semiampleness and the fixed-tube criterion, it yields Theorem 19.18.22. This supplies the integral PL bridge while leaving the general continuous predecessor problem unchanged.

For CLD25 Proposition 8.1.11, Theorem 11.6.4 supplies a fixed-compact selection theorem but not the required sheaf globalization. The valid indexed inclusion is Proposition 11.6.6. The neighbouring Proposition 8.1.12 uses inherited local integrability and is proved in Proposition 11.6.7; it does not share the missing continuity step.

A.6 The local envelope comparison in Wu's preprint

Remark A.6.1 (A negative barrier and the background metric). In [Wu26], Proposition 3.28, p. 22, the displayed Fubini–Study maximum includes the branch $\log|s_0|$ with weight zero. Its local weight in the frame s_0 is therefore at least zero, whereas the next step requires $v(x) < 0$ and uses negative sublevel sets. Formula (11.7.4) retains a negative constant branch and is consistent with those sublevels. The proof also changes from the initial background ω to a constructed background θ on the same line bundle. These backgrounds need not be equal. Proposition 11.7.3 records $\theta = \omega + d'd''b$ and the compensating global term b; Lemma 11.7.4 supplies a fixed collar for the decreasing tail. These corrections concern the displayed construction, not a counterexample to the proposition's conclusion.

Remark A.6.2 (Version matching before importing the comparison). Wu's Appendix A cites prepublication Propositions 5.20 and 5.22 of the work called APW26b there. The fixed October version [APW26] used in this book has a Lemma 5.20 instead; localization and local PL approximation are treated in Propositions 5.19 and 5.21. Thus the printed locators cannot simply be copied into a reference to the October version. Appendix A also uses Theorem 6.13 of a second, mixed-calculus preprint. The comparison in Corollary 11.7.6 is attributed to Wu's Theorem A.3; it is not presented as a newly completed proof from those external ingredients. The content-level matching and a uniform bounded-family repair are now given in Section A.7; the mixed-calculus theorem and full common-domain passage remain unverified here.

A.7 APW localization and the common-domain audit

The following table matches the sources by mathematical content. The locators in Wu26 refer to a version preceding the October APW file; they are not the theorem numbers in the latter. No new numbering is assigned to the unavailable mixed-calculus theorem.

Locator in Wu26	Fixed source	Content and status
APW26b, 5.20	APW26 v1, 5.19	Localization. Local part repaired in Proposition 11.8.6; extra majorization on all U is false.
APW26b, 5.22	APW26 v1, 5.21	Local PL approximation. Its proof uses the local identity and global PL density, not the false whole-domain inequality.
APW26a, 6.13	Mixed-calculus preprint	Analytic comparison invoked in Appendix A; primary full text not retrieved in this revision.

Remark A.7.1 (The scope of the counterexample). Example 11.8.5 satisfies the hypotheses of the fixed Proposition 5.19 in every positive dimension and contradicts its added whole-domain inequality. The failure is already visible from boundedness in Definition 4.3. Restricting the maximum construction to an inner compact domain repairs the local conclusion and gives uniform constants for a bounded family. We do not label strong semipositivity or Proposition 5.21 false on this basis. Its global PL density and bounded-growth correspondence remain source inputs.

Remark A.7.2 (What a generator comparison still needs). Wu26 Appendix A proves approximation assertions for individual generators before concluding the two inclusions of first-stage classes. Lemma 11.8.1 supplies the numerical diagonal step once a common domain is known; Proposition 11.8.2 states one sufficient domain hypothesis. Corollary 11.8.3 verifies it for sequences with the displayed finite logarithmic presentations. These are concrete cases, not a proof that arbitrary local presentations can be placed on one prescribed neighbourhood. Likewise, the corrected affine support in Lemma 11.8.4 fixes the centred differential and the tropical point, without establishing the analytic pullback comparison assigned to the other APW preprint. Theorem 11.9.5 now supplies one projective rational PL direction; Theorem 11.12.5 now supplies the relative rational PL identification, while the reverse comparison remains.

A.8 Convexity and model approximation in the local comparison

The comparison in Section 11.9 requires genuine convex PL generators and a nontrivially valued model category. Two source checks prevent those requirements from being lost in the notation.

Example A.8.1 (A missing contact condition)**.** In [Wu26, Definition 3.1(3), p. 14], the function is named f, but the displayed supporting-affine condition uses u and reads

$$u(x) = \ell(x) + C, \qquad u(x') \geq \ell(x') \quad (x' \in V).$$

Even after correcting u to f, unrestricted C makes this condition insufficient to define convexity. On an ordinary real open set every finite continuous function satisfies it locally. For example $f(t) = -|t_1|$ on $\mathbf{R}^n$, $n \geq 1$, satisfies it but is not convex.

Proof. Near any point x, continuity supplies a constant c with $f \geq c$. Take $\ell = c$ and $C = f(x) - c$. These satisfy the displayed condition. For the example, the midpoint inequality along the first coordinate would require $0 = f(0) \leq (f(e_1) + f(-e_1))/2 = -1$, which is false. □

The intended touching-support condition is $f(x) = \ell(x)$ and $f \geq \ell$ near x. Equivalently, retain C in the first equation and put $f(x') \geq \ell(x') + C$ in the second. The source's supporting-function argument in Appendix A uses equality at the point, consistent with this repair. We treat the displayed definition as a typographical defect. It neither refutes the intended convex theory nor supplies the missing smooth-to-PL comparison. Whenever this book uses convex PL generators, it uses the actual convexity condition, and Section 11.9.1 gives an explicit local maximum presentation.

Remark A.8.2 (Separating current positivity from common-domain regularization). The proof of [Wu26, Theorem 3.23] passes from its local PL approximation class to CLD current positivity and then to model nefness. For a finite continuous endpoint, Lemma 11.9.1 justifies the first passage directly by uniform convergence on each compact test support. It does not require Theorem A.3 or a resolution of the general predecessor-replacement issue in CLD25 Proposition 8.1.11. For the extensions in Proposition 11.9.3 even this endpoint step is unnecessary: their local weights already have convex

finite-max presentations. The remaining model nefness theorem is an explicit geometric input.

The following table records the source boundaries of the new assembly. In particular, the trivial-valuation statements of Wu are not obtained by applying a theorem whose hypothesis says nontrivially valued.

Input	Verified use	Restriction retained
[GM, Theorem 1.1]	PL/formal/model identification	Proper algebraic analytification, nontrivial valuation; rational version by a tensor power
[CLD25, Corollary 18.8.8]	Positive model curvature gives model nefness	Original current tests and the cited residual geometry
[BE21, Theorem 7.8]	Uniform pure FS approximation of a nef model metric	Ample line bundle, nontrivially valued field
Theorem 20.8.1	Original smooth approximation on the same line bundle	Exact deletion at zeros and transition compatibility
Theorem 11.9.5	One ordinary neighbourhood for a decreasing sequence	Explicit $\mathcal{G}$ generators; one inclusion only

For the generator class, Section 11.10 expands Wu's Remark 3.2 and the Lipschitz step of Remark 3.9. The relative-image definitions 3.6–3.7 are kept separate; Example 11.10.9 explains the failure of a same-chart extension shortcut. For the numerical step, Proposition 11.10.7 uses [GK19, Theorem 0.1] over an algebraically closed field and then the model descent results of [GM, Lemmas 3.3, 3.7 and Proposition 3.5]. The GM locators have been checked in the published 2019 text: an earlier author-hosted PDF carries the v2 arXiv stamp and must not be described as the v3 file.

Theorem 11.10.11 gives an alternative model-theoretic route. Its additional inputs are [GJR, Proposition 7.1 and Definition 7.2], [GRPL, Corollary 5.5], and the local-to-global semipositivity in [GM, Theorem 1.1 and Proposition 3.11]. The maximum result is used for rational PL functions on good strictly analytic spaces. It is not a theorem about arbitrary real-coefficient presentations or positive currents.

Section 11.11 gives a direct curvewise route from the corrected relative support condition to classical psh, and supplies the model-compatible PL identification on unit charts. Its source comparison fixes a locator mismatch: the companion-paper references Proposition 3.1 and Corollary 3.5 in the proof of GRPSH Theorem 4.3 (v2) correspond to Proposition 5.1 and Corollary 5.4 in the consulted GRPL v1. This is a reference-version issue, not a counterexample to either result. Section 11.12 subsequently establishes the extended-image PL identification under its common-branch and coefficient hypotheses, using Martin's finite polyhedral image theorem.

Two further APW proof details are repaired in Section 11.11.5. Proposition 11.11.10 restores the interior lower-support hypothesis omitted from the auxiliary claim in APW Proposition 3.3. Lemma 11.11.11 replaces the erroneous zero set $\{f - \ell_f \leq 0\}$ in the proof of APW Proposition 5.10 by $\{g-\ell_g \leq 0\}$; equation (11.11.4) gives the required modification bound. These repairs retain the intended local conclusions. Separately, Remark 11.11.12 explains why APW Proposition 5.21, stated for its real PL class, is not by itself a value-group-compatible Q-PL approximation theorem. Proposition 11.11.13 supplies a replacement for finite ambient convex maxima on bounded unit charts.

For coefficient-dependent closure statements, Proposition 11.12.11 reconciles GJR v3 Remark 7.11 with its own Example 5.3 and with GRPL v1 Corollary 5.5 and Theorem 5.6. Maximum and limit statements must retain PL membership; integral and Q-PL maxima satisfy that requirement automatically. This is a correction to the overbroad R-PL maximum wording, not a failure of classical psh maximum stability.

The additional model audit in Section 11.14 checks the published GM Lemma 4.12 on p. 353 and the cofinal blowup construction on p. 337. The counterexample concerns the fixed source model; Proposition 11.14.4 recovers the formal-model domination needed by BE21 Theorem 7.8. Corollary 11.14.5 supplies finite simultaneous polarizations; Proposition 11.14.8 supplies the exact Picard relation and bounded correction, with formal-domination scope. Section 11.13 instead uses GJR Corollary 5.9, Lemmas 6.11 and 6.14, and Proposition 7.10; its original-psh comparison additionally uses GRPSH Theorem 4.3.

Model morphisms and reduced fibres. The source chain in Section 11.15 is FK arXiv v5, 0.9.2.7 and 0.8.5.25(2), then I.10.3.1 for algebraic

morphisms; BE21 Theorem 4.20 for reduced fibres over algebraically closed fields; and the published BGM22 Lemma 1.1, p. 6223. BE21 Lemmas 6.1–6.3 and Theorem 6.4 distinguish exact norm equalities from bounded errors. None of these inputs uses a general positive-current BT theorem.

A separate convention audit is now given in Section 11.21.4. The last finite-selection step of APW Proposition 5.12 preserves ordinary real-polyhedral PL functions, but rational face directions and a rational simplicial presentation require additional control. Examples 11.21.9 and 11.21.11 separate these requirements. They identify a gap in applying that proof under the literal source conventions; they do not disprove the existence conclusion of Proposition 5.21. The germ and transport results in Section 11.21 provide specified forward interfaces, not an unconditional replacement for the source regularization theorem.

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Appendix B

Recovered native algebraization source

B.1 Polarization and algebraization of a supplied formal curve

Let K be a complete nontrivially valued field with a real-valued rank-one valuation v. Put $O = K^\circ$, $\mathfrak{m} = K^{\circ\circ}$, and $k = O/\mathfrak{m}$, and fix $0 \neq \pi \in \mathfrak{m}$. The value group need not be discrete. All schemes and invertible sheaves below are ordinary algebraic schemes and sheaves until a formal completion is explicitly taken.

A proper flat finitely presented curve tower over O/π^n admits a compatible polarization, even when only its residue curve is initially polarized. The reason is topological: the residue curve and all levels of the tower have the same Noetherian underlying space of dimension at most one. We first record the precise lifting statement that uses this fact. The algebraization step will then be an application of the non-Noetherian theorem Stacks Tag 0D4T.

B.1.1 Lifting across a nil thickening

Lemma B.1.1 (Picard lifting in topological dimension at most one)**.** *Let $i : Z \hookrightarrow Y$ be a closed immersion of schemes inducing a homeomorphism of*

underlying spaces. Suppose that this common space is Noetherian and has dimension at most one. Then

$$i^* : \mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(Z)$$

is surjective. If $\mathcal{L}_Z$ is ample, every invertible sheaf on Y whose restriction is isomorphic to $\mathcal{L}_Z$ is ample. No finite-generation or nilpotence bound on the ideal of Z is required.

Proof. Identify the two underlying spaces, and put $\mathcal{J} = \ker(\mathcal{O}_Y \to i_*\mathcal{O}_Z)$. On an affine open of Y, every prime contains the ideal defining Z, so that ideal is nil. In particular its stalk ideals are nil. An element of a ring whose image is a unit modulo a nil ideal is a unit: a lift of its inverse has product $1+j$ with j nilpotent, and $1+j$ has a finite geometric-series inverse. Applied to stalks, this gives an exact sequence of abelian sheaves on the common space,

$$1 \longrightarrow 1 + \mathcal{J} \longrightarrow \mathcal{O}_Y^\times \longrightarrow i_*\mathcal{O}_Z^\times \longrightarrow 1.$$

Here $1 + \mathcal{J}$ has its multiplicative group law. There is no identification with the additive sheaf $\mathcal{J}$ in this argument.

Grothendieck vanishing for a Noetherian topological space of dimension at most one, Stacks Tag 02UZ, applies to every abelian sheaf. Hence $H^2(Y, 1 + \mathcal{J}) = 0$. The long exact cohomology sequence of Stacks Tag 01DZ and the identification of the Picard group with $H^1(\mathcal{O}^\times)$, Stacks Tag 09NU, show that i^* is surjective. Finally, Stacks Tag 09MW identifies ampleness of an invertible sheaf with ampleness of its restriction along any closed immersion inducing a homeomorphism. This proves the last assertion without requiring a nilpotence bound on $\mathcal{J}$. □

B.1.2 A compatible ample bundle on the tower

Proposition B.1.2 (Polarization of a supplied curve tower)**.** *For $n \geq 1$, put $R_n = O/\pi^n O$ and $S_n = \operatorname{Spec} R_n$. Suppose given schemes $X_n \to S_n$, proper, flat, and of finite presentation, together with transition morphisms $i_n : X_n \hookrightarrow X_{n+1}$ for which the squares over $S_n \hookrightarrow S_{n+1}$ are cartesian. Suppose that*

$$C = X_1 \times_{R_1} k$$

has dimension at most one. Then C has an ample invertible sheaf. For any prescribed ample invertible sheaf $\mathcal{L}_C$ on C, there exist ample invertible sheaves $\mathcal{L}_n$ on X_n, an isomorphism $\mathcal{L}_1|_C \simeq \mathcal{L}_C$, and isomorphisms

$$\varphi_n : i_n^* \mathcal{L}_{n+1} \xrightarrow{\sim} \mathcal{L}_n \qquad (n \geq 1).$$

Proof. The scheme C is proper and of finite type over k, hence is separated and has a Noetherian underlying space. A separated finite-type scheme over a field of dimension at most one has an ample invertible sheaf, by Stacks Tag 0A25 and its proof.

For $a \in \mathfrak{m}$, the rank-one valuation gives an integer $N \geq 1$ with $Nv(a) \geq v(\pi)$, unless $a = 0$, when no choice is needed. Thus $a^N \in \pi O$, and $\mathfrak{m}/\pi O$ is a nil ideal of R_1. Its extension to any R_1-algebra is still nil: each element is a finite sum of multiples of nilpotent elements in a commutative ring. Consequently the closed immersion $C \hookrightarrow X_1$ induces a homeomorphism. This is the same radical comparison used in the established formal-fibre argument 3.17.2; it does not assert that the rings $\mathcal{O}_{X_1}$ are Noetherian.

The underlying space of X_1 is therefore Noetherian of dimension at most one. Lemma B.1.1 lifts the prescribed $\mathcal{L}_C$ to an ample invertible sheaf $\mathcal{L}_1$. For each $n \geq 1$, the kernel of $R_{n+1} \to R_n$ is square-zero, since $2n \geq n + 1$. The cartesian square shows that i_n is a square-zero closed immersion and a homeomorphism. All X_n thus have the same underlying space as C.

Inductively apply Lemma B.1.1 to i_n, choosing $\mathcal{L}_{n+1}$ and an isomorphism φ_n as stated. The lemma also gives ampleness at each step. Composites of the chosen φ_n supply the compatibilities between any two levels of the tower. The construction proves existence and makes no assertion of uniqueness of these choices. □

B.1.3 Algebraization and the nodal conclusion

Theorem B.1.3 (Algebraization over the complete rank-one valuation ring)**.** *Under the hypotheses of Proposition B.1.2, there exist a projective, flat, finitely presented O-scheme Y, an ample invertible sheaf $\mathcal{L}$ on Y, and compatible isomorphisms*

$$Y \times_O R_n \simeq X_n, \qquad \mathcal{L}|_{X_n} \simeq \mathcal{L}_n \quad (n \geq 1),$$

where $(\mathcal{L}_n, \varphi_n)$ can be any compatible system provided by that proposition. In particular $\mathcal{L}|_C$ is the prescribed $\mathcal{L}_C$.

If C/k is at-worst-nodal of pure dimension one, then Y/O is at-worst-nodal of relative dimension one. No splitting condition on its nodes or branches is imposed.

If the tower is the system of quotients of a supplied π-adic formal O-scheme $\mathfrak{X}$, and $\mathfrak{X}$ is recovered from these quotients with their transition maps, then the compatible isomorphisms give

$$\widehat{Y}_{\pi} \simeq \mathfrak{X}.$$

This formal scheme is admissible, and its generic fibre satisfies $\mathfrak{X}_{\eta} \simeq (Y_K)^{\mathrm{an}}$.

Proof. We first check the base ring in the algebraization theorem. Since $v(\pi) > 0$, the sequence $nv(\pi)$ tends to infinity. The ideals $\pi^n O$ therefore define the valuation topology on O. The ring O is closed in the complete field K, so it is complete for this topology. More explicitly, representatives of a compatible system of classes modulo π^n form a Cauchy sequence in O. Its limit has the prescribed residues, since every $\pi^n O$ is closed. Also $\bigcap_n \pi^n O = 0$. Hence

$$O \simeq \varprojlim_{n} R_n.$$

Each transition $R_{n+1} \to R_n$ is surjective with square-zero kernel. The cartesian tower, its proper flat finitely presented structure morphisms, and the compatible invertible sheaves with $\mathcal{L}_1$ ample now satisfy every hypothesis of Stacks Tag 0D4T. That theorem supplies Y, $\mathcal{L}$, and all the displayed compatible isomorphisms, with Y/O proper, flat, and finitely presented and $\mathcal{L}$ ample on Y. The affine base $\operatorname{Spec} O$ has an ample invertible sheaf. Therefore properness together with the ample sheaf $\mathcal{L}$ implies projectivity by Stacks Tag 0B45, the equivalence of conditions (5) and (1).

Suppose next that C/k is at-worst-nodal of pure dimension one. For the flat finitely presented morphism $Y \to \operatorname{Spec} O$, Stacks Tag 0DSC gives a maximal open $U \subseteq Y$ on which it is at-worst-nodal of relative dimension one; this open commutes with arbitrary base change. Since $Y_k \simeq C$, it follows that $Y_k \subseteq U$. If the closed complement $Y \setminus U$ were nonempty, its image in $\operatorname{Spec} O$ would be nonempty and closed, by properness. Every nonempty closed subset of the spectrum of a local ring contains its closed point. This would force $Y \setminus U$ to meet Y_k, a contradiction. Thus $U = Y$.

Finally, the compatible isomorphisms at every level identify both the common underlying spaces and their inverse systems of structure sheaves. Taking the inverse limit gives the stated isomorphism of π-adic formal schemes. The

established projective-completion result 4.13.2 applies to the projective, flat, finitely presented scheme Y. It says that $\widehat{Y}_\pi$ is admissible and that its generic fibre is $(Y_K)^{\mathrm{an}}$. Transporting this identification through the formal isomorphism, using the functoriality in 3.17.2, gives the assertion for $\mathfrak{X}$. □

B.1.4 The formal interface and the nondiscrete boundary

The input is the supplied cartesian tower, including properness, flatness, and finite presentation at every level. For a formal model, these hypotheses must be checked for its actual quotients by π^n. For example, on a flat formal affine chart with a finite restricted-power-series presentation

$$B = O\langle T_1, \dots, T_r\rangle/(f_1, \dots, f_s),$$

reduction modulo π^n has the finite presentation $R_n[T_1, \dots, T_r]/(\bar{f}_1, \dots, \bar{f}_s)$; restricted series become polynomials at this level. Flatness descends by base change. Properness of the reductions is the separate global input. Reconstruction from the quotient sheaves is the stated completeness condition on the formal model.

The distinction between k and $O/\pi O$ cannot be dropped. If the value group is nondiscrete, then $\mathfrak{m}^2 = \mathfrak{m}$: for $a \in \mathfrak{m} \setminus \{0\}$ choose $b \in K$ with $0 < v(b) < v(a)$ and write $a = b(a/b)$. Both factors lie in $\mathfrak{m}$. It follows that

$$\varprojlim_n O/\mathfrak{m}^n = k,$$

which does not recover O. Moreover $\mathfrak{m}/\pi O$ is nonzero, nil, and idempotent, hence is not nilpotent. Its nonzero elements exist because one can choose a positive value less than $v(\pi)$. Lemma B.1.1 handles this first thickening without replacing it by finitely many square-zero steps.

The theorem algebraizes the specified tower and preserves its residue nodes. It does not construct a tower from an arbitrary analytic curve, choose a polarization canonically, or algebraize a supplied formal group action. Those are separate assertions.

Appendix C

Native scalar extension and finite restriction of constants

The ordinary integral formulas in this chapter give the two calibrations used in the finite-place calculation following Proposition 23.7.1.

C.1 Scalar extension and restriction of constants for ordinary integration

Throughout, K is complete for its specified real-valued non-Archimedean absolute value, and L/K is a complete isometric extension. Trivial valuation and arbitrary positive affinoid radii are allowed. Write $p : X_L \to X$ for actual completed scalar extension. All moment coordinates are $-\log|f_i|$ with the given absolute value.

The notation $\mathcal{A}^{a,b}_{X/K}$ means the ordinary smooth moment sheaf of v0.37 [5.52.3, 5.52.4]: a germ has one smooth ambient superform on an ordinary affinoid neighbourhood, modulo tangential equality on every face of the actual joint moment image. Its compactly supported top integral $\int_{X/K,\mathcal{F}}$ is the coherent-module functional of that source. The coefficient $\mathcal{F}$ weights the integration cycle; the differential form itself has real scalar coefficients. Purity means purity of analytic dimension on ordinary affinoid neighbourhoods, as in [5.50.1, 5.50.3, 5.50.4, 5.50.6].

Theorem C.1.1 (Native scalar-extension invariance)**.** *Let X be a good Hausdorff K-analytic space pure of analytic dimension d, and let $\mathcal{F}$ be coherent on X_G. Then X_L is good, Hausdorff and pure of analytic dimension d. Coherent pullback $\mathcal{F}_L = p^*\mathcal{F}$ and ordinary moment pullback are defined on actual completed scalar extensions, and for every $\alpha \in \mathcal{A}^{d,d}_{X/K,c}(X)$,*

$$\int_{X_L/L,\mathcal{F}_L} p^*\alpha = \int_{X/K,\mathcal{F}} \alpha. \tag{C.1.1}$$

The pullback has compact support. These assertions respect complete extension towers, coherent short exact sequences, and ordinary open restriction containing the support.

The proof is in Section C.1.3. It uses the frozen moment sheaf and integral, and does not require a comparison with another form sheaf. For finite L/K, the distinct formula

$$\int_{(X_L)_{[K]}/K,\mathcal{F}_L} p^*\alpha = [L:K] \int_{X/K,\mathcal{F}} \alpha \tag{C.1.2}$$

is proved in Section C.1.4. Here $Y_{[K]}$ denotes the same analytic space after restriction of constants; this notation is used only for finite L/K.

C.1.1 Generic lengths and componentwise purity

Lemma C.1.2 (The generic coefficient under scalar extension)**.** *Let A be a generalized K-affinoid algebra, $B = A_L$, and M a finite A-module. Then $M_L = B \otimes_A M = M\widehat{\otimes}_K L$, with equivalent canonical complete norms, and $A \to B$ is faithfully flat. If $\mathfrak{q} \in \mathrm{Min}(B)$ and $\mathfrak{p} = \mathfrak{q} \cap A$, then $\mathfrak{p} \in \mathrm{Min}(A)$ and*

$$\mathrm{length}_{B_\mathfrak{q}}(M_L)_\mathfrak{q} = \mathrm{length}_{A_\mathfrak{p}} M_\mathfrak{p}\ \mathrm{length}_{B_\mathfrak{q}}(B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q}). \tag{C.1.3}$$

In particular the generic coefficient can acquire a nontrivial length even when the original component is reduced.

Proof. The completed-module identification and faithful flatness are the exact assertions of [2.33.5, 2.33.6]; its proof uses closed finite presentations and strict exactness, not a formal replacement of completed tensors by algebraic tensors. Flat going down [StacksAlg, Tag 00HS, Lemma 10.39.19] implies that a prime strictly below $\mathfrak{p}$ would lift to a prime strictly below $\mathfrak{q}$. Thus $\mathfrak{p}$ is minimal.

The two local rings are Noetherian of dimension zero, hence Artinian [StacksAlg, Tag 00KH, Lemma 10.60.5]; their finite modules have finite length [StacksAlg, Tag 00JB, Lemma 10.53.6]. The localized homomorphism $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ is flat. A composition series of $M_{\mathfrak{p}}$ has every simple quotient equal to $\kappa(\mathfrak{p})$. Tensoring this series with $B_{\mathfrak{q}}$ is exact, and every successive quotient is $B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}}$. Additivity of length gives (C.1.3). The quotient is nonzero because the map is local. □

For later use, if $R \to S$ is a flat map of Noetherian rings and $\mathfrak{p} \in \operatorname{Min}(R)$, every prime minimal over $\mathfrak{p}S$ is minimal in S. Indeed, flat going down for $R/\mathfrak{p} \to S/\mathfrak{p}S$ makes its contraction $\mathfrak{p}$. Choose a minimal prime $\mathfrak{q}_0$ of S below it, using [StacksAlg, Tag 00E0, Lemma 10.17.2, part (4)]. Its contraction is contained in the minimal prime $\mathfrak{p}$, so equals $\mathfrak{p}$; minimality over $\mathfrak{p}S$ forces equality with $\mathfrak{q}_0$.

Lemma C.1.3 (Purity after an arbitrary complete extension)**.** *If $\mathcal{M}_K(A)$ is nonempty and pure of analytic dimension d, then $\mathcal{M}_L(A_L)$ is nonempty and pure of analytic dimension d.*

Proof. **Strict integral algebras.** Suppose first that A is strict and integral. Finite normalization [1.12.2] gives a bounded finite injection $R = K\langle T_1, \dots, T_d\rangle \hookrightarrow A$. Its finite-module norm is equivalent to the given affinoid norm, by the norm argument in [3.21.1, 3.21.6]: for nontrivial valuation use the bounded lifting theorem on a finite-generator surjection, while for trivial valuation both strict quotient norms take only the values zero and one.

Put $F = \operatorname{Frac}(R)$. The torsion-free finite R-module A embeds into R^m: embed it into the finite-dimensional vector space $A \otimes_R F$, choose a basis, and clear the finitely many denominators of the coordinates of a finite generating list. By the finite-module theorem [1.18.3], this embedding is norm-strict onto a closed submodule. Strict exactness of completed scalar tensors [2.33.3] preserves its injectivity. The module comparisons in [2.33.5, 2.33.6] therefore give an injection

$$A_L \hookrightarrow R_L^m.$$

Consequently A_L is torsion free over the domain $R_L = L\langle T_1, \dots, T_d\rangle$ and is finite over it.

Let $\mathfrak{q}$ be a minimal prime of A_L. No nonzero element of R_L can lie in $\mathfrak{q}$. To check this without a reducedness assumption, localize at $\mathfrak{q}$. This is an

Artinian local ring; an element of its maximal ideal is nilpotent. If a nonzero $r \in R_L$ lay in $\mathfrak{q}$, some power of r would annihilate a nonzero element after localization. Clearing its denominator would give a nonzero element of A_L killed by that power of r, contradicting torsion freeness. Thus

$$R_L \hookrightarrow A_L/\mathfrak{q}$$

is a finite injection. The dimension theorem for strict normalization [3.21.1, 3.21.6] makes every such quotient have dimension d.

Strict pure algebras. Now let A be strict and pure, allowing nilpotents. For $\mathfrak{q} \in \operatorname{Min}(A_L)$, Lemma C.1.2 makes $\mathfrak{p} = \mathfrak{q} \cap A$ minimal. The actual quotient identity

$$A_L/\mathfrak{p}A_L = (A/\mathfrak{p})_L$$

is part of faithful scalar extension [2.33.5, 2.33.6]. The prime $\mathfrak{q}/\mathfrak{p}A_L$ is minimal there. The strict integral case, applied to $A/\mathfrak{p}$, gives dimension d for $A_L/\mathfrak{q}$.

Arbitrary radii. Choose a finite free-radius strictification K_s/K by [1.17.6]. The exact component theorem [5.35.1] makes A_{K_s} pure of dimension d. Choose a common complete extension E of K_s and L by [3.21.7]. The strict case shows that $A_E = (A_{K_s})_E$ is pure of dimension d.

Set $B = A_L$ and fix $\mathfrak{q} \in \operatorname{Min}(B)$. Faithful flatness makes $(B/\mathfrak{q})_E$ nonzero. The observation preceding this lemma shows that its minimal components are minimal components of $B_E = A_E$, so all have dimension d. The analytic dimension of a nonempty affinoid is the maximum of the dimensions of its minimal components: each analytic point kernel contains a minimal prime, passing to that quotient preserves its completed residue field, and [3.22.7] identifies dimension with the maximum of residual dimensions at points. Hence $\dim_E \mathcal{M}((B/\mathfrak{q})_E) = d$. Scalar invariance of analytic dimension [3.21.8] now gives $\dim_L \mathcal{M}(B/\mathfrak{q}) = d$. This holds for every minimal prime of B. Nonvanishing follows from faithful flatness. □

Proposition C.1.4 (Finite generic pushforward coefficients)**.** *Suppose $n = [L : K] < \infty$ and $B = A_L$. For $\mathfrak{p} \in \operatorname{Min}(A)$ one has*

$$\sum_{\substack{\mathfrak{q} \in \operatorname{Min}(B) \\ \mathfrak{q} \cap A = \mathfrak{p}}} \operatorname{length}_{B_{\mathfrak{q}}}(M_L)_{\mathfrak{q}} [\kappa(\mathfrak{q}) : \kappa(\mathfrak{p})] = n \operatorname{length}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}. \tag{C.1.4}$$

Thus the algebraic generic cycle pushforward of the pullback coefficient is $n[M]$, with the indicated full residue-field degrees.

Proof. Finite-dimensional norm equivalence [1.12.1] gives bounded coordinate maps for a K-basis $e_1, \dots, e_n$ of L. Tensoring these maps with the identity of any Banach K-space identifies its completed tensor with L with its n-fold direct sum. In particular

$$A_L = A \otimes_K L = \bigoplus_{i=1}^{n} A e_i.$$

The finite-dimensional $\kappa(\mathfrak{p})$-algebra $C = B \otimes_A \kappa(\mathfrak{p})$ has dimension n. Its local factors are $B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}}$ for the primes in the displayed sum. Indeed they lie over the minimal prime $\mathfrak{p}$ in a finite flat map; a prime over it is minimal, since $A_{\mathfrak{p}}$ is Artinian and its finite algebra $B_{\mathfrak{p}}$ has finite length as an $A_{\mathfrak{p}}$-module, so is Artinian as a ring. A composition series of each factor gives

$$n = \sum_{\mathfrak{q}\mid\mathfrak{p}} \operatorname{length}_{B_{\mathfrak{q}}}(B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}})[\kappa(\mathfrak{q}) : \kappa(\mathfrak{p})].$$

Multiply by $\operatorname{length}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$ and apply Lemma C.1.2. □

C.1.2 The entire Gauss module

For a positive real polyradius r, let

$$R_{K,r} = K\{r^{-1}T, rU\}/(T_iU_i - 1), \qquad H_{K,r} = \widehat{\operatorname{Frac}(R_{K,r})}$$

for the multiplicative Laurent Gauss norm. Empty tuples give $R = H = K$. These are the actual algebras and fields of [5.36.2, 5.37.1].

Lemma C.1.5 (Module base change at a complete Gauss field)**.** *Let D be a finite canonical $R_{K,r}$-algebra and P a finite D-module, with its canonical norm. Then $D_L = D\widehat{\otimes}_K L = D \otimes_{R_{K,r}} R_{L,r}$ and $P_L = P\widehat{\otimes}_K L = P \otimes_{R_{K,r}} R_{L,r}$, with bounded inverse comparisons. There is a canonical isomorphism*

$$P_L \otimes_{R_{L,r}} H_{L,r} \simeq (P \otimes_{R_{K,r}} H_{K,r}) \otimes_{H_{K,r}} H_{L,r}. \tag{C.1.5}$$

The native vector-space dimension is unchanged. If $E = D \otimes_{R_{K,r}} H_{K,r} = \prod_j E_j$ and $Q = P \otimes_{R_{K,r}} H_{K,r}$, then

$$\dim_{H_{K,r}} Q = \sum_j \operatorname{length}_{E_j} Q_j \, [E_j/\mathfrak{m}_j : H_{K,r}]. \tag{C.1.6}$$

The corresponding formula after extension has the same total.

Proof. Composing finite free quotient presentations makes P a finite canonical $R_{K,r}$-module; the quotient norm is the given canonical D-module norm up to uniform equivalence. The actual scalar pushouts and finite tensor comparison [2.33.18, 2.33.19, 1.18.7] give the two identifications with $R_{L,r}$. Equivalently one may apply the algebra statement [5.36.2, 5.37.1] to the square-zero finite algebra $D \oplus P$ and take its P summand. The coefficient embedding $H_{K,r} \hookrightarrow H_{L,r}$ is isometric by [5.36.2, 5.37.1]. Associativity of these finite tensors gives (C.1.5). A basis of a finite-dimensional vector space remains a basis after field extension, so the dimensions agree. The finite algebra E is the product of its Artinian local factors [StacksAlg, Tag 00JB, Lemma 10.53.6]; the simple factors of Q_j are its full residue field. Taking dimensions proves (C.1.6). No separable degree or reduced quotient enters this computation. □

Proposition C.1.6 (Native global chains under complete extension)**.** *Let $X = \mathcal{M}_K(A)$ be pure of analytic dimension d, M a finite A-module, and f any finite tuple of analytic units. Then, in the same real moment space and saturated integral lattices,*

$$\mathcal{T}^{L}_{p^*f}(M_L) = \mathcal{T}^{K}_{f}(M). \tag{C.1.7}$$

Consequently their integrals against any smooth ambient (d,d)-form defined near the compact actual moment image agree.

Proof. Scalar projection is surjective [2.33.13], and the absolute values of the pulled-back units agree pointwise. Thus the actual moment images agree. Lemma C.1.3 supplies purity upstairs. Use Theorem 5.39.3 for the finite actual images and their boundary images. Use Lemma 5.14.2 to subdivide the common image and both boundary images simultaneously. Use the closed finite-module chains on this subdivision. Every d-cell interior avoids those boundary images. At a point w there, the full moment fibres in both spaces are interior, and the local image lies in the same saturated rank-d affine tangent plane.

Let D_w be the actual algebra of the entire simultaneous level $|f_i| = e^{-w_i}$. The domain base-change theorem [2.36.9] identifies the *entire* upstairs level algebra with $(D_w)_L$. Choose the same integral transverse projection Q on the tangent lattice N, and put $e_Q = [\mathbf{Z}^d : Q(N)]$. The flat-germ theorem in [5.36.2, 5.37.1] makes D_w finite canonical over $R_{K,\exp(-Qw)}$. Put $P =$

$M \otimes_A D_w$. All the completed module restrictions are the actual finite tensors by [1.18.7]. Lemma C.1.5 therefore gives

$$\frac{\dim_{H_{L,\exp(-Qw)}}(P_L \otimes_{R_{L,\exp(-Qw)}} H_{L,\exp(-Qw)})}{e_Q} = \frac{\dim_{H_{K,\exp(-Qw)}}(P \otimes_{R_{K,\exp(-Qw)}} H_{K,\exp(-Qu}}{e_Q}$$

By the exact module-weight theorem [5.40.1, 5.40.2], these are the two generic coefficients. That theorem also expresses them through generic Artinian lengths as in Lemma C.1.2. Equality on every top cell proves (C.1.7). Lower-dimensional cells carry no d-chain term on either side. The same real form, cell, orientation and lattice measure are used in the finite-cell integral, so the finite integral sums agree. Boundary-image codimension is used only to read generic coefficients; no differential boundary term is being discarded. □

C.1.3 Global pullback and compact support

Proof of Theorem C.1.1. **Spaces and coefficients.** Actual scalar extension and inverse images of domains are supplied by [2.40.12, 2.36.9]. If V is an ordinary affinoid neighbourhood of x, $p^{-1}(V) = V_L$ is affinoid and contains the open set $p^{-1}(\mathrm{Int}_{\mathrm{top}}(V, X))$. Hence these inverse images give affinoid neighbourhoods at every point upstairs, proving goodness. Points with different images can be separated downstairs. Points with the same image lie in the open interior of one such affinoid neighbourhood and can be separated there. Thus X_L is Hausdorff. Lemma C.1.3 proves its local purity.

On $V = \mathcal{M}(A)$, set $\mathcal{F}_L|_{V_L}$ to be the coherent module represented by $A_L \otimes_A \Gamma(V, \mathcal{F})$. The exact finite-module equivalence [2.39.6] and actual iterated tensors [2.33.9] make these constructions agree on restrictions; hence they glue. Faithful flatness [2.33.5, 2.33.6] preserves coherent exact sequences.

Ordinary form pullback. Pull a representative (V, f, η) back to $(V_L, p^* f, \eta)$ near any point over $\mathrm{Int}_{\mathrm{top}}(V, X)$. If two representatives agree on a smaller ordinary affinoid neighbourhood W, their joint moment images after extension equal the original joint images, by [2.33.13]. Tangential equality therefore remains true on W_L. These local representatives define $p^*\alpha$ independently of choices and glue on ordinary opens. The same description proves compatibility with d', d'', J, wedge and composition of extensions. This is also the mixed-ground-field version of the representative argument in [5.52.3, 5.52.4].

Support and integration. Put $S = \mathrm{supp}(\alpha)$. Vanishing near x pulls back to vanishing near every point above x, so $\mathrm{supp}(p^*\alpha) \subset p^{-1}(S)$. Choose

finitely many ordinary affinoid neighbourhoods V_i whose interiors cover S, and put $Q = \bigcup_i V_i$. The full inverse image $Q_L = \bigcup_i (V_i)_L$ is a finite union of compact affinoids. Since S is closed in the Hausdorff space X, $p^{-1}(S)$ is a closed subset of Q_L, hence compact. It contains the closed support of $p^*\alpha$. The interiors of the $(V_i)_L$ cover that support.

Use the finite chart data induced by α on Q in [5.52.3, 5.52.4, 5.50.1, 5.50.3, 5.50.4, 5.50.6]. Refine the cover so each member has one presenting unit tuple and ambient form. Its complete inverse-image cover gives the pulled-back data on Q_L. Every intersection satisfies $(V_I)_L = \bigcap_{i\in I}(V_i)_L$, with the actual algebra and coefficient module supplied by [2.36.9]. Proposition C.1.6 applies on every such affinoid intersection. It equates the corresponding terms in the two finite inclusion–exclusion formulas [5.50.1, 5.50.3, 5.50.4, 5.50.6]. Thus their compact integrals agree. The compact-support theorem [5.52.3, 5.52.4] identifies these with (C.1.1). The finite unions need not themselves be good; they lie in the good Hausdorff pure ambient spaces just established.

For a tower $K \subset L \subset E$, all local pullbacks compose by [2.33.9]. Applying (C.1.1) twice gives its assertion for E/K, with factor one at each step. Exact-sequence compatibility uses the flat coherent pullback and additivity of the given integral. Ordinary restriction containing the support uses the same compact neighbourhoods and the support independence in [5.52.3, 5.52.4]. □

C.1.4 Finite restriction of constants

Lemma C.1.7 (The analytic space with finite constants restricted)**.** *Suppose $n = [L : K] < \infty$. Every generalized L-affinoid algebra B is a generalized K-affinoid algebra with the same Banach topology. The two analytic spectra, domains, structure sheaves and completed residue fields agree. On a good Hausdorff L-analytic space Y this gives $Y_{[K]}$, with the same ordinary moment sheaf and supports. Purity of analytic dimension d is unchanged.*

Proof. Fix a basis $e_1, \dots, e_n$ of L/K. By [1.12.1] there are $c, C > 0$ such that

$$c \max_i |a_i| \le \left| \sum_i a_i e_i \right|_L \le C \max_i |a_i| \quad (a_i \in K).$$

Applying the same bounds to every power-series coefficient identifies $L\{r^{-1}T\}$ with $K\{r^{-1}T\} \otimes_K L$, finite free with an equivalent norm. Finite canonical

algebras and their quotients are actual generalized affinoids [2.33.18, 2.33.19, 1.18.3]. This gives the asserted K-affinoid realization of B. Every bounded K-seminorm on B restricts to the unique extended absolute value on L [1.12.1]. The spectra therefore consist of the same seminorms with the same evaluation topology. Evaluated fraction fields and their completions are unchanged.

Here is the domain comparison, including its norm condition. A K-affinoid algebra C with a bounded structural map $B \to C$ has an induced bounded L-action. A norm-admissible presentation $K\{s^{-1}U\} \twoheadrightarrow C$ extends coefficientwise to $L\{s^{-1}U\} \twoheadrightarrow C$. Bounded multiplication gives a bounded map; the original K-lifts give a uniform quotient-norm bound, so it is norm-admissible. Thus C is also L-affinoid. An L-affinoid test object is already K-affinoid by the first paragraph. A K-test map into B conversely gives its source this L-structure. Hence the two domain universal properties are equivalent: each tests exactly the same bounded factorizations over B. Domain algebras and restriction maps agree by uniqueness. Gluing affinoid charts gives the stated global space and structure sheaf.

The ordinary neighbourhoods, analytic units, absolute values and joint moment images used in the definition [5.52.3, 5.52.4] now coincide literally; thus so do the germs and their relations. At each point y, the valued-field tower $K \subset L \subset \mathcal{H}(y)$ has $d(\mathcal{H}(y)/K) = d(\mathcal{H}(y)/L)$, since L/K is algebraic [3.22.2]. The maximum formula [3.22.7] gives equality of analytic dimensions, also on every minimal-prime quotient. This proves the purity assertion. □

Lemma C.1.8 (The full finite Gauss degree)**.** *For finite L/K and every real positive r,*

$$H_{K,r} \otimes_K L = H_{L,r}, \qquad [H_{L,r} : H_{K,r}] = [L : K] = n.$$

The natural maps are bounded isomorphisms with bounded inverses.

Proof. The coefficient bounds in Lemma C.1.7 give

$$R_{L,r} = \bigoplus_i e_i R_{K,r}, \qquad c \max_i \|a_i\|_r \le \left\| \sum_i e_i a_i \right\|_r \le C \max_i \|a_i\|_r.$$

Both tails and uniqueness are controlled by these inequalities. Both Laurent algebras are domains by multiplicativity of their Gauss norms [5.36.2, 5.37.1]. Localizing $R_{L,r}$ over $F = \mathrm{Frac}(R_{K,r})$ gives a finite-dimensional F-domain, hence a field, necessarily $\mathrm{Frac}(R_{L,r})$. The e_i remain a basis. Clearing

a common denominator extends the two bounds to coefficients in F. Completion consequently gives $\bigoplus_i e_i H_{K,r}$, with the tensor-product multiplication, and this completion is $H_{L,r}$. This proves both the isomorphism and the full degree. □

Theorem C.1.9 (Finite change of calibration)**.** *Let L/K be finite of degree n. If Y is good Hausdorff and pure of dimension d over L, $\mathcal{G}$ is coherent, and $\beta \in \mathcal{A}^{d,d}_{Y/L,c}(Y)$, then*

$$\int_{Y_{[K]}/K,\mathcal{G}} \beta = n \int_{Y/L,\mathcal{G}} \beta. \tag{C.1.8}$$

For $Y = X_L$ and $\mathcal{G} = \mathcal{F}_L$, this gives (C.1.2)*. The factors multiply correctly in finite towers.*

Proof. First let $Y = \mathcal{M}_L(B)$ be affinoid with one unit tuple f, and let N be the finite B-module representing $\mathcal{G}$. By Lemma C.1.7, its actual level algebra D_w, module $P = N \otimes_B D_w$ on that level, real moment image and tangent lattice are identical in the two field interpretations. Choose a common subdivision avoiding both boundary images at generic top-cell points, as in Proposition C.1.6. There D_w and P are finite canonical over $R_{L,r}$. They are also finite canonical over $R_{K,r}$ by the finite free coefficient decomposition and composed finite presentations. Lemma C.1.8 gives

$$P \otimes_{R_{K,r}} H_{K,r} = P \otimes_{R_{L,r}} H_{L,r}$$

as the same module with different scalar fields. Its dimension over $H_{K,r}$ is n times its dimension over $H_{L,r}$. Dividing by the same lattice index proves $\mathcal{T}^K_f(N) = n\mathcal{T}^L_f(N)$, using the closed finite-module chains. Integrating by the finite-cell functional and then applying the identical finite covers and inclusion–exclusion formula proves (C.1.8) for the ordinary compact integrals.

There is also a direct check of (C.1.2). The basis of L/K gives

$$p_* p^* \mathcal{F} \simeq \mathcal{F}^{\oplus n}$$

on every full affinoid inverse image, compatibly with restriction. The map $p : (X_L)_{[K]} \to X$ is finite with its canonical module norm. The spaces have the required purity and Hausdorffness by the preceding lemmas. Thus the exact finite projection theorem [5.54.1] and coherent additivity give

$$\int_{(X_L)_{[K]}/K,\mathcal{F}_L} p^*\alpha = \int_{X/K,p_*\mathcal{F}_L} \alpha = n \int_{X/K,\mathcal{F}} \alpha.$$

This uses the actual coherent direct image [2.35.15], with no local-freeness assumption on $\mathcal{F}$. Finite tower factors multiply because $[E : L][L : K] = [E : K]$, while Theorem C.1.1 has factor one in every tower. $\square$

Supplement to Chapter 8, Section 8.4.4.

Cancellation for finite affinoid cuts

Proposition. *Let X be good, Hausdorff and analytically pure of dimension $d \geq 1$ over a complete real-valued non-Archimedean field K, with its specified absolute value and the norm-admissible generalized affinoid atlas. Let $\mathcal{F}$ be coherent on X_G and let β be an ordinary smooth moment form with compact germ support S. Use $D = d'$ in bidegree $(d-1, d)$ and $D = d''$ in bidegree $(d, d-1)$. Let $V_1, \dots, V_s$ be affinoid domains of X such that*

$$S \subseteq \mathrm{Int}_{\mathrm{top}} \left(\bigcup_{j=1}^{s} V_j, X \right).$$

For a nonempty index set L, put $V_L = \bigcap_{j \in L} V_j$ and $\epsilon_L = (-1)^{|L|+1}$. For an affinoid domain Z write $m_Z = (i_Z)_* \mu^D_{Z, \mathcal{F}|_Z, \beta|_Z}$, where $i_Z : Z \hookrightarrow X$, and set $m_\varnothing = 0$. Then

$$\sum_{L \neq \varnothing} \epsilon_L m_{V_L} = \mu^D_{X, \mathcal{F}, \beta}. \tag{C1}$$

In particular, on every Borel subset of $X \setminus (S \cap \partial_K X)$ the signed sum is zero. Here $\partial_K X = X \setminus \mathrm{Int}(X/K)$ is the analytic boundary of the structure map. The ordinary interior in the covering hypothesis is taken after forming the union.

Proof. First take a pure affinoid Y and any finite affinoid cover (B_j) of all of Y. For an ordinary top form ω on Y, choose a finite presenting cover (W_i) by NAG-002424. Compute its integral by NAG-002416 on the intersections W_I. On each W_I one restricted tuple represents ω. Apply NAG-002401 there to the cover by $W_I \cap B_j$. For fixed L, the $W_i \cap B_L$ present the restricted form on B_L, so summing first over I gives

$$I^N_{Y, \mathcal{F}}(\omega) = \sum_{L \neq \varnothing} \epsilon_L I^N_{B_L, \mathcal{F}|_{B_L}}(\omega|_{B_L}).$$

All the mixed pieces are actual affinoid domains, pure of dimension d when nonempty, and both restriction paths give the same finite coefficient module (NAG-002411, 002413, 002414). This computation uses the whole intersection even when its ordinary interior in Y is empty. Apply it to $\omega = D(u\beta)$ for *every* ordinary smooth scalar u on Y. Restriction commutes with product

and differential by NAG-002424. Thus (S1) and uniqueness of the signed measure give (C1) on Y for this cover. Every multiplier here is a restriction of the same test when its contributions are summed.

Now choose compact affinoid neighbourhoods A_i inside $\mathrm{Int}_{\mathrm{top}}(\bigcup_j V_j, X)$ whose ordinary interiors cover S; use goodness, the neighbourhood basis of NAG-002424 and compactness. The zero primitive is immediate if S is empty. Write $A_I = \bigcap_{i\in I} A_i$. By (S5), grouped into the coherent measures on A_I using (S4) and (S1), the ambient measure is $\sum_I \epsilon_I m_{A_I}$. Here and below the sums run over nonempty index sets. Each A_I is covered by $A_I \cap V_j$, so the affinoid calculation expresses the ambient measure as

$$\sum_{I,L} \epsilon_I \epsilon_L \, m_{A_I \cap V_L}.$$

Conversely, the support of $\beta|_{V_L}$ lies in $S \cap V_L$. For x there, some A_i contains an ordinary neighbourhood of x in X, hence $A_i \cap V_L$ contains one in V_L. The grouped (S5) on the good affinoid V_L sums the I-terms to m_{V_L}, after pushforward to X. Interchanging the finite sums proves (C1). For $D = d''$ the sign relating each measure to the d'-measure of $J\beta$ is $(-1)^d$ on every nonempty piece, including X. The cover signs are the same in both formulas. Taking the mass in (C1) gives $I^N_{X,\mathcal{F}}(D\beta)$. If $S \subseteq \mathrm{Int}(X/K)$, the signed sum vanishes as a measure. Both support assertions follow from the genuine-boundary support theorem in Section 8.4.4. □

For two families (V_j) and (H_i) satisfying the hypothesis, the common sum is also $\sum_{L,I} \epsilon_L \epsilon_I m_{V_L \cap H_I}$. Indeed (C1) applies on V_L to $V_L \cap H_i$: the relative ordinary interior of their union contains $S \cap V_L$. For three cuts the signs are $+, +, +$ on single members, minus on all three pair intersections and plus on the triple intersection. On each piece the coefficient of an integral component is its generic module length by (S4). These lengths belong to that piece.

On one common moment chart the normal convention can also be read off before integration. Subdivide the chains in NAG-002401 together. For each top cell σ, its coherent weight equals the alternating sum of the restricted weights on that cell. In the formula of NAG-002389 each incidence with a facet τ multiplies this coefficient by the same contraction: use a basis of the facet lattice in both slots, a primitive inward lift in the last double-primed position, the covolume-one facet measure and the sign $(-1)^{d(d+1)/2}$. Thus the weighted facet functionals agree on the fixed chart. Variable presentations in

(C1) are compared by testing the whole $D(u\beta)$. Equality of separate unsigned incidences over a projected face is unnecessary.

Annular overlap and the role of the ordinary germ. Take the closed annulus $a \le t = -\log|T| \le b$, $a < b$, with its structure module. For the radial test $\beta = h(t)d''t$ and scalar $u = q(t)$, put $F = qh$, with h, q smooth. On a subannulus $[s, r]$ of positive width its contribution is $F(r) - F(s)$: the whole exact-radius level for the coordinate T is the target polytorus itself, of Gauss degree and projection index one (NAG-002362, 002405). Integrate $F'(t)d't \wedge d''t$. For $a < c < e < b$, the cuts $[a, e]$, $[c, b]$ and their actual overlap $[c, e]$ give

$$[F(e) - F(a)] + [F(b) - F(c)] - [F(e) - F(c)] = F(b) - F(a).$$

If $c = e$, the intersection is the entire level affinoid; its one-coordinate image is a point and the restricted primitive is zero. The two endpoint terms at c cancel. For $\beta = h(t)d't$ and $D = d''$, each displayed endpoint difference has the opposite sign. These computations test radial scalars; (C1) gives the signed measure identity for arbitrary ordinary smooth scalars on the annulus.

The same touching cuts explain the germ hypothesis. Use $+d''t$ on $[a, c]$ and $-d''t$ on $[c, b]$. Both restrictions are zero on the joint level. For a radial smooth q supported in (a, b) with $q(c) = 1$, the two d' integrals of the scalar products sum to 2, and the intersection contributes zero. The two generators T, T^{-1} of the annulus presentation satisfy both radius bounds strictly over (a, b), placing this compactly supported test inside the analytic interior (NAG-000214, 001704). If the two primitives were restrictions of one ordinary smooth primitive, (C1) tested on q would give zero there. Thus agreement of tangential restrictions on a closed level alone cannot supply the common ordinary germ.

Continuous pullback of ordinary smooth forms

Supplement to Chapter 8.

This is the pullback counterpart to the fixed multiplier in Proposition 8.15.11. It uses the same intrinsic affine estimate (8.15.1), the single-chart and fixed-family topologies of Definitions 8.15.7 and 8.15.8, and the support and ordinary specialization of Definitions 8.15.9 and 8.15.10. The support stage here uses the inverse image of the given closed support.

Historical category. Use refined G-tropical spaces and their given affine structures. The spaces are Hausdorff, locally compact and have the historical countability-at-infinity neighbourhood condition. Their G-domains are locally closed, include opens, admit finite intersections and are G-covered by their compact subdomains. A G-cover is tested at a point by finitely many domains containing that point whose union contains a neighbourhood in the covered domain. Finite unions of compact domains are compact domains. The historical compact-image axiom makes the image of a compact domain under a chart a compact finite polyhedral image. A morphism is continuous, takes inverse images of G-domains to G-domains and pulls back paralinear functions. In the refined category it also pulls back affine functions. These are the scope-defining data in the historical definitions.

A chart is specified by its coordinate map and, where needed, a paralinear target containing the image in an ambient affine space. Use the historical form presheaf of smooth real (p,q)-coefficients on these targets, with the relation given by affine pullback to common dominating charts, and its associated G-sheaf. Ordinary smooth forms are sections locally represented on ordinary neighbourhoods by tropical charts, whose coordinates are affine functions in the refined structure. A support is the complement of the largest ordinary open on which the section vanishes.

Reader 33 R2, §8.T4.RA. *Let $u : Y \to X$ be a morphism in this category preserving the given affine structures, and let Φ, Ψ be the respective closed support families, with $u^{-1}F \in \Psi$ for each $F \in \Phi$. For $p, q \geq 0$ and $m \in \mathbb{N} \cup \{\infty\}$, the map*

$$u^* : \mathcal{A}^{p,q}_{\Phi}(X) \longrightarrow \mathcal{A}^{p,q}_{\Psi}(Y)$$

on ordinary smooth real forms is linear and continuous for the historical C^m topologies. Taking all closed supports gives the ordinary smooth-form state-

ment without support restriction. For a compact morphism, taking compact support families gives the compact-support statement.

Affine compatibility and pullback of sections. For a chart $f : V \to E$ on X, put $V' = u^{-1}V$ and $g = f \circ u|_{V'}$. The morphism preserves the domain and the coordinate type. When f is tropical, g is tropical, with the identity affine comparison on E. A specified coefficient target of f also contains $g(V')$. Pullback of a representative uses the same coefficient on that target and the composed chart g. A relation in the chart presheaf remains a relation: the dominating chart composes with u and the affine transition maps are unchanged. This preserves restriction and sums, the latter computed on joint charts.

A G-cover pulls back to a G-cover. At y, take the finitely many covering domains containing $u(y)$ whose union is a relative neighbourhood there, and take their inverse images. Thus local representatives for a sheaf section and the local equalities on overlaps pull back to compatible representatives. Gluing in the associated G-sheaf gives a unique section, agreeing with the representative rule. For an ordinary local tropical presentation the inverse image of its ordinary neighbourhood is ordinary open; hence this construction preserves ordinary smoothness. If a section vanishes on an ordinary open, its pullback vanishes on the inverse image there. In particular

$$\operatorname{supp}(u^*\omega) \subset u^{-1} \operatorname{supp} \omega.$$

Compact coefficient uniqueness. We prove the nullity test used by the topology. First let $\pi : P \to Q$ be an affine surjection between compact finite polyhedral sets. If $\pi^*b = 0$ tangentially, then for every source cell C the coefficient restricts to zero on $\pi(C)$. In fact the linear part maps the tangent space of C onto that of $\pi(C)$, so all primed and double-primed tangent vectors used to test b at a point of $\pi(C)$ can be lifted at a preimage in C. The equality on relative interiors holds also on the closed cell by continuity. The finitely many images $\pi(C)$ cover Q. A common subdivision adapted to those images shows that $b = 0$, since on each refined cell the tangential restriction is a further restriction of a zero form. This uses the same intrinsic finite-cell restriction as the surviving calculation in III-TOP-C.

Now take a chart f on a compact G-domain K and an intrinsic coefficient b on $f(K)$. Suppose the presheaf class of f^*b is zero. At a common dominating chart h the two representatives agree, so for the affine comparison π with

$f = \pi h$ the pullback $\pi^* b$ restricts to zero on $h(K)$. The actual image $h(K)$ is compact finite polyhedral and projects onto $f(K)$. The affine test just proved yields $b = 0$.

If the associated sheaf section of $f^* b$ is zero, its presheaf class vanishes on a G-cover, by the local equality relation defining the associated sheaf. Refine that cover by compact subdomains. Transitivity and the finite subcover property for the underlying quasi-net give finitely many compact domains K_i covering K. In particular each lies in a domain where the presheaf class vanishes. The same presheaf test for $f|_{K_i}$ gives $b|_{f(K_i)} = 0$. Those finitely many compact images cover $f(K)$; a common subdivision again gives $b = 0$. This proves injectivity for a compact chart restriction in the actual sheaf. Consequently the intrinsic coefficient representing a fixed section on this image, when it exists, is unique, and the coefficient on a smaller compact domain is its tangential restriction. This argument uses the topological part of NAG-001619/001620 for the quasi-net step.

The fixed-stage estimate. An input stage consists of an ordinary interior-covering family $\mathcal{D}$ on X and support in one closed $F \in \Phi$. For each $f : V \to E$ in this family take the chart g above. The ordinary interiors of the V' cover Y, since each contains $u^{-1} \operatorname{Int}_X V$. Their family $\mathcal{E}$ is fixed by $\mathcal{D}$ and u, independently of the varying section. The target support label is $u^{-1}F$.

Fix a compact domain $W \subseteq V'$. The compact subset $u(W)$ is contained in a compact G-domain $K \subseteq V$: at each point of V the compact-domain G-cover supplies a finite union of compact domains containing a relative neighbourhood. Finitely many such neighbourhoods cover $u(W)$, and the union of their compact domains is a compact domain. This step uses a containing domain and puts no requirement on $u(W)$ itself to be a domain. Let $b_{f,K}$ denote the input coefficient on $f(K)$. For any fixed affine comparison ℓ with $f \circ u|_{V'} = \ell \circ g$, the representative rule gives on $g(W)$ the coefficient

$$\big(\ell|_{g(W)}\big)^* b_{f,K}.$$

For the pulled-back family just chosen, ℓ is the identity. Uniqueness identifies the displayed intrinsic form with the coefficient of $u^*\omega|_W$. It is independent of K: coefficients for two choices are restrictions from their compact domain union. For each finite $r \le m$ the native finite-cell affine pullback estimate bounds the r-seminorm of this coefficient by $c_r \|b_{f,K}\|_{f(K),r}$, with geometry fixed before ω varies. The estimate uses only derivatives along the cells. For $m = \infty$ apply it at each finite order.

These seminorms generate the fixed-family topology. The linear map therefore lands continuously in the target stage with the subspace topology for support in $u^{-1}F$. Compose with that stage's inclusion in the target final space. This composition is continuous for every input stage, hence continuous for the source locally convex final topology. Both support specializations follow from the support inclusion and the respective support-family condition. In the compact case this is exactly compactness of the inverse image of a compact set.

Descent of graded valuation charts

Supplement to Section 3.19, following Theorem 3.19.22.

Let $G = \mathbb{R}_{>0}$, written multiplicatively. All field embeddings preserve degrees. For a graded field F over a graded field k, $P(F) = \mathcal{P}_k(F)$ is the space of graded valuation subrings containing k, with basic opens prescribed by finitely many homogeneous membership conditions. A graded field need not be an ordinary field. Let $H \leq G$ be a subgroup containing $\mathrm{supp}(k)$, allowing equality, and set $F_H = \bigoplus_{h \in H} F_h$. The analytic application uses $H = \mathrm{supp}(k)$. No restriction on the rank of a valuation is made.

Proposition 3.HS.1. *For every such subgroup H, restriction $p_F : P(F) \to P(F_H)$ is an open surjection with connected fibres. If $X = X_H \times_{P(F_H)} P(F)$ and $Y = Y_H \times_{P(L_H)} P(L)$ are given strict structures on graded birational spaces (the structure maps downstairs are local homeomorphisms), every morphism $(f, i) : X \to Y$, with $i : L \hookrightarrow F$, descends uniquely over $i|_{L_H}$. It is not assumed that f preserves any strict charts.*

Proof. Image of a basic open. If $A = k[a_1, \ldots, a_m] \subset F$ with a_j nonzero homogeneous, write $A_H = A \cap F_H$. Then

$$p_F(P(F)\{A\}) = P(F_H)\{A_H\}.$$

Indeed, fix O on the right and form $B = AO$. Its degree-H summand for the quotient grading by G/H is $A_H O = O$; the matching summand of $\mathfrak{m}_O B$ is $\mathfrak{m}_O$. The ideal is proper. Choose an ordinary maximal ideal containing it (Stacks 00E0(2)); the sum of its homogeneous components lying in the ideal is a homogeneous prime. Its homogeneous contraction to O is exactly $\mathfrak{m}_O$, since every homogeneous element of O outside $\mathfrak{m}_O$ is a unit. Homogeneously localize B there and apply the graded domination clause of NAG-001964 to this graded local subring of F. The resulting valuation contains A, O; any nonzero homogeneous $b \in F_H \setminus O$ is excluded because $b^{-1} \in \mathfrak{m}_O$ remains a nonunit. This proves the formula and surjectivity (take $A = k$).

For openness, A_H has finitely many homogeneous k-algebra generators. Indeed, the monomials needed have exponent vectors in $M = \{e \in \mathbb{N}^m : \prod_j \deg(a_j)^{e_j} \in H\}$: coefficients have degrees in H, and projecting a polynomial expression onto its H-summand retains only such monomials. Distinct minimal nonzero vectors in M are incomparable and form a finite set. In fact any infinite sequence in $\mathbb{N}^m$ has an infinite componentwise nondecreasing subsequence: for one coordinate choose an infinitely repeated value or a strictly

increasing subsequence; do this successively in the finitely many coordinates. If $0 \neq v \leq e$ in M, then $e - v \in M$. Induction on $\sum e_j$ now proves the finite generation assertion, including $M = \{0\}$. The right-hand side of the formula is therefore open. Restriction is continuous directly by the definition of the basic opens.

Fibre reduction. Fix $O \in P(F_H)$. The fibre Z is closed in $P(F)\{O\}$: its complement there is the union of the tests $b \in V$ for homogeneous $b \in F_H \setminus O$. Hence Z is quasi-compact by NAG-001965 applied to the possibly infinitely generated graded subalgebra O. We use only this existing compactness statement. Write $H^s = \{g : g^n \in H \text{ for some } n > 0\}$. Restriction from F_{H^s} to F_H is a homeomorphism on valuation spaces: it is an open surjection as just shown, and membership of each nonzero homogeneous u upstairs is equivalent to membership of u^n for $\deg(u)^n \in H$. The equivalence follows from the valuation alternative: if u is absent, its inverse is a nonunit. We may start at the unique extension of O to $E = F_{H^s}$ instead.

Were Z disconnected, its two complementary nonempty open pieces would be quasi-compact. Each would be a finite union of relative basic opens. Adjoin their finitely many nonzero tests to E as a graded field, obtaining F'. Every graded valuation of F' over the chosen restriction extends to F: apply NAG-001964 to that valuation ring in F; domination forces exact contraction by the inverse-nonunit test above. The two same finite families would thus disconnect the fibre for F'/E. The relative degrees $\operatorname{supp}(F')/\operatorname{supp}(E)$ form a finitely generated torsion-free subgroup of G/H^s (via its natural inclusion). It has a basis: embed such a group in its rational vector space, take rational coordinates, and clear the denominators of the finite generators. A subgroup of $\mathbb{Z}^r$ is free by projecting onto one axis, lifting a generator of the cyclic image when it is nonzero, and applying induction to the kernel. Choose homogeneous lifts of basis degrees. By restriction to the subgroups successively generated by these degrees and $\operatorname{supp}(E)$ we obtain a tower of one-free-degree extensions. Each step is $D[T, T^{-1}]$ as a graded ring over the preceding graded field D: each homogeneous term is cT^n, and distinct exponents have different degree classes.

One free degree. Fix a graded valuation W of D. On the multiplicative group of nonzero homogeneous elements of D, let v be the quotient by the homogeneous units of W, ordered additively by $v(c) \geq 0 \Leftrightarrow c \in W$. This is a totally ordered abelian group Γ; for same-degree summands the sum has value at least their minimum (divide by a summand of smaller value and use closure

of W). For every extending valuation the group Γ embeds order-preservingly into its value group and cT^n belongs to it precisely when $v(c) + nt \geq 0$, where t is the value of T. Thresholds for finitely many tests with $n \neq 0$ are $-v(c)/n \in \Gamma_{\mathbb{Q}}$. The rational hull is obtained by fractions with positive integer denominators and comparison by clearing denominators; embeddings of ordered groups induce embeddings of their hulls.

Conversely all positions relative to this finite threshold list occur. In $\Gamma_{\mathbb{Q}} \oplus \mathbb{Q}$ with lexicographic order put t at a threshold, between consecutive thresholds, or beyond the extreme thresholds (use the second coordinate for the unbounded intervals if necessary). Taking the direct sum of the homogeneous terms with $v(c) + nt \geq 0$ gives a graded valuation ring. Indeed products add, same-degree sums have the same exponent and satisfy the preceding minimum bound, and one of a homogeneous monomial and its inverse is admitted. These rings restrict to W. For a finite list of tests, membership in any finite union of finite intersections of tests depends only on the listed position; tests with $n = 0$ are constant. At a threshold all positive membership tests true in either adjacent interval remain true there, since the inequalities are weak. If the two sides of a separation were given by such finite families, the point at a threshold would have to be on the same side as each adjacent interval. The finite ordered list of intervals and thresholds is a chain, so every position is on the same side; with no thresholds there is only one pattern. This rules out any separation, because quasi-compactness of the fibre has already reduced it to finite families. This argument includes $\Gamma = 0$ and non-Archimedean ordered value groups; it does not use connectedness of a real interval.

Returning up the tower. Each restriction is continuous, open, surjective, and has connected fibres for a single free degree. On the inverse image of a fixed subset of the base its restriction remains open onto that subset, by the formula for the image of a relative open. A separation of the inverse image of a connected fibre would partition it into two saturated nonempty opens, since each single-step fibre is connected. Their images would disconnect that fibre. Starting at the base point proves connectedness up the entire finite tower, contradicting the supposed separation of Z.

Descent. The projections $q_X : X \to X_H$, $q_Y : Y \to Y_H$ are open surjections: locally they are pullbacks of p_F, p_L over open charts, or use that an open surjection remains so after base change. The fibre of q_X is a fibre of p_F. On it the composite $q_Y f$ takes values in one fibre of $Y_H \to P(L_H)$, since restriction of valuations commutes with i. This latter fibre is discrete by local

homeomorphy. Its connected image is one point, defining f_H uniquely. The open surjection q_X is a quotient map, so f_H is continuous. With the fixed field map $i|_{L_H}$ it is the required morphism, and its pullback is the original map by the fibre product description of Y. □

Corollary 3.HS.2 (The strict subdomain germ test)**.** *Let K be complete and nontrivially valued, and let (X, x) be a strictly K-analytic germ with its given strict atlas, not necessarily good. An analytic subdomain germ U_x is strict if and only if its graded birational reduction admits an H-strict structure, where $k = \widetilde{K}^{\mathrm{gr}}$ and $H = |K^\times|$. Every analytic morphism between strictly K-analytic spaces admits a unique strict representative as a morphism of the given strict spaces.*

Proof. For $F = \widetilde{\mathcal{H}(x)}^{\mathrm{gr}}$, $P(F_H)$ identifies with $\mathcal{P}_{k_1}(F_1)$: in a degree $h \in H$ choose $c_h \in k_h^\times$, write $F_h = c_h F_1$, and send a degree-one valuation V_1 to $\bigoplus_{h\in H} c_h V_1$. This is independent of the choices, since their ratios are in $k_1^\times$. Products, inverses and same-degree sums give the valuation property, and each homogeneous membership test is exactly a degree-one test. Thus the maps are inverse homeomorphisms.

NAG-001985 gives, on the supplied strict ambient atlas, the ordinary reduction X_0 and its graded pullback X_g; its nonempty quasi-compact opens classify the strict subdomain germs and pull back to their graded reductions. In particular a strict germ has H-strict reduction. Conversely let $U_g \subset X_g$ have a supplied abstract H-strict structure U_0, not assumed compatible with the embedding. Apply the proposition to $U_g \hookrightarrow X_g$ (identity on the field). The descended map $U_0 \to X_0$ is injective, since two points with the same image lie over the same valuation and can be paired with a *single* lift along p_F, where injectivity holds. Its image is open: its full inverse image under q_X is precisely the image of U_g, by the pullback description and the identity field map; q_X is quotient. The map is an open embedding: the same reasoning applied to any open in U_0, or the local homeomorphism charts over the same base, gives openness onto its image. Its image is nonempty and quasi-compact because U_0 is a birational object (also the continuous image of U_g). The strict subdomain classified by this image via NAG-001985 has graded reduction U_g. NAG-001973, for arbitrary analytic subdomain germs, reflects equality, so this is the original U_x. This proof works on non-good germs: NAG-001985 provides the comparison by gluing finitely many strict charts

and their strict overlaps. It is not an assertion that all analytic subdomains of a strict space are strict.

For an arbitrary analytic $f : Y \to X$ between such spaces take a strict target subdomain U and $y \in f^{-1}U$, $x = f(y)$. By the all-germs inverse-image clause of NAG-001972 the graded reduction V_g of the preimage germ is precisely the inverse image of U_g under the graded morphism. Descend that morphism by the proposition using the already-given strict structures on Y_g, X_g, not functoriality in the strict category. Write $U_g = q_X^{-1}U_0$ with U_0 the corresponding quasi-compact open; then $V_g = q_Y^{-1}(f_0^{-1}U_0)$. Here $f_0^{-1}U_0$ is open and quasi-compact: it is the continuous image of V_g by the surjection q_Y and V_g is a quasi-compact open by NAG-001973. This avoids assuming that continuous maps always preserve quasi-compactness under inverse image. The strict classification in NAG-001985 therefore gives a strict germ equal to that preimage germ. Outside the preimage there is no germ to test.

Equality of domain germs gives equality after shrinking an ordinary neighbourhood. The local strict representatives therefore furnish a strict chart quasi-net of the entire inverse image: at each of its points take finitely many strict affinoid pieces of the strict representative covering a relative neighbourhood, shrinking them into the equality neighbourhood by strict rational neighbourhoods inside the pieces (the construction in the proof of NAG-001985). Transitivity of quasi-nets then gives the domain assertion. Pull this back for the strict affinoid charts of the target. On a strict chart of the resulting source refinement mapping into a strict target chart the analytic morphism already supplies a bounded homomorphism of their affinoid algebras. These very maps commute with restrictions; the chart quasi-net condition and uniqueness under refinement are the locally bounded morphism definition and NAG-001612. Thus they define the strict morphism. Two strict morphisms with the same underlying analytic morphism have the same bounded algebra maps on these charts and hence agree. No test merely on the underlying point maps is used. □

Proof inputs. Proposition 3.HS.1 uses graded domination (Lemma 3.19.1), quasi-compactness of valuation containment loci (Proposition 3.19.2), and Stacks, Tag 00E0(2) for the nonzero ordinary quotient in the first paragraph of its proof. Corollary 3.HS.2 uses the preceding proposition, the quasi-net convention and transitivity (Definition 2.39.1 and Lemma 2.39.2), and the domain, reduction and morphism statements in Definition 2.40.7 and Theorems 2.40.8, 3.19.9, 3.19.10 and 3.19.22.

Algebraic supplement to Chapter 23 and the local-height discussion in Section 22.2.

Supported finite projection for a meromorphic section. Let $f : Y \to X$ be a finite morphism of schemes of finite type over a field K. Let $W \subset Y$ be an integral closed subscheme of dimension $m \geq 1$, and endow its image V with the reduced structure. Let M be an invertible sheaf on X, and suppose that a meromorphic section s is a generator off a closed set $T \subset X$, with $V \not\subset T$. Write s_V for a supplied nonzero meromorphic section on V agreeing with the generator on $V \setminus T$. Put $h = f|_W : W \to V$ and $e = [K(W) : K(V)]$, where $K(W)$ and $K(V)$ denote function fields. Then

$$h_* \operatorname{div}_{h^*(M|_V)}(h^* s_V) = e \operatorname{div}_{M|_V}(s_V) \quad \text{in } Z_{m-1}(T \cap V).$$

Here the pushforward is the restriction of h to the indicated supports. All cycle coefficients are signed. For an effective cycle with several prime components satisfying the same condition, the formula holds on each component and hence on their sum with its given coefficients.

Proof. The map h is finite and dominant between integral schemes of finite type over K. Both function fields have the same transcendence degree over K because their extension is finite. Thus the dimensions for the dimension function over $\operatorname{Spec} K$ agree. The nonzero rational section pulls back to a nonzero rational section through the field map, so its divisor is defined. Each divisor is zero off the indicated closed support, since the section there is a regular generator and pullback preserves that property. Apply Stacks Project, Lemma 42.26.3 (Tag 02ST), to h, $M|_V$ and s_V. It gives equality of cycles on V. Both sides come from the cycles of $T \cap V$, whose push into $Z_{m-1}(V)$ is injective because different prime cycles of the closed subset remain different prime cycles of V. The equality therefore holds on the support.

To make the degrees and lengths explicit, take a point $v \in V$ of codimension one and trivialize M near v. Write $s_V = a/b$ in that frame with a, b nonzero in $\mathcal{O}_{V,v}$. Every point w of W above v has codimension one, since h is finite and the residue degrees are finite. The order of a/b in either one-dimensional Noetherian local domain is the length of the quotient by a minus the length of the quotient by b, as in Tags 02MD and 02RJ. In particular the coefficient on the left is

$$\sum_{w \mapsto v} [\kappa(w) : \kappa(v)] \left(\ell_{\mathcal{O}_{W,w}}(\mathcal{O}_{W,w}/a) - \ell_{\mathcal{O}_{W,w}}(\mathcal{O}_{W,w}/b) \right).$$

Tag 02R4 uses full algebraic degrees in this pushforward. Its sum ranges over all points above v, and it includes inseparable degrees both at v and at the generic point. Tensoring the section and the line bundle by the same positive power multiplies each order by that power, by the order formula for fractions and the length-additivity sequence $0 \to A/(a) \xrightarrow{b} A/(ab) \to A/(b) \to 0$ in each local domain A. $\square$

Interior Stokes with coherent coefficients

Supplement to Section 8.4.4 and the closed-cut supplement C1.

This section restores the historical interior-support proof and its two localization inputs. Their full proof dependencies remain under review. The ordinary support collars and the absolute analytic interiors are separate conditions. In the weighted wall proof used upstream, the two projected radii have their own Gauss fields $\mathcal{H}(s_+)$ and $\mathcal{H}(s_-)$. The collar gives the same Gauss dimension at the two levels.

Fix a complete real-valued non-Archimedean field K, with its original absolute value. All generalized-affinoid radii below are positive real numbers; the valuation may be trivial. Use the ordinary smooth moment sheaf `v037:forms:ordinarysmoothgerm`, `v037:forms:germsheaf`, and the compact-support integral `v037:forms:compactintegral`. A moment tuple consists of analytic units and $t_f = (-\log|f_1|, \dots, -\log|f_n|)$. All intersections and restrictions are actual analytic domains.

(AFF), accepted affinoid input. If $V = \mathcal{M}(A)$ is pure of analytic dimension $d \geq 1$, M is a finite A-module, and $\beta = t_f^*\alpha$ is one global unit-moment representative of bidegree $(d-1, d)$, smooth near $t_f(V)$, then

$$\beta = 0 \text{ on an ordinary neighborhood of } \partial_K V \quad \Longrightarrow \quad \mathcal{I}_{M,f}(d'\alpha) = 0. \tag{AFF}$$

This is exactly `f7v38:annihilation`; no ordinary-section version on a general space is assumed.

Theorem NAG-002577 (Global Stokes with interior compact support)**.** *Let X be a good Hausdorff analytic space, pure of analytic dimension $d \geq 1$, and let $\mathcal{F}$ be coherent on X_G. For ordinary smooth moment sections*

$$\beta \in \mathcal{A}_c^{d-1,d}(X), \qquad \gamma \in \mathcal{A}_c^{d,d-1}(X)$$

with

$$\operatorname{supp}(\beta),\ \operatorname{supp}(\gamma)\ \subseteq \operatorname{Int}(X/K),$$

one has

$$\int_{X,\mathcal{F}} d'\beta = 0, \qquad \int_{X,\mathcal{F}} d''\gamma = 0.$$

In particular the assertion holds for every such compactly supported section if $\partial_K X = \varnothing$. No global paracompactness, reducedness, strictness, or connected-fiber hypothesis is added.

The single-representative affinoid input

The displayed implication (AFF) is recovered in Theorem NAG-002495, including its componentwise generic lengths.

The only newly used v0.38 theorem is

NAG-002455, v038:pointed:fullstabilization.

Its hypotheses are a generalized affinoid $V = \mathcal{M}(A)$, a point $x \in \mathrm{Int}(V/K)$, and a unit tuple f. Put $w = t_f(x)$ and

$$s_f(x) = \mathrm{trdeg}_{\widetilde{K}^{\mathrm{gr}}} \widetilde{K}^{\mathrm{gr}}\big(\widetilde{f_1(x)}, \dots, \widetilde{f_n(x)}\big).$$

It supplies a finite pure $s_f(x)$-dimensional rational fan C_x and an ordinary affinoid neighborhood W_0 of x. For every ordinary neighborhood $U \subseteq W_0$ of x, some $\varepsilon_U > 0$ satisfies

$$t_f(U) \cap B_{\varepsilon_U}(w) = (w + C_x) \cap B_{\varepsilon_U}(w). \tag{F7.1}$$

The theorem permits nonreduced or nonpure A. It gives neither weights nor a fan at a general boundary point. We use it twice at interior points: first on V, and then on an actual closed subspace of V.

The target is as follows. Let $V = \mathcal{M}(A)$ be pure of analytic dimension $d \geq 1$, let M be a finite A-module, and let α be a smooth ambient $(d-1,d)$-superform on an open neighborhood of the compact actual image $t_f(V)$. Set $\beta = t_f^*\alpha$, as an ordinary smooth moment section. We seek

$$\beta = 0 \text{ on an ordinary neighborhood of } \partial_K V \quad \Longrightarrow \quad \mathcal{I}_{M,f}(d'\alpha) = 0. \tag{F7.2}$$

The global chain and integral in this formula are v037:integration:globalchain, not an integral defined only after already knowing (F7.2).

Where a nonzero interior germ can occur

For an integral affinoid algebra A, call a point $x \in \mathcal{M}(A)$ *analytic generic* if the evaluation seminorm has kernel 0. This does not say that x is an isolated point of a moment fiber, nor that the fiber is disconnected.

Lemma NAG-002487 (Interior germs of degree d have generic support)**.** *Let A be integral and $V = \mathcal{M}(A)$ pure of analytic dimension $d \geq 1$. Let $x \in \mathrm{Int}(V/K)$, and let $\beta = t_f^*\alpha$ be a single global moment representative of bidegree $(d-1, d)$. If $\beta_x \neq 0$ in the ordinary stalk, then $s_f(x) = d$ and* $\ker(x) = 0$.

Proof. Apply (F7.1) on V. Taking $U = W_0$ and then a smaller ordinary affinoid neighborhood of x whose moment image lies in $B_{\varepsilon_{W_0}}(w)$ gives a neighborhood whose entire actual image is contained in the $s_f(x)$-dimensional fan. Such a neighborhood exists by continuity of t_f and the ordinary affinoid-neighborhood basis used in `v037:forms:germsheaf`. If $s_f(x) < d$, the second alternating degree d forces every tangential restriction of α on this image to vanish. The ordinary germ definition `v037:forms:ordinarysmoothgerm` would then give $\beta_x = 0$. Thus $s_f(x) \geq d$. The finite-image dimension bound `v037:globaltrop:finiteimage`, applied to V, and the equality (F7.1) imply $s_f(x) \leq d$.

Suppose that $0 \neq a \in \ker(x)$. The actual closed subspace

$$Z = \mathcal{M}(A/(a))$$

is nonempty because it contains the same point x. The ideal (a) is contained in no minimal prime of the integral ring A. Therefore `v037:degree:exceptionalimage` gives $\dim_K Z \leq d-1$. The map $Z \to V$ is finite with its canonical quotient norm, hence everywhere relatively interior by `fnd:generalizedfinitealgebras`. Since $x \in \mathrm{Int}(V/K)$, the composition inclusion `fnd:affinoidinteriorcalculus` gives $x \in \mathrm{Int}(Z/K)$.

The completed residue field at this point of Z is the same $\mathcal{H}(x)$: the quotient is by an ideal in the evaluation kernel. The restricted units $f_i|_Z$ have exactly the same values and initial forms. Applying `NAG-002455` on Z therefore gives a pointed fan of dimension the same number $s_f(x)$. On the other hand, the entire actual image $t_f(Z)$ has dimension at most $d-1$, by the same exceptional-image theorem. Equality (F7.1) on Z implies $s_f(x) \leq d-1$, contradicting $s_f(x) = d$. Thus $\ker(x) = 0$. □

Lemma NAG-002488 (Separation of a compact generic interior support)**.** *Let A be integral, and let S be a nonempty compact subset of* $\mathrm{Int}(V/K)$*, $V=\mathcal{M}(A)$, such that* $\ker(x)=0$ *for every $x\in S$. Given any global unit tuple f on V, there are a single affinoid domain $W\subseteq V$ containing an ordinary neighborhood of S and a finite tuple of units g on W, obtained by restricting nonzero elements of A, such that for $F=(f|_W,g)$,*

$$t_F(S)\cap t_F(\partial_K W)=\varnothing. \tag{F7.3}$$

More precisely, every point of W with the same F-moment as a point of S is interior in W/K.

Proof. By the affinoid definition `fnd:relativeinterior`, for each $x\in S$ there is an admissible presentation of the *whole* algebra A,

$$K\{r_{x,1}^{-1}T_1,\dots,r_{x,m_x}^{-1}T_{m_x}\}\twoheadrightarrow A,\qquad T_j\longmapsto a_{x,j},\qquad |a_{x,j}(x)|<r_{x,j}. \tag{F7.4}$$

Delete any coordinate whose image is 0 in A. Setting these coordinates to zero is a contractive projection of the Tate algebra and does not change its map to A. Every lift can be so projected without increasing its norm; hence the induced quotient norm and admissibility of the remaining presentation are preserved. Thus every listed $a_{x,j}$ may be taken nonzero.

The strict loci

$$G_x=\{y\in V:\ |a_{x,j}(y)|<r_{x,j}\text{ for every }j\}$$

are open, contained in $\mathrm{Int}(V/K)$, and cover S. Choose a finite subcover $G_{x_1},\dots,G_{x_q}$. Let $g_1,\dots,g_N$ list all nonzero generators in these finitely many presentations; repetitions may be retained. Every g_l is nonzero at *every* point of S, since these points have kernel 0. Compactness and continuity give

$$0<\delta_l<\min_{x\in S}|g_l(x)|.$$

The actual Laurent domain

$$W=\bigcap_{l=1}^{N}\{y\in V:\ |g_l(y)|\geq\delta_l\} \tag{F7.5}$$

is affinoid by the generalized rational point, universal-property, and joint-presentation results `fnd:generalizedrationalpoints`, `fnd:generalizedrationaluniversal`, and `fnd:generalizedjointpresentations`. Its

algebra is the actual joint quotient adjoining inverses $S_l = g_l^{-1}$ with bounds $|S_l| \le \delta_l^{-1}$. Thus the restrictions of the g_l are units on W. The strict lower-bound locus contains S, so W contains an ordinary neighborhood of S.

Let $x \in S$, $y \in W$, and $t_F(y) = t_F(x)$. Then $|g_l(y)| = |g_l(x)| > \delta_l$ for every l. Choose i with $x \in G_{x_i}$. Equality of the generator values puts y in the same strict presentation locus G_{x_i}, and hence $y \in \mathrm{Int}(V/K)$. The strict lower bounds also put y in the ordinary interior of W in V. By `fnd:affinoiddomainordinaryinterior`, $y \in \mathrm{Int}(W/V)$. The composition inclusion now gives $y \in \mathrm{Int}(W/K)$. This proves the stronger assertion and (F7.3). □

Proposition NAG-002492 (The integral affinoid case)**.** *Under the hypotheses of* (F7.2)*, assume that A is integral. Then* (F7.2) *holds for every finite A-module M.*

Proof. Let $S = \mathrm{supp}(\beta)$. This support is the complement of the open zero-germ locus in the ordinary sheaf; it is compact because V is compact and Hausdorff. The hypothesis implies $S \subset \mathrm{Int}(V/K)$. If S is empty, the ordinary section and its differential are zero, and `v037:forms:compactintegral` gives the assertion.

Assume $S \neq \varnothing$. Each point of S has nonzero germ. Lemma NAG-002487 shows that every such point has kernel 0. Apply Lemma NAG-002488. The two compact sets

$$K_0 = t_F(S), \qquad K_\partial = t_F(\partial_K W)$$

are disjoint. Compactness of the boundary follows from compactness of W and openness of affinoid interior.

Let O be an open neighborhood of $t_f(V)$ on which α is defined, and let π denote projection to the original f coordinates. On $\pi^{-1}O$ put $\widetilde{\alpha} = \pi^*\alpha$. Ordinary real bump functions give a single

$$\chi \in C_c^\infty\big(\pi^{-1}O \setminus K_\partial\big), \qquad \chi = 1 \text{ on a neighborhood of } K_0.$$

For example, a finite collection of real balls with closures in that open set covers K_0; standard smooth bumps and their finite product complement give such a function. Extend $\eta = \chi\widetilde{\alpha}$ by zero to the entire enlarged moment space. It is a smooth compactly supported superform, and

$$\beta|_W = t_F^*\eta \tag{F7.6}$$

as an ordinary section: near S the factor is 1; off S the germ of β is zero, so multiplication preserves zero. Unit-adjunction faithfulness and the ordinary germ sheaf justify this identity without any equality of ambient coefficients away from the actual source support.

The domain W is pure of analytic dimension d by v036:analytic:domainpurity. Put $M_W = M \otimes_A \mathcal{O}(W)$. The support of $d'\beta$ is contained in S, and W is an ordinary neighborhood of S in V. Compact-neighborhood and chart independence in v037:forms:compactintegral give

$$\mathcal{I}_{M,f}(d'\alpha) = \int_{V,\widetilde{M}} d'\beta = \int_{W,\widetilde{M_W}} d'(\beta|_W) = \mathcal{I}_{M_W,F}(d'\eta). \tag{F7.7}$$

The support of η is compact in $\mathbb{R}^{n+N} \setminus t_F(\partial_K W)$. Write $U_F = \mathbb{R}^{n+N} \setminus t_F(\partial_K W)$. For the primitive η of bidegree $(d-1,d)$, the support of $d'\eta$ is also compact in U_F. Use the finite-module chain and the finite-cell sum for its weights and for its integral. On U_F the local module cycle of v037:analytic:interiorcycle uses the same weights: on each common top cell the generic point lies in the same flat tangent plane and its entire F-fibre in W is absolutely interior. Both coefficients are the weight of M_W on that fibre by v037:degree:moduleweight. A finite refinement on the compact part of the actual image meeting the test support therefore gives identical top-cell integrals, with the same saturated tangent lattices and no contribution from smaller cells (v037:integration:polyhedralcalculus). Thus the last quantity in (F7.7) equals the whole-interior integral $I_{M_W,F}(d'\eta)$, which is zero by v037:integration:momenttests. □

Theorem NAG-002495 (Affinoid boundary-germ annihilation)**.** *For every pure d-dimensional generalized affinoid $V = \mathcal{M}(A)$, $d \geq 1$, every finite A-module M, every global unit tuple f, and every smooth ambient $(d-1,d)$-representative α near $t_f(V)$, implication (F7.2) holds. Nonreduced and reducible A, connected actual moment fibers, trivial valuation, and arbitrary positive radii are allowed.*

Proof. For $\mathfrak{p} \in \operatorname{Min}(A)$ let

$$V_{\mathfrak{p}} = \mathcal{M}(A/\mathfrak{p}), \quad i_{\mathfrak{p}} : V_{\mathfrak{p}} \longrightarrow V, \quad \ell_{\mathfrak{p}}(M) = \operatorname{length}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}.$$

These are actual complete quotient algebras by fnd:generalizedaffinoidquotients. Each is integral and pure of analytic dimension d. The finite

map $i_{\mathfrak{p}}$ is everywhere relatively interior. The exact formula `fnd:affinoidinteriorcompositionequality` yields

$$\operatorname{Int}(V_{\mathfrak{p}}/K) = i_{\mathfrak{p}}^{-1}\operatorname{Int}(V/K), \qquad \partial_K V_{\mathfrak{p}} = i_{\mathfrak{p}}^{-1}(\partial_K V). \tag{F7.8}$$

Thus $i_{\mathfrak{p}}^{*}\beta$ vanishes near $\partial_K V_{\mathfrak{p}}$.

For completeness the componentwise chain identity is

$$\mathcal{T}_f(M) = \sum_{\mathfrak{p}\in\operatorname{Min}(A)} \ell_{\mathfrak{p}}(M)\,\mathcal{T}_{f|V_{\mathfrak{p}}}(A/\mathfrak{p}). \tag{F7.9}$$

To see this, choose a finite common subdivision of the actual images of V, all $V_{\mathfrak{p}}$, and their boundary images, using `v037:globaltrop:finiteimage`. At a generic point of a d-cell the whole relevant fibers are interior and their images lie in the same affine d-plane. The coefficient equality is exactly `v037:degree:moduleweight`. Together with the finite-module chain and weights, this gives (F7.9) on all top cells. Cells of smaller dimension carry no d-chain term.

For each $\mathfrak{p}$, apply Proposition NAG-002492 to $A/\mathfrak{p}$ with its structure module and the tuple $i_{\mathfrak{p}}^{*}f$. Use the same α near the compact image of this tuple. Its ordinary pullback is $i_{\mathfrak{p}}^{*}\beta$, which vanishes near $\partial_K V_{\mathfrak{p}}$ by (F7.8). Each component integral therefore vanishes. The finite-cell functional applied to $d'\alpha$ in (F7.9) gives

$$\mathcal{I}_{M,f}(d'\alpha) = \sum_{\mathfrak{p}} \ell_{\mathfrak{p}}(M)\,\mathcal{I}_{A/\mathfrak{p},f|V_{\mathfrak{p}}}(d'\alpha) = 0.$$

All generic lengths, including nilpotent contributions, are retained. These closed components are used for a finite linear identity, not as an affinoid-domain cover. □

Finite localization on actual intersections

Lemma NAG-002578 (Minimal evaluation kernels on the relevant support)**.** *Let $V = \mathcal{M}(A)$ be pure of analytic dimension $d \geq 1$, let $x \in \operatorname{Int}(V/K)$, and let $t_f^{*}\alpha$ have bidegree $(d-1, d)$. If its ordinary germ at x is nonzero, then*

$$s_f(x) = d, \qquad \ker(x) \in \operatorname{Min}(A),$$

where $s_f(x)$ is the graded transcendence degree of the field generated by the initial forms of the $f_i(x)$.

Proof. The interior-only theorem NAG-002455, v038:pointed:fullstabilization, gives the germ of the actual moment image as a pure $s_f(x)$-dimensional fan. After shrinking an ordinary affinoid neighborhood into a small inverse moment ball, its entire image has this dimension. If $s_f(x) < d$, all tangential restrictions of the second alternating degree d vanish, so the ordinary germ is zero. The global finite-image dimension bound v037:globaltrop:finiteimage gives the reverse bound $s_f(x) \le d$. Thus $s_f(x) = d$.

Put $\mathfrak{q} = \ker(x)$. The algebra A is Noetherian by fnd:generalizedclosedsubmodules, so it has finitely many minimal primes, say $\mathfrak{p}_1, \dots, \mathfrak{p}_r$. We use the elementary prime-existence and finiteness facts Stacks 00E0(4) and Stacks 00FR. Suppose that $\mathfrak{q}$ is none of the $\mathfrak{p}_i$. Then $\mathfrak{q} \not\subseteq \mathfrak{p}_i$ for every i; otherwise minimality, using a minimal prime in $\mathfrak{q}$, would force equality. Choose $a_i \in \mathfrak{q} \setminus \mathfrak{p}_i$. By incomparability of the minimal primes choose

$$c_i \in \bigcap_{j \neq i} \mathfrak{p}_j \setminus \mathfrak{p}_i$$

as a product of elements of $\mathfrak{p}_j \setminus \mathfrak{p}_i$; use $c_i = 1$ when $r = 1$. Then $a = \sum_i a_i c_i$ belongs to $\mathfrak{q}$ and to none of the minimal primes: modulo $\mathfrak{p}_i$, only the nonzero product $a_i c_i$ remains.

The actual quotient $Z = \mathcal{M}(A/(a))$ contains x and has analytic dimension at most $d - 1$, by v037:degree:exceptionalimage. The finite closed immersion $Z \to V$ is everywhere relatively interior, and composition gives $x \in \mathrm{Int}(Z/K)$. Here we use fnd:generalizedfinitealgebras and fnd:affinoidinteriorcalculus. The completed residue field at this point of Z, and all initial forms of the restricted f_i, are unchanged. The same interior pointed-fan theorem, which permits nonpure Z, therefore gives a pointed image dimension $s_f(x) = d$ on Z. This contradicts the dimension bound for $t_f(Z)$ in the exceptional-image theorem. Hence $\mathfrak{q}$ is a minimal prime. □

Lemma NAG-002579 (A compact bump with one moment presentation)**.** *Let X be good and Hausdorff, let $V = \mathcal{M}(A)$ be an ordinary affinoid neighborhood of x, and suppose*

$$x \in \mathrm{Int}(V/K), \qquad \ker(x) = \mathfrak{p} \in \mathrm{Min}(A).$$

Given an ordinary open neighborhood O of x in X, there exist an affinoid domain $W \subseteq V$, containing an ordinary neighborhood of x in X, analytic

units g on W, and a smooth real function Φ on their moment space such that the scalar section $b_W = t_g^*\Phi$ has the following properties:

$$0 \le b_W \le 1, \qquad b_W = 1 \text{ near } x,$$
$$\operatorname{supp}_W(b_W) \subseteq C, \qquad C \text{ compact},$$
$$C \subseteq O \cap \operatorname{Int}_{\mathrm{top}}(W, X) \cap \operatorname{Int}(W/K).$$

It extends by zero to a global $b \in \mathcal{A}_X^{0,0}(X)$. The restriction of this extension to all of W is b_W. If a form is represented by one tuple on V, it remains represented by that tuple on W.

Proof. Choose $h \in A \setminus \mathfrak{p}$ lying in every other minimal prime, using the same finite product construction; take $h = 1$ if there are no other minimal primes. Then $|h(x)| > 0$. Choose $0 < \delta < |h(x)|$ and put

$$V_0 = \{y \in V : |h(y)| \ge \delta\}.$$

This is an actual Laurent domain. Every evaluation kernel on V_0 contains $\mathfrak{p}$: it contains a minimal prime, and the other ones contain h. Thus every element of $\mathfrak{p}$ has value zero *throughout* V_0.

Choose a basic ordinary neighborhood G of x in V, given by finitely many strict upper bounds and positive strict lower bounds on elements of A, with

$$G \subseteq O \cap \operatorname{Int}_{\mathrm{top}}(V, X) \cap \operatorname{Int}(V/K).$$

The evaluation-topology neighborhood basis and openness of affinoid interior justify this choice. If a function in these conditions vanishes at x, it lies in $\mathfrak{p}$; its only possible condition is a positive upper bound, automatically satisfied on V_0. Delete these conditions when working in V_0. Every remaining function is nonzero at x.

Adjoin the condition $|h| > \delta$ and positive lower bounds smaller than the values at x for all remaining functions. Adding finite upper bounds larger than their values at x, if necessary, gives a list $g_1, \dots, g_N$ of elements of A, nonzero at x, and real intervals (l_j, u_j) with

$$0 < l_j < |g_j(x)| < u_j < \infty.$$

Their simultaneous strict locus G' contains x, lies in V_0, and satisfies $G' \subseteq G$. For each j choose

$$l_j < L_j < |g_j(x)| < U_j < u_j.$$

Set

$$W = \{y \in V : L_j \le |g_j(y)| \le U_j \text{ for every } j\}.$$

The actual rational-domain point and joint-presentation theorems make W an affinoid domain. It is an ordinary neighborhood of x in X, since its inequalities are strict at x and $x \in \mathrm{Int}_{\mathrm{top}}(V, X)$. Each $g_j|_W$ is a unit, with inverse supplied by its Laurent presentation. Moreover $W \subseteq G$.

Choose $0 \le \Phi_j \le 1$ smooth with compact support in $(-\log U_j, -\log L_j)$, equal to 1 near $-\log|g_j(x)|$, and put $\Phi = \prod_j \Phi_j$. These real profiles are those constructed in v037:forms:smoothcutoff. The compact set

$$C = t_g^{-1}(\operatorname{supp}\Phi) \subseteq W$$

contains the support of $b_W = t_g^*\Phi$. Every point of C satisfies all defining inequalities of W strictly. Thus $C \subseteq \mathrm{Int}_{\mathrm{top}}(W, X)$: use the strict locus in V and $W \subseteq \mathrm{Int}_{\mathrm{top}}(V, X)$. Also $C \subseteq \mathrm{Int}(V/K)$. The domain-interior theorem and composition give $C \subseteq \mathrm{Int}(W/K)$.

Finally glue b_W on $\mathrm{Int}_{\mathrm{top}}(W, X)$ to zero on $X \setminus C$. These are an ordinary open cover of X and the sections agree on the overlap. The ordinary sheaf theorem gives a global smooth moment function b. On $W \setminus C$ both its restriction and b_W are zero locally, so they agree on all of W. Compactness of C and Hausdorffness of X are essential for this zero-extension argument. □

Lemma NAG-002580 (Extension with a compact collar)**.** *Let $W \subseteq X$ be an affinoid domain in a good Hausdorff space, and let $\theta \in \mathcal{A}^{p,q}(W)$ have support in a compact set $C \subseteq \mathrm{Int}_{\mathrm{top}}(W, X)$. There is a unique extension $e_!\theta$ on X with support contained in C and restriction θ on W. It is zero outside W and satisfies*

$$d' e_!\theta = e_! d'\theta, \qquad d'' e_!\theta = e_! d''\theta, \qquad J e_!\theta = e_! J\theta.$$

All displayed supports lie in C. If X is pure of dimension d, $\mathcal{F}$ is coherent, and θ has bidegree (d, d), then

$$\int_{X,\mathcal{F}} e_!\theta = \int_{W,\mathcal{F}|_{W_G}} \theta.$$

Proof. Use the open cover $\mathrm{Int}_{\mathrm{top}}(W, X)$, $X \setminus C$, and glue θ to zero. On their overlap θ has zero ordinary germs. Every differential and J is local on the ordinary sheaf and preserves the zero section, by v037:forms:germsheaf;

hence the same gluing proves the three commutations. At a point of W outside this ordinary interior, the compact set C is absent from a neighborhood, so the resulting restriction still agrees with θ. Uniqueness follows from the same open cover. The integral equality is compact-neighborhood independence in `v037:forms:compactintegral`. No distributional boundary derivative is introduced: the section is already zero on a neighborhood of every newly crossed boundary point. □

Proposition NAG-002581 (Finite single-representative decomposition)**.** *Let X be good, Hausdorff and pure of analytic dimension $d \geq 1$, and let $\beta \in \mathcal{A}_c^{d-1,d}(X)$ have support $S \subseteq \mathrm{Int}(X/K)$. There exist finitely many affinoid domains $W_i \subseteq X$ and global functions $0 \leq b_i \leq 1$ such that:*

1. *the loci on which $b_i = 1$ cover an ordinary neighborhood of S;*
2. *on W_i, both β and b_i have one global unit-moment representative, and $\mathrm{supp}(b_i) \subseteq C_i$ for a compact $C_i \subseteq \mathrm{Int}_{\mathrm{top}}(W_i, X) \cap \mathrm{Int}(X/K)$;*
3. *for $W_I = \bigcap_{i \in I} W_i$, $I \neq \varnothing$, the section*

$$\theta_I = \left(\prod_{i \in I} b_i|_{W_I} \right) \beta|_{W_I}$$

has one global unit-moment representative, with compact support in both ordinary interior and absolute analytic interior:

$$\mathrm{supp}(\theta_I) \subseteq \mathrm{Int}_{\mathrm{top}}(W_I, X) \cap \mathrm{Int}(W_I/K).$$

4. *as ordinary sections on X,*

$$\beta = \sum_{\varnothing \neq I \subseteq \{1,\dots,m\}} (-1)^{|I|+1} e_! \theta_I. \tag{F7.10}$$

Empty domains and zero terms are omitted.

Proof. If $S = \varnothing$, use the empty sum. For $x \in S$, choose an ordinary affinoid neighborhood V_x on which $\beta = t_{f_x}^* \alpha_x$. It is pure of dimension d by `v037:integration:puritylocality`. The point is interior in V_x/K by `fnd:goodinteriorlocaltesting`, since it is interior in X/K and V_x is an ordinary neighborhood. The germ is nonzero, so Lemma NAG-002578 applies. Choose the bump of Lemma NAG-002579, with its ordinary support neighborhood

contained in $\mathrm{Int}(X/K)$. Compactness of S gives finitely many such bumps, with domains W_i, which equal one on neighborhoods covering S. Write

$$\beta|_{W_i} = t_{f_i}^*\alpha_i, \qquad b_i|_{W_i} = t_{g_i}^*\Phi_i.$$

Each W_I is an actual affinoid domain by `v037:integration:separatedintersections`, and every nonempty W_I is pure of dimension d. Choose any $i_0 \in I$. The joint tuple

$$F_I = (f_{i_0}, (g_i)_{i\in I})|_{W_I}$$

consists of units on W_I. A single ambient representative of θ_I is

$$\alpha_I = \left(\prod_{i\in I} \pi_{g_i}^*\Phi_i\right) \pi_{f_{i_0}}^*\alpha_{i_0}, \tag{F7.11}$$

defined near its actual compact image. No extension of an analytic unit from one domain to another has been made; all tuples are restricted to the intersection. The ordinary sheaf and unit-adjunction faithfulness ensure that this representative is the displayed section, independently of the choice of i_0.

Put $C_I = \bigcap_{i\in I} C_i$. It is compact and contains $\mathrm{supp}_{W_I}(\theta_I)$. Every point of C_I lies in all the ordinary interiors of the W_i, and hence in $\mathrm{Int}_{\mathrm{top}}(W_I, X)$. It is also interior in X/K. Applying `fnd:goodinteriorlocaltesting` to this ordinary affinoid neighborhood gives

$$C_I \subseteq \mathrm{Int}(W_I/K).$$

Consequently θ_I is zero on an ordinary neighborhood of $\partial_K W_I$. Indeed C_I is compact and closed and is disjoint from that boundary.

The function

$$P = 1 - \prod_{i=1}^{m}(1 - b_i)$$

equals one on an ordinary neighborhood of S. Thus $\beta = P\beta$ everywhere as an ordinary section: off S its germs are zero. Expand the finite product to obtain (F7.10). The global product $(\prod_{i\in I} b_i)\beta$ is $e_!\theta_I$, by Lemma NAG-002580. This proves all four assertions. □

The global derivative integral

The proof applies the actual finite decomposition (Proposition NAG-002581), compact-collar zero extension (Lemma NAG-002580), and the affinoid input (Theorem NAG-002495).

Proof of Theorem NAG-002577. Put $S = \operatorname{supp}(\beta)$ and apply Proposition NAG-002581. On each nonempty intersection W_I, the resulting $\theta_I = t_{F_I}^{*}\alpha_I$ has compact support inside both the ordinary interior in X and the absolute analytic interior of W_I. Locality keeps the support of $d'\theta_I$ in the same collar. All sums are finite. The commutation in Lemma NAG-002580 gives

$$d'\beta = \sum_{I\neq\varnothing}(-1)^{|I|+1}e_!(d'\theta_I).$$

Let $M_I = \Gamma(W_I, \mathcal{F}|_{(W_I)_G})$. The affinoid coherent-module equivalence `fnd:fullgkiehltheoremb` makes this a finite module with its actual restriction interpretation. Each W_I is a compact carrier for its extended derivative. The compact-carrier and single-chart comparisons in `v037:forms:compactintegral`, with these same restricted coefficients, give

$$\int_{X,\mathcal{F}} d'\beta = \sum_{I\neq\varnothing}(-1)^{|I|+1}\int_{W_I,\widetilde{M}_I} d'\theta_I = \sum_{I\neq\varnothing}(-1)^{|I|+1}\mathcal{I}_{M_I,F_I}(d'\alpha_I).$$

Every summand is zero by Theorem NAG-002495 (AFF). The compact support of θ_I in $\operatorname{Int}(W_I/K)$ gives an ordinary zero neighborhood of the *actual absolute analytic boundary* of W_I. Here W_I is pure of dimension d, F_I is the restricted unit tuple, and α_I is defined near its actual compact moment image. All generic module lengths are retained.

Put $P = 1 - \prod_i(1-b_i)$. If $b_I = \prod_{i\in I} b_i$, the scalar product rule gives

$$d'(b_I\beta) = d'b_I \wedge \beta + b_I\, d'\beta.$$

The scalar b_I has degree zero, so the sign is plus. Summing with $(-1)^{|I|+1}$ gives $d'P \wedge \beta + P\, d'\beta$. Near S, $P = 1$ and $d'P = 0$; off S, both β and $d'\beta$ are locally zero. This is exactly $d'\beta$, as used above. The same check applies to d''.

For the second assertion use the frozen convention `v037:integration:conventions`: all one-form generators anticommute, J exchanges the two copies and fixes functions, and $d'' = Jd'J$. The positive interleaved top generator

$$\omega_N = d'x_1 \wedge d''x_1 \wedge \cdots \wedge d'x_d \wedge d''x_d$$

satisfies $J\omega_N = (-1)^d\omega_N$. On each top cell J changes the coefficient in this same generator by $(-1)^d$ and leaves its weight and lattice measure fixed. Thus the finite-chart integral satisfies $\int J\zeta = (-1)^d\int\zeta$ for top forms ζ. The

section $J\gamma$ has bidegree $(d-1,d)$ and the same support as γ. The proved assertion yields

$$0=\int_{X,\mathcal{F}} d'(J\gamma)=\int_{X,\mathcal{F}} J(d''\gamma)=(-1)^d\int_{X,\mathcal{F}} d''\gamma.$$

This proves the d'' formula with the exact frozen sign. $\square$

Proper direct image on historical ordinary compact tests

Supplement to Chapter 8.

The direct image below uses the pullback of Supplement 8.T4.RA on the support families of Definition 8.15.9 and the ordinary tests of Definition 8.15.10.

Work in the historical refined G-tropical category of 8.T4.RA, with its ordinary smooth real forms and its locally convex final topology for compact supports and interior-covering affine chart families. Write $E_r^{p,q}(Z)$ for the vector space of ordinary smooth (p,q)-forms of compact support on Z, with its C^r test topology. The index $r \in \mathbb{N}_0 \cup \{\infty\}$ specifies the topology on smooth tests. On a fixed stage $(\mathcal{D}, K)$ it uses the subspace topology generated by $\rho_{f,W,m}$ for $f \in \mathcal{D}$, compact chart domains W and $m \le r$, all finite m when $r = \infty$. A current of bidimension (p,q) is an element of $D_{p,q}(Z) = (E_\infty^{p,q}(Z))'$, the continuous real dual. It has order at most $r < \infty$ if its functional is continuous for $E_r^{p,q}(Z)$.

Proposition (Proper direct image)**.** *Let $u : Y \to X$ be a morphism as in* 8.T4.RA*. Suppose $u^{-1}K$ is compact for every ordinary compact subset K of X. For $T \in D_{p,q}(Y)$ the formula*

$$\langle u_* T, \eta \rangle = \langle T, u^* \eta \rangle, \qquad \eta \in E_\infty^{p,q}(X),$$

defines a current of bidimension (p,q) on X. The map u_* is linear and continuous for pointwise convergence on smooth compact tests. It preserves each finite upper bound on order.

Proof. Use all ordinary compact sets as the support families in 8.T4.RA. They contain the empty set and are closed under finite unions and closed subsets. The inverse-support condition follows from the hypothesis on u. A section vanishing on an open has pullback zero on the inverse open, so $\operatorname{supp}(u^*\eta) \subset u^{-1} \operatorname{supp} \eta$. For an input stage $(\mathcal{D}, K)$, 8.T4.RA fixes the interior-covering pullback chart family $\mathcal{E}$ on Y and the compact carrier $u^{-1}K$. The estimate of 8.T4.RA bounds each output seminorm of order m by a constant times an input seminorm of the same order, with the chart and compact restriction fixed before η varies. This gives a continuous linear map into that output stage for every r.

If T is continuous for $E_r^{p,q}(Y)$, its restriction to this output stage is bounded by a constant times the maximum of finitely many defining seminorms.

Substituting the pullback estimates gives a continuous functional on $(\mathcal{D}, K)$ in the input C^r topology. The linear final-topology property gives continuity on $E_r^{p,q}(X)$. For $r = \infty$, the orders in the finite bound may depend on the stage. For finite r, every seminorm used still has order at most r. The formula has the same test bidegree on both spaces, hence preserves bidimension. For pointwise continuity, evaluation of u_*T at a fixed η is evaluation of T at the single fixed test $u^*\eta$. □

Transport of a closed carrier. If $S \subset Y$ is closed and T vanishes on all compact tests supported in $Y \setminus S$, then $u(S)$ is a closed carrier for u_*T in the same sense. Indeed, for $x \notin u(S)$, take a compact ordinary neighbourhood K of x, using local compactness in 8.T4.RA. The set $u(S \cap u^{-1}K)$ is compact and is closed in the Hausdorff space X. Its complement inside the interior of K is an open neighbourhood of x disjoint from $u(S)$. For a test supported in $X \setminus u(S)$ its pullback is compactly supported in $Y \setminus S$. Apply the defining vanishing of T. In a current convention in which ordinary localization identifies the support as a least closed carrier, taking $S = \operatorname{supp} T$ gives $\operatorname{supp}(u_*T) \subset u(\operatorname{supp} T)$.

Compatibility with the usual chart differentials. For the two exterior differentials d' and d'' on the ordinary chart coefficients, tangential restriction and affine pullback commute with the operators by the constant-coefficient chain rule on the two factors. Computing on each cell shows that this descends to intrinsic coefficients: if a tangential coefficient vanishes there so do its tangential derivatives. Applying a differential preserves the support carrier and the chart family. Each order-m seminorm after application is bounded by a constant times the order-$(m+1)$ seminorm before application, by the coordinate formula for exterior differentiation. This gives continuity on the smooth compact tests and defines the two transposed differentials on currents. The transpose uses the Reader sign $(-1)^{a+b+1}$ on a current of bidegree (a, b). In bidimension (p, q) the sign is $(-1)^{p+q+1}$, since a space of level l has $a + b = 2l - p - q$. For a test of degree $(p-1, q)$ the two pairings for $d'u_*T$ and $u_*d'T$ are therefore

$$(-1)^{p+q+1}\langle T, u^*(d'\eta)\rangle \quad \text{and} \quad (-1)^{p+q+1}\langle T, d'(u^*\eta)\rangle.$$

The chart chain rule makes them equal. With a test of degree $(p, q-1)$ the same argument applies to d''; a negative test degree denotes the zero space.

Degree-zero descent for a finite rational cover

Let A be a norm-admissible generalized affinoid algebra over a complete real-valued non-Archimedean field K, including the trivially valued case. Take a finite rational cover of $\mathcal{M}_K(A)$ with defining rows $(g_i, f_{ia}; \rho_{ia})$, where $(g_i, (f_{ia})_a) = A$ and every $\rho_{ia} > 0$. For each one- or two-index list J, use the actual quotient B_J obtained by imposing all rows in the list, with separate variable blocks and equations $g_i T_{ia} = f_{ia}$. Use matched coefficient and quotient norms for the proof, and transport boundedness to any prescribed equivalent admissible norms. For the free rank-one coefficient $M = A$, the sequence

$$0 \longrightarrow A \xrightarrow{u} C^0 = \prod_i B_i \xrightarrow{\delta} C^1 = \prod_{i<j} B_{ij}, \qquad \delta(z)_{ij} = z_j|_{ij} - z_i|_{ij},$$

is exact through C^0, and u has bounded inverse onto its closed image. Products carry the maximum norm. In particular u identifies the structure algebra with the bounded equalizer of the two restrictions on every actual pair intersection.

Terms and arrows. The weighted coefficient algebra and its actual closed relation ideals use the selected NAG-001554 coefficient clause with NAG-001552 and NAG-001560. The restrictions $B_i \to B_{ij}$ send the variables to the corresponding block. Quotient infima give contractions with the matched norms. The actual two-block joint is the bounded Banach pushout: compatible bounded maps from the two rows supply scaled bounds on powers of all their variable images. Substitution by NAG-001398, the equations and quotient infima give the unique bounded joint map. This also identifies the joint with the separated completed algebra tensor, if that convention is used, as in NAG-001578. On finite balanced sums the product of the two structural maps is bounded for the projective seminorm, and the variable substitutions give the inverse on the actual joint and on dense pure tensors. All these comparisons retain the two structural maps.

For rank one the module terms are just the same actual Banach algebras. Multiplication identifies $B_J \otimes_A A$ with B_J, with inverse $b \mapsto b \otimes 1$. Both maps are contractive for the matched projective and coefficient norms, by the finite tensor estimate and the pure tensor representatives, respectively. Since B_J is complete, the separated completion gives the same term and the

same restrictions. Equivalent norms give the bounded version of the same comparison.

One free scalar extension and simultaneous powered rows. Use the reviewed extra-radii clause of NAG-001557 with one closed presentation of A and all the radii in the finite cover. Over the resulting free field $L = K_r$, put $R = A_L$. It is strict if nonzero. There are $N \geq 1$ and $c_{ia} \in L^\times$ with $|c_{ia}| = \rho_{ia}^N$. The reviewed NAG-001580 algebra and existing-map clause identifies $(B_J)_L$ with the actual weighted quotient of the same lists over R. It carries the bounded restrictions and augmentation just defined to their coefficient extensions.

In the weighted quotient each g_i with $i \in J$ is a unit by the row's unit-ideal identity. In the upper unit-radius quotient P_J, use the simultaneous rows

$$g_i^N W_{ia} = c_{ia}^{-1} f_{ia}^N \qquad (i \in J).$$

Each powered tuple generates the unit ideal: in a sufficiently high power of the original unit-ideal identity every monomial contains an N-th power of some generator. Hence every g_i^N, and thus g_i, is a unit in P_J. Sending W_{ia} to T_{ia}^N/c_{ia} gives a contraction to the weighted joint. In the other direction put $b_{ia} = f_{ia}/g_i$ in P_J. For $m = Nq + j$ with $0 \leq j < N$,

$$\|b_{ia}^m\| \leq \rho_{ia}^m \max_{0 \leq j < N} \rho_{ia}^{-j} \|b_{ia}^j\|.$$

Indeed $b_{ia}^N = c_{ia} W_{ia}$. Bounded substitution descends through the actual relations and sends T_{ia} to b_{ia}. The composites are identities by the equations and density. These are the NAG-001507 maps on the actual single and joint rows. They commute with every block inclusion because they fix R and use the same formula for each shared variable. The zero quotients satisfy the same identities. Every powered row has the original inequalities over R by NAG-001444. The upper family therefore covers because each upper spectral point restricts to a point of A. If the original cover is empty, NAG-001452 and the isometric constant embedding force $A = 0$. We treat a zero base or coefficient quotient by its zero algebra.

The upper ordered equalizer, including nonzero intersections. For a nontrivially valued L, use the finite product refinement of NAG-001474 on the powered unit-radius rows. Write the powered row as (p_{i0}, p_{ia}), with

$p_{i0} = g_i^N$. Index the standard tuple by λ having at least one zero slot, with $F_\lambda = \prod_i p_{i,\lambda_i}$. At a point of R, choose a coarse member containing it, use the denominator in that slot, and choose a nonvanishing entry of each other unit-ideal tuple. This gives a nonvanishing F_λ. NAG-001452 gives the unit-ideal condition for the standard tuple, so its standard quotients S_λ cover by taking maxima.

Choose once a zero slot $k(\lambda)$ of each λ. At a point of S_λ, if a containing coarse row is $i \neq k$, replace slot i by zero and slot $k = k(\lambda)$ by any numerator index a. The defining maximal inequality gives

$$\left|\frac{p_{i0}}{p_{i,\lambda_i}}\right| \left|\frac{p_{ka}}{p_{k0}}\right| \leq 1.$$

The factors divided by are nonzero since F_λ is nonzero. The first factor is at least one since the point lies in row i. When $i = k$ the desired inequality already holds. Thus the point lies in the chosen row k. The corresponding bounded map $P_k \to S_\lambda$ follows on the actual strict S_λ by NAG-001452 for invertibility of the denominator, the selected NAG-001457(1)⇒(4) for scaled ratios of radius one, and the reviewed NAG-001443/001442 supplied-power substitution. For a zero target the map is the unique zero map.

All strict mixed and pair restrictions are the concatenations of their actual relation lists, or the successive quotients with the same norms and arrows by NAG-001434 and NAG-001437. In the unit-radius product version of NAG-001471 the additional pair-product variable maps to the product of the two separate variables. Each original denominator is a unit after imposing its own row, and the product denominator is a unit in the product version. The substitutions in both directions are contractive and preserve the original blocks and all equations. In particular two owner maps into a standard joint give a bounded map from the actual coarse joint to that standard joint. There is generally a nonzero joint when the owners differ. For the mixed term P_i, S_λ, keep both lists even when $i \neq k(\lambda)$. If the owner is i, its coarse row is redundant by the already supplied power bounds, with bounded inverse to adjoining it. The restricted standard family over P_i is the image of the same unit-ideal tuple (F_λ) in that actual algebra.

Here is the degree-zero part of the standard-family argument NAG-001473 used over R and over each nonzero P_i. For a unit tuple, apply selected NAG-001462 to the fixed signed list of pair ratios at radius one. Each leaf maps boundedly from a standard member with all its ratios of norm at most one by the directed-path calculation of NAG-001467. The leaf maps remain valid

under further quotients. A matching standard tuple thus gives a matching signed tuple, since on a signed pair both standard blocks can be adjoined and eliminated by bounded substitution. Glue by the signed ordered equalizer. To verify each standard entry, test the resulting section on the signed family over that very standard algebra. The signed augmentation there is injective by NAG-001462 applied afresh. This gives a bounded inverse, and injectivity follows by the signed tests.

For a general unit-ideal tuple F_λ, choose $\sum a_\lambda F_\lambda = 1$ and $0 < |\kappa|$ with $\theta = |\kappa| \max_\lambda \|a_\lambda\| < 1$. Take all signs on F_λ/κ. On a leaf write its relations as $F_p = \kappa X_p$ for positive indices and $F_n X_n = \kappa$ for negative ones. For a positive denominator F_p, in its joint with that leaf set $T_p = 1$. The relation ideal contains $1 - \kappa X_p \sum_\lambda a_\lambda T_\lambda$, a unit by the geometric series (NAG-000184) since the subtracted term has norm at most θ. An all-positive leaf is itself zero by the same argument with $1 - \kappa \sum a_\lambda X_\lambda$. For a negative denominator F_n, its inverse is X_n/κ, and each positive numerator gives $F_p/F_n = X_p X_n$ of norm at most one. Eliminating precisely these redundant variables gives the actual standard family of the remaining units. These norm-one substitutions persist on every further joint leaf with its other blocks retained. This replays the NAG-001468 small-scalar calculation.

A matching standard tuple glues on each leaf by the unit case just proved. On a joint of two leaves the two glued values restrict to the same standard entries. The standard family on that joint is again a unit family after the same eliminations from either one of its leaves. Its injective augmentation gives equality on the actual joint. Glue the leaf tuple by NAG-001462. Now apply its augmentation injection to the signs over each actual standard member to check the prescribed standard values. The inverse bound is finite: take the maximum of the finitely many leaf inverse bounds and multiply by the signed inverse bound on the base. This proof supplies the needed degree-zero equalizer and injection over each actual strict base used above, without applying coefficient change to an exact complex.

Given a matching coarse tuple, restrict its owner entries to the standard members. On a joint of two standard members the entries match by the map from the actual coarse pair, including when that pair is nonzero. Glue by the standard equalizer over R. On each P_i, compare its prescribed entry with the glued section on every mixed P_i, S_λ. Equality follows from the original coarse matching condition after mapping $P_{i,k(\lambda)}$ to that mixed joint. For repeated owners both restrictions come from the same entry. Injectivity of

the restricted standard augmentation over P_i then gives the prescribed coarse value. Global injectivity follows by testing over the standard members, and the inverse bound is the standard bound times the maximum owner-map bound. All bounds concern the fixed finite data.

If L is trivially valued, use a defining strict quotient norm. Unit-radius Tate coefficient rings then consist of polynomials, with norm one on nonzero classes. Write d_i for the powered denominators. The unit-ideal identities identify P_i and P_{ij} with $R[1/d_i]$ and $R[1/(d_i d_j)]$, respectively, with the ordinary restriction maps. At every maximal ideal the trivial residue seminorm is bounded for this defining norm. The cover implies that some d_i is nonzero there, so the d_i generate the unit ideal. To glue a matching tuple, write its entries a_i/d_i^m with a common exponent $m \geq 1$. Pair equality allows clearing all pair differences by a common power of $d_i d_j$. Increasing the exponent and the numerators accordingly yields entries b_i/d_i^n with $d_j^n b_i = d_i^n b_j$ in R. Choose $\sum e_i d_i^n = 1$. Then $\sum e_i b_i$ has all the prescribed localizations. Injectivity follows because an element vanishing in each localization is killed by fixed powers of all d_i, which still generate the unit ideal. The inverse is bounded for the discrete norms. Equivalent admissible norms preserve the finite inverse bound.

Kernel and norm-controlled descent. Use the coefficient description NAG-001549 with the fixed free tuple r. Finite products commute isometrically with the weighted Laurent c_0 coefficient spaces. The upper complex, before power conversion, has differential δ_L acting as δ on each Laurent coefficient, including the displayed sign. Indeed NAG-001580 identifies both actual maps into each actual pair algebra, and the power comparisons commute with these maps. For $E = \ker \delta$, give E its induced norm. It is closed because δ is bounded. In coefficient norms the equality

$$E \widehat{\otimes}_K L \;\simeq\; c_0(\mathbf{Z}^{|r|}, r; E) \;=\; \ker \delta_L$$

is isometric: a coefficient family has zero difference exactly when every coefficient belongs to E, and the weighted norm and finite-tail condition are inherited. Transport to the powered diagram by the bounded inverse comparison maps.

For $z \in E$, the constant coefficient family is an upper matching tuple. The upper equalizer yields a preimage $w \in A_L$ with $\|w\| \leq Q\|z\|$, where Q includes the finite norm comparison factors. Naturality of the contractive

linear zeroth coefficient gives $u(p_0 w) = z$ and $\|p_0 w\| \leq Q\|z\|$. Upper injectivity and isometric constant embedding give injectivity of u. Thus the image is the closed kernel E and the augmentation has the asserted inverse norm. The inverse is an algebra map because it inverts the algebra map $u : A \to E$.

Final-space and coefficient estimates

Example C.1.10 (Finality cannot be omitted)**.** Let $E = \mathbb{R}^{(\mathbb{N})}$ be the finitely supported sequences, but give it the ℓ^2 norm topology. Let E_N consist of sequences supported in $\{1, \dots, N\}$ and let $D_m(x) = x_m$. Every D_m is continuous; $D_m|_{E_N} = 0$ for $m > N$; and $D_m(x)$ is eventually zero for each x. Nevertheless $D_m \not\to 0$ in the strong dual of this normed space.

Proof. The unit vectors form a bounded set $B = \{e_m : m \geq 1\}$ and $\sup_{b \in B} |D_m(b)| = 1$ for every m. The norm topology is strictly coarser than the locally convex final topology of the inclusions E_N: the seminorm $x \mapsto \sum_{m \geq 1} m|x_m|$ is continuous on every E_N, hence final-continuous, but is unbounded on the norm-bounded set B. With the final topology, this B is not bounded. □

Example C.1.11 (No presentation-independent extension constant)**.** Even integral finite polyhedral complexes with weight one can require unbounded constants in estimates of an ambient quotient seminorm by finitely many intrinsic seminorms. For $k \geq 1$, put

$$v_k = (k, 1), \quad w_k = (k{+}1, 1), \qquad P_k = \{tv_k : |t| \leq (k{+}1)^{-1}\} \cup \{tw_k : |t| \leq (k{+}1)^{-1}\}.$$

Subdivide at the origin and use the primitive integral tangent vectors on the segments. In scalar degree, let a_k be the restriction of $x - ky$. Every intrinsic derivative seminorm of order at least one is at most one, but every ambient representative has coefficient C^1 seminorm at least k on any compact neighbourhood of the origin.

Proof. On the v_k-segment the restriction is zero, and on the w_k-segment it is t. The values are bounded by $(k+1)^{-1}$, first tangential derivatives have absolute value at most one, and all higher derivatives vanish. If F is any ambient representative, its gradient at zero satisfies

$$kF_x(0) + F_y(0) = 0, \qquad (k+1)F_x(0) + F_y(0) = 1.$$

Thus $F_x(0) = 1$ and $F_y(0) = -k$. Taking the quotient infimum does not remove this lower bound. The two vectors are primitive and their matrix determinant is -1; each line has two opposite rays, so unit weights balance at the central vertex. The outer endpoints are simply truncation boundaries. □

Closed-fibre lengths

Lemma C.1.12 (Closed-fibre length; F8ST-L)**.** *Let $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a flat local map of Noetherian rings. Suppose $S/\mathfrak{m}S$ is Artinian. For every finite-length R-module M,*

$$\operatorname{length}_S(M \otimes_R S) = \operatorname{length}_R(M)\, \operatorname{length}_S(S/\mathfrak{m}S). \tag{C.1.9}$$

If the map is regular and the closed fibre is zero-dimensional, its closed fibre is a field and the last factor is one. In particular a regular local map between Artinian rings preserves these lengths.

Proof. Tensor a composition series of M with S. Flatness preserves its injections and quotients; every quotient is $S/\mathfrak{m}S$. Additivity of finite length gives (C.1.9). A zero-dimensional regular local ring has zero maximal ideal, hence is a field. Thus $\mathfrak{m}S = \mathfrak{n}$ in the regular case. A local map is faithfully flat when it is flat, so a nonzero composition factor is never lost. □

Lemma C.1.13 (Regular elements avoid minimal primes; F8ZP-N)**.** *Let A be a nonzero ring and suppose multiplication by $f \in A$ is injective. Then $f \notin \mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Min}(A)$.*

Proof. Localization preserves the injection. If $f \in \mathfrak{p}$, its image in the nonzero ring $A_{\mathfrak{p}}$ belongs to the unique prime of that ring and is therefore nilpotent, by the nilradical criterion. A nilpotent element cannot act injectively on a nonzero ring: iterating an injective multiplication would make the zero multiplication injective. This contradiction proves the assertion. No associated-prime or S_1 assertion is used. □

Proposition C.1.14 (Full lengths at a regular principal prime; F8CR-L)**.** *Let A be Noetherian, let $f \in A$ be a non-zero-divisor, and let $\mathfrak{p}$ be minimal over (f). Then $R = A_{\mathfrak{p}}$ has Krull dimension one, f is a regular nonunit in R, and R/fR is Artinian. For $\mathfrak{q}_i \in \operatorname{Min} A$ contained in $\mathfrak{p}$, put $m_i = \operatorname{length}_{A_{\mathfrak{q}_i}} A_{\mathfrak{q}_i}$. Then*

$$\operatorname{length}_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/fA_{\mathfrak{p}}) = \sum_{\mathfrak{q}_i \subset \mathfrak{p}} m_i \operatorname{length}_{(A/\mathfrak{q}_i)_{\mathfrak{p}/\mathfrak{q}_i}} \left((A/\mathfrak{q}_i)_{\mathfrak{p}/\mathfrak{q}_i}/f_i\right). \tag{C.1.10}$$

No S_1 hypothesis on A is made. The proof uses Stacks 00E0(7) with the zero ideal for the nilradical steps in the local rings and quotients, 00KV(1) for the height bound, and 02QG, second clause with $I = 0$, for the length formula.

Proof. A regular element avoids every minimal prime $\mathfrak{q}$: in the nonzero zero-dimensional local ring $A_{\mathfrak{q}}$, an element of its maximal ideal is nilpotent, whereas an injective multiplication map cannot be nilpotent. Localization preserves injectivity. By 00KV(1), $\operatorname{ht}\mathfrak{p} \leq 1$; height zero would make $\mathfrak{p}$ minimal in A, which was excluded. Thus $\dim R = 1$. Minimality over (f) says that the only prime of R containing f is its maximal ideal. Hence R/fR is zero-dimensional Noetherian local and Artinian. The element f is not a unit there and remains regular. In particular R has depth one and is Cohen–Macaulay, although the second clause of 02QG already uses exactly the hypotheses just checked.

The minimal primes of R are precisely $\mathfrak{q}_i R$ with $\mathfrak{q}_i \subset \mathfrak{p}$. Indeed, localization of a minimal prime contained in $\mathfrak{p}$ is minimal, and contraction identifies all such primes. Each has one-dimensional quotient in R: a zero-dimensional local quotient would force $\mathfrak{q}_i R$ to be the maximal ideal, contrary to $f \notin \mathfrak{q}_i R$. Also $R_{\mathfrak{q}_i R} = A_{\mathfrak{q}_i}$. Apply 02QG with this Noetherian local R, $x = f$, and $I = 0$. Its generic factors are exactly m_i. Its other factors are $\operatorname{length}_R R/(f, \mathfrak{q}_i R)$. Computing this length over the quotient local ring $R/\mathfrak{q}_i R$ changes no composition factors; both have residue field $\kappa(\mathfrak{p})$. This gives (C.1.10), with no extra residue-degree factor. This argument says nothing about embedded primes in other localizations of A. □

Let K be a complete non-Archimedean field with its specified absolute value. Trivial valuation, positive characteristic, and generalized affinoid radii are allowed. All rings are commutative with identity; all modules in the assertions are finite over the indicated Noetherian ring. "Regular function" means non-zero-divisor, not smooth function. A regular ring map is flat with geometrically regular algebraic fibres. We use ordinary local rings on good analytic spaces. A finite module M satisfies S_1 if

$$\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \geq \min\{1, \dim \operatorname{Supp}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}\} \quad (\mathfrak{p} \in \operatorname{Supp} M).$$

This is a condition at all localizations, including the localizations of an ordinary analytic stalk.

Local algebra used in Serre comparison. Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a nonzero finite module. If an element r kills a nonzero element of M, choose a nonzero element v in its kernel whose annihilator is maximal among those annihilators. This annihilator is prime. Indeed, if $abv = 0$ and $bv \neq 0$, then bv is in the same kernel, and its annihilator contains

both the annihilator of v and a. Maximality forces a into the annihilator of v. Thus the zero divisors on M are the union of its associated primes.

These primes are finite. Repeatedly choose an element with prime annihilator in the current nonzero quotient module, and add its cyclic submodule to a filtration. The process terminates by Noetherianity. An embedded copy of $R/\mathfrak{p}$ in a module either meets a given submodule nontrivially, in which case the intersection has a nonzero element with annihilator $\mathfrak{p}$, or maps injectively to the quotient. Applying this to the filtration shows that all associated primes occur among its cyclic factors.

If M has depth zero, every element of $\mathfrak{m}$ is a zero divisor on M. Discard contained primes from the finite union and write the remaining incomparable primes as $\mathfrak{p}_i$. If $\mathfrak{m}$ is contained in none of them, choose $a_i \in \mathfrak{m} \setminus \mathfrak{p}_i$ and $b_i \in (\bigcap_{j\neq i} \mathfrak{p}_j) \setminus \mathfrak{p}_i$. One can take b_i to be a product of one element from each $\mathfrak{p}_j \setminus \mathfrak{p}_i$, with the empty product one. Then $\sum_i a_i b_i$ is in $\mathfrak{m}$ and in none of the primes, a contradiction. Thus $\mathfrak{m}$ is associated and $R/\mathfrak{m}$ embeds in M.

A nonzero zero-dimensional Noetherian local ring has nilpotent maximal ideal. Its finitely many generators are nilpotent by the nilradical criterion, and their degree bound makes the ideal nilpotent. The ideal-power filtration has finite-dimensional residue-field factors, so the ring is Artinian. The last nonzero power (or the field itself) supplies a nonzero socle. A finite module supported only at the closed point has finite length by the same argument over the quotient by its annihilator. Indeed its support is the vanishing set of that annihilator, as is seen by clearing the local denominators on a finite set of generators.

Lemma C.1.15 (Serre comparison for finite modules; F8ST-S)**.** *Let $R \to S$ be a regular map of Noetherian rings and M a finite R-module. If M is S_1, then $M \otimes_R S$ is S_1. If the map is also faithfully flat, the converse holds. The converse in fact needs only faithful flatness, not regular fibres.*

Proof. For ascent, fix $\mathfrak{q} \in \operatorname{Supp}(M \otimes_R S)$, put $\mathfrak{p} = \mathfrak{q} \cap R$, and work with the flat local map $R_\mathfrak{p} \to S_\mathfrak{q}$. Write $M_0 = M_\mathfrak{p}$ and $N = M_0 \otimes S_\mathfrak{q}$. We need a non-zero-divisor on N in its maximal ideal only when $\dim \operatorname{Supp} N > 0$. If $\dim \operatorname{Supp} M_0 > 0$, the S_1 condition supplies such an element $a \in \mathfrak{p}R_\mathfrak{p}$ on M_0; flatness preserves its injectivity on N.

Otherwise M_0 is nonzero of finite length. Its composition series tensors to a filtration of N by copies of $F = S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$. Hence $\operatorname{Supp} N = \operatorname{Spec} F$. The ring F is regular local. If its dimension is positive, its maximal ideal contains a

nonzero element that is regular on F, because F is a domain (STACKS-00NP). Lift that element to $\mathfrak{q}S_{\mathfrak{q}}$. It is injective on each filtration quotient, and therefore on N. This proves ascent.

For descent, suppose M is not S_1. Choose a prime $\mathfrak{p}$ for which $M_{\mathfrak{p}}$ has positive support dimension and depth zero. There is an injection $\kappa(\mathfrak{p}) \hookrightarrow M_{\mathfrak{p}}$: equivalently, the maximal ideal is associated to this finite local module. In $(R \setminus \mathfrak{p})^{-1}S$, choose a prime minimal over the extension of $\mathfrak{p}$. Such a prime exists by faithful flatness; let $\mathfrak{q}$ be its preimage in S. Its contraction is $\mathfrak{p}$, and $F = S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}$ is a nonzero Artinian local ring. Tensoring the injection gives $F \hookrightarrow N$. The socle of F is nonzero, so $\operatorname{depth} N = 0$.

Choose $\mathfrak{p}_0 \subsetneq \mathfrak{p}$ in $\operatorname{Supp} M$. Flat going down (STACKS-00HS) gives $\mathfrak{q}_0 \subsetneq \mathfrak{q}$ over $\mathfrak{p}_0$. The localized map is faithfully flat, so $(M \otimes_R S)_{\mathfrak{q}_0} \neq 0$. Consequently $\dim \operatorname{Supp} N > 0$, contradicting S_1 upstairs. □

Finite extensions of complete discretely valued fields

Let K be complete for a specified nontrivial discretely valued non-Archimedean absolute value. Put

$$R = K^\circ, \qquad k = R/\mathfrak{m}_R, \qquad \lambda_K = -\log|\pi| > 0$$

for a uniformizer π of R.

Lemma C.1.16 (The valuation-ring factors)**.** *Let L/K be a finite field extension of degree n, with the unique extension of the given absolute value, and put $R' = L^\circ$ and $k' = R'/\mathfrak{m}_{R'}$. Then R' is a complete DVR, finite free of rank n over R. If ϖ is its uniformizer,*

$$\pi = u\varpi^e, \quad u \in (R')^\times, \qquad f = [k' : k], \qquad n = ef, \qquad \lambda_L = -\log|\varpi| = \lambda_K/e.$$

No separability assumption is required.

Proof. The extending absolute value, completeness of L, and equivalence with a coordinate maximum norm follow from Lemma 1.12.1. Write the value groups additively inside $\mathbb{R}$. Elements of $L^\times$ whose values represent distinct cosets modulo $\lambda_K\mathbb{Z}$ are linearly independent over K: in a nontrivial linear relation the nonzero terms would have distinct values, so the smallest could not cancel. Hence the index of $\lambda_K\mathbb{Z}$ in the value group of L is finite, at most n. A subgroup of $\mathbb{R}$ containing this lattice with finite index e is $(\lambda_K/e)\mathbb{Z}$: multiplication by the order of the finite quotient first embeds it in a discrete

cyclic group. Choose ϖ with its least positive value. Every nonzero ideal of R' is generated by an element of least value, so R' is a DVR, and $\pi = u\varpi^e$.

Fix a K-basis of L. The coordinate norm bound puts the unit ball R' inside $\pi^{-N}R^n$ for some integer N. It is an R-submodule and is finite free: for a submodule of R^n, project to one coordinate, whose image is a principal ideal, lift a generator, and apply induction to the kernel in R^{n-1}. This also proves finite generation, without assuming it beforehand. Scaling the chosen basis by powers of π puts it inside R', so the rank is n. The ϖ-adic topology is the induced valuation topology; the closed unit ball is complete.

Consequently $\dim_k R'/\pi R' = n$. Its filtration by $\varpi^j R'/\varpi^e R'$ has e quotients isomorphic to k'. The residue extension is finite (it is a quotient of this finite k-space), and comparison of dimensions proves $n = ef$. The absolute-value formula follows from the displayed equation for π, since a unit of R' has absolute value one. □

Ordinary height-one rings and normalization coefficients

Lemma C.1.17 (Ordinary height-one model local rings)**.** *Let $\mathcal{T}$ be a Noetherian normal integral scheme and x a point of height one. The ordinary algebraic local ring $A = \mathcal{O}_{\mathcal{T},x}$ is a DVR. Its normalized order* $\mathrm{ord}_x : K(\mathcal{T})^\times \to \mathbf{Z}$ *satisfies*

$$\mathrm{length}_A(A/fA) = \mathrm{ord}_x(f) \qquad (0 \neq f \in A).$$

For a rational local equation f/g, its signed divisor coefficient is therefore $\mathrm{ord}_x(f/g) = \mathrm{ord}_x(f) - \mathrm{ord}_x(g)$.

Exact external input: Tag 00PD, Lemma 10.119.7, implication (5) ⇒ (1) and its normalized uniformizer expansion.

Proof. The ring A is local and Noetherian, and normality makes it a normal domain. Height one means $\dim A = 1$. These are exactly condition (5) of Tag 00PD, so condition (1) gives a DVR. For a generator t of its maximal ideal, every nonzero element is ut^n with u a unit. The successive quotients of the t-adic filtration of $A/(t^n)$ are copies of the residue field; hence their total length is n. Subtracting orders gives the assertion for a rational equation. □

For an integral model $\mathcal{T}$ and a codimension-one point v, write $A = \mathcal{O}_{\mathcal{T},v}$. A Cartier divisor with local rational equation $f = a/b \in \mathrm{Frac}(A)^\times$ has coherent cycle coefficient

$$\mathrm{ord}_A(f) = \mathrm{length}_A(A/aA) - \mathrm{length}_A(A/bA).$$

A unit contributes length zero. These lengths are finite, and the formula is independent of a/b: for nonzero a, b the exact sequence $0 \to A/aA \xrightarrow{b} A/abA \to A/bA \to 0$ gives additivity of lengths.

Lemma C.1.18 (Normalization preserves the coefficient with branch weights)**.** *Let A be a one-dimensional Noetherian local domain whose integral closure B in $F = \mathrm{Frac}(A)$ is finite over A. Let $\mathfrak{q}_1, \dots, \mathfrak{q}_r$ be the maximal ideals of B. Their contractions are the maximal ideal $\mathfrak{m}$ of A; each $B_{\mathfrak{q}_j}$ is a DVR. For every $f \in F^\times$,*

$$\mathrm{ord}_A(f) = \sum_{j=1}^{r} [\kappa(\mathfrak{q}_j) : \kappa(\mathfrak{m})] \, \mathrm{ord}_{B_{\mathfrak{q}_j}}(f). \tag{C.1.11}$$

In particular, for nonzero $t \in \mathfrak{m}$,

$$\operatorname{length}_A(A/tA) = \sum_j [\kappa(\mathfrak{q}_j) : \kappa(\mathfrak{m})] \operatorname{length}_{B_{\mathfrak{q}_j}}(B_{\mathfrak{q}_j}/t). \tag{C.1.12}$$

All residue degrees are full degrees; no separable-degree substitution or reduction of the quotient rings is permitted.

Proof input. Finite module length, the affine integral-closure construction, and the proposed Tag 00PD; no determinant or pending F9 claim.

Proof. The finite algebra $B/\mathfrak{m}B$ is Artinian. A prime of B over (0) is (0), since localization by $A \setminus \{0\}$ gives the field F. A prime over $\mathfrak{m}$ is maximal, because its quotient is a finite domain over $A/\mathfrak{m}$, hence a field. A maximal ideal cannot lie over (0): this would make B a field finite over A; a monic equation for $1/t$, with $0 \neq t \in \mathfrak{m}$, would then imply $1/t \in A$. Thus B is Noetherian and semilocal, and each of its maximal localizations is a normal local domain of dimension one. Tag 00PD gives the DVR assertion. The residue extensions are finite because B is finite over A.

The finite A-module $Q = B/A$ is torsion and has finite length. Indeed, clearing the denominators of finitely many generators kills Q by a nonzero element of A; in dimension one its support is contained in $\{\mathfrak{m}\}$. Multiplication by t is injective on both A and B. The kernel-cokernel sequence gives

$$0 \longrightarrow Q[t] \longrightarrow A/tA \longrightarrow B/tB \longrightarrow Q/tQ \longrightarrow 0.$$

On a finite-length module the kernel and cokernel of any endomorphism have the same length. Hence $\operatorname{length}_A(A/tA) = \operatorname{length}_A(B/tB)$. The finite-length B-module B/tB has a composition series with simple factors $\kappa(\mathfrak{q}_j)$, their numbers being the lengths after localization at $\mathfrak{q}_j$. The A-length of such a factor is its full dimension over $\kappa(\mathfrak{m})$. This proves (C.1.12). The same equality for a nonzero unit has both sides zero. Apply it to a and b and subtract to obtain (C.1.11). □

Finite restriction of scalars

This is the proof of Lemma 1.14.6.

Finite restriction of scalars (NAG-001485). For the finite valued extension L/K use a K-basis (ℓ_i). NAG-001450 gives bounded coordinate maps λ_i and the completeness of the finite extension. For every Banach K-space E, sending (e_i) to $\sum e_i \otimes \ell_i$ is bounded for the projective seminorm. On any finite tensor expression $1_E \otimes \lambda_i$ is bounded by $\|\lambda_i\|$; taking its projective infimum bounds the inverse. Consequently the ordinary tensor is already complete with the finite coordinate norm, up to equivalence. Applying the same estimates to each Tate coefficient identifies $L\langle T\rangle$ with $K\langle T\rangle \otimes_K L$ as an actual finite free algebra with equivalent norm. The selected finite strict algebra theorem NAG-001483 makes it a strict K-affinoid. A strict L-affinoid is an actual closed quotient of this algebra. Composing its quotient with a strict K-presentation yields a strict K-presentation with equivalent admissible norm, using successive quotient infima. Bounded L-maps are bounded K-maps with the same inequality. This assertion uses the finite extension with its given absolute value; there is no separability hypothesis.

Proposition C.1.19 (The source-factor inclusion for relative interior)**.** *Let $Z \xrightarrow{g} Y \xrightarrow{f} X$ be composable affinoid morphisms over a specified complete non-Archimedean field. Then*

$$\mathrm{Int}(Z/X) \subseteq \mathrm{Int}(Z/Y).$$

Proof. Write $X = \mathcal{M}(A)$, $Y = \mathcal{M}(B)$, and $Z = \mathcal{M}(C)$. For $z \in \mathrm{Int}(Z/X)$, choose an admissible presentation $q_A : A\{r^{-1}T\} \twoheadrightarrow C$ with $|q_A(T_i)(z)| < r_i$ for every i. Give C the equivalent quotient norm from q_A. Then $\|q_A(T_i)^n\| \leq r_i^n$. The bounded coefficient map $B \to C$ and these bounds give a bounded map $q_B : B\{r^{-1}T\} \to C$ by Proposition 1.9.3. The coefficientwise map $j : A\{r^{-1}T\} \to B\{r^{-1}T\}$ is bounded by the bound for $A \to B$. The identity $q_B j = q_A$ holds on polynomials and hence on the completions. In particular q_B is surjective.

Write $\|c\|_{q_B} = \inf_{q_B(v)=c} \|v\|$. Applying j to lifts through q_A and taking infima gives $\|c\|_{q_B} \leq \|j\|\|c\|$. Boundedness of q_B gives $\|c\| \leq \|q_B\|\|c\|_{q_B}$. Thus q_B is admissible. It has the same variable values and radii as q_A, so Definition 2.7.1 gives $z \in \mathrm{Int}(Z/Y)$. □

Strict zero-space purity

The strict ring in NAG-002944 is catenary

The native affinoid-ring step of NAG-001783 uses the ring clause of NAG-001760 for a generalized Tate algebra, followed by Stacks 00NQ, 00NM, and 0ECE. The application in NAG-002944 takes $R = (A/\mathfrak{p})_{\mathfrak{m}/\mathfrak{p}}$. If $A = T/I$ is strict, with $T = K\langle X_1, \dots, X_e\rangle$ and $\mathfrak{n}$ the inverse image of $\mathfrak{m}$, then R is a quotient of $T_{\mathfrak{n}}$. Thus only the regularity of this prescribed maximal Tate localization is needed. Here is an algebraic replacement of that input in the same strict scope.

For any specified maximal ideal $\mathfrak{n}$ of T, use NAG-001452 to take the bounded quotient $T \to E = T/\mathfrak{n}$. The complete specified extension E/K is finite, and the image a_i of X_i has $|a_i| \le 1$. Put $S = E\langle X_1, \dots, X_e\rangle$. By the coordinate norm equivalence of NAG-001450, a restricted coefficient family in E is exactly a tuple of separated restricted coefficient families in K, after choosing a finite K-basis. This gives the ordinary ring isomorphism $S = T \otimes_K E$ and finite freeness over T. Evaluation $S \to E$ at a restricts to the original quotient $T \to E$: the bounded maps agree on the dense polynomial subalgebra. The kernel in S is $\mathfrak{l} = (X_1 - a_1, \dots, X_e - a_e)$, as follows also from the translation and constant-term calculation below. Hence $\mathfrak{l} \cap T = \mathfrak{n}$.

The substitutions $X_i \mapsto X_i \pm a_i$ are mutually inverse by NAG-000232. In translated coordinates $U_i = X_i - a_i$, the kernel of total degree $< k$ truncation of S is the ordinary ideal $(U)^k$: assign each monomial of degree at least k to one chosen degree-k divisor, and divide the finitely many resulting groups by their divisors. Each shifted restricted series still converges. The truncated ring has nilpotent maximal ideal and residue field E, so localizing at this ideal does not change it. Thus the maximal ideal completion of $S_{\mathfrak{l}}$ is $E[[U_1, \dots, U_e]]$, using exactly the truncation argument of NAG-001717. The rings T and S are Noetherian by NAG-001403 at K and E, respectively. Stacks 032C makes the completion regular, and 07NY makes $S_{\mathfrak{l}}$ regular. Its maximal ideal modulo its square has dimension e by the same degree-< 2 truncation. In particular $\dim S_{\mathfrak{l}} = e$, by the definition of a regular local ring.

The map $T_{\mathfrak{n}} \to S_{\mathfrak{l}}$ is flat and local: S is free, localization of a flat module is flat, and the maximal ideal contracts to the specified $\mathfrak{n}$. Stacks 00OF gives regularity of $T_{\mathfrak{n}}$. Applying 00NQ and 00NM to this Noetherian regular local ring gives universal catenarity. The actual quotient $R = T_{\mathfrak{n}}/I_{\mathfrak{p}}T_{\mathfrak{n}}$, where $I_{\mathfrak{p}}$ is the inverse image of $\mathfrak{p}$ in T, is an algebra of finite type over it, so 0ECE

applies. This establishes the exact catenarity needed in the strict proof, for trivial and nontrivial valuations and without separability of E/K.

The prescribed maximal dimensions in the same proof

The preceding construction also supplies the strict integral application of NAG-002292 used twice in NAG-002944. Along $T \to S$, use flat going down (00HS) to lift a chain ending at $\mathfrak{n}$ to one ending at $\mathfrak{l}$. Conversely use integrality of the finite map (00GK) and incomparability (00GT) to contract every strict chain ending at $\mathfrak{l}$ to a strict chain. It follows that $\dim T_{\mathfrak{n}} = e$ for each specified maximal ideal $\mathfrak{n}$, and $\dim T = e$.

For a nonzero strict integral affinoid B and a specified maximal ideal $\mathfrak{m}_B$, take the actual finite injection $T \hookrightarrow B$ of NAG-001451. Write $\mathfrak{n}_B = \mathfrak{m}_B \cap T$. The image of T in $B/\mathfrak{m}_B$ is a finite-dimensional K-domain by NAG-001452. Multiplication by a nonzero element of this domain is injective and therefore surjective, so the image is a field and $\mathfrak{n}_B$ is maximal. The strict Tate domain T is normal by NAG-001454; the integral map is between domains. Stacks 00H8 now lifts chains starting from the specified prime $\mathfrak{m}_B$ downwards. Incomparability contracts strict chains. Thus $\dim B_{\mathfrak{m}_B} = \dim T_{\mathfrak{n}_B} = e$. The finite injective map has $\dim B = \dim T = e$ by 00GK and 00OK.

For NAG-002944 apply this calculation to the actual strict integral quotients $A/\mathfrak{p}$ and $A/\mathfrak{q}$, with maximal ideals $\mathfrak{m}/\mathfrak{p}$ and $\mathfrak{m}/\mathfrak{q}$. The strict quotient follows from NAG-001403: the inverse image ideal in the defining Tate algebra is closed, and the successive quotient norm is the same as the direct quotient norm. Choose $\mathfrak{m}$ by 00E0(2) on $A/\mathfrak{q}$. The existing chain calculation now uses the two prescribed dimensions and catenarity of R. The dimension of a strict affinoid is the Krull dimension of its algebra by Definition 2.8.1. Purity here means that every actual minimal-prime quotient has this dimension equal to the given n.

Proposition C.1.20 (Componentwise strict dimension; F8ZP-E)**.** *Let A be a nonzero strict K-affinoid algebra pure of dimension n. For primes $\mathfrak{p} \in \operatorname{Min}(A)$ and $\mathfrak{q} \supseteq \mathfrak{p}$,*

$$\operatorname{ht}_{A/\mathfrak{p}}(\mathfrak{q}/\mathfrak{p}) + \dim(A/\mathfrak{q}) = n. \tag{C.1.13}$$

In particular, for each $\mathfrak{q} \in \operatorname{Spec}(A)$ the ordinary algebraic localization $A_{\mathfrak{q}}$ is equidimensional, of dimension $n - \dim(A/\mathfrak{q})$. This assertion concerns $A_{\mathfrak{q}}$, not an identification with an ordinary analytic stalk.

Proof. Choose a maximal ideal $\mathfrak{m} \supseteq \mathfrak{q}$. Both $A/\mathfrak{p}$ and $A/\mathfrak{q}$ are strict integral affinoids. Apply the prescribed maximal-dimension calculation to these two rings separately. With $d = \dim(A/\mathfrak{q})$ this gives

$$\dim(A/\mathfrak{p})_{\mathfrak{m}/\mathfrak{p}} = n, \qquad \dim(A/\mathfrak{q})_{\mathfrak{m}/\mathfrak{q}} = d.$$

The first equality uses the dimension of the particular component $A/\mathfrak{p}$, not merely $\dim A_{\mathfrak{m}} = n$. The integral local ring $R = (A/\mathfrak{p})_{\mathfrak{m}/\mathfrak{p}}$ is catenary by the strict quotient calculation. Write $\mathfrak{r} = (\mathfrak{q}/\mathfrak{p})R$ and let $\mathfrak{n}$ be its maximal ideal. A saturated chain from 0 to $\mathfrak{r}$, followed by one from $\mathfrak{r}$ to $\mathfrak{n}$, is saturated from 0 to $\mathfrak{n}$. Finite dimension permits such chains. Catenarity makes their lengths respectively h, d, and n, where h is the height of $\mathfrak{r}$. Thus $h + d = n$.

Localization at $\mathfrak{m}$ does not remove any prime contained in $\mathfrak{q}$, so $h = \mathrm{ht}_{A/\mathfrak{p}}(\mathfrak{q}/\mathfrak{p})$. This proves (C.1.13). The minimal primes of $A_{\mathfrak{q}}$ are exactly $\mathfrak{p}A_{\mathfrak{q}}$ for $\mathfrak{p} \in \mathrm{Min}(A)$ contained in $\mathfrak{q}$: use prime correspondence for localization and choose a minimal prime below a contracted prime. Each corresponding component localization has dimension $h = n - d$, independently of $\mathfrak{p}$. Their prime-chain spectra cover the spectrum of $A_{\mathfrak{q}}$, proving the last assertion. Equidimensionality has therefore been derived at the precise algebraic localizations used, rather than assumed for analytic stalks. □

Proposition C.1.21 (Strict principal zero purity; F8ZP-P)**.** *Let A be a nonzero strict K-affinoid algebra pure of dimension $n \geq 1$. Suppose f avoids every minimal prime of A. Then A/fA is zero or every one of its minimal-prime quotients has Krull dimension $n-1$. For every prime $\mathfrak{q}$ minimal over fA and every $\mathfrak{p} \in \mathrm{Min}(A)$ contained in $\mathfrak{q}$,*

$$\mathrm{ht}_{A/\mathfrak{p}}(\mathfrak{q}/\mathfrak{p}) = 1.$$

If instead $n = 0$, such an f is a unit. These conclusions apply in particular when f is a non-zero-divisor.

Proof. If f is a unit, the quotient is zero. Otherwise choose $\mathfrak{q} \in \mathrm{Min}_A(fA)$ and a minimal prime $\mathfrak{p} \subseteq \mathfrak{q}$. The image of f in the Noetherian domain $A/\mathfrak{p}$ is nonzero. The prime $\mathfrak{q}/\mathfrak{p}$ is minimal over this element: a smaller prime containing it would lift to a prime strictly below $\mathfrak{q}$ containing f. Tag 00KV applies and bounds its height by one. It is not the zero prime, because it contains that nonzero element. Its height is therefore exactly one. Equation (C.1.13) gives $\dim(A/\mathfrak{q}) = n - 1$ for every such $\mathfrak{q}$. These are exactly the minimal-prime quotients of A/fA.

When $n = 0$, the same reasoning would give height one in a zero-dimensional component, an impossibility. Thus no prime can contain f, and f is a unit. Lemma C.1.13 proves the final clause. Components on which f is a unit simply contribute no zero component; no assumption that f is a nonunit on every component was made. Nilpotents and embedded primes have not been removed. □

Finite polyhedral ends

Write $N = \mathbb{Z}^r$, $E = N_{\mathbb{R}} \times \mathbb{R}$, with coordinates (y, t), projection $\pi : E \to N_{\mathbb{R}}$, and ambient lattice $N \oplus \mathbb{Z}$. A polyhedron below is closed and given by finitely many affine linear inequalities, with rational directions and arbitrary real constants. For its tangent space L_σ, its lattice is the saturated lattice $N_\sigma = L_\sigma \cap (N \oplus \mathbb{Z})$.

Definition C.1.22 (Finite graph datum)**.** Fix $n \geq 1$, a compact set $B \subset N_{\mathbb{R}}$, and $a \in \mathbb{R}$. A finite graph datum is a finite polyhedral complex in $B \times [a, \infty)$, with an n-chain

$$C = \sum_{\dim \sigma = n} m_\sigma [\sigma], \qquad m_\sigma \in \mathbb{Z}.$$

Zero-weight cells may be discarded. The analytic application uses nonnegative integral weights. All cycle statements in this patch are restricted to this integer-weight scope. For a top coefficient expression γ, write its tangential restriction in lattice coordinates as $\gamma|_\sigma = c_\sigma \omega_{N_\sigma}$, and set

$$\|\gamma\|_{C,1} = \sum_\sigma |m_\sigma| \int_\sigma |c_\sigma| \, d\mu_\sigma, \qquad I_C(\gamma) = \sum_\sigma m_\sigma \int_\sigma c_\sigma \, d\mu_\sigma$$

only when the indicated scalar cell integrals have first been defined and the first sum is finite. On a bounded polytope we use the grid integral constructed for continuous coefficients in the proof of NAG-002389. On an unbounded cell we use that construction for continuous coefficients with compact tangential support, cutting outside that support; an identically zero coefficient has integral zero. All expressions integrated below meet these conditions. In particular, finiteness of a formal domination bound is not a definition of an integral for arbitrary measurable coefficients. This absolute cell norm is attached to this presentation. It is not a newly constructed analytic Radon measure.

Lemma C.1.23 (Eventual cylinders; F8LA-P1)**.** *There is $H > a$ such that all bounded cells lie below H, and every unbounded cell has the form*

$$\sigma \cap \{t > H\} = Q_\sigma \times (H, \infty),$$

where $Q_\sigma \subset B$ is a compact polytope. For an unbounded n-cell, $\dim Q_\sigma = n - 1$, and

$$N_\sigma = N_{Q_\sigma} \oplus \mathbb{Z}e_t. \tag{2.1}$$

In fact $\pi|_{L_\sigma}$ has rank $n - 1$ on the whole unbounded cell, not just on its tail.

Proof. Represent a nonempty unbounded cell by inequalities

$$\ell_j(y) + b_j t \le c_j.$$

Include equalities as pairs of inequalities. Since y is bounded and $t \ge a$, unboundedness gives a sequence with $t \to +\infty$. Each inequality then implies $b_j \le 0$. Let Q be the set defined by the inequalities with $b_j = 0$. For any $y \in Q$, all remaining inequalities can be satisfied by taking t sufficiently large. Thus $Q = \pi(\sigma)$, so Q is a closed polyhedron contained in B, hence compact. The inequalities with $b_j < 0$ have the form $t \ge L_j(y)$. Their finitely many affine lower bounds have a common finite upper bound on Q. Above that bound the cell is precisely the displayed cylinder.

Its tangent space is $L_Q \oplus \mathbb{R}e_t$. Intersecting this direct sum with $N \oplus \mathbb{Z}$ gives (2.1). This also proves the dimension and rank assertions. Choose a common H above these finitely many bounds and above the maxima of t on every bounded cell. □

Theorem C.1.24 (Uniform top-degree excision; F8LA-P2R)**.** *Let C be a finite graph datum, and let α be a smooth (n, n)-superform on a neighbourhood of B. There is a constant $A_C < \infty$, depending only on the fixed weighted cells and coordinate lattices, such that for every continuous function $v : \mathbb{R} \to \mathbb{R}$,*

$$\|v(t)\pi^*\alpha\|_{C,1} \le A_C \sup_{a \le s \le H} |v(s)| \max_{I,J} \sup_{y \in B} |\alpha_{I,J}(y)|. \tag{2.3}$$

The coefficient bound is taken to be zero when the coefficient index set is empty (in particular when $n > r$), or when B is empty. The integral is the continuous-coefficient cell integral just specified, and the displayed supremum is finite by continuity. On an unbounded cell the horizontal top pullback is

identically zero, so its product with the finite-valued v is identically zero. In particular

$$\|t\pi^*\alpha\|_{C,1} < \infty. \tag{2.4}$$

If $v_R : \mathbb{R} \to \mathbb{R}$ is continuous and $v_R(t) = v(t)$ for $t \leq R$, then for every $R \geq H$,

$$I_C(v_R(t)\pi^*\alpha) = I_C(v(t)\pi^*\alpha). \tag{2.5}$$

The same H works for every α in this fixed presentation. No balancing assumption is needed.

Proof. On an unbounded n-cell the projection has rank $n-1$, by lemma C.1.23. Both exterior factors of a pulled-back (n,n) form therefore restrict to zero. All remaining n-cells are bounded and lie in $B \times [a, H]$.

Choose an integral tangent basis on each bounded cell. Write the restriction of $\pi^*\alpha$ in the positive lattice generator. Its coefficient is a finite sum of terms $\alpha_{I,J}(\pi(y,t))\Delta_{\sigma,I}\Delta_{\sigma,J}$, up to the fixed exterior signs, where the Δ's are minors of the matrix of π in that tangent basis. Put

$$A_C = \sum_{\sigma \text{ bounded}, \dim\sigma = n} |m_\sigma| \operatorname{vol}_{N_\sigma}(\sigma) \sum_{I,J} |\Delta_{\sigma,I}\Delta_{\sigma,J}|. \tag{2.6}$$

This is finite. On each bounded cell the product coefficient and its absolute value are continuous on a neighbourhood of the cell. Thus their grid integrals exist by NAG-002389 before the estimate is used. The pointwise determinant bound, positivity of grid sums, and finite addition give (2.3). The unbounded cells have zero coefficient throughout. For (2.4), use $\sup_{[a,H]} |t| < \infty$. The integrands in (2.5) agree on every bounded contributing cell and are both zero on every unbounded cell. □

Further proofs for Gauss fields and Banach spaces

Proof of 3.24.4.

NAG-001598. Coordinate changes in a free Gauss field. Let K be complete, let $r = (r_i)$ be free over K, and let $S_i \in K_r$ have $|S_i| = r_i$. Then $T_i \mapsto S_i$ induces an isometric automorphism of K_r. In particular the complete subfield generated by the S_i over K is K_r.

The weighted-null Laurent expansion has a unique dominant monomial. By freeness the dominant monomial of S_i is $\alpha_i T_i$ with $\alpha_i \in K$ and $|\alpha_i| = 1$. Composing with a coordinate scaling, assume $\alpha_i = 1$. Choose $q < 1$ with $|S_i - T_i| \le q r_i$ for all i. Such a bound follows from weighted nullity and finiteness of the tuple. The inverse difference has $|S_i^{-1} - T_i^{-1}| \le q/r_i$. A telescoping product for each signed monomial ν therefore gives $|S^\nu - T^\nu| \le q r^\nu$. For any Laurent polynomial f this gives $|f(S) - f(T)| \le q|f(T)|$. Consequently substitution is isometric and extends to a K-algebra endomorphism U of K_r. The operator $I - U$ has norm at most q, so for each a the convergent sum $\sum_{j\ge 0}(I-U)^j a$ maps to a. This gives surjectivity. Laurent polynomials in the new coordinates are dense.

Proof of 3.24.7.

NAG-001601. A discrete field with its coefficient field. Let E be complete with nontrivial cyclic value group, and let $K \subset E$ be trivially valued with reduction $K \simeq \widetilde{E}$. For a uniformizer τ, set $r = |\tau| < 1$. Then $T \mapsto \tau$ defines an isometric isomorphism $K_r = K((T)) \simeq E$. Indeed, the terms of each nonzero Laurent polynomial in τ have distinct norms, so substitution extends isometrically. Weighted nullity at negative exponents is exactly the requirement of finitely many negative terms. For $a_0 \in E^\circ$ choose $c_j \in K$ recursively with the residue of a_j, and set $a_{j+1} = (a_j - c_j)/\tau$. Cyclicity ensures $|a_{j+1}| \le 1$. The remainder after N steps has norm at most r^N. Multiplying any element of E by a power of τ brings it into E°, which gives surjectivity.

Proof of 3.21.2.

Coordinates on a Banach space of countable type (NAG-001658). Enumerate a dense linear spanning set. Fix $0 < c < 1$ and choose $0 < c_j < 1$ with product $> c$. At each new vector v_j outside the previous finite span

W, let $\delta = \operatorname{dist}(v_j, W) > 0$. This span is closed by finite-dimensional norm equivalence. Choose $w_j \in W$ with $\|v_j - w_j\| < \delta/c_j$ and set $e_j = v_j - w_j$. If $\|w\| > |a|\|e_j\|$, then $\|w + ae_j\| = \|w\|$. Otherwise for $a \neq 0$ it is at least $|a|\delta$, hence at least $c_j \max(\|w\|, |a|\|e_j\|)$; the zero case is immediate. Inductively the finite sums have lower bound c times their weighted maximum and upper bound equal to that maximum. The sum map on the complete weighted-null families therefore has closed image containing the dense spanning set. It is an isomorphism, and the bounds give uniqueness and bounded coordinate maps. The zero space has the empty family.

Three norm sanity checks

Proof of 3.21.3.

NAG-001659. Geometric reducedness of the strict Gauss completion. Let $R = K\langle T_1, \dots, T_n\rangle$ be the unit-polyradius Tate algebra over the complete field K. Set $F = \operatorname{Frac} R$ with its Gauss absolute value and $H = \widehat{F}$. For every field extension E/F, $H \otimes_F E$ is reduced.

In characteristic zero, each element of the tensor comes from a finitely generated subfield E_0/F of E, and this smaller tensor maps injectively to the full tensor. For a transcendence basis t of E_0/F, $H \otimes_F F(t)$ embeds into $H(t)$ by polynomial localization. Tensoring this injection over $F(t)$ with E_0 preserves injectivity. The resulting ambient algebra $H(t) \otimes_{F(t)} E_0$ is reduced. In fact adjoin finitely many algebraic generators of $E_0/F(t)$ in succession. At each step the monic minimal polynomial is relatively prime to its derivative over the preceding field, so this relation persists after mapping that field to each current field factor. Quotienting by this polynomial gives a finite product of fields at the next step.

In characteristic $p > 0$ endow each p-root field with the absolute value $|u| = |u^p|^{1/p}$. Taking pth powers shows that $K^{1/p}$ and $H^{1/p}$ are complete and identifies $R^{1/p} = K^{1/p}\langle S_1, \dots, S_n\rangle$ with $S_i^p = T_i$. For an F-independent family $u_1, \dots, u_m \in F^{1/p}$, first multiply all its members by a common nonzero element of R so they lie in $R^{1/p}$. If $u_i^p = a_i/b_i$, $\prod_i b_i$ suffices. The K-algebra generated by all coefficients of the resulting series is a field, since $a^p \in K$ and $a^{-1} = a^{p-1}/a^p$ for its nonzero elements. Its closure K' in $K^{1/p}$ is a field by continuity of inversion. Monomials in the countably many coefficients have dense K-span in K'.

Use NAG-001658 to take bounded K-linear coordinate maps $\lambda_j : K' \to K$ separating its points. For $0 \le \mu_i < p$ the maps

$$L_{j,\mu}\left(\sum_{\nu} a_\nu S^\nu\right) = \sum_{\eta \in \mathbf{N}^n} \lambda_j(a_{\mu+p\eta}) T^\eta$$

are bounded with norm at most $\|\lambda_j\|$. They are R-linear, first for polynomial multipliers and then by density and continuity. They separate all series. The columns $(L_{j,\mu}(u_i))_{j,\mu}$ are independent over F: clear denominators in a putative column relation and apply separation and R-linearity. Some m rows give an invertible $m \times m$ matrix over F, since the rows span the full dual of F^m. Its inverse and the bounds on the rows give $C > 0$ with

$$\max_i |b_i| \le C \left|\sum_i b_i u_i\right| \qquad (b_i \in F).$$

For coefficients in R use the bounds directly and for fractions clear a common denominator and divide the inequality by its Gauss absolute value. Passing to limits in $H^{1/p}$ gives the same inequality for $b_i \in H$. Thus $H \otimes_F F^{1/p} \to H^{1/p}$ is injective.

If $a_i \in H$ are F-linearly independent then the a_i^p are also independent. In fact an F-linear relation on the pth powers yields, on taking the pth root, a relation on the a_i with coefficients in $F^{1/p}$. The proved injection followed by linear independence in the scalar-extended vector space kills all coefficients. For any E/F, express a nonzero tensor as $\sum_i a_i \otimes e_i$, with the a_i independent over F and some $e_i \neq 0$. Its pth power is nonzero by independence of the a_i^p. Iterating excludes all nilpotents.

Proof of 2.37.1.

Matching kernels before any sheaf theorem (NAG-001826). For a bounded map $d : E \to F$ of Banach K-spaces, $N = \ker d$ is closed and carries its induced norm. With any one free tuple, NAG-001549 represents extension as weighted Laurent coefficients, and the extended map acts coefficientwise. A weighted-null family is in its kernel exactly when each coefficient is in N, with the same norm on that family. Over a finite product the maximum and the coefficient supremum commute, and the finite union of the exceptions to a fixed tail bound is finite. These identifications are isometric and natural. Module actions agree on finite Laurent sums and then by continuity, using the coefficient algebra and constant maps of NAG-001559. This statement places no closed-image condition on d.

Points, finite residue extensions, and exact spectral comparisons

Finite self-tensor over a complete field

For the application in Lemma 1.14.8 with base a complete field, use the following proof of the comparison in Proposition 1.14.5 with $A = K$ and $C = B$. The norm comparisons are Proposition 1.14.4 and Proposition 1.18.7.

The actual finite strict tensor (NAG-001484). Let $A \to B$ and $A \to C$ be bounded maps of strict affinoids, with B finite over A. By the selected finite-algebra norm comparison NAG-001483, the given norm on B is equivalent to its canonical finite-A-module norm. Apply accepted NAG-001565 to that module and the actual bounded map to C. The ordinary tensor $B \otimes_A C$, finite over C, is already complete with its canonical norm and identifies with the separated completion with bounded inverse. Its ordinary multiplication is the projective multiplication by the estimate on finite tensor expressions. The resulting finite algebra is strict by NAG-001483. The two structural maps are bounded for the projective seminorm and thus for its equivalent norm on the finite tensor. This is the printed finite-base-change scope using the accepted generalized tensor comparison.

The tame discrete ramified coefficient tensor

Let K be a complete nontrivially real-valued field, with additive value group $\Gamma = v(K^\times)$, valuation ring O, maximal ideal $\mathfrak{m}$, and residue field k. Fix $e \geq 2$, invertible in O, and a primitive e-th root $\zeta \in O$ whose reduction also has order e. Let $L = K(s)$, with its extended valuation, satisfy

$$s^e = c \in K^\times, \qquad \gamma = v(s) > 0, \qquad i\gamma \notin \Gamma \quad (1 \leq i < e). \tag{K}$$

Thus the e value cosets are distinct. In particular the degree is exactly e, and $G = \langle\sigma\rangle$, $\sigma(s) = \zeta s$, is the cyclic Galois group. Put $O' = L^\circ$, its *actual* valuation ring. No equality $O' = O[s]$, and no module finiteness of O'/O, is assumed.

For $0 \leq i < e$, define the fractional O-module

$$I_i = \{a \in K : v(a) + i\gamma \geq 0\}, \qquad v(0) = +\infty. \tag{1}$$

Here $I_0 = O$; an I_i need not be finitely generated. The distinct value cosets give the orthogonal expansion

$$v\left(\sum_{i=0}^{e-1} a_i s^i\right) = \min_{a_i \neq 0}\big(v(a_i) + i\gamma\big), \qquad O' = \bigoplus_{i=0}^{e-1} I_i s^i. \tag{2}$$

Indeed nonzero summands have distinct valuations, so none can cancel a summand of least valuation. The same argument proves linear independence of $1, s, \ldots, s^{e-1}$, hence the degree assertion above. Completeness follows coefficientwise from (2). Only the $i = 0$ summand can have value zero, so O' has residue field k, and G acts trivially on that field. The value index and degree are both e; this is the totally ramified, defectless tame class considered here.

For this section let R be a complete discrete valuation ring with uniformizer π, residue field k, and valuation $v(\pi) = 1$. Assume e and ζ as above, and take

$$R' = R[s], \qquad s^e = \pi, \qquad R' = \bigoplus_{i=0}^{e-1} R s^i, \tag{5}$$

where R' is the valuation ring of $K(s)$. These are discrete hypotheses; (5) is not used for general O'.

Lemma C.1.25 (The ramified coefficient map is not a torsor map)**.** *In (5), consider the natural map*

$$\theta : R' \otimes_R R' \longrightarrow \prod_{j=0}^{e-1} R', \qquad a \otimes b \longmapsto (a\sigma^j(b))_j. \tag{6}$$

It is injective. In Fourier coordinates its image is $\bigoplus_{i=0}^{e-1} s^i R'$*, so*

$$\operatorname{coker}(\theta) \simeq \bigoplus_{i=1}^{e-1} R'/(s^i), \qquad \operatorname{length}_{R'} \operatorname{coker}(\theta) = e(e-1)/2. \tag{7}$$

In particular (6) is not an isomorphism.

Proof. Using $u = 1 \otimes s$, the source is $R'[u]/(u^e - s^e)$, free over R' on $1, u, \ldots, u^{e-1}$. The column for u^i in (6) is $s^i(\zeta^{ji})_{j=0}^{e-1}$. The Fourier matrix (ζ^{ji}) is invertible over R', with inverse given by $e^{-1}(\zeta^{-ij})$. Thus (6) is equivalent to the diagonal map with entries $1, s, \ldots, s^{e-1}$. These are nonzero in the DVR R', proving all assertions. □

Algebraic unit groups

Let K be an algebraically closed field and A a nonzero finitely generated integral K-algebra. Put $U = \operatorname{Spec} A$, $F = \operatorname{Frac} A$, and $M_A = A^\times/K^\times$. A lattice is a finite-rank free abelian group. A subgroup is saturated when its quotient is torsion-free.

Lemma C.1.26 (Subgroups of an integer lattice)**.** *Every subgroup of* $\mathbb{Z}^s$ *is free of rank at most* s *and is finitely generated.*

Proof. Induct on s. The image of a subgroup H under the last coordinate is either zero or $d\mathbb{Z}$ for some $d > 0$. In the first case apply the induction hypothesis in $\mathbb{Z}^{s-1}$. In the second, choose $h \in H$ with last coordinate d. Subtracting integer multiples of h gives $H = (H \cap \mathbb{Z}^{s-1}) \oplus \mathbb{Z}h$. The induction proves both assertions, including $s = 0$. □

The algebraic closedness assumption also gives a direct torsion check: if $u^n = c \in K^\times$, then $T^n - c$ splits over K, and in the domain A one linear factor vanishes, so $u \in K^\times$.

Proposition C.1.27 (Two different saturation questions)**.** *For chosen units $u_1, \dots, u_r \in A^\times$, the relation kernel*

$$R_u = \{a \in \mathbb{Z}^r : \textstyle\prod_i u_i^{a_i} \in K^\times\}$$

is saturated in $\mathbb{Z}^r$. Thus $\mathbb{Z}^r/R_u$ is the actual subgroup $H_u \subset M_A$ generated by these units. The inclusion $H_u \subset M_A$ need not be saturated, even when A is normal and the chosen units and their inverses generate A.

Proof. The relation kernel is the kernel of $\mathbb{Z}^r \to M_A$. Its quotient embeds in the torsion-free group M_A, proving saturation of the kernel. For the last assertion take, over an algebraically closed field of characteristic different from 2,

$$A = K[t, t^{-1}, (t-1)^{-1}], \qquad u_1 = t^2, \quad u_2 = t - 1.$$

Factorization of polynomials over K gives $A^\times = K^\times t^{\mathbb{Z}}(t-1)^{\mathbb{Z}}$: a unit cannot have a zero or pole at any other finite point, and conversely these monomials are units. The ring is normal, as a fraction integral over this localization cannot have a pole at a non-inverted linear factor: a monic equation would have a unique term of lowest order there. In the basis $[t], [t-1]$ of M_A, the two chosen units generate $2\mathbb{Z} \oplus \mathbb{Z}$, of index 2. Their relation kernel is zero. Nevertheless they generate the algebra with their inverses, since

$$t = u_2 + 1, \qquad t^{-1} = (u_2 + 1)u_1^{-1}.$$

The corresponding closed embedding in $\mathbb{G}_m^2$ has equation $u_1 = (u_2+1)^2$. This proves the asserted distinction. □

Units on the unit-radius Laurent affinoid

In this section only, assume that K is complete, algebraically closed and nontrivially valued with its fixed real-valued non-Archimedean absolute value. Put

$$\mathcal{B} = K\langle T, T^{-1}\rangle = \left\{ \sum_{k\in\mathbb{Z}} a_k T^k : |a_k| \longrightarrow 0 \ (|k| \longrightarrow \infty) \right\}, \qquad \|f\| = \max_k |a_k|.$$

The coefficient space is complete and convolution is submultiplicative; finite Laurent polynomials are dense. The actual Banach presentation $K\langle X, Y\rangle/(XY - 1) \simeq \mathcal{B}$ with this norm is the $d = 1$, $r = 1$, $E = K$ case of frozen `NAG-001553`. Its spectrum is the Laurent affinoid $|T| = 1$ in

$\mathbb{G}_m^{\mathrm{an}}$: a bounded multiplicative seminorm has $|T|, |T^{-1}| \le 1$, and conversely evaluation with $|T| = 1$ extends continuously from Laurent polynomials. The Laurent presentation used here is the one-variable, unit-radius instance of Theorem 1.17.2. The bound at a point uses Lemma 1.6.1.

Proposition C.1.28 (The full analytic unit group is not a finite lattice)**.** *The group $\mathcal{B}^\times/K^\times$ is not finitely generated. In fact, for any prime ℓ with $|\ell| = 1$, it contains a subgroup isomorphic to $\mathbb{Z}[1/\ell]$, represented by units whose absolute value is identically 1 on $\mathcal{M}(\mathcal{B})$.*

Proof. Choose $c \in K$ with $0 < r = |c| < 1$. Such a prime ℓ exists by taking one different from the residue characteristic when that characteristic is positive. For $m \ge 0$ set $n = \ell^m$. On the closed ball $\|v\| \le r$ consider

$$\Phi_n(v) = n^{-1}\left(cT - \sum_{j=2}^{n} \binom{n}{j} v^j\right).$$

The sum is empty when $n = 1$. Since $|n| = 1$ and integer coefficients have norm at most 1, this map preserves the ball. Moreover

$$\|\Phi_n(v) - \Phi_n(w)\| \le r\|v - w\|,$$

using $v^j - w^j = (v - w)\sum_{a=0}^{j-1} v^{j-1-a}w^a$. Iteration from zero is Cauchy, because the successive differences are bounded by a geometric sequence with ratio $r < 1$; completeness gives a unique fixed point v_m in the ball. Set $u_m = 1 + v_m$. The fixed-point equation says

$$u_m^{\ell^m} = 1 + cT. \tag{8}$$

The geometric series in $-v_m$ proves that u_m is a unit. Also u_{m+1}^{ℓ} lies within distance r of 1 and satisfies the same equation as u_m, so uniqueness yields $u_{m+1}^{\ell} = u_m$.

Consequently $a/\ell^m \mapsto [u_m^a]$ defines a homomorphism $\mathbb{Z}[1/\ell] \to \mathcal{B}^\times/K^\times$. If its value is zero for $a \neq 0$, raising to the ℓ^mth power would make $(1 + cT)^a$ constant. This is impossible: for positive a its top coefficient c^a is nonzero, and negative a reduces to the positive case by inversion. Uniqueness of Laurent coefficients justifies this test inside $\mathcal{B}$. Thus the homomorphism is injective.

Every subgroup of a finitely generated abelian group is finitely generated: lift it to a subgroup of a finite-rank free group and apply Lemma C.1.26.

But a finite collection of elements of $\mathbb{Z}[1/\ell]$ has bounded denominator and cannot generate this group. This proves non-finite-generation. Finally frozen `NAG-000655` gives $|v_m(x)| \le \|v_m\| < 1$ at every analytic point x, so $|u_m(x)| = 1$. All represented tropical logarithms therefore vanish. □

Minimal primes of a product

For a Noetherian ring A, write $\mathrm{Min}_A(a)$ for the primes of A minimal over (a), and set $\mathrm{Min}_A(a) = \varnothing$ when a is a unit. A finite collection of primes is an antichain if no two distinct members are comparable by inclusion.

Lemma C.1.29 (Minimal primes of a product; F8CA-M)**.** *For any $a, b \in A$, $\mathrm{Min}_A(ab)$ is exactly the set of inclusion-minimal members of $\mathrm{Min}_A(a) \cup \mathrm{Min}_A(b)$. Consequently*

$$\mathrm{Min}_A(ab) = \mathrm{Min}_A(a) \cup \mathrm{Min}_A(b) \tag{C.1.14}$$

holds if and only if that union is an antichain. This assertion is in the one actual common ring A; it is not an identification of unrelated chart components.

Proof. A prime containing ab contains a or b, and therefore contains a prime minimal over one of those two ideals. If it is itself minimal over (ab), it must equal that smaller prime, which is then an inclusion-minimal member of the union. Conversely, let p be an inclusion-minimal member of the union and suppose $q \subset p$ contains ab. Choose inside q a prime r minimal over a factor contained in q. Then $r \subset q \subset p$ and r belongs to the union, so its minimality gives $r = p$, hence $q = p$. Thus p is minimal over (ab). This proves both the description and the equivalence. Units cause no exception. □

Minimal-prime choices in the two factor quotients. For a factor c in the specified prime Q, put $B = A/(c)$. The prime-filtration assertion of Lemma 5.19.1 applies to the finite B-module B, since B is Noetherian. If its nonzero filtration factors are $B/\mathfrak{p}_i$, the product of the $\mathfrak{p}_i$ annihilates B: successively multiplying by each factor prime takes a filtration step into the preceding one. It follows that every prime of B contains a factor prime. Choose an inclusion-minimal factor prime contained in $Q/(c)$. Any smaller prime would itself contain a smaller factor prime. The inverse image is therefore minimal over (c) and lies in Q. This gives both minimal-prime choices in the proof.

Product boundary sign

The following display records the sign convention for the boundary terms used in the polyhedral calculation.

$$\int_A u\, d'd''\eta_1 - \int_A d'd''u \wedge \eta_1 = \int_{\partial A} u\, d''\eta_1 - \int_{\partial A} d''u \wedge \eta_1. \tag{C.1.15}$$

Proposition C.1.30 (Product boundary sign; F8RC-S)**.** *For a calibrated integral $(n-1)$-chain P, with the standard product lattice, the facet convention in* (C.1.15) *gives*

$$\int_{\partial(P\times[\sigma,\tau])} d''x \wedge \chi(x)\alpha = -\int_P \alpha, \tag{C.1.16}$$

provided $\chi = 1$ near σ, $\chi = 0$ near τ, and α is pulled back from the horizontal factor. This is a polyhedral identity and does not supply R-GREEN.

Proof. Subdivide into products of $(n-1)$-cells with the interval. On the upper face the coefficient is zero. On a lateral face the horizontal primed degree is $n-1$ but its horizontal tangent space has dimension at most $n-2$, so the restriction is zero. On the lower face the outward primitive interval normal is $-\partial_x$. With the same superform boundary convention as (C.1.15), contraction leaves $-\alpha$ with its original horizontal calibration. Equivalently, on one product cell the one-variable boundary contribution is $(\chi(\tau)-\chi(\sigma))\int_C \alpha = -\int_C \alpha$. For an explicit sign check, put $d = n-1$ and $s_d = (-1)^{d(d-1)/2}$. If $\alpha = s_d A\, d'y_1 \wedge \cdots \wedge d'y_d \wedge d''y_1 \wedge \cdots \wedge d''y_d$, then the vertical term of $d'(d''x \wedge \chi\alpha)$ is $\chi'(x)d'x \wedge d''x \wedge \alpha$. Its coefficient in the positive product generator is $\chi'(x)A$, since $s_{d+1} = (-1)^d s_d$. One-dimensional integration gives the displayed endpoint difference. Weights and shared facets add linearly. This checks both the outward normal and the cutoff sign; no limiting operation is involved. □

Proposition C.1.31 (Two precise failures of weaker analytic premises)**.** *The following elementary examples are topology and convergence obstructions, not asserted native analytic spaces.*

First, let $C = \{0\} \cup \{1/k : k \geq 1\}$ *and let* E_N *be the functions on* C *constant on* $\{0\} \cup \{1/k : k > N\}$*. Put* $E = \bigcup_N E_N$ *with its final locally convex topology from the finite-dimensional supremum-norm spaces* E_N*. Then*

$$F(f) = \sum_{k \geq 1} (f(1/k) - f(0))$$

is well-defined and has order-zero bound $|F(f)| \leq 2N\|f\|_C$ *on* E_N*, but it has no finite signed Radon representation on* C*. Indeed there are* $f_N \in E_N$ *with* $0 \leq f_N \leq 1$ *and* $F(f_N) = N$*.*

Second, on $[0, 1]$ *the signed measures* $\sigma_m = m(\delta_{1/m^2} - \delta_0)$ *satisfy*

$$|\sigma_m(h)| \leq \|h'\|_\infty / m \quad (h \in C^1([0, 1])), \qquad |\sigma_m|([0, 1]) = 2m,$$

while $\sigma_m(\sqrt{x}) = 1$*. Thus convergence to zero against all smooth tests does not imply convergence against a fixed continuous multiplier without a suitable variation bound.*

Proof. Every $f \in E_N$ makes only its first N summands nonzero, proving the first bound and continuity by finality. Let f_N be one on the first N nonzero points and zero on the remaining points and on zero. This is a continuous member of E_N and gives $F(f_N) = N$. A finite signed representing measure ν would instead give $|F(f_N)| \leq |\nu|(C)$ for every N, a contradiction. The space E is uniformly dense in $C(C)$, by uniform continuity at zero, so lack of smooth-density analogues is not the issue.

For the second assertion, the fundamental theorem gives $|h(1/m^2) - h(0)| \leq \|h'\|_\infty / m^2$. The two atoms are distinct with opposite masses of size m, so their total variation is $2m$. Finally $m\sqrt{1/m^2} = 1$. All claimed inequalities and failures follow directly. □

Lemma C.1.32 (A finite vertical module with zero closed fibre vanishes)**.** *Let R be a valuation ring of a nontrivially real-valued field F, with maximal ideal $\mathfrak{n}$. Let S be any R-algebra, and let C be a finitely generated S-module. If*

$$C \otimes_R F = 0, \qquad C = \mathfrak{n}C, \tag{1}$$

then $C = 0$. Neither S nor R is required to be Noetherian.

Proof. Choose finitely many S-module generators $c_1, \dots, c_q$. The generic vanishing means that each c_j is annihilated by some nonzero $a_j \in R$. The product $a = \prod_j a_j \neq 0$ therefore annihilates all of C. If there are no generators, or a is a unit, the assertion follows.

For each j, write c_j as a finite sum

$$c_j = \sum_{\ell} \beta_{j\ell} d_{j\ell}, \qquad \beta_{j\ell} \in \mathfrak{n}, \quad d_{j\ell} \in C.$$

If every coefficient in these finitely many expressions is zero, then $C = 0$. Otherwise choose a nonzero coefficient λ having least valuation among this *finite* list. Every $\beta_{j\ell}/\lambda$ belongs to R, so $c_j \in \lambda C$ for all j. Since these elements generate over S, this proves $C = \lambda C$, with $v(\lambda) > 0$.

Rank one now supplies an integer $N \geq 1$ with $Nv(\lambda) \geq v(a)$. Thus $\lambda^N/a \in R$. Iteration gives

$$C = \lambda^N C = (\lambda^N/a)\, aC = 0.$$

Only a finite list of coefficients was minimized. No least positive element of $v(F^\times)$, and no generator of $\mathfrak{n}$, was used. □

Lemma C.1.33 (Finite type makes the additive descent defect finite)**.** *Let A' be a commutative algebra, let a finite group G of order e act on A', and let $R' \subset A'$ be a subring. Put $B = (A')^G$ and let $S = R'[B] \subset A'$ be the subalgebra generated by R' and B. If $A' = R'[z_1, \dots, z_r]$, then A' is a finite S-module, generated by the e^r monomials*

$$z_1^{j_1} \cdots z_r^{j_r}, \qquad 0 \leq j_\nu < e. \tag{2}$$

Consequently the additive quotient A'/S is a finite S-module. No Noetherian hypothesis and no assumption that G fixes R' pointwise are required.

Proof. For each z_ν, the orbit polynomial

$$\prod_{g\in G}(T-g(z_\nu))$$

is monic of degree e, vanishes at z_ν, and has all coefficients in B: every group element permutes its factors. Since $B\subset S$, powers of z_ν of exponent at least e reduce to lower powers with coefficients in S. The equality $A'=S[z_1,\dots,z_r]$ follows from the definition of S. Repeated reduction in each variable proves (2). The subring S is an S-submodule of A', so its additive quotient is a finitely generated S-module as well. □

Lemma C.1.34 (Uniform norm preservation)**.** *Let B be a unital non-Archimedean Banach K-algebra with submultiplicative norm, and let $R\subset B$ be a norm-dense K-subalgebra. Given $f_1,\dots,f_m\in B^\times$, there are $g_i\in R$ such that*

$$\|g_i-f_i\|<\varepsilon,\qquad 0<\varepsilon<\min_i\|f_i^{-1}\|^{-1}.$$

For every $z\in\mathcal{M}(B)$, $|g_i(z)|=|f_i(z)|$. Each g_i is also a unit of B.

Proof. Choose one positive ε satisfying the displayed finite list of inequalities and then approximate each f_i by density. By contractivity of bounded multiplicative seminorms (NAG-001496),

$$|f_i(z)|\geq\|f_i^{-1}\|^{-1}>\varepsilon>|g_i(z)-f_i(z)|.$$

The ultrametric inequality gives equality of the first two absolute values. The element $u_i=f_i^{-1}(g_i-f_i)$ has norm less than one. The convergent series $\sum_{j\geq0}(-u_i)^j$ is the inverse of $1+u_i$, proving invertibility of g_i. For an empty tuple there is nothing to approximate. □

A smooth superform on a polyhedral subset means an ambient smooth form modulo restriction to the tangent space of every cell at every point of that cell. Both alternating factors are restricted.

Lemma C.1.35 (Detection through an affine image)**.** *Let Q be a finite union of compact convex polytopes in a real affine space, let π be an affine map, and put $P=\pi(Q)$. If α is a smooth ambient superform near P, then*

$$\pi^*\alpha|_Q=0\quad\Longleftrightarrow\quad\alpha|_P=0.$$

Moreover, if $P=\bigcup_{i=1}^r P_i$ is a finite union of compact polyhedral subsets and $\alpha|_{P_i}=0$ for every i, then $\alpha|_P=0$.

Proof. Take a finite common polyhedral subdivision of the images of the polytopes of Q, then triangulate it compatibly on its faces. Each resulting simplex τ is contained in the image of one convex polytope $\sigma \subset Q$. Choose a lift in σ of each vertex of τ. Affine interpolation in barycentric coordinates gives a section $j_\tau : \tau \to \sigma$ of π. Convexity ensures that its image stays in σ. Pullback along j_τ proves $\alpha|_\tau = 0$ if $\pi^*\alpha|_Q = 0$. This detects both alternating factors and applies also to zero-dimensional simplices. Vanishing on the full-dimensional pieces of any original cell extends to its boundary by continuity, giving all-face vanishing on P. For the converse, subdivide each source polytope by the inverse images of target cells; affine pullback carries its tangent vectors into the target tangent vectors.

For the last assertion, a finite common subdivision makes every P_i a subcomplex of P. Every cell of the resulting union lies in some P_i, so the same restriction and continuity argument applies. The common subdivision is obtained from the finitely many face hyperplanes. A compatible triangulation can be obtained by induction on faces, joining an interior point of each remaining polytope to the already triangulated boundary. No integral splitting or saturation is used. □

Let A be a nonzero Noetherian ring. Set

$$S = A \setminus \bigcup_{\mathfrak{p} \in \operatorname{Min}(A)} \mathfrak{p}, \qquad T(A) = \ker\left(A \longrightarrow \prod_{\mathfrak{p} \in \operatorname{Min}(A)} A_{\mathfrak{p}} \right), \qquad A^{\mathrm{u}} = A/T(A). \tag{1.1}$$

We call $T(A)$ the embedded torsion ideal. No reduction of the generic Artinian local rings is performed.

Lemma C.1.36 (Algebraic quotient and regular elements; F8U-A)**.** *The ideal $T(A)$ is finite and nilpotent, and $T(A)_{\mathfrak{p}} = 0$ at every minimal prime. The ring A^{u} has the same minimal primes and generic local rings as A, and has no embedded associated primes; hence it satisfies S_1. If f is a non-zero-divisor in A, it is a non-zero-divisor in both $T(A)$ and A^{u}, and*

$$0 \longrightarrow T(A)/fT(A) \longrightarrow A/fA \longrightarrow A^{\mathrm{u}}/fA^{\mathrm{u}} \longrightarrow 0 \tag{1.2}$$

is exact.

Proof. All ideals are finite. If $a \in T(A)$, then for each minimal $\mathfrak{p}$ some element outside $\mathfrak{p}$ kills a. Finite prime avoidance applied to $\operatorname{Ann}(a)$ produces

$s \in S$ with $sa = 0$. Conversely such an s is a unit in every $A_{\mathfrak{p}}$, so $sa = 0$ implies $a \in T(A)$. Thus $T(A)$ is exactly the S-torsion, and A^{u} embeds in $S^{-1}A$. Each $a \in T(A)$ belongs to every minimal prime of A; hence $T(A)$ lies in the nilradical and is nilpotent. Localizing (1.1) at a minimal prime gives $T(A)_{\mathfrak{p}} = 0$, including for a finite generating set, so the generic local rings are unchanged.

Every $s \in S$ acts injectively on A^{u}. An associated prime $\mathfrak{q}$ of A^{u} is therefore disjoint from S. Finite prime avoidance implies $\mathfrak{q} \subseteq \mathfrak{p}$ for some minimal prime $\mathfrak{p}$, so $\mathfrak{q} = \mathfrak{p}$. This proves absence of embedded primes. The equivalence with S_1 is the Noetherian algebra criterion Stacks 031Q; no analytic claim is hidden in this step.

A regular f avoids every minimal prime, so $f \in S$ and its action on A^{u} is injective. It acts injectively on $T(A) \subseteq A$ as well. Moreover $fa \in T(A)$ implies $a \in T(A)$. Thus $T(A) \cap fA = fT(A)$, which proves (1.2). $\square$

Affine coefficient comparison in NAG-002060. Take the finite separable extension in the original statement, with its actual monic lift $L = K(\alpha)$ from NAG-002050. Use the original coefficient maximum calculation in lines 45–51. It makes L° finite free over O with basis $1, \alpha, \dots, \alpha^{d-1}$ and gives $L^{\circ\circ} = K^{\circ\circ} L^\circ$.

For an actual flat finitely presented affine chart with completed algebra A_0, the formal clause of NAG-002037 and the finite free coefficient basis identify its coefficient extension after completion with $B_0 = A_0 \otimes_O L^\circ$. The maps are those given on the affine coefficients. The finite basis makes tensor extension commute with each quotient modulo π^n, with the completion, and with the completed principal restrictions. Also

$$B_0[1/\pi] = A \otimes_K L, \qquad A = A_0[1/\pi].$$

Apply NAG-001918 on the actual admissible affine charts to obtain the strict generic norms. In NAG-001702 take the finite algebra to be L over K, and the bounded target to be A. NAG-001483 gives the strict affinoid structure on the finite K-algebra L. Its actual absolute value is the canonical coefficient maximum norm, by the calculation above. Over A the finite free maximum norm on $A \otimes_K L$ has closed unit ball $\bigoplus_i A_0 \alpha^i = B_0$. The defining strict norm of $B_0[1/\pi]$ has the same closed unit ball by NAG-001918. For any nonzero element, scale by an integer power of π to put its norm in $(|\pi|, 1]$ for either norm. Applying the common unit-ball inclusion in both directions compares those norms with bound $|\pi|^{-1}$. Thus the natural algebra identification is bounded for the defining norms and gives

$$B_0[1/\pi] \simeq A \widehat{\otimes}_K L.$$

On the target it is an isomorphism of affinoid algebras, with bounded inverse. It commutes with every displayed principal restriction because the maps agree on the affine coefficients and on α. This supplies the operation at lines 64–67 of NAG-002060 that uses the finite-coordinate part of NAG-002023. It also gives the displayed special-fibre tensor formula by the equality of the open unit ideals. The clause concerns supplied affine models and their actual restrictions. The global generic glue and the separate reducedness assertion retain their original premises.

Lemma C.1.37 (Schematic closure of the generic graph)**.** *Let K be complete and nontrivially real-valued, put $O = K^\circ$, and fix $0 \neq \pi \in K^{\circ\circ}$. Let C/K be a smooth projective geometrically connected curve. Let $\mathcal{X}_1, \mathcal{X}_2$ be projective flat finitely presented O-models of C. The schematic closure $\mathcal{G}$ of the generic diagonal in $\mathcal{X}_1 \times_O \mathcal{X}_2$ is integral, projective, flat and finitely presented over O. Its two projections*

$$p_i : \mathcal{G} \longrightarrow \mathcal{X}_i$$

are proper and induce the identity on C. If $\mathcal{X}_i$ is integral, p_i is birational and surjective.

Proof. On an affine open $\operatorname{Spec} B$ of the product, let $I \subset B[1/\pi]$ be the ideal of the generic diagonal. Its closure has ring

$$B/(I \cap B).$$

These ideals commute with localization and hence define a closed subscheme of the product. A nonempty such affine closure has coordinate ring contained in the coordinate ring of a nonempty open of the integral curve C. It is therefore a domain and has no O-torsion. The generic diagonal is dense, so these affines give an integral scheme. Torsion-freeness over a valuation ring gives flatness by [StacksAlg, Tag 0539]. Each affine ring is of finite type over O, as a quotient of the coordinate ring of the product. It is consequently finitely presented by [StacksAlg, Tag 053E(1)]. The closure is projective as a closed subscheme of the projective product; in particular it is quasi-compact and separated. The local finite presentation just proved is therefore a finite presentation.

The projections are proper, being restrictions of the proper product projections to a closed subscheme. On the generic fibre they are the given identities. If a target is integral, their closed images contain its dense generic fibre, proving surjectivity and birationality. □

Lemma C.1.38 (The full finite flat cycle)**.** *Let $q : Y \to Z$ be finite locally free of constant rank n, with Z integral and Noetherian. Every irreducible component of Y dominates Z. For its generic points η_j, write*

$$m_j = \operatorname{length}_{\mathcal{O}_{Y,\eta_j}} \mathcal{O}_{Y,\eta_j}.$$

Then, with full field degrees,

$$\sum_j m_j[\kappa(\eta_j) : K(Z)] = n, \qquad q_*[Y] = n[Z]. \tag{C.1.17}$$

Here $[Y] = \sum_j m_j[\overline{\{\eta_j\}}]$ is the full fundamental cycle.

Proof. On an affine open put $Z = \operatorname{Spec} B$ and $Y = \operatorname{Spec} C$. The finite locally free B-module C is torsion free. If a nonzero $b \in B$ belonged to a minimal prime $\mathfrak{q}$ of C, its image in the zero-dimensional Noetherian local ring $C_{\mathfrak{q}}$ would be nilpotent; this ring is Artinian (Tag 00KH). Clearing a denominator gives $sb^r = 0$ for some $s \notin \mathfrak{q}$, contradicting torsion freeness. Thus every minimal prime lies over (0). Conversely a prime of C over (0) is minimal: after localization it is a prime in the finite-dimensional $K(Z)$-algebra $C \otimes_B K(Z)$, whose primes are all maximal. Its local factors are precisely $C_{\mathfrak{q}}$.

This finite-dimensional algebra has dimension n and is the product of these Artinian local factors (Tag 00JB). A composition series of one factor has m_j quotients equal to $\kappa(\eta_j)$. Taking dimensions proves the sum in (C.1.17). The definition of proper cycle pushforward, Tag 02R4, proves the second identity. In particular, inseparable degrees and generic nilpotent lengths are separate factors and both are retained. □

Finite tropical graphs and Gauss fibres

A finite logarithmic graph

Fix a complete field K with its original real-valued non-Archimedean absolute value, possibly trivial. Fix $k \geq 1$, $d \geq 0$, and $n = k + d$. Choose real numbers

$$A_i > 0, \quad B_i > 0 \quad (1 \leq i \leq k), \qquad q_\nu^- < q_\nu^+ \quad (1 \leq \nu \leq d), \qquad Q = \prod_{\nu=1}^{d} [q_\nu^-, q_\nu^+].$$

An empty product Q is a point. Put

$$V_0 = \left\{ \begin{array}{ll} |T_i| \leq e^{A_i}, & |T_i - 1| \geq e^{-B_i} \quad (1 \leq i \leq k), \\ e^{-q_\nu^+} \leq |Z_\nu| \leq e^{-q_\nu^-} & (1 \leq \nu \leq d). \end{array} \right\}. \tag{1.1}$$

This is the actual generalized affinoid rational domain in the corresponding closed polydisc. In particular $T_i - 1$ and Z_ν are analytic units on V_0.

Let $M \in M_{r,n}(\mathbb{Z})$ have column rank n. Choose $m_i \in \mathbb{Z}_{>0}$, $b = (b_1, \ldots, b_d) \in \mathbb{Z}^d$, and $a, c_1, \ldots, c_r \in K^\times$. Define

$$f_0 = a \prod_{i=1}^{k} T_i^{m_i} \prod_{\nu=1}^{d} Z_\nu^{b_\nu}, \tag{1.2}$$

$$g_{0,\ell} = c_\ell \prod_{i=1}^{k} (T_i - 1)^{M_{\ell i}} \prod_{\nu=1}^{d} Z_\nu^{M_{\ell,k+\nu}} \qquad (1 \leq \ell \leq r). \tag{1.3}$$

Thus f_0 is analytic, with zeros allowed, and the $g_{0,\ell}$ are units. Put

$$\lambda = -\log|a|, \quad \kappa_\ell = -\log|c_\ell|, \quad y_i = -\log|T_i - 1|, \quad x_\nu = -\log|Z_\nu|, \quad s_i = -\log|T_i|.$$

The coordinates s_i are used only where all $T_i \neq 0$. Write $v = (y_1, \ldots, y_k, x_1, \ldots, x_d)$.

For each i, let $\Gamma_i \subset \mathbb{R}^2_{y_i,s_i}$ consist of the following three edges, including their endpoints:

$$D_i = \{(u,u) : -A_i \leq u \leq 0\}, \quad H_i = \{(u,0) : 0 \leq u \leq B_i\}, \quad E_i = \{(0,u) : u \geq 0\}. \tag{1.4}$$

Let

$$P = \Gamma_1 \times \cdots \times \Gamma_k \times Q \subset \mathbb{R}^{n+k}_{v,s}, \qquad \mathcal{P} = \sum_{\dim \sigma = n} [\sigma], \tag{1.5}$$

using its product complex, with weight one on every n-cell. Every cell has its saturated tangent lattice in $\mathbb{Z}^{n+k}$. Define the integral affine map

$$F(v,s) = \left(\kappa + Mv,\ \lambda + \sum_i m_i s_i + b\cdot x\right) =: (Y,t). \tag{1.6}$$

Translations here come from the specified constants in $K^\times$; the absolute value is unchanged.

For a real cutoff S, use the restrictions

$$V_{0,S} = \{z \in V_0 : |f_0(z)| \geq e^{-S}\}, \qquad P_S = P \cap \left\{\lambda + \sum_i m_i s_i + b\cdot x \leq S\right\}. \tag{1.7}$$

Write $C_0 = F_*\mathcal{P}$, counting full-rank cells with their lattice index and discarding lower-rank images.

For an integral linear map $L : N_\sigma \to \mathbb{Z}^{r+1}$ of rank n, the pushforward index used throughout is

$$\delta(L) = [\, L(N_{\sigma,\mathbb{R}}) \cap \mathbb{Z}^{r+1} : L(N_\sigma)\,]. \tag{2.1}$$

Lemma C.1.39 (Actual source image; F8GP-2)**.** *On $V_0 \setminus \{f_0 = 0\}$, the moment tuple*

$$h = (T_1 - 1, \dots, T_k - 1, Z_1, \dots, Z_d, T_1, \dots, T_k)$$

has image exactly P. Its restriction to $V_{0,S}$ has image exactly P_S. These are statements about all analytic points over the original field.

Proof. For one variable, the strong triangle inequality applied to $T = (T-1)+1$ gives three possibilities. If $|T| > 1$, then $|T-1| = |T|$. If $|T| < 1$, then $|T-1| = 1$. If $|T| = 1$, then $|T-1| \leq 1$. Since $T \neq 0$ and $T-1$ is a unit, these are precisely the three edges in (1.4), with the stated outer bounds.

Conversely, every point of an edge is attained by a Gauss seminorm: on D_i or E_i, use the variable T_i with radius e^{-s_i}; on H_i, use $T_i - 1$ with radius e^{-y_i}. At the common vertex, the radius-one Gauss seminorm in T_i has both $|T_i| = |T_i - 1| = 1$. Together with radii e^{-x_ν} for the Z_ν, the multivariable Gauss seminorm is multiplicative over K. Its values satisfy all inequalities

of (1.1), so the actual rational domain property gives a point of V_0. All T_i have positive value, as required. This proves both inclusions for P. Finally $-\log|f_0| = \lambda + \sum_i m_i s_i + b \cdot x$ on every analytic point. Imposing its upper bound S proves the second equality. □

Lemma C.1.40 (Purity and regularity; F8GP-3)**.** *The space V_0 is nonempty and pure of dimension n. The function f_0 is a non-zero-divisor in its affinoid algebra. Every nonempty $V_{0,S}$ is pure of dimension n.*

Proof. A closed generalized n-polydisc is integral because its Gauss norm is multiplicative and has zero kernel. Its Gauss point has valued-field dimension n by `fnd:gaussvaluedfielddimension`. At every point, polynomials are dense in the evaluated Tate algebra; approximation of a nonzero denominator also approximates its inverse. The completed residue field is consequently the completion of a field generated by at most n coordinate elements over K. The upper bound follows from `fnd:valuedfieldabhyankar`, including passage to completion. The maximum theorem `fnd:affinoidabhyankarmaximum` therefore gives analytic dimension n. It has one minimal component.

The inequalities in (1.1) define a nonempty rational domain in a closed polydisc: a Gauss point from lemma C.1.39, or the point $T_i = 0$ with a torus Gauss point, witnesses nonemptiness. Domain purity gives pure dimension n. Multiplication by each T_i is injective in the ambient Tate algebra, as seen by shifting its coefficient indices. Its rational-domain algebra is flat over that algebra. Consequently multiplication by every T_i, and by their positive powers and product, remains injective. The factors a and Z^b are units on V_0, proving regularity of f_0.

The domain $|f_0| \geq e^{-S}$ is an actual generalized Laurent domain in V_0. Domain purity gives the last assertion. There is also a direct compact bound useful below: from P_S,

$$-A_i \leq s_i \leq U_i(S) := \frac{S - \lambda - \min_Q b \cdot x + \sum_{j \neq i} m_j A_j}{m_i}. \tag{3.1}$$

Thus every nonempty truncation lies in a closed n-polyannulus, allowing exact-radius intervals when an endpoint coincides. □

Lemma C.1.41 (A proper finite polyhedral pushforward; F8GP-5)**.** *The map $F : P \to \mathbb{R}^{r+1}$ is proper. The chain $C_0 = F_*\mathcal{P}$ has a finite polyhedral presentation, and for every real S,*

$$F_*\big(\mathcal{P}|_{\{t\circ F\le S\}}\big) = C_0|_{\{t\le S\}}. \tag{3.3}$$

Proof. The v-coordinates range over one fixed compact box. On the inverse image of a compact target set, its t-coordinate has an upper bound S. Equation (3.1) bounds each s_i above, while $s_i \ge -A_i$. Thus the inverse image is a closed subset of a bounded box, hence compact. Positivity of every m_i is used in these separate upper bounds.

There are only 3^k product n-cells before subdivision. The image of each is a polyhedron: eliminate the finitely many linear inequalities, or use the injective affine map on its span when the rank is n. The finitely many image faces admit a common finite polyhedral refinement. On a full-rank cell, F is an affine isomorphism onto its image, with the fixed finite index (2.1). Sum these indices over all cells above each image cell. Rank-deficient cells contribute zero. This constructs a finite nonnegative integral chain.

On a full-rank cell, intersecting its image with $t \le S$ is exactly the image of its intersection with $t \circ F \le S$. The tangent lattice and index are unchanged on surviving n-dimensional pieces. If the whole cell is at $t = S$, both operations retain the whole cell. Rank-deficient pieces remain rank-deficient after cutting. These facts prove (3.3) coefficient by coefficient, after finite refinement. □

For a full-dimensional product cell σ of P, let $J(\sigma)$ be the set of indices for which its factor is E_i. If $J = \varnothing$, use v as its lattice coordinates and write

$$\ell_\sigma = (\epsilon_1 m_1, \dots, \epsilon_k m_k, b_1, \dots, b_d), \qquad \epsilon_i = \begin{cases} 1, & \text{the factor is } D_i, \\ 0, & \text{the factor is } H_i. \end{cases}$$

The matrix of F on its tangent lattice is

$$L_\sigma = \begin{pmatrix} M \\ \ell_\sigma \end{pmatrix}. \tag{4.1}$$

Proposition C.1.42 (The three cell cases; F8GP-6)**.** *The coefficients before common refinement are as follows.*

(i) If $J = \varnothing$, the image is bounded, has dimension n, and has coefficient $\delta(L_\sigma)$.

(ii) If $J = \{j\}$, let $M_{\widehat{j}}$ be M with its j-th column removed and put

$$\delta_j = [\,\mathrm{span}_{\mathbb{R}}(M_{\widehat{j}}) \cap \mathbb{Z}^r : M_{\widehat{j}}\mathbb{Z}^{n-1}\,].$$

The image has dimension n, is unbounded, and has coefficient $m_j\delta_j$. For $n = 1$, the empty-lattice index is one.

(iii) If $|J| \geq 2$, the image has rank $n - |J| + 1 < n$; it contributes zero to C_0.

Coincident image pieces are counted by summing their coefficients. Moreover, one may take

$$H = \lambda + \max_{x \in Q} b \cdot x + 1 \tag{4.2}$$

as a common strict upper bound for all bounded image cells and a cylindrical-tail threshold for the unbounded image cells.

Proof. When $J = \varnothing$, M is injective over $\mathbb{R}$, so (4.1) has rank n, and the definition of pushforward gives its index. All coordinates of the source cell are bounded.

For $J = \{j\}$, the base coordinate y_j is zero and the source variable s_j is free in $[0, \infty)$. The target tangent space is

$$\mathrm{span}_{\mathbb{R}}(M_{\widehat{j}}) \oplus \mathbb{R}e_t,$$

with saturated lattice $(\mathrm{span}_{\mathbb{R}}(M_{\widehat{j}}) \cap \mathbb{Z}^r) \oplus \mathbb{Z}e_t$. Projection of the image lattice to the first factor has image $M_{\widehat{j}}\mathbb{Z}^{n-1}$, of index δ_j. Its intersection with the vertical lattice is exactly $m_j\mathbb{Z}e_t$: independence of the remaining columns of M forces all their coefficients to vanish in a vertical combination. Counting cosets first in the base and then vertically gives index $m_j\delta_j$.

For general J, the base has $n - |J|$ independent directions. All $|J|$ vertical source directions map into the same line, and at least one maps nontrivially since all $m_i > 0$. The rank is therefore $n - |J| + 1$.

On every bounded cell, $s_i \le 0$, so $t \le \lambda + \max_Q b\cdot x$. On a cell with unique vertical branch j, its image lies above the lower graph

$$t = \lambda + \sum_{i\neq j} \epsilon_i m_i y_i + b\cdot x,$$

where $y_i \le 0$ whenever $\epsilon_i = 1$. This lower graph is everywhere at most $\lambda + \max_Q b\cdot x$. Above (4.2), any t is obtained by choosing $s_j \ge 0$. The image is thus a vertical cylinder there. Common refinements may be taken after cutting at this same height, without introducing a new larger threshold. □

Whole Gauss fibres on finite charts

Let K be complete for its specified real-valued non-Archimedean absolute value, possibly trivial. All affinoid algebras and domains are actual generalized affinoid objects with their given Banach structures. A finite map is required to have the canonical finite-module topology. Fibre algebras retain their nilpotents. Purity refers to analytic dimension.

For a full-dimensional closed rectangular box $P \subset \mathbb{R}^d$, write $\mathbb{T}_P = \mathcal{M}(R_P)$ for its polyannulus, with coordinates $T_1, \dots, T_d$ and $t_T = -\log|T| \in P$. For $s \in P$, let η_s be its Gauss point, R_s the algebra of the entire exact-radius polytorus $\mathbb{T}_s = t_T^{-1}(s)$, and $H_s = \mathcal{H}(\eta_s)$. The Laurent presentations, density, and kernel-zero Gauss norms are those of 5.35.4. Radii need not belong to $|K^\times|$.

Lemma C.1.43 (Whole fibres outside finitely many walls)**.** *Let $s_0 \in \mathrm{int}(P)$ and let O be an ordinary open neighbourhood of η_{s_0} in $\mathbb{T}_P$. There are an open Euclidean neighbourhood $B_0 \subset \mathrm{int}(P)$ of s_0 and a finite union H of proper affine hyperplanes, with integral normals, such that*

$$\mathbb{T}_s \subset O \qquad (s \in B_0 \setminus H).$$

This includes every analytic point of each exact-radius polytorus.

Proof. The evaluation topology gives finitely many strict upper and lower bounds on $|g_j|$, $g_j \in R_P$, whose common locus contains η_{s_0} and lies in O. Discard identically zero functions, which impose no effective condition. Every remaining $r_j = |g_j(\eta_{s_0})|$ is positive by the kernel-zero Gauss norm. An absent lower or upper bound is interpreted as zero or infinity; thus r_j lies in an open interval I_j.

Choose finite Laurent polynomials $p_j = \sum_m c_{jm} T^m$ whose uniform errors on $\mathbb{T}_P$ are less than $\epsilon_j < r_j$, using Banach-norm density. Combine equal exponents and remove zero coefficients. Then $|p_j(\eta_{s_0})| = r_j$. The finite maximum

$$G_j(s) = \max_m |c_{jm}| e^{-\langle m, s \rangle}$$

is continuous. Shrink B_0 until $G_j(s) \in I_j$ and $G_j(s) > \epsilon_j$ for every j. Let H contain the equality hyperplane for every pair of distinct exponent terms in every p_j. These are finitely many proper hyperplanes with integral normals; a monomial contributes none.

For $s \notin H$ there is a unique largest term in each p_j at every point $y \in \mathbb{T}_s$. The strong triangle inequality gives $|p_j(y)| = G_j(s)$. The approximation error is smaller, so $|g_j(y)| = G_j(s) \in I_j$. Every point satisfies all the selected bounds and belongs to O. □

Definition C.1.44 (Finite Gauss chart with monomial moments)**.** Fix $d \geq 1$, $s_0 \in \operatorname{int}(P)$, and a norm-admissible finite morphism

$$p : Y = \mathcal{M}(B) \longrightarrow \mathbb{T}_P$$

with Y nonempty and pure of analytic dimension d. Assume that the entire fibre over the *analytic* Gauss point is $p^{-1}(\eta_{s_0}) = \{x\}$ as a set. Its finite algebra may have nontrivial length and residue-field degree. This does not assert that the tropical fibre $p^{-1}(\mathbb{T}_{s_0})$ is a singleton.

Let $A \in M_{n \times d}(\mathbb{Z})$ have rank d, with rows a_i, and let $c_i \in K^\times$. The moment tuple is

$$f_i = c_i p^* T^{a_i}, \qquad t_f = b + A\, t_T \circ p, \qquad b_i = -\log |c_i|.$$

Put $w_0 = b + A s_0$, $N = A\mathbb{R}^d \cap \mathbb{Z}^n$, and $j_A = [N : A\mathbb{Z}^d]$. Let M be a finite B-module, restricted to affinoid subdomains by the actual domain tensor.

Set $F = \operatorname{Frac}(R_P)$ and $D_M = \dim_F(M \otimes_{R_P} F)$, a finite integer. An admissible generic projection for these monomial moments is an integral $Q : \mathbb{Z}^n \to \mathbb{Z}^d$ injective on N; put $e_Q = [\mathbb{Z}^d : Q(N)]$. At $v = b + As$, let $R_{Q,v}$ be the exact-radius target algebra for the actual units $f^{q_1}, \dots, f^{q_d}$, and let $H_{Q,v}$ be its Gauss field.

Write $D_s = B \widehat{\otimes}_{R_P} R_s$, the algebra of the entire exact-radius inverse image by 2.33.19. For the original logarithmic normalization write $v = b + As$ and choose the normalizers only after the complete isometric extension in the statement.

Corollary C.1.45 (Lengths, field degrees, and normalization)**.** *Put* $E_s = B \otimes_{R_P} H_s$ *and* $M_s = M \otimes_{R_P} H_s$*. For every* $s \in P$*,*

$$D_M = \sum_{\mathfrak{q} \in \operatorname{Min} B} \operatorname{length}_{B_{\mathfrak{q}}}(M_{\mathfrak{q}})[\operatorname{Frac}(B/\mathfrak{q}) : F], \tag{C.1.18}$$

$$= \sum_{\mathfrak{m} \in \operatorname{Max} E_s} \operatorname{length}_{(E_s)_{\mathfrak{m}}}((M_s)_{\mathfrak{m}})[\kappa(\mathfrak{m}) : H_s]. \tag{C.1.19}$$

At $s = s_0$ *the second sum has one term, with residue field* $\mathcal{H}(x)$*. All degrees are full degrees, including inseparable degrees. The structure module retains nilpotent length.*

After any complete isometric extension realizing the needed radii, the normalized Gauss dimensions and lattice divisors give the same coefficient on the full base-changed fibre. All its factors must be retained; a selected lift of x *is not that full fibre.*

Proof. Every minimal prime of B contracts to zero in R_P, by 5.35.4. Apply the generic-length calculation 5.40.1 to $B \to D_s$ with its finite R_s-map. For a minimal prime $\mathfrak{q}$, its reduced generic factor over F is $\operatorname{Frac}(B/\mathfrak{q})$; extending to H_s preserves vector-space dimension. This proves (C.1.18). The finite-dimensional algebra E_s is Artinian. Every simple quotient in a composition series over a local factor is its residue field. Adding dimensions gives (C.1.19). By 2.33.11, the singleton analytic fibre means that E_{s_0} has one local factor; its residue field is the finite complete extension $\mathcal{H}(x)$ of H_{s_0}.

The actual finite tensor identities of 5.36.2 apply to finite modules as well: a vector-space basis remains a basis after extension of the Gauss field. For normalizers $|a_i| = e^{-v_i}$ of f_i, use $\prod_i a_i^{q_{ji}}$ for f^{q_j}. Changing these normalizers is an isometric target coordinate change. The model formula 5.36.3 retains the generic lengths and full field degrees. Only finite fibre normalization is asserted here, not scalar-extension compatibility of a global integral. □

Puncture products

The ground field K is complete for its specified real-valued non-Archimedean absolute value, which may be trivial. Every finite analytic map below is required to carry the canonical finite-module norm, up to uniform equivalence. All fibres are actual analytic fibres over K, with their full algebras and nilpotents. No scalar extension is used in the construction.

Definition C.1.46 (Puncture-product data and positive projections)**.** Choose integers $p \geq 1$, $k \geq 0$, and $m_\ell \geq 1$; put $d = p + k$ and $n = \sum_\ell m_\ell + k$. For $1 \leq \ell \leq p$, choose $c_{\ell j} \in K^\circ$ such that

$$|c_{\ell j} - c_{\ell h}| = 1 \qquad (j \neq h).$$

Thus the punctures in each block have distinct residues. Set

$$Z = \prod_{\ell=1}^{p} \big(\mathbb{A}^1_K \setminus \{c_{\ell 1}, \dots, c_{\ell m_\ell}\}\big) \times (\mathbb{G}_{m,K})^k.$$

Write t_ℓ, z_b for its coordinates, and use the actual unit tuple

$$g = ((t_\ell - c_{\ell j})_{\ell,j}, (z_b)_b).$$

Let $\eta \in Z^{\mathrm{an}}$ be the unit Gauss point in the variables $t_1, \dots, t_p, z_1, \dots, z_k$; then $t_g(\eta) = 0$. For $\varepsilon > 0$ put

$$Y_\varepsilon = \{y \in Z^{\mathrm{an}} : |\log |g_\nu(y)|| \leq \varepsilon \text{ for every } \nu\}.$$

For an array $a = (a_{\ell j})$ of positive integers, define

$$A_\ell = \sum_j a_{\ell j}, \qquad P_{\ell,a}(T) = \prod_j (T - c_{\ell j})^{a_{\ell j}}, \qquad D_a = \prod_{\ell=1}^{p} A_\ell.$$

The projection $Q_a : \mathbb{Z}^n \to \mathbb{Z}^d$ sends the ℓ-th block to $\sum_j a_{\ell j} v_{\ell j}$ and is the identity on the final k coordinates. Its actual analytic map is

$$q_a = (P_{1,a}(t_1), \dots, P_{p,a}(t_p), z_1, \dots, z_k) : Z^{\mathrm{an}} \longrightarrow (\mathbb{G}_m^d)^{\mathrm{an}}.$$

For a full-dimensional closed rectangular box P with $0 \in \mathrm{int}(P)$, $\mathbb{T}_P$ denotes the complete closed polyannulus $t^{-1}(P)$, and $R_P = \mathcal{O}(\mathbb{T}_P)$.

Lemma C.1.47 (Explicit base collars and their whole Gauss fibre)**.** *The domain Y_ε is a nonempty generalized affinoid, pure of dimension d, and $\eta \in \mathrm{Int}(Y_\varepsilon/K)$. For every positive array a, q_a is finite, with*

$$q_a^{-1}(\mathbb{T}_P) = \mathcal{M}\big(R_P[T_1, \dots, T_p]/(P_{\ell,a}(T_\ell) - S_\ell)_{\ell=1}^{p}\big), \tag{1}$$

where S_ℓ are the first p target coordinates. The displayed algebra is free of rank D_a over R_P and has its canonical finite-module norm.

Put $b_\ell = \min_j a_{\ell j}$. *If* P *satisfies*

$$|u_\ell| < b_\ell \varepsilon \ (1 \le \ell \le p), \qquad |u_{p+b}| < \varepsilon \ (1 \le b \le k) \quad \textit{for every } u \in P, \tag{2}$$

then the entire *inverse image in (1) lies in the ordinary interior of* Y_ε *in* Z^{an}. *Consequently it is exactly the saturated collar* $(Q_a t_g|_{Y_\varepsilon})^{-1}(P)$.

If η_a *is the target unit Gauss point, then*

$$q_a^{-1}(\eta_a) = \{\eta\}. \tag{3}$$

For its actual Artinian fibre algebra E_a,

$$\dim_{\mathcal{H}(\eta_a)} E_a = D_a = \sum_{\mathfrak{m} \in \operatorname{Max} E_a} \operatorname{length}((E_a)_{\mathfrak{m}})[\kappa(\mathfrak{m}) : \mathcal{H}(\eta_a)]. \tag{4}$$

Equation (3) concerns points; (4) retains all length and field-degree data and does not replace E_a *by its reduction.*

Proof. Set $R = e^\varepsilon > 1$. A presentation of Y_ε uses $t_\ell, z_b, z_b^{-1}, (t_\ell - c_{\ell j})^{-1}$, all with radius R, and their polynomial inverse relations. Indeed $|t_\ell| \le R$ and $|c_{\ell j}| \le 1 < R$ give every upper bound $|t_\ell - c_{\ell j}| \le R$; conversely one such upper bound gives $|t_\ell| \le R$. The inverse coordinates give the lower bounds. This is an actual generalized affinoid presentation. All its coordinates have value $1 < R$ at η, so the presentation witnesses absolute interiority there. It is an affinoid domain of the closed polyannulus in $(t_\ell - c_{\ell 1}, z_b)$, cut out by the additional puncture bounds. That polyannulus is integral and pure of dimension d by 5.35.4; domain purity 5.35.3 proves the assertion for Y_ε.

The algebraic coordinate ring of Z is

$$K[S_1^{\pm 1}, \dots, S_p^{\pm 1}, Z_1^{\pm 1}, \dots, Z_k^{\pm 1}][T_1, \dots, T_p]/(P_{\ell,a}(T_\ell) - S_\ell)_\ell.$$

To check the inverse coordinates in this identity, use

$$(T_\ell - c_{\ell j})^{-1} = S_\ell^{-1}(T_\ell - c_{\ell j})^{a_{\ell j}-1} \prod_{h \ne j} (T_\ell - c_{\ell h})^{a_{\ell h}}.$$

Successive division by the separate monic polynomials gives the free basis $\prod_\ell T_\ell^{i_\ell}$, $0 \le i_\ell < A_\ell$. The frozen analytification theorem 2.33.26 identifies the *whole* inverse image of any target affinoid with its ordinary tensor algebra and canonical finite-module norm. It gives (1).

Here is the estimate which supplies saturation. At any analytic point, fix a block ℓ. If $|t_\ell| > 1$, all its puncture distances are equal to $|t_\ell|$. If $|t_\ell| \le 1$,

all those distances are at most 1, and at most one is less than 1, because the difference of two punctures has norm 1. Thus its weight vector belongs to

$$F_\ell = \bigcup_{j=1}^{m_\ell} \mathbb{R}_{\geq 0} e_{\ell j} \ \cup \ \mathbb{R}_{\geq 0}(-e_{\ell 1} - \cdots - e_{\ell m_\ell}). \tag{5}$$

On the positive ray j, $u_\ell = a_{\ell j} v_{\ell j}$; on the last ray, $u_\ell = -A_\ell s$ with all $v_{\ell j} = -s$. In either case

$$|v_{\ell j}| \leq |u_\ell| / b_\ell. \tag{6}$$

The final k weights are unchanged. Bounds (2) therefore put every inverse-image point in the strict moment inequalities. This proves the collar assertion without selecting any branch.

We prove (3) over the original field. The unit Gauss point maps to η_a: after reduction, the separate nonconstant monic polynomials $\overline{P}_{\ell,a}(\overline{t}_\ell)$, together with the $\overline{z}_b$, are algebraically independent over $\widetilde{K}$. For example, their distinct leading multidegrees prove the injectivity of polynomial substitution. Scaling a largest coefficient of any Laurent polynomial then proves the unit Gauss norm identity.

Conversely let y map to η_a. Equation (6), with $u = 0$, gives $|t_\ell - c_{\ell j}| = |z_b| = 1$ at y, hence $|t_\ell| \leq 1$. In the residue field of $\mathcal{H}(y)$, the elements $\overline{P}_{\ell,a}(\overline{t}_\ell), \overline{z}_b$ are algebraically independent: any relation would contradict the target Gauss seminorm after lifting its finitely many coefficients. Each $\overline{t}_\ell$ is integral over the field generated by these elements. Consequently the field generated by $\overline{t}_1, \ldots, \overline{t}_p, \overline{z}_1, \ldots, \overline{z}_k$ has transcendence degree d. These d elements are therefore algebraically independent. For any nonzero polynomial in t, z, divide by a largest coefficient. Its reduction is nonzero at y, so its value is the maximum coefficient norm. The same holds at η. Inverting the puncture coordinates and the z_b proves equality on the entire algebraic coordinate ring, hence equality of the analytic points $y = \eta$. No separability or defectlessness assertion enters this argument.

Finally the finite tensor comparison 1.18.7 identifies E_a with the ordinary base change of the free module in (1). Its dimension is D_a. An Artinian composition series gives the second equality in (4), with full residue-field degrees. □

Finite local lengths and a nodal example

Example C.1.48 (A regular parameter without S_1; F8SC-X)**.** For any field K, set

$$R = \big(K[u,v,\epsilon]/(\epsilon^2, u\epsilon)\big)_{(u,v,\epsilon)}. \tag{4.1}$$

The element v is a non-zero-divisor in the maximal ideal, but R does not satisfy S_1. Its unique minimal component has dimension two. For the unlocalized algebra $A = K[u,v,\epsilon]/(\epsilon^2, u\epsilon)$, the morphism $\operatorname{Spec} A \to \mathbf{A}^1_K$ with parameter v has pure one-dimensional fibres; R is its local ring at the origin.

Proof. Before localization the algebra is the direct sum $K[u,v] \oplus \epsilon K[v]$ as a $K[u,v]$-module. Multiplication by v is injective on both summands and remains so after localization. The unique minimal prime is (ϵ). The prime $\mathfrak{q} = (u,\epsilon)$ is embedded, with $\operatorname{Ann}(\epsilon) = \mathfrak{q}$. In $R_{\mathfrak{q}}$, the element ϵ is nonzero, $\dim R_{\mathfrak{q}} = 1$, and every member of the maximal ideal annihilates ϵ. Its depth is zero, contradicting S_1. Every fibre of $\operatorname{Spec} A \to \mathbf{A}^1_K$ has underlying ring $L[u]$ after reduction, where L is the residue field of the parameter; hence it is pure of dimension one. The same local calculation is available in the convergent analytic germ at the origin over a complete nontrivially valued K, replacing the polynomial summands by convergent power-series germs. □

Proposition C.1.49 (Finite-presentation accounting; F8ST-F)**.** *For a norm-admissible finite map $p : \mathcal{M}(C) \to \mathcal{M}(A)$, $R = \mathcal{O}_{\mathcal{M}(A),x}$, and a finite C-module M, there are canonical identifications*

$$C \otimes_A R = \prod_{y \in p^{-1}(x)} \mathcal{O}_{\mathcal{M}(C),y}, \qquad M \otimes_A R = \bigoplus_{y \in p^{-1}(x)} \widetilde{M}_y.$$

If the module on the right has finite R-length, then

$$\operatorname{length}_R(M \otimes_A R) = \sum_{y \in p^{-1}(x)} [\kappa(y) : \kappa(x)] \operatorname{length}_{\mathcal{O}_{\mathcal{M}(C),y}} \widetilde{M}_y. \tag{C.1.20}$$

For a finite algebra E over a field H, correspondingly,

$$\dim_H E = \sum_{\mathfrak{q} \in \operatorname{Spec} E} [\kappa(\mathfrak{q}) : H] \operatorname{length}_{E_{\mathfrak{q}}} E_{\mathfrak{q}}. \tag{C.1.21}$$

These formulas do not assert that a finite map preserves each local length.

Proof. The ring product is `NAG-001735`. Tensor a finite presentation of M with that product to obtain the module formula; a finite direct product is

a direct sum as modules. Each local factor is finite over R. A composition factor at y is $\kappa(y)$, which has R-length $[\kappa(y) : \kappa(x)]$. Additivity proves (C.1.20); the same argument over H proves (C.1.21). Finite tensor comparison `NAG-001565` identifies the ordinary and completed finite-module tensors with their canonical norms. It does not by itself assert flatness or a reduced closed fibre. □

Here K is algebraically closed, and $M_A = A^\times/K^\times$ for the indicated K-algebra A. For its localization use the same quotient by $K^\times$.

Example C.1.50 (Nonsaturation on an irreducible very affine nodal curve)**.** Assume $\operatorname{char} K \neq 2$. There is an integral, reduced, connected very affine curve $U = \operatorname{Spec} A$ and a nonempty principal open U_h such that $M_A \to M_{A_h}$ has a quotient containing an element of exact order 2. Explicitly, take

$$A = K[x^{\pm 1}, z^{\pm 1}]/\big((z-1)^2 - x(x-1)^2\big), \qquad h = x - 1. \tag{6}$$

Proof. Inside $K(t)$ put $x = t^2$ and $z = 1 + t(t^2 - 1)$. Before inverting z, division by the monic relation in $y = z - 1$ writes every element uniquely as $a(x) + b(x)y$ with $a, b \in K[x, x^{-1}]$. Its image is $a(t^2) + t(t^2-1)b(t^2)$, whose even and odd powers of t cannot cancel. This proves injectivity, and localization by the nonzero element z preserves it. Hence (6) is an integral domain. Every nonzero prime contains a nonzero Laurent polynomial in x: choose a nonzero element of the prime, take its algebraic equation over $K(x)$ with nonzero constant term, and clear denominators. In the quotient, algebraic closedness then forces x to be a scalar; the defining equation forces z to be a scalar as well. Thus every nonzero prime is maximal. The nonzero prime $(x-1, z-1)$ exists, so the dimension is exactly one.

Its fraction field is $K(t)$ because $t = (z-1)/(x-1)$ there. Its normalization is

$$B = K[t, t^{-1}, (1 + t^3 - t)^{-1}] = A[t].$$

Indeed $t^2 = x$, so t is integral over A, and $t^{-1} = t/x$ shows the displayed ring equals $A[t]$, hence is finite over A. Factorization in $K[t]$ shows that this localization is integrally closed, by the pole argument used in Proposition C.1.27. It follows that it is the normalization in the common fraction field.

The two evaluations at $t = 1$ and $t = -1$ are defined on B and agree on A: both give $x = z = 1$. They differ on t, since the characteristic is not 2. Thus $t \notin A$. After inverting h the formula $t = (z-1)/(x-1)$ gives $A_h = B_h$. In

particular $t \in A_h^\times$, while its class modulo constants does not come from $A^\times$: an equality $t = cu$ would put t in A. But $t^2 = x \in A^\times$. Its image in $A_h^\times/A^\times$ therefore has exact order 2.

The presentation gives a closed embedding of U in $\mathbb{G}_m^2$, and A_h is generated by the units x, z, h and their inverses, so U_h is also very affine. Both rings are integral and $h \neq 0$, giving irreducibility, reducedness and connectedness. At $x = z = 1$, with local coordinates $X = x - 1$ and $Y = z - 1$, the equation is $Y^2 - X^2 - X^3$; its two distinct tangent lines are $Y = \pm X$. Thus this is an ordinary node. □

Very affine charts and the smooth test kernel

Throughout the new arguments, K is complete, algebraically closed and nontrivially valued, with its specified real-valued absolute value. Put $v = -\log|\cdot|$, $\Gamma = v(K^\times)$, $O = K^\circ$, $\mathfrak{m} = K^{\circ\circ}$ and $k = O/\mathfrak{m}$. Let X be an irreducible, reduced, separated scheme of finite type over K, and let $W \subset X^{\mathrm{an}}$ be ordinary open. The dimension may be zero. An analytic moment tuple is a finite tuple of units in $\mathcal{O}(W)$. A very affine chart starts with an actual closed embedding of a Zariski open $U \subset X$ into a split torus. An intrinsic chart, when used, is exactly the one in GK [GK17, 4.1–4.4]: it uses $\mathcal{O}(U)^\times/K^\times$ and its original universal pullback kernel.

Proposition C.1.51 (Simultaneous algebraic replacement)**.** *Let $x \in W$ and let $f_1, \dots, f_m \in \mathcal{O}(W)^\times$. There exist a compact affinoid domain $C \subset W$, an ordinary open neighborhood V of x in C, a very affine Zariski open $U \subset X$ with $C \subset U^{\mathrm{an}}$, and $g_i \in \mathcal{O}(U)^\times$ such that*

$$|f_i(z)| = |g_i(z)| \quad (z \in C,\ 1 \le i \le m).$$

Moreover, V is the inverse image of an open real set under an explicit algebraic closed embedding of U into a torus. All analytic coordinates and all finite germ relations specified in advance may use this same C.

Proof. Choose an affine Zariski neighborhood of the scheme point underlying x and shrink W within its analytification. Write it as $\operatorname{Spec} A$. By the evaluation topology in `NAG-001707`, an ordinary neighborhood of x inside W is specified by finitely many strict bounds on $|a|$, $a \in A$. We replace them by bounds on functions nonzero at x.

If $|a(x)| > 0$, choose a smaller positive lower and upper bound around this value. If $a(x) = 0$, the only relevant constraint is an upper bound $|a| < b$. Choose $c \in K^\times$ with $|c| < b$ and numbers $0 < s < |c| < r < b$. Then

$$s < |a - c| < r \quad \Longrightarrow \quad |a| < b,$$

and the new inequalities hold at x. Also choose algebra generators for A, translating any generator vanishing at x by a nonzero scalar. Translated generators still generate A.

Combining these finite lists gives $y_1, \dots, y_N \in A$ which generate A and satisfy $|y_j(x)| > 0$. We can choose

$$0 < s_j < |y_j(x)| < r_j, \qquad C = \{z \in (\operatorname{Spec} A)^{\mathrm{an}} : s_j \le |y_j(z)| \le r_j \ \forall j\} \subset W.$$

Choose all bounds strictly inside the earlier bounds. Since K is algebraically closed and nontrivially valued, $|K^\times|$ is dense in $\mathbb{R}_{>0}$: the rational powers of any element of absolute value different from one already have this property. Hence the s_j, r_j may belong to $|K^\times|$.

If $A = K[T_1, \dots, T_N]/I$, with $T_j \mapsto y_j$, the domain has the actual affinoid presentation

$$B = K\{r^{-1}T, sS\}/(I(T), T_jS_j - 1)_j. \tag{C.1.22}$$

Here the bound on S_j is s_j^{-1}. The affine analytification and rational-domain results `NAG-001707`, `NAG-001576` and `NAG-001577` identify its spectrum with C. Actual ideals are closed by `NAG-001562`; compactness is `NAG-001496`. Set

$$V = \{z : s_j < |y_j(z)| < r_j \ \forall j\}.$$

This is ordinary open, contains x and lies in C. The displayed strict inequalities on both sets of generators also put V in the analytic interior of this presentation of C over K.

The image of $A[(y_1 \cdots y_N)^{-1}]$ is dense in B: truncate a lift in the weighted Tate algebra in (C.1.22); the resulting polynomial in T, S is such a Laurent polynomial. This uses the actual quotient norm and does not require a Runge theorem. By `NAG-001610`, restriction of the analytic units and their inverses gives units of B. Apply Lemma C.1.34, writing each approximant as

$$g_i = p_i/(y_1 \cdots y_N)^{a_i}, \qquad p_i \in A.$$

The equality of absolute values holds on all of C, so every p_i is nonzero at every point of C. Define

$$U = \operatorname{Spec} A[(y_1 \cdots y_N p_1 \cdots p_m)^{-1}].$$

By NAG-001708, its analytification contains C. Its coordinate algebra is generated by the units y_j, p_i and their inverses, so these coordinates give an explicit closed embedding $\psi : U \hookrightarrow \mathbb{G}_m^{N+m}$. The open set V is exactly $\psi_{\mathrm{trop}}^{-1}(\Omega)$, where Ω imposes the strict bounds on the first N coordinates. Its remaining coordinates are unrestricted.

For a finite list of germ relations, first intersect their ordinary witness neighborhoods with W and include all analytic units occurring in them in one tuple. The preceding construction then applies once. This proves the last assertion without a compatibility choice between separate approximations. □

An ambient superform is tangentially zero when its restriction to the tangent space of every cell and face vanishes at every point of that cell or face.

Lemma C.1.52 (The universal GK kernel restricted to smooth forms)**.** *Fix an original GK tropical chart (V, φ_U), with tropical image P in a real open set $\widetilde{\Omega}$. Assume the canonical cycles of this chart and each compatible test chart have their actual tropical supports, finite pure polyhedral structures and strictly positive top-dimensional weights. Let $\mathcal{N}$ be GK's kernel from Definition 4.4, testing every compatible algebraic morphism and chart. For an ambient smooth (p, q)-form α,*

$$\alpha \in \mathcal{N} \quad \Longleftrightarrow \quad \alpha \text{ is tangentially zero on every face of } P.$$

This equality is independent of the numerical values of the positive weights.

Proof. For the reverse implication, take any compatible algebraic morphism and chart. Its integral affine map F carries its actual tropical image into P. Subdivide so that each source cell maps into a target cell. The differential of F maps the source tangent into the target tangent; hence $F^*\alpha$ is tangentially zero. Its product with the integration current of the source tropical cycle is zero, which is exactly the smooth instance of restriction in GK [GK17, 3.4 and 4.4]. This proves membership in every test kernel, including rank-dropping maps and maps whose source dimension differs from that of U.

Conversely, the identity morphism is one of the tests. It says $\alpha \wedge \delta_P = 0$. If a tangential coefficient of α were nonzero in the relative interior of a maximal cell, choose a small ball meeting no other maximal cell and a complementary constant superform which isolates that coefficient. Multiplication by a nonnegative smooth bump supported in the ball gives a nonzero pairing, because

the coefficient has fixed sign on a smaller ball and the weight is positive. Thus all coefficients vanish in every maximal relative interior. Continuity and purity give vanishing on every face. If p or q exceeds the cell dimension, the tangential restriction is already zero by exterior degree. The zero-dimensional case is the same assertion for functions on a finite set. □

Tame coefficient descent and split nodes

Kummer coefficients and effective descent

Let K be a complete nontrivially real-valued field, with additive value group $\Gamma = v(K^\times)$, valuation ring O, maximal ideal $\mathfrak{m}$, and residue field k. Fix $e \geq 2$, invertible in O, and a primitive e-th root $\zeta \in O$ whose reduction also has order e. Let $L = K(s)$, with its extended valuation, satisfy

$$s^e = c \in K^\times, \qquad \gamma = v(s) > 0, \qquad i\gamma \notin \Gamma \quad (1 \leq i < e). \tag{K}$$

Thus the e value cosets are distinct. In particular the degree is exactly e, and $G = \langle\sigma\rangle$, $\sigma(s) = \zeta s$, is the cyclic Galois group. Put $O' = L^\circ$, its *actual* valuation ring. No equality $O' = O[s]$, and no module finiteness of O'/O, is assumed.

For $0 \leq i < e$, define the fractional O-module

$$I_i = \{a \in K : v(a) + i\gamma \geq 0\}, \qquad v(0) = +\infty. \tag{1}$$

Here $I_0 = O$; an I_i need not be finitely generated. The distinct value cosets give the orthogonal expansion

$$v\left(\sum_{i=0}^{e-1} a_i s^i\right) = \min_{a_i \neq 0}\big(v(a_i) + i\gamma\big), \qquad O' = \bigoplus_{i=0}^{e-1} I_i s^i. \tag{2}$$

Indeed nonzero summands have distinct valuations, so none can cancel a summand of least valuation. The same argument proves linear independence of $1, s, \ldots, s^{e-1}$, hence the degree assertion above. Completeness follows coefficientwise from (2). Only the $i = 0$ summand can have value zero, so O' has residue field k, and G acts trivially on that field. The value index and degree are both e; this is the totally ramified, defectless tame class considered here.

An affine semilinear G-algebra means an O'-algebra B' with an O-algebra action of G inducing this action on O'. An *effective descent* means an O-algebra D and a G-equivariant isomorphism $O' \otimes_O D \simeq B'$, where G acts only on the first factor on the left.

Theorem C.1.53 (Character modules give an exact integral test)**.** *Assume (K), and let B' be flat over O'. Set*

$$B = (B')^G, \qquad M_i = \{b \in B' : \sigma(b) = \zeta^i b\}, \qquad \Lambda_i = s^{-i} M_i \subset B_K, \quad B_K = K \otimes_O B. \tag{3}$$

The displayed inclusion is well-defined. In the expression $I_i B \subset B_K$, products mean finite sums *of ab with $a \in I_i$ and $b \in B$. Then multiplication*

$$\mu : O' \otimes_O B \longrightarrow B' \tag{4}$$

is injective, and the following conditions are equivalent:

1. *the given semilinear algebra B' has effective descent;*
2. *μ is an isomorphism;*
3. *$\Lambda_i = I_i B$ for every $0 \le i < e$.*

If B' is generated as an O'-algebra by $z_1, \dots, z_r$, condition 3 can equivalently be checked by the finitely many membership conditions

$$s^{-i} P_i(z_j) \in I_i B \qquad (0 \le i < e,\ 1 \le j \le r), \tag{4a}$$

where P_i is the Fourier projector defined in the proof. When these conditions hold, B is the uniquely identified invariant descent. It is flat over O. If B' is finitely presented over O', then B is finitely presented over O; if its structural morphism is a nodal curve, so is that of B. Effective descent necessarily makes the action on $B'/\mathfrak{m}'B'$ the identity.

Proof. The Fourier operators

$$P_i = \frac{1}{e} \sum_{j=0}^{e-1} \zeta^{-ij} \sigma^j$$

are O-linear orthogonal idempotents summing to the identity. The usual finite geometric sums prove $B' = \bigoplus_i M_i$ and $M_0 = B$. Flatness of B' over the domain O' makes B', and hence B, torsion free over O. Thus B is O-flat by Stacks Tag 0539.

Localizing O' at $O \setminus \{0\}$ gives L: multiply any element of L by a sufficiently high power of a fixed positive element of O. Invariants commute with this localization. Explicitly, if $b/d \in B'_L$, with $b \in B'$ and $d \in O \setminus \{0\}$, is invariant, then $(\sigma(b) - b)/d = 0$, so torsion-freeness gives $\sigma(b) = b$. Consequently $(B'_L)^G = B_K$. For $m \in M_i$, $s^{-i} m$ is invariant in B'_L, proving (3).

Tensor (2) with B. Flatness of B makes $I_i \otimes_O B \longrightarrow K \otimes_O B = B_K$ injective, with image exactly I_iB. Under this identification, (4) on the i-th summand is multiplication by s^i, with image $s^iI_iB \subset M_i$. This is injective since B' embeds in B'_L. Images of different summands are independent by the Fourier decomposition. This proves injectivity of μ and the equivalence of conditions 2 and 3. Each test (4a) says that $P_i(z_j)$ belongs to the image of μ. If all the tests hold, their sums put every z_j in that image. The image is an O'-subalgebra, so then equals B'. Necessity is immediate.

Condition 2 is an effective descent, so implies condition 1. Conversely, for any effective presentation $O' \otimes_O D \simeq B'$, the Fourier projectors on the left act on O' alone. Their weight-zero summand is $I_0 \otimes_O D = D$. Thus the isomorphism identifies D with B, and its multiplication map is (4). This proves condition 2, including the asserted uniqueness.

The valuation-ring extension $O \to O'$ is faithfully flat: it is torsion free and hence flat by Stacks Tag 0539, and $\mathfrak{m}O' \subset \mathfrak{m}'$ makes this local flat extension faithful. Its affine spectrum map is quasi-compact, so it is an fpqc cover. Once (4) is an isomorphism, finite presentation descends by Stacks Tag 02L0, and the nodal relative dimension-one property descends by Stacks Tag 0C5E. Finally the special fibre of an effective presentation is $k \otimes_O D$; the action on $k = O'/\mathfrak{m}'$ is the identity. Thus its entire special-fibre algebra has trivial action. □

Discrete specialization

For this section let R be a complete discrete valuation ring with uniformizer π, residue field k, and valuation $v(\pi) = 1$. Assume e and ζ as above, and take

$$R' = R[s], \qquad s^e = \pi, \qquad R' = \bigoplus_{i=0}^{e-1} Rs^i, \tag{5}$$

where R' is the valuation ring of $K(s)$. These are discrete hypotheses; (5) is not used for general O'. In (1) each I_i is now R.

Corollary C.1.54 (The affine special-fibre criterion over a DVR)**.** *For a flat R'-algebra B' with the specified semilinear cyclic action, effective descent over R holds if and only if G acts identically on B'/sB'. Under these equivalent conditions, finite presentation and nodality descend as in Theorem C.1.53. No smoothness, branch-splitting, or diagonalization of B' is required for this criterion.*

Proof. Necessity was proved in the theorem. Suppose the special-fibre action is the identity, and retain the Fourier eigenspaces M_i. For $1 \le i < e$ and $a \in M_i$, $(\zeta^i - 1)a \in sB'$. The coefficient $\zeta^i - 1$ is a unit, since its reduction is nonzero. Hence $a = sb$ for a unique $b \in B'$. Applying σ and cancelling the nonzerodivisor s gives $\sigma(b) = \zeta^{i-1}b$. Induction, with the indices $i, i-1, \dots, 1$, gives $M_i = s^i M_0$. The opposite containment is immediate. Thus $\Lambda_i = B = I_i B$, and the theorem applies. □

Split diagonal nodes

Proposition C.1.55 (Branch characters and the quotient of a node)**.** *In (5) let*

$$B' = R'[x,y]/(xy - s^n), \qquad n \ge 1, \qquad \sigma(x) = \zeta^a x, \quad \sigma(y) = \zeta^b y, \tag{8}$$

where $0 \le a, b < e$ and $a + b \equiv n \pmod e$. Let $B = (B')^G$. The images in $\operatorname{Spec}(B/\pi B)$ of the components $y = 0$ and $x = 0$ have generic multiplicities, respectively,

$$\frac{e}{\gcd(e,a)}, \qquad \frac{e}{\gcd(e,b)}. \tag{9}$$

Moreover the following conditions are equivalent:

1. *the given chart (8) effectively descends over R;*
2. *$a = b = 0$;*
3. *$\operatorname{Spec} B \to \operatorname{Spec} R$ is a semistable affine nodal chart.*

When these conditions hold, $e \mid n$ and

$$B = R[x,y]/(xy - \pi^{\,n/e}). \tag{10}$$

All these statements allow arbitrary k subject to the root-of-unity and tameness hypotheses. They concern the given split diagonal chart, without an assertion that every nonsplit node has such coordinates over its original residue field.

Proof. The ring B' is free over R' on $1, x^i, y^j$ for $i, j \ge 1$. Reduction of mixed monomials using $xy = s^n$ gives spanning; their distinct Laurent monomials in $L[x, x^{-1}]$ give independence. For an integer q, let $[q]_e$ denote its representative in $\{0, \dots, e-1\}$. The invariant ring is free over R on

$$1, \quad s^{[-ai]_e} x^i \ (i \ge 1), \quad s^{[-bj]_e} y^j \ (j \ge 1). \tag{11}$$

This follows by expanding each coefficient in the basis (5) and imposing the character equation. The elements in (11) with $1 \le i, j \le e$ generate the algebra, since multiplying by the invariant x^e or y^e handles the remaining exponents. Thus B is flat and of finite type, hence finitely presented, over the Noetherian ring R.

Set $d = \gcd(e, a)$, $h = e/d$. If $h > 1$, choose an integer v with $(a/d)v \equiv 1 \pmod{h}$; if $h = 1$, take $v = 0$. On the invariant open where x^e is invertible, put

$$z = s^d x^{-v}, \qquad w = x^h.$$

Both are invariant and w is a unit. We claim

$$B[1/x^e] \simeq R[z, w, w^{-1}]/(z^h w^v - \pi). \tag{12}$$

Indeed $B'[1/x^e] = R'[x, x^{-1}]$, and invariants commute with localization at the invariant nonzerodivisor x^e. An invariant monomial $s^r x^q$, $0 \le r < e$, satisfies $r + aq \equiv 0 \pmod{e}$. Thus $r = dj$ with $0 \le j < h$ and $q = -vj + h\ell$ for an integer ℓ. It is $z^j w^\ell$, which proves surjectivity in (12). The monic relation in z gives a spanning set $z^j w^\ell$, $0 \le j < h$. Its images have distinct pairs of s-exponent modulo e and x-exponent, so are R-linearly independent. This proves (12).

Reducing (12) modulo π gives $k[z, w, w^{-1}]/(z^h)$. At its sole generic point the local ring has length h, proving the first multiplicity in (9). Interchanging x, a with y, b proves the other. These are the generic points of the two component images: x^e, respectively y^e, is nonzero there and vanishes on the other component. The two images are therefore distinct. The finite invariant map introduces no ambiguity here: B' is finite over B, since s, x, y are integral and generate it as a B-algebra.

On $B'/sB' = k[x, y]/(xy)$, the action is the identity exactly when $a = b = 0$. The preceding corollary proves the equivalence of conditions 1 and 2. If either weight is nonzero, its multiplicity in (9) exceeds one, so $B/\pi B$ is not reduced on the corresponding open and condition 3 fails. If both weights vanish, the congruence in (8) gives $e \mid n$; (11) directly gives (10), which is flat with reduced nodal geometric special fibre and smooth generic fibre. □

Corollary C.1.56 (Invariant branches and integral length do not suffice)**.** *Take $e = 2, n = 2, a = b = 1$, with residue characteristic different from two. Then (8) fixes both branches setwise and every point of its geometric skeleton. Its length measured by $v(\pi) = 1$ is $1 \in v(K^\times)$, but its given semilinear chart has no effective descent. The quotient has multiplicity two along each component image.*

Proof. The congruence $a + b \equiv n \pmod 2$ holds. Multiplication of coordinates by -1 preserves every annular radius, and fixes the two branch labels. The interval in the x-valuation is $[0, v(s^2)] = [0, 1]$. But both weights are nonzero, so the proposition applies. More precisely, the invariant generic coordinate is $T = x/s$; its endpoint values are $-1/2, 1/2$, outside $v(K^\times) = \mathbb{Z}$. Thus an integral total length does not make the individual vertices integral over the original field. □

The ramified coefficient map is computed in Lemma C.1.25.

Three applications of analytic dimension drop

Purity and minimal primes

For a Noetherian ring A, write $\operatorname{Min}_A(a)$ for the primes of A minimal over (a), and set $\operatorname{Min}_A(a) = \varnothing$ when a is a unit. A finite collection of primes is an antichain if no two distinct members are comparable by inclusion.

Lemma C.1.57 (Analytic purity supplies incomparability; F8CA-P)**.** *Let A be a generalized affinoid over a complete non-Archimedean field with its specified absolute value, and let $a, b \in A$. Suppose the nonempty actual zero spaces $\mathcal{M}(A/(a))$ and $\mathcal{M}(A/(b))$ are both pure of one common analytic dimension d. Empty zero spaces are allowed. Then their minimal-prime union is an antichain, (C.1.14) holds, and the product zero space is empty or pure of dimension d.*

Proof. Every prime in the union defines an integral affinoid component of analytic dimension d. If distinct members $q \subsetneq p$ were comparable, the nonzero ideal p/q in the integral pure d-dimensional algebra A/q would, by frozen NAG-002374, give

$$\dim \mathcal{M}(A/p) \leq d - 1$$

(or an empty quotient when $d = 0$). This contradicts its dimension d. Apply Lemma C.1.29. The listed primes are exactly the irreducible components of the product zero space, so all its components have dimension d. This argument uses analytic dimension, not an unproved equality with the Krull dimension of a generalized affinoid ring. □

Dimension of the ordinary local ring

Fix a complete non-Archimedean field K with its specified absolute value. Arbitrary positive real radii, trivial valuation, and positive characteristic are allowed. Dimensions of analytic spaces are analytic dimensions. Ordinary analytic local rings are denoted by $\mathcal{O}_{X,x}$. Put

$$d_K(x) = \operatorname{trdeg}_{\widetilde{K}} \widetilde{\mathcal{H}(x)} + \dim_{\mathbb{Q}}\big((|\mathcal{H}(x)^\times|/|K^\times|) \otimes_{\mathbb{Z}} \mathbb{Q}\big).$$

Lemma C.1.58 (Ordinary local dimension bound; F8RC-D)**.** *Let X be good, Hausdorff, and pure of analytic dimension n. If $d_K(x) \geq n-1$, then* $\dim \mathcal{O}_{X,x} \leq 1$.

Proof. The ordinary local ring $R = \mathcal{O}_{X,x}$ is Noetherian. If its dimension were at least two, choose primes $P_0 \subsetneq P_1 \subsetneq \mathfrak{m}_R$, and choose $a \in P_1 \setminus P_0$ and $b \in \mathfrak{m}_R \setminus P_1$. Represent both germs on one ordinary affinoid neighbourhood $U = \mathcal{M}(C)$ of x, using NAG-001698. Contract the three primes to $q_0 \subsetneq q_1 \subsetneq q_2$ of C; strictness is witnessed by a, b, and $q_2 = \ker(C \to \mathcal{H}(x))$. Choose a minimal prime $p \subset q_0$. The affinoid C/p is integral and pure of dimension n, by purity of U. Twice applying the dimension-drop theorem NAG-002374, first to $C/p \to C/q_1$ and then to the integral algebra $C/q_1 \to C/q_2$, gives $\dim_K \mathcal{M}(C/q_2) \leq n-2$. The intermediate integral affinoid is pure of its own analytic dimension, so the second application has its stated hypothesis. The point x has the same completed residue field on this quotient. Its Abhyankar dimension is therefore at most $n-2$, a contradiction. This also covers $n = 1$, where a second nonempty strict dimension drop is impossible. □

The prime choice uses Stacks Tag 00E0(4) in C, with the zero ideal and the specified prime $\mathfrak{q}_0$.

The discarded Cartier contribution

Let A be a nonzero Noetherian ring. Set

$$S = A \setminus \bigcup_{\mathfrak{p} \in \operatorname{Min}(A)} \mathfrak{p}, \qquad T(A) = \ker\left(A \longrightarrow \prod_{\mathfrak{p} \in \operatorname{Min}(A)} A_{\mathfrak{p}} \right), \qquad A^{\mathrm{u}} = A/T(A). \tag{1.1}$$

We call $T(A)$ the embedded torsion ideal. No reduction of the generic Artinian local rings is performed.

Now let A be generalized K-affinoid and $V = \mathcal{M}(A)$ pure of analytic dimension $n \geq 1$. All dimensions in the following lemma are *analytic* dimensions. We do not identify $\dim_{\mathrm{Krull}} A$ with n for arbitrary generalized radii.

The exact sequence (1.2) is in Lemma C.1.36.

Lemma C.1.59 (The discarded Cartier contribution; F8U-D)**.** *The coherent module $T = T(A)$ has analytic support of dimension at most $n-1$. If $f \in A$ is regular, then*

$$\dim_K \operatorname{Supp}(T/fT) \le n-2, \tag{2.1}$$

where the right side means empty support when $n = 1$. Consequently

$$[A/fA]_{n-1} = [A^{\mathrm{u}}/fA^{\mathrm{u}}]_{n-1} \tag{2.2}$$

as integer cycles, with generic local lengths, on the common reduced zero support.

Proof. Since T is finite and $T_{\mathfrak{p}} = 0$ at all minimal primes, $\operatorname{Ann}(T)$ is contained in none of them. The frozen exceptional-image theorem NAG-002374, applied to this ideal, gives $\dim_K \mathcal{M}(A/\operatorname{Ann}(T)) \le n-1$.

Let $\mathfrak{q}$ be a minimal prime of $\operatorname{Supp}(T)$. The nonzero module $T_{\mathfrak{q}}$ has finite length. If $f \in \mathfrak{q}$, some power of f annihilates this module, contradicting injectivity of f on $T_{\mathfrak{q}}$. Hence $f \notin \mathfrak{q}$. Each reduced irreducible support component $Z_{\mathfrak{q}} = \mathcal{M}(A/\mathfrak{q})$ is pure of its analytic dimension $d_{\mathfrak{q}} \le n-1$. Apply NAG-002374 on this component to the nonzero image of f. Its zero set has dimension at most $d_{\mathfrak{q}} - 1 \le n-2$, and is empty if $d_{\mathfrak{q}} = 0$. The finite union of these sets contains $\operatorname{Supp}(T/fT)$, proving (2.1).

At every generic point of an $(n-1)$-dimensional zero component, the first term in (1.2) localizes to zero. The other two localized quotients are therefore isomorphic, not just equal after reduction. Their Artinian lengths agree. This proves (2.2). □

For the finite-length step, apply Stacks Tag 00E0(7) with the zero ideal in $A_{\mathfrak{q}}/\operatorname{Ann}(T_{\mathfrak{q}})$. Its sole prime is the maximal ideal. Finite generation makes this nil ideal nilpotent, so its powers give a finite filtration of $T_{\mathfrak{q}}$ with finite-dimensional residue-field quotients.

Ordinary compact partitions

Theorem C.1.60 (Finite smooth partitions near a compact set)**.** *Let X be good and Hausdorff, let $S \subseteq X$ be compact, and let $O_1, \dots, O_m$ be ordinary open sets covering S. There are compactly supported sections $\chi_i \in \mathcal{A}_X^{0,0}(X)$ such that*

$$0 \le \chi_i \le 1, \qquad \operatorname{supp}(\chi_i) \subseteq O_i,$$
$$\sum_i \chi_i = 1 \quad \text{on an ordinary neighbourhood of } S.$$

Proof. The case $S = \varnothing$ uses zero functions. Otherwise, apply Lemma 5.53.1 at each point of S, with one O_i containing it. Compactness gives finitely many resulting functions b_j whose loci where they equal one cover S by ordinary neighbourhoods. Group them by the chosen index and put

$$b_i' = \sum_{j \text{ assigned to } i} b_j, \qquad s = \sum_i b_i'.$$

Thus $s \ge 1$ on S and each b_i' has compact support contained in O_i. Choose a smooth real function η with $0 \le \eta \le 1$, equal to zero for $t \le 1/4$ and to one for $t \ge 1/2$, by the profile construction in that lemma. The function

$$h(t) = \begin{cases} \eta(t)/t, & t > 0, \\ 0, & t \le 0 \end{cases}$$

is smooth, since it is identically zero for $t \le 1/4$. Set $\chi_i = h(s) b_i'$. Smooth composition and the sheaf algebra of Theorem 5.52.3 make these global smooth moment sections. Since $0 \le b_i' \le s$, they take values in $[0, 1]$, and their supports are contained in those of b_i'. Their sum is $\eta(s)$, which equals one on the open neighbourhood $\{s > 1/2\}$ of S. This proves the partition assertion. □

Finite Gauss overlaps and algebraic Cartier coefficients

Whole fibres in smaller ordinary neighbourhoods

Use the polyannulus notation of Lemma C.1.43 and the finite Gauss charts of Definition C.1.44.

Proposition C.1.61 (A cofinal pointed overlap retains whole fibres)**.** *In Definition C.1.44, for every ordinary neighbourhood U of x in Y there are B_U and H_U as in Lemma C.1.43 such that*

$$p^{-1}(\mathbb{T}_s) \subset U \qquad (s \in B_U \setminus H_U).$$

This also holds for an ordinary affinoid neighbourhood inside U, and for a common smaller ordinary affinoid neighbourhood inside any pair of such neighbourhoods.

Proof. First replace U by an ordinary open neighbourhood of x contained in it. The compact set $C = Y \setminus U$ has compact image in the Hausdorff space $\mathbb{T}_P$. This image misses η_{s_0}, because its entire analytic fibre is $\{x\}$. Thus $O = \mathbb{T}_P \setminus p(C)$ is an ordinary open neighbourhood of η_{s_0} with $p^{-1}(O) \subset U$. Lemma C.1.43 gives the assertion.

The ordinary affinoid-neighbourhood basis used in 5.52.3 supplies an affinoid neighbourhood in U, or in the ordinary intersection of two given neighbourhoods. Apply the same argument there. Compactness was used on the whole source; no chosen sheet replaces $p^{-1}(O)$. □

Additivity over one Noetherian ring

For a Noetherian ring A, write $\mathrm{Min}_A(a)$ for the primes of A minimal over (a), and set $\mathrm{Min}_A(a) = \varnothing$ when a is a unit. A finite collection of primes is an antichain if no two distinct members are comparable by inclusion.

Theorem C.1.62 (Algebraic Cartier additivity under an antichain)**.** *Let A be Noetherian and let a, b be non-zero-divisors. Assume $\mathrm{Min}_A(a) \cup \mathrm{Min}_A(b)$ is an antichain. Define the effective cycle of a regular equation by its actual components and the ambient lengths*

$$\mathrm{div}(a) = \sum_{p \in \mathrm{Min}_A(a)} \mathrm{length}_{A_p}(A_p/aA_p)[\mathrm{Spec}(A/p)].$$

Then $\operatorname{div}(ab) = \operatorname{div}(a) + \operatorname{div}(b)$.

Proof. Lemma C.1.29 identifies all components on both sides, including the converse inclusion missing from the original proof. At a prime p in this union, each of $A_p/(a)$ and $A_p/(b)$ is either zero (when the factor is a unit in A_p) or Artinian (when p is minimal over that factor). It is minimal over every factor it contains: otherwise a smaller minimal prime of that factor would contradict the antichain condition. The regularity of b makes the sequence

$$0 \longrightarrow A/(a) \xrightarrow{b} A/(ab) \longrightarrow A/(b) \longrightarrow 0$$

exact. Localize at p and add finite lengths. This gives the coefficient equality on the now identical component sets. □

For a factor $e = a$ or b contained in p, apply Stacks Tag 00E0(4) in A to $(e) \subseteq p$. The antichain forces the resulting minimal prime to equal p. In the nonzero quotient $A_p/(e)$, apply Stacks Tag 00E0(7) to the zero ideal. The maximal ideal is the unique prime, so each of its finitely many generators is nilpotent. A power of the maximal ideal vanishes and the filtration by its powers has finite-dimensional residue-field quotients.

Operations on native ordinary currents

Native conventions. Let K be complete for its specified real-valued non-Archimedean absolute value, and let X be good and Hausdorff. The valuation may be trivial. Use the ordinary smooth moment sheaf $\mathcal{A}_X^{p,q}$ of ordinary smooth germ sheaf and its compactly supported test space

$$\mathcal{E}^{p,q}(U) = \mathcal{A}_{X,c}^{p,q}(U), \qquad \tau = \tau_{\mathrm{cut}} = \tau_{\mathrm{ord}}$$

on every ordinary open $U \subset X$, with the equality and fixed-stage descriptions of ordinary test topology. A cutoff stage is

$$Q_i\eta = \mathrm{ext}_0(\chi t_f^*\eta), \quad i = (V, f, \Omega, \chi), \quad t_f = -\log|f|,$$

where V is affinoid, f is a tuple of analytic units, Ω contains the actual compact image $t_f(V)$, and χ has compact support inside $\mathrm{Int}(V, X)$. Ambient seminorms are

$$p_{L,m}(\eta) = \max_{I,J,\,|a|\le m} \sup_L |\partial^a \eta_{I,J}|, \qquad L \Subset \Omega.$$

Ordinary stages $R_s : H_s \to \mathcal{E}^{p,q}$ use a fixed finite compatible chart family and a fixed compact support. All one-form generators anticommute, and d', d'' differentiate coefficients and wedge on the left. The positive top generator is $\bigwedge_{j=1}^n (d'x_j \wedge d''x_j)$.

Write $\mathcal{D}_{r,s}(U) = \mathcal{E}^{r,s}(U)'$ for the continuous real dual, called currents of bidimension (r, s). Negative test degrees and their current spaces are zero. If a pure analytic dimension n is specified, this is bidegree $(n - r, n - s)$. We use the transposition signs

$$(d'T)(\alpha) = (-1)^{r+s+1} T(d'\alpha), \qquad (d''T)(\alpha) = (-1)^{r+s+1} T(d''\alpha). \quad \text{(C.1.23)}$$

A current is closed when both transposed differentials vanish. Its support is the complement of the largest ordinary open set on which its restriction is zero. We do not identify closedness of the native integration current with a coefficientwise identity.

Theorem C.1.63 (Continuous test operations and gluing of currents)**.** *On every ordinary open $U \subset X$, the maps d', d'' and multiplication $\eta \mapsto \beta \wedge \eta$ by a fixed ordinary smooth form β are continuous on the native compact-support test spaces in their indicated degrees. Extension by zero from an ordinary open subset is continuous.*

The spaces $\mathcal{D}_{r,s}$ form sheaves of real vector spaces. Formula (C.1.23) defines currents and gives

$$(d')^2 = (d'')^2 = 0, \qquad d'd'' + d''d' = 0, \qquad (d'd''T)(\alpha) = T(d'd''\alpha).$$

For smooth h, $(hT)(\alpha) = T(h\alpha)$ defines a current. For a smooth form β of bidegree (a, a), set

$$(\beta \wedge T)(\alpha) = T(\beta \wedge \alpha).$$

These operations commute with ordinary restriction. Differentiation does not enlarge support; multiplication has support contained in $\operatorname{supp} T \cap \operatorname{supp} \beta$ (with $\beta = h$ for scalar multiplication). The statement concerns fixed smooth multipliers; it asserts no product of arbitrary currents or joint continuity on varying coefficient stages.

Proof. The algebraic operations and support locality exist by ordinary smooth germ sheaf. On an ordinary stage H_s, differentiation of each ambient representative maps continuously to the corresponding ordinary stage in the new degree: compatibility and vanishing off the fixed support are preserved. Coefficient order m is bounded by order $m+1$. The final topology of ordinary test topology therefore makes both differentials continuous.

For multiplication, fix a cutoff stage i and put $C = \operatorname{supp} \chi$. Cover C by finitely many affinoid neighborhoods W_j inside $\operatorname{Int}(V, X)$ on which the fixed forms χ and β have ordinary smooth representatives in fixed tuples g_j. This uses the neighborhood and restriction construction in ordinary smooth germ sheaf. Adjoin $f|_{W_j}$ to g_j. On the joint ambient open, the output is represented by

$$L_j\eta = c_j\, \pi_g^* \beta_j \wedge \pi_f^* \eta,$$

where c_j represents χ. For every compact L and integer m,

$$p_{L,m}(L_j\eta) \le C_{j,L,m}\, p_{\pi_f(L),m}(\eta).$$

The constant involves only the fixed coefficient functions and their derivatives. The representatives agree on ordinary overlaps and vanish off C. Thus they

belong to one fixed ordinary stage in the output degree, and depend continuously on η. The final-topology criterion proves continuity. All refinements and constants were chosen before the variable η.

For an open inclusion $j : W \hookrightarrow U$, a compactly supported cutoff on W extends smoothly by zero to U: its support is closed in the Hausdorff space and is contained in W. The extended cutoff gives exactly the same generating stage on U. This proves continuity of $j_!$ and defines restriction of a current by $T|_W(\eta) = T(j_!\eta)$.

To glue compatible currents T_j on an open cover, choose for each compact test support a finite subcover and a finite smooth partition χ_j equal to one in sum near that support, as supplied by compact partition. Define

$$T(\alpha) = \sum_j T_j(\chi_j \alpha).$$

Independence follows by multiplying two such partitions: each double term is supported compactly in an overlap, where the two currents agree. This also proves linearity, restriction and uniqueness. For continuity on one cutoff stage, its support is fixed, so choose the finite subcover and partition once. Each map $\eta \mapsto \chi_j Q_i \eta$, viewed as a test on the corresponding open set, has the fixed joint-chart factorization just proved. Each summand is continuous, so the glued current is continuous. No locally finite partition on all of X is required.

The transposed maps are continuous linear functionals because the test maps are continuous. If $r + s = N$, successive transpositions give the sign $(-1)^{N+1}(-1)^N = -1$. The test identity $d''d' = -d'd''$ therefore gives $(d'd''T)(\alpha) = T(d'd''\alpha)$; the other differential identities follow in the same way. The multiplication formulas are transposes of continuous fixed test multipliers. Locality gives restriction compatibility and the support assertions. □

Theorem C.1.64 (Closed-current harmonic annihilation)**.** *Let $r, s \geq 1$, let $T \in \mathcal{D}_{r,s}(U)$ be closed, and let $h \in \mathcal{A}^{0,0}(U)$. Then*

$$d'd''(hT) = (d'd''h) \wedge T \quad \text{in } \mathcal{D}_{r-1,s-1}(U). \tag{C.1.24}$$

In particular $d'd''(hT) = 0$ for every smooth harmonic function

$$h \in \mathcal{H}(U) := \ker(d'd'' : \mathcal{A}^{0,0}(U) \to \mathcal{A}^{1,1}(U)).$$

If g is an analytic unit, then $\log|g| \in \mathcal{H}(U)$ and hence $d'd''(\log|g|\,T) = 0$. This applies to an intermediate closed current without any Bedford–Taylor assumption or comparison of current sheaves.

Proof. All terms are currents by Theorem C.1.63. For $\alpha \in \mathcal{E}^{r-1,s-1}(U)$, closedness and the ordinary graded Leibniz rules imply

$$\begin{aligned} T(d''h \wedge d'\alpha) &= T(d'd''h \wedge \alpha), && \text{from } T(d'(d''h \wedge \alpha)) = 0,\\ T(d'h \wedge d''\alpha) &= -T(d'd''h \wedge \alpha), && \text{from } T(d''(d'h \wedge \alpha)) = 0. \end{aligned}$$

Every test here has compact support contained in $\operatorname{supp}\alpha$. Also

$$d'd''(h\alpha) = d'd''h \wedge \alpha - d''h \wedge d'\alpha + d'h \wedge d''\alpha + hd'd''\alpha.$$

Applying T gives zero on the left, and the two preceding identities yield $T(hd'd''\alpha) = T(d'd''h \wedge \alpha)$. By the positive double-transpose sign in Theorem C.1.63, this is precisely (C.1.24). This proof uses closedness of T and algebra on tests, not integration by parts on X.

For an analytic unit g, take the ordinary moment coordinate $t_g = -\log|g|$ on sufficiently small affinoid neighborhoods. The function $\log|g|$ has ambient representative $-x$, whose second derivatives vanish. The frozen differential sheaf ordinary smooth germ sheaf gives $d'd''\log|g| = 0$ on U. The first assertion now supplies the distributional conclusion for closed T. □

An admissible multiplication interface. Fix r, s and $p \le \min(r, s)$. Let $\mathcal{P}$ be a sheaf of real vector spaces of continuous functions containing $\mathcal{A}^{0,0}$. For $0 \le j \le p$, supply sheaves $\mathcal{C}_j$ of real vector spaces consisting of closed native currents of bidimension $(r-j, s-j)$. For $j < p$, suppose there are bilinear maps, commuting with ordinary restrictions,

$$M_j : \mathcal{P} \times \mathcal{C}_j \longrightarrow \mathcal{D}_{r-j,s-j}, \qquad R_j(u, S) = d'd''M_j(u, S),$$

with the following exact requirements: $M_j(h, S) = hS$ for every ordinary smooth h, and $R_j(u, S) \in \mathcal{C}_{j+1}$. Thus M_j supplies an actual continuous native functional on all tests, not only a candidate pairing or values on radial tests. Its existence and representative independence are hypotheses of this interface. Closedness of R_j as a raw current follows automatically from Theorem C.1.63; membership in any more restrictive admissible class remains a separate hypothesis.

Theorem C.1.65 (Frame descent and the ordinary smooth class)**.** *Under the multiplication interface, the ordered recursion*

$$S_0 = T \in \mathcal{C}_0, \qquad S_{j+1} = R_j(u_{j+1}, S_j)$$

is multilinear and depends on each u_i only modulo $\mathcal{H}$. If local potentials of metrized analytic line bundles lie in $\mathcal{P}$, these ordered currents glue independently of all local frames. They are symmetric in the metrized factors provided, in addition, the successive commutator identities

$$R_{j+1}(u, R_j(v, S)) = R_{j+1}(v, R_j(u, S)) \tag{C.1.25}$$

hold for every indicated $S \in \mathcal{C}_j$ and $j + 1 < p$.

There is an unconditional realization of this interface for $\mathcal{P} = \mathcal{A}^{0,0}$ and $\mathcal{C}_j$ equal to all closed currents of the stated bidimension: take $M_j(u, S) = uS$. For a supplied closed T, it gives a symmetric, multilinear, frame-independent product for ordinary smooth metric potentials on every native test, locally equal to

$$(d'd''u_p) \wedge \cdots \wedge (d'd''u_1) \wedge T. \tag{C.1.26}$$

It respects ordinary restriction. For positive integers e_i, replacing a power-metric potential u_i by its positive real root potential u_i/e_i scales the result by $\prod_i e_i^{-1}$. No closedness of a native integration current is assumed unless that particular current is supplied as the closed input T.

Proof. Theorem C.1.64 gives

$$R_j(u + h, S) - R_j(u, S) = d'd''M_j(h, S) = d'd''(hS) = 0 \qquad (h \in \mathcal{H}).$$

Induction proves independence modulo $\mathcal{H}$ in every slot, including those preceding further multiplications. Bilinearity gives multilinearity. Local frames $e_a = g_{ab}e_b$ have potentials

$$u_a = -\log \|e_a\| = u_b - \log |g_{ab}|.$$

Their differences are in $\mathcal{H}$ by Theorem C.1.64. Restriction compatibility therefore identifies the ordered currents on all overlaps; Theorem C.1.63 glues them. Adjacent transpositions using (C.1.25) prove symmetry when that extra condition holds.

In the ordinary smooth class, Theorem C.1.63 constructs uS, and $d'd''(uS)$ is closed because $(d')^2 = (d'')^2 = 0$ and the differentials anticommute. Theorem C.1.64 identifies it with $(d'd''u) \wedge S$. Iteration gives (C.1.26). The

even-degree forms $d'd''u$ commute in the frozen exterior algebra; their fixed test actions are associative, proving (C.1.25). The resulting product is a current by the same fixed-multiplier continuity. Restriction is built into the construction. Finally the power metric has potential e_i times its root potential. Multilinearity yields the stated factor, with no extraction of roots in K and no change of its absolute value. □

Fixed-stage certificates for scalar pairings

Native tests, support, and signs. Fix a complete field K with its specified real-valued non-Archimedean absolute value, possibly trivial. Spaces are good and Hausdorff. Purity means analytic pure dimension; generalized-affinoid radii, quotient norms, and finite-module norms retain their frozen conventions. Use the ordinary smooth moment sheaf of ordinary smooth germ sheaf and

$$\mathcal{E}^{p,q}(W) = \mathcal{A}_c^{p,q}(W), \qquad \tau = \tau_{\mathrm{cut}} = \tau_{\mathrm{ord}}$$

from ordinary test topology. A generating cutoff stage is

$$Q_i\eta = \mathrm{ext}_0(\chi t_f^*\eta), \qquad i = (V, f, \Omega, \chi), \qquad t_f = -\log|f|,$$

where V is affinoid, the tuple f consists of analytic units, $\Omega \supset t_f(V)$ is open in $\mathbb{R}^N$, and $C_i = \operatorname{supp}\chi$ is compact inside the ordinary interior of V in W. The coefficient seminorms are

$$p_{L,m}(\eta) = \max_{I,J,\,|a|\le m} \sup_L |\partial^a \eta_{I,J}|, \qquad L \Subset \Omega.$$

The ordinary stages use a fixed finite compatible chart family and a fixed compact support. Scalar continuity is equivalent to continuity after every Q_i, or every ordinary stage. No estimate uniform over changing stages is implied.

Write $\mathcal{D}_{r,s}(W) = \mathcal{E}^{r,s}(W)'$ for the continuous real dual, of bidimension (r,s). On a pure n-space its bidegree is $(n-r, n-s)$. All one-form generators anticommute; the positive top generator is $\bigwedge_{k=1}^{d}(d'x_k \wedge d''x_k)$. Put

$$(d'T)(\alpha) = (-1)^{r+s+1}T(d'\alpha), \qquad (d''T)(\alpha) = (-1)^{r+s+1}T(d''\alpha). \quad \text{(C.1.27)}$$

Negative test degrees are zero. Successive transpositions and $d''d' = -d'd''$ give $(d'd''T)(\alpha) = T(d'd''\alpha)$. Ordinary differential maps are continuous: on a fixed ordinary stage they differentiate the same ambient coefficients, keep the same support and overlap relations, and increase the required coefficient order by one. This argument uses ordinary smooth germ sheaf and ordinary test topology, not the pending Chern-interface packet. Compactly supported ordinary forms on an open subspace extend smoothly by zero; the sheaf property and their closed compact support give this extension, with commuting differentials. On cutoff stages it is the same stage in the ambient space, so it is continuous.

Restricted class for the next step. Fix a pure-dimensional starting structure current or a supplied cycle, of bidimension (r,r). Let $\mathcal{P}$ be a specified sheaf of real vector spaces of continuous functions generated by the actual continuous model potentials to be used and by ordinary smooth scalars. For $0 \le j \le p \le r$, supply sheaves $\mathcal{C}_j$ of real vector spaces of closed currents of bidimension $(r-j, r-j)$ containing the required reachable currents. The word "reachable" restricts the multiplier problem to this supplied class; it is not an assertion that arbitrary continuous functions multiply arbitrary currents. Write $M_j(u,S)$ for a proposed multiplier of $S \in \mathcal{C}_j$ by $u \in \mathcal{P}$, and $R_j(u,S) = d'd''M_j(u,S)$. Closedness of the raw differentiated current and membership in a class on which the next multiplication is defined are different statements.

Proposition C.1.66 (Scalar-test certificates for multiplication, symmetry, and a limit)**.** *Suppose real scalar pairings $B_j(u,S;\alpha)$ are supplied, trilinear in u, S, and compact tests $\alpha \in \mathcal{E}^{r-j,r-j}$, and compatible with ordinary restriction. Require smooth normalization*

$$B_j(h,S;\alpha) = S(h\alpha) \qquad (h \in \mathcal{A}^{0,0}). \tag{C.1.28}$$

Then these pairings define actual native currents $M_j(u,S)$ if and only if, for every fixed u, S and every cutoff stage, a continuous coefficient seminorm bound holds. Equivalently, one may require, for suitable compact $L_i \Subset \Omega_i$, integer $m_i \ge 0$, and constant A_i,

$$|B_j(u,S;Q_i\eta)| \le A_i p_{L_i,m_i}(\eta). \tag{C.1.29}$$

All these choices may depend on u, S, i. The raw current $R_j(u,S)$ is then closed. To iterate, require separately $R_j(u,S) \in \mathcal{C}_{j+1}$.

For such currents, the exact adjacent symmetry identity is equivalent to the following equality on every *$\alpha \in \mathcal{E}^{r-j-2,r-j-2}$:*

$$B_{j+1}(u,R_j(v,S);d'd''\alpha) = B_{j+1}(v,R_j(u,S);d'd''\alpha). \tag{C.1.30}$$

Testing only a terminal measure, selected radial functions, or one order of multiplication does not supply this criterion.

A sufficient, stronger certificate for (C.1.29) is a family of finite signed Radon measures $\nu_{S,\alpha}$, linear in S,α, supported in $\operatorname{supp}\alpha$*, compatible with restriction, and satisfying*

$$S(h\alpha) = \int h\, d\nu_{S,\alpha} \quad (h \text{ ordinary smooth}), \qquad \int |u|\, d|\nu_{S,Q_i\eta}| \le A_i p_{L_i,m_i}(\eta).$$

In that case $B_j(u,S;\alpha)=\int u\,d\nu_{S,\alpha}$ *works.*

For a limit through such measures on a fixed compact set C*, the following is a sufficient tested convergence condition:* $u_a\to u$ *uniformly on* C*,* $\nu_a\to\nu$ *weakly against* every *continuous function on* C*, and* $\sup_a|\nu_a|(C)<\infty$*. Then*

$$\int_C u_a\,d\nu_a\longrightarrow\int_C u\,d\nu. \tag{C.1.31}$$

To obtain a native current from a family of limits, require in addition linearity, locality and a bound (C.1.29) *uniform in* a *on each fixed stage. To pass symmetry from approximants, verify convergence of* both *ordered pairings in* (C.1.30) *on every indicated test.*

Proof. For fixed u,S, the scalar functional $\alpha\mapsto B_j(u,S;\alpha)$ is continuous exactly when each composition with Q_i is continuous, by the final topology of ordinary test topology. A continuous scalar functional on the ambient coefficient space is bounded by a finite maximum of defining seminorms. Taking the union of their compact sets and the maximum of their derivative orders gives (C.1.29), after absorbing finitely many constants. The converse follows directly from that inequality. Thus it is an exact continuity criterion, not a demand for an unnecessarily stronger variation norm on all intermediate currents.

The differential check in the native conventions makes $d'd''M_j(u,S)$ a current. The identities $(d')^2=(d'')^2=0$ and $d'd''=-d''d'$ show that it is closed; they do not imply membership in a stronger supplied class. The positive double-transpose sign gives

$$R_{j+1}(u,R_j(v,S))(\alpha)=B_{j+1}(u,R_j(v,S);d'd''\alpha).$$

Equality of currents is equality on all tests, proving the stated equivalence. The measure certificate gives finiteness, smooth normalization, trilinearity and locality, while its absolute-integral bound implies (C.1.29).

For the convergence assertion use the explicit estimate

$$\left|\int_C u_a\,d\nu_a-\int_C u\,d\nu\right|\le\|u_a-u\|_C\,|\nu_a|(C)+\left|\int_C u\,d(\nu_a-\nu)\right|.$$

Its first term tends to zero by the uniform variation bound; its second does so by the stated continuous-test weak convergence. This proof works for a sequence or a net with the given uniform bound. For current-valued limits,

the additional fixed-stage bound passes to the pointwise limit and proves continuity by finality; the stated linearity and locality assumptions pass the appropriate identities. If both ordered scalar pairings converge, their equality for every approximant passes to their two limits. Without those two convergence assertions there is no argument identifying the limits with the desired products. □

Supported scalar measures for positive tests

Native scope and signs. Let K be complete for its specified real-valued non-Archimedean absolute value. The analytic arguments allow trivial valuation and generalized radii. Let X be good, Hausdorff, and pure of analytic dimension $n \geq 2$, and work on ordinary opens $U \subseteq \mathrm{Int}(X/K)$. Use the ordinary moment sheaf of ordinary smooth germ sheaf and the topology $\tau_{\mathrm{cut}} = \tau_{\mathrm{ord}}$ of ordinary test topology. For $0 \leq p \leq n$, write

$$\mathcal{E}^{p,p}(U) = \mathcal{A}_c^{p,p}(U), \qquad \mathcal{D}_{p,p}(U) = \mathcal{E}^{p,p}(U)'.$$

A cutoff stage is $Q_i\eta = \mathrm{ext}_0(\chi t_f^*\eta)$, with V affinoid, f a tuple of units, $t_f = -\log|f|$, an ambient open $\Omega \supset t_f(V)$, and $C_i = \operatorname{supp}\chi \Subset \mathrm{Int}_{\mathrm{top}}(V, U)$. Put

$$p_{L,m}(\eta) = \max_{I,J,\,|a|\leq m} \sup_L |\partial^a \eta_{I,J}|.$$

All constants and refinements in a stage estimate are fixed before η. A scalar linear functional is continuous exactly when its compositions with all cutoff stages are continuous.

All one-form generators anticommute; the positive top generator on a real d-plane is $\bigwedge_{k=1}^{d}(d'x_k \wedge d''x_k)$. Currents have bidimension (r, s) and transpose signs $(-1)^{r+s+1}$ for both differentials. Thus

$$(d'd''T)(\alpha) = T(d'd''\alpha). \tag{C.1.32}$$

The exact frozen theorem current operations supplies continuous test differentiation, fixed smooth multiplication, open zero extension, the transpose signs, support locality and finite-partition gluing of currents. Its scope is the ordinary topology above. Its fixed-stage mechanism differentiates coefficients with one extra derivative, or adjoins the fixed multiplier moments, and chooses compact partitions before variable coefficients. No varying-stage uniform bound, nonsmooth multiplier, or arbitrary current product is part of that input.

A real (p, p)-form will be called nonnegative on p-planes if its restriction to every p-plane inside every actual tangent plane in a common moment refinement has nonnegative coefficient in the indicated positive generator. Tangential equality and pullback make this criterion independent of representatives. Only this explicit criterion, including its lineality, is used below; no equivalence of external positivity conventions is assumed. For $p = 0$ it means a nonnegative scalar.

Theorem C.1.67 (A scalar-measure construction with explicit domination)**.** *Let $S : \mathcal{E}^{p,p}(U) \to \mathbb{R}$ be linear and nonnegative on every compact form nonnegative on p-planes. Then S is a native current. For every $\alpha \in \mathcal{E}^{p,p}(U)$ there is a unique finite signed Radon measure $\nu_{S,\alpha}$ with compact support in* $\operatorname{supp}\alpha$*, such that*

$$\int h\, d\nu_{S,\alpha} = S(h\alpha) \quad \text{for every ordinary smooth scalar } h. \tag{C.1.33}$$

It is linear in α, positive when α is nonnegative on p-planes, local under ordinary open restriction, and supported in $\operatorname{supp} S \cap \operatorname{supp}\alpha$*.*

For each cutoff stage there are a fixed compact form θ_i nonnegative on p-planes and a constant $c_i \geq 0$ such that

$$|\nu_{S,Q_i\eta}| \leq c_i p_{t_f(V),0}(\eta)\nu_{S,\theta_i}. \tag{C.1.34}$$

The assertions extend linearly and canonically to a linear functional whose ordinary local restrictions are finite signed sums of such S. In that case the right side is replaced locally by the sum of the absolute coefficients times the corresponding positive measures.

For such an S and any continuous real v, the formula

$$(vS)(\alpha) = \int v\, d\nu_{S,\alpha}$$

defines a native current, independently of its signed presentation, with support contained in $\operatorname{supp} S \cap \operatorname{supp} v$*. This conclusion is derived from the stated sign and domination hypotheses; order zero alone is not a premise sufficient for it.*

Proof. Fix a stage with N moment coordinates. By compact partition, choose $0 \leq \rho \leq 1$ compactly supported in $\operatorname{Int}_{\mathrm{top}}(V,U)$ and equal to one near C_i. Put $M = \sup|\chi|$ and

$$\Theta_p = \frac{1}{p!}\left(\sum_{k=1}^{N} d'x_k \wedge d''x_k\right)^p, \qquad \theta_i = M\,\operatorname{ext}_0(\rho t_f^* \Theta_p).$$

For $p = 0$ put $\Theta_0 = 1$; for $N < p$ the stage is zero. On a projected decomposable p-vector z, Θ_p has coefficient $|z|^2$ and an arbitrary representative η has coefficient $z^{\mathrm{t}} A_\eta z$ in the positive convention. The antisymmetric part contributes zero. A finite matrix bound gives, on every actual joint refinement, including rank drop,

$$c_i p_{t_f(V),0}(\eta)\theta_i \pm Q_i\eta \;\geq 0 \quad \text{on } p\text{-planes}. \tag{C.1.35}$$

The scalar cutoffs cause no difficulty: $\rho = 1$ on C_i, $|\chi| \le M$, and $Q_i\eta$ vanishes off C_i. Positivity of S gives $|S(Q_i\eta)| \le c_i S(\theta_i) p_{t_f(V),0}(\eta)$. Finality in ordinary test topology proves continuity.

Every compact form α is a finite sum of cutoff-stage images, by compact partition and the ordinary sheaf. Applying (C.1.35) to these fixed summands gives a compact form θ nonnegative on p-planes with $\theta \pm \alpha \ge 0$. Let $C = \operatorname{supp}\theta \cup \operatorname{supp}\alpha$. For every ordinary smooth h, the forms $\|h\|_C\theta \pm h\alpha$ are nonnegative on p-planes. Therefore

$$|S(h\alpha)| \le S(\theta)\|h\|_C. \tag{C.1.36}$$

This is a bound by one scalar supremum on a fixed compact set, stronger than an abstract ambient-stage order-zero statement.

We give the measure argument explicitly. Restrictions of ordinary smooth scalars are uniformly dense in $C(C,\mathbb{R})$: cover C by finitely many opens on which a given continuous function oscillates by less than δ, and combine sample values with the nonnegative finite partition from compact partition. For a nonnegative target the approximants may be chosen nonnegative. Thus (C.1.36) extends $h \mapsto S(h\alpha)$ uniquely to a bounded real linear functional Λ on $C(C)$.

Here is the elementary representation construction, so no abstract measure assertion is being inferred from the word "current." For a positive bounded functional P on $C(C)$ define

$$a(O) = \sup\{P(f) : 0 \le f \le 1,\ \operatorname{supp} f \subset O\}, \qquad a^*(E) = \inf_{O \supset E \text{ open}} a(O).$$

Continuous compact cutoffs and finite partitions on C are obtained by restricting the native cutoffs just used. Partitions on the compact support of f prove countable subadditivity of a on opens; disjoint opens give the reverse finite inequality by summing disjointly supported f. Hence a^* is an outer measure. To check that an open O is measurable, choose $V \supset E$ with $a(V) \le a^*(E)+\delta$ and $0 \le g \le 1$ supported in $V \cap O$, with $P(g) \ge a(V \cap O) - \delta$. Write $F = \operatorname{supp} g$. Tests supported in $V \setminus F$ have disjoint support from g, so

$$a(V \setminus F) \le a(V) - P(g), \qquad a^*(E \cap O) + a^*(E \setminus O) \le a^*(E) + 2\delta.$$

Let δ decrease to zero. The reverse inequality is subadditivity. The sets satisfying this splitting identity form a sigma-algebra: for disjoint such sets, iterate the identity finitely, then use the supremum of the finite sums and

outer subadditivity for the countable union. The same argument proves countable additivity on that sigma-algebra. Thus a^* restricts to a finite Borel measure μ on C.

If f is an admissible test for $a(O)$, then $P(f) \le \mu(\operatorname{supp} f)$ by taking the infimum over open neighborhoods of its support. It follows that $\mu(O) = a(O)$ is the supremum of $\mu(F)$ over compact $F \subset O$. Outer regularity is built into a^*; inner regularity on all Borel sets follows by complements and finiteness. Hence μ is Radon. To verify $P(u) = \int u \, d\mu$ for continuous $u \ge 0$, use lower step sums $\delta \sum_{k=1}^{m} \mu\{u > k\delta\}$ when $u \le m\delta$. For the upper estimate use the continuous functions $v_k = \min(1, \max(0, (u - k\delta)/\delta))$ with $u = \delta \sum_{k=0}^{m-1} v_k$. The resulting upper and lower bounds differ from the corresponding integral by at most $2\delta\mu(C)$. Let $\delta \downarrow 0$. Linearity gives the representation, and compact cutoffs give uniqueness.

For a bounded real Λ, put $\Lambda^+(u) = \sup_{0 \le v \le u} \Lambda(v)$ for $u \ge 0$. This is bounded, positive and additive: split $v \le u_1 + u_2$ into $\min(v, u_1)$ and its remainder for one inequality, and add near maximizing choices for the other. Extend linearly; both Λ^+ and $\Lambda^+ - \Lambda$ are positive. Their Radon representations give a signed representation of Λ. Positive uniqueness implies signed uniqueness. Extending by zero from C gives (C.1.33) on U.

Tests supported outside $\operatorname{supp} \alpha$, or in an open where S vanishes, have zero scalar pairing. Smooth density and Radon regularity on compact subsets of such opens prove the claimed support. The same uniqueness on a compact union of supports proves linearity, locality and independence from C or θ. If $\alpha \ge 0$ on p-planes, nonnegative smooth scalar tests give a positive measure. Apply this positivity to (C.1.35); the measure inequalities imply (C.1.34). Indeed $-a\lambda \le \nu \le a\lambda$ implies $|\nu| \le a\lambda$ by summing over finite Borel partitions.

Finite signed sums inherit scalar measures, the variation bound, and uniqueness. For local signed presentations, choose a fixed finite partition near each compact test support; summing the measures for the partitioned tests gives the same representation and hence the same measure on overlaps. For continuous v, compactness and (C.1.34) give on each stage

$$|(vS)(Q_i\eta)| \le c_i \sup_{C_i} |v| \, \nu_{S,\theta_i}(U) p_{t_f(V),0}(\eta)$$

in the positive case, and the corresponding finite sum in the signed case. A fixed local partition gives the same kind of bound. This proves native continuity, and locality proves the stated support. □

Algebraic unit lattices and intrinsic chart refinements

Throughout the algebraic unit and localization results below, K is an algebraically closed field and A is a nonzero finitely generated integral K-algebra. Put $U = \operatorname{Spec} A$ and $F = \operatorname{Frac} A$. Neither a valuation nor normality of A is assumed. A lattice means a free abelian group of finite rank. An inclusion of lattices is *saturated* if its quotient has no torsion.

Theorem C.1.68 (Boundary construction of the actual unit lattice)**.** *The group*

$$M_A = A^\times / K^\times$$

is a lattice. More precisely, there is a normal integral proper K-variety Y with function field F, a dense open $V = \operatorname{Spec} B \subset Y$ where B is the normalization of A in F, and a finite set S of codimension-one points of $Y \setminus V$, such that

$$M_A \longrightarrow \mathbb{Z}^S, \qquad [u] \longmapsto (\operatorname{ord}_D(u))_{D \in S} \tag{1}$$

is injective. No saturation of a group generated by chosen coordinates is used in this construction.

Proof links. The final finite-freeness step uses the subgroup lemma.

Proof. Choose generators $a_1, \dots, a_m$ of A. To construct a projective closure explicitly, take the homogeneous kernel of

$$K[X_0, \dots, X_m] \longrightarrow A[T], \qquad X_0 \longmapsto T, \quad X_i \longmapsto a_i T.$$

The image is an integral graded algebra. Its Proj is an integral closed subscheme $\overline{U}$ of $\mathbf{P}^m_K$ whose $X_0 \neq 0$ chart is U; that chart is dense. Projectivity gives properness by Stacks 01WC.

Normalize $\overline{U}$. By Stacks 0BXR, the normalization $\nu : Y \to \overline{U}$ is finite and birational, with Y a normal integral variety. It is proper: a finite ring map is integral by the determinant argument applied to multiplication on a finite generating module, so its associated finite morphism is universally closed by Stacks 01WM; it is finite type and separated because its affine diagonal is induced by a surjective multiplication map. Composition with $\overline{U} \to \operatorname{Spec} K$ is proper by Stacks 01W3. The inverse image V of U is the affine normalization $\operatorname{Spec} B$. Here normalization restricts to an affine open by

taking the integral closure in F; localization commutes with this operation by clearing the denominators of a monic integral equation.

All these schemes are Noetherian by Stacks 00FN. The closed set $Y \setminus V$ has finitely many irreducible components. Any codimension-one point in it is the generic point of one of those components, since Y is integral and the complement misses its generic point. Thus S is finite. At a codimension-one point D, the normal Noetherian local ring $\mathcal{O}_{Y,D}$ is a discrete valuation ring by Stacks 00PD, and we use its order normalized by $\operatorname{ord}_D(\pi_D) = 1$. Every unit of A is a unit on V, so its order vanishes at every codimension-one point outside S. Constants have order zero everywhere. This defines (1).

If $[u]$ maps to zero, then u and u^{-1} belong to every codimension-one local ring of Y. On every affine open of Y, Stacks 031T(2) identifies its normal coordinate ring with the intersection of its height-one localizations inside F. Thus u and u^{-1} are regular everywhere on Y.

A regular function f on this proper integral Y is constant. Indeed, the associated map $Y \to \mathbf{A}^1_K \subset \mathbf{P}^1_K$ is proper by Stacks 01W6, since $\mathbf{P}^1_K$ is separated. Its image is a nonempty closed irreducible subset of $\mathbf{P}^1_K$ missing infinity, hence one closed point. Since K is algebraically closed it is a K-point, and reducedness of Y implies f equals that scalar as a regular function. Applying this to u proves injectivity. Lemma C.1.26 now proves finite freeness. □

Proposition C.1.69 (Exact boundary image and conductor descent)**.** *For the Y, V, B, S of Theorem C.1.68, set*

$$E_S = \{f \in F^\times : \operatorname{ord}_D(f) = 0 \text{ for every codimension-one } D \notin S\},$$
$$P_S = \{(\operatorname{ord}_D f)_{D \in S} : f \in E_S\} \subset \mathbb{Z}^S.$$

Then $E_S = B^\times$ and $B^\times/K^\times \simeq P_S$. The exact image of M_A is

$$L_A = \{(\operatorname{ord}_D f)_{D \in S} : f \in B^\times,\ f, f^{-1} \in A\}. \tag{2}$$

If $A \neq B$, let $I = \{a \in A : aB \subset A\}$ be the conductor. It is a nonzero common ideal of A and B, and (2) can equivalently be specified by

$$f \in B^\times, \qquad \bar{f} \in (A/I)^\times \subset (B/I)^\times. \tag{3}$$

For $A = B$, the descent condition is vacuous and $L_A = P_S$.

Proof links. The boundary and the constant-kernel calculation use the same proper model.

Proof. The height-one points of V are exactly the height-one points of Y lying in V, hence those outside S. Apply Stacks 031T(2) to B, both to f and to f^{-1}. This proves $E_S = B^\times$. The kernel of the order map is $K^\times$ by the same properness argument as above. Formula (2) follows because a function and its inverse descend to A precisely when the function is a unit of A.

Since B is finite over A and contained in F, a common nonzero denominator of finite A-module generators of B belongs to I. If $a \in I$ and $b \in B$, then $ab \in A$ and $(ab)B \subset A$, so I is also an ideal of B. There is an equality of subrings of B,

$$A = B \times_{B/I} A/I.$$

Indeed, a residue class of $b \in B$ lies in A/I if and only if b differs from an element of A by an element of $I \subset A$. For $f \in B^\times$, condition (3) therefore puts both f and f^{-1} in A, and the converse is immediate. Multiplication by a constant does not change this condition, so it is well-defined on boundary vectors. □

Proposition C.1.70 (Intrinsic torus and factorization)**.** *Choose a basis of M_A and lifts of its elements to $A^\times$. Extending multiplicatively gives a section $s_A : M_A \to A^\times$, a torus*

$$T_A = \operatorname{Spec} K[M_A], \qquad N_A = \operatorname{Hom}_{\mathbb{Z}}(M_A, \mathbb{Z}),$$

and a morphism $\varphi_A : U \to T_A$. Changing the lifts translates this morphism by a point of $T_A(K)$. Every algebraic moment morphism $U \to \mathbb{G}_m^r$ factors through φ_A by an affine homomorphism of tori, meaning a torus homomorphism followed by a constant translation. The morphism φ_A is a closed immersion if and only if A is generated as a K-algebra by its units; this is equivalent to U being very affine.

For a K-algebra map $A \to C$ between integral finite-type K-algebras, the induced map $M_A \to M_C$ gives an affine torus map $T_C \to T_A$ compatible with the moment morphisms. No injectivity of this lattice map is asserted for an arbitrary map $A \to C$.

Proof links. The construction starts with the finite free algebraic unit lattice for each integral algebra.

Proof. Finite freeness supplies the section: lift a basis and use the unique integral expression of any lattice element. The resulting map $K[M_A] \to A$ sends its basis character χ^m to $s_A(m)$. For another section s'_A, the ratio $s'_A/s_A : M_A \to K^\times$ is a character-valued constant, that is, a point of $T_A(K)$; this gives the translation assertion.

An algebraic moment morphism is a finite tuple $u_i \in A^\times$. Each u_i is uniquely a constant times a monomial in the chosen basis lifts. This proves its factorization. The image of $K[M_A] \to A$ is the subalgebra generated by all units of A, so the map is surjective exactly under the stated generation condition, equivalently the morphism is a closed immersion. A closed embedding into a finite-dimensional torus implies generation by the coordinate units and their inverses. Conversely, if all units generate A, finite generation of A lets one select finitely many units generating it and gives a closed embedding into a torus.

A K-algebra map preserves units and constants. Its induced lattice map and the chosen sections express every pulled-back basis lift as a constant times a monomial in the target basis lifts. These expressions define the asserted affine homomorphism and prove the compatibility on coordinate rings. □

Theorem C.1.71 (Exact stability, with a normality condition for saturation)**.** *Let $A \subset C \subset F$ be a finitely generated K-subalgebra of F containing A. Then $M_C = C^\times/K^\times$ is a lattice, and there is an exact sequence*

$$0 \longrightarrow M_A \longrightarrow M_C \longrightarrow C^\times/A^\times \longrightarrow 0. \tag{4}$$

If A is normal, the first inclusion is saturated. This applies in particular to $C = A_h$ for $h \in A \setminus \{0\}$ and to

$$C = A[q_1^{\pm 1}, \dots, q_s^{\pm 1}], \qquad q_i \in F^\times. \tag{5}$$

If A is generated by units, so is the ring in (5), including A_h as the case $q_1 = h$. Thus these rings are very affine. In (5) the morphism $\operatorname{Spec} C \to U$ is not asserted to be an open immersion. When one inverts finitely many nonzero numerators and denominators $a_i, b_i \in A$, the ring is the genuine principal-open localization A_h, where $h = \prod_i a_i b_i$.

Proof links. The exact sequence starts from the actual finite free unit groups.

Proof. The ring C is an integral finite-type K-algebra, so Theorem C.1.68 applies to it. The inclusion of rings induces an injection of units, and it

induces an injection modulo constants because the constant field in both quotients is the same embedded K. Taking these successive group quotients gives (4).

Suppose now that A is normal and that $n[g] \in M_A$ for $g \in C^\times$ and $n > 0$. Then $g^n = cu$ for some $c \in K^\times$ and $u \in A^\times$. Both g and g^{-1} are integral over A, by the monic equations $X^n - cu$ and $X^n - (cu)^{-1}$. They lie in F, so normality puts both in A. Therefore $[g] \in M_A$, which is exactly saturation. The generation assertion follows directly by adjoining the specified units and their inverses. The final identification follows because making each nonzero a_i, b_i invertible is equivalent to inverting their product. □

Proposition C.1.72 (One boundary model for a principal localization)**.** *For $0 \neq h \in A$, retain the model Y of Theorem C.1.68. Put $V_h = \operatorname{Spec} B_h \subset V$, and let S_h be the codimension-one points of $Y \setminus V_h$. Then B_h is the normalization of A_h, $S \subset S_h$ is finite, and the exact lattice M_{A_h} is obtained by the construction of Proposition C.1.69 on this same Y, using S_h. The conductor of $A_h \subset B_h$ is I_h. Thus the inclusion $M_A \to M_{A_h}$ is realized by extending boundary vectors by zero on $S_h \setminus S$, with the respective conductor descent conditions retained.*

If A is normal, restriction of orders to the new boundary points gives the additional exact sequence

$$0 \longrightarrow M_A \longrightarrow M_{A_h} \longrightarrow \mathbb{Z}^{S_h \setminus S}, \qquad [f] \longmapsto (\operatorname{ord}_D f)_{D \in S_h \setminus S}.$$

There is no assertion of surjectivity onto the final group.

Proof links. The first paragraph uses the same boundary model and its conductor test. The final paragraph uses the known inclusion of unit groups.

Proof. Integral closure commutes with localization by clearing denominators in a monic equation, so the normalization is B_h. The same finiteness argument for the codimension-one boundary of a dense open in Y gives finiteness of S_h. Apply Proposition C.1.69 to V_h; properness, the function field and the normalized orders are unchanged.

Clearly I_h is contained in the conductor of $A_h \subset B_h$. Conversely, choose finite A-module generators $b_1, \dots, b_q$ of B. If a/h^m belongs to that conductor, then each ab_j lies in A_h. A common power h^N clears all these denominators, so $h^N aB \subset A$ and $h^N a \in I$. Therefore $a/h^m \in I_h$. This proves the conductor

identity. If the localized rings agree, the descent condition is simply vacuous. Units of A have zero orders at the additional points because those points lie in V, proving the extension-by-zero description of their boundary vectors.

For the last sequence assume $A = B$. A unit f of A_h with zero orders on $S_h \setminus S$ has zero order at every height-one point of V: the remaining points lie in V_h, where f is already a unit. By Stacks 031T(2), both f and f^{-1} then belong to A. This proves exactness at M_{A_h}, and injectivity at the first arrow was proved in Theorem C.1.71. □

Actual joint moment relations and intrinsic chart refinements

Fix a complete, algebraically closed, nontrivially valued non-Archimedean field K, with its specified real absolute value and natural logarithm. Let X be an integral separated scheme of finite type over K. The dimension is arbitrary, including zero. All sheaves below are on the ordinary topology of X^{an}, which is good and Hausdorff. Every use of the unit lattice is algebraic.

Lemma C.1.73 (The diagonal relation)**.** *Suppose f and g are two equal-length moment tuples on one compact affinoid C and $|f_i| = |g_i|$ everywhere on C. For every ambient smooth superform α defined near their common moment image, the two pullbacks of α to the actual joint image $t_{(f,g)}(C)$ agree tangentially on every cell and face. The assertion remains true after adjoining any finite tuple of further moment coordinates.*

Proof. The actual joint image lies in the diagonal $\{(u, u)\}$. On this diagonal, the two coordinate projections agree on points and on tangent vectors. Their pullbacks therefore agree in both exterior factors of a superform. Every cell tangent of the joint image lies in the diagonal tangent. Adjoining coordinates preserves the equality of the first two projections. Finite actual-image subdivisions are available by NAG-002358. Thus the assertion applies to the finite-image germ relations of NAG-002424. □

Proposition C.1.74 (Finite analytic relations survive replacement)**.** *For the simultaneous replacement in Proposition C.1.51, every finite ordinary moment-germ relation has the same value after replacing the analytic coordinates by the algebraic coordinates. After passing to a given compatible intrinsic chart contained in the witness neighborhood, a zero relation belongs to the full smooth part of GK's universal kernel.*

Proof links. The first paragraph uses one simultaneous replacement on the full compact witness. The intrinsic-chart step uses factorization through the actual algebraic unit lattice. For the last step, the given compatible chart and tests have actual tropical supports, finite pure polyhedral structures and positive top-dimensional weights, as in the smooth-kernel lemma. The application is the implication from tangential vanishing to membership in the smooth universal kernel.

Proof. Use the single C for the complete tuple of coordinates and apply Lemma C.1.73. A relation tangentially zero on the original actual joint image is therefore zero on the replaced joint image. On the explicit very affine embedding, the replaced tuple is given by monomials and constants in the selected unit coordinates. The preimage V of the strict box is contained in the original witness. Pullback by the resulting integral affine map preserves tangential vanishing on each refined cell.

For a given intrinsic chart, the selected units factor through its torus by their classes in $\mathcal{O}(U)^\times/K^\times$; shrinking its real open set by the inverse image of the strict box gives a chart still contained in V. The same argument applies to this affine pullback. Lemma C.1.52 then gives vanishing under every compatible algebraic test. Two choices of approximation are compared by putting their tuples and the original tuple on one smaller ordinary witness and applying the same diagonal argument. The resulting assertion is local and hence a statement about germ relations. □

The simultaneous replacement construction starts with one compact witness for the entire finite tuple. For finitely many analytic units on an ordinary neighborhood of x, including all coordinates of finitely many witnessed germ relations, there are

$$x \in V \subset C \subset U^{\mathrm{an}}, \qquad C \subset W,$$

where C is a compact Laurent affinoid domain, V is an ordinary open neighborhood of x, and U is a very affine Zariski open, together with $g_i \in \mathcal{O}(U)^\times$ satisfying

$$|f_i(z)| = |g_i(z)| \quad (z \in C). \tag{C.1.37}$$

Here V is the inverse image of an open real box under an explicit closed algebraic unit embedding of U.

For precision, the reviewed construction uses translated algebra generators y_j nonzero at x and the actual Banach algebra

$$B = K\{r^{-1}T, sS\}/(I(T), T_jS_j - 1)_j, \qquad C = \mathcal{M}(B).$$

The image of $A[(\prod_j y_j)^{-1}]$ is dense in B by truncation of a lift in the displayed Tate algebra. A single error bound

$$\|g_i - f_i\| < \varepsilon < \min_i \|f_i^{-1}\|^{-1}$$

forces (C.1.37) everywhere on C. The nonzero numerator polynomials of the g_i are inverted in A to obtain U. The strict box defines V and puts it in the analytic interior of this presentation of C over K. Thus the same actual domain contains all relations before any comparison is made. This is the application of the reviewed R1 argument, not a new assertion of algebraic equality $f_i = g_i$.

Proposition C.1.75 (Intrinsic realization and local common refinements)**.** *The preceding V is an intrinsic chart for the actual algebraic lattice M_U. Every selected approximant is a constant times a character of T_U, and* (C.1.37) *holds on the entire original C. Intrinsic charts form a basis of ordinary neighborhoods in X^{an} and admit common intrinsic refinements near every point of a finite intersection.*

Proof links. The compact witness comes from uniform norm preservation and simultaneous replacement.

Proof. Apply `F7U44R1-L1` to the integral finite-type coordinate ring of U, and `F7U44R1-T1` to its selected units. Their tropicalizations are integral affine functions of t_U. In particular the explicit unit embedding ψ of R1 factors through φ_U by an affine torus map, with tropical affine map F. If $V = \psi_{\mathrm{trop}}^{-1}(\Omega_0)$, then

$$V = t_U^{-1}\big(F^{-1}(\Omega_0)\big).$$

The inverse image is real open. This does not shrink or change C and does not change the approximants.

Applying R1 to the empty tuple inside any prescribed ordinary neighborhood constructs a chart V there. For common refinements of charts (V_j, U_j) at x, choose an affine Zariski neighborhood of the scheme point underlying x contained in $\bigcap_j U_j$. Apply R1 to the ordinary neighborhood $\bigcap_j V_j$ within

this affine analytification. Its output U' lies in every U_j and its V' lies in every V_j. Restriction of algebraic units gives the compatible maps $M_{U_j} \to M_{U'}$ and hence affine torus maps by F7U44R1-T1. Their tropical maps are the required refinement maps. This construction also refines any prescribed ordinary neighborhood of x.

All unit groups here are $\mathcal{O}(U)^\times/K^\times$ or the corresponding groups on genuine algebraic localizations. The stability used is F7U44R1-L3; no normality or saturation assertion is required. In particular neither a coordinate-generated subgroup nor the unit group of the completed algebra B is substituted for M_U. □

A graph closure and a common semistable model

On the affine T-chart write $A = O[T]$. For a centre c and radius r, write $\zeta_{c,r}$ for the Gauss point of the $T - c$ coordinate with radius r.

Proposition C.1.76 (Normalization does not repair the generic graph)**.** *Let k be any field, $O = k[[\pi]]$, and $K = k((\pi))$, with its usual complete discrete absolute value. On $\mathbf{P}^1_O$, with affine coordinate T, set*

$$\mathcal{X}_1 = \mathrm{Bl}_{(T,\pi^2)}\,\mathbf{P}^1_O, \qquad \mathcal{X}_2 = \mathrm{Bl}_{(T-\pi,\pi^2)}\,\mathbf{P}^1_O, \tag{4}$$

extending each ideal by the unit ideal near infinity. These are projective semistable models of $\mathbf{P}^1_K$. The schematic closure $\mathcal{G}$ of their generic identity graph has an open neighbourhood of the pair of special nodes equal to $\operatorname{Spec} R$*, with*

$$R = \frac{O[T,u,v]}{(Tu-\pi^2,\ (T-\pi)v-\pi^2,\ uv+\pi u-\pi v)}. \tag{5}$$

The domain R is normal, but

$$R/\pi R = k[T,u,v]/(Tu, Tv, uv). \tag{6}$$

At the origin this special fibre has three geometric branches and is not nodal. The normalization of $\mathcal{G}$ is therefore not a semistable model.

Proof. The actual charts of the first blowup are $O[T,u]/(Tu-\pi^2)$, with $u = \pi^2/T$, and $O[z]$, with $T = \pi^2 z$. These are the blowup subrings $A[\pi^2/T]$ and $A[T/\pi^2]$. The first embeds in $K[T,T^{-1}]$ using its normal-form monomials $1, T^i, u^j$ for $i,j > 0$. Both charts are flat and finitely presented, with smooth generic fibres. Their special fibres form two projective lines meeting at the split node $Tu = \pi^2$. Replace T by $T - \pi$ for the second model. Both ideals are units where their first generators are units, so the extension near infinity is valid. Projectivity follows from Stacks Tag 02NS and Stacks Tag 0C4P.

On the product of the node charts, the generic diagonal has ring

$$K[T,T^{-1},(T-\pi)^{-1}], \qquad u = \pi^2/T, \quad v = \pi^2/(T-\pi).$$

The coordinate ring of its schematic closure is the image of the product chart ring in this domain. The relations (5) vanish there. They reduce every mixed monomial to an O-linear combination of

$$1, \quad T^i, \quad u^j, \quad v^h \qquad (i,j,h \geq 1), \tag{7}$$

since $Tu = \pi^2$, $Tv = \pi v + \pi^2$ and $uv = \pi v - \pi u$ decrease total degree in T, u, v. The elements (7) are linearly independent in $K(T)$: the u-terms have sole poles at 0, the v-terms at π, and the polynomial part is regular at both. Comparing principal parts at these distinct points eliminates all u- and v-coefficients, and then the polynomial coefficients. Thus (5) is exactly the image ring, a domain free over O on (7). Reduction gives (6).

Put $\mathfrak{n} = (\pi, T, u, v)$. After inverting π we have the displayed smooth K-algebra. At a prime containing π but not T, the local ring is a localization of $O[T, T^{-1}, (T-\pi)^{-1}]$. At a prime containing π but not u, the elements $u, u - \pi$ are units, and

$$T = \pi^2/u, \qquad v = -\pi u/(u - \pi).$$

The local ring is a localization of $O[u, u^{-1}, (u-\pi)^{-1}]$. For a prime containing π but not v, use

$$T = \pi + \pi^2/v, \qquad u = \pi v/(v + \pi).$$

Every local ring outside $\mathfrak{n}$ is therefore regular, by regularity of localized polynomial rings over a DVR (or Stacks Tag 07NF).

At $\mathfrak{n}$, the element π is a nonzerodivisor. In (6), $T + u + v$ is a nonzerodivisor: that ring injects into $k[T] \times k[u] \times k[v]$, and multiplication restricts to multiplication by T, u, v in the respective factors. Thus $\pi, T + u + v$ is a regular sequence in $R_{\mathfrak{n}}$. Its depth is at least two. The strict chain of primes

$$(0) \subsetneq (\pi, u, v) \subsetneq \mathfrak{n}$$

shows that $\mathfrak{n}$ has height at least two: $R/(\pi, u, v) = k[T]$. Every height-one prime is outside $\mathfrak{n}$, where regularity was proved. Regular local rings are Cohen–Macaulay by Stacks Tag 00NQ. The Noetherian ring R therefore satisfies R_1 and S_2, so is normal by Stacks Tag 031S.

Over an algebraic closure of k, the three axes in (6) remain distinct. The special local ring at the origin has dimension one and embedding dimension three, so is neither smooth nor an ordinary double point. Normalization is the identity on the normal open $\operatorname{Spec} R$ and cannot remove this three-branch point. □

Proposition C.1.77 (The component missing from the graph)**.** *For (4), a projective semistable common model is obtained by first blowing up (T, π), and then, in the chart $T = \pi z$, blowing up the two disjoint special-point ideals (z, π) and $(z - 1, \pi)$, extended by the unit ideal elsewhere. The resulting $\mathcal{Z}$ has component vertices*

$$\zeta_{0,1}, \quad w = \zeta_{0,|\pi|} = \zeta_{\pi,|\pi|}, \quad \zeta_{0,|\pi|^2}, \quad \zeta_{\pi,|\pi|^2}.$$

The component at w is a geometric projective line attached to the other three components at $0, 1, \infty$. The map $\mathcal{Z} \to \mathcal{G}$ contracts that entire component to the origin of (6).

Proof. Blowing up (h, π) at a smooth point with local parameter h has charts $O[h, t]/(ht - \pi)$ and $O[z]$, with $h = \pi z$. It is projective, flat, finitely presented and semistable, with one new projective-line component. After the first blowup the next two centres are distinct smooth points $z = 0, 1$ of its exceptional line; the old component attaches at infinity. These ideals agree with the unit ideal off their supporting points, so define actual global blowups. Their coordinates give the stated Gauss radii.

In the z-chart the ideals in (4) pull back to $\pi(z, \pi)$ and $\pi(z - 1, \pi)$, which are invertible after the two subsequent blowups. The other first-blowup chart is $O[T, r]/(Tr - \pi)$. On $D(1 - r)$ both ideals are (T), because

$$(T - \pi, \pi^2) = T(1 - r, Tr^2).$$

The open $D(r)$ lies in the z-chart, and $D(r) \cup D(1 - r)$ covers this other chart. Hence both pullback ideals are globally invertible, generated locally by nonzerodivisors since $\mathcal{Z}$ is integral. The blowup universal property Stacks Tag 0806 supplies the two generic-identity maps. Their pair factors through $\mathcal{G}$, because the generic fibre is schematically dense in the flat integral source.

At the generic point of w, z has nonconstant residue and

$$T = \pi z, \quad T - \pi = \pi(z - 1), \quad u = \pi/z, \quad v = \pi/(z - 1)$$

all reduce to zero. The image is the origin of (6). Its inverse image is closed and contains the generic point of that component, hence contains the entire component. □

Tame Kummer descent for supplied models

We retain the exact coefficient class of the passing theorem. Let K be complete and nontrivially real-valued, with valuation ring O, value group Γ, and residue field k. Let $e \geq 2$ be invertible in O, and let $\zeta \in O$ be an e-th root whose order and residual order are both e. Suppose

$$L = K(s), \qquad s^e = c \in K^\times, \qquad \gamma = v(s) > 0, \qquad i\gamma \notin \Gamma \quad (0 < i < e). \tag{K}$$

Write $O' = L^\circ$ for the actual valuation ring, $\mathfrak{m}'$ for its maximal ideal, and $G = \langle \sigma \rangle$, with $\sigma(s) = \zeta s$. The passing coefficient calculation gives

$$O' = \bigoplus_{i=0}^{e-1} I_i s^i, \qquad I_i = \{a \in K : v(a) + i\gamma \geq 0\}, \tag{3}$$

and identifies its residue field with k. No module finiteness of O' over O is assumed.

Let B' be a flat O'-algebra with a semilinear G-action inducing this coefficient action. Set

$$B = (B')^G, \qquad S = O'[B] \subset B', \qquad C = B'/S. \tag{4}$$

The quotient C is an additive S-module. It is not being regarded as a B'-module. The passing theorem identifies S with the image of the injective multiplication map

$$\mu : O' \otimes_O B \longrightarrow B'. \tag{5}$$

Theorem C.1.78 (Residual identity suffices in the nondiscrete class)**.** *Assume (K) and the stated flatness and semilinear action on B'. If the additive quotient $C = B'/O'[(B')^G]$ is a finitely generated $S = O'[(B')^G]$-module, then the following are equivalent:*

1. *G acts identically on the full special-fibre algebra $B'/\mathfrak{m}'B'$;*
2. *the given semilinear algebra has effective descent over O;*
3. *the multiplication map (5) is an isomorphism;*
4. *for every $0 \leq i < e$,*

$$s^{-i}\{b \in B' : \sigma(b) = \zeta^i b\} = I_i (B')^G \quad \text{inside } K \otimes_O (B')^G. \tag{6}$$

The finiteness assumption on C holds whenever B' is of finite type over O'. In particular, the equivalence applies to every flat finitely presented nodal affine chart in (K). For such a chart the invariant algebra B is flat, finitely presented and nodal over O, and (5) gives its effective descent with the prescribed action. No split-branch, diagonal-coordinate, or discrete-value-group hypothesis is added.

Proof. The zero algebra is immediate, so assume $B' \neq 0$. Flatness then makes $O' \to B'$ injective. The equivalence of conditions 2, 3 and 4, and the implication $2 \Rightarrow 1$, are the exact passing result C.1.53. We prove $1 \Rightarrow 3$ under the specified finite S-module assumption.

First $C \otimes_{O'} L = 0$. Here is the generic calculation using the same exact input, without any integral torsor identity. The Fourier projectors give $B' = \bigoplus_i M_i$, with $M_i = \{b : \sigma(b) = \zeta^i b\}$. The localization statement in the passing theorem gives

$$s^{-i} M_i \subset B_K, \qquad B_K = K \otimes_O B.$$

For $m \in M_i$, write $s^{-i} m = b/d$ with $b \in B$, $d \in O \setminus \{0\}$. Then $m = s^i b/d$ lies in $S \otimes_{O'} L$, because $s^i \in O' \subset S$. Summing the character components shows

$$S \otimes_{O'} L = B' \otimes_{O'} L.$$

Localization is exact, so $C \otimes_{O'} L = 0$.

Next, residual identity gives $C = \mathfrak{m}' C$. For every $z \in B'$, the average

$$P_0(z) = \frac{1}{e} \sum_{g \in G} g(z)$$

belongs to $B \subset S$. Since $z - g(z) \in \mathfrak{m}' B'$ for every g, we have $z - P_0(z) \in \mathfrak{m}' B'$. Consequently

$$B' = S + \mathfrak{m}' B', \qquad C = \mathfrak{m}' C.$$

The ring S contains the coefficient ring O', so these are precisely the module and scalar actions needed in Lemma C.1.32. Apply that lemma with $R = O'$, $F = L$ to obtain $C = 0$. Thus $S = B'$, proving surjectivity of (5); its injectivity is already supplied by the passing criterion. This proves condition 3.

If B' is of finite type over O', choose a finite set of algebra generators. Lemma C.1.33 applies with $R' = O'$ and gives the required finite S-module C. There is no claim that B' or C is finite over O'. For B' finitely presented

and nodal, the flatness, finite presentation and nodality of B follow from the persistence part of C.1.53, now that the actual base-change map is an isomorphism. □

We work with the exact tame Kummer class of the passing C.1.53. Thus K is complete and nontrivially real-valued, $O = K^\circ$, $e \geq 2$ is invertible in O, and $\zeta \in O$ and its reduction both have order e. We assume

$$L = K(s), \quad s^e = c \in K^\times, \quad \gamma = v(s) > 0, \quad i\gamma \notin v(K^\times) \quad (0 < i < e). \tag{K}$$

The ring $O' = L^\circ$ is the actual valuation ring, $\mathfrak{m}'$ is its maximal ideal, and $G = \langle\sigma\rangle$, $\sigma(s) = \zeta s$. The residue field of O' is $k = \widetilde{K}$, on which G acts trivially. The extension $O \to O'$ is faithfully flat and affine; it need not be finite.

A semilinear action on an O'-scheme Y' means an O-scheme action inducing the specified action on $\operatorname{Spec} O'$. An effective descent is an O-scheme Y with a G-equivariant isomorphism $Y \times_O O' \simeq Y'$, where G acts on the coefficient factor on the left. Full residual identity means that the induced action on the entire scheme $Y'_k = Y' \times_{O'} k$ is the identity.

Lemma C.1.79 (Fixed special points and generic finite orbits)**.** *Let Y' be a separated O'-scheme, locally of finite presentation, with a semilinear G-action. Suppose that every point of Y'_k is fixed by G. Assume also that Y'_L is of finite type over L and has dimension at most one. Then Y' has a cover by G-invariant affine open subschemes. If Y' is quasi-compact, the cover can be chosen finite.*

Alternatively, properness of $Y' \to \operatorname{Spec} O'$ may replace the hypothesis on the generic fibre. Every finite intersection of the affine opens just obtained is affine, invariant, and of finite presentation over O'.

Proof. Choose $0 \neq \pi \in \mathfrak{m}$. Rank one gives $O'[1/\pi] = L$ and $\sqrt{\pi O'} = \mathfrak{m}'$: any positive real valuation eventually dominates any other after multiplication by a positive integer. Hence the generic fibre is the open $D(\pi) \subset Y'$, and its complement is the underlying special fibre.

Let $x \in Y'_k$, and choose an affine open $V \subset Y'$ containing x. Every translate gV contains x, because the point is fixed. The intersection

$$U_x = \bigcap_{g \in G} gV \tag{2}$$

is invariant and still contains x. It is affine by Stacks Tag 01KP(1)(a), since Y' is separated over the affine base $\operatorname{Spec} O'$.

For $x \in Y'_L$, its orbit is a finite subset of the separated finite-type L-scheme Y'_L. By Stacks Tag 0A25, this scheme is a locally closed subscheme of some projective space over L. Therefore Stacks Tag 01ZY places the entire finite orbit in an affine open $V \subset Y'_L$. This is an affine open of Y', since Y'_L is open. Formula (2) again gives an invariant affine neighborhood of x. This covers every point of Y'. Quasi-compactness supplies a finite subcover when required.

For the alternative proper case, let U be the union of the invariant affine neighborhoods constructed at special-fibre points. The closed complement $Z = Y' \setminus U$, with its reduced closed subscheme structure, is proper over O'. Its image is closed and misses the closed point of $\operatorname{Spec} O'$. Every nonempty closed subset of a local spectrum contains its closed point, so Z is empty. A finite subcover again exists by properness.

Finally, finite intersections remain invariant and affine by Stacks Tag 01KP. Their structural morphisms are locally of finite presentation, and a morphism between affine schemes is quasi-compact and quasi-separated. They are therefore of finite presentation. □

Proof inputs. C.1.53.

Proposition C.1.80 (Affine overlaps descend as open immersions)**.** *Let Y' be separated, flat and locally of finite presentation over O', with a semilinear G-action which is the identity on Y'_k. Let $U'_1, \dots, U'_r$ be invariant affine opens. For every nonempty subset $I \subset \{1, \dots, r\}$, set*

$$U'_I = \bigcap_{i \in I} U'_i, \qquad A'_I = \Gamma(U'_I, \mathcal{O}_{Y'}), \qquad A_I = (A'_I)^G, \qquad U_I = \operatorname{Spec} A_I. \tag{3}$$

Empty intersections are allowed as schemes, with coordinate ring zero. Then:

1. *U_I is flat and finitely presented over O, and multiplication identifies $(U_I)_{O'}$ equivariantly with U'_I. If Y' is nodal of relative dimension one, then U_I/O is nodal of relative dimension one.*
2. *If $I \subset J$, restriction defines an open immersion $j_{J,I} : U_J \hookrightarrow U_I$. Its base change is exactly the given inclusion $U'_J \hookrightarrow U'_I$.*
3. *If $\varnothing \neq H \subset I \cap J$, the canonical map is an isomorphism*

$$A_I \otimes_{A_H} A_J \xrightarrow{\sim} A_{I \cup J}. \tag{4}$$

These identifications respect every further restriction. Thus pairwise and triple intersections downstairs are the descents of the corresponding actual intersections upstairs.

Proof. Separatedness makes every U'_I affine by Stacks Tag 01KP. Its restricted structural morphism is flat and locally of finite presentation. Since its source and target are affine, it is also quasi-compact and quasi-separated, and hence finitely presented. The restricted action has full residual identity. Apply C.1.78 to obtain

$$\mu_I : O' \otimes_O A_I \simeq A'_I. \tag{5}$$

The same result supplies flatness, finite presentation and, when applicable, nodality downstairs.

Restriction of functions is G-equivariant and hence maps A_I to A_J. The multiplication maps (5) commute with these restrictions: both composites send $a' \otimes b$ to $a'\, b|_{U'_J}$. Consequently the base change of the actual morphism $j_{J,I}$ is the specified open immersion. By Stacks Tag 02L3, open immersion is fpqc local on the target. The relevant target cover is

$$(U_I)_{O'} \longrightarrow U_I,$$

the base change of the faithfully flat affine morphism $\operatorname{Spec} O' \to \operatorname{Spec} O$. Thus $j_{J,I}$ is an open immersion. This application does not identify $O' \otimes_O O'$ with a product indexed by G.

The map in (4) is induced by multiplying the two restriction maps. After tensoring over O with O', it becomes

$$A'_I \otimes_{A'_H} A'_J \longrightarrow A'_{I \cup J}.$$

This is an isomorphism because U'_I and U'_J are affine open subschemes of U'_H, and their fibre product is their intersection. Faithful flatness over O now makes (4) an isomorphism: tensoring is exact and detects zero kernels and cokernels. All maps were formed from restriction and multiplication, so their compositions have the asserted compatibility. □

Proof inputs. C.1.53.

Theorem C.1.81 (Effective descent of a supplied separated nodal curve)**.** *Assume (K). Let $Y' \to \operatorname{Spec} O'$ be a separated, flat, finitely presented scheme which is at-worst-nodal of relative dimension one, equipped with a semilinear G-action. The following are equivalent:*

1. *G acts identically on the entire special fibre Y'_k.*
2. *There is a separated, flat, finitely presented nodal O-scheme Y of relative dimension one and a G-equivariant isomorphism*

$$Y \times_O O' \simeq Y'. \tag{6}$$

The descent with identification (6) is unique up to a unique isomorphism compatible with that identification. It is obtained from the invariant coordinate rings of a finite invariant affine cover, with the invariant overlap rings in (3). If Y' is proper over O', then Y is proper over O.

Proof. Assume condition 1. The generic fibre of Y' is a separated finite-type L-scheme of dimension one. Lemma C.1.79 gives a finite invariant affine cover $Y' = \bigcup_i U'_i$. Apply Proposition C.1.80 and write $U_i = U_{\{i\}}$, $U_{ij} = U_{\{i,j\}}$.

Each U_{ij} embeds openly in both U_i and U_j. Use these embeddings to identify their images. Formula (4) identifies, inside U_i, the intersection of the images of U_{ij} and U_{ik} with $U_{\{i,j,k\}}$. The transition maps on this intersection satisfy the cocycle condition: their coordinate maps are the same restrictions to $A_{\{i,j,k\}}$. Thus these are genuine open-gluing data. By Stacks Tag 01JB they glue to a locally ringed space Y. Since its open pieces U_i are affine schemes, Y is a scheme. Their O-structure maps glue as well.

The isomorphisms (5) agree on every overlap. They therefore glue to (6), and it is G-equivariant because each μ_I is equivariant. In particular the construction uses the full given algebraic scheme and its action.

For completeness, separatedness can be checked directly, without an assertion about quotients preserving it. On the pair of charts U_i, U_j, consider

$$A_i \otimes_O A_j \longrightarrow A_{ij}. \tag{7}$$

After tensoring with O', this is the restriction map $A'_i \otimes_{O'} A'_j \to A'_{ij}$, which is surjective because Y'/O' is separated, by Stacks Tag 01KP. Faithful flatness makes (7) surjective. Hence the diagonal of Y/O is a closed immersion on every affine open $U_i \times_O U_j$ of its target, so Y/O is separated. Flatness

and the nodal relative-dimension-one property hold on the charts by Proposition C.1.80. Each chart is finitely presented over O; the finite cover makes Y quasi-compact, and separatedness makes it quasi-separated. Thus Y/O is finitely presented.

The construction is independent of the invariant affine cover. For two such covers, their pairwise intersections give a common invariant affine refinement. Proposition C.1.80 identifies its invariant rings and open embeddings on both sides. The resulting opens cover each descended chart: they do so after base change, and that base change is surjective. They therefore produce a canonical isomorphism between the gluings. For any other effective descent Z/O with a specified equivariant identification $Z_{O'} \simeq Y'$, pull back an affine cover of Z. These are invariant affine opens of Y'. Their invariant rings recover the original coordinate rings of Z: this is the weight-zero recovery in C.1.53 for an effective presentation. The same common-refinement argument identifies Z with Y, compatibly with (6). Uniqueness follows because equality of maps can be checked after a faithfully flat base change, the faithful part of Stacks Tag 02W0. Here only faithfulness is used; a semilinear action has not been presumed to be an integral fpqc descent datum.

If Y' is proper, (6) and Stacks Tag 02L1 imply that Y/O is proper. Conversely, any effective presentation (6) has special fibre $Y \times_O k$. The action on this scheme is the identity because the coefficient action on k is trivial. This proves condition 1. □

Retain the coefficient class of C.1.53. Let K be complete and nontrivially real-valued, with valuation ring O, value group Γ, and residue field k. Let $e \geq 2$ be invertible in O, and let $\zeta \in O$ have order e, with residue also of order e. Assume

$$L = K(s), \qquad s^e = c \in K^\times, \qquad \gamma = v(s) > 0, \qquad i\gamma \notin \Gamma \quad (0 < i < e). \tag{K}$$

Write $O' = L^\circ$, $\mathfrak{m}'$ for its maximal ideal, and $G = \langle \sigma \rangle$, where $\sigma(s) = \zeta s$. The cited criterion gives a faithfully flat map $O \to O'$, identifies both residue fields with k, and gives identity coefficient action on k. In particular, its character-zero summand is $(O')^G = O$. The value group need not be discrete, and O' is the actual valuation ring; it need not be finite as an O-module.

Proposition C.1.82 (Residual identity for finite coefficient modules)**.** *Assume (K). Let B' be a flat finite-type O'-algebra with a semilinear G-action which is the identity on $B'/\mathfrak{m}'B'$. Set $B = (B')^G$. Let M' be a finitely generated B'-module, flat over O', with a compatible semilinear action, meaning $g(bm) = g(b)g(m)$. Put $M = (M')^G$. The following conditions are equivalent:*

1. *G acts identically on $M'/\mathfrak{m}'M'$.*
2. *The natural multiplication map is an equivariant isomorphism*

$$B' \otimes_B M \longrightarrow M'. \tag{1}$$

Here $B' \simeq O' \otimes_O B$ by C.1.78, so (1) is also the map $O' \otimes_O M \to M'$, with coefficient action on the left. If these conditions hold and M' is finite projective of constant rank r over B', then M is finite projective of rank r over B. In particular this applies to rank one.

Proof. Assume condition 1. Form the commutative square-zero algebra

$$E' = B' \oplus M', \qquad (b, m)(d, n) = (bd, bn + dm). \tag{2}$$

It is flat over O', since both summands are flat. It is a finite-type O'-algebra: take finitely many algebra generators of B' and finitely many B'-module generators of M'. These elements, placed in their respective summands, generate E' as an algebra. This does not assert finite generation of M' over O'.

The given actions define a semilinear algebra action on E', and the full residual algebra action is the identity because

$$E'/\mathfrak{m}'E' = (B'/\mathfrak{m}'B') \oplus (M'/\mathfrak{m}'M')$$

with the same square-zero multiplication. Moreover, $(E')^G = B \oplus M$. Apply the finite-type assertion of C.1.78 to E'. Its canonical multiplication isomorphism is

$$O' \otimes_O (B \oplus M) \xrightarrow{\sim} B' \oplus M'. \tag{3}$$

It preserves both summands. On the first it is the known algebra isomorphism, and on the second it is $O' \otimes_O M \to M'$. Thus the second map is an isomorphism. The algebra identification $B' = O' \otimes_O B$ rewrites it as (1). Equivariance follows from the multiplication formula and the fact that M is fixed. The square-zero algebra is generally not nodal; this use is legitimate because the antecedent proves effectiveness for every flat finite-type algebra, with no nodality or reducedness assumption.

Conversely, under (1) the reduction of M' is $k \otimes_O M$, on which the coefficient action is the identity. This proves condition 1.

The map $B \to B'$ is faithfully flat, being the base change of $O \to O'$. If M' is finite projective, Stacks Tag 058S applied to (1) makes M finite projective. For each prime $\mathfrak{p} \subset B$, choose a prime of B' above it. The dimension of the fibre of M at $\mathfrak{p}$ is unchanged by extending to the corresponding residue field upstairs; it is therefore r. This proves the rank assertion. □

Proof inputs. C.1.53.

Proposition C.1.83 (Line bundles and relative ampleness)**.** *Assume (K). Let Y be a flat, locally finitely presented O-scheme, put $Y' = Y \times_O O'$, and let $p : Y' \to Y$ be the projection, with its coefficient G-action. Let $\mathcal{L}'$ be a G-linearized invertible sheaf on Y'. The following conditions are equivalent:*

1. *The restricted action on $\mathcal{L}'|_{Y'_k}$ is the identity.*
2. *The sheaf $\mathcal{L} = (p_*\mathcal{L}')^G$ is invertible, and evaluation gives a G-equivariant isomorphism*
$$p^*\mathcal{L} \simeq \mathcal{L}'. \tag{4}$$

This descent is unique up to a unique isomorphism compatible with (4). If the conditions hold, $\mathcal{L}$ is relatively ample over O if and only if $\mathcal{L}'$ is relatively ample over O'.

Proof. Assume condition 1. For any affine open $V = \operatorname{Spec} B$ of Y, its inverse image is the invariant affine $V' = \operatorname{Spec} B'$, where $B' = O' \otimes_O B$. The algebra B' is flat and finitely presented over O'. Its coefficient action has full residual identity, and $(B')^G = B$ by the character-zero recovery in C.1.53 for an effective presentation.

By the affine equivalence Stacks Tag 01IB and the finite-projective characterization Stacks Tag 00NX, the module $M' = \Gamma(V', \mathcal{L}')$ is finite projective of rank one over B'. It is flat over O', since it is a direct summand of a finite free B'-module and B' is O'-flat. Its residual action is the identity: reduction of sections on this affine identifies it with the sections of $\mathcal{L}'|_{V'_k}$, by Stacks Tag 01I9. Proposition C.1.82 therefore gives a finite projective rank-one B-module $M = (M')^G$ and the equivariant isomorphism (1).

These affine modules describe the asserted invariant sheaf. Indeed, for $f \in B$, the averaging projector

$$P_0 = \frac{1}{e} \sum_{g \in G} g$$

is B-linear and commutes with localization at f. Its image is the invariant module, both before and after localization. Hence

$$(M'_f)^G = M_f. \tag{5}$$

On the principal-open basis of V, formula (5) identifies $(p_*\mathcal{L}')^G|_V$ with $\widetilde{M}$. Thus the invariant sheaf is quasi-coherent and invertible. The local multiplication maps (1) are the restrictions of the canonical evaluation map (4), so they glue and make (4) an equivariant isomorphism. No independent assumption of an integral fpqc descent datum has been made.

For any other invertible sheaf $\mathcal{D}$ on Y with an equivariant identification $p^*\mathcal{D} \simeq \mathcal{L}'$, local trivializations of $\mathcal{D}$, together with $(O' \otimes_O B)^G = B$, identify $\mathcal{D} \simeq (p_*p^*\mathcal{D})^G$. Taking invariants of the specified identification gives its unique compatible isomorphism with $\mathcal{L}$. Conversely, (4) gives identity on the special fibre because the coefficient action on k is the identity.

Finally, $\operatorname{Spec} O' \to \operatorname{Spec} O$ is a faithfully flat affine morphism, hence an fpqc cover. Once the actual invertible sheaf $\mathcal{L}$ and (4) have been constructed, Stacks Tag 0D2P applies to this cover and proves the equivalence of relative ampleness. Its hypotheses require neither a Noetherian base nor a finite extension of valuation rings. □

A symmetric insertion and the arithmetic of its midpoint

Let K be a complete non-Archimedean field with a specified nontrivial real-valued absolute value. Write

$$O = K^\circ, \qquad k = \widetilde{K}, \qquad v(c) = -\log|c|, \qquad \Gamma = v(K^\times) \subset \mathbb{R}.$$

Let C/K be a smooth projective geometrically connected curve. A model means a projective flat finitely presented nodal O-curve with generic fibre identified with C and geometrically connected, geometrically reduced special

fibre, as in 4.45.3. A marking D, when present, is a fixed finite etale O-scheme embedded in the relative smooth locus. All model morphisms below induce the identity on C.

Fix an actual model $\mathcal{X}$. Choose a finite unramified Galois residue lift L/K splitting its nodes. Put $l = \widetilde{L}$ and $G = \mathrm{Gal}(L/K)$. Here *unramified* means the actual finite free etale coefficient extension of 4.43.1; in particular $v(L^\times) = \Gamma$. The splitting field and rational pointed etale nodal charts exist by 4.18.2.

For each node e of $\mathcal{X}_l$, choose an ordering of its two branches and a pointed chart $x_e y_e = a_e$, with $0 < |a_e| < 1$. Put $\ell_e = v(a_e)$. By 4.39.1 and 4.39.3, its entire tube is the standard open annulus

$$|a_e| < |x_e| < 1, \qquad s = v(x_e) \in (0, \ell_e).$$

The coordinate s increases from the chosen first end to the second. These are two distinct *branch ends* even if their component vertices coincide.

Use an algebraically closed complete isometric comparison field $\Omega \supset L$ with residue field $\overline{k}$, as in 4.45.1, and write $p : C_\Omega^{\mathrm{an}} \to C^{\mathrm{an}}$. The geometric component vertices of a model are exactly $p^{-1}(V(\mathcal{X}))$, by 4.45.2. Write $\zeta_e(r)$ for the Gauss point of radius r in the selected annular tube over Ω.

Lemma C.1.84 (An original-ring symmetric insertion)**.** *Let q be a closed node of $\mathcal{X}_s$, and let ℓ be the common length of its split geometric tubes. For $\delta \in \Gamma$ with $0 < \delta \le \ell/2$, there is a projective semistable modification of $\mathcal{X}$, an isomorphism away from q, inserting precisely $\delta, \ell - \delta$ in each geometric tube over q. A coincident pair is counted once. The modification preserves D and all old vertices and is the blowup of an explicit finite-type ideal on $\mathcal{X}$.*

Proof. Choose $\lambda \in K$ with $v(\lambda) = \delta$. On a Zariski neighbourhood U of q omitting the other special nodes, set

$$\mathcal{I}_U = (\lambda) + \mathrm{Fitt}_1\, \Omega_{U/O}.$$

The differential module is finitely presented, so its Fitting ideal is of finite type. At a smooth point that module is locally free of rank one, and its first Fitting ideal is the unit ideal. On the generic fibre λ is invertible. Thus $\mathcal{I}_U$ equals the unit ideal on $U \setminus \{q\}$ and glues with that ideal on $\mathcal{X} \setminus \{q\}$. We have constructed a quasi-coherent finite-type ideal $\mathcal{I}_q$ on the original model. It contains (λ); for any fixed $0 \neq \pi \in K^{\circ\circ}$, some power of π belongs to (λ), since the valuation has rank one.

Extend to $O' = L^\circ$. Differentials and Fitting ideals commute with coefficient extension and the indicated etale charts (Stacks tags Stacks Tag 04BX, Stacks Tag 0C3D, Stacks Tag 0C3E). The ideal in every selected chart is (λ, x_e, y_e). Blowup commutes with these flat base changes by Stacks Tag 0805. The accepted 4.27.1 applies because $c_e = a_e/\lambda^2 \in O'$. Its three actual torsion-free charts are

$$O'[u,z]/(uz - c_e), \qquad O'[x,w]/(xw - \lambda), \qquad O'[y,w']/(yw' - \lambda).$$

We reuse that calculation, including its gluing and component count. It inserts two smooth geometric projective lines if $2\delta < \ell$, and one if $2\delta = \ell$, at exactly the asserted radii. They form a chain between the two old branch ends.

Shrink each auxiliary pointed etale neighbourhood so its fibre over each selected node in either target is exactly the specified rational point. These charts cover the exceptional fibre. With the unchanged complement, they prove flatness and local finite presentation as in 4.27.2, using Stacks Tag 036K and Stacks Tag 036N. Nodality descends through the finite faithfully flat coefficient extension by Stacks Tag 0C5E. The actual global blowup is projective by Stacks Tag 02NS; composition over the affine base is projective by Stacks Tag 0C4P. Quasi-compactness, separatedness and local finite presentation give finite presentation. Replacing each geometric edge by a connected chain preserves geometric connectedness and reducedness. The generic fibre is unchanged, and the smooth marking is disjoint from the centre.

This constructs the integral model before using geometric vertex invariance. The Fitting ideal is defined even at a nonsplit node and when both branches belong to one global component. □

Proof inputs. 4.18.2, 4.39.1, 4.39.3, 4.43.1.

Proposition C.1.85 (An invariant midpoint over a dense value group)**.** *There is a complete nontrivially real-valued field K, with algebraically closed residue field of characteristic zero and dense value group $\Gamma = \mathbb{Z}[1/3]$, and a split projective semistable model $\mathcal{X}$ of $\mathbb{P}^1_K$, for which the following holds. Its geometric skeleton is an interval of length 1. Adding its midpoint gives a semistable geometric vertex set fixed by every isometric K-automorphism of $\Omega = \widehat{K}$, but no semistable original-O model has that geometric vertex set. The midpoint is realized after the ramified quadratic extension $K(\sqrt{t})/K$.*

Proof. Fix an algebraically closed field k of characteristic zero. Take compatible Laurent-series fields

$$F_n = k((t^{1/3^n})), \qquad F_n \hookrightarrow F_{n+1}, \qquad v(t^{1/3^n}) = 3^{-n},$$

and let K be the completion of $F = \bigcup_{n\geq 0} F_n$. For nonzero $a \in K$, choose $f \in F$ with $|a - f| < |a|$; then $|a| = |f|$. Thus completion introduces no new nonzero values and $v(K^\times) = \mathbb{Z}[1/3]$, which is dense in $\mathbb{R}$. For $a \in K^\circ$, approximation with error less than 1 identifies its residue with a residue from some F_n°. Hence the residue field is k.

On $\mathbb{P}^1_O$, with affine coordinate T, blow up the ideal (T, t), extended as the unit ideal outside its support. The two affine charts over the T-line are

$$\operatorname{Spec} O[T, u]/(Tu - t), \qquad \operatorname{Spec} O[z], \quad T = tz.$$

Their standard monomial bases make these rings free over O, hence flat; the displayed presentations are finite. The first is a nodal chart and the second is smooth. Together with the unchanged infinity chart they give a projective semistable model $\mathcal{X}$, by Stacks Tag 02NS and Stacks Tag 0C4P, with special fibre two copies of $\mathbb{P}^1_k$ meeting at one split node. Its geometric vertices have $v(T) = 0, 1$, and the node tube is $|t| < |T| < 1$, of length 1.

Choose $b \in \Omega$ with $b^2 = t$, and put $L = K(b)$. Since $1/2 \notin \Gamma$, $b \notin K$ and $[L : K] = 2$. The two terms in $a + cb$, for $a, c \in K$, have distinct valuations when nonzero, since $\Gamma \cap (\Gamma + 1/2) = \varnothing$. Thus $|a + cb| = \max(|a|, |c||b|)$; completeness follows coefficientwise from completeness of K. Since $v(b) = 1/2$, the extension is ramified. Lemma C.1.84, using the accepted charts 4.27.1, at $\delta = 1/2$ on $\mathcal{X}_{L^\circ}$ inserts exactly the midpoint. Consequently its base change to Ω° realizes the three-element semistable geometric vertex set

$$\{\zeta(1),\ \zeta(|t|^{1/2}),\ \zeta(|t|)\}.$$

Each of these Gauss seminorms is fixed by every isometric K-automorphism of Ω: its radius is fixed and its formula on $\Omega[T]$ uses only the absolute values of the coefficients.

If an original semistable model realized this set, then 4.45.2 would make the image w of the midpoint a component vertex satisfying $v(\mathcal{H}(w)^\times) = \Gamma$. But T is a nonzero element of $\mathcal{H}(w)$ with $v(T(w)) = 1/2$, a contradiction. Thus invariance, even with already split branches and residue field, is insufficient. The use of L° above is a ramified construction, not an etale descent step. □

Proposition C.1.86 (A tame valuation-ring extension that is not finite)**.** *Use the audited field*

$$K = \bigcup_{r\geq 0} \widehat{k((t^{1/3^r}))}, \qquad \operatorname{char}(k) = 0, \quad k \text{ algebraically closed}, \qquad v(t) = 1, \quad \Gamma = \mathbb{Z}[1/3],$$

and $L = K(b)$, $b^2 = t$. *For* $O = K^\circ$, $O' = L^\circ$,

$$O' = O \oplus Mb, \qquad M = \{c \in K : v(c) > -1/2\}. \tag{13}$$

The O*-module* M*, and therefore* O'*, is not finitely generated. In particular* $O[b] \neq O'$*, although* L/K *is tame quadratic with unchanged residue field.*

Proof. The audited midpoint proof gives the degree, completeness, value group, residue field, and orthogonal equality $v(a+cb) = \min(v(a), v(c)+1/2)$. Hence (13) follows. The weak inequality for c is strict because $-1/2 \notin \Gamma$.

A nonzero O-submodule of K generated by finitely many elements is principal: choose a generator of least valuation, and divide every other generator by it in O. But the value set $\Gamma \cap (-1/2, \infty)$ of M has no least element, since Γ is dense. Thus M is not finitely generated. Projection onto the second summand in (13) is O-linear; finiteness of O' would imply finiteness of M.

Finally $O[b] = O \oplus Ob$, whereas $t^{-1/3}b \in O'$ has value $1/6 > 0$ and its coefficient $t^{-1/3}$ does not belong to O. This proves the strict inequality of the rings. The degree equals the value index two, the residue field is unchanged, and its characteristic is zero, giving the stated tameness. □

Proof inputs. C.1.85.

Coordinate zero spaces and fixed-test cutoffs

Let K be complete for its specified non-Archimedean absolute value, which may be trivial. All dimensions below are analytic. For a function f on an ordinary open U, the conditions S1–S3 are the following.

S1. Every ordinary local ring of U satisfies S_1.

S2. Multiplication by f is injective on $\mathcal{O}_{U_G}$. Thus "regular" means an actual non-zero-divisor, not merely a nonzero function on one component.

S3. The actual zero space $D = V(f) \subset U$, with its quotient structure, is empty or pure of analytic dimension $n-1$.

Proposition C.1.87 (An arbitrary-dimensional class with checked purity; F8SA-M)**.** *On the strict n-polydisc $V = \mathcal{M}(K\langle T_1, \dots, T_n\rangle)$, let $f = u\prod_{j=1}^{r} T_j^{m_j}$, where u is an analytic unit, $1 \le r \le n$, and m_j are positive integers. Then S1–S3 hold on V. The irreducible components of its actual zero space are the coordinate hyperplanes, and its module length at the generic prime (T_j) is m_j.*

Proof. The polydisc algebra is a domain, and its ordinary local rings and all actual affinoid-domain algebras are regular by NAG-001760; in particular they satisfy S_1. Multiplication by f is injective, also after domain restriction by the flatness of NAG-001832. A prime containing the displayed product contains one T_j. Each (T_j) is prime, its quotient is a strict $(n-1)$-polydisc algebra, and these distinct coordinate primes are incomparable. They are therefore exactly the minimal primes over (f). Their analytic dimensions are $n-1$ by NAG-001657/NAG-001664. This verifies S3 without an unqualified prime-union rule in a mixed-component ring.

At (T_j), the other coordinates and u are units. The maximal ideal is generated by the non-zero-divisor T_j; successive powers in the quotient by $T_j^{m_j}$ have exactly m_j residue-field factors. The length is thus m_j. □

Cutoffs under prescribed support exclusion

Lemma C.1.88 (A fixed-test substitute; F8SC-L)**.** *Let X be good Hausdorff and pure of dimension n, let f be regular analytic, and let $\eta \in \mathcal{A}_c^{n-1,n-1}(X)$, with compact support K_0 in $\mathrm{Int}(X/K)$. Let B be a closed analytic set such that $X \setminus B$ satisfies S_1. Assume*

$$\dim(D \cap B) \le n-2, \qquad \operatorname{supp}\eta \cap (D \cap B) = \varnothing, \tag{4.2}$$

where $D = \{f = 0\}$; for $n = 1$, require $D \cap B = \varnothing$. Let W be an ordinary open neighbourhood of $K_0 \cap D$ on which f is pure of relative dimension $n-1$. There is a smooth cutoff χ, equal to one near D, such that $\chi\eta$ has compact support in $W \setminus B$, and $(1-\chi)\eta$ has compact support in $X \setminus D$.

Proof. The compact set $K_0 \cap B$ is disjoint from D, as is $K_0 \setminus W$. On either nonempty set the continuous function $|f|$ has a positive minimum. Choose $r > 0$ below both minima, ignoring an empty set. Choose a smooth function of $\log|f|$ equal to one for sufficiently small $|f|$, and zero for $|f| \ge r' < r$. Extend it by one across D. This is an ordinary smooth scalar: it is constant near D, and off D it uses the unit f. Its two products with η have the asserted compact supports. □

The proof uses compact smooth cutoffs and the ordinary sheaf of smooth moment forms.

Fixed-stage tests and matched tangential topologies

Let X be good and Hausdorff over a complete non-Archimedean field with its given absolute value, which can be trivial. Use the ordinary compact test space $\mathcal{E}^{p,q}(X) = \mathcal{A}_c^{p,q}(X)$ and the cutoff and ordinary coefficient presentations of Definition 8.15.3 and Theorem 8.15.4.

Proposition C.1.89 (Sequences in a fixed coefficient presentation)**.** *For a linear functional $T : \mathcal{E}^{p,q}(X) \to \mathbb{R}$, the following are equivalent:*

(i) T is continuous for either topology of Theorem 8.15.4.

(ii) For every s and every sequence $\eta^{(k)} \in H_s$ converging to zero in all coefficient C^∞ seminorms, $T(R_s\eta^{(k)}) \to 0$.

This is a criterion for scalar functionals. It does not characterize all convergent sequences in the full final test space.

Proof. Continuity implies (ii). For the converse, choose compact exhaustions of the finitely many Ω_j. Maxima of coefficient derivatives through order m on the first m compact sets give an increasing countable family P_m defining the topology of H_s. If the linear functional TR_s were not continuous, then for every m the neighbourhood $\{P_m < 1/m\}$ would contain v_m with $|TR_s(v_m)| \geq 1$. Indeed, a bound by one on any neighbourhood, followed by scalar dilation, would imply continuity. But $P_l(v_m) < 1/m$ for $m \geq l$, so $v_m \to 0$, contrary to (ii). Thus each TR_s is continuous, and the final seminorm definition and Theorem 8.15.4 imply (i). □

Proposition C.1.90 (Separation for functions)**.** *The space $(\mathcal{E}^{0,0}(X), \tau_{\mathrm{cut}})$ is Hausdorff.*

Proof. For $x \in X$, evaluation on a cutoff stage is either zero or $\chi(x)\eta(t_f(x))$. It is continuous in the ambient coefficient space. Thus $\alpha \mapsto |\alpha(x)|$ is one of the defining continuous seminorms. By `v037:forms:smoothcutoff`, ordinary scalar sections embed in continuous functions, so these seminorms separate distinct sections. Given a nonzero difference, disjoint intervals around its two values pull back to disjoint neighbourhoods. This proves Hausdorffness. □

Matched coefficient spaces and compact face sectors

Write U for a good Hausdorff analytic space over a complete non-Archimedean field, with arbitrary valuation. For an application with interior tests, take U to be an ordinary open subset of the true analytic interior of the ambient space. Put $E = \mathcal{A}_c^{p,q}(U)$, with the frozen ordinary moment sheaf and actual ordinary compact supports. The two final topologies of NAG-002472 are equal by NAG-002473; denote their common value by τ_a. A cutoff generator has the form

$$Q_i : E^{p,q}(\Omega_i) \longrightarrow E, \qquad Q_i\eta = \mathrm{ext}_0(\chi_i t_{f_i}^* \eta), \qquad t_{f_i} = -\log|f_i|.$$

Here V_i is affinoid, f_i is a unit tuple, Ω_i contains $t_{f_i}(V_i)$, and $\operatorname{supp}\chi_i$ is compact in $\mathrm{Int}_{\mathrm{top}}(V_i, U)$. The coefficient space has all compact C^∞ seminorms. A seminorm on E is τ_a-continuous exactly when all its composites with Q_i are continuous. The images of the Q_i span E. We now define a matched intrinsic model, so that no external sheaf identification is hidden. For a finite compact polyhedral set $P \subset \Omega \subset \mathbb{R}^N$, let

$$C(P) = E^{p,q}(\Omega)/N_P,$$

where N_P consists of forms whose tangential restrictions to all faces vanish. The underlying vector space consists of ambiently smooth restrictions. Its *intrinsic* topology is generated by the suprema $r_{\sigma,m}$ of tangential coefficient derivatives through order m on the compact faces σ, in fixed affine tangent bases. Using all compact subfaces gives the same model after the corresponding restrictions. This definition does not add arbitrary piecewise smooth forms that lack an ambient representative.

For a finite ordinary presentation s, set $P_j = t_{f_j}(V_j)$ and use these $C(P_j)$. We require the usual native nullity implication: zero tangential restriction on P_j pulls back to the zero ordinary section. Let $J_s \subset \prod_j C(P_j)$ consist of the tuples whose pullbacks agree on ordinary open overlaps and vanish off the fixed compact support $K_s \subset \bigcup_j \mathrm{Int}_{\mathrm{top}}(V_j, U)$. Give J_s the intrinsic subspace topology. Its gluing map is G_s. Let τ_t be the final locally convex topology for these G_s. The ambient preimage of J_s is exactly the frozen space H_s. Thus both topologies in this comparison are on the same native test space, with the same compact supports and overlap relations.

Proposition C.1.91 (The exact local-to-final comparison)**.** *For the matched stages above, $\tau_t \subseteq \tau_a$. Choose an increasing cofinal sequence of ambient seminorms p_m and put*

$$q_m([\eta]) = \inf_{\nu \in N_P} p_m(\eta + \nu).$$

If, on every coefficient space used in a generating family, for each m there are finitely many intrinsic seminorms r_l and a constant C_m with

$$q_m(a) \le C_m \max_{l \le s_m} r_l(a), \tag{C.1.38}$$

then $\tau_t = \tau_a$ for that generating family. Constants and derivative orders may depend on the presentation. No uniform estimate over all presentations is needed for this equality of final topologies.

Proof. Tangential restriction and differentiation are fixed linear combinations of ambient coefficient derivatives. They annihilate N_P and give continuous maps from ambient coefficients to $C(P)$ with its intrinsic topology; this is the exact direction of NAG-002476. Consequently $H_s \to J_s \to (E, \tau_t)$ is continuous. The final property of τ_a gives the asserted inclusion.

The q_m generate the quotient locally convex topology. Thus (C.1.38), together with the preceding direction, makes that quotient topology equal to the intrinsic one. The finite product quotient map π onto $\prod_j C(P_j)$ is open. Its restriction from $H_s = \pi^{-1}(J_s)$ onto J_s is still open: the image of $O \cap \pi^{-1}(J_s)$ is $\pi(O) \cap J_s$ for every open O. The continuous gluing map on H_s therefore descends continuously from J_s to (E, τ_a). Applying its final property to all J_s gives $\tau_a \subseteq \tau_t$. □

Proposition C.1.92 (Extension away from polyhedral junctions)**.** *Let $P \subset \Omega$ be a finite compact polyhedral complex. Choose finitely many maximal faces σ and compact subsets $K_\sigma \subset \operatorname{relint}(\sigma)$, and put $K = \bigcup_\sigma K_\sigma$. Let $C_K(P) \subset C(P)$ consist of the forms supported in K, and give its ambient preimage the coefficient subspace topology. There is a linear extension*

$$S : C_K(P) \longrightarrow E^{p,q}(\Omega), \qquad [Sa] = a,$$

with compact support in a fixed neighbourhood of K and, for every ambient compact L and every m, a bound

$$p_{L,m}(Sa) \le C_{L,m} \max_{\sigma} r_{\sigma,m}(a). \tag{C.1.39}$$

Hence the quotient topology on this support sector equals its intrinsic topology. The same conclusion holds for final spaces generated by compatible tuples of these sectors.

Proof. Write L_σ for the affine span of σ, choose a fixed affine projection π_σ onto it and complementary normal coordinates. The tangential coefficients of $a|_\sigma$ are smooth on $\operatorname{relint}(\sigma)$ and vanish off K_σ. Extending them by zero to L_σ is smooth, since K_σ stays away from the relative boundary. Extend the tangential covectors by π_σ^*.

The distance from K_σ to the other maximal faces and to the complement of Ω is positive. Choose a normal smooth cutoff ζ_σ, equal to one near zero, with sufficiently small fixed support, and form

$$S_\sigma a = \zeta_\sigma \, \pi_\sigma^*(a|_{L_\sigma}).$$

Its support lies in a fixed compact tube over K_σ; shrink the normal cutoff so that this tube misses every other maximal face and stays in Ω. Define $S = \sum_\sigma S_\sigma$. On P this is exactly a, including the lower-dimensional faces, where a vanishes. For a zero-dimensional face the same construction is the constant coefficient times a bump; inadmissible bidegrees are zero.

The product and chain rules express each derivative of order at most m using derivatives of $a|_{L_\sigma}$ of order at most m. All cutoff derivatives and all covector/projection matrices are fixed. Their finitely many constants give (C.1.39). The section S gives the reverse quotient bound on this sector; the forward bound was already proved. For compatible tuples, apply the sections componentwise. Their pullbacks have not changed, so all ordinary overlap and support constraints remain satisfied. The two continuous factorizations give the asserted final-topology comparison. □

The comparison uses the final topology on matched stages.

Finite algebra and length formulas

Proposition C.1.93 (Finite locally free intersection lengths)**.** *Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring and let B be a finite locally free A-algebra of constant rank $n \geq 1$. Let $a_1, \dots, a_r$ be an A-regular sequence, and put $I = (a_1, \dots, a_r)$. Assume A/I is nonzero and has finite length. Then the sequence remains regular on B, and*

$$\sum_{\mathfrak{q}|\mathfrak{m}} [\kappa(\mathfrak{q}) : k] \operatorname{length}_{B_{\mathfrak{q}}}(B_{\mathfrak{q}}/IB_{\mathfrak{q}}) = n \operatorname{length}_A(A/I). \tag{C.1.40}$$

Every maximal ideal of B is included. Residue degrees are full degrees, including inseparable degrees; the quotient algebras are not reduced before taking lengths.

Proof. Since A is local, the finite locally free module B is free of rank n. Tensoring the injections defining regularity with this flat module, and then with its successive free quotients over $A/(a_1, \dots, a_i)$, proves regularity on B. The final quotient is nonzero since it is, as an A-module, $(A/I)^n$. In particular

$$\operatorname{length}_A(B/IB) = n \operatorname{length}_A(A/I). \tag{C.1.41}$$

The finite algebra $C = B/IB$ has finite A-length and hence finite length over itself, since its ideals are also A-submodules. Tag 00JB therefore makes it Artinian and identifies it with the product of its finitely many localizations $C_{\mathfrak{q}}$ at the maximal ideals of B above $\mathfrak{m}$. Each composition factor of $C_{\mathfrak{q}}$ as a $B_{\mathfrak{q}}$-module is $\kappa(\mathfrak{q})$. Viewed as an A-module this field is annihilated by $\mathfrak{m}$ and has length $[\kappa(\mathfrak{q}) : k]$, its vector-space dimension over k. Additivity of length along each composition series and then the finite product gives the left side of (C.1.40) as $\operatorname{length}_A(C)$. Equation (C.1.41) proves the result. □

Let L/K be a finite field extension and let $n = [L : K]$.

Corollary C.1.94 (Two scalar conventions for finite intersection algebras)**.** *If C is any finite-dimensional K-algebra, then*

$$\dim_L(C \otimes_K L) = \dim_K C, \qquad \dim_K(C \otimes_K L) = n \dim_K C.$$

For any finite-dimensional algebra D over a field H,

$$\dim_H D = \sum_j \operatorname{length}_{D_j}(D_j)\, [D_j/\mathfrak{m}_j : H], \qquad D = \prod_j D_j.$$

Proof. A K-basis of C becomes an L-basis after tensoring. Restriction of scalars then multiplies the dimension by n. For the last formula, use a composition series in each Artinian local factor; every simple factor is its residue field. This is the same length calculation as in Proposition C.1.93. □

Lemma C.1.95 (A finite algebra over a DVR)**.** *Let $(A, (\pi), k)$ be a DVR, and let B be a commutative finite free A-algebra of rank $n \geq 1$. If $b \in B$ is a non-zero-divisor, then $D = \det_A(m_b)$ is nonzero, B/bB has finite A-length, and*

$$\operatorname{ord}_A D = \operatorname{length}_A(B/bB) = \sum_{\mathfrak{q} \in \operatorname{Max} B} [\kappa(\mathfrak{q}) : k] \operatorname{length}_{B_{\mathfrak{q}}}(B_{\mathfrak{q}}/bB_{\mathfrak{q}}). \quad \text{(C.1.42)}$$

All maximal ideals of B lie above (π). Terms with b a unit are zero. Full residue-field degrees and unreduced quotient rings are used. The assertion does not require B to be a domain or reduced, or any residue extension to be separable.

Proof. Choose an A-basis of B, and let T be the matrix of m_b. Its kernel is zero. Over $\operatorname{Frac} A$, a nonzero kernel vector could be multiplied by one nonzero common denominator to give a nonzero kernel vector in A^n. Thus T is invertible over that field and $D \neq 0$.

Here is the required matrix reduction over A. Choose a nonzero entry of smallest order and move it to the upper-left corner by row and column permutations. Multiply its row by a unit to make this entry π^a. It divides every matrix entry. Elementary row operations therefore clear its column; elementary column operations then clear its row. These operations are invertible over A. The remaining square block has nonzero determinant, so induction reduces T to a diagonal matrix $\operatorname{diag}(\pi^{a_1}, \dots, \pi^{a_n})$ with $a_i \geq 0$. It follows that

$$B/bB \simeq \bigoplus_i A/(\pi^{a_i}), \qquad \operatorname{ord}_A D = \sum_i a_i.$$

The successive powers of π give a composition series of length a_i for $A/(\pi^{a_i})$. Length is additive in a short exact sequence: intersect a chain with the submodule and project it to the quotient for one inequality, and concatenate a submodule chain with inverse images of a quotient chain for the other. This proves the first equality and finite length.

A finite algebra is integral: on its finite generating module, the adjugate of the matrix of multiplication gives a monic equation for each element. If $\mathfrak{q}$

is maximal in B, integrality makes its contraction maximal in A. Indeed, if a field is integral over a subdomain, then that subdomain is a field: a monic equation for the inverse of a nonzero element, multiplied by a suitable power of the element, puts its inverse in the subdomain. Therefore $\mathfrak{q} \cap A = (\pi)$, and $\kappa(\mathfrak{q})/k$ is finite.

The ring $C = B/bB$ has finite length over itself, since every chain of its ideals is a chain of its A-submodules. Tag 00JB decomposes C into its finitely many Artinian local factors. In a $B_{\mathfrak{q}}$-composition series for $C_{\mathfrak{q}}$, every simple factor is $\kappa(\mathfrak{q})$. Its A-length is its full k-dimension: the action factors through k, so its A-submodules are precisely its k-subspaces. Adding the lengths of all factors gives the second equality in (C.1.42). If $C = 0$, the sum is zero and the same equality holds. Neither step removes nilpotents or counts embeddings instead of field dimensions. □

Common normal models and the supported projection interface

Let R be a complete DVR with uniformizer π, fraction field K, and residue field k. A model of a fixed normal integral proper K-variety X is a proper flat finite-type R-scheme $\mathcal{X}$ with a specified isomorphism $\mathcal{X}_K \simeq X$. Write $d = \dim X$; the applications of interest have $d \geq 2$. Maps of models must respect these actual identifications. The special fibre may be nonreduced. We impose no projectivity or normality on the initial models.

We use the affine construction of normalization: for an integral scheme with function field F, normalize $\operatorname{Spec} A$ by the integral closure $\overline{A} \subset F$ and glue. Integral closure commutes with localization. Indeed, if z is integral over $A[1/t]$, multiplying z by a sufficiently high power of t clears all coefficients of its monic equation and makes that multiple integral over A. The same argument works for an arbitrary multiplicative set. Consequently normalization is an isomorphism on any normal open. A normal integral scheme mapping birationally to an integral scheme factors uniquely through its normalization: each integral element in the common function field is regular in every normal local ring upstairs, and these sections glue.

Properness is preserved by composition, base change and closed immersion, by Stacks, Tag 01W3, Stacks, Tag 01W4 and Stacks, Tag 01W5. A morphism from a proper S-scheme to a separated S-scheme is proper by Stacks, Tag 01W6. Finite morphisms are proper: they are affine, separated and of finite type, and are universally closed by Stacks, Tag 01WM. These facts concern schemes and do not invoke formal-model domination.

Theorem C.1.96 (The normalized diagonal gives a common model)**.** *Let $\mathcal{X}_1, \dots, \mathcal{X}_m$ be a finite nonempty family of models of X. Each $\mathcal{X}_i$ is integral. In $P = \prod_{i=1}^m \mathcal{X}_i$ over R, let C be the scheme-theoretic closure of the generic diagonal $X \hookrightarrow P_K$. Then C is integral and proper flat of finite type over R, with $C_K = X$. Its normalization $\nu : \mathcal{M} \to C$ is finite, and $\mathcal{M}$ is a normal integral proper flat finite-type model of X. The maps*

$$p_i : \mathcal{M} \longrightarrow \mathcal{X}_i$$

are proper birational maps inducing the identity on X. The normalization map is finite; the maps p_i need not be finite.

The same construction provides a common normal refinement of any finite collection of normal models already mapping to a fixed model; the new maps commute with those maps to the fixed model.

Proof. On a nonempty affine open $\operatorname{Spec} A \subset \mathcal{X}_i$, flatness makes $A \to A[1/\pi]$ injective. Its target is the coordinate ring of a nonempty affine open of the integral variety X. It is nonempty because $A[1/\pi] = 0$ would make 1 into π-torsion. Thus A is a domain. Every nonempty open meets X, so X is dense and its irreducibility makes the entire model irreducible. The model is therefore integral, even though its special fibre need not be reduced.

The diagonal is closed in P_K because X is separated. Its scheme-theoretic closure is locally cut out by the kernel of the map to its generic function field $K(X)$ on the affines which meet the closure. That kernel is prime. These kernels are coherent since P is Noetherian, and they restrict to the diagonal ideal after inverting π. Hence C is integral with the stated generic fibre. It is a closed subscheme of the proper finite-type scheme P, hence proper and finite type. Each of its affine rings contains R and is a domain, so is torsion-free over R and is flat by Stacks, Tag 0539.

The complete DVR is Noetherian and complete local. By the exact parts (2) and (5) of Stacks, Tag 0335, every finite-type R-algebra is Nagata. Thus C is Nagata, and Stacks, Tag 035S makes its normalization finite. Its affine rings are integral closures inside $K(X)$, so $\mathcal{M}$ is integral and normal. It is proper and finite type, and is flat by the same torsion-free argument. Since $C_K = X$ is normal, normalization is unchanged on this open; hence $\mathcal{M}_K = X$. The projections p_i are proper and are the identity on the dense open X, which proves birationality.

For the refinement assertion apply the same construction to the models in question. Two resulting morphisms to the fixed separated model agree on the dense generic fibre. They agree everywhere: the inverse image of its diagonal is a closed subscheme containing that dense open, and an integral scheme has no nonzero ideal vanishing on a dense open. This proves commutation, not a least-model assertion. □

Theorem C.1.97 (Divisor and special-fibre pushforward from the common model)**.** *Let $p : \mathcal{M} \to \mathcal{T}$ be one of the maps of Theorem C.1.96, or any proper birational map from a normal model to a model of the same X, inducing the identity on X. For each codimension-one point v of $\mathcal{T}$, the prime divisors of $\mathcal{M}$ mapping onto $\overline{\{v\}}$ correspond exactly to the branches $\mathfrak{q}_j$ of the finite*

normalization of $\mathcal{O}_{\mathcal{T},v}$. Their local DVRs and residue fields are those branch DVRs and fields, embedded in the common function field.

Consequently, for every rational Cartier section σ of a line bundle on $\mathcal{T}$,

$$p_* \operatorname{div}(p^*\sigma) = \operatorname{div}(\sigma) \quad \text{as coherent divisor cycles on } \mathcal{T}. \tag{C.1.43}$$

For the special fibres, of pure dimension d,

$$p_*[\mathcal{M}_k]_d = [\mathcal{T}_k]_d, \qquad b_V = \sum_{E \mapsto V} [\kappa(E) : \kappa(V)] b_E. \tag{C.1.44}$$

Every other prime divisor upstairs is vertical and has image of codimension at least two, so contributes zero to these cycle pushforwards. If $\mathcal{T}$ is normal, the branch over each prime divisor is unique, its residue degree is one, and its DVR and b_V are unchanged. In that case p is an isomorphism over an open containing all codimension-one points of $\mathcal{T}$. These formulas apply both to $\nu : \mathcal{M} \to C$ and to every projection p_i in Theorem C.1.96.

Proof. The target normalization $n : \mathcal{T}^\nu \to \mathcal{T}$ is finite by the same Nagata argument. Normality of $\mathcal{M}$ gives a factorization $p = n \circ q$. The map q is proper by Stacks, Tag 01W6, applied over $\mathcal{T}$, since $\mathcal{T}^\nu$ is separated over $\mathcal{T}$ and p is proper. It is birational. Localization of the integral closure at v is exactly the finite algebra B in Lemma C.1.18. Its maximal ideals give the codimension-one points of $\mathcal{T}^\nu$ over v. The morphism q is an isomorphism near every such point by Stacks, Tag 0BFP(2): the target is Noetherian, the source is reduced, and the relevant normal height-one local ring is a DVR. This proves the precise branch correspondence, including equality of the embedded local rings. It does not assert that p is finite.

At each v, choose a local frame for the given line. The coefficient of the left side of (C.1.43) is the branch sum in (C.1.11), because the pulled-back rational function is the same element of the common function field. That sum is exactly the coherent coefficient downstairs. All other source divisors are contracted in codimension-one cycle pushforward. They cannot be horizontal because the generic-fibre morphism is an isomorphism.

The special fibres are pure d-dimensional by Stacks, Tag 00QK, applied to the integral affine coordinate algebras containing R. Their generic multiplicities are the lengths of the local quotients by π. Apply (C.1.12) with $t = \pi$ at each V to obtain (C.1.44). A contracted special component has image dimension less than d: its image is contained in the special fibre and

contains none of its generic component points. Thus its cycle pushforward is zero. Normality of $\mathcal{T}$ reduces its finite local normalization to its own DVR, giving the final assertions directly from Stacks, Tag 0BFP. □

A *Cartier datum on a prescribed closed support* Z consists of an invertible sheaf $\mathcal{L}$ on an integral scheme and a nonzero rational section σ which is a regular generating section off Z. Equivalently its Cartier divisor D is zero off Z. A *signed presentation on* Z is a pair of effective Cartier divisors A, B with $D = A - B$ as Cartier divisors and $|A| \cup |B| \subset Z$. Equivalently it consists of two line bundles with nonzero global sections, an actual identification of their ratio with $(\mathcal{L}, \sigma)$, and the same support condition. This is stronger than equality of ordinary Chow classes.

Call a proper birational map of models *generic-identity* when it is the identity on the specified X. The next result constructs presentations; it does not assume the pending supported-calculus operations or prove descent of those operations on Chow groups.

Theorem C.1.98 (Exact generic criterion for acquiring presentations)**.** *Let $\mathcal{M}$ be a normal model of X, and let $(\mathcal{L}_i, \sigma_i; Z_i)$ be finitely many Cartier data on it. There exists a normal model $\mathcal{M}'$ and a proper generic-identity map $q : \mathcal{M}' \to \mathcal{M}$ such that every pulled-back datum admits a signed presentation on $q^{-1}Z_i$ if and only if every generic Cartier datum admits a signed presentation on $Z_i \cap X$. When these equivalent conditions hold, q can be constructed by finitely many blowups of nonzero coherent ideals, followed by finite normalization. It is an isomorphism off $\bigcup_i Z_i$. Each individual presentation retains its own prescribed inverse-image support.*

In particular the condition holds if signed presentations are already supplied on another model and pulled back to $\mathcal{M}$. It also holds for every collection of rational Cartier data when X is locally factorial. No factoriality of $\mathcal{M}$ or $\mathcal{M}'$ follows or is required.

Proof. Necessity follows by restriction to X: the map is the identity there, and each effective Cartier factor remains effective Cartier because its nonzero section stays nonzero on the integral generic fibre.

For sufficiency first consider one datum, with divisor D and a generic presentation $D|_X = A_K - B_K$ on $Z \cap X$. Let $\mathcal{I}_+$ and $\mathcal{I}_-$ be the ideals of the scheme-theoretic closures of A_K and B_K in $\mathcal{M}$. They are coherent nonzero ideals, are invertible on X, and are the unit ideal off Z. For example,

on $\operatorname{Spec} A \subset \mathcal{M}$ the closure ideal is the contraction of its ideal in $A[1/\pi]$; Noetherianity gives finite generation and localization gives gluing.

Blow up $\mathcal{I}_+$ and then the extended ideal $\mathcal{I}_-$. These blowups are projective and proper by Stacks, Tag 02NS and Stacks, Tag 01WC. Their affine charts for an ideal $I = (a_1, \dots, a_s)$ are $A[I/a_j] \subset \operatorname{Frac}(A)$, for nonzero a_j. On such a chart the extended ideal is (a_j), an invertible ideal generated by a non-zero-divisor. Thus the blowups are integral, make the indicated ideals invertible, and are isomorphisms wherever those ideals were invertible. In particular they leave X and $\mathcal{M} \setminus Z$ unchanged. Write $u : Y \to \mathcal{M}$ for the composite, and let A_0, B_0 be the effective Cartier divisors of the two extended ideals. Their supports lie in $u^{-1}Z$, and their generic restrictions are A_K, B_K.

The Cartier divisor

$$V = u^*D - (A_0 - B_0)$$

is zero on the generic fibre and off $u^{-1}Z$. As subsheaves of the constant sheaf of rational functions on the integral Y, form the coherent nonzero ideal

$$\mathcal{J} = \mathcal{O}_Y(-V) \cap \mathcal{O}_Y.$$

Coherence follows locally by clearing one denominator and intersecting two finitely generated modules over a Noetherian ring. It is the unit ideal on X and off $u^{-1}Z$. Blow up $\mathcal{J}$, with map $v : Y' \to Y$, and write $\mathcal{J}\mathcal{O}_{Y'} = \mathcal{O}_{Y'}(-E)$. The inclusion $\mathcal{J} \subset \mathcal{O}_Y(-V)$ gives a nonzero map between invertible sheaves

$$\mathcal{O}_{Y'}(-E) \longrightarrow \mathcal{O}_{Y'}(-v^*V).$$

Its section has an effective Cartier divisor F satisfying $E - v^*V = F$. This can also be checked in a local frame: if $\mathcal{O}_Y(-V) = f\mathcal{O}_Y$ and $\mathcal{J}\mathcal{O}_{Y'} = (h)$, then h/f is regular and $F = \operatorname{div}(h/f)$. Both E and F vanish off $(uv)^{-1}Z$. Thus

$$(uv)^*D = (v^*A_0 + E) - (v^*B_0 + F)$$

is the required signed presentation. This third blowup leaves the generic fibre unchanged because $\mathcal{J}|_X = \mathcal{O}_X$.

For finitely many data repeat these steps, pulling earlier effective Cartier factors along subsequent dominant maps. Their sections remain nonzero, and their supports remain inside the respective inverse images of Z_i. All stages are integral proper finite-type schemes over R, with generic fibre X, and are flat by Stacks, Tag 0539. Normalize the final stage. Its normalization is finite by Theorem C.1.96's Nagata argument, is the identity over normal

X, and preserves all effective Cartier factors under its dominant pullback. Outside $\bigcup_i Z_i$ the previous stages were the original normal $\mathcal{M}$, so normalization changes nothing there either. This gives q and all the asserted support properties.

An existing presentation pulls back to any common integral model, with its actual line and section identities. No factoriality is needed for this step. Finally on a locally factorial integral Noetherian X, decompose the finite Weil cycle of a Cartier divisor into its positive and negative parts. Local unique factorization makes both parts effective Cartier: their local equations differ by units on overlaps. Their difference is the original Cartier divisor, and their supports lie in its support. This proves the last sufficient condition, independently of any pending intersection claim. □

Proposition C.1.99 (A principal divisor with no presentation on its support)**.** *Over any field K, let*

$$X = \{ad - bc = 0\} \subset \mathbf{P}^4_K, \qquad [a : b : c : d : t], \qquad o = [0 : 0 : 0 : 0 : 1].$$

This is a normal integral proper threefold. Put

$$P_0 = \{a = b = 0\}, \qquad P_1 = \{c = d = 0\}, \qquad Z = P_0 \cup P_1.$$

The rational section $f = a/c$ of $\mathcal{O}_X$ has Cartier divisor with Weil cycle $P_0 - P_1$ and is a regular unit off Z. It has no signed presentation on Z, even in a neighbourhood of o. Hence no proper generic-identity modification of any normal DVR model carrying this datum can acquire such a presentation on the inverse image of the prescribed support. In particular this obstruction applies to the constant equation $ad - bc = 0$ over a complete DVR.

Proof. On the chart $t = 1$ the coordinate algebra is $A = K[a, b, c, d]/(ad - bc)$. It identifies with the weight-zero subring of $B = K[u_0, u_1, v_0, v_1]$, where u_0, u_1 have weight 1, v_0, v_1 have weight -1, and

$$a = u_0 v_0, \quad b = u_0 v_1, \quad c = u_1 v_0, \quad d = u_1 v_1.$$

Every weight-zero monomial is a product of these four generators. Their only relation is the displayed determinant: after inverting a one solves $d = bc/a$, and a, b, c are algebraically independent under this substitution. The determinant polynomial is prime by its primitive linear form in d and unique factorization, so this localization is injective.

The polynomial ring B is normal by Stacks, Tag 0BC1 and Stacks, Tag 0AFV. If $z \in \operatorname{Frac}(A)$ is integral over A, it is integral over B, so belongs to B. Under the weight substitution with an independent Laurent variable, z is fixed because it is a fraction of weight-zero elements. Comparing Laurent powers shows that its polynomial expression has weight zero, hence lies in A. Thus A is normal. Its localizations are normal by Stacks, Tag 00GY. The remaining projective charts, where one of a, b, c, d is nonzero, solve for a variable and are polynomial three-dimensional charts. This proves normality and integrality of X; projectivity gives properness.

The hyperplane sections satisfy $\operatorname{div}(a) = P_0 + Q$ and $\operatorname{div}(c) = P_1 + Q$, where $Q = \{a = c = 0\}$. Each coefficient is one, as is seen by solving the determinant equation at the corresponding generic point. Their difference is $P_0 - P_1$. Off Z, the two rows are nonzero; where a/c is not directly a unit, it equals b/d, which is a unit there. Thus the support is exactly as claimed. The presentation using the two hyperplane sections adds Q, which is not contained in Z, and is not an allowed presentation.

We prove the obstruction without a class-group computation. Over the affine chart $t = 1$, consider the incidence variety

$$Y = \{((a, b, c, d), [u_0 : u_1]) : au_1 = cu_0,\ bu_1 = du_0\} \longrightarrow \operatorname{Spec} A.$$

It is closed in $\operatorname{Spec} A \times \mathbf{P}^1$ and hence proper over $\operatorname{Spec} A$. Its two charts are

$$(a, b, z), \quad c = za,\ d = zb; \qquad (c, d, w), \quad a = wc,\ b = wd.$$

They are affine three-spaces. The map is an isomorphism off the origin, since a nonzero rank-one matrix determines its column line. Its exceptional fibre is the zero section $C \simeq \mathbf{P}^1$, which has codimension two. In particular there is no exceptional prime divisor. The strict transforms F_0, F_1 of P_0, P_1 are fibres of $Y \to \mathbf{P}^1$, over $[0 : 1]$ and $[1 : 0]$.

Suppose an effective Cartier divisor H near o is supported in $P_0 \cup P_1$, with multiplicities $m, n \geq 0$. Shrink to an affine neighbourhood of o on which H is principal. Its pullback to Y over that neighbourhood has divisor $mF_0 + nF_1$: off C this follows from the isomorphism, and no prime divisor is contained in C. Equality of these Cartier divisors follows on the local UFD charts. The associated line bundle restricts to the trivial line on C, because it was pulled back from a principal divisor. But the restriction of $\mathcal{O}_Y(mF_0 + nF_1)$ is $\mathcal{O}_{\mathbf{P}^1}(m + n)$. In the standard two affine trivializations its transition function is z^{m+n}. A trivialization of the line on each affine chart can change this

only by invertible polynomials, which are constants. Thus triviality forces $m + n = 0$, and hence $m = n = 0$. Both effective factors of a hypothetical signed presentation would therefore have zero coefficients at P_0, P_1 near o, contradicting the coefficients $1, -1$ of $\operatorname{div}(f)$.

The same weight-zero argument over the UFD R shows that the constant affine cone is normal over R; the other charts are polynomial rings over R. This gives a normal proper model, flat by torsion-freeness. The ratio a/c is a unit outside the two corresponding closed row planes, so defines the prescribed model datum. Any proposed generic-identity presentation would restrict to the impossible presentation on X, by Theorem C.1.98. □

Let K be complete and nontrivially discretely valued, with valuation ring R, residue field k, uniformizer π, and the fixed number $\lambda = -\log|\pi| > 0$. Let X be an integral proper K-variety of dimension $d \geq 2$. A model in this note is proper, flat, integral and normal over R, of finite type, with generic fibre identified with X. Retain explicitly the ordinary DVR local rings at all height-one points used below. Its special components have dimension d.

For a proper model $\mathcal{T}$ and closed support S, suppose there are groups $C_r(\mathcal{T}; S)$ of r-dimensional cycles modulo equivalences *retaining that support*. Suppose model pseudo-divisors D act by operators $D\cdot : C_r(\mathcal{T}; S) \to C_{r-1}(\mathcal{T}; S \cap |D|)$. For the objects used, require the following four properties:

1. The operators are additive and commute; the first operation on an integral model is its Cartier divisor cycle when that is defined.
2. Proper pushforward retains the prescribed supports and satisfies $p_*(p^*D\cdot \alpha) = D \cdot p_*\alpha$.
3. On vertical cycles, D acts by the ordinary first Chern operation of its model line, even if its chosen section restricts identically to zero. Moreover $\operatorname{div}(\pi) \cdot [\mathcal{T}] = \sum_V b_V[V]$.
4. Vertical zero-cycles have degree over k, with actual local lengths and full residue-field degrees, and this degree is preserved by proper pushforward.

These are interface hypotheses. In particular, the second property is the supported algebraic projection formula, not F6's analytic theorem.

For a special component V, write b_V for the order of π at its ordinary height-one DVR, hence for its full fibre multiplicity. For the complementary d model lines, the model-measure mass in the following formula is the sum

of the weighted mixed component degrees divided by the complementary product of root exponents. The closing blowup sentence recalls a separate historical illustration.

Proposition C.1.100 (Common domination and the uniformizer test)**.** *Assume the four interface properties for all cycles and operations below. Let $p : \mathcal{Y} \to \mathcal{X}$ be a proper common-generic-fibre domination with $p_*[\mathcal{Y}] = [\mathcal{X}]$. Let D_i be model pseudo-divisors with lines extending $L_i^{\otimes e_i}$ and specified sections $s_i^{\otimes e_i}$, $0 \le i \le d$. Suppose only that*

$$\bigcap_{i=0}^{d} |D_i| \cap X = \varnothing. \tag{C.1.45}$$

No pairwise or intermediate proper intersection is required. With $E_0 = \prod_{i=0}^{d} e_i$, define the supported model value by

$$J_{\mathcal{X}}(D_0, \dots, D_d) = \frac{\lambda}{E_0} \deg_k(D_0 \cdots D_d \cdot [\mathcal{X}]).$$

Then it is symmetric and

$$J_{\mathcal{Y}}(p^*D_0, \dots, p^*D_d) = J_{\mathcal{X}}(D_0, \dots, D_d). \tag{C.1.46}$$

Hence two model presentations have equal values if their divisor data pull back compatibly to a common domination satisfying the same properties. This requires the actual pulled-back lines, sections and supports, not just their generic line-bundle classes.

Replacing s_i by πs_i at a fixed metric gives

$$\Delta J = +\frac{\lambda}{\prod_{j \ne i} e_j} \sum_V b_V \deg_{(\mathcal{L}_j|_V)_{j \ne i}, k}(V). \tag{C.1.47}$$

This equals $+\lambda$ times the corresponding model-measure mass. It is $+\lambda$ times the generic degree only when the relevant specialization-of-degree identity has also been established.

Proof. The support after all $d+1$ operations lies in $\bigcap_i |D_i|$. By (C.1.45) this is vertical, so the degree in the definition is available, even when the total support has positive dimension. The operations produce a zero-dimensional class there. Their commutativity proves symmetry. Iterating the second property gives

$$p_*\big(p^*D_0 \cdots p^*D_d \cdot [\mathcal{Y}]\big) = D_0 \cdots D_d \cdot p_*[\mathcal{Y}] = D_0 \cdots D_d \cdot [\mathcal{X}].$$

The retained vertical degree and identical root exponents prove (C.1.46). Applying this equality to each side of a common domination proves the next assertion. This proof does not manufacture the supported operators or their projection formula.

The model divisor changes by $e_i \operatorname{div}(\pi)$ when s_i is multiplied by π. Additivity, commutativity and the third property turn the change in the final class into

$$e_i \sum_V b_V \left(\prod_{j \neq i} c_1(\mathcal{L}_j|_V) \right) [V].$$

Taking its degree and dividing by E_0/λ proves (C.1.47). The generic support condition is unchanged; the model support is enlarged by the special fibre when needed. On a model with only the standard projective component and degrees m_j, this gives $+\lambda \prod_{j \neq i} m_j$, as required by the point normalization. The blowup computation gives the same mass for pulled-back data, so it is also a common-model regression test. □

Intersections on prescribed supports over a DVR

Let R be a DVR, $S = \operatorname{Spec} R$, η its generic point, and k its residue field. For analytic assertions assume that R is the valuation ring of a complete nontrivially discretely valued field K with its fixed absolute value. Use the dimension function $\delta(\eta) = 1$, $\delta(s) = 0$ at the closed point. A DVR is Noetherian and Cohen–Macaulay: its uniformizer is a regular sequence of length one, and its only other localization is its fraction field. Thus Stacks, Tag 00NM gives the universally catenary hypothesis of Stacks, Tag 02QL. On a finite-type S-scheme, dimensions below are δ-dimensions. In particular, a horizontal integral model of relative dimension d has dimension $d + 1$, and a vertical integral k-scheme has its usual dimension.

For a closed subset T of a finite-type S-scheme $\mathcal{X}$ set

$$C_r(\mathcal{X}; T) = \mathrm{CH}_r(T_{\mathrm{red}}).$$

Rational equivalences use integral subschemes *inside* T. This group is not defined as the image in $\mathrm{CH}_r(\mathcal{X})$. Reduction identifies Chow groups: integral subschemes, their rational functions, and their divisor relations are unchanged. This observation does not replace the fundamental cycle of a nonreduced scheme by that of its reduction; its generic lengths remain coefficients of the cycle. We freely use a scheme-theoretic intersection to construct a cycle and the canonical reduction identification to place its class in the group on its closed support.

Proper pushforward uses the full function-field degree on an integral cycle whose image has the same dimension, and zero when the image has smaller dimension (Stacks, Tag 02R4). It descends to Chow groups by Stacks, Tag 02S2. Its composition law follows on prime cycles from the tower law for field degrees; a dimension decrease remains a decrease under further pushforward. All field degrees include their inseparable parts. If T is vertical and proper over R, then T_{red} is proper over k. Its zero-cycle degree is pushforward to $\operatorname{Spec} k$ (Stacks, Tag 0AZ1); on closed points it is $\sum n_x[\kappa(x) : k]$ (Stacks, Tag 0AZ2). Hence proper pushforward preserves this degree while retaining the prescribed vertical support.

For an invertible sheaf $\mathcal{L}$ and a global section s on $\mathcal{X}$, let $D = Z(s)$. A section here is allowed to be zero or a zero-divisor; “global section” does not mean a regular section in the non-zero-divisor sense. Apply the Gysin

construction of Stacks, Tag 02T8 on T_{red} and then identify its target with the reduced support. Write the resulting operation as

$$G_{(\mathcal{L},s),T} : C_r(\mathcal{X}; T) \longrightarrow C_{r-1}(\mathcal{X}; T \cap |D|).$$

Theorem C.1.101 (Supported Gysin operations on Chow groups)**.** *The maps $G_{(\mathcal{L},s),T}$ are well-defined on Chow groups and satisfy all the following assertions.*

1. *For an integral $W \subset T$, if $s|_W$ is nonzero, the result is the divisor cycle of $s|_W$, with its actual local lengths. If $s|_W = 0$, it is the image of $c_1(\mathcal{L}|_W) \cap [W]$ in the stated target, not an instruction to discard W.*
2. *Let $p : \mathcal{Y} \to \mathcal{X}$ be proper, let $T' \subset \mathcal{Y}$ and $T \subset \mathcal{X}$ be closed with $p(T') \subset T$, and pull back the actual line and section. Then*

$$p_* G_{(p^*\mathcal{L},p^*s),T'}(\alpha) = G_{(\mathcal{L},s),T}(p_*\alpha) \quad \text{in } C_{r-1}(\mathcal{X}; T \cap |D|). \tag{C.1.48}$$

   This includes naturality under enlargement of the prescribed support.
3. *Two such operations commute in the group on $T \cap |Z(s)| \cap |Z(t)|$. No proper-intersection hypothesis is needed.*
4. *For $(\mathcal{M}, t)$ a second pair, set $B = |Z(s)| \cup |Z(t)|$. After pushing the three targets into $T \cap B$,*

$$G_{(\mathcal{L}\otimes\mathcal{M},s\otimes t),T} = G_{(\mathcal{L},s),T} + G_{(\mathcal{M},t),T}. \tag{C.1.49}$$

   After pushing into T, the operation is $c_1(\mathcal{L}|_T) \cap -$.

Proof. The definition gives (1), and Stacks, Tag 02TO proves descent through rational equivalence. Apply Stacks, Tag 02TA to the proper map $T'_{\mathrm{red}} \to T_{\mathrm{red}}$ and the restricted line and section. Its target is the zero scheme of the restricted section on T, so its pushforward identity is exactly (C.1.48) after reduction. Applying this also to closed immersions proves the support-enlargement statement. It does not pass through the image of $\mathrm{CH}(T)$ in the ambient Chow group.

For clarity, the proper-pushforward calculation retains the factor $[K(W') : K(p(W'))]$ when dimensions agree. If an integral W' is contracted, choose a rational section of the relevant line on its integral image W. Its pullback divisor is supported over a proper closed subset of W. Every component then

has image dimension less than $\dim_\delta W' - 1$, and its cycle pushforward is zero. This also handles the contained-component Chern term by its divisor representative. Thus dimension-decreasing images do not create an unrecorded coefficient.

Apply Stacks, Tag 0B73 on T_{red}. The rational equivalence in its proof lies on the intersection of both zero schemes; in particular its target is precisely the support in (3). Reduction and the naturality just proved identify the successive targets used here with those of that lemma.

To prove (4), work on an integral cycle $[W]$. If neither restricted section is zero, divisors of their tensor product add as cycles. Locally this is $\mathrm{ord}(fg) = \mathrm{ord}(f) + \mathrm{ord}(g)$; at a one-dimensional local domain one can see the length equality from $0 \to A/(f) \xrightarrow{g} A/(fg) \to A/(g) \to 0$. If one or both sections vanish identically on W, then $W \subset B$. The product contributes $c_1((\mathcal{L} \otimes \mathcal{M})|_W)[W]$; each nonzero restricted section represents its own first Chern class on W. Additivity of that class, Stacks, Tag 02SP, proves equality already in $\mathrm{CH}(W)$, and hence in the required target. Linearity proves (C.1.49). Finally Stacks, Tag 02T9 identifies the push into T with the first Chern operation. □

A *signed presentation on Z* of an invertible sheaf $\mathcal{L}$ with a nonzero rational section σ on an integral $\mathcal{X}$ consists of two pairs $(\mathcal{L}^+, s^+)$, $(\mathcal{L}^-, s^-)$ with nonzero global sections, an isomorphism $\mathcal{L} \simeq \mathcal{L}^+ \otimes (\mathcal{L}^-)^{-1}$ taking σ to s^+/s^-, and $|Z(s^+)| \cup |Z(s^-)| \subset Z$. For every T, define

$$I_{(\mathcal{L},\sigma;Z),T} = G_{(\mathcal{L}^+,s^+),T} - G_{(\mathcal{L}^-,s^-),T} : C_r(\mathcal{X};T) \longrightarrow C_{r-1}(\mathcal{X};T \cap Z), \quad \text{(C.1.50)}$$

using the closed-immersion pushforwards to this common target.

Proposition C.1.102 (Independence with no extra support)**.** *The operation* (C.1.50) *is independent of the signed presentation on the* same prescribed *Z. It is additive under tensor product with union of supports, commutes with other such operations on their intersection of supports, and satisfies* (C.1.48) *for pulled-back presentations. These assertions remain meaningful when a pulled-back factor section vanishes identically. Its image in $C_{r-1}(\mathcal{X};T)$ is the first Chern operation of $\mathcal{L}|_T$.*

Every rational Cartier section on a locally factorial integral Noetherian $\mathcal{X}$ has a signed presentation on exactly the support of its divisor. Thus the result applies, in particular, to arbitrary rational Cartier sections on locally factorial proper DVR models. On other integral models the existence of the stated presentation is a hypothesis, not a conclusion obtained by enlarging Z.

Proof. Suppose $(A,a)/(B,b)$ and $(A',a')/(B',b')$ present the same $(\mathcal{L},\sigma)$. The specified line identifications give $A\otimes B'\simeq A'\otimes B$, and the global sections $a\otimes b'$ and $a'\otimes b$ agree: they agree at the generic point, and an invertible sheaf on an integral scheme has no section supported off that point. Equation (C.1.49), with all targets pushed to $T\cap Z$, gives $G_a+G_{b'}=G_{a'}+G_b$. Subtraction proves independence. The equality of these global sections persists under every pullback, even if both pull back to zero. Consequently the same calculation proves independence for pulled-back presentations on arbitrary source schemes.

Tensor additivity follows from (C.1.49). Proper compatibility follows by subtracting the two identities (C.1.48). For commutation, expand into four compositions and use Theorem C.1.101(3), then push to the common intersection of the prescribed supports. Naturality for the intermediate closed immersions justifies this last step. The Chern assertion follows by subtraction and Stacks, Tag 02SP.

For existence in the locally factorial case, write the Cartier divisor as the finite Weil sum $D=\sum n_P P$. Let $D^+=\sum \max(n_P,0)P$ and $D^-=\sum \max(-n_P,0)P$. At each point, unique factorization makes each of these effective sums principal. These local equations glue as effective Cartier divisors: on an overlap their quotient has zero exponent at every prime factor of each local UFD, and so is a unit. Their canonical sections of $\mathcal{O}(D^+)$ and $\mathcal{O}(D^-)$ have union of zero supports $|D|$. The usual local identification of $\mathcal{O}(D)$ with $\mathcal{L}$ carries their ratio to σ. These are the required data. This argument does not replace a prescribed support by the support of an auxiliary ample divisor. □

Let R be a complete DVR, with uniformizer π, fraction field K, and residue field k. Equip K with a nontrivial non-Archimedean absolute value whose valuation ring is R. Put $\lambda=-\log|\pi|>0$. Let X be a normal integral proper K-variety of dimension $d\geq 2$. A model is a proper flat finite-type R-scheme with a specified identification of its generic fibre with X. Maps called generic-identity respect these identifications. Such models are integral by Theorem C.1.96; their special fibres may be nonreduced.

A root model presentation of a metric h on a line bundle L is $(\mathcal{T},\mathcal{L},e,\alpha)$, with $e\geq 1$ and $\alpha:\mathcal{L}|_X\xrightarrow{\sim}L^{\otimes e}$ specified. If s is a nonzero rational section of L, its model section is $\sigma=\alpha^{-1}(s^{\otimes e})$. The identification α is part of the datum. An abstract isomorphism of the underlying line bundles does not replace it.

Fix line bundles L_i, root algebraic model metrics h_i, nonzero rational sections s_i, and prescribed closed subsets $S_i \subset X$, for $0 \le i \le d$. Require that s_i is a regular generator off S_i and

$$\bigcap_{i=0}^{d} S_i = \varnothing. \tag{C.1.51}$$

A computing datum consists of an integral model $\mathcal{T}$, extensions $\mathcal{A}_i|_X \simeq L_i^{\otimes e_i}$ inducing h_i, and signed presentations of their model sections on closed sets $Z_i \subset \mathcal{T}$ such that $Z_i \cap X = S_i$. Every factor zero support must lie in its own Z_i. A support is not replaced by the image of its Chow group in the Chow group of $\mathcal{T}$.

Use the dimension function of the supported-calculus packet: $\delta(\operatorname{Spec} K) = 1$ and $\delta(\operatorname{Spec} k) = 0$. Thus $[\mathcal{T}]$ has dimension $d+1$. For a closed support Z put $C_r(\mathcal{T}; Z) = \mathrm{CH}_r(Z_{\mathrm{red}})$. Write I_i for the signed operation supplied by Theorem C.1.101 and Proposition C.1.102. Then

$$\beta_{\mathcal{T}} = I_0 \cdots I_d[\mathcal{T}] \in \mathrm{CH}_0(T_{\mathrm{red}}), \qquad T = \bigcap_i Z_i.$$

By (C.1.51), T_{red} is vertical and proper over k, even if it has positive dimension. Define

$$J_{\mathcal{T}} = \frac{\lambda}{\prod_{i=0}^{d} e_i} \deg_k(\beta_{\mathcal{T}}). \tag{C.1.52}$$

Degree is proper pushforward to $\operatorname{Spec} k$: a closed point contributes its full residue degree, including its inseparable part. The original coherent and divisor lengths stay in the cycle coefficients. There is no extra factorial and no normalization of λ to one.

Lemma C.1.103 (The supported comparison rules needed here)**.** *For computing data, the following changes leave* (C.1.52) *unchanged:*

1. *pullback of the actual data along a proper generic-identity map of integral models, using the inverse-image supports;*
2. *replacement of* $(\mathcal{A}_i, \sigma_i, e_i)$ *by* $(\mathcal{A}_i^{\otimes m_i}, \sigma_i^{\otimes m_i}, m_i e_i)$*, with* $m_i \ge 1$*;*
3. *replacement of signed presentations on one common support;*
4. *enlargement of model supports, provided their total generic intersection is still empty.*

In (1), before degree one has equality in $\mathrm{CH}_0(T_{\mathrm{red}})$ *under the proper map of the final supports. The source need not be normal.*

Proof. Let $q : \mathcal{Y} \to \mathcal{T}$ be as in (1). Repeated supported projection gives

$$q_*\big(I_{q^*0} \cdots I_{q^*d}[\mathcal{Y}]\big) = I_0 \cdots I_d q_*[\mathcal{Y}] = I_0 \cdots I_d[\mathcal{T}].$$

The last equality uses the actual function-field degree one. At every step the target is the prescribed intersection, and the final map is between proper vertical supports. Its zero-cycle pushforward preserves degree over k, including residue degrees. This proves (1).

Signed tensor additivity on the same support gives $I_{\sigma_i^{m_i}} = m_i I_{\sigma_i}$. The factors $\prod_i m_i$ in the numerator and denominator of (C.1.52) cancel, proving (2).

For (3), present one datum as $(A, a)/(B, b)$ and $(A', a')/(B', b')$. The actual line identities and equality of their rational ratios give the global equality $a \otimes b' = a' \otimes b$ on the integral model. Supported Gysin additivity therefore gives

$$G_a + G_{b'} = G_{a'} + G_b$$

in the group on the common prescribed support, and subtraction identifies the signed operations. Naturality for each intermediate closed support permits their iteration with all other operations. The comparison precedes degree; it is not merely equality of ambient Chow classes. Finally the same naturality proves (4). The larger final support is still proper over k, so its inclusion also preserves degree. □

Proposition C.1.104 (A support-sensitive counterexample in every dimension)**.** *Let* $\mathcal{X} = \mathbf{P}^d_R$*,* $d \geq 2$*,* $V = \mathcal{X}_k$*, and take* $(\mathcal{L}, s) = (\mathcal{O}(1), \pi Z_0)$*. With prescribed support* $T = V$*,*

$$G_{(\mathcal{L},s),V}[V] = c_1(\mathcal{O}_V(1)) \cap [V] \neq 0 \quad \text{in } \mathrm{CH}_{d-1}(V).$$

After another $d-1$ *hyperplane operations its degree is one. Thus an operation that discards a component when its restricted section is zero fails the required vertical Chern compatibility.*

More strongly, let $D = Z(\pi Z_0)$*, with closed support* $V \cup H_0$*, where* $H_0 = \{Z_0 = 0\}$*. The same nonzero class maps to zero in* $\mathrm{CH}_{d-1}(D)$*. Knowing only its image in the ambient zero scheme therefore does not determine the operation on the original support* V*.*

Proof. The restricted section is identically zero on V, so the definition gives the first Chern class. Its representative $V \cap H_0$ followed by $d-1$ other coordinate hyperplanes is one rational point with local quotient length one. Successive coordinate quotients compute its degree as one, proving nonvanishing on V. Inside D, the horizontal integral $H_0 \simeq \mathbf{P}_R^{d-1}$ is an allowed scheme for rational equivalence. Its rational function π has divisor $V \cap H_0$ with coefficient one. The image of our class is consequently zero in $\mathrm{CH}_{d-1}(D)$. Inside V that horizontal rational-equivalence relation is unavailable. This is why the operation must be constructed on each T, not recovered from an ambient image after the first intersection. □

For a proper d-dimensional integral k-scheme W and invertible sheaves $M_1, \dots, M_d$, use the mixed degree

$$\delta(W; M_1, \dots, M_d) = \deg_k\big(c_1(M_1) \cdots c_1(M_d) \cap [W]\big).$$

By Stacks, Tag 0BFI this agrees with the usual numerical mixed intersection, with no extra factorial. No ampleness or nefness is required in this mixed formula.

Theorem C.1.105 (Projection for algebraic component degrees)**.** *Let $q : W \to V$ be a proper morphism of proper integral k-schemes, with $\dim W = d$ and $\dim V \le d$, and let $M_1, \dots, M_d$ be invertible sheaves on V. If $\dim q(W) < d$, then*

$$\delta(W; q^*M_1, \dots, q^*M_d) = 0.$$

If q is dominant and $\dim V = d$, then

$$\delta(W; q^*M_1, \dots, q^*M_d) = [k(W) : k(V)]\delta(V; M_1, \dots, M_d). \tag{C.1.53}$$

Proof. Use the global pairs $(M_i, 0)$ on V. Their Gysin operations are exactly $c_1(M_i)$, since their zero schemes are all of V. Iterating (C.1.48) gives

$$q_*\big(c_1(q^*M_1) \cdots c_1(q^*M_d) \cap [W]\big) = c_1(M_1) \cdots c_1(M_d) \cap q_*[W].$$

In the contracted case $q_*[W] = 0$. In the dominant equal-dimensional case it is $[k(W) : k(V)][V]$, with the full field degree. Proper pushforward preserves degree over k, proving both formulas. □

Model metrics and intersections on prescribed supports

Use the complete DVR, model identifications and prescribed supports in Lemma C.1.103, with empty total generic intersection (C.1.51) and supported value (C.1.52).

For an analytic point x, use its valued residue field F_x and valuation ring F_x°. The morphism $\operatorname{Spec} F_x \to X$ has a unique extension to $\operatorname{Spec} F_x^\circ \to \mathcal{T}$ by Stacks, Tag 0BX5. A frame of the pulled-back model line is assigned norm one. For $v \in L(x)$, if $v^{\otimes e} = a\,\alpha(\ell)$ in such a frame, the root norm is $|a|^{1/e}$. A change of frame is a unit of F_x°, of absolute value one. This is the pointwise bounded-frame convention for the supplied algebraic model metrics. No new theorem about existence of arbitrary formal models or continuity of general metrics is assumed.

On a normal model $\mathcal{M}$, the local ring at the generic point of a special component E is an ordinary algebraic DVR. Set $b_E = \operatorname{ord}_E(\pi) > 0$. The corresponding analytic point ξ_E has generic scheme support and

$$|f(\xi_E)| = |\pi|^{\operatorname{ord}_E(f)/b_E} \quad (f \in K(X)^\times).$$

Here the normalized valuation extends $|\cdot|$ on K, and NAG-001707 realizes its restriction on affine K-algebras as an analytic point. The factor b_E is the multiplicity in the full scheme special fibre $\operatorname{div}(\pi)$, not the coefficient of its reduction.

The height-one assertion uses Lemma C.1.17; the minimal primes over the nonzero uniformizer have height one by Stacks, Tag 00KV(1). At ξ_E the restriction of the valuation ring $F_{\xi_E}^\circ$ to $K(X)$ is $\mathcal{O}_{\mathcal{M},\eta_E}$. The map from that ordinary local ring into $F_{\xi_E}^\circ$ is local, because a positive order has norm less than one. Thus the induced map $\operatorname{Spec} F_{\xi_E}^\circ \to \mathcal{M}$ has centre η_E. The uniqueness in Stacks, Tag 0BX5 identifies it with the centre used in the frame definition.

Theorem C.1.106 (Metric equality and compatible powered extensions)**.** *Root model norms are unchanged by generic-identity pullback. Consider two root model presentations $(\mathcal{T}, \mathcal{L}, e, \alpha)$ and $(\mathcal{T}', \mathcal{L}', e', \alpha')$ of L, with norms h, h'. Let $p : \mathcal{M} \to \mathcal{T}$ and $p' : \mathcal{M} \to \mathcal{T}'$ be any common normal model, and let n be a positive common multiple of e, e'. Put*

$$\mathcal{A} = p^*\mathcal{L}^{\otimes n/e}, \qquad \mathcal{B} = p'^*\mathcal{L}'^{\otimes n/e'}.$$

The generic identifications define $\theta_K : \mathcal{B}|_X \xrightarrow{\sim} \mathcal{A}|_X$ *and its rational section* τ *of* $\mathcal{A} \otimes \mathcal{B}^{-1}$*. The following are equivalent:*

1. $h = h'$ *on* L *at every analytic point of* X*;*
2. $h = h'$ *at the finitely many* ξ_E *for the special components of* $\mathcal{M}$*;*
3. θ_K *extends to a unique isomorphism* $\theta : \mathcal{B} \xrightarrow{\sim} \mathcal{A}$ *on* $\mathcal{M}$*.*

In this case θ *identifies the rational sections induced by* $s^{\otimes n}$ *for every nonzero rational* s *of* L*. The conclusion is a compatible isomorphism of actual extensions after pullback and tensor powers. It is not equality of prescribed closed supports.*

Proof. For a generic-identity map of models, composition of the centre on the source with that map is the centre on the target, by uniqueness in Stacks, Tag 0BX5. Pulling back the actual line gives the same lattice in the fibre over F_x; hence the same norm. This also proves pointwise compatibility with tensor powers and duals.

Clearly (1) implies (2). At a special component, a frame of $\mathcal{A} \otimes \mathcal{B}^{-1}$ and the local rational equation of τ give

$$-\log \|\tau(\xi_E)\| = \lambda \frac{\operatorname{ord}_E(\tau)}{b_E}.$$

The norm of τ is the ratio of the two nth-power norms. Thus (2) implies $\operatorname{ord}_E(\tau) = 0$ for every vertical prime divisor. At a horizontal height-one point, τ already generates the line on the generic fibre, so its order is zero there too.

The model is Noetherian by Stacks, Tag 01T6. In a local frame at any point y of $\mathcal{M}$, write $\tau = f$. Its local ring is a Noetherian normal domain; all its height-one localizations are the relevant model height-one DVRs. Both f and f^{-1} belong to all of them. Their intersection is the local ring by Stacks, Tag 031T(2). Hence f is a unit. These local extensions agree on the dense generic fibre and glue, proving (3) and uniqueness. This is the exact additional normal-ring input; the codimension-one vanishing alone was not silently treated as an isomorphism.

Finally an isomorphism in (3) identifies the pulled-back lattices at every analytic point. The frame rule proves (1). Its generic restriction is the specified identity on $L^{\otimes n}$, so it also identifies the stated rational sections. □

Write $D_i = \operatorname{div}(s_i)$. The generic power-presentation condition is:

$$\text{for some } r_i \geq 1, \quad r_i D_i = A_i - B_i, \qquad A_i, B_i \text{ effective Cartier}, \quad |A_i| \cup |B_i| \subset S_i. \tag{C.1.54}$$

All divisor equalities here are actual Cartier equalities, with the associated line and rational-section identifications.

Theorem C.1.107 (Exact acquisition criterion, allowing tensor powers)**.** *Computing data exist if and only if* (C.1.54) *holds for every i. More precisely, start with any finite supplied family of model presentations of the metrics and any model supports for their Cartier data whose generic restrictions are the S_i. After taking suitable positive tensor powers, there is a common normal generic-identity refinement carrying signed presentations on the respective inverse images of those supports. No additional support is needed in this refinement assertion.*

The condition holds for all rational Cartier data on their divisor supports when X is locally factorial. It also holds when every Weil divisor on X has a positive Cartier multiple, in particular for a normal $\mathbf{Q}$*-factorial X.*

Proof. Restriction of any computing presentation to the identified generic fibre gives (C.1.54) with $r_i = e_i$. Nonzero factor sections remain nonzero on that dense integral open. This proves necessity, including after any generic-identity refinement.

For sufficiency take a common normal model of the supplied model presentations. Choose the new exponent for each i to be a common multiple of its model exponents and r_i. The tensor-powered generic data have the required presentations on S_i. Apply precisely Theorem C.1.98 to the actual pulled-back lines, rational sections and inverse-image supports. It produces effective Cartier factors with no extra support and leaves X unchanged. The frame rule preserves the metrics. If no model supports were supplied, one may choose

$$Z_i = \overline{S_i}^{\mathcal{M}} \cup |\mathcal{M}_k|.$$

This is an explicit allowed support containing the entire special fibre, and the datum is a regular generator on its complement. It is an existence choice, not an enlargement of a previously restricted support.

For the last assertion, split the Weil cycle of D_i into its positive and negative parts. Under the stated hypothesis some common multiple makes both parts Cartier. In the locally $\mathbf{Q}$-factorial case, a finite open cover gives such

a common multiple. They are effective Cartier: in every normal local ring their local equations have nonnegative orders at all height-one points, hence are regular by Stacks, Tag 031T(2). The difference is the chosen multiple of D_i, since a rational local equation of order zero everywhere is a unit by the same argument as in Theorem C.1.106. This gives the required presentation. Local factoriality is the case with no powers needed. □

Theorem C.1.108 (Model independence on the prescribed generic supports)**.** *Fix the data of* (C.1.51) *and* (C.1.54)*. The value* (C.1.52) *is a well-defined element of* $\lambda\mathbf{Q}$*, independent of the computing model, all generic-identity refinements, model exponents, signed presentations, and model supports with the fixed generic restrictions* S_i*. It is also independent of the chosen model extensions inducing the same pointwise metrics* h_i*, with their specified generic identifications.*

The value is symmetric under simultaneous permutation of the inputs and supports. It is additive under tensor product in any input when the other inputs are fixed, using the union of the two generic supports and requiring the displayed total-intersection condition. If two systems of generic supports S_i, S_i' *both describe the same metric and rational-section data, their values agree whenever* $\bigcap_i (S_i \cup S_i') = \varnothing$ *and both systems satisfy* (C.1.54)*.*

Proof. Theorem C.1.107 makes the collection of computations nonempty. Compare two of them on a common normal model $\mathcal{N}$. For each input choose a common multiple n_i of their exponents and pull back and tensor-power the actual factor presentations. These steps preserve each value by Lemma C.1.103.

Theorem C.1.106 gives the unique compatible isomorphism between the two powered line extensions on $\mathcal{N}$. It identifies their rational sections, because both restrict to $s_i^{\otimes n_i}$. It does not identify their allowed supports. Instead set

$$W_i = q^{-1}Z_i \ \cup \ q'^{-1}Z_i', \qquad W = \bigcap_i W_i.$$

Here $W_i \cap X = S_i$, so W_{red} is proper over k. Both presentations are allowed on W_i. Push to these common supports, apply the cross-addition identity of Lemma C.1.103, and iterate the identified operations. The resulting zero-classes agree in $\mathrm{CH}_0(W_{\mathrm{red}})$. Taking degree there proves equality of the original values. This also proves independence of the chosen common refinement.

Symmetry is the supported commutation in the reviewed calculus. For tensor additivity choose simultaneous computing data and common exponents. Signed additivity on the union support makes the numerator additive; the common denominators give the assertion. The same comparison works for S_i, S_i' when their componentwise union still has empty total intersection. In each case it is this last condition that makes the final degree comparison legitimate. □

Proposition C.1.109 (Failures of weaker replacement statements)**.** *The following limitations already occur in the present dimensional scope.*

1. *Equal induced model metrics need not make the specified generic isomorphism extend on a nonnormal initial model, even when both underlying model lines are trivial.*
2. *Abstractly equal model extensions, without their generic identifications, can induce different metrics and different values in* (C.1.52)*.*
3. *Normality of X does not imply the generic power-presentation condition on its prescribed divisor support. The quadric example of Proposition C.1.99 fails it for every positive power.*

Proof. For (1) take $R = \mathbf{Q}[[\pi]]$, $K = \mathbf{Q}((\pi))$, $u^2 = 2$, $B = \mathbf{Q}(u)[[\pi]]$ and

$$A = R + \pi B = R \oplus R(\pi u) \simeq R[z]/(z^2 - 2\pi^2) \subset B.$$

This is a finite free integral R-algebra with $A[1/\pi] = K(u)$, and $u \notin A$ is integral over A. The model $\mathcal{T} = \mathbf{P}^d_A$ is proper flat of finite type over R; its normal integral generic variety is $\mathbf{P}^d_{K(u)}$. The special fibre is allowed to be nonreduced: $A/\pi A \simeq \mathbf{Q}[z]/(z^2)$.

Use two copies of $\mathcal{O}_\mathcal{T}$ as extensions of $\mathcal{O}_X$, with generic identifications $1 \mapsto 1$ and $1 \mapsto u$. At every analytic point $|u|^2 = |2| = 1$, so both induced norms of 1 are one. But the compatible generic isomorphism between these two model lines is multiplication by u in one direction. It cannot extend on an affine chart with ring $A[t_1, \dots, t_d]$, since u does not belong to that ring. On the normal model $\mathbf{P}^d_B$ it is a unit and does extend. Normality of this model follows from normality of the DVR B and Stacks, Tag 030A. Thus passing to a common normal model in Theorem C.1.106 is substantive.

For (2) work on $\mathbf{P}^d_R$. Take $L_0 = \mathcal{O}_X$, $s_0 = 1$, $e_0 = 1$, and the two trivial extensions with identifications $1 \mapsto 1$ and $1 \mapsto \pi^{-1}$. Their root norms of 1

are respectively 1 and $|\pi|$. For $1 \le i \le d$ take the standard extension of $\mathcal{O}_X(1)$ with section Z_i. Set $S_0 = \varnothing$ and $S_i = \{Z_i = 0\}$; the total generic intersection is empty. The first computation has model section $\sigma_0 = 1$ and value zero. In the second $\sigma_0 = \pi$. Its first intersection is the full special fibre. Intersecting with $Z_1, \dots, Z_d$ gives the k-rational point $[1 : 0 : \cdots : 0]$ with length one. The second value is therefore $+\lambda$. This verifies the sign without using the pending logarithmic induction theorem.

For (3) use the proper normal quadric $ad - bc = 0$ in $\mathbf{P}^4_K$ and $f = a/c$, with prescribed support $P_0 \cup P_1$, $P_0 = \{a = b = 0\}$, $P_1 = \{c = d = 0\}$. The reviewed obstruction proof establishes that every effective Cartier divisor supported on this union near the vertex has zero coefficients at both planes: its pullback to the displayed small resolution restricts as $\mathcal{O}_{\mathbf{P}^1}(m+n)$ on the exceptional curve, while a principal local divisor restricts trivially; $m, n \ge 0$ force $m = n = 0$. A signed presentation of $r\,\mathrm{div}(f)$ would thus have zero coefficients at both planes, contradicting $r, -r$ for every $r > 0$. □

Purely inseparable base change

In the following example use the supported operation of Theorem C.1.101 and the pulled-back presentations of Proposition C.1.102. Use the dimension function of Lemma C.1.103. All root exponents are one. The native scales are $\lambda_K = -\log c$ and $\lambda_L = -\log |u| = \lambda_K / p$.

Example C.1.110 (Reduction loses a factor)**.** Let $k = \mathbf{F}_p$, $K = k((t))$ and $L = k((u))$ with $u^p = t$. Keep $|t| = c$ with $0 < c < 1$, so $|u| = c^{1/p}$. Take $R = k[[t]]$, $R' = k[[u]]$, and regard $\mathcal{X} = \mathbf{P}^d_{k[[v]]}$ as an R-model via $t = v^p$, where $d \ge 2$. Grouping power-series exponents modulo p identifies $k[[v]]$ with the free R-module on $1, v, \dots, v^{p-1}$. Thus $\mathcal{X}$ is integral, proper and flat over R, and its generic fibre is normal integral over K.

Take exponent one throughout, $(\mathcal{L}_0, \sigma_0) = (\mathcal{O}, v)$ with vertical support, and use d coordinate hyperplane sections of $\mathcal{O}(1)$ for the other factors. Their intersection with $\mathrm{div}(v)$ is one k-point of length one, so $J_R(\mathcal{X}) = \lambda_K$. The full base change is

$$\mathcal{Y} = \mathbf{P}^d_{R'[v]/(v^p - u^p)} \simeq \mathbf{P}^d_{R'[\epsilon]/(\epsilon^p)}, \qquad v = u + \epsilon.$$

It has generic length p along its one reduced component. After the hyperplane operations the quotient by v is $R'/(u^p)$, of length p; multiplication by v is

injective because $v^p = u^p$ and the algebra is R'-free. Equivalently, compute on the full fundamental cycle $p[\mathbf{P}^d_{R'}]$. Thus

$$\deg_{k'} \beta_{\mathcal{Y}} = p, \qquad J_{R'}(\mathcal{Y}) = (\lambda_K/p)p = \lambda_K.$$

Replacing $\mathcal{Y}$ by its reduction before forming its fundamental cycle gives degree one and the incorrect value λ_K/p. Its reduction is a different scheme, not a permissible deletion of generic multiplicity.

For the supported-cycle computation, the sole reduced component upstairs is $\mathbf{P}^d_{R'}$, with coefficient p in the full input cycle. On that component the d coordinate hyperplanes cut out the section $\operatorname{Spec} R'$. Intersecting this section with $v = u$ gives the single $k' = k$ point of length one. Applying the supported operation linearly gives degree p in the actual vertical support. The direct quotient above records the same multiplicity.

Supplied model maps and supported induction

Let K be complete and nontrivially discretely valued, with valuation ring R, residue field k, uniformizer π, and the fixed number $\lambda = -\log|\pi| > 0$. Let X be an integral proper K-variety of dimension $d \geq 2$. A model in this note is proper, flat, integral and normal over R, of finite type, with generic fibre identified with X. Retain explicitly the ordinary DVR local rings at all height-one points used below. Its special components have dimension d.

For a vertical component V use the ordinary local ring $A_V = \mathcal{O}_{\mathcal{X},\eta_V}$ and write $b_V = \mathrm{ord}_V(\pi) > 0$.

A *displayed modification* is a proper morphism $p : \mathcal{Y} \to \mathcal{X}$ inducing the identity on X, together with an open $U \subseteq \mathcal{X}$ containing X and every height-one point of $\mathcal{X}$, such that $p^{-1}(U) \to U$ is an isomorphism. This open is part of the hypotheses. We do not infer it merely from the equality of generic fibres. The complement has codimension at least two. A component W of $\mathcal{Y}_k$ is called exceptional here if it does not dominate a component of $\mathcal{X}_k$.

Lemma C.1.111 (Ordinary divisor coefficients under a supplied modification)**.** *Let p be a displayed modification. For each prime divisor V of $\mathcal{X}$ there is exactly one prime divisor $\widetilde{V}$ of $\mathcal{Y}$ dominating it, and their ordinary local DVRs are identified inside $K(X)$. For a Cartier divisor D with rational section presentation,*

$$p_* \,\mathrm{cyc}(p^*D) = \mathrm{cyc}(D). \tag{C.1.55}$$

The pushforward uses full residue-field degrees on equal-dimensional images and zero on dimension-decreasing images. In particular, for special components, $b_{\widetilde{V}} = b_V$. Every exceptional special component has image of dimension at most $d-1$.

The codimension-one isomorphism hypothesis is checkable for a blowup of a finite-type ideal I with $I|_X = \mathcal{O}_X$: it holds wherever I is locally principal and nonzero, hence on an open containing every height-one point of the normal model. This assertion concerns the blowup itself; any additional normalization or domination needs its own compatibility check.

Proof. Over the generic point of a prime divisor V, the morphism is an isomorphism. Taking the closure of its inverse-image point gives $\widetilde{V}$ and identifies its local ring and residue field with those of V. No second divisor can dominate V, since it would also meet this isomorphic inverse image. Any other

prime divisor lies over the complement of U, so its image has smaller dimension. Properness makes that image closed. The coefficient of p^*D at $\widetilde{V}$ is the coefficient of D at V, because the same rational function is measured in the same DVR. Its pushforward degree is one, and the other coefficients push forward to zero. This proves (C.1.55). Applying the same ordinary DVR calculation to π gives $b_{\widetilde{V}} = b_V$.

At a height-one point v, I_v is a nonzero finitely generated ideal in a DVR and thus is principal, say (a). After shrinking a neighbourhood, a is a section of I and every chosen finite generator of I is a multiple of a; clearing the finitely many localization denominators proves this assertion on an open. On that open, the Rees algebra is $\bigoplus_{n\geq 0}(a^n)T^n \simeq A[t]$, since a is a non-zero-divisor. Its Proj is $\operatorname{Spec} A$. The union of these opens and the generic fibre gives the required U for the blowup. □

Finite measures with supplied atoms and centres

Assume the following actual analytic data on the common Hausdorff X^{an}. For every special component on either model the point ξ_V has generic scheme support, valuation $|f(\xi_V)| = |\pi|^{\operatorname{ord}_V(f)/b_V}$ on $K(X)$, and its indicated centre on that model. The points attached to distinct components of each model are distinct. The points for V and $\widetilde{V}$ agree, and the downstairs centres of the exceptional atoms lie outside U. Assume the two centres of each point are compatible under p and use the prescribed root frame rule on their actual reducing domains, with positive integral root exponents.

For lines $\mathcal{L}_i$ extending $L_i^{\otimes e_i}$, $1 \leq i \leq d$, write

$$E = \prod_{i=1}^{d} e_i, \qquad \delta_V = \deg_{\mathcal{L}_1|_V,\dots,\mathcal{L}_d|_V,k}(V), \qquad \mu_{\mathcal{X}} = \sum_V \frac{b_V \delta_V}{E}\delta_{\xi_V}. \quad \text{(C.1.56)}$$

This is the specified finite model measure, with the ordinary mixed component degree over k. All root exponents, nilpotent lengths and full residue-field degrees are retained. It can be signed. The root model metric has $\|s\| = |a|^{1/e}$ when $s^{\otimes e} = ae_{\mathcal{M}}$ in a model frame of $\mathcal{M}$.

Theorem C.1.112 (Component-degree criterion with supplied model atoms)**.** *Under the point, centre and frame hypotheses above, use $p : \mathcal{Y} \to \mathcal{X}$ and the pulled-back model lines $p^*\mathcal{L}_i$ with the same root exponents. Then the following are equivalent:*

1. *The two finite measures on the common* X^{an} *are equal.*
2. *Their component degrees satisfy*

$$\delta_{\widetilde{V}} = \delta_V \quad \text{for every } V, \qquad \delta_W = 0 \quad \text{for every exceptional } W. \tag{C.1.57}$$

Under this criterion all continuous tests, and all Borel tests finite on the common finite support, have equal integrals. In particular, if M *has a model* $\mathcal{M}$ *extending* $M^{\otimes e_0}$*, the pullback model metric is the same metric on* M*. Every nonzero rational section* s *satisfies*

$$\int -\log\|s\|\, d\mu_{\mathcal{Y}} = \int -\log\|s\|\, d\mu_{\mathcal{X}}. \tag{C.1.58}$$

The logarithm is used off its zero/pole support and may be extended arbitrarily across that null set for these atomic measures. Both are finite total-variation integrals. If $a_V = \mathrm{ord}_V(s^{\otimes e_0})$ *in a model frame, the common downstairs expression is*

$$\frac{\lambda}{e_0 E}\sum_V a_V \delta_V, \qquad -\log\|s(\xi_V)\| = \frac{\lambda a_V}{e_0 b_V}. \tag{C.1.59}$$

Replacing s *by* πs *changes it by* $+\lambda\mu_{\mathcal{X}}(X^{\mathrm{an}})$*.*

Proof. The strict-transform atoms agree by hypothesis, and their b's agree by Lemma C.1.111. An exceptional atom cannot equal one of these: its centre on $\mathcal{X}$ is in $\mathcal{X}\setminus U$, whereas the strict-transform atom has centre $\eta_V \in U$. Distinct components of $\mathcal{Y}$ also give distinct atoms. Comparing the coefficients of each Dirac mass in (C.1.56), and using $b_W > 0$, proves precisely the equivalence with (C.1.57). This argument allows signed degrees; no cancellation between distinct atoms is possible. Equality of the finite measures proves equality for every stated test.

A frame of $\mathcal{M}$ pulls back to a frame of $p^*\mathcal{M}$ at every point reducing over that frame. The coefficient of a generic section is the same rational function, evaluated at the same analytic point. The root metric formula therefore proves equality of these metrics on all of X^{an}. The prescribed centres are compatible under p, so the comparison uses the same reducing domains. Every atom has generic underlying scheme point, so a nonzero rational section is regular and nonvanishing there. The root norm gives the second formula in (C.1.59); multiplication by its mass $b_V\delta_V/E$ gives the first. This proves finiteness without any singular limiting argument. Finally a_V becomes $a_V + e_0 b_V$ for πs, giving the positive change $\lambda\sum_V b_V\delta_V/E$. □

Algebraic induction on prescribed supports

Use the dimension function and supported Gysin operations of Theorem C.1.101 and the signed operations of Proposition C.1.102. Use the mixed degree convention preceding Theorem C.1.105. The ordinary height-one DVRs are the rings of Lemma C.1.17. In an integral finite-type flat model, a minimal prime over the nonzero uniformizer has height one by Stacks, Tag 00KV(1).

Now let K be complete and nontrivially discretely valued with ring R, and put $\lambda = -\log|\pi| > 0$. Let $\mathcal{X}$ be an integral normal proper flat model of relative dimension $d \geq 2$. For $0 \leq i \leq d$, let $\mathcal{L}_i$ extend $L_i^{\otimes e_i}$, $e_i \geq 1$, and let $\sigma_i = s_i^{\otimes e_i}$ be a nonzero rational model section. Specify signed presentations on closed supports Z_i satisfying

$$\bigcap_{i=0}^{d} Z_i \cap X = \varnothing, \qquad X = \mathcal{X}_K. \tag{C.1.60}$$

Write I_i for the supported operations, $E = \prod_{i=0}^{d} e_i$, and $T = \bigcap_i Z_i$. Define

$$J_{\mathcal{X}} = \frac{\lambda}{E} \deg_k\big(I_0 \cdots I_d[\mathcal{X}]\big), \qquad I_0 \cdots I_d[\mathcal{X}] \in \mathrm{CH}_0(T_{\mathrm{red}}). \tag{C.1.61}$$

The value is defined on this model data. This construction also applies to the horizontal integral cycles occurring below, using the remaining pulled-back factor presentations, even when a factor vanishes identically.

Theorem C.1.113 (Supported projection and algebraic horizontal–vertical induction)**.** *The value* (C.1.61) *is independent of the permitted signed presentations and symmetric in all $d+1$ inputs. Let $p : \mathcal{Y} \to \mathcal{X}$ be a supplied proper dominant map of integral proper flat models of the same relative dimension, and put $N = [K(\mathcal{Y}) : K(\mathcal{X})]$. With the actual pulled-back factor data and supports on $\mathcal{Y}$, one has*

$$J_{\mathcal{Y}} = N J_{\mathcal{X}}. \tag{C.1.62}$$

Normality of $\mathcal{Y}$ is not needed for this supported equality.

At the ordinary height-one DVRs of $\mathcal{X}$, write

$$\mathrm{div}_{\mathcal{L}_0}(\sigma_0) = \sum_{H \text{ horizontal}} a_H[H] + \sum_{V \text{ vertical}} a_V[V].$$

Let J_H be the construction (C.1.61) on H using the remaining d pulled-back presentations and denominator $\prod_{i=1}^{d} e_i$. Put $\delta_V = \delta(V; \mathcal{L}_1|_V, \dots, \mathcal{L}_d|_V)$. Then

$$J_{\mathcal{X}} = \frac{1}{e_0} \sum_H a_H J_H + \frac{\lambda}{e_0 \prod_{i=1}^{d} e_i} \sum_V a_V \delta_V. \tag{C.1.63}$$

Thus the horizontal coefficients are exactly those of $\operatorname{div}(s_0)$ on the generic fibre. At fixed model lines, identifications and root exponents, replacing s_i by πs_i and enlarging its support by the special fibre when needed changes the supported value by

$$\Delta J = \frac{\lambda}{\prod_{j \neq i} e_j} \sum_V b_V \, \delta\big(V; (\mathcal{L}_j|_V)_{j \neq i}\big).$$

Only (C.1.60) is required; no pairwise proper intersection or generic nonvanishing on every intermediate cycle is imposed.

Proof. The operations produce a zero-class on T after $d+1$ steps. Condition (C.1.60) makes T_{red} a proper k-scheme, so its degree is defined even if T has positive dimension. Presentation independence and commutation prove the first assertions. For the supplied proper map, repeated supported projection gives

$$p_*(I_{p^*0} \cdots I_{p^*d}[\mathcal{Y}]) = I_0 \cdots I_d p_*[\mathcal{Y}] = N I_0 \cdots I_d[\mathcal{X}].$$

The supports remain the displayed inverse images; degree compatibility proves (C.1.62). This proves a statement about a supplied map and compatible data, not existence of a common model.

Apply I_0 first. Each factor section in its signed presentation is nonzero on $\mathcal{X}$. Its Gysin image of $[\mathcal{X}]$ is therefore its divisor cycle, and subtraction gives the displayed cycle of σ_0 on Z_0. A positive coefficient at a height-one point is the length of the quotient by a local equation in the ordinary DVR; differences give the signed coefficient. Apply the remaining operators to each component, using proper compatibility for its closed immersion. On a horizontal H, the remaining generic supports have empty total intersection: $H \subset Z_0$ and (C.1.60) applies. This contribution is $(a_H/e_0)J_H$. The value J_H uses the restricted factor presentations, not a purported restriction of a rational section that vanishes identically on H.

On a vertical V, push the remaining supported intersection class into $\mathrm{CH}_0(V)$. The support-forgetting first Chern assertion of Theorem C.1.101,

its signed extension in Proposition C.1.102, and proper compatibility show that the result is $c_1(\mathcal{L}_1|_V)\cdots c_1(\mathcal{L}_d|_V)[V]$. This push stays inside the proper vertical scheme V, so it preserves the relevant degree. It gives $a_V\delta_V$ and proves (C.1.63). Localization at π identifies the horizontal DVR with its generic-fibre DVR. Since $\sigma_0 = s_0^{e_0}$, its horizontal coefficient is e_0 times the coefficient of $\operatorname{div}(s_0)$.

Finally $s_i \mapsto \pi s_i$ changes the model divisor by $e_i \operatorname{div}(\pi)$ and may enlarge its support by the special fibre. Its first supported operation on $[\mathcal{X}]$ gives $e_i \sum_V b_V[V]$. Commutation and the same vertical calculation give

$$\Delta J = \frac{\lambda}{\prod_{j\neq i} e_j} \sum_V b_V \delta(V; (\mathcal{L}_j|_V)_{j\neq i}),$$

with positive sign. □

Local criteria for affinoid neighbourhoods and collars

Theorem C.1.114 (An ordinary affinoid neighbourhood gives affine reduction)**.** *If x has an ordinary affinoid neighbourhood in an analytic space U, the graded birational reduction of (U, x) is affine.*

Proof. Choose such a neighbourhood $V = \mathcal{M}(B)$. By reduction on actual analytic germs and ordinary affinoid neighbourhood invariance, $\mathcal{R}(U, x)$ is the reduction computed on V in the same graded completed residue field. Choose an actual admissible absolute weighted Tate presentation for B. The fixed absolute presentation makes its graded reduction finite over the image of the graded polynomial algebra. Its further evaluated image in the graded residue field is therefore a finitely generated graded algebra. Its valuation locus is affine by the definition of the reduction of the affinoid representative. □

Theorem C.1.115 (Absolute interior on an actual affinoid domain)**.** *For an affinoid domain D of an affinoid Y,*

$$\mathrm{Int}(D/K) = \mathrm{Int}_{\mathrm{top}}(D, Y) \cap \mathrm{Int}(Y/K).$$

Proof. The source-factor inclusion applied to $D \to Y \to \mathcal{M}(K)$ sends an absolutely interior point of D into $\mathrm{Int}(D/Y)$. The affinoid-domain criterion makes D an ordinary neighbourhood there. The ordinary local test then gives absolute interior in Y. Conversely the same domain criterion together with forward composition gives absolute interior in D. □

C.2 The exact whole-fibre boundary and capture condition

Fix a complete non-Archimedean field K with its specified real-valued absolute value, possibly trivial. All affinoids are generalized affinoids, with arbitrary positive real presentation radii. Finite maps are norm-admissible finite: their given Banach structures are equivalent to the canonical finite-module structures. All domains, fibres and modules below have their actual analytic structures; nilpotents are retained. Write $t_g = -\log|g|$ for a tuple of analytic units. For a closed rectangular box $P \subset \mathbb{R}^m$, $\mathbb{T}_P$ is the entire inverse image of P in the analytic m-torus. The point η_s is its weighted Gauss point at $s \in P$.

The frozen local finite criterion 2.49.1 already reconstructs Berkovich's ordinary-neighbourhood theorem. It does not assert that a neighbourhood of η_s contains the entire exact polytorus $\mathbb{T}_s$. The next criterion makes the missing quantifier explicit: every point of $t_g^{-1}(s)$, not just the points above η_s, must be controlled.

Theorem C.2.1 (Compact ambient separation and completion over a source)**.** *Let $Y = \mathcal{M}(A)$ be a generalized affinoid. Let $U, V \subset Y$ be affinoid domains, and let $g_i \in (A^\times)^{m_i}$, $1 \le i \le r < \infty$, be actual unit tuples. Put $q_i = t_{g_i}$, choose $s_i \in q_i(Y)$, and define*

$$K_i = q_i^{-1}(s_i), \qquad G = \mathrm{Int}(Y/K) \cap \mathrm{Int}_{\mathrm{top}}(U, Y) \cap \mathrm{Int}_{\mathrm{top}}(V, Y), \qquad B = Y \setminus G. \tag{1}$$

The following are equivalent.

1. *For every i, $s_i \notin q_i(B)$, or equivalently the* entire *K_i is contained in G.*
2. *There are full-dimensional closed rectangular boxes P_i, with $s_i \in \mathrm{int}(P_i)$, such that*

$$C_i = q_i^{-1}(P_i) \subset U \cap V, \qquad g_i : C_i \longrightarrow \mathbb{T}_{P_i} \quad \textit{is norm-admissible finite.} \tag{2}$$

   The inverse image defining C_i is taken in all of Y.
3. *Every K_i is contained in $\mathrm{Int}_{\mathrm{top}}(U, Y) \cap \mathrm{Int}_{\mathrm{top}}(V, Y)$, and the analytic map $g_i : Y \to (\mathbb{G}_m^{m_i})^{\mathrm{an}}$ is finite at* every point of *K_i in the ordinary-neighbourhood sense.*

When these conditions hold, the P_i may be chosen inside any prescribed real neighbourhoods of their centres, and one may require $C_i \subset G$. In particular,

$$(q_i|_U)^{-1}(P_i) = (q_i|_V)^{-1}(P_i) = q_i^{-1}(P_i) \tag{3}$$

as actual analytic domains. Over every analytic target point of $\mathbb{T}_{P_i}$, these identities retain the entire analytic fibres in Y, U, V, including all local factors. Compatible finite modules on U and V give the same complete module on these fibres.

There is a separate criterion when Y is only a domain in a larger intended source. More generally, let Ω be a good Hausdorff analytic space, let $Z \subset \Omega$ be an affinoid domain, and suppose that actual unit tuples on Ω restrict to norm-admissible finite saturated collars over boxes P_i^0 in Z, with $s_i \in \mathrm{int}(P_i^0)$. There exist smaller such boxes $P_i \subset P_i^0$ with

$$q_i^{-1}(P_i) \text{ in } \Omega = (q_i|_Z)^{-1}(P_i) \tag{4}$$

if and only if

$$s_i \notin \overline{q_i(\Omega \setminus Z)} \quad \text{for every } i. \tag{5}$$

The closure is taken in $\mathbb{R}^{m_i}$. *The equality in (4) is again an equality of analytic domains and entire fibre algebras, not just of selected Gauss lifts.*

Proof. The set G is ordinary open in the compact Hausdorff space Y. Indeed absolute interior is open, and the other two sets are topological interiors. Hence B and $q_i(B)$ are compact. The domain-interior and composition identities give, as subsets of Y,

$$G = \operatorname{Int}(U/K) \cap \operatorname{Int}(V/K). \tag{6}$$

Here 2.44.3 identifies relative interior of a domain with its topological interior, and C.1.115 is applied to $U \to Y \to \mathcal{M}(K)$ and to $V \to Y \to \mathcal{M}(K)$.

Assume condition 1. Since $K_i \subset \operatorname{Int}(Y/K)$, the frozen whole-moment-fibre theorem 5.38.4, applied to the tuple g_i, supplies a finite saturated box about s_i. Because the compact image $q_i(B)$ misses s_i, shrink that box to avoid $q_i(B)$ and to lie in the prescribed real neighbourhood. Finite base change preserves the canonical finite norm. Its whole inverse image C_i is contained in G. This can be done for all the finitely many i; the ambient source and the two intended domains stay fixed. This proves condition 2 and the stated simultaneous choices. The finiteness theorem itself is inherited, not reproved here.

Conversely assume condition 2 and let $y \in K_i$. The target point $g_i(y)$ lies in $\operatorname{Int}(\mathbb{T}_{P_i}/K)$: all its defining polyannulus bounds are strict because $s_i \in \operatorname{int}(P_i)$. A finite map is everywhere relatively interior by 2.33.18, so composition gives $y \in \operatorname{Int}(C_i/K)$. Since $C_i \subset U, V$, the domain restriction theorem makes C_i an affinoid domain of both. Applying the composition identity to $C_i \to U \to \mathcal{M}(K)$ and to $C_i \to V \to \mathcal{M}(K)$ places y in both $\operatorname{Int}(U/K)$ and $\operatorname{Int}(V/K)$. Equation (6) gives $y \in G$, proving condition 1. Thus merely placing K_i in the closed subsets U, V would not suffice; the necessary interior condition has been deduced from finiteness.

We verify the ordinary-local formulation as well. Choose a large closed polyannulus $\mathbb{T}_{R_i}$ so that all coordinate bounds strictly exceed the norms of g_i and g_i^{-1}, and the whole image $g_i(Y)$ is in its ordinary and absolute interior. It gives a bounded map $Y \to \mathbb{T}_{R_i}$. At a point where g_i is finite in the ordinary sense, the frozen local criterion and ordinary local testing give relative interiority for this affinoid map. Composing with $\mathbb{T}_{R_i} \to \mathcal{M}(K)$

gives absolute interiority in Y. Thus condition 3 implies 1. Under condition 2 the whole analytic fibre over any point with moment s_i is a fibre of the finite map $C_i \to \mathbb{T}_{P_i}$, and is therefore finite and discrete. In particular every point of K_i is isolated in its analytic fibre in Y. The already proved condition 1 gives absolute interiority, hence relative interiority over $\mathbb{T}_{R_i}$. The frozen local finite criterion gives ordinary local finiteness at every such point. This proves condition 3. No flatness hypothesis was used.

The set equalities (3) follow from $C_i \subset U \cap V$ and saturation in Y. The actual domain universal property and domain base change identify their analytic structures. Further base change to any analytic target point identifies the entire fibre algebras. For the module assertion, fix an actual isomorphism between the restrictions of the two finite modules to $U \cap V$. Restrict it to C_i, then to the fibre. Finite-module tensor comparison makes these the complete canonical modules, and every such finite-module isomorphism is bounded with bounded inverse. Nothing in these operations takes a reduction or discards a field or local Artinian factor.

Finally consider Ω and Z. If (5) holds, choose $P_i \subset \operatorname{int}(P_i^0)$ about s_i disjoint from $\overline{q_i(\Omega \setminus Z)}$. Every inverse-image point then belongs to Z, proving (4). The resulting map is finite by restriction of the given finite collar in Z. Conversely, (4) implies $P_i \cap q_i(\Omega \setminus Z) = \varnothing$. The open set $\operatorname{int}(P_i)$ contains s_i and is disjoint from that image, so s_i is outside its closure. Domain base change gives the analytic assertion as before. No compactness of $\Omega \setminus Z$ was assumed; this is exactly why the closure in (5) is retained. □

The ordinary neighbourhood step uses the local finite criterion and the ordinary local test on the larger torus chart.

Finite base extension of supported model intersections

Let K be complete for a specified nontrivial discretely valued non-Archimedean absolute value. Put

$$R = K^\circ, \qquad k = R/\mathfrak{m}_R, \qquad \lambda_K = -\log|\pi| > 0$$

for a uniformizer π of R. Let X be a normal integral proper K-variety of dimension $d \geq 2$, and let $\mathcal{X}$ be an integral proper flat finite-type R-model of X. The initial model need not be normal.

For $0 \leq i \leq d$, fix a line bundle L_i on X, a nonzero rational section s_i, and a closed set S_i off which s_i is a regular generator. Assume $\bigcap_i S_i = \varnothing$.

A computing datum consists of positive integers a_i, invertible sheaves $\mathcal{L}_i$, specified identifications $\mathcal{L}_i|_X \simeq L_i^{\otimes a_i}$, and signed presentations of the induced rational sections $\sigma_i = s_i^{\otimes a_i}$. Thus

$$(\mathcal{L}_i, \sigma_i) \simeq (\mathcal{L}_i^+, s_i^+) \otimes (\mathcal{L}_i^-, s_i^-)^{-1}, \qquad |Z(s_i^+)| \cup |Z(s_i^-)| \subset Z_i, \qquad Z_i \cap X = S_i,$$

where the factor sections are nonzero global sections. All line identifications and all supports are part of the data.

We use the supported projection formula (2) and the signed-presentation construction for the following operations.

$$C_r(\mathcal{X}; T) = \mathrm{CH}_r(T_{\mathrm{red}}), \qquad I_{i,T} : C_r(\mathcal{X}; T) \longrightarrow C_{r-1}(\mathcal{X}; T \cap Z_i).$$

The dimension convention assigns 1 to the generic point of $\operatorname{Spec} R$ and 0 to its closed point; thus $[\mathcal{X}]$ has dimension $d+1$. Set

$$A = \prod_{i=0}^{d} a_i, \qquad T = \bigcap_i Z_i, \qquad \beta_{\mathcal{X}} = I_0 \cdots I_d[\mathcal{X}] \in \mathrm{CH}_0(T_{\mathrm{red}}), \qquad J_R(\mathcal{X}) = \frac{\lambda_K}{A} \deg_k \beta_{\mathcal{X}}. \tag{C.2.1}$$

The support T_{red} is proper over k, although it need not be zero-dimensional. Degree uses the full residue-field degrees. Passing to a reduced support identifies the Chow groups; it never changes the coefficients of an input fundamental cycle.

A local model frame has norm one, and the specified line identification gives the a_i-th root of its coefficient norm. The proper centre is the unique extension from the fraction field of the relevant valuation ring. Stacks Tag 0BX5

Choose a finite extension as in the valuation-ring calculation. Keep the entire fibre product $\mathcal{Y} = \mathcal{X} \times_R R'$ with projection $q : \mathcal{Y} \to \mathcal{X}$. The schemes and all prescribed inverse-image supports carry their actual structures.

Lemma C.2.2 (Pullback of the data and of the root norms)**.** *The scheme $\mathcal{Y}$ is proper, flat and of finite type over R'; q is finite locally free of rank n. Its generic fibre is $X_L = X \times_K L$. The actual pulled-back lines, sections and identifications have the same exponents a_i. With $Z_i' = q^{-1}Z_i$ and $T' = q^{-1}T$, their supported operators I_i' define*

$$\beta_{\mathcal{Y}} = I_0' \cdots I_d'[\mathcal{Y}] \in \mathrm{CH}_0(T'_{\mathrm{red}}).$$

The support T'_{red} is proper over k'. Their pointwise root model norms are the pullbacks of the original norms, using the original extending absolute value.

Proof. On an affine chart B of $\mathcal{X}$ the fibre product has ring $B \otimes_R R'$, locally free of rank n over B by Lemma C.1.16. Properness and finite type persist by base change. Flatness over R' follows by tensoring an exact sequence of R'-modules with the flat R-module B. Localizing at ϖ gives $B \otimes_R L$; these identifications glue to the stated generic fibre.

Pullback preserves all specified line and cross-product identities, and the zero schemes of the factor sections pull back to their scheme-theoretic inverse images. This is exactly the arbitrary-source pullback scope of `f9sc:signed`, not a new assertion about arbitrary rational data on a nonreduced scheme. Every component of $\mathcal{Y}$ dominates $\mathcal{X}$ by Lemma C.1.38; hence it is horizontal and has dimension $d+1$. The total generic support is empty because it is the inverse image of $\bigcap_i S_i$. Thus the final reduced support is vertical and proper over k', and the supported operations give the stated zero-class.

For the norm statement let $y \in X_L^{\mathrm{an}}$ and $x = q(y)$. The valued residue field at y contains the valued residue field at x isometrically. The centre on $\mathcal{X}$ extended to the valuation ring at y, together with its map from R', gives the centre on $\mathcal{Y}$ by the fibre-product property. Uniqueness is the valuative criterion for properness, Tag 0BX5, as in the proper-centre argument. A local model frame pulls back to a frame. If a vector to the a_ith power has coefficient c in this frame, both norms are $|c|^{1/a_i}$. Units have norm one. This also handles the actual generic identifications and their tensor powers. □

The degree calculation uses this full pullback of the model and its data, and the full finite flat cycle.

Theorem C.2.3 (Finite base extension and numerical descent)**.** *For the full pulled-back computing datum,*

$$q_*\beta_{\mathcal{Y}} = n\beta_{\mathcal{X}} \quad \textit{in } \mathrm{CH}_0(T_{\mathrm{red}}), \qquad \deg_{k'} \beta_{\mathcal{Y}} = e \deg_k \beta_{\mathcal{X}}. \tag{C.2.2}$$

Consequently its native R'-normalized value is

$$J_{R'}(\mathcal{Y}) := \frac{\lambda_L}{A} \deg_{k'} \beta_{\mathcal{Y}} = J_R(\mathcal{X}). \tag{C.2.3}$$

If the same upstairs data are instead regarded over R, with the R-normalization, then

$$J_{\mathcal{Y}/R} := \frac{\lambda_K}{A} \deg_k \beta_{\mathcal{Y}} = nJ_{R'}(\mathcal{Y}) = nJ_R(\mathcal{X}). \tag{C.2.4}$$

These formulas hold for nonreduced $\mathcal{Y}$ and nonreduced X_L, respect finite extension towers, and are unchanged by tensor-power replacement of any root presentation.

Proof. View both models over R in order to apply proper projection. Their dimension functions agree with those over R': the generic and closed residue extensions are algebraic, so they add no transcendence degree. At every stage use the actual inverse-image intersection of supports. The proper map between these reduced supports is the restriction of q. Repeatedly applying `f9sc:refined`(2) to the two global factors and subtracting as in `f9sc:signed` gives

$$q_*(I'_0 \cdots I'_d[\mathcal{Y}]) = I_0 \cdots I_d(q_*[\mathcal{Y}]) = nI_0 \cdots I_d[\mathcal{X}],$$

where the last equality is Lemma C.1.38. This proves the first identity in (C.2.2) on the prescribed support. Neither a flat-pullback identity for Chow groups nor a proper-intersection hypothesis has been used.

For any zero-cycle class γ on the proper k'-scheme T'_{red}, the tower law at each closed point gives

$$\deg_k(q_*\gamma) = \deg_k \gamma = f \deg_{k'} \gamma.$$

Here the first degree treats T'_{red} as a k-scheme; proper pushforward and degree are the frozen conventions Tag 02S2, Tag 0AZ2. Thus $f \deg_{k'} \beta_{\mathcal{Y}} = n \deg_k \beta_{\mathcal{X}}$. By Lemma C.1.16, $n/f = e$ and $\lambda_L = \lambda_K/e$, which prove (C.2.3). Using the original base k and λ_K instead gives (C.2.4). The factor n in that formula is not a factor in the native value.

Replacing $(\mathcal{L}_i, \sigma_i, a_i)$ by $(\mathcal{L}_i^{\otimes b_i}, \sigma_i^{\otimes b_i}, b_i a_i)$ multiplies its signed operation by b_i, by the inherited tensor additivity on the same support. The product of these integers cancels the denominator. In a finite tower the full base-change models associate, the fundamental-cycle transfer ranks multiply, and both ramification and residue degrees multiply. Applying the identities twice therefore gives the direct identities. □

Finite analytic descent and direct images

Fix a complete field K with its specified real-valued non-Archimedean absolute value. Trivial valuation and generalized affinoid radii are allowed. Spaces in this supplement are good and Hausdorff. Coherent modules live on the analytic G-site. A line bundle means a coherent module which is free of rank one on ordinary open neighbourhoods. The symbol q^* on coherent modules denotes ordinary tensor pullback; it is not a derived functor.

For a norm-admissible finite morphism $q : Y \to X$, the full inverse image of $V = \mathcal{M}(A)$ is $\mathcal{M}(B)$, where B is finite over A with its canonical finite-module norm. If N is finite over B, compose controlled finite free presentations $A^a \twoheadrightarrow B$ and $B^b \twoheadrightarrow N$. Their lifting bounds compare the iterated quotient norm with the canonical finite-A-module norm. Thus restriction of scalars also preserves the canonical topology on N. Every tensor used below between finite modules has its canonical topology. The comparison with the completed analytic tensor is provided by NAG-001565 and NAG-001702. Algebraic finiteness with an unrelated Banach norm would not suffice.

Let L/K be a finite field extension and let $n = [L : K]$. Give L the unique extending absolute value, without rescaling it. Write $p : Y = X_L \to X$. In every direct image and every descent fibre product below, Y is regarded as a K-analytic space. This use of restriction of constants keeps the same Banach algebra; it is not the Weil restriction functor.

Lemma C.2.4 (The finite tensor bridge)**.** *For a generalized K-affinoid algebra A, put $B = A\widehat{\otimes}_K L$. There are natural algebra identifications*

$$B = A \otimes_K L, \qquad \underbrace{B\widehat{\otimes}_A \cdots \widehat{\otimes}_A B}_{r \text{ factors}} = A \otimes_K L^{\otimes_K r} \quad (r \geq 1). \tag{C.2.5}$$

Every algebra on the right has its canonical finite-A-module topology. In particular $A \to B$ is faithfully flat and finite free of rank n. For a finite A-module M, $B \otimes_A M = M \otimes_K L$ is the actual analytic pullback. These identifications commute with affinoid restrictions.

Direct inputs: NAG-001450, NAG-001561, NAG-001565, NAG-001685, NAG-001689, NAG-001701, NAG-001702.

Proof. Choose a K-basis of L containing 1. The finite-dimensional norm theorem makes its coordinate maps bounded in both directions. Tensoring

these maps with A identifies $A\widehat{\otimes}_K L$ with the complete space A^n, with equivalent norms. The maps are inverse on elementary tensors and hence on their completions. Their multiplication is the ordinary tensor multiplication. The scalar-presentation theorem gives the native L-affinoid structure, and the finite-algebra theorem gives the K-affinoid structure with the same Banach topology. A bounded K-seminorm on B restricts to the specified absolute value on L by uniqueness. Thus the two spectra have the same points and evaluation topology. All sheaves and fibre products in this supplement are taken on the finite K-analytic model just constructed.

Finite base change now identifies the completed tensors over A with ordinary tensors, proving (C.2.5) by iteration. The same finite-module comparison applies to M. A free A-module of positive rank is faithfully flat: tensoring an A-module with B gives its n-fold direct sum. The formulas and their inverses commute with the maps $A \to A_V$, which proves restriction compatibility. □

Full inverse images of ordinary affinoid neighbourhoods in X are affinoid by finite base change. Their interiors contain the inverse images of the downstairs interiors, so Y is good. Points with distinct images are separated by the Hausdorff property of X. Two distinct points above the same x lie in the open inverse image of the interior of one affinoid neighbourhood of x. This open set is a subspace of a Hausdorff affinoid inverse image and separates the points. Thus Y is Hausdorff as well.

The lemma is only the finite coefficient bridge needed for descent. No Gauss-field degree or native integral comparison is derived here. In particular the double space for descent is

$$Y^{[2]} = Y \times_X Y, \qquad \mathcal{O}(Y^{[2]}|_V) = A \otimes_K (L \otimes_K L),$$

not a product formed by replacing $L \otimes_K L$ by L. Write $q_1, q_2 : Y^{[2]} \to Y$ for its two projections.

Definition C.2.5. A coherent descent datum is a coherent module $\mathcal{F}$ on Y_G and an isomorphism $\delta : q_1^*\mathcal{F} \xrightarrow{\sim} q_2^*\mathcal{F}$ whose pullbacks to $Y^{[3]} = Y \times_X Y \times_X Y$ satisfy $\delta_{23}\delta_{12} = \delta_{13}$. Morphisms commute with δ. The isomorphism and its cocycle are identities of modules on the full, possibly nonreduced, fibre products.

Theorem C.2.6 (Effective coherent and line-bundle descent)**.** *Pullback along p gives an equivalence between $\mathrm{Coh}(X_G)$ and coherent descent data on Y/X. It restricts to an equivalence of line bundles with descent data. The descent construction commutes with affinoid restriction and tensor products.*

Direct inputs: Lemma C.2.4, NAG-001561, NAG-001624, NAG-001752, NAG-001753, and the already registered STACKS-023N(1).

Proof. First work on $V = \mathcal{M}(A)$. Let N represent $\mathcal{F}$ on $\mathcal{M}(B)$. Lemma C.2.4 converts the analytic datum into an ordinary $B \otimes_A B$-linear isomorphism

$$\varphi : N \otimes_A B \xrightarrow{\sim} B \otimes_A N$$

with the ordinary triple-tensor cocycle. The exact effectiveness statement of Stacks Tag 023N(1) applies because $A \to B$ is faithfully flat. It provides an effective A-module.

For completeness, identify the inverse and morphisms rather than using the parts of Tag 023N whose proofs are omitted there. Set

$$M = \{z \in N : \varphi(z \otimes 1) = 1 \otimes z\}. \tag{C.2.6}$$

Choose the A-linear map $r : B \to A$ induced by a K-linear map $L \to K$ taking 1 to 1. For a canonical effective datum $N = B \otimes_A M_0$, invariance says, on writing $z = \sum_i b_i \otimes m_i$,

$$\sum_i b_i \otimes 1 \otimes m_i = \sum_i 1 \otimes b_i \otimes m_i.$$

Applying r to the first factor shows $z = 1 \otimes \sum_i r(b_i) m_i$. Conversely every $1 \otimes m$ is invariant, and the map $m \mapsto 1 \otimes m$ is injective by the same retraction. Thus (C.2.6) recovers M_0 and $B \otimes_A M \to N$ is an isomorphism.

Since A is Noetherian and N is finite over the finite A-algebra B, the submodule M is finite. The difference defining (C.2.6) is a map of finite A-modules. Its kernel is closed and has its canonical complete norm by the finite-module theorem. The effective isomorphism is bounded with bounded inverse.

A morphism of data preserves the equalizer (C.2.6). For canonical data it is therefore the base change of exactly one map of the descended modules. This proves full faithfulness as well as the specified inverse. For a domain algebra A_V, the module $M \otimes_A A_V$ has the restricted datum as its pullback;

the just proved uniqueness identifies it with the equalizer constructed there. Thus no unsupported commutation of a kernel with arbitrary completion is used. Coherent equivalence and these canonical restriction maps glue the construction on ordinary affinoid neighbourhoods of X. On overlaps the isomorphisms agree by full faithfulness. Tensor products are compatible by associativity and the same uniqueness.

It remains to check line bundles. An invertible coherent module on an affinoid corresponds to an invertible finite module: tensor it with its dual, and use affinoid coherent equivalence to identify evaluation and coevaluation. The resulting finite dual-basis identity makes its module finite projective. Explicitly, if coevaluation is $\sum_i u_i \otimes \lambda_i$, then $u = \sum_i \lambda_i(u) u_i$ splits a surjection from a finite free module.

Thus, for a line bundle upstairs, N is finite projective over B, hence finite projective over A, because B is finite free over A. The identification $N = B \otimes_A M$ and the retraction r make M an A-module direct summand of N. Consequently M is finite projective over A. For every prime $\mathfrak{p}$ of A choose a prime of the nonzero finite algebra $B \otimes_A \kappa(\mathfrak{p})$. After its residue-field extension, $M \otimes_A \kappa(\mathfrak{p})$ has dimension one, because N has rank one. The same dimension is therefore one before that field extension. This proves rank one at every prime.

The associated coherent module has flat ordinary stalks; by NAG-001753 it is free on ordinary neighbourhoods. Its rank is one, as may also be checked after evaluation in $\mathcal{H}(x)$. This proves the assertion for line bundles. □

Lemma C.2.7 (The topological coequalizer)**.** *The map $p : Y \to X$ is a closed surjection and hence a quotient map. Every pair $y_1, y_2 \in Y$ with $p(y_1) = p(y_2)$ lifts to a point of $Y^{[2]}$ with projections y_1, y_2. Consequently pullback identifies $C^0(X, \mathbb{R})$ with*

$$\{u \in C^0(Y, \mathbb{R}) : q_1^* u = q_2^* u \text{ on } |Y^{[2]}|\}.$$

Direct inputs: Lemma C.2.4, NAG-001450, NAG-001496, NAG-001692, NAG-001702, and STACKS-00JB.

Proof. At $x \in X$, put $H = \mathcal{H}(x)$. The actual fibre algebra is $C = H \otimes_K L$ by the finite-tensor and affinoid-fibre theorems. It is a nonzero n-dimensional H-algebra. Each residue field of an Artinian local factor is finite over the

complete field H and therefore has the unique extending absolute value. It defines a point of the fibre. Thus p is surjective.

If F is closed in Y and V is an affinoid neighbourhood in X, then $F \cap p^{-1}(V)$ is closed in the compact affinoid $p^{-1}(V)$. Its image is compact, hence closed in the Hausdorff space V. That image is exactly $p(F) \cap V$. The ordinary interiors of such V cover X, so $p(F)$ is closed. A continuous closed surjection is a quotient map.

For the pair assertion, let $E_i = \mathcal{H}(y_i)$, a finite extension of H. The nonzero finite-dimensional H-algebra $E_1 \otimes_H E_2$ has a field quotient F'. Both E_i embed in F'. Uniqueness of the extending absolute values makes these embeddings isometric. The compatible evaluations to F' are bounded; by the finite analytic tensor universal property they define a point of $Y^{[2]}$ with the required projections. This uses the entire tensor, whether or not it is reduced.

The equalizer condition on u now says exactly that u is constant on every fibre of p. It defines a unique function v on X with $u = v \circ p$. For an open set $O \subset \mathbb{R}$, the preimage of $v^{-1}(O)$ is $u^{-1}(O)$; the quotient property proves continuity of v. The converse follows by composition. □

Definition C.2.8. A continuous metric on a line bundle $\mathcal{M}$ assigns a norm on the one-dimensional $\mathcal{H}(x)$-vector space $\mathcal{M}(x)$ for every x. For each ordinary local frame e, the function $\varphi_e(x) = -\log \|e(x)\|$ must be finite and continuous. Scalar homogeneity uses the specified absolute value on $\mathcal{H}(x)$. An isometry of metrics on $Y^{[2]}$ is checked after pullback to the completed residue field of each of its points.

Theorem C.2.9 (Descent of continuous metrized line bundles)**.** *Let $(\mathcal{M}_Y, \delta)$ be a line bundle with the full analytic descent datum of Theorem C.2.6. Let $\|\cdot\|_Y$ be a continuous metric. It is the pullback of a unique continuous metric on the descended line bundle, relative to its specified descent isomorphism, if and only if*

$$\delta : (q_1^*\mathcal{M}_Y, q_1^*\|\cdot\|_Y) \longrightarrow (q_2^*\mathcal{M}_Y, q_2^*\|\cdot\|_Y)$$

is an isometry. Tensor products, duals and isometric isomorphisms are compatible with this descent.

Direct inputs: Theorems C.2.6 and Lemma C.2.7.

Proof. Descend the analytic line bundle first. On an ordinary open set U with descended frame e, let $\widetilde{e}$ be its frame on $p^{-1}(U)$ under the specified descent isomorphism. The function $u = -\log \|\widetilde{e}\|_Y$ is continuous. The cocycle sends the two pullbacks of $\widetilde{e}$ to each other. Its isometry condition therefore gives $q_1^* u = q_2^* u$ above U. Lemma C.2.7, also valid on every open restriction, gives a unique continuous function v_e on U with $p^* v_e = u$.

If $e' = he$ on an overlap, then upstairs $u_{e'} = u_e - \log |p^* h|$. Surjectivity of p implies $v_{e'} = v_e - \log |h|$. Hence $\|ae(x)\| = |a| \exp(-v_e(x))$ is independent of frame and defines the required continuous metric. The necessity and uniqueness follow by pulling back frame weights. Sums and negatives of weights give tensor and dual compatibility. An analytic isomorphism commuting with descent data descends by Theorem C.2.6; whether it is an isometry is checked by the surjective map p. □

The result is a statement about continuous metrics. It does not prove that a continuous descended weight is an ordinary smooth moment function, a model function, or a semipositive potential. Those are distinct regularity conditions on representatives.

Rank for norm-admissible finite maps

Here $q : Y \to X$ may be any norm-admissible finite morphism between the fixed K-spaces, with $q_* \mathcal{O}_Y$ locally free of constant rank $n \geq 1$ on ordinary opens. In particular this applies to the finite constant-extension map above. We work with algebraic determinants of coherent modules, before introducing any metric on a norm bundle.

Lemma C.2.10 (Rank of the finite direct image)**.** *For an analytic line bundle $\mathcal{L}$ on Y, the module $q_* \mathcal{L}$ is locally free of rank n on ordinary opens of X.*

Direct inputs: NAG-001624, NAG-001752, NAG-001753, STACKS-00JB; the finite dual-basis argument in Theorem C.2.6.

Proof. Choose an affinoid neighbourhood on which $B = \Gamma(q^{-1}V, \mathcal{O}_Y)$ is free of rank n over $A = \Gamma(V, \mathcal{O}_X)$. The module $P = \Gamma(q^{-1}V, \mathcal{L})$ is invertible over B. The dual-basis argument makes P a direct summand of a finite free B-module, so it is finite projective over A. For any residue field k of A, the algebra B_k is Artinian of k-dimension n. An invertible module on each local

Artinian factor is free of rank one: lift a nonzero residue generator and apply Nakayama; projectivity splits the resulting surjection, and its kernel has zero residue and hence is zero. It follows that $\dim_k P_k = \dim_k B_k = n$. Thus P is projective of constant rank n. Its coherent sheaf has flat ordinary stalks, and NAG-001753 gives ordinary local freeness of that rank. □

Pointed weighted germs and maximal-rank excision

The finite-module chain and its weights. Let $X = \mathcal{M}(A)$ be pure of analytic dimension d over a complete non-Archimedean field with its specified absolute value, and let a unit tuple f define $t_f : X \to \mathbb{R}^n$. For every finite A-module M there is a canonical finite nonnegative integral d-dimensional polyhedral chain

$$\mathcal{T}_f(M).$$

Its support is contained in $t_f(X)$. On any common finite subdivision of the actual image and $t_f(\partial_K X)$, the weight of a d-cell is the module weight $m_{M,f}(v)$ at any point of its relative interior. Cells of dimension less than d give no d-chain term. The chain is additive in short exact sequences, independent of subdivision. For $M = A$ every occurring d-cell has positive weight.

Proof. Theorem 5.39.3 gives a finite polyhedral image of dimension at most d and a finite boundary image of dimension at most $d - 1$. Choose one finite common subdivision with the boundary image as a subcomplex. At a point in the relative interior of a d-cell, the whole moment fibre is therefore interior, and the local image is contained in that cell's affine d-plane. Theorem 5.40.2 defines the nonnegative integer weight there. It is locally constant, by Theorem 5.42.1 on small convex neighbourhoods with the actual local fan presentations of Theorem 5.38.6. The relative interior of the cell is convex, so the weight is constant there. For the structure module it is positive by Theorem 5.37.1.

On a common subdivision of two choices, the generic weights compute the same whole-fibre module invariant. Thus their chains agree. Additivity follows from the exact-sequence formula of Theorem 5.40.2. □

The finite-cell functional. Let $X = \mathcal{M}(A)$ be a pure generalized affinoid of dimension d over the specified complete field, f a finite tuple of analytic units, and M a finite A-module. Use the canonical chain from the finite-module chain and weights and its weights m_σ. For a smooth (d, d)-superform α on an open neighbourhood of the actual compact image, define

$$\mathcal{I}_{M,f}(\alpha) = \sum_\sigma m_\sigma \int_\sigma \alpha.$$

This is independent of subdivision and is nonnegative when the tangential top coefficients of α are nonnegative.

Proof. All cells with nonzero weights are compact polytopes: the actual moment image is compact, and the complex is finite. Consequently the integral is a finite sum of the integrals constructed in Lemma 5.46.2. That lemma gives subdivision independence and positivity. □

Adjoining unit coordinates on the same affinoid. With the same pure generalized affinoid $X = \mathcal{M}(A)$, finite A-module M, and finite unit tuple f, let g be another finite unit tuple on X. Let a be the projection from the combined coordinates to the f-coordinates. For the canonical finite chains and finite-cell functionals, and every smooth top superform α near the actual f-moment image,

$$a_*\mathcal{T}_{(f,g)}(M) = \mathcal{T}_f(M), \qquad \mathcal{I}_{M,(f,g)}(a^*\alpha) = \mathcal{I}_{M,f}(\alpha).$$

Proof. Take the finite map $p = \mathrm{id}_X$, with $f_B = (f, g)$ and $f_A = f$. Each target unit is the corresponding coordinate of the joint tuple. In the following calculation the two chains and integrals are given by the finite-module chain and weights and the finite-cell functional. For this finite morphism, take finite global subdivisions of the two actual images and their boundary images by finite polyhedrality. Refine along the images of all source cells, using the finite linear elimination and hyperplane construction in the finite refinement calculation. Outside a finite union of polyhedra of dimension at most $d-1$ in each target d-cell, the target whole fibre is interior, and every source weight above it lies in a full-rank d-cell. The entire inverse weight set is finite, with at most one point per such source cell. For this finite identity, relative interior and composition give absolute interior at every point of those source fibres.

The local star of a whole interior fibre supplies pure local images near the target weight and at each source weight above it. Finite patching retains the complete source inverse support in a finite pure presentation on a small convex target neighbourhood in the target local image presentation. Theorem 5.42.2 then identifies the finite lattice-index sum with the target weight of M as an A-module. Its empty-inverse- image case gives zero on both sides. Thus the generic weight of every target d-cell is exactly its pushforward weight. The resulting equality of finite chains is $a_*\mathcal{T}_{f_B}(M) = \mathcal{T}_{f_A}(M)$. This proves the chain formula for the coordinate projection on the same affinoid.

The full actual source support is compact, so this pushforward is proper. On each full-rank source cell, Lemma 5.46.2 gives the factor $[N_\tau : a(N_\sigma)]$

in the integral. On a rank-dropping cell the top-degree pullback is zero. Grouping these finite cell integrals by their target cells proves the integral identity. No ambient support cutoff is needed here, because all the integration polytopes are compact. This proves the integral formula on the same affinoid. □

Fix a complete field K with its specified real-valued non-Archimedean absolute value; the valuation may be trivial. Affinoid algebras are generalized, with arbitrary positive real presentation radii. Finite maps and finite modules carry their canonical Banach structures. Let $X = \mathcal{M}(A)$ be nonempty and pure of analytic dimension $d \geq 1$, let $f = (f_1, \ldots, f_n) \in (A^\times)^n$, and let M be a finite A-module. Write $\mathcal{F}$ for its coherent sheaf and put

$$t = t_f = -\log|f|, \qquad F_w = t^{-1}(w), \qquad r_f(x) = \operatorname{trdeg}_{\widetilde{K}^{\mathrm{gr}}} \widetilde{K}^{\mathrm{gr}}(\widetilde{f_1(x)}, \ldots, \widetilde{f_n(x)}).$$

The graded field is taken inside $\widetilde{\mathcal{H}(x)}^{\mathrm{gr}}$. Thus $0 \leq r_f(x) \leq d$. Set

$$S_f(X) = \{x : r_f(x) = d\}, \qquad S_w = S_f(X) \cap F_w.$$

For an affinoid domain $U = \mathcal{M}(A_U) \subset X$, write $M_U = M \otimes_A A_U$ and use the already constructed finite integral d-chain $\mathcal{T}_{f|U}(M_U)$ of the finite-module chain. Domain purity is 5.35.3. Two such chains have the same *germ at* w when their restrictions to some real neighbourhood of w agree after a common finite subdivision. Equality is determined by the weights on the relative interiors of the d-cells. Only the chain and generic-weight clauses of that frozen theorem are used here, not its integral or pushforward clauses.

Let $T^d = (\mathbf{G}^d_{m,K})^{\mathrm{an}}$, let η_s be its Gauss point of weight $s \in \mathbf{R}^d$, and put $H_s = \mathcal{H}(\eta_s)$ and $\mathbb{T}_s = t_T^{-1}(s)$. The last space is the entire exact-radius polytorus. An ordinary affinoid neighbourhood contains an ordinary open neighbourhood; it need not contain the entire polytorus at the weight of the chosen point. All interiors without a topological subscript are analytic interiors. Compactness and Hausdorffness of affinoids are 2.24.1; the ordinary rational-neighbourhood basis is 2.41.1. Actual domain intersections and finite-module restrictions use 2.36.9 and 2.38.4.

Lemma C.2.11 (Gauss branches and the finite maximal-rank set)**.** *For any analytic map $\psi : X \to T^d$, the underlying set of $\psi^{-1}(\eta_s)$ is finite. If x belongs to this fibre and $x \in \mathrm{Int}(X/K)$, there are arbitrarily small ordinary affinoid neighbourhoods $Y = \mathcal{M}(B)$ of x and $N = \mathcal{M}(C)$ of η_s such that $\psi : Y \to N$ is norm-admissible finite and $\psi^{-1}(\eta_s) \cap Y = \{x\}$. One can take C integral. The ordinary stalk $\kappa_s := \mathcal{O}_{T^d,\eta_s}$ is a field, and, with*

$$E = B\widehat{\otimes}_C H_s = B \otimes_C H_s, \qquad P = M \otimes_A E,$$

E is the actual local Artinian fibre algebra at x, with residue field $\mathcal{H}(x)$. There are natural identifications

$$E = \mathcal{O}_{X,x} \otimes_{\kappa_s} H_s, \qquad P = \mathcal{F}_x \otimes_{\kappa_s} H_s,$$

and the finite integer

$$\delta_\psi(M, x) := \dim_{H_s} P = \dim_{\kappa_s} \mathcal{F}_x = \mathrm{length}_E(P)[\mathcal{H}(x) : H_s] \tag{C.2.7}$$

is independent of the isolating neighbourhoods. After shrinking N ordinarily and taking the inverse image in Y, the direct images of $\mathcal{O}_Y$ and $\mathcal{F}|Y$ are free, of ranks $\delta_\psi(A, x) > 0$ and $\delta_\psi(M, x)$ respectively. Moreover, writing $\beta_{\mathfrak{p}}(M) = \mathrm{length}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$, one has

$$\delta_\psi(M, x) = \sum_{\mathfrak{p} \in \mathrm{Min}(A)} \beta_{\mathfrak{p}}(M) \dim_{H_s}(E/\mathfrak{p}E). \tag{C.2.8}$$

For every unit tuple f and every w, the set S_w is finite, without an interior assumption on its points.

Proof. Choose a compact polyannulus containing $\psi(X)$. The actual fibre over η_s is affinoid over H_s; 2.33.11 identifies its points and completed residue fields with the points over η_s and their original completed residue fields. The Gauss dimension, tower formula, and Abhyankar bound give

$$0 \le d_{H_s}(y) = d_K(y) - d_K(\eta_s) \le d - d = 0$$

for every such point. Here the dependencies are 3.22.4, 3.22.2, and 3.22.7. The maximum formula therefore makes the fibre zero-dimensional. A zero-dimensional generalized affinoid has finitely many points: strictify it, use dimension invariance and strict finite normalization to make the strictified algebra finite over its field, and project its finite spectrum surjectively back.

These are exactly 1.17.6, 3.21.8, 1.12.2, 3.21.1, 1.14.4, 2.35.1, and 2.33.13. No finiteness assertion over the whole $\mathbb{T}_s$ follows.

We give the field-stalk argument over the original K. If a generalized affinoid D has analytic dimension at most d and $z \in \mathcal{M}(D)$ has $d_K(z) = d$, then $\mathfrak{p} = \ker(D \to \mathcal{H}(z))$ is minimal. Otherwise choose a minimal prime $\mathfrak{q} \subsetneq \mathfrak{p}$. Simultaneously strictify $D/\mathfrak{q}$ and $D/\mathfrak{p}$ by a finite K-free extension, making the new field nontrivially valued as well. The domain-preservation and quotient statements of 5.35.1, together with faithful flatness 1.18.8, preserve the strict inclusion of these primes. The strictified domain has Krull dimension at most d; quotienting by a nonzero prime lowers it to at most $d - 1$, since every prime chain in the quotient can be preceded by the zero prime of the domain. Analytic dimension invariance gives $\dim(D/\mathfrak{p}) \leq d - 1$, contradicting the Abhyankar bound applied to z on that quotient. Here equality of strict analytic and Krull dimensions is the strict dimension theory in 3.21.8 and 1.12.2.

An affinoid neighbourhood of η_s in T^d is an affinoid domain of affine d-space. It is regular by 2.35.23, and is a finite product of integral affinoid algebras by 2.52.2. Pass to the open-and-closed component containing η_s. It is still an ordinary neighbourhood. Its evaluation kernel is minimal by the preceding paragraph, hence zero. If a germ at η_s evaluates to zero, represent it on such an integral neighbourhood inside its domain of definition. It is zero there. Evaluation on the ordinary stalk is therefore injective. The description of its maximal ideal in 2.33.16 makes that ideal zero, so this local ring is the asserted field κ_s. This argument does not identify κ_s with its completion.

The torus is boundaryless over K by 2.33.27, and absolute interior at x implies relative interior for ψ there: test on ordinary affinoid neighbourhoods and use 2.44.6 and the source-factor implication. The finite set already proved makes x isolated in its analytic fibre. The local finite criterion 2.49.1, applied inside an ordinary neighbourhood disjoint from the other fibre points, gives ordinary open finite neighbourhoods. Restrict the target to an ordinary affinoid neighbourhood, and then to its integral component at η_s. Finite base change gives the asserted $Y \to N$; Y remains an ordinary neighbourhood of x.

The finite direct-image and stalk identities 2.35.15 and 2.35.17 give

$$B \otimes_C \kappa_s = \mathcal{O}_{X,x}, \qquad (M \otimes_A B) \otimes_C \kappa_s = \mathcal{F}_x.$$

There is only one summand because the isolated analytic fibre has only x. The finite tensor comparison 2.33.19 then gives the displayed identifications with E and P, retaining their actual canonical complete norms. A finite algebra over the complete field H_s has one analytic point for each local Artinian factor. Thus E is local, and its residue field is precisely $\mathcal{H}(x)$ by 2.33.11. A composition series of P has copies of this residue field as its simple factors, proving (C.2.7). Since κ_s is a field, both finite direct-image stalks are free. Apply 2.35.16 to each, and intersect the resulting ordinary neighbourhoods. This derives freeness from the Gauss stalk; no generic-flatness assumption has been made.

For (C.2.8), the fraction-field argument applies: $C \hookrightarrow H_s$ factors through $\mathrm{Frac}(C)$, so H_s is flat over C. The domain map $A \to B$ is flat by ordinary single-domain flatness; hence $A \to E$ is flat. The algebraic composition-series argument of the local Artinian calculation now applies verbatim to this actual Artinian algebra: flat going down makes its prime contract to a minimal prime of A, and tensoring a composition series at that prime gives the displayed sum. Every other minimal-prime quotient of E is zero. In particular neither nilpotent length nor the full residue-field degree is discarded.

Finally, if $r_f(x) = d$, some d of the n homogeneous elements $\widetilde{f_i(x)}$ are algebraically independent. For that coordinate subset I, the map $\psi_I = (f_i)_{i\in I}$ sends x to η_{w_I}: for every Laurent polynomial the nonzero initial terms of maximal norm cannot cancel, which is exactly the weighted Gauss norm. There are finitely many choices of I, and each $\psi_I^{-1}(\eta_{w_I})$ is finite by the first paragraph. Their union contains S_w. If $n < d$, S_w is empty. □

Theorem C.2.12 (Cofinal pointed weighted germs)**.** *Let $x \in \mathrm{Int}(X/K)$ and $w = t(x)$. There is an ordinary affinoid neighbourhood Y_0 of x such that the germs at w of*

$$\mathcal{T}_{f|U}(M_U) \qquad (x \in \mathrm{Int}_{\mathrm{top}}(U, X),\ U \subset Y_0)$$

are identical for all ordinary affinoid neighbourhoods U. They define a germ $\mathcal{C}_{x,f}(M)$, independent of all auxiliary admissible generic projections and their finite Gauss neighbourhoods. It is additive in M. If $r_f(x) < d$, this germ is zero. If $r_f(x) = d$, it is a nonnegative integral weighted rational d-fan germ on the pure pointed fan $w + C_x$ of 5.55.8; for $M = A$ its support is the whole of that fan germ.

In the case $r_f(x) = d$, a projection $Q : \mathbb{Z}^n \to \mathbb{Z}^d$ is admissible for this proof if it is injective on the span of every d-cone of C_x. For each such Q a finite ordinary Gauss chart as in Lemma C.2.11 exists at x for $\psi = f^Q$. In each such chart Y_Q, for every ordinary affinoid neighbourhood $U \subset Y_Q$ of x, there are a real neighbourhood B of w and finitely many proper affine hyperplanes $H \subset \mathbf{R}^d$ such that

$$t^{-1}(v) \cap Y_Q = t^{-1}(v) \cap U \quad (v \in B,\ Qv \notin H).$$

Both full fibres are then interior in their respective source domains. The inverse images of H have dimension at most $d-1$ on each d-cone.

If C_x is the d-plane $N_{\mathbb{R}}$ for a saturated rank-d lattice $N \subset \mathbb{Z}^n$, the germ has constant weight

$$\mu_{x,f}(M) = \frac{\delta_{fQ}(M, x)}{[\mathbb{Z}^d : Q(N)]} \in \mathbb{Z}_{\geq 0} \tag{C.2.9}$$

for every integral Q injective on N. No analogous single quotient of the total projected degree is asserted at a nonflat fan.

Proof. Write $r = r_f(x)$. The full pointed-fan theorem supplies W_0 such that every ordinary neighbourhood of x inside W_0 has moment-image germ $w+C_x$, of pure dimension r. If $r < d$, no such image has a d-dimensional cell near w, and the support clause of finite-module chain support proves that the chain germ is zero. This includes the case $n < d$.

Suppose $r = d$. Choose an integral Q injective on the span of every maximal cone: the failure for each of the finitely many rational d-planes is a proper determinant-zero condition on a $d \times n$ matrix; a rational matrix outside their finite union exists, and clearing denominators gives the required integer matrix. For every sufficiently small ordinary neighbourhood of x, its f-image contains the pointed d-cones, and their Q-images have dimension d. Apply the full pointed-fan theorem again, now to the d units f^Q. It follows that $r_{fQ}(x) = d$. As in the last paragraph of Lemma C.2.11, this means $f^Q(x) = \eta_{Qw}$.

That lemma gives a finite ordinary affinoid chart $\psi : Y_0 \to N_0$, with central Gauss fibre $\{x\}$. Choose it inside W_0 and inside the inverse image of a sufficiently small real ball where the pointed upper bound holds. Consequently

$$t(Y_0) \subset w + C_x. \tag{C.2.10}$$

These source restrictions are allowed by the arbitrarily small neighbourhood clause of the local finite criterion. They do not replace N_0 by the entire central polytorus.

Fix an ordinary affinoid neighbourhood $U \subset Y_0$ of x. The compact set

$$D = Y_0 \setminus \mathrm{Int}_{\mathrm{top}}(U, Y_0)$$

does not meet the Gauss fibre. Its image is compact and closed in the Hausdorff torus, so

$$O = \mathrm{Int}_{\mathrm{top}}(N_0, T^d) \setminus \psi(D)$$

is an ordinary open neighbourhood of η_{Qw}. Choose a compact full polyannulus with Qw in the interior of its weight box. Apply the frozen whole-torus overlap lemma C.1.43 to O intersected with this polyannulus. There are a real neighbourhood B' of Qw and finitely many proper affine hyperplanes H such that

$$\mathbb{T}_s \subset O, \qquad \psi^{-1}(\mathbb{T}_s) \cap Y_0 \subset \mathrm{Int}_{\mathrm{top}}(U, Y_0) \quad (s \in B' \setminus H). \tag{C.2.11}$$

This assertion retains every analytic point over every such $\mathbb{T}_s$, including all of its Gauss lifts in Y_0.

For v near w with $Qv \in B' \setminus H$, equation (C.2.11) identifies the entire f-fibre in Y_0 with the entire f-fibre in U. These are identical analytic domains, and the restricted modules are identical. The target polytorus is contained in the ordinary interior of N_0 in the boundaryless torus. Its inverse image is therefore absolutely interior in Y_0, by finite relative interior and composition. Its points lie in the ordinary interior of U in Y_0, so they are absolutely interior in U as well. The precise locality inputs are 2.33.18, 2.44.3, and forward composition with ordinary target testing.

The inverse image of every hyperplane in H cuts each d-cone of C_x in dimension at most $d-1$, since Q is injective on its span. Make a common finite subdivision of the two image germs, these cuts, and their faces. At generic points of every d-cell, the local image lies in its affine d-plane and the whole fibre is interior on both sides. The exact-domain identification above and 5.40.2 therefore give identical weights, using the saturated tangent lattice of that cell. This proves equality of the two d-chain germs. Finite polyhedrality makes each germ a weighted rational fan after shrinking at w; for the structure module, positivity in finite-module chain positivity and full stabilization give exactly the asserted support. Restriction $M \mapsto M_U$

is exact by ordinary single-domain flatness. Additivity then follows from the finite-module chain.

For clarity, the comparison just proved first compares each $U \subset Y_0$ with Y_0. Two such U's can be compared through a common ordinary affinoid neighbourhood of x inside their ordinary interiors. Charts constructed using a different admissible projection can likewise be compared through a common smaller ordinary affinoid neighbourhood, applying the same argument in each chart. Thus both the cofinal germ and the full-fibre comparison off the indicated cuts are independent of the chosen projection. Different nearby Gauss lifts have not been identified or removed; they occur in the same whole fibre being compared.

Finally suppose $C_x = N_{\mathbb{R}}$. Use the free direct-image shrinking in Lemma C.2.11, retaining (C.2.10). For $s = Qv$ near Qw outside the hyperplanes, the finite inverse image of $\mathbb{T}_s$ in this chart equals the single original moment fibre at v: Q is injective on the entire plane containing the moment image. Its actual Gauss module has dimension $\delta_{fQ}(M, x)$, because the direct image is free of that rank. The interior flat-germ formula 5.40.2 divides this number by exactly $[\mathbb{Z}^d : Q(N)]$. It determines the weight on a dense subset of every d-cell, and hence the constant weight germ. Equation (C.2.7) keeps the actual local length and full field degree, while (C.2.8) keeps the generic component lengths. □

Theorem C.2.13 (Excision of lower-rank central loci in top degree)**.** *Let $w \in t(X)$ and suppose only that*

$$S_w \subset \operatorname{Int}(X/K). \tag{C.2.12}$$

Choose pairwise disjoint ordinary affinoid neighbourhoods U_x of all $x \in S_w$, sufficiently small for Theorem C.2.12, and contained in $\operatorname{Int}(X/K)$. Then

$$[\mathcal{T}_f(M)]_w = \sum_{x \in S_w} \mathcal{C}_{x,f}(M). \tag{C.2.13}$$

More precisely, there are a real neighbourhood B of w and a finite union E of rational polyhedral fan germs translated to w, each of dimension at most $d-1$, such that for $v \in B \setminus E$ the entire original source fibre satisfies

$$F_v = \coprod_{x \in S_w} (F_v \cap U_x) \subset \bigcup_{x \in S_w} \operatorname{Int}_{\mathrm{top}}(U_x, X). \tag{C.2.14}$$

The assertion permits positive-dimensional analytic projection fibres and boundary points elsewhere in the central fibre F_w. If S_w is empty, the entire image germ has dimension at most $d-1$ and the d-chain germ is zero.

Proof. Lemma C.2.11 makes S_w finite. Hausdorffness, the ordinary affinoid-neighbourhood basis, and the openness of analytic interior allow the indicated disjoint U_x. Their ordinary interiors in X cover every point of S_w. The residual compact set

$$C = F_w \setminus \bigcup_{x \in S_w} \mathrm{Int}_{\mathrm{top}}(U_x, X)$$

consists entirely of points y with $r_f(y) < d$. For each such y, the pointed upper bound 5.55.7 supplies an ordinary affinoid neighbourhood V_y, a real neighbourhood of $w = t(y)$, and a finite rational fan C_y of dimension $r_f(y)$ such that $t(V_y)$ in that neighbourhood is contained in $w + C_y$. This bound requires neither interior nor purity at y. Choose finitely many V_y whose ordinary interiors cover C, and let E be the union of their translated fans near w. It has dimension at most $d-1$.

Let G be the union of these ordinary interiors and the ordinary interiors of all U_x. It is an ordinary open set containing F_w. The complement $X \setminus G$ is compact; its image under t is compact and does not contain w. Shrink a real neighbourhood B to miss that image and to satisfy all the finitely many pointed upper bounds. Every point of F_v, for $v \in B \setminus E$, must then lie in an ordinary interior of some U_x. Pairwise disjointness gives (C.2.14). Notice that this is a statement about the whole fibre in the fixed original X, not merely an inverse image within one selected local branch.

At such a weight the partition into $F_v \cap U_x$ is a finite disjoint partition by relatively open and closed analytic domains. The degree-zero equalizer on this disjoint cover and the actual domain restrictions give

$$\mathcal{O}(F_v) = \prod_{x \in S_w} \mathcal{O}(F_v \cap U_x), \qquad M|_{F_v} = \bigoplus_{x \in S_w} M|_{F_v \cap U_x},$$

with zero factors omitted. These are the actual fibre algebras and modules, including every local Artinian factor. Each point of the fibre is absolutely interior in X, since $U_x \subset \mathrm{Int}(X/K)$, and in U_x, since it is ordinarily interior there. The last assertion uses the ordinary domain test and forward composition.

Take a common finite polyhedral subdivision for $t(X)$, the images of the U_x, the exceptional set E, and their faces, and then work at generic points of its

d-cells. All applicable local image germs lie in the same affine d-plane, with the same saturated tangent lattice N; absent local images contribute zero. Choose an integral projection injective on N. The interior module-weight theorem 5.40.2 applies to X and every nonempty U_x fibre. Tensoring the displayed finite product decomposition with the projected Gauss field retains all factors, so the dimensions of the actual Gauss modules add. Dividing every term by the common index $[\mathbb{Z}^d : Q(N)]$ proves equality of the generic weights. By the finite-module chain, this proves the chain-germ equality (C.2.13) by Theorem C.2.12. If S_w is empty, the same compact-cover argument puts the whole image near w inside E, giving the final assertion. □

Principal zero spaces on generalized affinoids

Theorem C.2.14 (Generalized affinoid principal zero purity; F8ZP-G)**.** *Let A be a nonzero generalized K-affinoid algebra pure of analytic dimension $n \geq 1$. If f avoids every minimal prime of A, then $\mathcal{M}(A/fA)$ is empty or pure of analytic dimension $n-1$. In particular this holds for every non-zero-divisor $f \in A$. For $n = 0$, every element avoiding the minimal primes is a unit. Moreover, in the original algebra A, for every $\mathfrak{q} \in \mathrm{Min}_A(fA)$ and $\mathfrak{p} \in \mathrm{Min}(A)$ contained in $\mathfrak{q}$, the ordinary height of $\mathfrak{q}/\mathfrak{p}$ in $A/\mathfrak{p}$ is one.*

Proof. Put $B = A/fA$. This is an actual affinoid quotient by NAG-001561–001562. If $B = 0$, there is nothing to prove. Choose one finite K-free polyradius r, and put $L = K_r$, such that A_L and B_L are strict; NAG-001557 permits this simultaneous choice. If K is trivially valued, include any additional radius other than 1 in the strictification request, so that L is nontrivially valued. The quotient comparison and faithful flatness in NAG-001566 give

$$B_L = (A/fA)_L \simeq A_L/fA_L \neq 0. \tag{C.2.15}$$

By NAG-002291, the minimal primes of A_L are exactly $\mathfrak{p}A_L$ for $\mathfrak{p} \in \mathrm{Min}(A)$, with contraction $\mathfrak{p}$ and with the same componentwise analytic dimension. Thus A_L is pure of dimension n, and f avoids each of its minimal primes. Analytic and Krull dimensions agree for these strict algebras by NAG-001664. Proposition C.1.21 makes B_L pure of dimension $n-1$ (and rules out $B \neq 0$ when $n = 0$).

Apply NAG-002291 again, now to the full quotient B. For every $\mathfrak{b} \in \mathrm{Min}(B)$, the prime $\mathfrak{b}B_L$ is minimal and

$$\dim_K \mathcal{M}(B/\mathfrak{b}) = \dim_L \mathcal{M}\big(B_L/\mathfrak{b}B_L\big) = n-1.$$

This is the required descent of componentwise analytic purity. Nonzero affinoids have nonempty spectra by NAG-001572, so the empty-space and zero-algebra alternatives agree. The regular case follows from Lemma C.1.13; its regularity also persists upstairs by the flatness in NAG-001566. The ordinary height assertion in the original A follows by the minimal-prime and Tag 00KV argument of Proposition C.1.21: it uses only that $A/\mathfrak{p}$ is a Noetherian domain and that f is nonzero there. It is the strictification argument, not this ordinary height statement alone, that supplies the asserted analytic dimension.

Only this finite free-radius extension, with its exact minimal-prime bijection, was used. No assertion of purity ascent by arbitrary flat extension, of freeness over completed residue fields, or of equality between generalized analytic dimension and ordinary Krull dimension has entered the argument. □

Theorem C.2.15 (Actual refinements and zero-space purity; F8ZP-R)**.** *Let X be a good Hausdorff K-analytic space pure of analytic dimension $n \geq 1$, let $U \subseteq X$ be open, and let $f \in \mathcal{O}_{U_G}(U)$ act injectively on $\mathcal{O}_{U_G}$. Let $D = V(f) \hookrightarrow U$ be the actual principal closed analytic subspace. Then:*

(i) D is empty or pure of analytic dimension $n-1$.

(ii) For every actual affinoid domain $V = \mathcal{M}(B) \subseteq U$, including a domain that is not an ordinary neighbourhood in U, its restricted zero space is the full quotient $D \cap V = \mathcal{M}(B/fB)$, empty or pure of analytic dimension $n-1$.

(iii) These conclusions hold on every nonempty finite intersection $V_I = \bigcap_{i \in I} V_i$ of actual affinoid domains in U, and on further actual affinoid-domain refinements of V_I.

The same conclusion holds for an already given effective Cartier closed subspace locally defined by such regular sections.

Proof. If V is nonempty, NAG-002413 makes B pure of analytic dimension n. The kernel sheaf is formed on sections: kernels of compatible sections glue uniquely. Thus a section on the G-domain V killed by f belongs to this zero kernel sheaf. This gives an injection $B \xrightarrow{f} B$. Theorem C.2.14 applies to this ring and section. The defining quotient is actual by NAG-001562. Its restriction from an affinoid chart is the principal quotient just written, by NAG-001846 and the finite tensor comparison NAG-001565. This proves (ii), with the zero-algebra convention covering empty domains.

For clarity, on a further actual affinoid domain $W = \mathcal{M}(C) \subseteq V$ these same interfaces give bounded canonical identifications

$$C \widehat{\otimes}_B (B/fB) \simeq C \otimes_B (B/fB) \simeq C/fC. \tag{C.2.16}$$

The first tensor has a finite canonical second factor. It is not an assertion about arbitrary Banach modules. Moreover the restriction $B \to C$ is flat by ordinary single-domain flatness, so f remains a non-zero-divisor in C even if regularity was originally specified on B alone. The quotient in (C.2.16) is the actual inverse-image domain in $D \cap V$, with its nilpotents. Both purity of C and purity of its zero quotient follow from the statements already applied.

The intersections V_I are actual affinoid domains by NAG-002411; hence (ii) and (C.2.16) apply to them and to their refinements. This proves (iii). Each point of D has an ordinary affinoid neighbourhood obtained by intersecting D with an affinoid neighbourhood in U. Assertion (ii) proves purity on these neighbourhoods, which is exactly the good-space definition. Thus (i) follows. In the effective Cartier variant apply the same argument on its defining open charts. On overlaps two local equations differ by a unit, hence define the same ideal and actual quotient. Purity is local in these neighbourhoods, so the conclusions agree on overlaps. No global irreducible-component matching is required. □

A specified complete-intersection slice

Corollary C.2.16 (Determinants after a DVR slice)**.** *Let $(R, \mathfrak{m}, k)$ be Noetherian local, let S be a finite free commutative R-algebra of rank $n \geq 1$, and let $f_1, \ldots, f_c \in \mathfrak{m}$ be an R-regular sequence (the empty sequence is allowed). Put $I = (f_1, \ldots, f_c)$, and assume $A = R/I$ is a DVR. For $b \in S$, assume its image $\bar{b}$ is a non-zero-divisor in $B = S/IS$, and assume $B/\bar{b}B \neq 0$. Put $D = \det_R(m_b : S \to S)$. Then*

$$\begin{aligned} \operatorname{length}_R R/(I, D) &= \operatorname{ord}_A(\bar{D}) \\ &= \sum_{\mathfrak{q} \in \operatorname{Max} S} [\kappa(\mathfrak{q}) : k] \operatorname{length}_{S_{\mathfrak{q}}} S_{\mathfrak{q}}/(I, b)S_{\mathfrak{q}}. \end{aligned} \tag{C.2.17}$$

All lengths are finite. The sequence $(f_1, \ldots, f_c, D)$ is regular in R. On every local factor where the final quotient is nonzero, $(f_1, \ldots, f_c, b)$ is regular in $S_{\mathfrak{q}}$. Thus these are ordinary local complete-intersection lengths whenever this ring data describe the chosen geometric intersection.

Direct inputs: Lemma C.1.95; free tensors, determinant polynomials and the regular-sequence argument below.

Proof. Because $S \simeq R^n$ as an R-module, $B \simeq A^n$. Reducing the multiplication matrix modulo I gives $\bar{D} = \det_A(m_{\bar{b}}) \neq 0$ by Lemma C.1.95. This also proves that $\bar{D}$ is a non-zero-divisor in A, giving the regular sequence in R. The quotients $R/(I, D)$ and $S/(I, b)$ are annihilated by I, so their submodule lattices, and hence their lengths, can be computed over the corresponding quotient rings. Apply the lemma to $A \to B$ and identify $B_{\bar{\mathfrak{q}}}/(\bar{b})$ with $S_{\mathfrak{q}}/(I, b)$. Every maximal ideal of S contracts to $\mathfrak{m}$, by the finite-integral argument in the lemma, so contains I. Their residue fields are unchanged by this quotient. This proves (C.2.17).

At each step of the R-regular sequence, the injection given by multiplication by f_i remains injective on the free quotient module $S/(f_1, \ldots, f_{i-1})S \simeq (R/(f_1, \ldots, f_{i-1}))^n$. Localization preserves injectivity. The final multiplication by $\bar{b}$ is injective by assumption. On a factor with nonzero final quotient, all elements of the sequence belong to its maximal ideal and all intermediate quotients are nonzero. This is the stated regular-sequence assertion. □

Appendix D

Polarized towers with compatible cyclic actions

A *polarized isomorphism* $(X, \mathcal{L}) \to (Y, \mathcal{M})$ is a pair (f, λ) consisting of an isomorphism of schemes and a specified isomorphism $\lambda : f^*\mathcal{M} \simeq \mathcal{L}$. Thus its line-bundle map is part of the datum.

Proposition D.0.1 (Orbit tensor and its canonical linearization)**.** *Let a cyclic group $G = \langle \tau \rangle$ of order e act on a scheme X, and let $\mathcal{A}$ be an invertible sheaf on X, with no chosen linearization. The orbit tensor*

$$\mathcal{N} = \bigotimes_{j=0}^{e-1} (\tau^j)^* \mathcal{A} \tag{6}$$

has a canonical G-linearization. If $i : Z \hookrightarrow X$ is an invariant closed subscheme and G acts identically on Z as a scheme, then this linearization acts identically on $i^*\mathcal{N}$. If X is separated over an affine scheme S and $\mathcal{A}$ is relatively ample over S, then so is $\mathcal{N}$; the group action need not be S-linear.

Proof. The factors of $\tau^*\mathcal{N}$ have indices $1, 2, \dots, e$. Using $\tau^e = \mathrm{id}$, reorder them cyclically into indices $0, 1, \dots, e-1$. This gives $\lambda : \tau^*\mathcal{N} \simeq \mathcal{N}$. Iterating this construction e times returns every factor to its original position. The ordinary tensor symmetry is the permutation of pure tensors, so the composite is the identity. The iterates of λ therefore satisfy the cyclic group law and define the claimed linearization.

On Z, the equality $\tau^j \circ i = i$ gives canonical identifications

$$i^*(\tau^j)^*\mathcal{A} = i^*\mathcal{A}, \qquad i^*\mathcal{N} \simeq (i^*\mathcal{A})^{\otimes e}. \tag{7}$$

The restricted λ is the cyclic permutation of these identical line-bundle factors. On an open of Z where $i^*\mathcal{A}$ has a generator a, it fixes $a^{\otimes e}$. It fixes every coefficient as well, because $\tau|_Z$ is the identity morphism. Hence it fixes every local section. This proves identity of the full module, including its behaviour over any nilpotent functions on Z. It also proves identity on the associated total line-bundle scheme.

For ampleness, Stacks Tag 01VK turns relative ampleness over the affine S into ampleness on X. Each automorphism τ^j preserves ampleness, by Stacks Tag 0B3E. For two ample invertible sheaves $\mathcal{H}_1, \mathcal{H}_2$ on X, Stacks Tag 0DNK makes $\mathrm{pr}_1^*\,\mathcal{H}_1 \otimes \mathrm{pr}_2^*\,\mathcal{H}_2$ ample on $X \times_S X$. Here S is quasi-separated because it is affine. Pulling back along the closed diagonal by Stacks Tag 0B3E shows that $\mathcal{H}_1 \otimes \mathcal{H}_2$ is ample on X. Induction gives ampleness of (6), and Stacks Tag 01VK converts it back to relative ampleness over S. □

Lemma D.0.2 (Full faithfulness for polarized isomorphisms)**.** *Let R be a ring, let $\pi \in R$, and suppose that the canonical map $R \to \varprojlim_{n\geq 1} R/\pi^n R$ is an isomorphism. Write $R_n = R/\pi^n R$. Let X, Y be proper, flat, finitely presented R-schemes, and let $\mathcal{L}, \mathcal{M}$ be relatively ample invertible sheaves on them. Reduction gives a bijection*

$$\mathrm{Isom}_R^{\mathrm{pol}}((X,\mathcal{L}),(Y,\mathcal{M})) \xrightarrow{\sim} \varprojlim_n \mathrm{Isom}_{R_n}^{\mathrm{pol}}((X_n,\mathcal{L}_n),(Y_n,\mathcal{M}_n)). \tag{1}$$

Here the inverse limit means compatible polarized isomorphisms with respect to the specified reduction maps. The analogous statement holds for polarized isomorphisms covering a fixed automorphism b of $\operatorname{Spec} R$ that preserves the closed subscheme $\operatorname{Spec} R_n$ for every n. No Noetherian hypothesis on R is imposed.

Proof. Apply Stacks Tag 0D4F to each polarized scheme, allowing the identity thickening of its affine base, and enlarge the two bounds to a common $d_0 \geq 1$. For every $d \geq d_0$, the modules

$$P_d = H^0(X, \mathcal{L}^{\otimes d}), \qquad Q_d = H^0(Y, \mathcal{M}^{\otimes d})$$

are finite projective over R. The arbitrary-base-change assertion in the proof of that lemma identifies their reductions with $H^0(X_n, \mathcal{L}_n^{\otimes d})$ and $H^0(Y_n, \mathcal{M}_n^{\otimes d})$. Moreover the graded algebras

$$A_X = R \oplus \bigoplus_{d\geq d_0} P_d, \qquad A_Y = R \oplus \bigoplus_{d\geq d_0} Q_d \tag{2}$$

functorially reconstruct the pairs as $(\operatorname{Proj} A_X, \mathcal{O}(1))$ and $(\operatorname{Proj} A_Y, \mathcal{O}(1))$.

For any finite projective R-module P, the canonical map $P \to \varprojlim_n P/\pi^n P$ is an isomorphism. Indeed, write P as the image of an idempotent matrix E on R^m. The assertion becomes $ER^m = \varprojlim_n \overline{E} R_n^m$, which follows from the assumed identity for R. This is the finite-projective completeness argument in the proof of Stacks Tag 0D4B.

Let (f_n, λ_n) be a compatible system on the right of (1). For every $d \geq d_0$, its maps on sections are

$$u_{n,d} : Q_d/\pi^n Q_d \longrightarrow P_d/\pi^n P_d, \qquad s \longmapsto \lambda_n^{\otimes d}(f_n^* s). \tag{3}$$

Taking the inverse limit in each degree gives $u_d : Q_d \to P_d$. The same construction applied to the inverse polarized isomorphisms gives inverse maps. The u_d preserve products: after reduction modulo every π^n the required equality holds by (3), and its values lie in the separated module $P_{d+d'}$. Together with the identity on degree zero, they therefore give a graded isomorphism $A_Y \simeq A_X$.

Functoriality of Proj, including the maps on all twists in Stacks Tag 01MY, gives an isomorphism of schemes and an isomorphism of their $\mathcal{O}(1)$ sheaves. The canonical identifications from Stacks Tag 0D4F turn this into (f, λ) on the original pairs. Its reductions are exactly (f_n, λ_n), by the functoriality of those identifications. Conversely, any polarized isomorphism inducing the prescribed reductions has the same maps (3), hence the same u_d by completeness and separation. The same functorial reconstruction makes it equal to (f, λ). This proves both directions of (1).

The limit was taken degree by degree; it was not interchanged with an infinite direct sum. All degrees $d \geq d_0$ were retained, so the reconstruction recovers the actual invertible sheaf and its map, not merely a selected tensor power. Finally, an isomorphism covering b is an R-isomorphism to the pullback pair $b^*(Y, \mathcal{M})$. This twist commutes with reduction by the stated hypothesis on b. Applying (1) to the twisted target proves the last assertion. □

Theorem D.0.3 (Equivariant algebraization of a supplied nodal tower)**.** *Let F be a complete nontrivially real-valued field, $R = F^\circ$, $\mathfrak{m}_R = F^{\circ\circ}$, and $k_R = R/\mathfrak{m}_R$. Let a cyclic group G of order $e \geq 1$ act isometrically on F, and put $b_g = \operatorname{Spec}(g^{-1}|_R)$. Fix $0 \neq \pi \in \mathfrak{m}_R$, set $R_n = R/\pi^n R$, and suppose given proper, flat, finitely presented R_n-schemes X_n with cartesian*

transition closed immersions $i_n : X_n \hookrightarrow X_{n+1}$. Suppose compatible semilinear G-actions on these schemes are specified, covering the induced $b_{g,n}$. Assume that $C = X_1 \times_{R_1} k_R$ is at-worst-nodal of pure dimension one and that the action on the entire scheme *C is the identity.*

For any ample invertible sheaf $\mathcal{A}_C$ on C, one can choose compatible ample lifts $\mathcal{A}_n$ and form the orbit tensors

$$\mathcal{N}_n = \bigotimes_{j=0}^{e-1} (\tau_n^j)^* \mathcal{A}_n, \tag{4}$$

where τ_n is the action of a generator of G. These form a compatible G-linearized ample tower, with identity action on $\mathcal{N}_n|_C \simeq \mathcal{A}_C^{\otimes e}$. There exist a projective, flat, finitely presented nodal R-scheme Y' of relative dimension one, an ample invertible sheaf $\mathcal{N}'$ on Y', a semilinear G-action on Y', and a compatible linearization of $\mathcal{N}'$, together with compatible equivariant polarized identifications

$$(Y', \mathcal{N}') \times_R R_n \simeq (X_n, \mathcal{N}_n). \tag{5}$$

The actions on both Y'_{k_R} and $\mathcal{N}'|_{Y'_{k_R}}$ are the identity. For a fixed polarized tower (4), this algebraization is unique up to the unique compatible equivariant polarized isomorphism. Invertibility of e is not required here.

Proof. The isometries preserve R and every ideal $\pi^n R$: membership in this ideal is the inequality $v(a) \geq nv(\pi)$. Hence all indicated actions on quotient bases exist. The rank-one completeness argument in B.1.3 gives $R = \varprojlim R_n$.

The exact result B.1.2 gives the compatible ample $\mathcal{A}_n$, with transition isomorphisms $\varphi_n : i_n^* \mathcal{A}_{n+1} \simeq \mathcal{A}_n$ and a specified restriction to $\mathcal{A}_C$. It also supplies at least one ample $\mathcal{A}_C$, if none was prescribed. Apply D.0.1 at each level. Its hypotheses hold because X_n is separated over the affine base $\operatorname{Spec} R_n$. It gives ampleness of (4), its canonical permutation linearization, and identity on the full restricted line bundle on C.

We check the extra tower compatibility explicitly. Equivariance of i_n gives canonical identifications

$$i_n^* (\tau_{n+1}^j)^* \mathcal{A}_{n+1} \simeq (\tau_n^j)^* i_n^* \mathcal{A}_{n+1}.$$

Tensor the maps $(\tau_n^j)^* \varphi_n$ over all j to obtain $\nu_n : i_n^* \mathcal{N}_{n+1} \simeq \mathcal{N}_n$. Applying the cyclic permutation linearization before or after this tensor map permutes the same indexed factors and applies the same maps on them. Thus ν_n is

equivariant. Composing these maps gives compatibility at all levels. Their restrictions to C identify (4) with $\mathcal{A}_C^{\otimes e}$, with the identity action already given by the orbit-tensor result.

Now apply B.1.3 to the ordinary polarized tower $(X_n, \mathcal{N}_n, \nu_n)$. It gives the projective flat finitely presented pair $(Y', \mathcal{N}')$ and the underlying polarized identifications (5). The nodal hypothesis on C gives nodality of relative dimension one. At this stage no action on Y' has been asserted.

Let b be the base automorphism corresponding to the chosen generator. Via (5), its tower action and linearization are compatible polarized R_n-isomorphisms from $(Y', \mathcal{N}')_{R_n}$ to its b_n-twist. Both algebraic pairs $(Y', \mathcal{N}')$ and $b^*(Y', \mathcal{N}')$ satisfy Lemma D.0.2. That lemma gives a unique polarized R-isomorphism between them. Composing its scheme map with the first projection from the twist gives a semilinear automorphism $\tau : Y' \to Y'$ covering b, and its line-bundle map is $\lambda : \tau^* \mathcal{N}' \simeq \mathcal{N}'$.

The e-fold composite of this polarized semilinear automorphism covers $b^e = \mathrm{id}$. At every level it is the identity polarized automorphism, by the given action and its linearization. Injectivity in Lemma D.0.2 therefore makes it the identity on the algebraic pair. The iterates of (τ, λ) give the required G-action and linearization, and (5) is equivariant. Restricting (5) for $n = 1$ further to C proves the full residual identities. Finally, the same lemma gives a unique polarized comparison of any two algebraizations with (5). Its reductions commute with the specified actions, so injectivity makes the comparison equivariant. □

Appendix E

Projective descent in the tame Kummer class

Retain the coefficient class of C.1.53. Let K be complete and nontrivially real-valued, with valuation ring O, value group Γ, and residue field k. Let $e \geq 2$ be invertible in O, and let $\zeta \in O$ have order e, with residue also of order e. Assume

$$L = K(s), \qquad s^e = c \in K^\times, \qquad \gamma = v(s) > 0, \qquad i\gamma \notin \Gamma \quad (0 < i < e). \tag{K}$$

Write $O' = L^\circ$, $\mathfrak{m}'$ for its maximal ideal, and $G = \langle \sigma \rangle$, where $\sigma(s) = \zeta s$. The cited criterion gives a faithfully flat map $O \to O'$, identifies both residue fields with k, and gives identity coefficient action on k. In particular, its character-zero summand is $(O')^G = O$. The value group need not be discrete, and O' is the actual valuation ring; it need not be finite as an O-module.

Theorem E.0.1 (Projective descent in the exact tame Kummer class)**.** *Assume (K). Let Y'/O' be a supplied projective, flat, finitely presented nodal scheme of relative dimension one, equipped with a semilinear G-action. Then the following are equivalent:*

1. *G acts identically on the entire special-fibre scheme Y'_k.*
2. *There is a projective, flat, finitely presented nodal O-scheme Y of relative dimension one and a G-equivariant isomorphism $Y \times_O O' \simeq Y'$.*

The descent is unique with the specified equivariant identification. For every relatively ample invertible sheaf $\mathcal{A}'$ on Y', its canonically linearized orbit tensor

$$\mathcal{N}' = \bigotimes_{j=0}^{e-1} (\sigma^j)^* \mathcal{A}' \tag{8}$$

is the equivariant pullback of a relatively ample invertible sheaf $\mathcal{N}$ on Y.

Proof. Assume condition 1. The affine base $\operatorname{Spec} O'$ has an ample structure sheaf. By Stacks Tag 0B45, (1) $\Rightarrow$ (6), the supplied projective scheme Y'/O' is proper and has a relatively ample invertible sheaf. Fix any such $\mathcal{A}'$. Theorem C.1.81 supplies a separated, flat, finitely presented nodal O-scheme Y with the required equivariant identification and descends properness to Y/O.

Apply Proposition D.0.1 with $X = Y'$, $S = \operatorname{Spec} O'$, and $Z = Y'_k$. It makes (8) relatively ample and gives a canonical linearization whose restriction to the full special fibre is the identity. Under the supplied identification $Y' = Y \times_O O'$, Proposition C.1.83 therefore produces the invertible sheaf

$$\mathcal{N} = (p_* \mathcal{N}')^G$$

and its equivariant pullback identification; it also proves that $\mathcal{N}$ is relatively ample over O. The base $\operatorname{Spec} O$ has the ample invertible sheaf $\mathcal{O}$. Applying Stacks Tag 0B45, (6) $\Rightarrow$ (1) to the proper scheme Y/O with the descended relatively ample invertible sheaf $\mathcal{N}$ makes Y/O projective. The flatness, finite presentation, nodality, relative dimension and uniqueness are those of Theorem C.1.81.

Conversely, an equivariant presentation in condition 2 has special fibre $Y \times_O k$, with coefficient action. That action is the identity because G acts identically on k. This proves condition 1. □

Appendix F

Descent of an algebraized polarized tower

Corollary F.0.1 (A descended polarization in the exact tame Kummer class)**.** *Let K be complete and nontrivially real-valued, with valuation ring O, residue field k, and value group Γ. Assume the exact coefficient class of C.1.53: $e \geq 2$ is invertible in O, $\zeta \in O$ and its residue have order e, and*

$$L = K(s), \quad s^e = c \in K^\times, \quad \gamma = v(s) > 0, \qquad i\gamma \notin \Gamma \quad (0 < i < e). \tag{K}$$

Put $O' = L^\circ$ and $G = \langle \sigma \rangle$, where $\sigma(s) = \zeta s$. Choose $0 \neq \pi \in \mathfrak{m}_O$. Suppose a cartesian proper flat finitely presented tower over $O'/\pi^n O'$ with compatible semilinear G-actions satisfies the nodal and full special-fibre identity hypotheses of Theorem D.0.3. Then there exist a projective, flat, finitely presented nodal O-scheme Y of relative dimension one and a relatively ample invertible sheaf $\mathcal{P}$ on Y, with compatible equivariant polarized isomorphisms

$$(Y, \mathcal{P}) \times_O (O'/\pi^n O') \simeq (X_n, \mathcal{N}_n). \tag{6}$$

Here $\mathcal{N}_n$ is the chosen orbit-tensor tower (4). The algebraized bundle $\mathcal{N}'$ itself descends to $\mathcal{P}$; no further tensor power of $\mathcal{N}'$ is needed. In particular $\mathcal{P}|_{Y_k}$ identifies with $\mathcal{A}_C^{\otimes e}$ under the induced residue identification. The reductions $(Y, \mathcal{P}) \times_O O/\pi^n O$ form the corresponding descended polarized tower.

Proof. The coefficient criterion identifies the residue fields of O and O' with k, gives identity coefficient action on k, and uses the actual complete valued field L and its valuation ring. Thus Theorem D.0.3, applied over O', first supplies

an algebraic projective nodal pair $(Y', \mathcal{N}')$ with the specified semilinear action and full residual identities. This step uses no ramified descent. For clarity, with $b_g = \mathrm{Spec}(g^{-1})$ the induced left action on functions is $g \cdot a = \rho_{g^{-1}}^* a$, and on linearized sections it is $g \cdot t = \lambda_{g^{-1}}(\rho_{g^{-1}}^* t)$. Thus its coefficient action on O' is exactly g, as in the cited tame descent results.

Theorem E.0.1 applies to Y'/O'. It supplies a projective flat finitely presented nodal Y/O and an equivariant isomorphism $Y \times_O O' \simeq Y'$, with the prescribed coefficient action on the base change. Properness is retained: Y' is projective and proper, and the cited result descends properness before concluding projectivity.

Under this identification, $\mathcal{N}'$ is an already existing G-linearized invertible sheaf on $Y \times_O O'$ whose action on the entire special-fibre module is the identity. Apply C.1.83 to this bundle itself. For $p : Y \times_O O' \to Y$, it gives

$$\mathcal{P} = (p_* \mathcal{N}')^G, \qquad p^* \mathcal{P} \simeq \mathcal{N}', \tag{7}$$

and makes $\mathcal{P}$ relatively ample. There is no need to replace $\mathcal{N}'$ by a second orbit tensor. The scheme and bundle descents are unique with their specified identifications, by the two cited results.

Reduce the algebraic isomorphisms just constructed modulo π^n and use the level comparisons (5) of Theorem D.0.3. This gives (6), with all transition maps and linearizations, and then the asserted residue polarization. The $O/\pi^n O$ reductions of the actual pair $(Y, \mathcal{P})$ are automatically cartesian, proper, flat and finitely presented. Their base changes to $O'/\pi^n O'$ are exactly the supplied polarized levels. In particular no valuation-ring descent theorem has been applied directly to the non-valuation quotient rings $O'/\pi^n O'$. □

F.1 Prescribed subdivisions of an existing model

Let K be a complete non-Archimedean field with a specified nontrivial real-valued absolute value. Write

$$O = K^\circ, \qquad k = \widetilde{K}, \qquad v(c) = -\log|c|, \qquad \Gamma = v(K^\times) \subset \mathbb{R}.$$

Let C/K be a smooth projective geometrically connected curve. A model means a projective flat finitely presented nodal O-curve with generic fibre identified with C and geometrically connected, geometrically reduced special fibre, as in 4.45.3. A marking D, when present, is a fixed finite etale O-scheme embedded in the relative smooth locus. All model morphisms below induce the identity on C.

Fix an actual model $\mathcal{X}$. Choose a finite unramified Galois residue lift L/K splitting its nodes. Put $l = \widetilde{L}$ and $G = \mathrm{Gal}(L/K)$. Here *unramified* means the actual finite free etale coefficient extension of 4.43.1; in particular $v(L^\times) = \Gamma$. The splitting field and rational pointed etale nodal charts exist by 4.18.2.

For each node e of $\mathcal{X}_l$, choose an ordering of its two branches and a pointed chart $x_e y_e = a_e$, with $0 < |a_e| < 1$. Put $\ell_e = v(a_e)$. By 4.39.1 and 4.39.3, its entire tube is the standard open annulus

$$|a_e| < |x_e| < 1, \qquad s = v(x_e) \in (0, \ell_e).$$

The coordinate s increases from the chosen first end to the second. These are two distinct *branch ends* even if their component vertices coincide.

Use an algebraically closed complete isometric comparison field $\Omega \supset L$ with residue field $\overline{k}$, as in 4.45.1, and write $p : C_\Omega^{\mathrm{an}} \to C^{\mathrm{an}}$. The geometric component vertices of a model are exactly $p^{-1}(V(\mathcal{X}))$, by 4.45.2. Write $\zeta_e(r)$ for the Gauss point of radius r in the selected annular tube over Ω.

Definition F.1.1 (Compatible cut data)**.** A cut datum consists of a finite set $S_e \subset (0, \ell_e) \cap \Gamma$ for each split node e. The canonical semilinear action of $g \in G$ takes the skeleton coordinate s either to s or to $\ell_e - s$ on the ge-tube, according as it preserves or exchanges the chosen branch orderings. Require respectively

$$S_{ge} = S_e \quad \text{or} \quad S_{ge} = \ell_e - S_e.$$

Empty sets are permitted.

These conditions concern labelled tubes of the *given* model. The coordinate-change result 4.19.1 makes them independent of normalized annular coordinates. A stabilizer exchanging the two branches forces reflection invariance of S_e; a stabilizer preserving both branches permits asymmetric cuts. A self-loop alone imposes no reflection condition: the issue is the branch action, not equality of endpoint vertices.

Theorem F.1.2 (Exact arithmetic subdivisions)**.** *Every compatible cut datum $S = (S_e)$ has a model $\mathcal{X}[S]$, preserving D, with geometric vertices exactly*

$$p^{-1}(V(\mathcal{X})) \ \cup \ \bigcup_e \{\zeta_e(\exp(-s)) : s \in S_e\}. \tag{*}$$

It dominates $\mathcal{X}$ and is unique up to the unique marked generic-identity isomorphism. It is independent of the splitting field and coordinates when the prescribed analytic points are fixed.

If $S_e \subset T_e$ for all e, there is a unique compatible morphism $\mathcal{X}[T] \to \mathcal{X}[S]$. The model $\mathcal{X}[S \cup T]$ is the least common dominating model of $\mathcal{X}[S]$ and $\mathcal{X}[T]$, even among models with vertices outside the original skeleton.

Proof. First form the reflection closure $S_e^{\#} = S_e \cup (\ell_e - S_e)$. For nodes above one original closed node, these are the same finite numerical set independently of branch ordering: G acts transitively above that node, and reflection no longer changes the set.

We construct an original-ring model $\mathcal{X}^{\#}$ with exactly these extra cuts. For a nonempty reflection-invariant set in $(0, \ell)$, let d be its smallest member. Then $d \in \Gamma$ and $d \le \ell/2$. Apply Lemma C.1.84. If $d < \ell/2$, all remaining cuts lie in the central annulus, of length $\ell - 2d$; their new coordinates $s - d$ again form a reflection-invariant set. If $d = \ell/2$, there was just one cut.

Over l, the new central nodes are rational and split by the displayed charts. They form a G-invariant collection, hence are the full inverse image of a collection of closed nodes of the actual current k-model. Perform the same original-ring operation at these nodes. At each nonempty step at least one member of the finite list is removed. All new distances are differences in Γ, so the required scalars belong to K. This finite process, applied at the finitely many original nodes, constructs $\mathcal{X}^{\#}$ and its actual map to $\mathcal{X}$. It uses no least positive value and no discreteness assumption.

Retain all old geometric components and exactly the new components labelled by S_e, discarding those in $S_e^{\#} \setminus S_e$. This selection is G-invariant.

The new components and their attachment points are split over l: they are the exceptional projective lines in the accepted chart calculation, and the pointed etale maps induce isomorphisms on the exceptional fibres. Thus the action of $\mathrm{Aut}(\overline{k}/k)$ on the new components factors through $\mathrm{Gal}(l/k)$. All old geometric components are retained. The selection is therefore invariant under the full residual action. By 4.49.1, it is the entire inverse image of a subset $V \subset V(\mathcal{X}^{\#})$ of actual original-field component vertices.

Each connected collection of discarded geometric components is a chain of smooth projective lines with no marking and two attaching branches on retained components. It cannot be a cycle: every original edge has retained endpoints, including the retained old vertex at both ends of a self-loop. If the two attaching branches meet the same retained component, they are counted twice. This is exactly the marked forest condition 4.45.4, with $b = 2, m = 0$. The actual original-ring theorem 4.45.5 now gives a projective model $\mathcal{X}[S]$ with precisely (*).

All old vertices remain, so 4.46.2 gives $\mathcal{X}[S] \to \mathcal{X}$. That accepted theorem and 4.46.3 also give uniqueness, marking compatibility and the morphisms for $S \subset T$. They prove independence of auxiliary choices, using 4.45.2 to identify original vertex sets from their entire geometric inverse images.

The union datum is compatible. Every model dominating both $\mathcal{X}[S]$ and $\mathcal{X}[T]$ contains both vertex sets, hence their union, which is $V(\mathcal{X}[S \cup T])$. Applying 4.46.2 gives its unique morphism to $\mathcal{X}[S \cup T]$, as required. □

Proposition F.1.3 (The full tubes of a subdivision)**.** *In a geometric node tube A_e, list the cuts $0 < s_1 < \cdots < s_m < \ell_e$, and set*

$$r_0 = 1, \qquad r_j = \exp(-s_j), \qquad r_{m+1} = |a_e|.$$

The closed-point tubes of $\mathcal{X}[S]_{\Omega^\circ}$ inside A_e are precisely the open annuli

$$\{r_{j+1} < |x_e| < r_j\}, \quad 0 \le j \le m, \tag{1}$$

and the open discs, for $1 \le j \le m$,

$$\{|x_e/b_j - \widehat{\alpha}| < 1\}, \quad \alpha \in \overline{k}^{\times}, \quad b_j \in K^{\times}, \quad v(b_j) = s_j. \tag{2}$$

Here $\widehat{\alpha} \in \Omega^\circ$ is any lift of α. The omitted point on $|x_e| = r_j$ is its Gauss vertex. The old smooth tubes are unchanged as analytic subsets.

The two ends of (1) carry consecutive component labels, allowing coincident endpoint labels at an original self-loop. Each disc (2) has the inserted

vertex as its sole boundary point. On an inserted geometric component the two annular directions are the normalization points $0, \infty$; the other residual points give (2).

Proof. On $|x_e| = r_j$, the coordinate $z = x_e/b_j$ gives $\mathcal{M}(\Omega\langle z, z^{-1}\rangle)$. Its canonical reduction is $\overline{k}[z, z^{-1}]$, with one component point, by 3.20.3. The inverse images of its closed points are exactly (2), by 3.20.1; these are open unit discs with coordinate $z - \widehat{\alpha}$. The intervals between cut circles give the standard open annuli (1). These open connected sets partition the complement of the inserted vertices in A_e, so they are its connected components.

No new vertex lies in an old smooth tube. The specialization partition 4.39.5, applied over Ω, identifies precisely these components with the closed-point tubes of $\mathcal{X}[S]$. The boundary assertions follow from 4.39.3. The two ends are retained before any identification of component labels, so this includes self-loops. The accepted contraction theorem only removes chains between these retained endpoints. □

Over the original field, specialization commutes with coefficient extension by 4.28.1. The inverse image of an original closed-point tube is therefore the union of the geometric tubes above that *residual closed point*. A finite unramified field splitting the nodes need not split every smooth residual closed point. An abstract edge and a list of radii do not record this information. The actual blowups and contractions supply the residual schemes and their morphisms. In a nonsplit situation a stabilizer can exchange $0, \infty$; its action on these points and on the other residual points must be retained. No assertion that the descended component or all its tubes are split is used.

Corollary F.1.4 (A literal-union common refinement)**.** *Let $\mathcal{X}, \mathcal{Y}$ be two given models of C, with the same D when marked. Let $\Sigma_{\mathcal{X},\Omega}$ be the embedded compact geometric skeleton of $\mathcal{X}$, including its component vertices. Assume*

$$V(\mathcal{Y}_{\Omega^\circ}) \subset \Sigma_{\mathcal{X},\Omega}. \tag{3}$$

There is then a least common dominating model $\mathcal{Z}$ with

$$V(\mathcal{Z}) = V(\mathcal{X}) \cup V(\mathcal{Y}).$$

It is the exact subdivision of $\mathcal{X}$ in Theorem F.1.2, with the full tube description of Proposition F.1.3. Condition (3) holds, in particular, if the two embedded geometric skeleta agree.

Proof. Choose a common finite unramified Galois splitting field for the nodes of both models. Every geometric vertex of $\mathcal{Y}$ not already a geometric vertex of $\mathcal{X}$ lies on the radial skeleton of one of the split $\mathcal{X}$-annuli. Its image $w \in C_L^{\mathrm{an}}$ is a component vertex of $\mathcal{Y}_{L^\circ}$, by 4.45.2. The annular coordinate is analytic on its entire L-tube, so its evaluation lies in $\mathcal{H}(w)$. Consequently

$$v(x_e(w)) \in v(\mathcal{H}(w)^\times) = v(L^\times) = \Gamma.$$

The finite cut sets obey the branch compatibility because both models are defined over K. The theorem constructs $\mathcal{X}[S]$ whose geometric vertices are their union. The entire-inverse-image assertion 4.45.2 identifies its original vertex set with the asserted original union. The two domination maps and least-common property follow from 4.46.2. Marking compatibility follows from 4.46.3. □

Residual action at the ramified midpoint

Use the field and model of Proposition C.1.85. Its quadratic extension is $L = K(b)$ with $b^2 = t$, and $O' = L^\circ$. The coefficient ring is the actual ring of Proposition C.1.86.

Proposition F.1.5 (The midpoint's residual action is nontrivial)**.** *Let $\mathcal{Y}/O'$ be the three-vertex projective model constructed in the audited midpoint proof by symmetric insertion in $TU = t = b^2$. The canonical action fixing T, U and sending b to $-b$ extends to $\mathcal{Y}$. On the new residual component it acts as*

$$\overline{u} \longmapsto -\overline{u}, \qquad u = T/b. \tag{14}$$

Its two branch ends $0, \infty$ are fixed. For $\alpha \in k^\times$, the geometric residual-direction disc labelled α is carried to that labelled $-\alpha$. Thus the invariant midpoint and its fixed branch ends do not carry trivial residual-scheme or tube action. This gives a residual-action obstruction to effective descent of this given *model with its fixed generic identification.*

Proof. The audited symmetric insertion uses the ideal (T, U, b) on the node chart. This ideal is invariant under the stated action, so its Rees algebra and blowup inherit the action; on the rest of the projective model the original coefficient action already exists. These actions agree on overlaps. The accepted symmetric blowup charts, used in the passing proof, give its middle chart as

$$O'[u, v]/(uv - 1), \qquad T = bu, \quad U = bv. \tag{15}$$

Here $\sigma(u) = -u$, $\sigma(v) = -v$. The two adjacent node charts add the missing ends $0, \infty$ of this residual $\mathbb{P}^1_k$, proving (14) on the whole component. Since $\mathrm{char}(k) = 0$, it is a nontrivial scheme automorphism.

The full tube description in F.1.3 identifies the smooth residual direction $\alpha \neq 0, \infty$ with

$$|T/b - \widehat{\alpha}| < 1.$$

Here one may use the constant lift $\widehat{\alpha} = \alpha \in k$. The action fixes this lift and changes T/b to $-T/b$. It therefore carries this disc to the disc for $-\alpha$, which is distinct because $\alpha \neq 0$ and the residue characteristic is zero.

Every actual base change from O has identity inertia on its special fibre, because $O'/\mathfrak{m}' = O/\mathfrak{m} = k$ and the coefficient action on k is the identity. An isomorphism giving descent with the prescribed generic identification would transport that action to the displayed action on $\mathcal{Y}$. Indeed extensions of a fixed generic action to a flat separated model are unique: their equalizer is closed by separatedness, and its defining ideal vanishes after inverting the nonzero elements of O', so vanishes by flatness. This contradicts (14). This argument diagnoses the residual mechanism of the already audited obstruction; it does not replace that earlier proof or assert a new model-existence theorem. □

Ordinary moment support and finite freeness

Moment norms near a supplied retraction

Let K be a complete non-Archimedean field with its given real-valued absolute value. We use $t_f = (-\log|f_j|)_j$ and write $\mathrm{Int}_{\mathrm{top}}(V,U)$ for the ordinary interior of V in U.

Lemma F.1.6 (Local equality of moment norms)**.** *Let U and Y be good Hausdorff K-analytic spaces, let $i : Y \hookrightarrow U$ be a closed immersion, and let $r : U \to Y$ be an analytic morphism with $r \circ i = \mathrm{id}_Y$. Assume that every affinoid domain of Y under consideration has analytic dimension at most d. Let $z = i(y)$, let $V \subset U$ be an ordinary affinoid neighbourhood of z, and let $f = (f_1, \dots, f_N)$ be analytic units on V. There is an ordinary affinoid neighbourhood $W \subset V$ of z such that*

$$\dim t_f(W) \le d.$$

The choice of W depends on the tuple, but not on any ambient superform subsequently pulled back by that tuple. Thus all such forms of bidegree (p,q) with $\max(p,q) > d$ induce zero ordinary sections on $\mathrm{Int}_{\mathrm{top}}(W,U)$.

Proof. Choose an ordinary affinoid neighbourhood B of y in Y contained in $i^{-1}(\mathrm{Int}_{\mathrm{top}}(V,U))$. Write $g_j = i^* f_j|_B$, which is a unit on B. On the ordinary open neighbourhood

$$O_0 = \mathrm{Int}_{\mathrm{top}}(V,U) \cap r^{-1}(\mathrm{Int}_{\mathrm{top}}(B,Y))$$

of z, the functions $a_j = r^* g_j$ are units. The analytic-domain restriction and neighbourhood basis used here are those in `v037:forms:germsheaf`. Since $ri = \mathrm{id}$, each $f_j - a_j$ has value zero at z, whereas $|a_j(z)| > 0$. By continuity, the open set

$$O_1 = \{x \in O_0 : |f_j(x) - a_j(x)| < |a_j(x)| \text{ for every } j\}$$

contains z. Choose an ordinary affinoid neighbourhood W of z contained in O_1. At every $x \in W$, the strict ultrametric inequality gives $|f_j(x)| = |a_j(x)|$: one inequality follows from $f_j = a_j + (f_j - a_j)$, and a strict reverse inequality would contradict the same triangle inequality applied to $a_j = f_j - (f_j - a_j)$. Consequently

$$t_f(W) = t_g(r(W)) \subseteq t_g(B).$$

By v037:globaltrop:finiteimage, the right-hand side is a finite polyhedral set of dimension at most d, and the left-hand side is also finite polyhedral. Hence its dimension is at most d. On every face its tangent space has dimension at most d, so one of the exterior factors of a form with $\max(p,q) > d$ restricts to zero. The ordinary moment definition then gives the asserted zero sections. □

Avoiding the regular zero space

Let X be good, Hausdorff, and pure of analytic dimension $n \geq 1$ over a complete real-valued non-Archimedean field K, possibly trivially valued. Let η be an ordinary smooth compactly supported $(n-1,n-1)$-form whose actual support is contained in the absolute analytic interior $\mathrm{Int}(X/K)$. Let f be a regular analytic function on X, with $D = V(f)$. On the affinoid neighbourhoods at issue its representative is a non-zero-divisor in the actual affinoid algebra.

Lemma F.1.7 (Support avoidance from the exact F7 support lemma; F8F7-S)**.** *For the test η above,*

$$\mathrm{supp}(d''\eta) \cap D = \varnothing. \tag{2.1}$$

In particular the extension by zero

$$\mathrm{ext}_0\big(\log|f|\, d'd''\eta|_{X\setminus D}\big) \tag{2.2}$$

is an ordinary smooth compactly supported top form on X.

Proof. At a point x of $\mathrm{supp}(d''\eta)$, choose an ordinary affinoid neighbourhood $V = \mathcal{M}(A)$ carrying one unit-moment representative of $d''\eta$, and on which $f \in A$ is a non-zero-divisor. It can be chosen inside the analytic interior around x. The exact F7 lemma NAG-002578, lines 165–225, applies to this $(n-1,n)$-form: its germ is nonzero, and the evaluation kernel $\ker(x)$ is a minimal prime of A.

A non-zero-divisor in a Noetherian ring avoids every minimal prime. Indeed, localization at a minimal prime is a nonzero zero-dimensional Noetherian local ring, hence Artinian by Stacks 00KH, forward direction; an element of its maximal ideal is nilpotent and cannot act injectively. Localization preserves injectivity. Thus $f \notin \ker(x)$, so $|f(x)| > 0$, proving (2.1). Here Noetherianity is the same NAG-001560 used in the F7 source.

Locality of d' gives $\operatorname{supp}(d'd''\eta) \subseteq \operatorname{supp}(d''\eta)$. This compact support is disjoint from D. On $X \setminus D$, the logarithm is a smooth function of the unit f; near D the tested top form is identically zero. Gluing gives the smooth extension (2.2). □

Freeness at defect at most one

Use the given complete field and the same convention for analytic dimension. Write $d_K(s)$ for the Abhyankar dimension of $\mathcal{H}(s)$ over K.

Lemma F.1.8 (Finite maps become free along the section; F8RC-F)**.** *Let $F : Y \to Z$ be finite between good Hausdorff spaces pure of analytic dimension n. Suppose the ordinary local rings of Z are regular, and every affinoid domain of Y has S_1 algebra. At every $s \in Z$ with $d_K(s) \geq n-1$, the finite coherent module $F_*\mathcal{O}_Y$ is free on an ordinary neighbourhood of s. Rank zero is allowed when the fibre is empty.*

Proof. Put $R = \mathcal{O}_{Z,s}$ and $M = (F_*\mathcal{O}_Y)_s$. By Lemma C.1.58, R is regular local of dimension zero or one. In dimension zero it is a field, and the finite module M is free. Suppose $\dim R = 1$, and choose a generator π of its maximal ideal. A regular local ring is a domain by Stacks 00NP, so $\pi \neq 0$.

Represent π by a on a sufficiently small ordinary affinoid neighbourhood $\mathcal{M}(C)$ of s. Write $U = \mathcal{M}(C)$ for this chosen domain. At every $x \in U$, including its domain boundary, the regular ordinary-stalk comparison makes $\mathcal{O}_{Z,x} \to \mathcal{O}_{U,x}$ a flat local map of Noetherian local rings with regular closed fibre. The assumed regularity of $\mathcal{O}_{Z,x}$ then gives regularity of $\mathcal{O}_{U,x}$ by Stacks 031E. Thus all ordinary stalks of this affinoid are regular. By `NAG-001861`, we may retain the integral, open-and-closed component containing s. Its inverse image is affinoid, say $\mathcal{M}(B)$, finite over C, and is either empty or pure of dimension n. The hypothesis makes B an S_1 ring. We claim that every minimal prime q of B contracts to zero in C.

Indeed, if $p = q \cap C$, then $C/p \to B/q$ is injective and finite. Such a map is surjective on analytic spectra: its finite module has support all of $\operatorname{Spec}(C/p)$, so every residue-field fibre is nonzero by Stacks 00DV(2) by contraposition at the corresponding local ring; extending to the complete residue field gives a nonzero finite-dimensional affinoid algebra and hence an

analytic point. Completed residue extensions at points of a finite map are finite, by NAG-001590. The maximum formula NAG-001591 therefore gives

$$\dim_K \mathcal{M}(C/p) = \dim_K \mathcal{M}(B/q) = n.$$

If $p \neq 0$, dimension drop (NAG-002374) in the integral pure n-dimensional algebra C contradicts this equality. Thus $p = 0$.

The element a is nonzero in C, since its germ is $\pi \neq 0$. It avoids every minimal prime of B, hence every associated prime, because B is S_1 by Stacks 031Q, (2) implies (1). Multiplication by a on B is injective. The ordinary comparison $C \to R$ is flat (NAG-001698, or NAG-001782 after localization), and $M = B \otimes_C R$ by finite coherent restriction. Thus π acts injectively on M.

Lift a basis of $M/\pi M$ to obtain $R^r \to M$, surjective by Nakayama (Stacks 00DV(2) on the finite cokernel). Its kernel N is finite. The resolution $0 \to R \xrightarrow{\pi} R \to R/(\pi) \to 0$ and injectivity of π on M give $\operatorname{Tor}_1^R(M, R/(\pi)) = 0$. Tensoring the presentation of M therefore gives $N/\pi N = 0$. Nakayama gives $N = 0$ by Stacks 00DV(2) on N, proving that M is free. Finally NAG-001753 spreads the free stalk to an ordinary neighbourhood. This proves actual local freeness, including its analytic base changes; mere torsion-freeness over an unspecified higher-dimensional ring has not been used. □

Serre comparison and Cartier coefficients

Fix a complete non-Archimedean field with its specified real-valued absolute value, possibly trivial. Use actual generalized affinoids with their canonical quotient and finite-module norms. All dimensions of analytic spaces in this section are analytic dimensions.

Serre S_1 for analytic stalks

Theorem F.1.9 (Analytic Serre comparison; F8ST-A)**.** *Let A be a generalized affinoid algebra, $X = \mathcal{M}(A)$, M a finite A-module, and $\widetilde{M}$ its coherent sheaf. For $x \in X$, put $\mathfrak{p} = \ker_A(x)$. Then*

$$M_{\mathfrak{p}} \text{ is } S_1 \iff \widetilde{M}_x \text{ is } S_1 \text{ as a module over } \mathcal{O}_{X,x}. \tag{F.1.1}$$

Thus M is S_1 if and only if all these analytic stalk modules are S_1. If V is a good analytic domain in a good analytic space X, and $\mathcal{F}$ is coherent, then at every $x \in V$

$$\mathcal{F}_x \text{ is } S_1 \iff (\mathcal{F}|_V)_x \text{ is } S_1. \tag{F.1.2}$$

The same comparison applies to a finite-type scheme over an affinoid base, its analytification, and an affinoid domain therein, with the corresponding algebraic and analytic stalks.

Proof. By NAG-001624/001666, $\widetilde{M}_x = M \otimes_A \mathcal{O}_{X,x}$. The map $A_{\mathfrak{p}} \to \mathcal{O}_{X,x}$ is regular by NAG-001782 and is local, hence faithfully flat. Lemma C.1.15 proves (F.1.1). Every prime of A is the kernel of an analytic point by NAG-001860, proving the assertion for all of M. For (F.1.2), use the tensor identity NAG-001666 and the regular local map NAG-001792. For the last assertion use the regular comparison square NAG-001791 and the same finite-presentation stalk calculation. All localizations required in Lemma C.1.15 are Noetherian. □

The ordinary stalk identifications are the finite-coherent equivalence and coherent restriction. The regular maps used above are affinoid comparison, good-domain restriction, and comparison over an affinoid base. The passage to all primes uses realization by analytic points.

Let X be pure of analytic dimension $n \geq 1$, and let $f \in A$ be regular. Write $f_i = f|_{X_i}$ for its component restrictions. For an affinoid $X = \mathcal{M}(A)$ pure of dimension n, write

$$\operatorname{Min} A = \{\mathfrak{q}_1, \dots, \mathfrak{q}_s\}, \qquad X_i = \mathcal{M}(A/\mathfrak{q}_i), \quad i_i : X_i \hookrightarrow X, \quad m_i = \operatorname{length}_{A_{\mathfrak{q}_i}} A_{\mathfrak{q}_i}. \tag{F.1.3}$$

Proposition F.1.10 (Component scope and support; F8CR-S)**.** *In* (F.1.3)*, every X_i is integral and pure of dimension n, every f_i is regular, and every actual good affinoid domain in X_i has an S_1 coordinate ring. For an ordinary compactly supported form η on X,*

$$\operatorname{Supp}(i_i^*\eta) \subseteq i_i^{-1} \operatorname{Supp} \eta.$$

If $\operatorname{Supp}\eta \subset \operatorname{Int}(X/K)$*, its pullback has compact support in* $\operatorname{Int}(X_i/K)$*. Moreover $i_i^*(d'd''\eta) = d'd''(i_i^*\eta)$. The support and functoriality assertions also hold for an arbitrary closed immersion of good Hausdorff analytic spaces.*

Proof. The quotients by minimal primes have their actual affinoid structures by NAG-001561 and NAG-001562. Purity of X gives their analytic dimension n. Proposition C.1.14's avoidance argument makes $f_i \neq 0$ in the domain $A/\mathfrak{q}_i$. Every domain is S_1: at a nonzero prime its local ring has a nonzero element of the maximal ideal which is regular, and the zero-prime localization is a field. The finite coherent comparison NAG-002765 applies to the structure module of this affinoid and then to its actual good analytic domains. It gives the asserted analytic stalk and affinoid S_1 statements. It is never applied to infer S_1 for the original possibly embedded-component ring.

Closed immersions are finite locally with the canonical quotient norm. By NAG-001701, every point is relatively interior; NAG-001704 and the atlas definition of relative interior give

$$i_i^{-1} \operatorname{Int}(X/K) \subseteq \operatorname{Int}(X_i/K).$$

Their underlying maps identify the source with a closed subspace. The preimage of the compact support is therefore compact. Pullback of a zero germ is zero, proving the support inclusion. Finally NAG-002424 gives pullback commuting with both differentials, with their fixed order unchanged. These statements are about the true analytic interior; compactness alone does not remove an analytic boundary. □

Proposition F.1.11 (Points computing full generic lengths; F8ST-G)**.** *Let $Y = \mathcal{M}(B)$ be nonempty and pure of analytic dimension d, $\mathfrak{p} \in \operatorname{Min} B$, and M a finite B-module. There is $x \in \mathcal{M}(B/\mathfrak{p})$ with $d_K(x) = d$. Every such point has kernel $\mathfrak{p}$ on B, belongs to no other irreducible component, and has the following property. For every affinoid domain $V \subset Y$ containing x, the ordinary local ring $S_V = \mathcal{O}_{V,x}$ is Artinian, and*

$$\operatorname{length}_{S_V}(M \otimes_B S_V) = \operatorname{length}_{B_{\mathfrak{p}}} M_{\mathfrak{p}}. \tag{F.1.4}$$

The same equality holds for the pointed analytic-domain colimit $\mathcal{O}_{Y_G,x}$ and the coherent module colimit. These colimit modules have finite length. Thus the analytic generic multiplicity can be computed by the algebraic generic length, including nilpotents.

Proof. The reduced component $\mathcal{M}(B/\mathfrak{p})$ is pure of dimension d. The maximum formula gives the required point. A nonzero element of $B/\mathfrak{p}$ has zero locus of dimension at most $d-1$, or empty when $d = 0$, by NAG-002374. Such a zero locus cannot contain a point of Abhyankar dimension d. Hence the kernel on B is exactly $\mathfrak{p}$. For another minimal prime $\mathfrak{p}'$, an element of $\mathfrak{p}' \setminus \mathfrak{p}$ proves that x is not on that component.

For any affinoid domain $W = \mathcal{M}(C)$ containing x, $\dim_K W \le d$ by NAG-001592. Its evaluation kernel $\mathfrak{q}$ is minimal in C. The prime choice below it uses Stacks 00E0(4). Indeed, if $\mathfrak{r} \subsetneq \mathfrak{q}$ were a minimal prime, a nonzero element of $\mathfrak{q}/\mathfrak{r}$ would cut the integral affinoid $\mathcal{M}(C/\mathfrak{r})$ down to dimension at most $d-1$, contradicting $d_K(x) = d$. This also covers $d = 0$ using the empty-locus assertion of NAG-002374.

Now let h be in the maximal ideal of S_V. By NAG-001698, it is represented by an analytic section on an affinoid ordinary neighbourhood W of x in V. Its image lies in the minimal prime $\ker_C(x)$. The localization $C_{\ker_C(x)}$ is Artinian. Its maximal ideal is nil by Stacks 00E0(7), so some power of h is annihilated in C by an element nonzero at x. That element is a unit in the ordinary stalk. Thus h is nilpotent in S_V. The stalk is Noetherian by NAG-001747; a Noetherian local ring with nil maximal ideal is Artinian.

The natural local map $B_{\mathfrak{p}} \to S_V$ is regular by NAG-001791; both rings are Artinian. The coherent restriction identity and Lemma C.1.12 prove (F.1.4). In particular $\mathfrak{p}S_V$ is the maximal ideal of S_V.

For completeness, use the definition of the pointed G-local ring as the filtered colimit over analytic domains containing x. Taking ordinary stalks

on those domains gives the same colimit: a representative section restricts to an ordinary neighbourhood in its domain, and that neighbourhood has an affinoid refinement containing x. Finite intersections admit such refinements (NAG-002411). Tensor a composition series of $M_{\mathfrak{p}}$ with every S_V, then pass to this filtered colimit. Each factor is the filtered colimit of the fields $S_V/\mathfrak{p}S_V$. The transition maps of these fields are unital and injective, so this colimit is a field. It is precisely the residue field of the colimit local ring, because every stage has maximal ideal $\mathfrak{p}S_V$. The resulting series has exactly $\operatorname{length}_{B_{\mathfrak{p}}} M_{\mathfrak{p}}$ nonzero simple factors. Taking $M = B$ also shows directly that the colimit ring is Artinian. No general Noetherianity theorem for G-local rings is needed here. □

Purity of the neighborhood used for an effective coefficient. Let $Z = \mathcal{M}(B)$, let $\mathfrak{p}$ be the minimal prime of the chosen reduced component $Y = \mathcal{M}(B/\mathfrak{p})$, and let $x \in Y$ be away from the other components. Take an ordinary affinoid neighborhood $W = \mathcal{M}(C)$ of x in Z disjoint from those components. Actual domain base change and the finite quotient tensor identify the inverse image of W in Y with the affinoid domain $\mathcal{M}(C/\mathfrak{p}C)$ of Y. Every analytic point of W maps into Y, hence its kernel contains $\mathfrak{p}C$. By realization of every prime of C as an analytic kernel, $\mathfrak{p}C$ is contained in every prime of C. Thus $\mathfrak{p}C$ is a nil ideal by Stacks 00E0(7). Consequently C and $C/\mathfrak{p}C$ have the same minimal-prime quotients. Domain purity on Y gives purity of the latter domain of dimension $\dim_K Y$, so the full W is pure of that dimension. Apply the generic Artin-length argument on the full W. Passing to $C/\mathfrak{p}C$ in the purity check changes none of the nilpotent structure used in computing the actual coefficient on W.

Theorem F.1.12 (Cartier coefficient transport with scoped gluing; F8ST-C-R1)**.** *Let A be generalized affinoid and $f \in A$ a regular element. Put $B = A/(f)$, with its actual quotient analytic structure. For a component $Y = \mathcal{M}(A/\mathfrak{p})$ of the Cartier zero space, $\mathfrak{p}$ minimal over (f), its coefficient is*

$$c_Y(f) = \operatorname{length}_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/fA_{\mathfrak{p}}) = \operatorname{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/f\mathcal{O}_{X,x}), \qquad X = \mathcal{M}(A), \tag{F.1.5}$$

where x can be chosen on Y, away from the other zero-space components, of maximal Abhyankar dimension in Y. Equivalently, the right side is the generic length of the analytic quotient sheaf. The lengths are finite positive integers. The assertion applies in particular to every codimension-one Cartier

component. It makes no claim about an arbitrary closed integral subspace of the zero locus.

The effective coefficients are compatible with actual affinoid-domain restriction and ordinary gluing. For regular a, b, product additivity is asserted when $\mathrm{Min}_A(a) \cup \mathrm{Min}_A(b)$ is an antichain: no two distinct primes in this union contain one another. Here $\mathrm{Min}_A(a)$ means primes of the actual common ring A minimal over (a), with the empty set for a unit. A sufficient condition is that the two actual zero spaces are empty or pure of one common analytic dimension.

For regular meromorphic presentations, the differences $\mathrm{div}(a) - \mathrm{div}(b)$ are independent and glue among atlases for which this antichain condition holds for the cross-products on every actual common affinoid refinement. In particular this holds when all local numerator and denominator zero spaces are empty or pure of one fixed analytic dimension. The effective coefficient formula above needs no such purity assumption. No reduction of A, B, or their finite fibres is permitted. The previously unrestricted product and meromorphic clauses are not claimed closed in this induced revision.

Proof. The prime $\mathfrak{p}/(f)$ is minimal in B, so $B_{\mathfrak{p}/(f)} = A_{\mathfrak{p}}/(f)$ is Artinian. Its length as a module over itself is its length as an $A_{\mathfrak{p}}$-module. Choose x on the integral component Y of maximal Abhyankar dimension. It has kernel $\mathfrak{p}/(f)$ on B and is on no other component, by the dimension-drop argument in Proposition F.1.11. Shrink to an affinoid ordinary neighbourhood in the zero space avoiding the other components. It is pure of dimension $\dim_K Y$ by the mixed-neighbourhood argument above and domain purity NAG-002413. The proof of Proposition F.1.11 makes its ordinary stalk at x, hence $\mathcal{O}_{\mathcal{M}(B),x}$, Artinian. The regular comparison NAG-001782 for B, followed by Lemma C.1.12, gives its length as length $B_{\mathfrak{p}/(f)}$. The finite closed-immersion case of NAG-001735 identifies

$$\mathcal{O}_{\mathcal{M}(B),x} = \mathcal{O}_{X,x} \otimes_A B = \mathcal{O}_{X,x}/f\mathcal{O}_{X,x}.$$

Length over a local ring is unchanged when computed over a quotient annihilating the module, proving (F.1.5). The same argument on the pointed domain colimit proves the generic-length interpretation.

For a domain $\mathcal{M}(C) \subset \mathcal{M}(B)$ and $\mathfrak{q} \in \mathrm{Min}\, C$, its contraction $\mathfrak{r}$ is minimal in B by flat going down (Stacks 00HS). Choose an analytic point of $\mathcal{M}(C)$ with kernel $\mathfrak{q}$, using NAG-001860. The upper arrow of NAG-001791 gives a regular local map $B_{\mathfrak{r}} \to C_{\mathfrak{q}}$. Both rings are Artinian, so their lengths

agree. The same calculation holds for any finite coherent module. Actual domain/quotient base change, NAG-001702/001846, identifies this quotient with the restriction of $A/(f)$. Flatness preserves the injectivity of multiplication by f, so the restricted equation is still regular. This proves compatibility on every actual domain and hence on affinoid refinements of overlaps.

We first supply the missing component inclusion under the stated antichain hypothesis. In any Noetherian ring, $\mathrm{Min}_A(ab)$ is the set of inclusion-minimal members of $\mathrm{Min}_A(a) \cup \mathrm{Min}_A(b)$. Indeed a prime containing ab contains a factor and therefore a prime minimal over that factor by Stacks 00E0(4). A prime minimal over ab must equal that smaller prime. Conversely, if p is inclusion-minimal in the union and $q \subset p$ contains ab, choose by Stacks 00E0(4) a factor-minimal prime $r \subset q$. Then $r \subset q \subset p$ forces $r = q = p$. Thus, for an antichain, every factor component occurs in the product:

$$\mathrm{Min}_A(ab) = \mathrm{Min}_A(a) \cup \mathrm{Min}_A(b).$$

The analytic purity criterion for this hypothesis uses only frozen NAG-002374. If the factor zero spaces are empty or pure of one common dimension d, all primes in the union define integral affinoids of dimension d. A strict inclusion $q \subsetneq p$ would make the nonzero ideal p/q in A/q cut out $\mathcal{M}(A/p)$ of dimension at most $d-1$, or the empty space when $d = 0$. This is impossible. Their union is therefore an antichain, and the product zero space is also empty or pure of dimension d.

For regular a, b, multiplication by b gives the exact sequence

$$0 \longrightarrow A/(a) \xrightarrow{b} A/(ab) \longrightarrow A/(b) \longrightarrow 0.$$

At each prime in the now identified component union, each factor quotient is either zero (a unit after localization) or Artinian. A prime in the union containing a factor is minimal over that factor; otherwise a smaller factor-minimal prime would violate the antichain condition. Add localized finite lengths to get the coefficient identity on every component, including all factor components.

For meromorphic representatives in the stated scope, equality $a/b = c/e$ gives $ae = bc$ on an actual common affinoid restriction: localization at regular elements is injective. The cross-product antichain hypotheses hold there by assumption. In the common pure-zero-space case they hold by the preceding dimension-drop argument: quotient/domain base change identifies the restricted zero spaces with actual domains in the original zero spaces,

and NAG-002413 preserves their purity. Additivity yields $\operatorname{div}(a) - \operatorname{div}(b) = \operatorname{div}(c) - \operatorname{div}(e)$. On a good space, local quotient representatives may be taken on ordinary neighbourhoods. If regularity is initially known only at a germ, the kernel of multiplication is coherent, so it vanishes after shrinking. For a unit of the local total quotient ring, a representative numerator is regular as well: multiplying by an inverse shows that it divides a regular element. These observations justify regular numerator and denominator representatives after the necessary restrictions. Domain compatibility just proved makes their local cycles glue. □

Theorem F.1.13 (Additivity in the verified scope; F8CA-C)**.** *Let A be Noetherian and let a, b be non-zero-divisors. Assume $\operatorname{Min}_A(a) \cup \operatorname{Min}_A(b)$ is an antichain. Define the effective cycle of a regular equation by its actual components and the ambient lengths*

$$\operatorname{div}(a) = \sum_{p \in \operatorname{Min}_A(a)} \operatorname{length}_{A_p}(A_p/aA_p)[\operatorname{Spec}(A/p)].$$

Then $\operatorname{div}(ab) = \operatorname{div}(a) + \operatorname{div}(b)$. For generalized affinoids, the same assertion holds for the actual analytic cycles and their verified effective coefficient transport. In particular it holds under Lemma C.1.57's hypotheses.

Proof. Lemma C.1.29 identifies all components on both sides, including the converse inclusion missing from the original proof. At a prime p in this union, each of $A_p/(a)$ and $A_p/(b)$ is either zero (when the factor is a unit in A_p) or Artinian (when p is minimal over that factor). It is minimal over every factor it contains: otherwise a smaller minimal prime of that factor would contradict the antichain condition. The regularity of b makes the sequence

$$0 \longrightarrow A/(a) \xrightarrow{b} A/(ab) \longrightarrow A/(b) \longrightarrow 0$$

exact. Localize at p and add finite lengths. This gives the coefficient equality on the now identical component sets, the selected algebraic antichain clause. The effective analytic comparison and actual-domain compatibility are the unchanged, independently verified portions of NAG-002772 effective coefficients and actual domain transport; no new length comparison is being inferred from flatness. □

Proposition F.1.14 (Ambient regular representatives; F8NM-R)**.** *Ambient S_1 is unnecessary for local regular numerator and denominator representatives of a unit in the total quotient sheaf. Such representatives remain regular after every actual affinoid restriction, and their pullbacks to the reduced components are nonzero and regular. Equality of two representations $a/b = c/d$ gives $ad = bc$ after an actual common ordinary refinement. Componentwise nonvanishing alone is not a substitute for ambient regularity.*

Proof. For a ring A, localization at its regular elements is injective. If a/b is a unit, write its inverse as c/d, where b, d are regular. Equality in the localization gives $ac = bd$ in A. The product bd is regular. If $az = 0$, then $bdz = acz = 0$, so $z = 0$. Thus a is regular; the same argument applies to c. No associated-prime purity hypothesis is used.

The good-space construction at the end of the proof of frozen NAG-002772 takes these quotients on ordinary neighbourhoods. When injectivity is first known at a germ, the multiplication kernel is coherent and vanishes after shrinking. A finite collection of numerators, denominators, inverse representatives and equalities can be handled on one smaller neighbourhood. This is the exact coherent shrinking scope of that input, not a claim about arbitrary meromorphic objects on a nongood space. Affinoid-domain restriction is flat (NAG-001832), so it preserves the regular multiplication maps.

A regular element avoids every minimal prime, as checked in reviewed F8CR-L. Its image in the domain $A/\mathfrak{q}_i$ is therefore nonzero and regular. F8CR-S and flat actual restrictions supply the corresponding analytic component regularity. The same injection of regular localization proves the cross-product identity. The failure of the last proposed weakening is shown in the algebraic example below. □

In $A = K\langle t, e\rangle/(e^2, te)$, every class has the unique module form $a(t) + ec$, with $a \in K\langle t\rangle$ and $c \in K$. The nilradical is (e), the underlying space is the pure disc, and $\operatorname{ann}(e) = (t, e)$ is an embedded associated prime. At the minimal prime (e), t is invertible and kills e, so the generic length is one. But $A/(t) = K[e]/(e^2)$ has length two at the rigid origin; the reduced component quotient is K. Since $te = 0$ with $e \neq 0$, t is not a Cartier equation. It also cannot be a unit of the ambient total quotient ring: if $t(c/d) = 1$ with d regular, then $tc = d$, and multiplication by e contradicts regularity of d. Componentwise meromorphic invertibility does not solve this.

Appendix G

Stable inertia and original-field models

Let K be a complete nontrivially real-valued field, let $O = K^\circ$, and write $\mathfrak{m}$ and k for its maximal ideal and residue field. The valuation has rank one and need not be discrete. An *F5 model* of a smooth projective geometrically connected curve C/K means a projective, flat, finitely presented nodal O-curve, together with its generic-fibre identification with C, whose special fibre is geometrically connected and reduced. The marking is empty throughout this note. Nodal means at worst nodal of pure relative dimension one; branches need not split over k.

Stable families and their generic-fibre identifications have the conventions of the admitted source 4.14. We use its exact results 4.14.4 and 4.14.5. The example reuses the valued field and extension from C.1.85 and C.1.86.

Lemma G.0.1 (A necessary inertia condition for original-field existence)**.** *Let C/K be as above, of genus $g \geq 2$. Let L/K be finite Galois, with group G, and give L its unique extending absolute value. Put $O' = L^\circ$, let k' be its residue field, and set*

$$I = \ker\bigl(G \longrightarrow \mathrm{Aut}(k'/k)\bigr).$$

Suppose a stable model $\mathcal{Z}/O'$ of C_L is supplied with its generic-fibre identification. Equip it with the unique semilinear G-action of 4.14.5. If C admits an F5 model over O, then every element of I induces the identity morphism on the entire k'-scheme $\mathcal{Z}_{k'}$. No tameness or residue-separability hypothesis is needed for this necessary condition.

Proof. Suppose $f : \mathcal{X} \to \operatorname{Spec} O$ is an F5 model. A nontrivial rank-one valuation ring has only its generic and closed points. Hence all geometric fibres of f are reduced, connected curves of dimension one: the generic fibre is smooth and geometrically connected, and the special fibre has the stated properties. The criterion in Stacks Tag 0E6T, with Stacks Tag 0E6L, makes f a prestable family. The same lemma gives a finite locally free $R^1 f_* \mathcal{O}_{\mathcal{X}}$ with arbitrary base change. Its rank is constant on the connected spectrum of O and equals g at the generic point. Thus this is a prestable family of genus g in the sense of Stacks Tag 0E6V.

Apply the arbitrary-base stabilization theorem Stacks Tag 0E8A. It gives a stable family $\mathcal{Y}/O$ and a morphism $\mathcal{X} \to \mathcal{Y}$ whose fibres contract rational tails and bridges. The smooth geometrically connected generic curve of genus at least two has none, so this morphism identifies $\mathcal{Y}_K$ with C. This use of the stack of stable families requires no new projectivity theorem for $\mathcal{Y}$.

For the scheme convention, the absence of rational tails and bridges on the stable closed fibre gives an ample relative dualizing sheaf on a neighbourhood of the closed point by Stacks Tag 0E74. As the base is local, that neighbourhood is the whole base. The ample sheaf makes $\mathcal{Y}$ a scheme by Stacks Tag 0D32.

Both $\mathcal{Y}_{O'}$ and $\mathcal{Z}$ are stable families of genus g. Their specified generic identifications give a unique isomorphism

$$\alpha : \mathcal{Y}_{O'} \xrightarrow{\sim} \mathcal{Z}$$

by 4.14.4, which applies to arbitrary valuation rings. The canonical semilinear action on the base change $\mathcal{Y}_{O'}$ extends the canonical action on C_L. Transporting it by α must therefore give the specified action on $\mathcal{Z}$, by the uniqueness in 4.14.5.

On the full special fibre $\mathcal{Y}_{k'} = \mathcal{Y}_k \times_k \operatorname{Spec} k'$, the canonical action of $\sigma \in G$ is $\mathrm{id}_{\mathcal{Y}_k} \times \operatorname{Spec}(\overline{\sigma}^{-1})$. For $\sigma \in I$, this is the identity morphism, including the structure sheaf. The equivariant isomorphism $\alpha_{k'}$ transfers that identity to $\mathcal{Z}_{k'}$, as required. □

Proposition G.0.2 (A tame nondiscrete curve with no original-field F5 model)**.** *Use the admitted field and quadratic extension*

$$K = \bigcup_{r \geq 0} \widehat{\mathbf{C}((t^{1/3^r}))}, \qquad v(t) = 1, \qquad L = K(s), \quad s^2 = t.$$

Then $v(K^\times) = \mathbf{Z}[1/3]$ is dense, L/K is tame Galois of degree two, and both residue fields are $\mathbf{C}$. The curve

$$C: \quad Z^4 = t^2(X^4 + Y^4) \ \subset \ \mathbf{P}^2_K$$

is smooth, projective and geometrically connected of genus three. It admits a smooth projective stable model over L°, but admits no F5 model over K°. The geometric dual graph of the supplied stable model has a single vertex and no edges, and is fixed by the Galois action. Thus invariant stable vertices do not imply even existence of some original-field F5 model.

Proof. The field, value group, completeness and extension data are exactly those of C.1.85 and C.1.86, with $k = \mathbf{C}$ and s in place of b. We reuse these facts. In particular $G = \{1, \sigma\}$ with $\sigma(s) = -s$, the residue field does not change, and $I = G$. No assertion that O'/O is finite or etale is being used; the admitted coefficient result in fact proves that O' is not finite over O.

For the homogeneous polynomial $F = Z^4 - t^2(X^4 + Y^4)$ the partial derivatives are

$$-4t^2X^3, \qquad -4t^2Y^3, \qquad 4Z^3.$$

They cannot vanish simultaneously at a projective point over any field extension of K. On each affine chart the dehomogenized partials likewise have no common zero on the curve, by the displayed formula and the degree-four Euler identity. Inverting a nonzero partial gives a standard smooth presentation as in Stacks Tag 00T6; Stacks Tag 00T7 gives smoothness and relative dimension one. Thus C is a geometrically smooth plane quartic. These smooth hypersurfaces are pure one-dimensional and have no embedded points, so Stacks Tag 0BYD applies over K and over every field extension. It gives $H^0(C_E, \mathcal{O}_{C_E}) = E$ and genus $(4-1)(4-2)/2 = 3$. In particular C is geometrically connected: a disconnection over an algebraic closure would give a nontrivial idempotent in that field of global functions. Projectivity is part of its displayed construction.

Let $D/\mathbf{C}$ be the plane quartic $W^4 = X^4 + Y^4$ and put

$$\mathcal{D} = D \times_{\mathbf{C}} \operatorname{Spec} O'.$$

The same computation makes D a smooth projective geometrically connected curve of genus three. Hence $\mathcal{D}$ is a projective smooth family with these geometric fibres. It is stable by Stacks Tag 0E6T and Stacks Tag 0E75: a

smooth connected genus-three curve has no rational tails or bridges. There is an L-isomorphism

$$\phi : C_L \xrightarrow{\sim} D_L, \qquad [X : Y : Z] \longmapsto [X : Y : Z/s].$$

Indeed $s^4 = t^2$, and multiplication of the third coordinate by s gives its inverse. This supplies the required stable model with generic-fibre identification.

Under ϕ, the canonical semilinear action of σ is the coefficient action of σ together with

$$\tau : [X : Y : W] \longmapsto [X : Y : -W].$$

For example, the pullback of $W = Z/s$ is $Z/(-s) = -W$. This action extends to $\mathcal{D}$, since τ is defined over $\mathbf{C}$ and preserves its equation. Uniqueness in 4.14.5 identifies it with the specified action on the stable model. On the special fibre it is τ, which sends $[1 : 0 : 1]$ to the distinct point $[1 : 0 : -1]$. It is therefore not the identity on the special fibre.

If an F5 model of C over O existed, Lemma G.0.1 would force this action to be the identity, a contradiction. Finally, the geometric special fibre of $\mathcal{D}$ is a smooth connected curve, hence has one irreducible component and no nodes. Its geometric dual graph therefore has one vertex and no edges; the action fixes that graph. This graph statement does not assert that its geometric component vertex descends to an original-field F5 vertex. □

Finite model measures and logarithmic integrals

Let K be complete and nontrivially discretely valued, with its specified absolute value, valuation ring R, residue field k, uniformizer π, and

$$\lambda = -\log|\pi| > 0.$$

We do not renormalize λ to one or pass to an algebraic closure. Let X be an integral proper K-variety of dimension $d \geq 2$. Take a proper flat integral normal finite-type R-model $\mathcal{X}$ with identified generic fibre X. Retain explicitly the following local data: every height-one local ring used below is a DVR, and the special fibre has finitely many irreducible components V, all of dimension d. These are hypotheses, not assertions about arbitrary analytic local rings. Smoothness or strict semistability of the model is not required.

Write $A_V = \mathcal{O}_{\mathcal{X},\eta_V}$ and normalize ord_V by a uniformizer of this DVR. Then

$$b_V = \mathrm{ord}_V(\pi) = \mathrm{length}_{A_V}(A_V/\pi A_V) > 0, \qquad [\mathcal{X}_k] = \sum_V b_V[V]. \tag{G.0.1}$$

Thus a nonreduced special fibre is allowed. Its b_V is not discarded. The point $\xi_V \in X^{\mathrm{an}}$ is specified on $K(X)$ by

$$|a(\xi_V)| = |\pi|^{\mathrm{ord}_V(a)/b_V} \quad (a \neq 0). \tag{G.0.2}$$

Distinct components give distinct points: the bounded valuation ring in (G.0.2) has centre η_V on $\mathcal{X}$, so equality of points would give equality of their centres. If the points were equal, their valuations would give the same valuation ring. The two maps from its spectrum to the separated model agree at the generic point. The closed diagonal and integrality of that spectrum force them to agree everywhere, including at the centre. The model ring A_V, the ordinary analytic stalk $\mathcal{O}_{X^{\mathrm{an}},\xi_V}$, and the completed residue field $\mathcal{H}(\xi_V)$ are different objects. Only the first is a DVR hypothesis here.

Choose line bundles $L_1, \ldots, L_d, M$ on X and model lines

$$\mathcal{L}_i|_X \simeq L_i^{\otimes e_i}, \qquad \mathcal{M}|_X \simeq M^{\otimes e_0}, \qquad e_i, e_0 \in \mathbf{Z}_{>0}.$$

Use their continuous root model metrics: a model frame has norm one, and if $s^{\otimes e_0} = ae$ in a model frame e, then $\|s\| = |a|^{1/e_0}$ on the domain reducing to that frame. The model metrics are supplied as part of these data. Put

$$E = \prod_{i=1}^{d} e_i, \quad \delta_V = \deg_{\mathcal{L}_1|_V,\ldots,\mathcal{L}_d|_V,k}(V), \quad \mu_{\mathcal{X}} = \sum_V \frac{b_V\delta_V}{E}\,\delta_{\xi_V}. \tag{G.0.3}$$

The degree is the mixed algebraic degree over k, with its usual full residue-field dimensions. It is not a degree over the field of constants of V followed by omission of that field degree. We use the specified integers δ_V and the finite measure above. The measure may be signed. Positivity requires $\delta_V \geq 0$. Its mass is exactly $E^{-1}\sum_V b_V\delta_V$; identifying this with a generic-fibre degree would use an additional algebraic specialization identity. The mixed convention contains no extra $d!$. For a proper d-dimensional integral k-scheme W and invertible sheaves $M_1,\dots,M_d$, use the mixed degree

$$\delta(W;M_1,\dots,M_d) = \deg_k\big(c_1(M_1)\cdots c_1(M_d)\cap[W]\big).$$

The selected supported operations in NAG-003112 compute the actual multiples of the coordinate divisor cycles, and their push into the ambient support gives the first Chern operation. The degree convention is the one used with NAG-003117.

Let s be a nonzero rational section of M. A model frame at η_V writes $s^{\otimes e_0} = a_V e_V$, and put $c_V = \operatorname{ord}_V(a_V)$. A change of model frame multiplies a_V by a DVR unit, so c_V is well-defined. Write $g_s = -\log\|s\|$ on the dense open where s is regular and nonvanishing.

Theorem G.0.3 (Exact logarithmic integral on a proper model)**.** *Under the standing finite-model data, every ξ_V lies in the analytification of every nonempty Zariski open subset of X. In particular g_s is finite at all the atoms of $\mu_{\mathcal{X}}$ and is integrable with respect to its total variation. The exact formulas are*

$$g_s(\xi_V) = \frac{\lambda c_V}{e_0 b_V}, \tag{G.0.4}$$

$$\int_{X^{\mathrm{an}}} g_s\, d\mu_{\mathcal{X}} = \frac{\lambda}{e_0 E}\sum_V c_V\delta_V, \tag{G.0.5}$$

$$\int_{X^{\mathrm{an}}} |g_s|\, d|\mu_{\mathcal{X}}| = \frac{\lambda}{e_0 E}\sum_V |c_V\delta_V|. \tag{G.0.6}$$

Every proper algebraic closed subset of X has $|\mu_{\mathcal{X}}|$-measure zero. For $a\in K^\times$, replacing s by as changes (G.0.5) by $-\log|a|\,\mu_{\mathcal{X}}(X^{\mathrm{an}})$.

Direct inputs NAG-001707, the standing finite-model data.

Proof. On an affine model neighbourhood of η_V, the valuation in (G.0.2) is nonnegative on the model coordinate ring and extends the given absolute

value on K. Indeed, $\mathrm{ord}_V(u) = 0$ for $u \in R^\times$, and $\mathrm{ord}_V(\pi) = b_V$. The valuation inequality and multiplicativity therefore give an analytic point by NAG-001707. Its kernel in the generic-fibre coordinate ring is zero, since the valuation of each nonzero rational function is finite. Its underlying scheme point is the generic point of X. Hence it belongs to every nonempty Zariski open, including the locus where s is regular and nonvanishing. Formula (G.0.2) and the root metric give (G.0.4). This also shows that an arbitrary value assigned on the omitted divisor does not affect the integral.

A finite sum of distinct Dirac masses has total variation obtained by taking the absolute values of its coefficients. Substituting (G.0.4) into (G.0.3) proves both integral formulas. The positive integer b_V cancels between the mass and the value; it has not been set to one. The zero-mass assertion follows from the generic scheme point of every atom. Finally

$$c_V(as) = c_V(s) + e_0 \,\mathrm{ord}_V(a) = c_V(s) + e_0 b_V \,\mathrm{ord}_R(a).$$

Since $-\log|a| = \lambda \,\mathrm{ord}_R(a)$, substitution proves the last assertion with its stated sign. □

Proposition G.0.4 (Weak convergence does not control the logarithm)**.** *For every $d \geq 2$ and K as in the standing finite-model data there are genuine semipositive product-model measures μ_j of mixed mass one on $X = (\mathbf{P}^1_K)^d$ and a fixed model-metrized line (M, h) with a nonzero regular section s such that*

$$\mu_j \Longrightarrow \delta_o, \qquad \int -\log \|s\|_h \, d\mu_j = j\lambda \longrightarrow +\infty, \qquad o \in Z(s).$$

Thus finite-model integrability and weak measure convergence, even with an unchanged metric on M, do not imply logarithmic integrability of the limit. This does not contradict convergence theorems that also require uniform convergence of the semipositive metrics defining the measures.

Direct inputs NAG-000182, NAG-001707, NAG-001709, NAG-001714, the standing finite-model data, the mixed algebraic degree convention.

Proof. For each $j \geq 1$ use the model $(\mathbf{P}^1_R)^d$ with generic-fibre identification $T_i = \pi^j S_i$, and the model lines $\mathrm{pr}_i^* \mathcal{O}(1)$. They are semipositive: on any special-fibre curve their degrees are nonnegative, since they are generated by global sections. The special fibre has one reduced component, $b = 1$,

and its mixed degree is one, by intersecting the d coordinate divisors at one k-rational product point. Its model measure is therefore $\mu_j = \delta_{\xi_j}$, where

$$\left| \sum_{m \in \mathbf{N}^d} a_m T^m \right| (\xi_j) = \max_m |a_m| \, |\pi|^{j(m_1 + \cdots + m_d)}.$$

Gauss multiplicativity gives the point, and its polynomial evaluations converge to $|a_0|$, the evaluations at the origin o. The evaluation topology in NAG-001707 gives $\xi_j \to o$ in the common affine chart and thus in X^{an}. Consequently the Dirac masses converge against every continuous analytic test, not just radial tests.

Use $M = \mathrm{pr}_1^* \mathcal{O}(\infty)$ with its standard model metric and its global section $s = T_1$ relative to the frame 1 on the finite chart. On that chart

$$\|s(x)\|_h = \frac{|T_1(x)|}{\max(1, |T_1(x)|)}.$$

This extends as a regular section nonzero at infinity, with zero divisor $\{T_1 = 0\}$. At ξ_j the norm is $|\pi|^j$, so the integral is $j\lambda$. At o it is zero and its negative logarithm is infinite. For fixed R, the tails are $(j\lambda - R)_+$ and are unbounded in j. The curvature metrics defining μ_j are moving; their uniform convergence is not claimed. □

Coherent tensor projection for finite analytic maps

Fix a complete field K with its specified real-valued non-Archimedean absolute value. Trivial valuation and generalized affinoid radii are allowed. Spaces in this supplement are good and Hausdorff. Coherent modules live on the analytic G-site. A line bundle means a coherent module which is free of rank one on ordinary open neighbourhoods. The symbol q^* on coherent modules denotes ordinary tensor pullback; it is not a derived functor.

For a norm-admissible finite morphism $q : Y \to X$, the full inverse image of $V = \mathcal{M}(A)$ is $\mathcal{M}(B)$, where B is finite over A with its canonical finite-module norm. If N is finite over B, compose controlled finite free presentations $A^a \twoheadrightarrow B$ and $B^b \twoheadrightarrow N$. Their lifting bounds compare the iterated quotient norm with the canonical finite-A-module norm. Thus restriction of scalars also preserves the canonical topology on N. Every tensor used below between finite modules has its canonical topology. The comparison with the completed analytic tensor is provided by NAG-001565 and NAG-001702. Algebraic finiteness with an unrelated Banach norm would not suffice.

Proposition G.0.5 (Coherent tensor projection)**.** *Let $q : Y \to X$ be norm-admissible and finite. For $\mathcal{F} \in \mathrm{Coh}(Y_G)$ and $\mathcal{E} \in \mathrm{Coh}(X_G)$ there is a canonical isomorphism*

$$q_*(\mathcal{F} \otimes_{\mathcal{O}_Y} q^*\mathcal{E}) \simeq q_*\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}. \tag{G.0.7}$$

It commutes with affinoid-domain restriction and analytic base change. Neither $\mathcal{F}$ nor $\mathcal{E}$ is required to be locally free.

Direct inputs NAG-001565, NAG-001624, NAG-001702, and NAG-001752.

Proof. On $V = \mathcal{M}(A)$ and its full inverse image $\mathcal{M}(B)$, let M and N represent $\mathcal{E}$ and $\mathcal{F}$. The map

$$N \otimes_B (B \otimes_A M) \longrightarrow N \otimes_A M, \qquad n \otimes (b \otimes m) \longmapsto bn \otimes m$$

has inverse $n \otimes m \mapsto n \otimes (1 \otimes m)$. These are maps of finite A-modules, so they are bounded, with bounded inverses, for the canonical norms. The finite-tensor theorem identifies the modules with the analytic completed constructions.

For a bounded affinoid base change $A \to C$, put $D = B \otimes_A C$. The same formulas identify both base changes with $N \otimes_A M \otimes_A C$. In particular they

respect domain restrictions. The finite coherent direct-image theorem and affinoid coherent equivalence therefore glue them to (G.0.7). This argument concerns an isomorphism of tensors, and asserts no exactness for tensoring with an arbitrary $\mathcal{E}$. □

Norm integrals and logarithmic limits for finite models

Theorem G.0.6 (Norm integration under a component-degree criterion)**.** *Let $q : \mathcal{Y} \to \mathcal{X}$ be finite locally free of constant rank $N \geq 1$, with both models in the class of the standing finite-model data and with integral generic fibres Y, X. Give q^*L_i and q^*M their pulled-back model metrics. For each V, set*

$$B_V = (q_*\mathcal{O}_{\mathcal{Y}})_{\eta_V}, \qquad A_V = \mathcal{O}_{\mathcal{X},\eta_V}.$$

Retain the ordinary algebraic local descriptions: B_V is the whole finite free rank-N algebra, and all its maximal localizations are the DVRs $\mathcal{O}_{\mathcal{Y},\eta_W}$ for the components W over V. Let t_V be a uniformizer of A_V, and put

$$r_W = \mathrm{ord}_W(t_V), \quad f_W = [\kappa(W) : \kappa(V)], \quad \delta_W = \deg_{q^*\mathcal{L}_1|_W,\dots,q^*\mathcal{L}_d|_W,k}(W).$$

Assume the finite list of component-degree identities

$$\delta_W = f_W \delta_V \quad (W \mid V). \tag{G.0.8}$$

For a nonzero rational section s of q^*M, use the determinant section $\mathrm{Nm}_q(s)$ of $\mathrm{Nm}_q(q^*M) \simeq M^{\otimes N}$, with the metric induced by the model line $\mathcal{M}^{\otimes N}$ and root exponent e_0. Then

$$q_*^{\mathrm{an}}\mu_{\mathcal{Y}} = N\mu_{\mathcal{X}}, \tag{G.0.9}$$

$$\int_{X^{\mathrm{an}}} -\log\|\mathrm{Nm}_q(s)\|\, d\mu_{\mathcal{X}} = \int_{Y^{\mathrm{an}}} -\log\|s\|\, d\mu_{\mathcal{Y}}. \tag{G.0.10}$$

Both logarithmic integrals are absolutely convergent with respect to total variation. All components, full residue degrees and unreduced DVR quotients are retained, including in inseparable cases. The criterion holds for the coordinate power map $T_i = S_i^{n_i}$ on $(\mathbf{P}^1_R)^d$, with $n_i \geq 1$, $\mathcal{L}_i = \mathrm{pr}_i^*\,\mathcal{O}(1)$, and $e_i = 1$.

Direct inputs `NAG-002543`, `NAG-002552`, Theorem G.0.3, the mixed algebraic degree convention.

Proof. Each local DVR extension gives

$$b_W = r_W b_V, \qquad \sum_{W|V} f_W r_W = N. \tag{G.0.11}$$

The first equality follows by writing $\pi = ut_V^{b_V}$ with u a unit. For the second, apply NAG-002543 to the regular sequence (t_V) in A_V and the full algebra B_V. Its summand at W is $f_W r_W$ because $\operatorname{length}_{\mathcal{O}_{\mathcal{Y},\eta_W}}(\mathcal{O}_{\mathcal{Y},\eta_W}/t_V) = r_W$. This proof does not require a separable residue extension. Equations (G.0.2) and (G.0.11) give $q^{\mathrm{an}}(\xi_W) = \xi_V$. Thus the mass pushed to ξ_V is

$$\sum_{W|V} \frac{b_W \delta_W}{E} = \frac{b_V \delta_V}{E} \sum_{W|V} r_W f_W = N \frac{b_V \delta_V}{E},$$

which proves (G.0.9).

Let $F = K(X)$ and $L = K(Y)$. The generic algebra $B_V \otimes_{A_V} F$ is L, a field extension of full degree N. For $a \in L^\times$, NAG-002552 gives

$$\operatorname{ord}_V \operatorname{Nm}_{L/F}(a) = \sum_{W|V} f_W \operatorname{ord}_W(a). \tag{G.0.12}$$

To include a rational a, choose a nonzero $z \in A_V$ with $za \in B_V$, possible because B_V spans L over F. Apply that lemma to za and to z, and subtract. Multiplication by a on L is invertible even for an inseparable extension; its determinant, rather than a product over distinct embeddings, is the norm used here.

In a local frame e of $\mathcal{M}$, write $s^{\otimes e_0} = aq^*e$. The coefficient of $(\operatorname{Nm}_q s)^{\otimes e_0}$ in $e^{\otimes N}$ is $\operatorname{Nm}_{L/F}(a)$. Indeed, exterior multiplication on the rank-N module gives $\det(m_a)$, while changing e by a unit u multiplies the determinant frame by u^N. These changes are exactly those of $\mathcal{M}^{\otimes N}$, so the construction glues and has the stated pulled-back tensor-power metric. Consequently, if $c_W = \operatorname{ord}_W(a)$, then the downstairs coefficient is $\sum_{W|V} f_W c_W$. Theorem G.0.3 now gives

$$\int -\log \|\operatorname{Nm}_q(s)\| \, d\mu_{\mathcal{X}} = \frac{\lambda}{e_0 E} \sum_V \delta_V \sum_{W|V} f_W c_W = \frac{\lambda}{e_0 E} \sum_W \delta_W c_W.$$

The last expression is the upstairs integral. Absolute convergence is part of the same finite-sum theorem.

For the coordinate power map, the exponent classes $\prod_i S_i^{a_i}$, $0 \le a_i < n_i$, are a free basis on every coordinate chart. They give finite flat rank $N = \prod_i n_i$ and the full special-fibre field degree N, including inseparable degrees. The single reduced component has mixed degree one downstairs. Upstairs the pullback of the ith coordinate line is $\mathrm{pr}_i^* \mathcal{O}(n_i)$. Choose the coordinate

section with divisor $n_i\{S_i = 0\}$ for each i. Their proper intersection at the rational product origin has local algebra

$$k[S_1, \dots, S_d]/(S_1^{n_1}, \dots, S_d^{n_d}),$$

whose monomial basis has length N. This computes the mixed degree as N and verifies (G.0.8) in the stated example. □

Assume in this section that X^{an} is compact Hausdorff; the projective case is supplied by NAG-001714. Let s be a nonzero global regular section of a line bundle M, $D = Z(s)$, and h a continuous metric. Extend $g = -\log \|s\|_h$ by $+\infty$ on D^{an}. It is bounded below. For each j, let h_j be a continuous metric and μ_j a positive finite model measure on X^{an}, with its generic divisorial atoms as above. Let $g_j = -\log \|s\|_{h_j}$ at those atoms, and assume

$$\varepsilon_j = \sup_{x \in X^{\mathrm{an}}} \left| \log \frac{\| \cdot \|_{h_j,x}}{\| \cdot \|_{h,x}} \right| \longrightarrow 0, \qquad \mu_j \Longrightarrow \mu, \tag{G.0.13}$$

where μ is a finite positive Radon measure and convergence is against all real continuous functions. The fibrewise ratio uses any nonzero vector and is independent of that vector.

Theorem G.0.7 (Uniform logarithmic tails suffice)**.** *In the preceding setup write*

$$T_j(R) = \int (g_j - R)_+ \, d\mu_j = \sum_V a_{j,V} \big(g_j(\xi_{j,V}) - R\big)_+, \qquad a_{j,V} = \mu_j(\{\xi_{j,V}\}) \geq 0.$$

Suppose

$$\lim_{R \to \infty} \sup_j T_j(R) = 0. \tag{G.0.14}$$

Then $\mu(D^{\mathrm{an}}) = 0$, $g \in L^1(\mu)$ and

$$\int g \, d\mu = \lim_{j \to \infty} \int g_j \, d\mu_j. \tag{G.0.15}$$

If h_j and the metrics defining μ_j have the common model data of the standing finite-model data, the hypothesis is the explicit coefficient condition

$$\lim_{R \to \infty} \sup_j \sum_V \frac{\delta_{j,V}}{e_{0,j} E_j} \big(\lambda c_{j,V} - e_{0,j} b_{j,V} R\big)_+ = 0. \tag{G.0.16}$$

In particular a common upper bound on $c_{j,V}/(e_{0,j}b_{j,V})$ suffices. More generally, if $g_j \geq C$ at all atoms and for some $\alpha > 0$ and $A < \infty$ one has

$$\sup_j \sum_V a_{j,V}(g_j(\xi_{j,V}) - C)^{1+\alpha} \leq A, \tag{G.0.17}$$

then $T_j(R) \leq A/(R-C)^\alpha$ for $R > C$, so the conclusion holds.

Direct inputs Theorem G.0.3, NAG-001714.

Proof. Because $\|s\|_h$ is continuous and bounded on the compact space, g is bounded below. Each function $g_R = \min(g, R)$ is a finite continuous function, including at D^{an}: it is the continuous capped function of the nonnegative quantity $\|s\|_h$. Metric convergence implies $|g_j - g| \leq \varepsilon_j$ off D^{an} and hence at every model atom by Theorem G.0.3. The sequence ε_j is bounded. Thus all these atom values have a common lower bound C, and g can also be taken at least C. Let $m_j = \mu_j(X^{\mathrm{an}})$. Weak convergence tested on 1 makes m_j bounded. The map $u \mapsto \min(u, R)$ is 1-Lipschitz, so

$$\left| \int \min(g_j, R)\, d\mu_j - \int g_R\, d\mu_j \right| \leq \varepsilon_j m_j. \tag{G.0.18}$$

For fixed R, its left integrals therefore have common limit $I_R = \int g_R\, d\mu$.

Put $I_j = \int g_j\, d\mu_j$ and $T(R) = \sup_j T_j(R)$. The exact identity

$$I_j = \int \min(g_j, R)\, d\mu_j + T_j(R)$$

gives

$$I_R \leq \liminf_j I_j \leq \limsup_j I_j \leq I_R + T(R). \tag{G.0.19}$$

For some fixed sufficiently large R_0, $T(R_0)$ is finite. Since g_{R_0} is bounded and the m_j are bounded, the I_j are bounded above; their lower bound follows from Cm_j. Hence I_R is bounded above as R grows. By the definition of the positive integral, $\sup_R \int (g_R - C)\, d\mu = \int (g - C)\, d\mu$: every bounded nonnegative simple function below $g - C$ lies below $g_R - C$ once $R - C$ exceeds its largest value. Thus g has finite integral. The simple functions $n\, 1_{D^{\mathrm{an}}} \leq g - C$ then force $\mu(D^{\mathrm{an}}) = 0$, and the lower bound gives $g \in L^1(\mu)$. Letting $R \to \infty$ in (G.0.19) proves (G.0.15).

On a common model, substitute $a_{j,V} = b_{j,V}\delta_{j,V}/E_j$ and $g_j(\xi_{j,V}) = \lambda c_{j,V}/(e_{0,j}b_{j,V})$. Since $b_{j,V}, e_{0,j} > 0$ and $\delta_{j,V} \geq 0$, the expression for $T_j(R)$

becomes exactly the sum in (G.0.16). A uniform upper bound on the indicated ratios makes it zero for large R. Finally, for $x \geq C$ and $R > C$,

$$(x - R)_+ \leq \frac{(x - C)^{1+\alpha}}{(R - C)^{\alpha}}.$$

Summing proves the final sufficient condition. □

Lemma G.0.8 (A finite-group orbit tensor with trivial inertia)**.** *Let S be affine, let X/S be separated, and let a finite group G act on X by automorphisms covering automorphisms of S. Let $I \triangleleft G$, and let $C \hookrightarrow X$ be a G-stable closed subscheme on which every element of I acts identically. If $\mathcal{A}$ is relatively ample on X/S, then*

$$\mathcal{N} = \bigotimes_{g \in G} \rho_g^* \mathcal{A} \tag{11}$$

is relatively ample and has its canonical permutation G-linearization. Its restricted I-action on $\mathcal{N}|_C$ is the identity. If $H = G/I$, and $\bar{\rho}_h$ denotes its induced action on C, the underlying restricted invertible sheaf is

$$\mathcal{N}|_C \simeq \bigotimes_{h \in H} (\bar{\rho}_h^*(\mathcal{A}|_C))^{\otimes |I|}. \tag{12}$$

The construction commutes with equivariant pullbacks and specified transition isomorphisms of the input sheaves. It therefore applies to compatible towers. No group order is inverted.

Proof. For $h \in G$, the factors of $\rho_h^* \mathcal{N}$ are $\rho_{gh}^* \mathcal{A}$, indexed by $g \in G$. Reindexing by $g \mapsto gh$ gives λ_h. Associativity of pullback and the usual symmetry of tensor products show that the two maps for any composite agree: both reindex the same factors in the same way. The identity gives the identity map. This proves the linearization, including all noncommutative group relations.

On C, the action factors through H. The factors indexed by a right coset gI are canonically the same invertible sheaf $\bar{\rho}_{gI}^*(\mathcal{A}|_C)$. Grouping these factors gives (12). For $i \in I$, right multiplication by i permutes factors within each such coset. Locally trivialize each of the finitely many coset sheaves. A permutation of copies of a rank-one module fixes the tensor of their chosen generator, and fixes every coefficient because $\rho_i|_C$ is the identity. Hence it is the identity of the full restricted module. This includes nilpotents; no conclusion is being drawn merely from pointwise characters.

Ampleness is unchanged by an isomorphism of schemes. Over the affine base, relative and absolute ampleness agree by Stacks Tag 01VK. Finite tensor products of these ample sheaves are ample: use the external-product assertion Stacks Tag 0DNK over the affine, hence quasi-separated, base, and pull back by the closed diagonal using Stacks Tag 0B3E; then induct on the number of factors. In the section argument for the external-product assertion, sections of powers n and m are first raised to powers m and n, so their product is a section of the common power nm. This is the same ampleness argument used in the audited cyclic orbit construction, with a finite group as index set.

Finally, an equivariant pullback identifies each indexed factor with the corresponding pulled-back factor. Tensoring the given maps on these factors commutes with every permutation above. This proves the asserted transition compatibility and the tower statement. □

Let A be a nonzero Noetherian ring. Set

$$S = A \setminus \bigcup_{\mathfrak{p} \in \operatorname{Min}(A)} \mathfrak{p}, \qquad T(A) = \ker\left(A \longrightarrow \prod_{\mathfrak{p} \in \operatorname{Min}(A)} A_{\mathfrak{p}} \right), \qquad A^{\mathrm{u}} = A/T(A). \tag{1.1}$$

We call $T(A)$ the embedded torsion ideal. No reduction of the generic Artinian local rings is performed.

Supplied hypothesis (SC). For a finite module over a generalized affinoid algebra, the S_1 property of the algebraic localizations is equivalent to the corresponding analytic property. The property is preserved on analytic domains.

Proposition G.0.9 (Coherent embedded torsion quotient; F8U-G)**.** *Assume (SC) and the stated v0.38 coherent-domain interfaces. On a good Hausdorff pure n-dimensional analytic X, the ideals $T(A)$ on affinoid domains glue to a coherent ideal $\mathcal{T}$. The closed subspace*

$$i : X^{\mathrm{u}} = (X, \mathcal{O}_X/\mathcal{T}) \longrightarrow X \tag{3.1}$$

is finite, is a homeomorphism on underlying points, is pure of analytic dimension n, and satisfies S_1. Every regular analytic local function on X remains regular on X^{u}. Pullback sends $\operatorname{Int}(X/K)$ *into* $\operatorname{Int}(X^{\mathrm{u}}/K)$.

Proof. Let $A \to B$ represent an actual nonempty affinoid domain of $\mathcal{M}(A)$. Restriction is exact on coherent sheaves, and NAG-001624 identifies it with $B \otimes_A -$ on finite modules. In particular $TB = T(A) \otimes_A B$ injects into B. The lower-dimensional support bound of NAG-002669 applies to the original torsion ideal. Its support is the restriction of $\operatorname{Supp} T(A)$, so has analytic dimension at most $n-1$, by domain base change NAG-001846 and dimension locality. The domain is pure of dimension n by NAG-002413. Therefore $TB \subseteq T(B)$: a module with smaller-dimensional support has zero localization at every minimal prime of B.

The quotient B/TB is the actual domain algebra obtained by base change from A^{u}, using NAG-001565 and NAG-001846. By Lemma C.1.36, A^{u} is S_1; (SC) makes B/TB S_1 as well. Its underlying space is unchanged by the nilpotent ideal TB and remains pure of dimension n. The image of $T(B)$ in B/TB has support of dimension at most $n-1$. A nonzero submodule of a Noetherian S_1 pure structure module has an associated prime among its

minimal primes, so cannot have smaller-dimensional support. The image is zero, proving

$$T(B) = T(A) \otimes_A B. \tag{3.2}$$

Thus the coherent ideals coincide on all actual affinoid intersections and satisfy the cocycle condition. The coherent sheaf equivalence NAG-001624 glues them, without assuming a finite global cover.

Each local quotient is by a finite nilpotent ideal, as in the actual closed quotient of NAG-001653, so (3.1) is a finite closed immersion and a homeomorphism. Its analytic dimension and purity agree with those of X; (SC) gives the asserted S_1 property. Regularity follows locally from Lemma C.1.36. Finite morphisms are relatively interior by NAG-001701; composition of interiors, NAG-001704, proves the last claim. In particular a boundaryless X gives boundaryless X^{u}. □

Finite logarithmic graph: weights and persistence.

Use the finite logarithmic graph and its notation from the setup preceding the actual source-image lemma. In particular the field, the tuples $h, (g_0, f_0)$, and the domains $V_0, V_{0,S}$ are those specified there.

Lemma G.0.10 (The whole-fibre weight is one; F8GP-4)**.** *For every S, the h-moment chain of $V_{0,S}$ is*

$$\mathrm{Trop}_{h,V_{0,S}}(\mathcal{O}_{V_{0,S}}) = \mathcal{P}|_{\{t\circ F\le S\}}. \tag{3.2}$$

Each full-dimensional product cell that survives truncation has weight one. Lower-dimensional pieces contribute no n-chain.

Proof. Apply the frozen compact-chain theorem to the pure space $V_{0,S}$ and tuple h. Its construction subdivides the actual image and its boundary image, the latter of dimension at most $n-1$. Choose w in the relative interior of an n-cell of P_S, away from that boundary image and from the product's proper faces. Thus the *entire* h-fibre over w is interior over K, and the local image is in one product cell's affine n-plane. These are precisely the two flat-germ hypotheses required by NAG-002362. This use of the frozen theorem is only its existing compact-domain scope. For a factor of type D_i or E_i, set $q_i = T_i$; for a factor of type H_i, set $q_i = T_i - 1$. Adjoin the Z_ν as the remaining coordinates. The tangent lattice of this product cell is carried isomorphically onto $\mathbb{Z}^n$ by these selected moment coordinates: its basis vectors are respectively $(1,1)$, $(0,1)$, $(1,0)$, and the torus coordinate vectors in disjoint coordinate blocks.

The algebra of the *entire* h-fibre over w is the exact-radius Laurent algebra $R_q = A_{K,r(w)}$. To check the assertion as an algebra statement, not just a set equality, express the omitted coordinates and their inverses in R_q. On a D_i-factor, $|q_i| > 1$, and

$$(T_i - 1)^{-1} = q_i^{-1} \sum_{\ell\ge 0} q_i^{-\ell}.$$

On an E_i-factor, $|q_i| < 1$, and

$$(T_i - 1)^{-1} = -\sum_{\ell\ge 0} q_i^{\ell}.$$

On an H_i-factor, $|q_i| < 1$, and

$$T_i^{-1} = (1+q_i)^{-1} = \sum_{\ell\ge 0} (-q_i)^{\ell}.$$

These series converge in the exact-radius norm. The simultaneous level relations give bounded homomorphisms in both directions; they fix all coordinate generators and hence are inverse. All inequalities defining $V_{0,S}$ have fixed values on this whole fibre, so no factor or sheet is removed by them.

The choice of w does not require $t \circ F < S$. If $t \circ F$ is constant on an n-cell and equals S, that whole cell is part of the actual image P_S. The frozen boundary-image dimension bound still permits the same generic interior-fibre choice on it. No equality-level cell is removed.

For the selected transverse coordinates, the lattice divisor is one and the actual Gauss algebra is

$$R_q \widehat{\otimes}_{R_q} H_q = H_q.$$

Its dimension over H_q is one. The frozen module-weight theorem therefore gives weight one at w. This determines the coefficient of every surviving n-cell. The finite-chain theorem, subdivision independence, and lemma C.1.39 give (3.2). □

Theorem G.0.11 (Persistent logarithmic graph; F8GP-1)**.** *For every real S, put*

$$V_{0,S} = \{z \in V_0 : |f_0(z)| \geq e^{-S}\}, \qquad P_S = P \cap \left\{ \lambda + \sum_i m_i s_i + b \cdot x \leq S \right\}. \tag{1.7}$$

Every nonempty $V_{0,S}$ is pure of analytic dimension n. The following chain is defined once, independently of S:

$$C_0 = F_* \mathcal{P}. \tag{1.8}$$

The pushforward counts each full-rank cell with its lattice index and discards cells on which F has rank below n. It is a finite nonnegative integral n-chain, and

$$\boxed{\mathrm{Trop}_{(g_0,f_0),V_{0,S}}(\mathcal{O}_{V_{0,S}}) = C_0|_{\{t \leq S\}}} \tag{1.9}$$

for every S, with zero chain for the empty domain. Its support lies in

$$\left(\kappa + M \left(\prod_i [-A_i, B_i] \times Q \right) \right) \times [t_{\min}, \infty), \quad t_{\min} = \lambda - \sum_i m_i A_i + \min_Q b \cdot x. \tag{1.10}$$

Thus the same finite weighted chain works at all levels, including levels where a full-dimensional cell lies in $\{t = S\}$.

Proof of theorem G.0.11. Purity is lemma C.1.40. On $V_{0,S}$, all entries of h and (g_0, f_0) are units. Their actual moment maps satisfy $t_{(g_0,f_0)} = F \circ t_h$. If $V_{0,S}$ is empty, the actual source image makes $\mathcal{P}|_{\{t \circ F \le S\}}$ zero. The closed-cutoff pushforward then gives the required zero chain. Otherwise adjoin the actual constant units $a, c_1, \dots, c_r$ to h. The whole level algebras and the transverse Gauss dimensions are unchanged, and the saturated tangent lattices embed with zero constant coordinates and index one. The translated formula is now a literal character map for the identity finite map of $V_{0,S}$. Use whole-interior projection off the target absolute boundary image. That lower-dimensional image does not contain any top-dimensional cell of a full-rank image, even at $t = S$. On a common subdivision the finite-module chain and weights identify the closed chains from their generic coefficients. This gives

$$\operatorname{Trop}_{(g_0,f_0),V_{0,S}}(\mathcal{O}) = F_* \operatorname{Trop}_{h,V_{0,S}}(\mathcal{O}).$$

Substitute (3.2), then (3.3), to obtain (1.9). The bounds in (1.10) follow directly from $y_i \in [-A_i, B_i]$, $x \in Q$, and $s_i \ge -A_i$. The finite chain is constructed before any S is chosen. □

Write $A_0 = \mathcal{O}(V_0)$ for the algebra of the specified graph domain.

Theorem G.0.12 (Persistence under covers and perturbations; F8GP-7)**.** *Let $p : V = \mathcal{M}(B) \to V_0 = \mathcal{M}(A_0)$ be finite with its canonical module norm, and suppose B is a locally free A_0-module of constant positive rank e. Assume V is pure of dimension n. Let $u_0, u_1, \dots, u_r \in B^\times$ satisfy*

$$|u_j(z)| = 1 \quad \text{for every } z \in V. \tag{5.1}$$

For

$$f = p^* f_0 \, u_0, \qquad g_\ell = p^* g_{0,\ell} \, u_\ell, \qquad V_S = \{|f| \ge e^{-S}\},$$

the function f is regular, every nonempty V_S is pure of dimension n, and one single chain

$$C = eC_0 \tag{5.2}$$

satisfies

$$\boxed{\operatorname{Trop}_{(g,f),V_S}(\mathcal{O}_{V_S}) = C|_{\{t \le S\}}} \tag{5.3}$$

for every real S. The same bound (4.2) applies. Neither separability, reducedness of V, nor global freeness of B is required.

A concrete sufficient condition for (5.1) is $u_j = 1 + \varepsilon_j$ with $\|\varepsilon_j\| < 1$ in a specified complete submultiplicative affinoid norm on B.

Proof. The norm identities give, on all points,

$$|f| = |p^* f_0|, \qquad |g_\ell| = |p^* g_{0,\ell}|, \qquad V_S = p^{-1}(V_{0,S}). \tag{5.4}$$

Thus no source sheets are selected or omitted. The map on V_S is the actual finite base change; its direct-image module is still locally free of rank e. Purity of V_S follows from domain purity applied to V. Flatness of B over A_0 preserves injectivity of multiplication by f_0. Multiplication by u_0 is invertible, proving regularity of f.

If $V_{0,S}$ is empty, (5.4) makes V_S empty and (1.9) gives (5.3). Otherwise apply whole-interior projection to $p|_{V_S}$ over the original field, using the literal pullbacks of the target tuple and the identity map on the moment coordinates. By (5.4) the norm-one twists have the same moment map and the same whole analytic level domains as those literal pullbacks. The relative interiors of target top cells avoid the target absolute boundary image after a common subdivision. The finite-module chain and weights give equality of the closed finite chains from the generic equalities. This gives

$$\mathrm{Trop}_{(g,f),V_S}(\mathcal{O}_{V_S}) = \mathrm{Trop}_{(g_0,f_0),V_{0,S}}(p_*\mathcal{O}_{V_S}). \tag{5.5}$$

At a generic target cell, let E_Q be its complete transverse Gauss algebra. The tensor of the locally free module $p_*\mathcal{O}_{V_S}$ with E_Q has constant rank e. The finite-dimensional Artinian algebra E_Q is a product of local factors; on each factor a finite locally free module of constant rank e is free of that rank. Consequently

$$\dim_{H_Q}\big(p_*\mathcal{O}_{V_S} \otimes E_Q\big) = e \dim_{H_Q} E_Q.$$

The same lattice divisor occurs in both module-weight formulas. Therefore the right side of (5.5) is $e\,\mathrm{Trop}_{(g_0,f_0),V_{0,S}}(\mathcal{O})$. Equation (1.9) proves (5.3). This dimension calculation retains all Artinian lengths and full residue degrees, including inseparable degrees.

A positive-rank locally free algebra has zero structural kernel, and the finite-image theorem makes p surjective. Thus the bounded unit-moment base in (1.10) is still the actual moment image of V. The scalar e changes weights, not that base or the height bound in (4.2).

Finally, if $\|\varepsilon\| < 1$, the Neumann series $\sum_{\ell\geq 0}(-\varepsilon)^\ell$ converges to the inverse of $1+\varepsilon$. Every bounded multiplicative seminorm satisfies $|\varepsilon(z)| \leq \|\varepsilon\| < 1$; this follows directly by applying boundedness to powers and taking roots. The strong triangle inequality then gives $|1+\varepsilon(z)| = 1$, proving the sufficient condition. □

Fix the original complete non-Archimedean field K and its absolute value; trivial valuation and generalized real radii are allowed. Let X be good, Hausdorff, and pure of analytic dimension $n \geq 1$. Fix $C \Subset \mathrm{Int}(X/K)$, an ordinary open neighbourhood $U \supset C$, and $f \in \mathcal{O}(U)$. Require:

S1. Every ordinary local ring of U satisfies S_1.

S2. Multiplication by f is injective on $\mathcal{O}_{U_G}$. Thus "regular" means an actual non-zero-divisor, not merely a nonzero function on one component.

S3. The actual zero space $D = V(f) \subset U$, with its quotient structure, is empty or pure of analytic dimension $n-1$.

The test η is an ordinary smooth moment form of bidegree $(n-1, n-1)$, with $\operatorname{Supp}\eta \subset C$. We use the structure sheaf coefficients in the scalar PL assertion. An arbitrary coherent coefficient version is not silently substituted. All dimensions here are analytic dimensions. In particular S3 is not a statement about ordinary Krull dimension alone.

Chart condition (IP). For a pure n-dimensional affinoid V, a regular analytic f, nonempty pure $(n-1)$-dimensional D_V, and a unit moment tuple g, assume every actual affinoid-domain algebra in V is S_1. Then for some H, all $H < a < b$ satisfy

$$\mathcal{T}_{(g,f)}\big(\mathcal{O}_{\{e^{-b}\leq|f|\leq e^{-a}\}}\big) = \mathcal{T}_g(\mathcal{O}_{D_V}) \times [a,b]$$

in the last coordinate $-\log|f|$, with full actual quotient weights and the standard interval lattice.

Lemma G.0.13 (Actual chart hypotheses; F8SA-C)**.** *Under S1–S3, choose finitely many affinoid domains $V_i \subset U$ whose ordinary interiors cover C, and unit moment representatives $\eta|_{V_i} = t_{g_i}^{*}\alpha_i$. For every nonempty intersection $V_I = \bigcap_{i\in I} V_i$, the hypotheses of IP hold whenever $D_I = D \cap V_I$ is nonempty. If $D_I = \varnothing$, a sufficiently small f-tube in V_I is empty.*

Proof. Goodness and the ordinary local definition of a smooth moment form [NAG-002424] supply such neighbourhoods; compactness gives a finite family. NAG-002411/NAG-002413 make each nonempty V_I an actual pure n-dimensional affinoid domain. Restricting the injective sheaf map in S2 and taking sections proves that f is a non-zero-divisor in its algebra. Quotient/domain base change NAG-001846 identifies D_I with the actual algebra quotient by f, not its reduction. It is an affinoid domain of D, hence is pure

of dimension $n-1$ by S3 and domain purity. R-S1 [NAG-002765] applied to every affinoid domain of V_I supplies IP's full affinoid-domain S_1 condition. A chosen g_i, $i \in I$, remains a tuple of units upon restriction. If D_I is empty, continuity on compact V_I gives $\min_{V_I} |f| > 0$. □

The ordinary cutoff presentations and the final topologies are those of the two generating families and their comparison. For an open $\Omega \subset \mathbb{R}^N$, let $E^{p,q}(\Omega)$ be the space of smooth ambient superforms with the coefficient seminorms

$$p_{L,m}(\eta) = \max_{I,J,\,|a|\le m} \sup_{u\in L} |\partial^a \eta_{I,J}(u)|, \qquad L \Subset \Omega.$$

Empty coefficient slots give the zero space.

Definition G.0.14 (Local retraction condition)**.** For a closed subset $Z \subset X$, condition (R_d) means that for every $z \in Z$ there are an ordinary neighbourhood $U_z \subset X$, a good Hausdorff space Y_z of local analytic dimension at most d, a closed immersion $i_z : Y_z \hookrightarrow U_z$ whose image contains z, and an analytic retraction $r_z : U_z \to Y_z$. The chosen image need only contain z; it need not contain all of $Z \cap U_z$. The dimension bound means that the smaller affinoid neighbourhoods used in Lemma F.1.6 have dimension at most d.

Theorem G.0.15 (A logarithmic current in the retraction scope)**.** *Let X be good, Hausdorff and pure of analytic dimension $n \ge 1$, and let $\mathcal{F}$ be a coherent $\mathcal{O}_{X_G}$-module. Let $g, h \in \mathcal{O}_X(X)$ be non-zero-divisors in every local ring and put*

$$Z = \{|g| = 0\} \cup \{|h| = 0\}, \qquad u = \log|g/h| \quad \text{on } X \setminus Z.$$

Assume (R_{n-1}) for Z. Then

$$T_{g/h,\mathcal{F}}(\alpha) = \int_{X\setminus Z,\,\mathcal{F}|_{X\setminus Z}} u\alpha \tag{G.0.20}$$

is well-defined and continuous on $(\mathcal{E}^{n,n}(X), \tau_{\mathrm{cut}}) = (\mathcal{E}^{n,n}(X), \tau_{\mathrm{ord}})$. The displayed functional is independent of the representatives of the ordinary test form. It is a current of bidimension (n,n) for this fixed regular quotient.

For each cutoff presentation Q_i, there is $C_i < \infty$ with

$$|T_{g/h,\mathcal{F}}(Q_i\eta)| \le C_i \max_{I,J} \sup_{t_{f_i}(V_i)} |\eta_{I,J}|. \tag{G.0.21}$$

There are smooth functions $u_M \in \mathcal{A}^{0,0}(X)$, $M \ge 0$, formed by constant smooth truncation of $\log|g|$ and $\log|h|$ near their zeros, such that for each i one number M_i satisfies

$$\int_{X,\mathcal{F}} u_M Q_i\eta = T_{g/h,\mathcal{F}}(Q_i\eta) \quad (M \ge M_i,\ \eta \in E^{n,n}(\Omega_i)). \tag{G.0.22}$$

The threshold is uniform over the whole fixed presentation, with no bound on η required.

Proof. Fix a cutoff presentation Q_i and put $K = \operatorname{supp}\chi_i$. For each $z \in K \cap Z$, Lemma F.1.6 applied to its fixed tuple makes every pulled-back (n,n)-form zero on one ordinary neighbourhood of z. A finite union O of these neighbourhoods covers $K \cap Z$. Therefore $K_i = K \setminus O$ is compact in $X \setminus Z$ and contains $\operatorname{supp} Q_i\eta$ for every η. The cutoff presentations span the test space, so every test has compact support in the complement. There g and h are locally analytic units, using NAG-001572 on smaller affinoids as in NAG-002427. Thus u is an ordinary smooth moment function. The product $u\alpha$ has compact support in the complement. The integral exists and is independent of its chart representatives by NAG-002425. Open restrictions are pure of dimension n by NAG-002413. These facts also prove linearity of (G.0.20).

For the fixed i and K_i, by the compact partition scope of NAG-002428, choose $\theta \in \mathcal{A}_c^{0,0}(X)$ supported in $X \setminus Z$ and equal to one near K_i. The function $b = \operatorname{ext}_0(\theta u)$ is ordinary smooth on X, and locality of the baseline integral gives

$$T_{g/h,\mathcal{F}}(Q_i\eta) = \int_{X,\mathcal{F}} b\, Q_i\eta.$$

Compute this integral on a finite affinoid cover of the fixed support of χ_i inside $\operatorname{Int}_{\mathrm{top}}(V_i, X)$, adjoining f_i to representatives of $b\chi_i$. The finite inclusion–exclusion formula the actual compact cover formula of NAG-002416 and the finite-module chain and the finite-cell functional give fixed cells, actual coherent module weights and fixed coefficient multipliers. On each cell, the absolute integral is at most its lattice-normalized volume times the absolute weight times a fixed constant times the right-hand coefficient supremum in (G.0.21), by NAG-002389. The projected coefficient arguments lie in $t_{f_i}(V_i)$. Summing these finitely many bounds proves (G.0.21) and continuity by the definition of the final topology. No positivity or integration by parts enters this estimate.

Choose a smooth real function ρ with $\rho = 0$ on $(-\infty, 0]$ and $\rho = 1$ on $[1, \infty)$, as explicitly constructed in NAG-002427. Put

$$\lambda(s) = s\rho(s+1), \qquad \lambda_M(t) = -M + \lambda(t+M).$$

Thus $\lambda_M(t) = t$ for $t \geq -M$ and is $-M$ for $t \leq -M-1$. For an analytic function a set

$$\ell_M(a)(x) = \begin{cases} \lambda_M(\log|a(x)|), & |a(x)| > 0, \\ -M, & |a(x)| = 0. \end{cases}$$

This is an ordinary smooth function: near a zero it is constant on $\{|a| < e^{-M-1}\}$, and on the nonzero locus it is a smooth function of a unit moment coordinate. The scalar sheaf in NAG-002427 glues these representatives. Set $u_M = \ell_M(g) - \ell_M(h)$.

If K_i is empty, every $Q_i\eta$ is zero and stationarity is immediate. Otherwise, the positive continuous functions $|g|$ and $|h|$ have positive minima on K_i. Choose $M_i \geq 0$ strictly larger than both negatives of the minimum logarithms there. For $M \geq M_i$, $u_M = u$ on a neighbourhood of K_i. Since every $Q_i\eta$ is supported in K_i, the two products agree as ordinary forms after extension by zero. This proves (G.0.22). It also gives absolute finiteness on every finite cell presentation: all coefficients and the logarithms are bounded on the relevant compact supports. No separate representation by Radon measures is required.

□

Proposition G.0.16 (A unit logarithm need not annihilate a positive current)**.** *For every $n \geq 2$, let $z_1, \dots, z_n$ be the coordinates on the analytic split torus over K, write $x_j = -\log|z_j|$ and $t = (x_1, \dots, x_n)$, and set*

$$\Omega = (1,2) \times (0,1)^{n-1}, \qquad U = t^{-1}(\Omega), \qquad h = x_1 = \log|z_1^{-1}|.$$

Then

$$T(\eta) = \int_{U,\mathcal{O}_U} x_1 \eta, \qquad \eta \in \mathcal{E}^{n,n}(U),$$

is a positive native current with an order-zero bound on every fixed coefficient stage. Nevertheless $d'd''h = 0$ and $d'd''(hT) \neq 0$. Thus positivity and these order-zero bounds do not suffice for harmonic frame invariance.

Proof. Closed polyannuli with boxes inside Ω are ordinary affinoid neighborhoods in U. They are integral and have analytic dimension n by NAG-002294; The affine analytification construction NAG-001707 and separated-space theorem NAG-001713 make the ambient torus good and Hausdorff. Thus U is good, Hausdorff and pure of dimension n. For the global units $g = z_1^{-1}$ and $h_0 = 1$, the zero set in NAG-002480 is empty. Its retraction condition is therefore vacuous. That frozen theorem constructs exactly T and its fixed-stage order-zero estimates, on all native top tests. Since $x_1 > 0$, positivity follows from NAG-002425. The unit calculation in Theorem C.1.64 gives $d'd''h = 0$ as a smooth form; its current conclusion is unavailable without closedness of T.

Choose a nonnegative nonzero $\psi \in C_c^\infty(\Omega)$ and use the native test

$$\alpha = \psi(x) \bigwedge_{j=2}^{n} (d'x_j \wedge d''x_j) \in \mathcal{E}^{n-1,n-1}(U).$$

Its support is compact: the inverse image of $\operatorname{supp}\psi$ is closed in a closed polyannulus whose box is compactly contained in Ω. For the coordinate moment map the actual whole-fibre Gauss module is the Gauss field itself, and the tangent lattice map is the identity. Thus its weight is one by NAG-002362, consistently with the degree-one identity case of NAG-002294. The global chain formula the finite-module chain and the finite-cell functional and the lattice convention identify coordinate top integrals with ordinary real integrals of their coefficients, without a factorial or a valuation-normalization factor.

Only the x_1 derivative survives in $d'd''\alpha$. Therefore Theorem C.1.63 gives

$$(d'd''(hT))(\alpha) = \int_\Omega x_1^2\,\partial_1^2\psi\,dx_1\cdots dx_n = 2\int_\Omega \psi\,dx_1\cdots dx_n > 0.$$

The second equality is the one-variable fundamental theorem applied twice, with vanishing boundary terms because ψ is compactly supported; iterated integration is the elementary calculation proved in NAG-002389. No analytic Stokes or Poincaré–Lelong theorem enters this computation. In particular T cannot be closed, by Theorem C.1.64. □

In the local Cartier setting of NAG-002772, use the full quotient $B = A/(f)$ by the given regular equation on the actual affinoid chart. The effective Cartier coefficients are those of that quotient. The purity of the zero space is supplied as stated below.

Theorem G.0.17 (Cartier integration comparison in the pure-zero-space scope; F8ST-I-R1)**.** *Let $D = \mathcal{M}(B)$ be a nonempty pure d-dimensional Cartier zero space as above, and let $g = (g_1, \dots, g_r)$ be analytic units on a neighbourhood of the chart. Put $t_g = -\log|g|$. For $\mathfrak{p} \in \operatorname{Min} B$, let $D_{\mathfrak{p}} = \mathcal{M}(B/\mathfrak{p})$ and $c_{\mathfrak{p}} = \operatorname{length}_{B_{\mathfrak{p}}} B_{\mathfrak{p}}$. Then the frozen integral chains and all their smooth top-form pairings satisfy*

$$\mathcal{T}_g(B) = \sum_{\mathfrak{p} \in \operatorname{Min} B} c_{\mathfrak{p}} \mathcal{T}_{g|D_{\mathfrak{p}}}(B/\mathfrak{p}), \tag{G.0.23}$$

$$\mathcal{I}_{B,g}(\alpha) = \sum_{\mathfrak{p} \in \operatorname{Min} B} c_{\mathfrak{p}} \mathcal{I}_{B/\mathfrak{p},g}(\alpha). \tag{G.0.24}$$

Here α is a smooth (d,d)-superform near the compact moment image. The same equality holds for ordinary compactly supported forms after chart localization and a finite partition. The coefficient $c_{\mathfrak{p}}$ is exactly the analytic Cartier coefficient of Theorem F.1.12.

In particular, on a good Hausdorff pure n-dimensional space, for a regular meromorphic f whose local numerator and denominator zero spaces are empty or pure of dimension $n-1$, the cycles formed from these presentations are independent among such presentations and glue. Their analytic Cartier-cycle pairing agrees with the frozen coherent-cycle pairing on every ordinary $(n-1,n-1)$-test with compact support in $\operatorname{Int}(X/K)$. This purity condition is retained explicitly to match the frozen integrator; it is not deduced here from a relative-dimension theorem. The local coefficient result in Theorem F.1.12 does not require it.

Proof. Take a common finite subdivision of the actual moment images and their boundary images for D and the $D_{\mathfrak{p}}$, using the finite-module chain and weights. At a relative interior point w of a top-dimensional cell, the full analytic moment fibre is interior to D. The local moment image is contained in the affine plane $w + N_{\mathbb{R}}$ for the saturated rank-d tangent lattice N. The boundary images have dimension at most $d-1$, so they miss these relative interiors. For an integral projection Q injective on the tangent lattice N, put

$$E_Q = D_w \widehat{\otimes}_{R_Q} H_Q, \qquad e_Q = [\mathbb{Z}^d : Q(N)].$$

Here D_w is the *actual algebra of the entire moment fibre* and H_Q is the projected polytorus Gauss field, precisely as in NAG-002362. That theorem, with structure module B, gives

$$m_{D,g}(w) = \frac{\dim_{H_Q} E_Q}{e_Q} = \sum_{\mathfrak{p} \in \operatorname{Min} B} c_{\mathfrak{p}}\, m_{D_{\mathfrak{p}},g}(w). \tag{G.0.25}$$

For an empty component fibre its weight is zero. This formula includes all local Artin lengths and residue degrees in E_Q; no separate residue-degree multiplier is attached to $c_{\mathfrak{p}}$. The lattice index is divided out exactly once.

The equality holds on every top cell. Lower-dimensional cells carry no d-chain term, so the finite-module chain and weights give (G.0.23). By the finite-cell functional, integration of the same tangential smooth coefficients on a common subdivision gives (G.0.24). This is a finite sum; no limiting or measurable-integration extension occurs. The ordinary compact integral and compact partitions give the ordinary-form assertion. Actual domain restriction of the coefficients is the effective domain transport in the Cartier theorem, so the local formulas agree on overlaps. For a regular meromorphic function in the stated pure-zero-space scope, subtract the two effective formulas. On each actual common affinoid refinement, quotient/domain base change and domain purity preserve the empty-or-pure dimension $n-1$ condition for all four equations. The antichain clause of the Cartier theorem therefore applies to both cross-products and proves representation independence. A compact test needs finitely many such representatives and partition terms, whose sum is consequently independent of these choices. Closed immersions have compact inverse support on compact sets and are finite and everywhere relatively interior (NAG-001701); interior composition (NAG-001704) therefore shows that a test supported in $\operatorname{Int}(X/K)$ has the required interior support on its zero spaces. □

Field steps in tame descent

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Throughout, a valued field is complete and nontrivially real-valued, with additive valuation v, actual valuation ring F°, maximal ideal $\mathfrak{m}_F$, value group Γ_F, and residue field $\widetilde{F}$. Finite extensions carry their unique extending absolute values. Their existence, uniqueness and completeness are NAG-001450. Simple residue roots lift by NAG-002049.

For a finite separable extension E/K, the *explicit tame hypotheses* in this patch mean all three conditions

$$\begin{gathered} e = [\Gamma_E : \Gamma_K], \qquad f = [\widetilde{E} : \widetilde{K}], \qquad [E : K] = ef, \\ e \in (K^\circ)^\times, \qquad \widetilde{E}/\widetilde{K} \text{ separable.} \end{gathered} \tag{G.0.26}$$

Thus defectlessness is an assumption, not a consequence of the other two conditions. An *actual unramified lift* means the finite separable residue lift constructed by NAG-002050; in the Galois case its valuation rings have the finite free etale torsor property of NAG-002262.

An *exact Kummer step* $F \subset F(s)$ of degree $n \geq 2$ means the precise coefficient class of NAG-002610:

$$\begin{gathered} n \in (F^\circ)^\times, \qquad \zeta \in F^\circ, \qquad \operatorname{ord}(\zeta) = \operatorname{ord}(\bar{\zeta}) = n, \\ s^n \in F^\times, \qquad v(s) > 0, \qquad jv(s) \notin \Gamma_F \quad (0 < j < n). \end{gathered} \tag{G.0.27}$$

Lemma G.0.18 (Corrected radicals and a value basis)**.** *Let E/F be finite with equal residue fields and $[E : F] = [\Gamma_E : \Gamma_F] = e$, where $e \in (F^\circ)^\times$. There is a tower*

$$F = F_0 \subset F_1 \subset \cdots \subset F_r = E, \qquad F_i = F_{i-1}(s_i), \qquad s_i^{n_i} \in F_{i-1}^\times, \tag{G.0.28}$$

with $n_i \geq 2$, $v(s_i) > 0$, and $[F_i : F_{i-1}] = [\Gamma_{F_i} : \Gamma_{F_{i-1}}] = n_i$. The residue fields are unchanged, $\prod_i n_i = e$, and the monomials $\prod_i s_i^{a_i}$, $0 \leq a_i < n_i$, have distinct value classes modulo Γ_F and form an F-basis of E. The empty tower is allowed when $e = 1$.

Every step is exact Kummer if either E/F is Galois or F contains all e-th roots of unity. Moreover, if N/F is finite with $\Gamma_N = \Gamma_F$ in a common valued

overfield, then $E \otimes_F N$ is the field EN, of degree and value index e over N and with residue field $\widetilde{N}$. If N contains all e-th roots of unity, the base-changed tower (G.0.28) *consists of exact Kummer steps of the same degrees.*

Direct inputs NAG-001450, NAG-002049, NAG-002610.

Proof. Suppose F_{i-1} has been constructed and its value group is not Γ_E. Choose $a \in E^\times$ whose value has order $n_i > 1$ in $\Gamma_E/\Gamma_{F_{i-1}}$. Choose $c \in F_{i-1}^\times$ with $v(c) = n_i v(a)$, and choose a unit $u \in F_{i-1}^\circ$ lifting the residue of a^{n_i}/c. This is possible because every intermediate residue field is the common residue field. The integer n_i divides e and is a unit. The simple root 1 of the reduction of $T^{n_i} - a^{n_i}/(cu)$ therefore lifts to $d \in 1 + \mathfrak{m}_E$. Set $s_i = a/d$. Then $s_i^{n_i} = cu$ and $v(s_i) = v(a)$. Multiplying s_i by a sufficiently large positive-value power of any fixed element of F makes its value positive and preserves all these assertions about its class and its power.

The powers $1, s_i, \dots, s_i^{n_i-1}$ have distinct value classes over F_{i-1}. A nonzero linear combination has a unique term of smallest value and cannot vanish. Thus its degree is at least n_i and, by the displayed polynomial, at most n_i. The fundamental inequality Stacks 0ASH now shows that its value index is exactly n_i and its residue degree is one. In particular Γ_{F_i} is exactly the subgroup generated by $\Gamma_{F_{i-1}}$ and $v(s_i)$.

These subgroups strictly increase in the finite quotient Γ_E/Γ_F, so the process stops. Products of the listed powers give distinct representatives of successive group cosets, hence distinct classes modulo Γ_F. At termination their number is e. Their degrees multiply to $e = [E : F]$, so the terminal field is E and they are the required basis. The unique-minimum argument also gives the value of every nonzero basis expansion.

If E/F is Galois, it is normal over every intermediate field, by Stacks 09DW. The minimal polynomial $T^{n_i} - s_i^{n_i}$ therefore splits in E; it has degree n_i and is separable. All n_i-th roots of unity belong to E. Reduction is injective on them: two with the same reduction coincide by uniqueness of simple-root lifts, Stacks 06RR. Their residues lie in $\widetilde{F_{i-1}}$, and Hensel lifting there, with the same uniqueness in E, places all these roots in F_{i-1}. The group of roots is cyclic by Stacks 09HX, since a degree-d polynomial has at most d roots. Hence a primitive root and its reduction both have order n_i. If instead $\mu_e \subset F$, the same conclusion follows because $n_i \mid e$, with reduction injective by Stacks 06RR. In either case the polynomial

splits in F_i, and its root multiplications give the cyclic Galois group. This proves (G.0.27).

Finally the same e monomials remain linearly independent over N, since $\Gamma_N = \Gamma_F$. The multiplication map $E \otimes_F N \to EN$ is therefore injective and has image of N-dimension e; its image is a finite-dimensional domain, hence a field and thus all of EN. Its value group contains Γ_E, of index e over Γ_N. Another application of Stacks 0ASH forces equality of the value index with e and equality of residue fields with $\widetilde{N}$. Applying the same basis argument to each initial segment of (G.0.28) preserves its degree and value index. If $\mu_e \subset N$, each segment has the required roots of unity with unchanged residue order, so is exact Kummer. □

Proposition G.0.19 (An actual unramified base change supplies the steps)**.** *Let E/K satisfy (G.0.26), without assuming it Galois. There is an actual unramified subextension M/K in E with $\widetilde{M} = \widetilde{E}$ and $[M : K] = f$. There is a finite Galois actual unramified extension N/K, containing this M in a common valued overfield, such that EN/N is a tower of exact Kummer steps, of total degree and value index e and with residue field $\widetilde{N}$. For $e = 1$ this tower is empty. Neither a Galois action on a supplied E-model nor a Galois extension EN/K is asserted by this conclusion.*

Direct inputs NAG-002049, NAG-002050, NAG-002052, NAG-002262, NAG-003143.

Proof. Write $k = \widetilde{K}$ and $l = \widetilde{E}$. Use the primitive residue generator and its lifted monic polynomial in the proof of NAG-002050, but lift its simple residue root inside E. This gives $\alpha \in E^\circ$ and $M = K(\alpha)$. The degree is at most f; the residue degree is at least f. The fundamental inequality Stacks 0ASH forces $[M : K] = f$, $\Gamma_M = \Gamma_K$ and $\widetilde{M} = l$. This is the same actual residue lift, with its chosen embedding in E. By (G.0.26), E/M has degree and value index e and unchanged residue field.

Choose a finite Galois residue extension k'/k containing l and all e-th roots of unity. It exists by taking the splitting field of a separable defining polynomial for l/k together with $T^e - 1$, using Stacks 09HU; the latter polynomial is separable because e is a unit. First lift k'/l over M by NAG-002050, obtaining N/M in a common algebraic overfield of E. Thus $[N : K] = [k' : k]$, $\widetilde{N} = k'$ and $\Gamma_N = \Gamma_K$.

To identify N/K as the actual Galois unramified lift, lift a primitive generator of k'/k inside N by Hensel. Its field over K has degree at least $[k' : k]$, by

its residue, and at most that degree, by its lifted polynomial. It is therefore N. All roots of the residue polynomial lift inside N, since its residue is k'; they are distinct and exhaust the roots of the lifted polynomial. Thus N/K is Galois and is precisely the lift of NAG-002052. Its valuation rings have the torsor property in NAG-002262. Hensel also lifts all roots of $T^e - 1$ from k' to N. Lemma G.0.18, applied to E/M and N/M, proves all the assertions about EN/N. The construction supplies field inclusions; it does not supply action maps on a model or assert normality over K. □

Theorem G.0.20 (A Galois-normal sequence of exact Kummer steps)**.** *Let L/K be finite Galois satisfying (G.0.26), with group G. Put $k = \widetilde{K}$ and $l = \widetilde{L}$. Then l/k is Galois, and its actual unramified lift $M \subset L$ satisfies*

$$1 \longrightarrow I = \mathrm{Gal}(L/M) \longrightarrow G \longrightarrow \mathrm{Gal}(l/k) \longrightarrow 1, \qquad |I| = e. \tag{G.0.29}$$

Here I is exactly the kernel of the action on l. It is abelian, and every subgroup of I is normal in G. There is consequently a chain

$$1 = J_0 \subset J_1 \subset \cdots \subset J_r = I, \qquad F_j = L^{J_j}, \qquad L = F_0 \supset \cdots \supset F_r = M, \tag{G.0.30}$$

whose nontrivial factors J_j/J_{j-1} have prime order and whose steps F_{j-1}/F_j are exact Kummer. All fields in (G.0.30) have residue field l. No roots of unity are assumed in K, and no cyclicity of I or splitting of (G.0.29) is assumed.

Direct inputs NAG-001450, NAG-002052, NAG-003143, NAG-003144.

Proof. Take the embedded $M = K(\alpha)$ constructed in Proposition G.0.19; let P be its lifted polynomial of degree f. It is minimal by the degree calculation there. Since L/K is normal, P splits in L. Every root is integral, by the unique extending absolute value and conjugacy. Reducing this factorization shows that the separable minimal polynomial of $\bar{\alpha}$ splits in l. Thus l/k is Galois. Every root of its polynomial lifts inside M, since $\widetilde{M} = l$; these lifts exhaust the roots of P. Consequently M/K is Galois, with reduction identifying its group with $\mathrm{Gal}(l/k)$, just as in NAG-002052. Restriction and Stacks 09DW give (G.0.29). An element fixing M fixes l. Conversely an element acting trivially on l fixes α by Stacks 06RR, hence fixes M. This identifies inertia and proves $|I| = [L : M] = e$.

We give the needed ramification argument to avoid importing a DVR-only cyclicity theorem. For $i \in I$ define

$$\theta_i : \Gamma_L/\Gamma_M \longrightarrow l^\times, \qquad \theta_i(v(a)) = \overline{i(a)/a}. \tag{G.0.31}$$

This is well defined: changing a by an element of the same value changes the ratio by $i(u)/u$ for a unit u, whose residue is one; changing the value by $v(b)$, $b \in M^\times$, has no effect. Multiplication shows that θ_i is a character. Since I fixes l, composition gives a homomorphism $I \to \mathrm{Hom}(\Gamma_L/\Gamma_M, l^\times)$. It is injective. Indeed use the radical tower of Lemma G.0.18 for L/M. If $\theta_i = 1$ and i fixes the previous field, then $i(s_j)/s_j$ is an n_j-th root of unity with residue one. It equals one by Stacks 06RR. Induction fixes every radical and hence all of L. The target of (G.0.31) is abelian, so I is abelian.

All values of (G.0.31) lie in $\mu_e(l)$, since the value quotient has order e. This finite group is cyclic by Stacks 09HX. For $g \in G$, valuation preservation gives

$$\theta_{gig^{-1}}(\gamma) = \bar{g}(\theta_i(\gamma)). \tag{G.0.32}$$

To see this directly, represent γ by a and apply g to $i(g^{-1}a)/(g^{-1}a)$; the value of $g^{-1}a$ is still γ. If $t = |\mu_e(l)|$, the automorphism $\bar{g}$ of this cyclic group is exponentiation by some integer q_g coprime to t. Thus injectivity of (G.0.31) implies $gig^{-1} = i^{q_g}$. Also (G.0.31) implies $i^t = 1$ for every i. Exponentiation by q_g therefore preserves every subgroup of I, proving its normality in G.

A finite abelian group has a series with prime cyclic factors: take a maximal proper subgroup and induct on its order; the quotient is simple abelian, so any nonidentity element generates it and its order is prime. All groups in the resulting series are G-normal by the preceding paragraph. The correspondence Stacks 09DW gives (G.0.30) and its cyclic prime-degree extensions. Their residue fields all equal l, since they lie between M and L. The product of their degrees and the product of their value indices both equal e. Each index is at most its degree by Stacks 0ASH, so equality holds at every step. For a prime-degree step its value quotient has prime order, hence is cyclic. The first radical in Lemma G.0.18 already generates that step; its Galois clause supplies a primitive root with primitive reduction. This is exactly (G.0.27) for each step. □

Proposition G.0.21 (A noncyclic obstruction surviving unramified base change)**.** *Give $\mathbb{Q}(x,y)$ the valuation trivial on $\mathbb{Q}$ with $v(x)=1$ and $v(y)=\sqrt{2}$, defined on polynomials by the least monomial value, and let K be its completion. Then $\Gamma_K=\mathbb{Z}+\sqrt{2}\mathbb{Z}$ and $\widetilde{K}=\mathbb{Q}$. For*

$$L=K(u,w),\qquad u^2=x,\qquad w^2=y, \tag{G.0.33}$$

the field extension is tame Galois, of degree and value index four, with unchanged residue field, and

$$\mathrm{Gal}(L/K)=I\simeq(\mathbb{Z}/2)^2,\qquad \Gamma_L/\Gamma_K\simeq(\mathbb{Z}/2)^2. \tag{G.0.34}$$

The same assertions hold over any actual finite unramified extension N/K after replacing L by LN, with residue field $\widetilde{N}$ and degree four over N. Consequently no such base change converts this into one full-value-index cyclic Kummer step. Two exact quadratic steps suffice. This is a complete rank-one nondiscrete example.

Direct inputs NAG-001450, NAG-002050, NAG-003143.

Proof. Irrationality makes the values of distinct monomials distinct. A polynomial has a unique lowest term, whose product with the lowest term of another polynomial is again uniquely lowest. This proves multiplicativity and extends the valuation to fractions. Its group is $\mathbb{Z}+\sqrt{2}\mathbb{Z}$. A value-zero fraction has numerator and denominator with the same leading exponents; its residue is their rational leading coefficient ratio. Hence its residue field is $\mathbb{Q}$. Completion changes neither value group nor residue field: for a nonzero completed element choose an approximation whose difference has larger value; the two values and their leading residues agree. For value-zero elements this also identifies ordinary residues. The value group is nondiscrete. Indeed the positive numbers $(\sqrt{2}-1)^m$ lie in $\mathbb{Z}+\sqrt{2}\mathbb{Z}$ and tend to zero. It is a nonzero ordered subgroup of $\mathbb{R}$, hence has rank one.

Take the two roots in a finite algebraic extension of K, with its unique extending valuation. The four monomials $1,u,w,uw$ have values in four distinct cosets modulo Γ_K: their values are $0,1/2,\sqrt{2}/2,(1+\sqrt{2})/2$. They are linearly independent by the unique-minimum argument. Their span is closed under multiplication using (G.0.33), and the extension has degree at most four, so they are a basis and $[L:K]=4$. Their values generate $\frac{1}{2}\Gamma_K$, of index four. The fundamental inequality Stacks 0ASH forces this

to be the whole value group and forces residue degree one. Independent sign changes of u, w are four automorphisms, giving (G.0.34); they are all inertia. Residue characteristic zero, equality of degree with value index, and unchanged residue verify all of (G.0.26).

The tower $K \subset K(u) \subset L$ has value index and degree two at each step: the new value classes are represented by $1/2$ and $\sqrt{2}/2$, respectively. Their positive generators and the root -1, of residue order two, satisfy (G.0.27). For any actual unramified N/K, its value group is still Γ_K. Lemma G.0.18 proves $L \otimes_K N = LN$, degree and value index four, and unchanged residue over N. The four sign changes extend on this tensor product, giving the same noncyclic inertia and value quotient. A single (G.0.27) step of degree four has a cyclic value quotient of order four and a cyclic Galois group. Both conflict with (G.0.34). This proves the asserted obstruction, even if N contains a primitive fourth root of unity. □

The toric Gauss section and smooth tests

Let K be algebraically closed, complete and nontrivially valued, with its fixed absolute value and $v = -\log|\cdot|$. Throughout,

$$d \geq 2, \qquad X = (\mathbf{P}^1_K)^d, \qquad T = (\mathbf{G}_{m,K})^d, \qquad L_i = \mathrm{pr}_i^*\,\mathcal{O}(m_i\infty), \quad m_i \geq 1.$$

We use the usual good Hausdorff analytification, pure of dimension d. Put $t_i = -\log|T_i|$ on T^{an}, and let $\bar{t} : X^{\mathrm{an}} \to \overline{\mathbf{R}}^{,d}$ be its continuous extension, where $\overline{\mathbf{R}} = [-\infty, +\infty]$; zero maps to $+\infty$ and infinity to $-\infty$. The lattices are $N = M = \mathbf{Z}^d$ in the coordinate dual bases, and dt has covolume one.

Choose $\phi_i \in C^\infty(\mathbf{R})$ convex, with exact tails

$$\phi_i(t) = -m_i t + c_i^- \quad (t \leq A_i), \qquad \phi_i(t) = c_i^+ \quad (t \geq B_i), \qquad A_i < B_i. \tag{G.0.35}$$

The frame $e_{i,0} = 1$ of L_i has weight $\phi_i(t_i)$, and $e_{i,\infty} = T_i^{m_i} e_{i,0}$ has weight $\phi_i(t_i) + m_i t_i$. Both are constant near their respective boundary points. These define global metrics h_i in both the ordinary moment-smooth convention and the primary algebraic tropical smooth convention. Explicit examples are $\phi_i(t) = m_i \int_t^\infty (s-t)\rho_i(s)\,ds$, with $\rho_i \in C_c^\infty((A_i, B_i))$, $\rho_i \geq 0$, $\int \rho_i = 1$.

Set

$$\theta_i = d'd''\phi_i(t_i), \quad \theta = \bigwedge_{i=1}^{d} \theta_i, \quad W(t) = \prod_i \phi_i''(t_i), \quad D = \prod_i m_i, \quad S = \prod_i \operatorname{supp}(\phi_i''). \tag{G.0.36}$$

Use the positive generator $\omega_d = \bigwedge_i (d't_i \wedge d''t_i)$, so that $\theta = t^*(W\omega_d)$ and $\int a\omega_d = \int a(t)dt$. There is no $d!$ in this mixed wedge and no rescaling of the valuation. The curvature is zero near $X^{\mathrm{an}} \backslash T^{\mathrm{an}}$. If a finite box B contains S in its interior, then the whole compact polyannulus $Q = t^{-1}(B)$ is an ordinary neighbourhood of its support. All frozen chart weights use full fibres and actual canonical Banach algebras.

For $w \in \mathbf{R}^d$, let $\rho(w)$ be the Gauss point

$$\left| \sum_{m \in \mathbf{Z}^d} a_m T^m \right|_{\rho(w)} = \max_m |a_m| e^{-\langle m, w \rangle} \quad \text{on finite Laurent polynomials.} \tag{G.0.37}$$

Write $\Gamma = \rho(\mathbf{R}^d)$.

In the primary algebraic tropical convention used below, the germ is read by tangential restriction on the faces of the actual tropical chart, after subdivision. The ordinary convention uses the actual smooth moment germs.

Lemma G.0.22 (Local cancellation forces rank drop)**.** *The map ρ extends continuously to a section $\bar{\rho} : \overline{\mathbf{R}}^{,d} \to X^{\mathrm{an}}$ of $\bar{t}$. Its image $\bar{\Gamma}$ is closed, and $\Gamma = \bar{\Gamma} \cap T^{\mathrm{an}}$. For every $x \in T^{\mathrm{an}} \setminus \Gamma$ there is an ordinary open neighbourhood $U_x \subset T^{\mathrm{an}}$ and a finite union H_x of proper integral-normal affine hyperplanes in $\mathbf{R}^d$ such that*

$$t(U_x) \subset H_x.$$

Consequently any smooth moment form $t^*\alpha$ with α of bidegree (d,d) has zero germ at x, in both the frozen ordinary moment convention and the primary algebraic tropical convention. In particular the two versions of θ in (G.0.36) satisfy

$$\operatorname{supp} \theta \subset \rho(S). \tag{G.0.38}$$

Proof. The weighted Gauss norm is multiplicative by NAG-000182. On a finite coordinate chart of X, allow some radii to be zero and first quotient by those coordinate variables. The Gauss norm in the remaining variables gives a point on that chart. These constructions agree under inversion of nonzero coordinates and give $\bar{\rho}$. Evaluation on a polynomial is a finite maximum of

continuous monomial functions of the radii, including zero radii. This proves continuity in each analytic coordinate chart, hence globally. The equality $\bar{t}\bar{\rho} = 1$ follows directly. The domain is compact and X^{an} is Hausdorff; thus $\bar{\rho}$ is a homeomorphism onto its closed image. A finite tropical coordinate has positive finite radius, so its image is in T^{an}, proving the assertion about Γ.

Fix $x \notin \Gamma$, and put $w = t(x)$. Distinct points of the torus analytification are distinct multiplicative seminorms on its Laurent coordinate ring. The ultrametric inequality always gives $|f(x)| \leq |f(\rho(w))|$. Hence some Laurent polynomial $f = \sum_m a_m T^m$, with like terms collected, satisfies

$$|f(x)| < M_f(t(x)), \qquad M_f(u) = \max_m |a_m| e^{-\langle m,u\rangle}.$$

Both sides are continuous and M_f is positive. The same strict inequality defines an ordinary open neighbourhood U_x. At every $y \in U_x$ the largest term in f cannot be unique: a unique largest summand has the absolute value of the entire sum in a valued field, and the same statement holds in the completed residue field at y. At least two distinct exponents m, n therefore satisfy

$$v(a_m) + \langle m, t(y)\rangle = v(a_n) + \langle n, t(y)\rangle.$$

These are finitely many proper affine hyperplanes with nonzero integer normal $m - n$. Their union is the required H_x.

For the frozen germs, choose an ordinary affinoid neighbourhood of x inside U_x, using the neighbourhood construction in NAG-002424. Its actual T-moment image is finite polyhedral. Subdivision along H_x puts every face in dimension at most $d - 1$. Restriction of an alternating d-covector to each tangent space is zero. Thus the representative $t^*\alpha$ is tangentially zero on every face, which is exactly a zero germ in the frozen definition.

For the primary convention the same argument can be placed on an explicit algebraic tropical chart. If $f(x) \neq 0$, put $g = f$. If $f(x) = 0$, choose $a \in K^\times$ with $|a| < M_f(w)$ and put $g = f - a$. Then $g(x) \neq 0$, $M_g(w) = M_f(w)$ and $|g(x)| < M_g(w)$. The very affine open $T \setminus V(g)$ has unit coordinates T_i, g, possibly augmented to its canonical unit tuple. On its tropical open subset defined by $|g| < M_g(t)$, projection to the T_i coordinates lands in the finite union of hyperplanes just proved. Subdivide its polyhedral faces accordingly. On every face the projected tangent space has rank at most $d - 1$, so the pullback of α is zero by polyhedral restriction. This gives the zero primary smooth germ without an approximation of analytic moment maps.

Finally θ is already zero off $t^{-1}(S)$ in the torus and near the toric boundary. The argument just given removes every remaining point outside Γ. Since $\rho(S)$ is compact and closed, the support inclusion (G.0.38) follows. □

Let $\mathcal{A}$ be the real algebra of global ordinary smooth moment functions on X^{an}. Let $\mathcal{A}_{\mathrm{com}}$ be its subalgebra of functions also smooth on primary algebraic tropical charts.

Lemma G.0.23 (A common smooth algebra is uniformly dense)**.** *The algebra $\mathcal{A}_{\mathrm{com}}$ separates points of X^{an} and contains the constants. Both $\mathcal{A}_{\mathrm{com}}$ and $\mathcal{A}$ are uniformly dense in $C(X^{\mathrm{an}})$.*

Proof. For any positive multi-degree $\ell = (\ell_1, \dots, \ell_d)$, the line $L(\ell) = \bigotimes_i \mathrm{pr}_i^* \mathcal{O}(\ell_i \infty)$ has a common smooth metric with weight $\sum_i (\ell_i/m_i)\phi_i(t_i)$ on the torus. If s is a nonzero global algebraic section and $\chi \in C^\infty(\mathbf{R})$ is constant at both sufficiently large positive and negative arguments, then

$$u_{s,\chi}(x) = \chi(-\log \|s(x)\|) \tag{G.0.39}$$

is a common smooth function. At a zero of s, use the constant value $\chi(+\infty)$. This extends continuously and is locally constant there. Off the zero set, in an algebraic frame $s = fe$ its argument is $-\log|f| - \log\|e\|$, a smooth function of algebraic unit moments. Near a zero, one may make a tropical neighbourhood explicit by replacing the vanishing regular coefficient f with $f - a$, where $a \neq 0$ is small, and imposing a strict upper bound on $|f - a|$ larger than $|a|$. After shrinking the other moment bounds, this lies in the region on which (G.0.39) is constant. At a coordinate boundary the same construction with $T_i - b$ or $T_i^{-1} - b$ describes the region where the metric weight has its constant tail. Removing the zeros of these finitely many nonvanishing algebraic units gives a very affine chart. Thus these functions have the required representatives in both conventions. Sums and products are computed using joint moment tuples.

Take distinct $x, y \in X^{\mathrm{an}}$. For each coordinate choose $a_i \in K$ different from any rigid projection of x or y to that factor. This is possible since at most two values are excluded and K is infinite. Then both points lie in $U = \prod_i (\mathbf{P}^1 \setminus \{a_i\}) \simeq \mathbf{A}^d$, with affine coordinates $z_i = (T_i - a_i)^{-1}$. Some $f \in K[z_1, \dots, z_d]$ has $|f(x)| \neq |f(y)|$, because points of this analytification are polynomial seminorms. Choose positive ℓ_i at least the degree of f in z_i. There is a section s_0 of $L(\ell)$ with divisor $\sum_i \ell_i \{T_i = a_i\}$, and $s_f = f s_0$ extends

to a global section by the degree bound. The section s_0 is nonvanishing at x and y. If its norms there differ, use it in (G.0.39), with χ taking different values at the two logarithmic norms. If its norms agree, the norms of s_f differ and the same construction with s_f separates the points. This also covers a zero value of f, using the constant tail of χ.

Here is a complete compact-space density argument. Let Y be a compact Hausdorff space and $A \subset C(Y, \mathbf{R})$ a unital point-separating algebra. Its uniform closure B is an algebra. It is closed under absolute values: on a compact interval, polynomial approximation of $|r|$ follows from the Bernstein polynomials

$$B_n h(t) = \sum_{k=0}^{n} h(k/n) \binom{n}{k} t^k (1-t)^{n-k}.$$

Indeed, for continuous h on $[0,1]$, the binomial mean and variance give, for every $\delta > 0$,

$$|B_n h(t) - h(t)| \le \omega_h(\delta) + \frac{\|h\|_\infty}{2n\delta^2},$$

uniformly in t. Choose $\delta = n^{-1/4}$ and use uniform continuity. Rescaling proves the required approximation of $|r|$. Polynomial composition therefore puts $|b|$ in B for every $b \in B$. Thus B is closed under finite maxima and minima.

Given $u \in C(Y)$ and $\epsilon > 0$, for any $x, y \in Y$ a suitable affine function of a separating element of A takes the values $u(x), u(y)$ at x, y; for $x = y$ use a constant. Denote it by a_{xy}. For fixed x, the open sets where $a_{xy} > u - \epsilon$ cover Y. Choose finitely many y and let b_x be the maximum of the corresponding a_{xy}. Then $b_x \in B$, $b_x > u - \epsilon$ everywhere and $b_x(x) = u(x)$. Hence $b_x < u + \epsilon$ on a neighbourhood of x. Finitely many such neighbourhoods cover Y. The minimum b of their b_x satisfies $u - \epsilon < b < u + \epsilon$ on Y and belongs to B. Thus $u \in B$. Apply this argument to the algebra generated by the functions (G.0.39) and the constants. It is contained in $\mathcal{A}_{\mathrm{com}} \subset \mathcal{A}$, proving both density claims. □

Radon determination on the toric Gauss section

Use the same standing product, curvature and chart conventions.

Theorem G.0.24 (When radial Radon data determines the full measure)**.** *Let ν be a finite positive Radon measure on $\overline{\mathbf{R}}^{d}$ and μ a finite positive Radon measure on X^{an} with $\bar{t}_*\mu = \nu$. Then*

$$\mu = \bar{\rho}_*\nu \quad \Longleftrightarrow \quad \mu(X^{\mathrm{an}} \setminus \bar{\Gamma}) = 0. \tag{G.0.40}$$

If ν gives zero mass to the tropical boundary and to every proper affine hyperplane in $\mathbf{R}^d$ with nonzero integer normal, the right-hand condition follows automatically. In particular it follows for $\nu = W(t)dt$ from (G.0.36). In this case for every $u \in C(X^{\mathrm{an}})$,

$$\int u\, d\mu = \int u(\rho(t))W(t)\, dt. \tag{G.0.41}$$

No countability condition on the value group or the set of analytic points is needed. Positivity and Radon regularity are hypotheses.

Proof. On $\bar{\Gamma}$ the map $\bar{\rho}\bar{t}$ is the identity. If μ is concentrated there, then for every continuous u,

$$\int u\, d\mu = \int u \circ \bar{\rho} \circ \bar{t}\, d\mu = \int u \circ \bar{\rho}\, d\nu.$$

This proves the forward identification of measures; the converse concentration is immediate from the image of $\bar{\rho}$.

Now impose the additional hypotheses on ν. The analytic toric boundary has μ-mass zero because its image is in the tropical boundary. For each $x \in T^{\mathrm{an}} \setminus \Gamma$, the neighbourhood in Lemma G.0.22 has mass zero: positivity gives

$$0 \le \mu(U_x) \le \mu(t^{-1}(H_x)) = \nu(H_x) = 0.$$

Every compact subset of the open set $T^{\mathrm{an}} \setminus \Gamma$ is covered by finitely many such U_x, hence has mass zero. Inner regularity therefore makes the entire open set have mass zero. This finite-cover argument is essential; it does not take a countable union over all Laurent polynomials. Equation (G.0.40) now applies. An absolutely continuous measure $W dt$ assigns zero mass to all the specified hyperplanes, which proves (G.0.41). □

Normalizers in exact Kummer descent

PDF-derived transcription. Historical English reader, Section 1.55, Lemma 1.55.4, printed pages 220–221. The standing context is from printed pages 216–217. The historical original TeX remains unlocated.

The lemma uses the valued-field convention, the exact Kummer step (K*) and the action convention below. The remaining definitions record the surrounding historical terminology.

Standing context

Throughout, a valued field is complete and nontrivially real-valued, with additive valuation v, actual valuation ring F°, maximal ideal $\mathfrak{m}_F$, value group Γ_F, and residue field $\widetilde{F}$. Finite extensions carry their unique extending absolute values. Their existence, uniqueness and completeness are NAG-001450. Simple residue roots lift by NAG-002049.

For a finite separable extension E/K, the *explicit tame hypotheses* in this patch mean all three conditions

$$\begin{gathered} e = [\Gamma_E : \Gamma_K], \qquad f = [\widetilde{E} : \widetilde{K}], \qquad [E : K] = ef, \\ e \in (K^\circ)^\times, \qquad \widetilde{E}/\widetilde{K} \text{ separable.} \end{gathered} \tag{Tg}$$

Thus defectlessness is an assumption, not a consequence of the other two conditions. An *actual unramified lift* means the finite separable residue lift constructed by NAG-002050; in the Galois case its valuation rings have the finite free etale torsor property of NAG-002262.

An *exact Kummer step* $F \subset F(s)$ of degree $n \geq 2$ means the precise coefficient class of NAG-002610:

$$\begin{gathered} n \in (F^\circ)^\times, \qquad \zeta \in F^\circ, \qquad \operatorname{ord}(\zeta) = \operatorname{ord}(\bar{\zeta}) = n, \\ s^n \in F^\times, \qquad v(s) > 0, \qquad jv(s) \notin \Gamma_F \quad (0 < j < n). \end{gathered} \tag{K*}$$

Its integral extension is faithfully flat but need not be finite. We use left scheme actions ρ_g covering $\operatorname{Spec}(g^{-1})$, and linearizations $\rho_g^*\mathcal{L} \simeq \mathcal{L}$, as in the frozen mixed packet. Identity on a special fibre always means identity on the entire scheme or restricted module, including its structure sheaf.

Lemma G.0.25 (Retaining a normalizer action at a Kummer step)**.** *Let F/E be an exact Kummer step with group Q. Let a finite group P act on F by isometric field automorphisms preserving E, with $Q \triangleleft P$, and suppose its restriction to Q is the specified Galois action. Let Z/F° be projective, flat, finitely presented and nodal of relative dimension one, with a compatible semilinear P-action. Suppose Q acts identically on the entire special scheme. Then the canonical Q-descent Z_E/E° carries a canonical semilinear P/Q-action, and its base-change comparison is P-equivariant.*

A P-linearized relatively ample invertible sheaf with identity Q-action on its full residue module descends to a relatively ample P/Q-linearized invertible sheaf on Z_E. Both descents are functorial for equivariant isomorphisms with their specified comparisons. The action of P/Q on E need not be faithful; there is no unramified hypothesis on any lower field.

Proof. The exact input NAG-002908 supplies Z_E and its projective nodal structure. Its construction is that of NAG-002683, from the invariant rings of a finite Q-invariant affine cover. Adjoin the finitely many P-translates and all their nonempty finite intersections. Normality of Q preserves invariance; separatedness over the affine base makes the intersections affine. Write

$$D_U = \Gamma(U, \mathcal{O}_Z), \qquad C_U = D_U^Q.$$

The cited gluing theorem gives $F^\circ \otimes_{E^\circ} C_U \simeq D_U$ by multiplication and identifies invariant overlap rings with the actual overlaps of the descended charts.

Pullback by $\rho_p : U \to \rho_p U$ maps $D_{\rho_p U}$ to D_U and $C_{\rho_p U}$ to C_U, because conjugation preserves Q. It restricts on E° to $(p|_E)^{-1}$ and commutes with restriction maps. These maps glue to an action on Z_E; its group laws hold already on the invariant rings. Every $q \in Q$ fixes each U and its invariant ring, so the action factors through P/Q. The multiplication comparisons commute with these maps and are therefore P-equivariant.

The input NAG-002906 descends the bundle and its relative ampleness. On the charts its modules are $\Gamma(U, \mathcal{L})^Q$. Linearized pullbacks transport these modules by the same conjugation argument and commute with restriction. They glue to a linearization factoring through P/Q, with the required comparison. An equivariant isomorphism transports both invariant rings and modules; intersecting covers gives the same maps, by the canonical overlap comparisons. This proves functoriality and uniqueness with a specified comparison.

Only invariant descent in the exact Kummer class was used; no integral ramified torsor was presumed. □

Appendix H

Quotient action in the mixed tame tower

All valuation rings below are the actual valuation rings, with the specified extending absolute values. Fix the following supplied field configuration, denoted (T):

$$K \subset M \subset L, \qquad O = K^{\circ}, \quad A = M^{\circ}, \quad B = L^{\circ}, \qquad k = \widetilde{K}, \quad l = \widetilde{M}. \tag{T}$$

The field K is complete and nontrivially real-valued. The extension M/K is the finite unramified Galois lift of a finite Galois extension l/k, in the exact sense of 4.43.1. The extension L/K is finite Galois, with group G, and restriction gives the exact sequence

$$1 \longrightarrow I = \mathrm{Gal}(L/M) \longrightarrow G \longrightarrow H = \mathrm{Gal}(M/K) \longrightarrow 1. \tag{8}$$

The extension L/M satisfies the exact class of C.1.53: for $e = |I| \geq 2$,

$$\begin{gathered} e \in A^{\times}, \quad \zeta \in A, \quad \mathrm{ord}(\zeta) = \mathrm{ord}(\bar{\zeta}) = e, \qquad L = M(s), \quad s^e = c \in M^{\times}, \\ \gamma = v(s) > 0, \qquad j\gamma \notin v(M^{\times}) \quad (0 < j < e). \end{gathered} \tag{9}$$

Thus I is generated by $s \mapsto \zeta s$, $\widetilde{L} = l$, and $A \to B$ is faithfully flat, possibly not finite. In contrast, $O \to A$ is finite free, faithfully flat and etale, and

$$A \otimes_O A \simeq \prod_{h \in H} A, \qquad a \otimes b \longmapsto (a\, h(b))_h. \tag{10}$$

These facts are outputs of the two exact coefficient results just cited. No splitting of (8), invertibility of $|H|$, or discreteness of the value group is assumed.

Scheme actions are left actions ρ_g covering $\mathrm{Spec}(g^{-1})$. Linearizations are isomorphisms $\lambda_g : \rho_g^*\mathcal{L} \simeq \mathcal{L}$ compatible with composition. The induced left action on functions and sections is formed using $\rho_{g^{-1}}^*$ and $\lambda_{g^{-1}}$, so its coefficient action is g. Identity on a special scheme or module always means scheme-theoretic identity on the entire restriction.

Lemma H.0.1 (The quotient action on the inertia descent)**.** *Assume (T). Let Y_B/B be projective, flat, finitely presented, and at-worst-nodal of relative dimension one, with a semilinear G-action. Suppose I acts identically on $C = (Y_B)_l$. The canonical I-descent Y_A/A carries a canonical semilinear H-action, and its comparison*

$$Y_A \times_A B \simeq Y_B \tag{13}$$

is G-equivariant, where g acts on Y_A through gI and on the coefficient ring B as specified. The model Y_A is projective, flat, finitely presented and nodal of relative dimension one.

If $\mathcal{L}_B$ is a G-linearized relatively ample invertible sheaf whose restricted I-action on C is the identity, then its inertia descent $\mathcal{L}_A$ carries a canonical H-linearization, is relatively ample, and (13) identifies its pullback equivariantly with $\mathcal{L}_B$. Both constructions are functorial for equivariant isomorphisms and retain their specified comparison maps.

Proof. Apply E.0.1 to the I-action over B/A. Its descent is the invariant-affine gluing of C.1.81, so it gives the asserted projective model and the I-equivariant comparison (13). We prove the extra equivariance rather than presuming an integral Galois torsor for B/A.

Choose the finite I-invariant affine cover furnished by that gluing theorem. Adjoin all its G-translates and all nonempty finite intersections. Normality of I makes every translated open I-invariant; separatedness makes the intersections affine. For a member U, put

$$D_U = \Gamma(U, \mathcal{O}_{Y_B}), \qquad E_U = D_U^I, \qquad U_A = \operatorname{Spec} E_U. \tag{14}$$

The gluing theorem identifies $B \otimes_A E_U$ with D_U by multiplication and identifies invariant overlap rings with the actual overlaps of the U_A.

For $g \in G$, pullback along $\rho_g : U \to \rho_g U$ maps $D_{\rho_g U}$ to D_U and maps $E_{\rho_g U}$ to E_U: conjugation by g preserves I. On A this map is $(g|_M)^{-1}$. It commutes with every restriction map and hence glues to an automorphism of

Y_A covering $\mathrm{Spec}((g|_M)^{-1})$. The multiplication comparisons commute with these pullbacks, which proves the claimed G-equivariance of (13). The group law follows already on the invariant coordinate rings. For $i \in I$, the map on each E_U is the identity and $\rho_i U = U$, so the action factors through H, independently of a choice of coset lifts.

The given bundle has the inertia descent $\mathcal{L}_A = (p_*\mathcal{L}_B)^I$ by C.1.83, with $p : Y_B \to Y_A$. That result supplies its invertibility, ampleness and comparison. On (14), its sections are $\Gamma(U, \mathcal{L}_B)^I$. The linearized pullbacks send these invariant sections to invariant sections by the same conjugation calculation, and commute with restriction. They therefore define a G-linearization of $\mathcal{L}_A$ over the just constructed action. Every $i \in I$ is the identity on these invariant modules, so the linearization factors through H as well.

An equivariant isomorphism transports the invariant-affine data and their invariant modules. Intersecting either cover with the transported other cover gives a common invariant affine refinement; the canonical gluing comparisons agree there. This proves the stated functoriality and independence of the cover. Uniqueness of a comparison can also be checked after the faithfully flat base change, using the faithful part of Stacks Tag 02W0 for schemes and the faithful part of Stacks Tag 023T for their sheaves. □

Theorem H.0.2 (The mixed inertia criterion)**.** *Assume (T), and let Y_B/B be a supplied projective, flat, finitely presented nodal curve of relative dimension one with a semilinear G-action. Then the following are equivalent:*

1. *I acts identically on the entire scheme $(Y_B)_l$.*
2. *There is a projective, flat, finitely presented nodal O-scheme Y of relative dimension one with a G-equivariant isomorphism $Y \times_O B \simeq Y_B$, for the coefficient action.*

This descent, with its comparison, is unique up to the unique compatible isomorphism. If a G-linearized relatively ample invertible sheaf $\mathcal{L}_B$ is supplied, it descends with its linearization to a relatively ample invertible sheaf on Y if and only if its entire special-fibre I-action is the identity. There is no requirement of identity for the residual H-action.

Proof. Assume condition 1. Lemma H.0.1 constructs Y_A and its semilinear H-action. Equation (10) turns this action into an actual descent datum for $\mathrm{Spec}\, A \to \mathrm{Spec}\, O$: the two pullbacks are identified on the components

indexed by H, and the group law is exactly the cocycle condition on the triple product. Since this base map is surjective finite locally free and Y_A/A is projective, Stacks Tag 0CCJ makes the datum effective as an ordinary scheme Y/O.

Properness and finite presentation descend by Stacks Tag 02L1 and Stacks Tag 02L0. Flatness and nodality of relative dimension one descend by Stacks Tag 0C5E. To retain projectivity, note that $q : Y_A \to Y$ is finite locally free and surjective, as a base change of $\operatorname{Spec} A \to \operatorname{Spec} O$. The affine base and projectivity give an ample invertible sheaf on Y_A. The finite locally free case of Stacks Tag 0BD4 gives an ample invertible sheaf on Y. Properness over the affine base now gives projectivity by Stacks Tag 0B45. No division by $|H|$ occurs.

Combining $Y \times_O A \simeq Y_A$ with (13) gives the required G-equivariant comparison over B. Conversely, in any such comparison the special fibre is $Y_k \times_k l$. The subgroup I acts trivially on l by (9), and acts on the coefficient factor only. Hence it is the identity on the whole special scheme. This proves necessity.

Suppose now that $\mathcal{L}_B$ has the stated residual I-identity. The preceding lemma gives an H-linearized relatively ample $\mathcal{L}_A$ on Y_A. Equation (10), after base change to Y, makes this a genuine fpqc sheaf descent datum. Effectivity and full faithfulness in Stacks Tag 023T give a quasi-coherent sheaf $\mathcal{L}$ on Y with the specified pullback. By Stacks Tag 05B2 it is finite locally free; its rank is one because rank is unchanged by residue-field extension and q is surjective. It is therefore invertible. Relative ampleness descends by Stacks Tag 0D2P. Conversely, a bundle pulled back from Y has identity I-action on its full special-fibre module, for the same coefficient reason as above. Thus the actual supplied bundle descends, with no further power.

The inertia comparison is unique by the cited invariant descent and isomorphism functoriality; unramified descent is fully faithful by Stacks Tag 02W0, and sheaf descent by Stacks Tag 023T. Any two final descents therefore have a unique compatible comparison. In particular the special scheme C has the resulting full semilinear H-descent datum over l/k, and $Y_k \times_k l \simeq C$ respects it. This datum, rather than identity of the H-action or mere invariance of vertices, is what is used in the unramified step. □

Lemma H.0.3 (Closed subspace neighbourhood; F8RC-N)**.** *Let X be an analytic space, $Z \hookrightarrow X$ a closed analytic subspace, and $W \subset X$ an analytic domain containing $|Z|$. Then W contains an ordinary open neighbourhood of $|Z|$.*

Proof. For $z \in Z$, the domain universal property identifies $Z \times_X W$ with Z, since the entire map $Z \to X$ factors through W.

For the closed immersion into W, one may check the pullback on actual affinoid domains of W subordinate to affinoid charts of X. On these charts the canonical finite tensor of the closed ideal quotient is the actual closed quotient. This verifies the closed immersion by the target-local criterion NAG-001653.

Both maps $(Z, z) \to (X, z)$ and $(Z, z) \to (W, z)$ are closed immersions of analytic germs. By NAG-001978, their maps on reductions are isomorphisms. Functoriality makes $\mathcal{R}(W, z) \to \mathcal{R}(X, z)$ an isomorphism.

The reduction functor NAG-001972 identifies the reduction of (W, z) with an open subspace of the reduction of (X, z). The isomorphism above makes this subspace the whole reduction. By the subdomain classification NAG-001973, (W, z) and (X, z) are the same subdomain germ of (X, z). Thus some ordinary open neighbourhood of z in X lies in W. Taking the union over z proves the assertion. □

Lemma H.0.4 (The ordinary moment-image tube)**.** *Let $V = \mathcal{M}(A)$ be affinoid over the specified complete non-Archimedean field, let $f \in A$, and let g be a finite unit tuple on V. Suppose the nonempty actual zero space $D = \mathcal{M}(A/(f))$ has analytic dimension at most d, and put $P_0 = t_g(D)$. Then for some $\rho > 0$,*

$$t_g(\{|f| < \rho\}) \subseteq P_0. \tag{H.0.1}$$

If D is empty, a sufficiently small tube is empty. Arbitrary positive affinoid radii and the trivial absolute value are allowed.

Proof. By NAG-002358, P_0 is a compact finite integral affine polyhedral set of dimension at most d. The inverse image of each compact cell is an actual affinoid Laurent domain: integral affine inequalities become bounds on monomials in the units g_i, with positive real radii. Their finite union $W_0 = t_g^{-1}(P_0)$ is an analytic domain, obtained from its finite quasi-net of affinoid pieces along their actual intersections by the domain calculus. It contains the closed

subspace D. Lemma H.0.3 gives an ordinary neighbourhood O of D contained in W_0. On the compact complement $V \setminus O$, $|f|$ has a positive minimum, if the complement is nonempty. This proves (H.0.1) for a sufficiently small positive ρ.

If D is empty and V is nonempty, $|f|$ has a positive minimum on the compact V. If V is empty, every positive ρ works. □

Proposition H.0.5 (Fixed-test logarithmic integrability; F8RC-L)**.** *Let X be good, Hausdorff, and pure of dimension n, and let f be a regular analytic function on an open neighbourhood of the compact support of an ordinary smooth form β. If one of its bidegrees is n, then β vanishes on an ordinary neighbourhood of $\{f = 0\}$. In particular, for an ordinary compact (n, n)-test α, the pairing*

$$L_f(\alpha) = \int_{X \setminus \{f=0\}} \log |f| \, \alpha$$

is well-defined by the frozen ordinary compact integral. For every fixed such test, bounded continuous truncations agreeing with $\log |f|$ above $-R$ give the same value for all sufficiently large R. If η is ordinary smooth of bidegree $(n-1, n-1)$, then $d''\eta$ and $d'd''\eta$ vanish near $\{f = 0\}$.

Proof. Work on the open set where f is given. At a zero outside $\operatorname{supp} \beta$ the form already vanishes nearby. At a zero in the support, choose a common ordinary affinoid neighbourhood V on which f is regular and β has one unit-moment representative $t_g^*\gamma$, using the ordinary moment sheaf. The affinoid is pure of dimension n by domain purity. Regularity avoids every minimal prime. Apply dimension drop to the nonzero image of f on each integral ambient component. The component zero spaces cover the support of the full actual zero quotient $D = \mathcal{M}(\mathcal{O}(V)/(f))$, so its dimension is at most $n - 1$; for $n = 0$ it is empty. The ordinary moment-image tube for this actual quotient puts the g-image of a sufficiently small strict sublevel of $|f|$ in $t_g(D)$, of dimension at most $n-1$ when nonempty. Intersect that sublevel with the ordinary interior of V in X. The restriction of γ to every face is zero when one exterior degree is n. Thus its ordinary pullback is zero on this neighbourhood. This is an assertion about the actual ordinary germ and its representative, not just about a vanishing integral. It is local at every point of the zero scheme.

For compactly supported α, its closed support is therefore disjoint from the zero scheme. On this compact support $|f|$ has a positive lower bound

and a finite upper bound. The logarithm is bounded there and is an ordinary smooth function where f is a unit. Adjoin the unit f to the chart tuple on the unit locus near the compact support. Multiplication gives a compact ordinary form, extended by zero across the neighbourhood where α is zero. Its integral is the displayed value by the ordinary compact integral. Choose R so that $\log|f| > -R$ on this support. Then the truncation agrees with the smooth unit logarithm on a neighbourhood of the support. The product agrees as an ordinary compact form and is zero where the fixed test is zero. The final assertions follow by applying the same argument to $d''\eta$ and $d'd''\eta$, whose bidegrees are $(n-1,n)$ and (n,n), on the open set where they and f are defined. Neither S_1 nor purity of the zero scheme is needed for this proposition. □

Appendix I

Descent of the polarized mixed tower

Use the valuation-ring tower (T) and the action conventions of Chapter H.

Theorem I.0.1 (Descent of a compatible mixed equivariant tower)**.** *Assume (T), choose $0 \neq \pi \in \mathfrak{m}_O$, and put $B_n = B/\pi^n B$. Suppose given a cartesian tower of proper, flat, finitely presented B_n-schemes X_n with compatible semilinear G-actions. Assume that $C = X_1 \times_{B_1} l$ is at-worst-nodal of pure dimension one and that I acts identically on the entire scheme C.*

Choose any ample $\mathcal{A}_C$, compatible ample lifts $\mathcal{A}_n$, and put $\mathcal{N}_n = \bigotimes_{g\in G} \rho_{g,n}^* \mathcal{A}_n$. There exist a projective, flat, finitely presented nodal O-scheme Y of relative dimension one and a relatively ample invertible sheaf $\mathcal{P}$, with compatible equivariant polarized identifications

$$(Y, \mathcal{P}) \times_O B_n \simeq (X_n, \mathcal{N}_n) \qquad (n \geq 1). \tag{15}$$

The chosen polarized tower determines this pair uniquely up to its unique compatible polarized isomorphism. Its residue polarization is characterized after extension to l by

$$\mathcal{P}|_{Y_k} \otimes_k l \simeq \bigotimes_{h\in H} (\bar{\rho}_h^* \mathcal{A}_C)^{\otimes e}, \tag{16}$$

with the H-descent datum induced by the orbit linearization. An equivariant realization of the underlying tower over O necessarily has the stated whole-special-scheme I-identity.

Proof. The exact result B.1.2 supplies $\mathcal{A}_C$ and its compatible ample lifts, or lifts any prescribed ample $\mathcal{A}_C$. Apply Lemma G.0.8 at every level. Its naturality makes the tensor transition maps G-equivariant, and the full I-action on $\mathcal{N}_n|_C$ is the identity. Formula (12) gives the right side of (16); it is not replaced by $\mathcal{A}_C^{\otimes|G|}$.

Apply B.1.3 to the ordinary polarized tower. This constructs a projective, flat, finitely presented nodal pair $(Y_B, \mathcal{N}_B)$ over the complete valuation ring B. Rank-one completeness gives $B = \varprojlim B_n$. For each $g \in G$, the given tower automorphism and its linearization are compatible polarized isomorphisms into the base twist. The passing exact lemma D.0.2 algebraizes them uniquely. For every g, h, their appropriate composite and the map for gh have the same reductions; faithfulness makes them equal. The identity relation is checked in the same way. Thus this is a full G-action and linearization, including relations in a possibly nonsplit extension (8). Reduction at level one shows the required entire residual I-identities on the scheme and the bundle.

Theorem H.0.2 now applies to the actual algebraic pair. It descends the scheme and the existing $\mathcal{N}_B$ to $(Y, \mathcal{P})$, without a second orbit tensor. Reducing the algebraic comparisons gives (15), all transition compatibilities, and (16). The corresponding $O/\pi^n O$ tower consists of the reductions of this actual proper flat finitely presented polarized scheme. Neither ramified residual descent nor a valuation-ring module theorem has been applied to the quotient rings B_n.

For uniqueness, two such pairs become algebraizations of the same polarized tower over B. Polarized full faithfulness gives the unique comparison there; it is G-equivariant because its equivariance equations hold at every level. Inertia functoriality and fully faithful unramified descent descend that comparison uniquely. Finally, if the underlying tower is already the equivariant base change of an O-model, its residue scheme is obtained by extending k to l, on which I acts trivially. This proves the last assertion. □

Primitive cuts on a projective DVR model

Let K be complete and nontrivially discretely valued, with valuation ring R, residue field k, uniformizer π, and fixed absolute value. Set

$$v = \operatorname{ord}_R : K^\times \longrightarrow \mathbf{Z}, \qquad \lambda = -\log|\pi| > 0, \qquad -\log|a| = \lambda v(a).$$

We put $v(0) = +\infty$ when taking contents. No algebraic closure or rescaling of the absolute value is used. Fix a free R-module E of rank $d+1$, $d \geq 2$. Our common model is $\mathbf{P}(E)$, with projective points interpreted as lines, so its linear sections are elements of E^*. This is a proper flat normal model with one reduced special component $V = \mathbf{P}(E_k)$. At every induction stage we reuse E for a free module of rank $r+1$, $0 \leq r \leq d$, and reuse the corresponding projective-model notation. The initial stage is $r = d$.

For $0 \leq i \leq r$, choose

$$m_i \in \mathbf{Z}_{>0}, \qquad c_i \in \mathbf{Q}, \qquad L_i = \mathcal{O}(m_i), \qquad s_i = \ell_i^{m_i}, \qquad \ell_i \in E_K^*.$$

The rows ℓ_i form an invertible matrix A in an R-basis of E. Thus every subfamily is independent over K; the generic divisors meet properly and all $r+1$ supports have empty total intersection. Use the metric

$$\|s_i([z])\|_i = |\pi|^{c_i} \frac{|\ell_i(z)|^{m_i}}{(\max_j |z_j|)^{m_i}}. \tag{I.0.1}$$

The homogeneous expression is interpreted on analytic coordinate charts, or with coordinates in the completed residue field of an analytic point. It is independent of rescaling z.

These are algebraic root model metrics. Choose $e_i \geq 1$ with $e_i c_i \in \mathbf{Z}$ and use $\mathcal{O}(e_i m_i)$ on $\mathbf{P}(E)$, whose standard model frame is identified generically with $\pi^{-e_i c_i}$ times the standard frame. Taking the e_ith root gives (I.0.1). The special-fibre line is $\mathcal{O}(e_i m_i)$ and is generated by global sections, hence has nonnegative degree on every vertical curve. Thus the metrics are semipositive. The rational shifts change the metric identification, not that special-fibre positivity. Changing the clearing denominator e_i does not change the metric.

For a nonzero row $\ell \in E_K^*$ write

$$a = \min_j v(\ell_j), \qquad u = \pi^{-a}\ell \in E^*.$$

The row u is primitive: at least one coefficient is a unit. Its integer content a is independent of the chosen R-basis.

Lemma I.0.2 (Primitive cuts and determinant recursion)**.** *Assume $r \geq 1$. There is an R-basis $e_0, \dots, e_r$ of E with $u(e_0) = 1$ and $E' = \ker u = Re_1 \oplus \cdots \oplus Re_r$. The horizontal model cut is $\mathbf{P}(E') \simeq \mathbf{P}_R^{r-1}$. For $r \geq 1$, a section $s = \ell^m$ with metric shift c and clearing exponent e as above has normalized model divisor*

$$\frac{1}{e}\,\mathrm{cyc}\big(\mathrm{div}_{\mathcal{O}(em)}(s^{\otimes e})\big) = m[\mathbf{P}(E')] + (ma + c)[V]. \tag{I.0.2}$$

The generic divisor is $m[\mathbf{P}(E'_K)]$. A negative $ma+c$ is allowed as a rational vertical coefficient. All coefficients are computed by ordinary algebraic DVR orders, with actual quotient lengths.

If $\ell = \ell_i$ is a row of an invertible $(r+1)$-row matrix A, the other rows restrict to an invertible r-row matrix A_i on E'_K, and

$$v(\det A) = a_i + v(\det A_i). \tag{I.0.3}$$

The restricted standard metric is the standard metric for the lattice E', with the same shift c_j and exponent m_j for every remaining line. No primitivity of these restricted rows is assumed.

Elementary integral row reduction; ordinary DVR length convention of NAG-002552. The proof does not invoke the pending model log/divisor packet or a general intersection product.

Proof. Permute the coordinates to make a unit coefficient first, divide that coordinate vector by the unit, and subtract multiples of it from the other coordinate vectors. These are invertible R-operations and give the displayed basis. The content is the fractional ideal generated by the row coefficients, so this ideal and a are invariant under an invertible R-basis change. The primitive equation is now $z_0 = 0$. Its model quotient is a polynomial projective hyperplane over R, which proves the claim about $\mathbf{P}(E')$.

In a model frame, the coefficient of $s^{\otimes e}$ has a factor $\pi^{e(ma+c)}u^{em}$. At the generic point of the horizontal hyperplane its order is em, and at the generic point of V its order is $e(ma+c)$. These are the only components of the divisor. At a DVR with parameter t, the successive powers in $A/(t^n)$ give length n; for a rational coefficient one subtracts the lengths of numerator and denominator. Thus no nilpotent layer in $u^{em} = 0$ is removed. Divide the coefficients by e to obtain (I.0.2).

Move row i to the first position and use the constructed basis. The matrix becomes

$$\begin{pmatrix} \pi^{a_i} & 0 \\ * & A_i \end{pmatrix}.$$

Row permutation and the basis matrix have determinant a unit up to sign, proving (I.0.3); its nonzero determinant also proves that A_i is invertible over K. Finally, a matrix in $\mathrm{GL}_{r+1}(R)$ and its inverse both have operator norm at most one for the maximum norm over every valued extension of K.

Consequently the maximum norm is unchanged. Restriction to the direct summand E' gives exactly the standard metric in its basis. □

DVR divisor coefficients and conditional model recursion

Let K be complete and nontrivially discretely valued, with its specified absolute value, valuation ring R, residue field k, uniformizer π, and

$$\lambda = -\log|\pi| > 0.$$

We do not renormalize λ to one or pass to an algebraic closure. Let X be an integral proper K-variety of dimension $d \geq 2$. Take a proper flat integral normal finite-type R-model $\mathcal{X}$ with identified generic fibre X. Retain explicitly the following local data: every height-one local ring used below is a DVR, and the special fibre has finitely many irreducible components V, all of dimension d. These are hypotheses, not assertions about arbitrary analytic local rings. Smoothness or strict semistability of the model is not required.

Write $A_V = \mathcal{O}_{\mathcal{X},\eta_V}$ and normalize ord_V by a uniformizer of this DVR. Then

$$b_V = \mathrm{ord}_V(\pi) = \mathrm{length}_{A_V}(A_V/\pi A_V) > 0, \qquad [\mathcal{X}_k] = \sum_V b_V[V]. \tag{I.0.4}$$

Thus a nonreduced special fibre is allowed. Its b_V is not discarded.

Choose line bundles $L_1, \ldots, L_d, M$ on X and model lines

$$\mathcal{L}_i|_X \simeq L_i^{\otimes e_i}, \qquad \mathcal{M}|_X \simeq M^{\otimes e_0}, \qquad e_i, e_0 \in \mathbf{Z}_{>0}.$$

Let s be a nonzero rational section of M. A model frame at η_V writes $s^{\otimes e_0} = a_V e_V$, and put $c_V = \mathrm{ord}_V(a_V)$. A change of model frame multiplies a_V by a DVR unit, so c_V is well-defined.

Proposition I.0.3 (Horizontal and vertical divisor coefficients)**.** *Assume the height-one DVR hypotheses given above also for the horizontal prime divisors. Then, as codimension-one cycles on $\mathcal{X}$ with rational coefficients,*

$$\frac{1}{e_0}\,\mathrm{cyc}\big(\mathrm{div}_{\mathcal{X}}(s^{\otimes e_0})\big) = \overline{\mathrm{cyc}(\mathrm{div}_X(s))} + \sum_V \frac{c_V}{e_0}[V]. \tag{I.0.5}$$

Here the bar means coefficientwise closure of the horizontal prime cycles. At any height-one point with DVR A, a local coefficient $f = a/b \in \mathrm{Frac}(A)^\times$, $a, b \in A \setminus \{0\}$, has order

$$\mathrm{ord}_A(f) = \mathrm{length}_A(A/aA) - \mathrm{length}_A(A/bA). \tag{I.0.6}$$

For an effective local equation f, its coefficient is $\operatorname{length}_A(A/fA)$*, using the actual quotient, including its nilpotents. This is algebraic Cartier-to-cycle compatibility; it does not identify an analytic current or an F8 tropical end cycle.*

Proof. If t is a uniformizer and $a = t^m u$ with u a unit, the chain of powers of t in $A/(t^m)$ has m simple quotients, each its residue field. Thus $\operatorname{length}_A(A/aA) = m$; a unit has quotient zero and length zero. Subtracting the two orders proves (I.0.6), independently of the chosen fraction. In particular replacing $A/(t^m)$ by its reduction would incorrectly replace m by one when $m > 0$.

A height-one prime of $\mathcal{X}$ is either a vertical component V or horizontal. For a horizontal prime H, the element π is already a unit in $\mathcal{O}_{\mathcal{X},\eta_H}$. Localizing the model to its generic fibre therefore leaves this DVR unchanged. If H_K is the corresponding prime divisor of X, the local coefficient of $s^{\otimes e_0}$ has order $e_0 \operatorname{ord}_{H_K}(s)$. At V its order is c_V by definition. Comparing these coefficients proves (I.0.5).

There is no special-fibre factor b_V in a horizontal coefficient. For a finite residue extension, a simple residue-field factor has length its full field dimension after restriction of scalars; this is exactly the convention in NAG-002543 and NAG-002552. It counts inseparable degrees as well as separable degrees, without counting embeddings. □

Let Z have dimension $r \geq 2$, and suppose a proposed class of normal proper models and sections supplies the following operations for the data in question:

- $\mathcal{H}_i Z$ is the normalized horizontal divisor cycle of s_i, with actual multiplicities, and the required model data on its components;
- $\mathcal{V}_Z(i; J)$ is the normalized vertical term for cutting with s_i and retaining the metrics indexed by J;
- $H_C(J)$ is already defined, linear on the required cycles in lower dimensions, symmetric in its remaining entries, and satisfies the same horizontal–vertical recursion there.

All restrictions are required to be defined; no component is silently reduced or deleted. The operations and the vertical term are extended linearly to

cycles. On a common normal DVR model the proposed vertical term, in integer model data, is

$$\mathcal{V}_Z(i;J)=\frac{\lambda}{e_i\prod_{j\in J}e_j}\sum_V \operatorname{ord}_V(s_i^{\otimes e_i})\deg_{(\mathcal{L}_j)_{j\in J},k}(V). \tag{I.0.7}$$

This display specifies the algebraic term to be compared; it does not assume the pending packet's general logarithmic identification.

Lemma I.0.4 (The interchange defect)**.** *Let $i\neq j$, let J index the other $r-1$ entries, and put*

$$R_i=H_{\mathcal{H}_iZ}(j,J)+\mathcal{V}_Z(i;j,J),\qquad R_j=H_{\mathcal{H}_jZ}(i,J)+\mathcal{V}_Z(j;i,J).$$

Under the preceding explicit operation and lower-dimensional recursion hypotheses, their difference is exactly

$$\begin{aligned}R_i-R_j=\;&H_{\mathcal{H}_j\mathcal{H}_iZ-\mathcal{H}_i\mathcal{H}_jZ}(J)\\&+\mathcal{V}_{\mathcal{H}_iZ}(j;J)-\mathcal{V}_{\mathcal{H}_jZ}(i;J)\\&+\mathcal{V}_Z(i;j,J)-\mathcal{V}_Z(j;i,J).\end{aligned} \tag{I.0.8}$$

Consequently equality of the horizontal codimension-two cycles and the vertical interchange identity

$$\mathcal{V}_{\mathcal{H}_iZ}(j;J)+\mathcal{V}_Z(i;j,J)=\mathcal{V}_{\mathcal{H}_jZ}(i;J)+\mathcal{V}_Z(j;i,J) \tag{I.0.9}$$

are sufficient for equality of the two first choices. Without that horizontal-cycle equality, vanishing of the full right side of (I.0.8) *is the exact necessary and sufficient condition. If the appropriate identities hold at every node, and the curve/point base cases are compatible, the recursion is independent of its order.*

Proof. Apply the lower-dimensional recursion to $H_{\mathcal{H}_iZ}(j,J)$ and to $H_{\mathcal{H}_jZ}(i,J)$. Their horizontal terms are respectively $H_{\mathcal{H}_j\mathcal{H}_iZ}(J)$ and $H_{\mathcal{H}_i\mathcal{H}_jZ}(J)$, and their vertical terms are the first two vertical expressions in (I.0.8). Subtract, use cycle linearity, and retain the two original vertical terms. This proves the exact formula and both stated criteria. Induct on dimension for every possible first choice, using the criteria to compare any two first choices and the induction hypothesis for later choices. The separately assumed curve and point cases start this induction. No equality of Chow classes is substituted for a required cycle or functional identity. □

Determinant norms and Cartier sections

Here $q : Y \to X$ may be any norm-admissible finite morphism between the fixed K-spaces, with $q_*\mathcal{O}_Y$ locally free of constant rank $n \geq 1$ on ordinary opens. In particular this applies to the finite constant-extension map above. We work with algebraic determinants of coherent modules, before introducing any metric on a norm bundle.

Define the *determinant norm line*

$$\mathrm{Nm}_q(\mathcal{L}) = \det(q_*\mathcal{L}) \otimes \det(q_*\mathcal{O}_Y)^\vee. \tag{I.0.10}$$

Here det is the nth exterior power. Exterior powers and the displayed line are constructed on ordinary local frames and glued by their determinant transition maps. This also defines the norm of an isomorphism.

Proposition I.0.5 (Norm projection)**.** *For line bundles $\mathcal{L}$ on Y and $\mathcal{M}$ on X, there is a canonical isomorphism*

$$\mathrm{Nm}_q(\mathcal{L} \otimes q^*\mathcal{M}) \simeq \mathrm{Nm}_q(\mathcal{L}) \otimes \mathcal{M}^{\otimes n}. \tag{I.0.11}$$

In particular $\mathrm{Nm}_q(q^*\mathcal{M}) \simeq \mathcal{M}^{\otimes n}$. The formulas commute with analytic base change. Consequently the kernel of $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is killed by n.

Direct inputs Proposition G.0.5, Lemma C.2.10, `NAG-001752`.

Proof. The tensor projection isomorphism gives $q_*(\mathcal{L} \otimes q^*\mathcal{M}) \simeq q_*\mathcal{L} \otimes \mathcal{M}$. For any rank-$n$ locally free module E, local exterior multiplication gives

$$\det(E \otimes \mathcal{M}) \simeq \det(E) \otimes \mathcal{M}^{\otimes n}.$$

Changing a frame of $\mathcal{M}$ by a multiplies the top wedge by a^n, and changing a frame of E by a matrix multiplies both sides by the same determinant. These local maps therefore glue canonically. Cancel $\det(q_*\mathcal{O}_Y)$ to obtain (I.0.11). The norm line of $\mathcal{O}_Y$ has its canonical trivialization, which gives the specialization.

Finite coherent direct image commutes with analytic base change, including nonflat base change. Exterior powers of locally free modules and the displayed frame maps do too. This proves the stated compatibility. If $q^*\mathcal{M}$ is trivial, apply its determinant norm and the specialization to get $\mathcal{M}^{\otimes n} \simeq \mathcal{O}_X$. No injectivity of the unadorned Picard pullback is asserted. □

Atomic calibration and projective model induction

Use the same projective DVR and root-metric data.

Write ξ_E for the divisorial point of V. On a coordinate chart $z_0 = 1$ it is the unit-radius Gauss point. For $r = 0$ it is the unique point of $\mathbf{P}^0_K$.

For these model data the finite model measure is specified by its atom at ξ_E. Its weight is the mixed special-fibre degree of the root model lines divided by the product of their clearing exponents, with the multiplicity of the special component included. Here that multiplicity is one.

The comparison with the primary induction refers to the supplied recursion below when it has the same cycle, finite model measure and dimension-zero value.

$$H_Z(s_i, \text{remaining data}) = H_{\mathrm{cyc}(\mathrm{div}_Z\, s_i)}(\text{remaining data}) + \int_{Z^{\mathrm{an}}} -\log \|s_i\|\, d\mu_{\text{remaining data}}. \tag{I.0.12}$$

Proposition I.0.6 (Atomic calibration on the projective model)**.** *For any set J of r remaining lines on $\mathbf{P}(E)$, the specified mixed model measure and its mass are*

$$\mu_J = D_J \delta_{\xi_E}, \qquad D_J = \prod_{j \in J} m_j = \deg_{(L_j)_{j\in J},K}(\mathbf{P}(E_K)). \tag{I.0.13}$$

The empty product in dimension zero is one. For a further nonzero row ℓ_i of content a_i,

$$\int_{\mathbf{P}(E_K)^{\mathrm{an}}} -\log \|s_i\|_i\, d\mu_J = \lambda(m_i a_i + c_i) D_J. \tag{I.0.14}$$

The logarithm is finite at the atom, and every proper algebraic closed subset has measure zero. For $r \geq 1$ the normalized horizontal coefficient is m_i. For $r = 0$ the generic divisor is empty and the horizontal part is zero. The normalized vertical coefficient is $m_i a_i + c_i$, and the special-fibre multiplicity is one. There is no additional factorial.

NAG-000182 and NAG-001707 for Gauss seminorms and analytification; Lemma I.0.2; the specified finite model-measure definition. The coordinate quotient calculation below calibrates the algebraic degrees and unreduced lengths directly.

Proof. For projective space, the dimension of homogeneous polynomials of degree nq is $\binom{nq+r}{r}$, with leading term $n^r q^r / r!$. Thus $\deg_{\mathcal{O}(n)} \mathbf{P}^r = n^r$. Polarizing this polynomial shows that the mixed degree of $\mathcal{O}(n_1), \ldots, \mathcal{O}(n_r)$ is

$\prod_j n_j$: the coefficient of $t_1 \cdots t_r$ in $(\sum_j n_j t_j)^r$ is $r! \prod_j n_j$. The same normalization has a direct complete-intersection check. Choose the r coordinate equations $z_j^{n_j} = 0$, $1 \le j \le r$. They meet at $[1 : 0 : \cdots : 0]$, and their local algebra is

$$k[t_1, \dots, t_r]/(t_1^{n_1}, \dots, t_r^{n_r}).$$

Its monomials with exponents smaller than n_j form a basis of size $\prod_j n_j$. These are regular successive coordinate powers, and the quotient retains all nilpotents. Its residue field is k, so no residue-degree factor is suppressed. The same calculation holds over K.

For the root model lines $\mathcal{O}(e_j m_j)$, this mixed special-fibre degree is $\prod_j e_j m_j$. The BPS defining atomic weight divides by $\prod_j e_j$ and multiplies by the special-fibre multiplicity, which is one. This proves (I.0.13) independently of the chosen clearing exponents. The constant shifts do not change these special-fibre lines.

The Gauss seminorm is multiplicative by NAG-000182. Its kernel on the affine coordinate ring is zero, so its scheme point is generic; this proves the zero-mass assertion. The Gauss value of the row ℓ_i is the maximum absolute value of its coefficients, namely $|\pi|^{a_i}$. The denominator in (I.0.1) is one in this chart. Hence $-\log \|s_i(\xi_E)\| = \lambda(m_i a_i + c_i)$. Integrating a finite value against the one-atom measure proves (I.0.14). For $r = 0$ the row is a nonzero scalar and the same calculation applies. □

Define a model functional recursively on these tuples. In dimension zero, define it to be the negative logarithm of the nonzero section's norm. In dimension $r \ge 1$, choose any index i, take the primitive horizontal model $\mathbf{P}(E_i)$ of Lemma I.0.2, and set

$$H_E^{(i)}((L_j, s_j)_{j=0}^r) = m_i H_{E_i}((L_j|_{E_i}, s_j|_{E_i})_{j \ne i}) + \lambda(m_i a_i + c_i) \prod_{j \ne i} m_j. \quad \text{(I.0.15)}$$

Every subsequent stage again chooses an index. The first term uses the actual generic divisor cycle $m_i[\mathbf{P}(E_{i,K})]$, and the second is the full model logarithmic integral, not a guessed correction.

Theorem I.0.7 (Symmetric projective model induction)**.** *The recursion is defined at every stage for every permutation of the indices. Every order gives*

$$H_E = \lambda \left(D\, v(\det A) + \sum_{i=0}^{r} c_i \frac{D}{m_i} \right), \qquad D = \prod_{i=0}^{r} m_i. \quad \text{(I.0.16)}$$

It is therefore symmetric in the labelled pairs (L_i, s_i), independent of the chosen integral bases and primitive complements, and independent of the clearing exponents e_i. The value changes by $\lambda q D/m_i$ if c_i is increased by $q \in \mathbf{Q}$. Replacing ℓ_i by $b_i\ell_i$, $b_i \in K^\times$, changes the value by $-D\sum_i \log|b_i|$.

This is a self-contained model recursion on the stated projective class. It identifies the restriction of the primary BPS induction to this class because its base value, divisor cycle and measure are the same. No native smooth-integral comparison, analytic adjointness, metric limit, model-change invariance or extension to arbitrary pseudo-divisors is part of the assertion.

Lemmas and formulas I.0.2–I.0.6; induction and determinant arithmetic. The reference to the primary induction is a comparison of specified definitions, not an imported symmetry theorem.

Proof. The dimension-zero value is $\lambda(m_0 v(\ell_0)+c_0)$, which is (I.0.16). Suppose the formula has been proved in dimension $r-1$, for all choices of data and orders. The matrix of remaining rows on E_i is invertible, so the induction hypothesis applies after any first choice i. Writing $D_i = D/m_i$, the first term of (I.0.15) is

$$m_i\lambda\left(D_i v(\det A_i) + \sum_{j\neq i} c_j \frac{D_i}{m_j}\right) = \lambda\left(Dv(\det A_i) + \sum_{j\neq i} c_j \frac{D}{m_j}\right).$$

Its second term is $\lambda(Da_i + c_i D/m_i)$. Equation (I.0.3) gives precisely (I.0.16). This proves the claim for every first index and every later order, without assuming order-independence in dimension r.

Permuting rows changes the determinant by a sign of absolute value one; a change of R-basis multiplies it by a unit. The norms and the child lattices have the basis compatibilities proved in Lemma I.0.2. The formula itself contains no e_i. The two change rules follow by replacing c_i and by multiplying the matrix rows by b_i, respectively. Finally, the primary induction (I.0.12), if evaluated on this class, has exactly the same finite base value and exactly the two terms in (I.0.15). Finite induction identifies its value with the displayed formula; the new proof of symmetry did not use that primary theory. □

Proposition I.0.8 (Generic avoidance does not make model cuts proper)**.** *Fix $d \geq 2$ and $n \geq 1$ on $\mathbf{P}^d_R$, with unshifted standard metrics and $m_i = 1$. Take*

$$\ell_0 = z_0, \qquad \ell_i = z_0 + \pi^n z_i \quad (1 \leq i \leq d). \tag{I.0.17}$$

Every model section avoids the generic point of the special component. Every generic-fibre subfamily meets properly, and the full generic intersection is empty. Nevertheless the simultaneous model zero scheme has special-fibre support $\{z_0 = 0\} \simeq \mathbf{P}^{d-1}_k$, so it is not a finite vertical intersection to which a final quotient-length formula could be applied. The symmetric model recursion is finite and has value $dn\lambda$.

After the first horizontal cut $z_0 = 0$, each remaining section has vertical order n. Its restriction to that vertical component is the zero section. Thus using the remaining restricted sections themselves to cut that vertical component fails; its contribution must use the mixed line-bundle degree, with the correct multiplicities.

Theorem I.0.7; explicit model equations. This is an obstruction to the proposed naive cut procedure, not to known refined intersection theory or to the primary local height.

Proof. Each row is primitive, so its order at the generic point of the special fibre is zero. Subtract row zero from all other rows. The determinant is π^{nd}, hence all rows are independent over K and their hyperplanes meet properly there. The simultaneous ideal is

$$(z_0, \pi^n z_1, \ldots, \pi^n z_d).$$

Modulo π, its support is the stated projective hyperplane; its dimension is $d - 1 > 0$. For instance on $z_1 = 1$ it contains the quotient $R[z_2, \ldots, z_d]/(\pi^n)$ after setting $z_0 = 0$, which has positive-dimensional special support. Thus a finite local intersection length is not supplied by the whole tuple.

On the horizontal model $z_0 = 0$, the remaining rows are $\pi^n z_i$. Each therefore contributes the vertical component with order n and vanishes identically when restricted to that component. Its line-bundle restriction is still $\mathcal{O}(1)$, whose required mixed degree is one; the zero section is not a Cartier equation for that degree. Equation (I.0.16) gives $H = \lambda v(\pi^{nd}) = dn\lambda$. Equivalently, the first logarithmic term is zero, and the later d stages, including the final point, each contribute $n\lambda$. □

Cartier sections under determinant norm

Use the same finite locally free rank and determinant norm conventions.

Corollary I.0.9 (Pulled-back Cartier sections)**.** *Let s be a section of a line bundle $\mathcal{M}$ which is a non-zero-divisor in every local trivialization. Then q^*s is again a Cartier section. The determinant of multiplication by q^*s gives a section of $\mathrm{Nm}_q(q^*\mathcal{M})$ which, under (I.0.11), is $s^{\otimes n}$. Its effective Cartier divisor is $n\ \mathrm{div}(s)$.*

Direct inputs Proposition I.0.5, `NAG-001624`; finite local freeness of q.

Proof. Choose an affinoid neighbourhood $V = \mathcal{M}(A)$ on which $\mathcal{M}$ and $q_*\mathcal{O}_Y$ are trivial. The frame coefficient a of s is a non-zero-divisor in A: its multiplication kernel sheaf has zero ordinary stalks, so the affinoid coherent equivalence makes its section kernel zero. The free A-algebra module $B = \Gamma(q^{-1}V, \mathcal{O}_Y)$ preserves this injection. The action of a on the rank-n module $q_*\mathcal{O}_Y$ is aI_n, with determinant a^n. These equations agree on changes of frame, proving the assertion on sections. The local Cartier equation a^n is precisely that of $n\ \mathrm{div}(s)$. □

Use the data X, C, U, f, D, η and the actual chart cover V_i of the preceding chart lemma. There is no claim that $\eta|_{V_i}$ has support in $\mathrm{Int}(V_i/K)$. It can reach the original boundary of that chart. The finite union $Q = \bigcup_i V_i$ is a compact ordinary neighbourhood of C; it need not itself be good. The frozen finite-chart integrator covers exactly this situation. No locally finite partition, global paracompactness, or extension of a chart unit outside its domain is needed. On an overlap select one restricted representative; joint-tuple comparison is made only on that actual overlap.

Supplied fixed-test interface (IL). For a fixed ordinary compact form with one bidegree n, regularity of f implies vanishing on an ordinary neighbourhood of its zero space. The scalar logarithmic pairing on top forms is then the frozen ordinary compact integral away from that space. In particular this applies to $d''\eta$ and $d'd''\eta$, and later to the cut test. This interface needs neither S1 nor S3.

Proposition I.0.10 (Fixed-test stabilization; F8SA-E)**.** *Assume IL. For every fixed compact ordinary top form β supported in C, there is R_β such that the continuous truncation $\ell_R = \max(\log|f|, -R)$, extended by $-R$ at D, satisfies*

$$\ell_R\beta = \log|f|\,\beta \quad (R \geq R_\beta),$$

where the right side is extended by zero across a neighbourhood on which $\beta = 0$. These sufficiently deep products are ordinary smooth compact forms, and their scalar integrals are exactly equal.

More specifically, fix a smooth χ equal to one sufficiently far left and zero for $x \geq b_0$. Set $\eta_1 = \chi(\log|f|)\eta$, extending $\chi(\log|f|)$ by one at D. Given $b > b_0$, there exists finite $a_* < b_0$ such that for every $a \leq a_*$,

$$L_f(d'd''\eta_1) = \int_{Q\cap\{e^a\leq|f|\leq e^b\}} \log|f|\,d'd''\eta_1. \tag{I.0.18}$$

Both $d''\eta_1$ and $d'd''\eta_1$ vanish on an ordinary neighbourhood of the removed lower cap.

Proof. IL [NAG-002806] makes the closed compact support of β disjoint from D. Hence $|f|$ has a positive minimum on that support. Choose the cut strictly below this minimum. Outside the support the form is locally zero, so the asserted equality is equality of ordinary forms after zero extension, not just of integrals.

The cut scalar is smooth near D, because it is locally constant there; away from D, f is a unit and its logarithm is an ordinary smooth moment function. Define its product with η on U, and extend by zero on $X \setminus C$. These definitions agree on their ordinary open overlap. All resulting supports remain in C. Apply IL [NAG-002806] to the two indicated derivatives. Their compact supports avoid D, so one positive lower bound for $|f|$ works for both. Choose a_* strictly below its logarithm and in the constant-one range of χ. They vanish on a neighbourhood of $Q \cap \{|f| \leq e^{a_*}\}$. They also vanish wherever $\log|f| > b_0$. Compact-neighbourhood independence of the frozen integral [NAG-002425] now gives (I.0.18), including its closed endpoints. □

Fix a complete non-Archimedean field K, possibly trivially valued, with its specified absolute value. Affinoids are generalized affinoids with their canonical quotient and finite-module norms. An analytic space is good and Hausdorff, and purity always means *analytic* pure dimension. A regular function means a non-zero-divisor in the relevant affinoid rings, or equivalently the stated injectivity of its coherent sheaf multiplication when working locally. The zero ring and empty spaces contribute zero. We take $n \geq 1$.

For an affinoid $X = \mathcal{M}(A)$ pure of dimension n, write

$$\operatorname{Min} A = \{\mathfrak{q}_1, \dots, \mathfrak{q}_s\}, \qquad X_i = \mathcal{M}(A/\mathfrak{q}_i), \quad i_i : X_i \hookrightarrow X, \quad m_i = \operatorname{length}_{A_{\mathfrak{q}_i}} A_{\mathfrak{q}_i}. \tag{I.0.19}$$

All m_i are positive integers. For regular $f \in A$, put $f_i = f|_{X_i}$, $D = \mathcal{M}(A/(f))$, and $D_i = \mathcal{M}(A/(\mathfrak{q}_i, f))$, retaining their actual quotient structures. The following interfaces are deliberately separate.

Proposition I.0.11 (Weighted component formula for native integrals; F8CR-N)**.** *For every ordinary compact (n,n)-form ω on the affinoid X of (I.0.19),*

$$\int_X \omega = \sum_{i=1}^{s} m_i \int_{X_i} i_i^* \omega. \tag{I.0.20}$$

The same formula holds with ω replaced by $\phi\omega$ for every ordinary smooth scalar ϕ on X. No boundary support assumption, zero-space purity, or scalar PL is used.

Proof. Here is the restriction check needed to keep the original m_i when different moment representatives are used. Let $V = \mathcal{M}(B)$ be an actual affinoid domain of X. The map $A \to B$ is flat by ordinary single-domain flatness. If $\mathfrak{r} \in \operatorname{Min} B$, then $\mathfrak{q} = \mathfrak{r} \cap A$ is minimal in A: flat going down (the already registered algebraic input Stacks 00HS) would otherwise give a prime properly below $\mathfrak{r}$. The local flat map $A_{\mathfrak{q}} \to B_{\mathfrak{r}}$ has Artinian source and target. Tensor a composition series of $A_{\mathfrak{q}}$ with $B_{\mathfrak{r}}$. Exactness gives

$$\operatorname{length}_{B_{\mathfrak{r}}} B_{\mathfrak{r}} = m_{\mathfrak{q}} \operatorname{length}_{B_{\mathfrak{r}}}(B_{\mathfrak{r}}/\mathfrak{q}B_{\mathfrak{r}}). \tag{I.0.21}$$

For every other $\mathfrak{q}_i \in \operatorname{Min} A$, choose an element of $\mathfrak{q}_i \setminus \mathfrak{q}$. It is invertible in $B_{\mathfrak{r}}$, so $(B/\mathfrak{q}_i B)_{\mathfrak{r}} = 0$. Consequently (I.0.21) is exactly

$$\operatorname{length}_{B_{\mathfrak{r}}} B_{\mathfrak{r}} = \sum_i m_i \operatorname{length}_{B_{\mathfrak{r}}}(B/\mathfrak{q}_i B)_{\mathfrak{r}}.$$

In particular no assertion that a flat fibre has length one has entered this calculation.

Apply the generic finite-module weight formula NAG-002362 to these modules on V, and then its global chain construction closed finite-module chain. For any chosen unit tuple g on V, it gives

$$\mathcal{T}_g(B) = \sum_i m_i \mathcal{T}_g(B/\mathfrak{q}_i B).$$

Actual domain base change NAG-001846 and finite tensor comparison NAG-001565 identify $\mathcal{M}(B/\mathfrak{q}_i B)$ with $V \times_X X_i$, retaining its full structure. This is an actual domain of X_i, hence pure of dimension n when nonempty by NAG-002293. The compact finite-pushforward formula, on this single actual affinoid domain and for this closed immersion and the identity map on the common moment coordinates, identifies its integral with the target finite-module integral. There is no lattice or residue factor in addition to the recorded chain weights.

Represent ω on a finite actual affinoid cover of a compact neighbourhood of its support. Use the same cover pulled back to each X_i, discarding empty pieces. All actual intersections have the same calculation. The finite inclusion–exclusion integral NAG-002416, followed by NAG-002425, gives (I.0.20). No unit tuple is extended beyond its chart; joint tuples and their projection pullbacks are used only on actual overlaps. The argument applies verbatim to the compact form $\phi\omega$. □

Fix a complete non-Archimedean field K, possibly trivially valued, with its specified absolute value. Affinoids are generalized affinoids with their canonical quotient and finite-module norms. An analytic space is good and Hausdorff, and purity always means *analytic* pure dimension. A regular function means a non-zero-divisor in the relevant affinoid rings, or equivalently the stated injectivity of its coherent sheaf multiplication when working locally. The zero ring and empty spaces contribute zero. We take $n \geq 1$.

For an affinoid $X = \mathcal{M}(A)$ pure of dimension n, write

$$\operatorname{Min} A = \{\mathfrak{q}_1, \dots, \mathfrak{q}_s\}, \qquad X_i = \mathcal{M}(A/\mathfrak{q}_i), \quad i_i : X_i \hookrightarrow X, \quad m_i = \operatorname{length}_{A_{\mathfrak{q}_i}} A_{\mathfrak{q}_i}. \tag{I.0.22}$$

All m_i are positive integers. For regular $f \in A$, put $f_i = f|_{X_i}$, $D = \mathcal{M}(A/(f))$, and $D_i = \mathcal{M}(A/(\mathfrak{q}_i, f))$, retaining their actual quotient structures. The following interfaces are deliberately separate.

Supplied hypothesis $\mathbf{ZP}_{\mathrm{loc}}$. In each affinoid localization used here, D and every D_i are empty or pure of analytic dimension $n-1$.

Proposition I.0.12 (Cartier cycles with full generic weights; F8CR-C)**.** *For X, f as above, assume ZP_{loc}. Then, with the actual analytic Cartier coefficients and native pure-cycle integrals,*

$$\operatorname{div}_X(f) = \sum_i m_i (i_i)_* \operatorname{div}_{X_i}(f_i), \qquad \int_D \eta|_D = \sum_i m_i \int_{D_i} (i_i^* \eta)|_{D_i} \tag{I.0.23}$$

for every compact ordinary $(n-1, n-1)$-form η. The effective local length formula itself does not require ZP; its use here as a common pure-cycle identity does.

Proof. First check that no extra component is hidden in the right side. If $\mathfrak{r}$ is minimal over $(\mathfrak{q}_i, f)$, choose $\mathfrak{p} \subseteq \mathfrak{r}$ minimal over (f) [Stacks 00E0(4)]. $\mathrm{ZP}_{\mathrm{loc}}$ makes both integral affinoids $\mathcal{M}(A/\mathfrak{p})$ and $\mathcal{M}(A/\mathfrak{r})$ have analytic dimension $n-1$. A strict inclusion would contain a nonzero element of the domain $A/\mathfrak{p}$; the dimension-drop bound NAG-002374 would give dimension at most $n-2$ for the latter zero locus (empty when $n-1=0$). This is impossible. Thus $\mathfrak{r} = \mathfrak{p}$. Conversely, if $\mathfrak{p} \in \operatorname{Min}_A(f)$ contains $\mathfrak{q}_i$, no prime strictly between them can contain f, so $\mathfrak{p}/\mathfrak{q}_i$ is minimal over (f_i). Every $\mathfrak{p}$ contains at least one minimal prime of A. This establishes the exact component correspondence, allowing the same Cartier component to occur for several i.

At each such $\mathfrak{p}$, Proposition C.1.14 gives the coefficient identity with every local Artin length. The frozen effective Cartier comparison NAG-002772

identifies each of these algebraic lengths with the length of the actual analytic quotient stalk at a maximal-Abhyankar point of that zero-space component. More explicitly, NAG-002769 applies to the pure quotient affinoids $A/(f)$ and $(A/\mathfrak{q}_i)/(f_i)$ and their finite structure modules; its ordinary and pointed-domain Artinian length comparisons retain the full closed fibre. Its regular local comparison, not flatness alone, is what justifies that bridge. No S_1 property of A is required by either of these effective coefficient assertions. On the common reduced component the closed immersion has identical quotient $A/\mathfrak{p}$ and generic residue field, hence generic degree one.

Finally NAG-002778 identifies the full quotient integration with these analytic Cartier coefficients and their intrinsic component chains. Its pure-zero-space hypothesis is exactly the one retained here. Finite chart localization and its ordinary compact-form clause give the second equality in (I.0.23). No additional tropical multiplicity, lattice index, or reduced-fibre replacement is allowed. □

The tame inertia criterion for the supplied model

PDF-derived transcription. Historical English reader, Section 1.55, Theorem 1.55.5, printed pages 221–222. The historical original TeX remains unlocated.

Theorem I.0.13 (The full tame inertia criterion for a supplied model)**.** *Let L/K be finite Galois satisfying (Tg), put $O = K^\circ$, $B = L^\circ$, and let G, I, l be as in Theorem 1.55.3. Let Y_B/B be a supplied projective, flat, finitely presented nodal scheme of relative dimension one with a semilinear G-action. There is a projective, flat, finitely presented nodal O-scheme Y of relative dimension one and a G-equivariant comparison*

$$Y \times_O B \simeq Y_B \tag{22}$$

if and only if I acts identically on the entire scheme $(Y_B)_l$. Such a descent is unique up to the unique isomorphism compatible with (22).

A supplied G-linearized relatively ample invertible sheaf $\mathcal{L}_B$ descends, with its specified linearization and comparison, if and only if its entire special-fibre I-action is the identity. Its descent is relatively ample and unique with that comparison. There is no identity requirement on the residue Galois quotient G/I and no further tensor power of $\mathcal{L}_B$ is needed.

Proof. Assume the whole-special-scheme identity. Take (19) and descend from F°_{j-1} to F°_{j} successively using Lemma 1.55.4, with $P = G/J_{j-1}$ and $Q = J_j/J_{j-1}$. Normality in G makes these groups and their actions on the corresponding fields well defined. At each step projectivity, flatness, finite presentation and nodal relative dimension one are retained. Every coefficient residue field is l. Reducing the actual equivariant comparison therefore identifies the new special scheme, with its remaining inertia action, with the previous one. The remaining inertia is still the identity. This argument uses the constructed comparison; it does not assume that invariants commute with reduction.

If $r \geq 1$, it suffices to perform the first $r-1$ steps. The last field configuration $K \subset M \subset F_{r-1}$ then satisfies exactly (T) of `NAG-003009`: M/K is its actual unramified Galois lift, F_{r-1}/K is Galois, and F_{r-1}/M is one exact Kummer step. That frozen theorem performs the final Kummer and unramified descents. Composing the comparisons proves (22).

For completeness, when $I = 1$ the field $L = M$ itself is the actual unramified lift. The torsor identity in `NAG-002262` turns the full G-action into an fpqc descent datum. It is effective for this projective scheme by Stacks 0CCJ. Properness and finite presentation descend by Stacks 02L1 and Stacks 02L0, and flatness and nodal relative dimension one by Stacks 0C5E. The finite locally free surjective pullback has an ample invertible sheaf; Stacks 0BD4 gives one downstairs. Properness over the affine base then gives projectivity by Stacks 0B45.

If $\mathcal{L}_B$ has full residual I-identity, apply the bundle part of Lemma 1.55.4 at these same steps. The new residual module is identified with the old one by the actual bundle comparison, so the required identity persists. For $r \geq 1$ the bundle clause of `NAG-003009` finishes the descent of this very bundle. In the unramified case $I = 1$, fpqc sheaf descent Stacks 023T is effective; finite local freeness descends by Stacks 05B2, and its rank is one by surjective base change. Relative ampleness descends by Stacks 0D2P.

Conversely, (22) identifies the special scheme with $Y_k \times_k l$. Inertia fixes l and acts only on coefficients, so acts identically on that whole scheme. The same calculation on a pulled-back line bundle gives identity on its full residue module. Uniqueness follows at each ramified step from the canonical invariant descent and its functoriality. At the unramified step it is full faithfulness in Stacks 02W0 and Stacks 023T. This also proves independence of the chosen

series and Kummer generators. The quotient G/I retains its semilinear action on the complete special scheme, which is the descent datum used over l/k. □

Descent of the supplied equivariant tower

PDF-derived transcription. Historical English reader, Section 1.55, Corollary 1.55.6, printed page 222. The historical original TeX remains unlocated.

Corollary I.0.14 (The supplied equivariant tower interface)**.** *Keep the field hypotheses and notation of Theorem 1.55.5, and choose* $0 \neq \pi \in \mathfrak{m}_K$. *Put* $B_n = B/\pi^n B$. *Let* X_n/B_n *be a supplied cartesian tower of proper, flat, finitely presented schemes, with compatible semilinear* G*-actions. Suppose* $C = X_1 \times_{B_1} l$ *is nodal of pure dimension one and* I *acts identically on the entire scheme* C. *Choose an ample* $\mathcal{A}_C$ *and compatible ample lifts* $\mathcal{A}_n$, *and put* $\mathcal{N}_n = \bigotimes_{g\in G} \rho_{g,n}^* \mathcal{A}_n$. *Then there is a projective, flat, finitely presented nodal pair* $(Y, \mathcal{P})$ *over* O, *with* $\mathcal{P}$ *relatively ample, and compatible equivariant polarized identifications*

$$(Y, \mathcal{P}) \times_O B_n \simeq (X_n, \mathcal{N}_n). \tag{23}$$

It is unique with these comparisons. The residue polarization, after extension to l, *is*

$$\mathcal{P}|_{Y_k} \otimes_k l \simeq \bigotimes_{h\in G/I} (\bar{\rho}_h^* \mathcal{A}_C)^{\otimes |I|}, \tag{24}$$

with the quotient descent datum. An equivariant realization over O *necessarily has the stated entire-special-scheme inertia identity.*

Proof. The exact input NAG-002910 supplies the indicated ample sheaf and any requested compatible lifts. The general finite-group lemma NAG-003007 makes $\mathcal{N}_n$ compatible and G-linearized, with identity I-action on its entire restriction to C; its coset formula is (24). This lemma does not assume cyclic inertia or invertibility of $|G/I|$.

Apply NAG-002911 to this ordinary polarized tower over the complete valuation ring B. It gives a projective flat finitely presented nodal pair over B. For each g, the maps of the polarized tower into its base twist algebraize uniquely by NAG-002985. The base twists preserve $\pi^n B$ because g fixes K. Full faithfulness makes all group and linearization equations hold: both sides have the same reduction at every level. Thus the actual algebraic pair has

its full G-action and full residual I-identities. Theorem 1.55.5 descends that pair and that existing polarization without another orbit tensor, giving (23) and (24).

Two such realizations become algebraizations of the same polarized tower over B. Polarized full faithfulness gives the unique comparison there and ensures its G-equivariance. The uniqueness and functoriality in Theorem 1.55.5 descend it. Necessity follows by reducing an equivariant O-realization to l, where inertia has trivial coefficient action. The ramified affine theorem has been used only over valuation rings after algebraization, never over the nonreduced rings B_n. □

Meromorphic pairings and line-bundle frames

Let K be complete and non-Archimedean with its specified absolute value; trivial valuation and generalized affinoid radii are allowed. Let X be good, Hausdorff, and pure of analytic dimension $n \geq 1$. Fix an ordinary open $U \subseteq \mathrm{Int}(X/K)$ on which all ordinary local rings are S_1. All identities below are on U. This also handles a single compact test on X: choose such a neighbourhood of its support and restrict the data there. No S_1 claim outside that neighbourhood is needed. Ordinary open subspaces retain purity by NAG-002413; NAG-001630 and NAG-001704 show that every point of U, and of its ordinary open subspaces, is interior over K. Compact supports in these subspaces are therefore genuine analytic-interior supports.

Use the ordinary moment sheaf NAG-002424, the native integral NAG-002425, and the interleaved positive generator $d't_1 \wedge d''t_1 \wedge \cdots \wedge d't_n \wedge d''t_n$ of NAG-002388. There is no additional factor of 2, 2π, or a change from $+\log|f|$. We use the following exact inputs.

R-SPL (supplied hypothesis) For a regular analytic a on an ordinary neighbourhood of the compact support of an ordinary $(n-1, n-1)$-test η,

$$L_a(d'd''\eta) = \int_{V(a)} \eta, \qquad L_a(\alpha) = \int_{U \setminus V(a)} \log|a|\, \alpha. \tag{I.0.24}$$

Regular means injective multiplication on the analytic G-structure sheaf. The full actual quotient $\mathcal{O}/(a)$ occurs on the right. The review retains the good Hausdorff pure S_1 category and ordinary compact interior tests.

R-ZP (supplied hypothesis) Actual zero spaces of such regular functions on pure affinoid refinements are empty or pure of dimension $n-1$, with the full quotient structure. This also applies to products and to every actual common refinement. Its proof does not require S_1, but no non-S_1 scalar PL is inferred here.

Frozen inputs. NAG-002772/002778 compare and glue Cartier coefficients and native cycle integrals in their pure-zero-space scope. NAG-002806 gives actual neighbourhood vanishing of an ordinary form with one bidegree n near the zeros of a regular analytic function. NAG-002427/002428 give ordinary smooth cutoffs and finite compact partitions.

A regular meromorphic function is a section locally represented on ordinary opens by a/b, with both a, b regular in the above G-sheaf sense. This

is the unit part of the meromorphic sheaf in the good-space scope of frozen NAG-002772: its proof supplies ordinary quotient representatives and shrinking of coherent kernels. We do not replace a regular denominator by an arbitrary nonzero element. A line bundle is a coherent $\mathcal{O}_{U_G}$-module free of rank one on ordinary open neighbourhoods, matching the established project category. A regular meromorphic section is a framewise regular meromorphic unit; its existence is data, not asserted for every line bundle.

Proposition I.0.15 (Quotient independence and full Cartier gluing; F8MG-Q)**.** *For a regular meromorphic f on U, the local native pairings $L_{a/b} = L_a - L_b$ define a unique linear pairing $L_f : \mathcal{A}^{n,n}_{U,c} \to \mathbb{R}$. It is additive under multiplication of regular meromorphic functions. The local full Cartier cycles* $\operatorname{div}(a) - \operatorname{div}(b)$ *glue to* $\operatorname{div}(f)$*, and their native pairings on compact ordinary $(n-1,n-1)$-tests glue independently of presentations. The cycle assertion uses R-ZP and the exact frozen restricted Cartier comparison; the logarithmic pairing does not need R-SPL or a new measure representation.*

For local regular quotient representatives, use the ordinary quotient charts. The logarithmic pairing argument uses actual compact top tests and supplied regular quotient factors. In that argument the compact integral and finite ordinary localization apply without an S_1 hypothesis. The Cartier comparison below retains the stated ambient and R-ZP hypotheses.

Proof. For a compact top test in one quotient neighbourhood, frozen NAG-002806 makes its closed support disjoint from the zeros of both a and b. On that support both logarithms are bounded, ordinary smooth functions, and their products with the test have well-defined native compact integrals, extended by zero where the test vanishes. If $a/b = c/d$, refine to actual common affinoid neighbourhoods. Localization at regular elements is injective, so $ad = bc$ there. The test vanishes near the zeros of all four factors, and on the remaining support $\log|a| - \log|b| = \log|c| - \log|d|$. Thus the pairings agree. This also proves logarithmic additivity for a product, after adjoining its finitely many regular factors.

For a general compact top test α, take a finite ordinary partition ρ_j subordinate to quotient neighbourhoods, summing to one near $\operatorname{Supp}\alpha$, and set $L_f(\alpha) = \sum_j (L_{a_j} - L_{b_j})(\rho_j\alpha)$. Two choices are compared by the finite product partition $\rho_j\rho'_k$. Its supports lie on actual common neighbourhoods, where the preceding equality applies. Their sums equal the original test as ordinary

sections. This proves independence and linearity; no countable partition or limiting interchange is used.

For cycles, R-ZP makes every numerator, denominator, and regular cross-product zero space empty or pure of one common dimension $n-1$ on each common affinoid. The antichain hypothesis of NAG-002772 is then satisfied: a strict containment between two component primes of that analytic dimension would contradict NAG-002374. Its scoped product additivity applied to $ad = bc$ gives equality of the divisor differences, including all generic lengths. Domain compatibility and ordinary gluing in that same theorem give the cycle; NAG-002778 supplies its native pairings with the full coherent coefficients. It does not replace these by reduced supports or count geometric points. The compact partition argument proves the asserted pairing independence. □

Theorem I.0.16 (Meromorphic scalar PL from the reviewed analytic input; F8MG-P)**.** *Under R-SPL and R-ZP in the category above, every regular meromorphic f satisfies, for every ordinary $\eta \in \mathcal{A}_{U,c}^{n-1,n-1}$,*

$$L_f(d'd''\eta) = \delta_{\mathrm{div}(f)}(\eta), \qquad \delta_D(\eta) := \int_D \eta. \tag{I.0.25}$$

This is first an equality of fixed-test pairings, before a continuity claim is made.

Apply the supplied scalar identity (I.0.24) to each regular factor on its actual compact quotient chart. The resulting differences refer to the two comparisons in Proposition I.0.15.

Proof. Use quotient neighbourhoods and a finite compact smooth partition ρ_j near $\operatorname{Supp}\eta$. Put $\eta_j = \rho_j\eta$. Each η_j has compact support in its ordinary quotient neighbourhood and in the true analytic interior. R-SPL applies separately to a_j and b_j on that neighbourhood. Subtract:

$$L_f(d'd''\eta_j) = \int_{V(a_j)} \eta_j - \int_{V(b_j)} \eta_j = \delta_{\mathrm{div}(f)}(\eta_j).$$

Here $\operatorname{Supp} d'd''\eta_j \subseteq \operatorname{Supp}\eta_j$, so the local pairing agrees with the global pairing of Proposition I.0.15. Since $\eta = \sum_j \eta_j$ as ordinary sections, differentiation gives $d'd''\eta = \sum_j d'd''\eta_j$. In particular the terms

$$d'd''(\rho_j\eta) = \rho_j d'd''\eta + d'\rho_j \wedge d''\eta - d''\rho_j \wedge d'\eta + d'd''\rho_j \wedge \eta$$

are all retained. Their sums, rather than individual terms, use $\sum \rho_j = 1$ near the support. Adding the finite identities proves (I.0.25). No component reduction, new annular Green assertion, or limit in a current topology enters this proof. □

Proposition I.0.17 (Frame gluing and its signs; F8MG-B)**.** *Let $\mathcal{L}$ have ordinary local frames e_i on U_i, with $e_j = u_{ij}e_i$ on overlaps, u_{ij} analytic units. A smooth metric means a norm on each one-dimensional $\mathcal{H}(x)$-fibre whose weights $h_i = -\log\|e_i\|$ are ordinary smooth scalar functions. The compatibility equation is*

$$h_j = h_i - \log|u_{ij}|. \tag{I.0.26}$$

For a regular meromorphic section $s = f_i e_i$, one has $f_j = u_{ij}^{-1} f_i$. Its full divisor D_s glues from $\operatorname{div}(f_i)$, its ordinary smooth curvature form glues as

$$c_1(\overline{\mathcal{L}})|_{U_i} = d'd''h_i,$$

and the local top-test pairings

$$G_{s,i}(\alpha) = \int_{U_i} h_i\alpha - L_{f_i}(\alpha) \tag{I.0.27}$$

glue independently of frames and compact partitions. Off the local zero and pole loci this is the potential $g_s = -\log\|s\| = h_i - \log|f_i|$.

Unit frame changes use the ordinary joint unit coordinates. Apply the ordinary zero-neighborhood vanishing to the actual top-test support on each regular quotient chart.

Proof. The metric rule $\|ae_i(x)\| = |a|\exp(-h_i(x))$ is independent of the frame precisely when (I.0.26) holds. On any smaller unit moment neighbourhood, $\log|u_{ij}|$ is the negative of one moment coordinate. Its mixed second differential is zero as an ordinary form. Thus $d'd''h_j = d'd''h_i$, and the sheaf property NAG-002424 glues the curvature; it is killed by both d' and d''. No PL theorem is needed to prove this unit identity.

The section transformation follows by comparing $f_i e_i = f_j e_j$. The divisor of a unit is zero, so Proposition I.0.15 and frozen Cartier additivity give the same full divisor on overlaps. For local top tests, logarithmic additivity gives $L_{f_j} = L_{f_i} - L_{u_{ij}}$, and $L_{u_{ij}}(\alpha) = \int \log|u_{ij}|\alpha$, since the unit has no zero. Equation (I.0.26) therefore proves equality of (I.0.27) on every admissible

local top test. In a common regular-quotient chart the actual support avoids the regular numerator and denominator zero loci, by the ordinary top-form vanishing used in the quotient pairing. The larger compact set used to choose the charts may meet those loci. Finite compact partitions and their product refinements glue it just as in Proposition I.0.15. This proves independence, rather than merely a pointwise equality off a divisor. □

We now specify the topology rather than infer continuity from the preceding fixed-test equalities. Write $\mathcal{E}^{r,s}(U) = \mathcal{A}^{r,s}_{U,c}$. A frozen cutoff presentation $i = (V, g, \Omega, \chi)$ has an actual affinoid $V \subset U$, analytic units g on V, an open $\Omega \supset t_g(V)$, and ordinary smooth compact χ with support in $\mathrm{Int}_{\mathrm{top}}(V, U)$. Its map is

$$Q_i\alpha = \mathrm{ext}_0(\chi t_g^*\alpha), \qquad t_g = -\log|g|.$$

The coefficient space $E^{r,s}(\Omega)$ has the usual seminorms $p_{L,m}(\alpha) = \max_{I,J,|a|\le m} \sup_L |\partial^a \alpha_{I,J}$ Definition NAG-002472 defines τ_{cut} by those seminorms P on $\mathcal{E}$ for which every PQ_i is continuous. Frozen NAG-002473 proves that this equals the final ordinary compatible-presentation topology τ_{ord}. We use that comparison without reproving it. The topology on interior tests is this construction on U; no unproved identification with a subspace topology on boundary-reaching tests of X is made.

Proposition I.0.18 (Order-zero coefficient representations; F8MG-O)**.** *For every cutoff presentation, each applicable pairing*

$$L_f \quad (r = s = n), \qquad G_s \quad (r = s = n), \qquad \delta_D \quad (r = s = n-1), \qquad T_\omega(\alpha) = \int \omega\wedge\alpha$$

(where ω is a fixed ordinary smooth complementary form) has a bound

$$|T(Q_i\alpha)| \le C_i p_{t_g(V),0}(\alpha), \qquad C_i < \infty. \tag{I.0.28}$$

Here D is any of the locally finite Cartier differences above. More precisely, the functional on each coefficient presentation has a finite representation

$$T(Q_i\alpha) = \sum_{\ell,I,J} \int_{\sigma_\ell} b_{\ell,IJ}(w)\alpha_{IJ}(P_\ell w)\, d\lambda_\ell(w), \qquad P_\ell(\sigma_\ell) \subseteq t_g(V), \tag{I.0.29}$$

with fixed compact polytopes, lattice-normalized real cell integrals, and fixed smooth bounded densities. They retain all actual integer cycle weights and finite-cover signs. This is a local order-zero representation on coefficient

spaces*; no native signed Radon measure on the analytic space is assumed or constructed here.*

For a globally represented quotient $f = a/b$, smooth constant truncations of the two logarithms give pairings exactly equal to L_f on the entire image of each fixed top-degree Q_i, once the truncation is sufficiently deep. The threshold depends on i. The logarithmic parts of this proposition require regular factors and the frozen logarithmic lemma, not ambient S_1 or scalar PL.

The logarithmic step for supplied regular factors uses the logarithmic pairing argument of NAG-002989. The Green pairing uses Proposition I.0.17 where applicable. The compact cells use the finite-cell functional and unit augmentation on the same affinoid, with exact finite-cover inclusion–exclusion.

Proof. First suppose $f = a/b$ on a neighbourhood of $\operatorname{Supp}\chi$. For every constant ambient basis form $e_{IJ} = d't_I \wedge d''t_J$ of bidegree (n,n), put $\beta_{IJ} = Q_i e_{IJ}$. There are finitely many of these fixed compact ordinary forms. By frozen NAG-002806, each vanishes on an ordinary neighbourhood of $V(a) \cup V(b)$. Hence

$$K_i = \bigcup_{I,J} \operatorname{Supp}\beta_{IJ}$$

is compact and disjoint from these zero loci. Every $Q_i\alpha = \sum_{I,J} \alpha_{IJ}(t_g)\beta_{IJ}$ has support in this *same* K_i, independently of the variable coefficients. This finite-basis argument is the uniform step; fixed-test integrability alone would not suffice. If there are no coefficient slots, the entire presentation is zero.

The logarithms are bounded on K_i. Choose a smooth compact cutoff θ, supported where both factors are units and equal to one near K_i, using NAG-002427/002428. The zero extension $b_0 = \operatorname{ext}_0(\theta \log|a/b|)$ is an ordinary smooth function and $L_f(Q_i\alpha) = \int b_0 Q_i \alpha$. For a meromorphic function given only locally, choose finitely many quotient neighbourhoods and a fixed smooth partition ρ_j near $\operatorname{Supp}\chi$. Apply the same argument to the finitely many fixed basis forms $\rho_j\beta_{IJ}$ on those neighbourhoods. It produces finitely many fixed smooth multipliers b_j and reduces $L_f Q_i$ to integrals of $b_j\rho_j Q_i\alpha$. These functions and their support bounds are chosen before α. In particular this argument uses no result from the unreviewed component packet.

For any fixed smooth multiplier or complementary form, cover the fixed compact support by finitely many actual ordinary affinoid neighbourhoods inside V, and take joint unit tuples representing that fixed data and g. The vari-

able coefficient appears only as $\alpha_{IJ}(Pw)$, where P is projection to the original g-coordinates. The native finite-chart integral NAG-002405/002416/002425 supplies finitely many compact cells. Restriction of the two covector factors to a lattice tangent basis contributes fixed minors. Multiplying those minors, the fixed coefficient functions, actual weights, and inclusion– exclusion signs gives the densities in (I.0.29). This proves that formula for L_f, T_ω, and therefore G_s after a finite frame partition. For δ_D, the same construction is made on the actual pure $(n-1)$-dimensional zero spaces, using R-ZP and NAG-002778. Closed-immersion pullback preserves compact support because its underlying image is closed. Its moment coordinates are the restricted g, and the full quotient weights are retained.

Every summand is bounded in absolute value by $\lambda_\ell(\sigma_\ell)\|b_{\ell,IJ}\|_\infty p_{t_g(V),0}(\alpha)$. The finite sum gives (I.0.28). These integrals also make sense for continuous coefficient functions on the compact moment image: the continuous real polytope integral is constructed in frozen NAG-002389, with a boundary-box estimate and uniform continuity. Smooth restrictions are uniformly dense in continuous functions on a compact subset of Ω: take finitely many Euclidean smooth bumps of small support, normalize their sum near the compact set, and use the value of the continuous function at one point in each small piece. Uniform continuity bounds the error. Thus (I.0.29) extends the coefficient functional uniquely in its zeroth-order norm. This is distinct from allowing arbitrary measurable multipliers in the frozen grid integral or from representing all these refinements by one analytic-space measure.

For the final assertion choose a smooth $\lambda_M : \mathbb{R} \to \mathbb{R}$ which equals t for $t \geq -M$ and is the constant $-M$ for $t \leq -M-1$; the real smooth profile in NAG-002427 supplies one. Define $\ell_M(a) = \lambda_M(\log|a|)$ on the nonzero locus and $-M$ at its zeros. It is ordinary smooth, being constant on a neighbourhood of the zeros and a smooth function of a unit moment coordinate elsewhere. Set $u_M = \ell_M(a) - \ell_M(b)$. On the common K_i the two logarithms have fixed finite lower bounds. For all sufficiently large M, $u_M = \log|a/b|$ there, so their products with every $Q_i\alpha$ are identical. The cutoff presentations span compact tests by the finite partition theorem; hence this also gives eventual equality on every individual top test. No threshold uniform over varying presentations is asserted. □

I.1 Whole fibres on the original source

Fix a complete non-Archimedean field K with its specified real-valued absolute value, possibly trivial. Affinoids may have arbitrary positive real presentation radii. Finite maps are norm-admissible finite, with their canonical finite-module topologies. All domains, analytic fibres, nilpotents and residue fields are retained.

Let Ω be a good Hausdorff K-analytic space, and let $f = (f_1, \dots, f_n)$, $n \geq 1$, be global analytic units. Write

$$\phi : \Omega \longrightarrow T = (\mathbb{G}_m^n)^{\mathrm{an}}, \qquad t = \mathrm{trop} \circ \phi = -\log |f|.$$

For a closed rectangular box $P \subset \mathbb{R}^n$, put $T_P = \mathrm{trop}^{-1}(P)$ and $\Omega_P = t^{-1}(P)$, with the latter inverse image taken in *all of the intended source* Ω. The exact polytorus is T_w, its Gauss point is η_w, and $F_w^{\Omega} = t^{-1}(w)$.

Interior on the actual charts. For an affinoid domain X of the good source Ω, the formulas used below are

$$\mathrm{Int}(\Omega/T) = \mathrm{Int}(\Omega/K), \qquad \mathrm{Int}(X/K) = X \cap \mathrm{Int}(\Omega/K) \cap \mathrm{Int}_{\mathrm{top}}(X, \Omega). \quad (1)$$

For the first formula take a Laurent affinoid neighbourhood of the target point with all bounds strict at that point, and an ordinary affinoid neighbourhood of the source point mapping into it. The target point is absolutely interior by this presentation. The source-factor inclusion and forward composition give the equivalence on these two affinoids. The ordinary local test then gives the asserted equivalence on the good spaces.

For the second formula, let $y \in X$, choose an ordinary affinoid neighbourhood W of y in Ω, and choose an affinoid neighbourhood V of y in X contained in the ordinary interior of W. Both domain presentations use the induced analytic structure, so V is an affinoid domain of W. Absolute interiority in X is equivalent to absolute interiority in V by the ordinary local test. The source-factor inclusion for $V \to W \to \mathcal{M}(K)$ then gives relative interiority in V/W. The affinoid domain criterion makes V an ordinary neighbourhood at y in W. Thus y is ordinary interior in X, and the local test on W and Ω gives absolute interiority in Ω. Conversely, if y is ordinary interior in X and absolutely interior in Ω, take one ordinary affinoid neighbourhood of y in Ω contained in X and use the local test on both spaces.

Theorem I.1.1 (A criterion for an affinoid containing the whole interior fibre)**.** *Assume $w \in t(\Omega)$, and suppose that for one closed full-dimensional box P_0, with $w \in \mathrm{int}(P_0)$, the whole domain Ω_{P_0} is compact Hausdorff. The following conditions are equivalent.*

1. *There is an affinoid domain $X \subset \Omega$ such that*
$$F_w^{\Omega} \subset \mathrm{Int}(X/K). \tag{2}$$
2. *Every point of F_w^{Ω} is in $\mathrm{Int}(\Omega/T)$, and for* every *analytic point $a \in T_w$ the underlying set of the entire analytic fibre Ω_a is finite. Empty fibres are allowed.*
3. *There is a closed full-dimensional box $P \subset \mathrm{int}(P_0)$, with $w \in \mathrm{int}(P)$, such that*
$$\phi : \Omega_P \longrightarrow T_P \quad \textit{is norm-admissible finite.} \tag{3}$$
In particular Ω_P is affinoid and $F_w^{\Omega} \subset \mathrm{Int}(\Omega_P/K)$.

The boxes in condition 3 may be made arbitrarily small about w. Restriction to any analytic point of T_P, or any exact polytorus $T_v \subset T_P$, retains the entire original-source analytic fibre with its actual algebra and canonical norm.

If Ω is boundaryless over K, the relative-interior part of condition 2 is automatic. Alternatively, if $\Omega_{P_0} \to T_{P_0}$ is analytically proper, it supplies both compactness and the required relative interiority at the central level. In either case the finite-fibre condition still quantifies over all of T_w, not only η_w. No flatness assumption is made.

Proof. Assume condition 1. The whole moment fibre in X at w is exactly F_w^{Ω}, since that entire set lies in X. The frozen theorem 5.38.4 therefore gives a finite collar in X for the original tuple f. For every $a \in T_w$, its fibre is the entire fibre in Ω; hence its underlying set is finite, by 2.35.1. Equation (1) propagates the absolute interiority from X to Ω, and identifies it with relative interiority for ϕ. This proves condition 2. The whole-fibre finite collar theorem itself is inherited rather than resubmitted.

Assume condition 2 and set $A_0 = \Omega_{P_0}$. This analytic domain is good: intersect an ordinary affinoid neighbourhood in Ω with the finitely many bounds on f_j and f_j^{-1}. The intersection is an actual affinoid domain and is an ordinary neighbourhood in A_0. Thus A_0 is a compact good Hausdorff analytic space. The map $p_0 : A_0 \to T_{P_0}$ is compact: every compact target

domain is closed in the Hausdorff target, and its inverse image is closed in the compact source.

Fix $a \in T_w$. Its fibre under p_0 is the entire Ω_a. At every point of that fibre, both $A_0 \subset \Omega$ and $T_{P_0} \subset T$ are ordinary neighbourhoods, since all box inequalities are strict at weight w. Ordinary local testing transfers the relative interiority in condition 2 to p_0. The frozen lemma 5.56.1 therefore supplies an ordinary open neighbourhood $N_a \subset T_{P_0}$ over which the *full* restriction of p_0 is finite. This includes the empty-fibre case. The union $N = \bigcup_{a \in T_w} N_a$ contains the entire exact polytorus, and $p_0^{-1}(N) \to N$ is finite by target locality. No branch or Gauss lift has been selected.

The compact set $T_{P_0} \setminus N$ has compact tropical image, and that image misses w. Choose a closed box P about w, inside $\mathrm{int}(P_0)$, avoiding that image and lying in any prescribed real neighbourhood of w. Then $T_P \subset N$. Finite base change yields the finite map

$$p_0^{-1}(T_P) = \Omega_P \longrightarrow T_P.$$

Its source is affinoid by the definition of a finite analytic morphism, and its Banach algebra has the canonical finite norm. For $y \in F_w^{\Omega}$, the target point is absolutely interior in T_P, and the finite map is relatively interior. The affinoid composition formula gives $y \in \mathrm{Int}(\Omega_P/K)$. This proves condition 3. Condition 3 immediately implies 1.

All source identifications are inverse-image domain identities, using 2.40.11. Actual finite base change and 2.33.11 give the stated analytic fibre identifications; finite tensor comparison preserves the complete algebras. In particular, neither reduction nor a selection of a local Artinian factor occurs.

If Ω is boundaryless, (1) proves the final interior assertion. For an analytically proper p_0, compactness is part of properness and every source point is relatively interior. At the central level ordinary local testing again transfers that interiority to Ω/T. This is exactly the project convention 2.53.1; topological compactness alone does not include boundarylessness. □

Theorem I.1.2 (Whole-fibre capture in a boundaryless affinoid exhaustion)**.** *Suppose that Ω is boundaryless over K and has increasing affinoid domains $U_1 \subset U_2 \subset \cdots$ whose ordinary interiors in Ω cover Ω. Let f be any tuple of global analytic units and let $w \in t(\Omega)$. If Ω_{P_0} is compact for some closed full-dimensional box P_0 with $w \in \mathrm{int}(P_0)$, then there are an index m and*

arbitrarily small closed full-dimensional boxes $P \subset \mathrm{int}(P_0)$*, with* $w \in \mathrm{int}(P)$*, such that*

$$\Omega_P \subset \mathrm{Int}_{\mathrm{top}}(U_m, \Omega), \qquad \Omega_P \longrightarrow T_P \quad \textit{is norm-admissible finite}, \qquad F_w^{\Omega} \subset \mathrm{Int}(\Omega_P/K). \tag{4}$$

The inverse images are full inverse images in Ω*. The fibre-finiteness assertion in condition 2 of Theorem I.1.1 is a consequence here, not an additional hypothesis.*

This applies, in particular, to the analytification of an affine finite-type K*-scheme, provided the whole weight-box inverse image is compact. The unit functions may be global analytic functions; no algebraicity of the tuple is required.*

Proof. The fibre F_w^{Ω} is a closed subset of the compact Ω_{P_0}. Finitely many of the interiors of the U_j cover it. Taking the largest index gives

$$F_w^{\Omega} \subset \mathrm{Int}_{\mathrm{top}}(U_m, \Omega).$$

Boundarylessness and (1) imply $F_w^{\Omega} \subset \mathrm{Int}(U_m/K)$. Theorem I.1.1 applies and gives a full finite collar in Ω. To put the collar inside the specified U_m, use the compact set

$$C = \Omega_{P_0} \setminus \mathrm{Int}_{\mathrm{top}}(U_m, \Omega).$$

Its tropical image is compact and misses w. Shrink the finite collar box to avoid $t(C)$, still keeping w in its interior. This proves every assertion of (4). In particular, the same compact argument controls escape outside U_m; no completeness over Ω is inferred merely from a collar inside U_m.

For $\Omega = (\mathrm{Spec}\, B)^{\mathrm{an}}$, choose a finite algebra presentation $B = K[X_1, \ldots, X_N]/I$. The exact radius-piece construction 2.33.23 gives affinoid domains with $|X_j| \leq R_m$, for any increasing sequence $R_m \to \infty$. Every point has all coordinates strictly below some R_m, so their ordinary interiors cover. The construction is good and Hausdorff, and 2.33.27 supplies absolute boundarylessness. The preceding argument uses only these properties and the stated compactness of the full inverse image. It places no extra restriction on the global analytic unit tuple. □

Proposition I.1.3 (A full-rank Gauss lift and a vertical projective fibre)**.** *Over any complete field K as above, put $T_{\mathrm{alg}} = \operatorname{Spec} K[T_1^{\pm 1}, T_2^{\pm 1}]$ and define the closed projective incidence surface*

$$Y = \{(T_1, T_2, [U : V]) \in T_{\mathrm{alg}} \times \mathbf{P}^1_K : (T_1 - 1)V = (T_2 - 1)U\}. \tag{5}$$

Let $\Omega = Y^{\mathrm{an}}$, and let $\phi : \Omega \to (\mathbb{G}_m^2)^{\mathrm{an}}$ be the projection, with moment tuple $f = (T_1, T_2)$ and weight $w = (0, 0)$. Then Ω is good, Hausdorff and boundaryless, ϕ is proper, and every whole weight-box inverse image is compact. The fibre over the torus Gauss point η_w consists of one point x, with unchanged completed residue field and graded moment rank two. Nevertheless no affinoid domain in Ω contains the entire F_w^{Ω}.

Moreover, there is a pure two-dimensional affinoid domain $X \subset \Omega$ containing x in its absolute interior for which

$$F_w^X \not\subset \operatorname{Int}(X/K). \tag{6}$$

It has the actual presentation, on the chart $U \neq 0$ with $z = V/U$,

$$X = \left\{ \tfrac{1}{2} \leq |T_1|, |T_2| \leq 2, \quad |z| \leq 2, \quad T_2 = 1 + (T_1 - 1)z \right\}. \tag{7}$$

Thus even a good moment chart about an interior full-rank Gauss lift in a proper boundaryless ambient source need not have an interior whole moment fibre. The restriction of the proper ambient map to this cut is compact but has relative boundary at weight w; it is not analytically proper.

Proof. On the chart $U \neq 0$, the equation eliminates T_2 and gives the affine ring

$$K[T_1, z, T_1^{-1}, (1 + (T_1 - 1)z)^{-1}]. \tag{8}$$

The chart $V \neq 0$ has the symmetric description. These charts give a finite-type separated scheme, and its projection to the torus is projective by (5). The exact analytification results 2.33.23, 2.33.29, and 2.33.27 give goodness, Hausdorffness, and boundarylessness of both the structural map and the projection. The compactness theorem 2.33.30 makes ϕ compact, hence proper. In particular the inverse image of each compact polyannulus T_P is compact. No proper-pushforward or integration result is used.

Away from $p = (1, 1)$, the inverse to the projection is explicitly

$$(T_1, T_2) \longmapsto (T_1, T_2, [T_1 - 1 : T_2 - 1]).$$

At η_w, both $|T_j - 1|$ are one, so this point is outside p, and ϕ is an analytic isomorphism on an ordinary neighbourhood of it. It follows that its entire Gauss fibre is $\{x\}$ and $\mathcal{H}(x) = \mathcal{H}(\eta_w)$. The initial forms of T_1, T_2 at the weighted Gauss point are algebraically independent over $\widetilde{K}^{\mathrm{gr}}$: a nonzero homogeneous Laurent polynomial has the prescribed Gauss norm and cannot vanish in the graded field. Thus the graded moment rank is two. In chart (8), $|z(x)| = |T_2 - 1|/|T_1 - 1| = 1$.

The fibre over p, by contrast, is the whole $E = (\mathbf{P}^1_K)^{\mathrm{an}}$, with its actual analytic structure: the incidence equation becomes zero. It is a closed analytic subspace of Ω and is contained in F^{Ω}_w. If an affinoid domain $A \subset \Omega$ contained all of F^{Ω}_w, the pullback of the closed immersion $E \hookrightarrow \Omega$ to A would identify E with a closed analytic subspace of an affinoid. Hence E would itself be affinoid. It is boundaryless over K, so 2.43.8 would make it finite over K. Its spectrum would then be a finite set. This is impossible: the weighted Gauss points in its affine line chart, indexed by all radii $\rho > 0$, are distinct because the value of the affine coordinate is ρ. This proves the failure of any affinoid container for the whole original moment fibre.

We now verify the cut, rather than presuming its boundary behaviour. The domain (7) has affinoid algebra

$$D = K\{2^{-1}T_1, 2^{-1}S_1, 2^{-1}T_2, 2^{-1}S_2, 2^{-1}z\}/(T_1S_1-1, T_2S_2-1, T_2-1-(T_1-1)z). \tag{9}$$

The domain universal property, applied in chart (8), identifies this with the actual indicated analytic domain, including its Banach structure. Equivalently, it is the generalized rational domain of the bidisc $|T_1|, |z| \leq 2$ cut out by the lower bound on T_1 and the two bounds on $1 + (T_1 - 1)z$. The Gauss norm makes the bidisc integral. Its completed residue fields are completions of fields generated by at most two coordinate classes, so 3.22.3 bounds their residual dimensions by two. The weighted Gauss field attains two by 3.22.4; 3.22.7 therefore gives analytic dimension two. The frozen domain-purity theorem 5.35.3 therefore makes the nonempty X pure of dimension two.

All inequalities in (7) are strict at x. The chart is ordinary open in Ω, so X contains an ordinary neighbourhood of x. Since Ω is boundaryless, (1) gives $x \in \mathrm{Int}(X/K)$. For $\rho > 0$, let y_ρ be the Gauss point of radius ρ on the exceptional affine chart $T_1 = T_2 = 1$, with coordinate z. Then $y_2 \in X$, whereas $y_\rho \notin X$ for $\rho > 2$. As $\rho \downarrow 2$, these points converge to y_2. Indeed polynomial evaluations are the continuous functions $\max_j |a_j|\rho^j$; on any fixed larger closed disc, uniform polynomial approximation gives the same

continuity for all analytic functions, which determine the spectrum topology. These are actual points over the original field, also when its absolute value is trivial.

Thus $y_2 \notin \mathrm{Int}_{\mathrm{top}}(X, \Omega)$. Equation (1) gives $y_2 \notin \mathrm{Int}(X/K)$, but $t(y_2) = (0, 0)$. This proves (6). The map from X to the target polyannulus of (7) is compact. At y_2 the target bounds are strict, so the affinoid composition formula identifies relative interiority there with absolute interiority in X. Consequently that map has relative boundary and is not proper. Likewise no full collar $(t|_X)^{-1}(P) \to T_P$, with w strictly inside P, can be finite: finite-map innerness and target innerness would make y_2 absolutely interior in that domain and, by composition, in X, a contradiction. □

Meromorphic Cartier coefficients

Quotient representatives must be regular on actual ordinary charts. Use the verified effective coefficient comparison and actual-domain transport. The pairing below uses the full pure zero-space integral.

Theorem I.1.4 (Meromorphic coherent-cycle comparison; F8CA-I)**.** *Let X be good, Hausdorff, and pure of analytic dimension $n \geq 1$. Let f be regular meromorphic and admit ordinary local presentations a/b, with a, b regular and their actual zero spaces empty or pure of dimension $n-1$. Restrict comparisons to such presentations. Then the differences of the effective Cartier cycles are independent of presentation, glue on actual common domains, and have the coherent-cycle integration pairing asserted in the pure-dimensional clause of F8ST-I, on ordinary $(n-1, n-1)$-tests with compact support in* $\mathrm{Int}(X/K)$.

Proof. Two representatives can be restricted to an actual common ordinary affinoid neighbourhood, say $\mathcal{M}(A)$, where their four equations a, b, c, e are defined and regular and $a/b = c/e$. The zero spaces of these restricted equations are actual affinoid domains in the original zero spaces, by quotient/domain base change NAG-001702, 001846. Frozen domain purity NAG-002413 makes each one empty or pure of dimension $n-1$. Thus Lemma C.1.57 applies to (a, e) and (b, c), and also to any further products used for a common denominator.

Localization at regular elements is injective, so equality of the fractions gives $ae = bc$ in this actual common ring. Theorem F.1.13 now gives

$$\mathrm{div}(a) + \mathrm{div}(e) = \mathrm{div}(ae) = \mathrm{div}(bc) = \mathrm{div}(b) + \mathrm{div}(c).$$

Subtracting proves independence. The already verified effective actual-domain comparison makes these equalities agree on refinements and gives ordinary gluing. For triple overlaps use a common further affinoid refinement; the same equality is transitive there.

The effective chain identity of F8ST-I retains the actual whole Gauss algebra, all generic lengths, full residue degrees, and one lattice-index division. Subtract its two effective formulas. The independence just proved now justifies the finite localization and partition sum for a compact test, using NAG-002425 and 002428. Its restriction to each zero space is compact and inside that space's analytic interior: a closed immersion is finite and relatively

interior, and NAG-001704 composes interiors. No boundary term, logarithmic limit, or test-space convergence is introduced. □

Full facet values and finite covers

The complete non-Archimedean field K keeps its specified absolute value. Trivial valuation and generalized positive real radii are allowed. Let X be good, Hausdorff, and pure of analytic dimension $n \geq 1$, and let $\mathcal{F}$ be coherent on X_G. All forms are ordinary smooth moment forms. All coefficients integrated in this packet are smooth or continuous coefficients on finitely many compact polyhedra. No measurable extension of the frozen grid integral is used.

The frozen convention NAG-002388 has all generators anticommuting. The involution J exchanges primed and double-primed generators and fixes functions; $d'' = Jd'J$. On an integral r-plane with coordinates $y_1, \dots, y_r$, the positive generator is

$$\omega_r = d'y_1 \wedge d''y_1 \wedge \cdots \wedge d'y_r \wedge d''y_r = s_r\, d'y_1 \wedge \cdots \wedge d'y_r \wedge d''y_1 \wedge \cdots \wedge d''y_r, \qquad s_r = (-1)^{r(r-1)/2}.$$

Thus $J\omega_r = (-1)^r \omega_r$. A symmetric (p,p)-form means $J\alpha = (-1)^p \alpha$ in this convention. The different convention

Let $V = \mathcal{M}(A)$ be pure of dimension n, M a finite A-module, and g a tuple of units. Put $C = \mathcal{T}_g(M)$. Choose a finite common subdivision of its actual moment image and boundary image. For every $(n-1)$-face τ, choose a basis $v_1, \dots, v_{n-1}$ of its tangent lattice. The primitive normal $u_{\sigma/\tau}$ points *into* the incident n-cell σ. Set

$$b_C(\tau) = \sum_{\sigma \supset \tau} m_\sigma u_{\sigma/\tau} \pmod{(N_\tau)_{\mathbb{R}}}, \qquad \epsilon_n = (-1)^{n(n+1)/2}.$$

For a smooth ambient $(n-1,n)$-form $\widetilde{\beta}$, define

$$\lambda_{C,\tau,w}(\widetilde{\beta}) = \epsilon_n \widetilde{\beta}_w(v'_1, \dots, v'_{n-1}; v''_1, \dots, v''_{n-1}, b_C(\tau)''), \tag{I.1.1}$$

$$\Phi_C(\widetilde{\beta}) = \sum_\tau \int_\tau \lambda_{C,\tau,w}(\widetilde{\beta})\, d\mu_\tau(w). \tag{I.1.2}$$

Here μ_τ gives the chosen tangent basis covolume one. Alternation removes dependence on the normal lift, and the two determinant factors remove orientation dependence. NAG-002389 gives the proved identity

$$\Phi_{\mathcal{T}_g(M)}(\widetilde{\beta}) = \mathcal{I}_{M,g}(d'\widetilde{\beta}). \tag{I.1.3}$$

Every facet is included, including unbalanced interior facets and original analytic-boundary images. Nothing has yet been set equal to zero.

The weights in this formula are $m_\sigma = \dim_{H_Q}(M \otimes_A E_Q)/[\mathbb{Z}^n : Q(N_\sigma)]$ on a complete transverse Gauss fibre, in the notation of NAG-002362. Thus full module lengths, nilpotents, and full residue degrees enter once. No selected geometric lift, reduced fibre, or separable degree is substituted. Coherent modules, and finite integer linear combinations of their chains, are allowed. No real-weight analytic cycle extension is asserted.

> **R-F7.** On a good Hausdorff pure n-dimensional analytic space, for coherent $\mathcal{F}$, an ordinary $(n-1,n)$-form β with compact support in $\mathrm{Int}(X/K)$ satisfies $\int_{X,\mathcal{F}} d'\beta = 0$; the corresponding d'' statement holds in bidegree $(n, n-1)$.

Theorem I.1.5 (The full ordinary boundary functional; F8AG-T)**.** *For $\beta \in \mathcal{A}^{n-1,n}_{X,c}(X)$, the finite facet construction below gives a canonical linear functional $\mathcal{B}_{X,\mathcal{F}}(\beta)$ with*

$$\mathcal{B}_{X,\mathcal{F}}(\beta) = \int_{X,\mathcal{F}} d'\beta. \tag{I.1.4}$$

With R-F7, it factors through the actual ordinary trace map

$$r_{\partial X} : \mathcal{A}^{n-1,n}_{X,c}(X) \longrightarrow \prod_{z \in \partial_K X} \mathcal{A}^{n-1,n}_{X,z}, \qquad \beta \longmapsto (\beta_z)_z. \tag{I.1.5}$$

That is, there is a unique functional on $\operatorname{im} r_{\partial X}$ whose composite is $\mathcal{B}_{X,\mathcal{F}}$. This is a full boundary-germ functional, not an assertion that a boundary trace vanishes.

Proof. Let $S = \operatorname{Supp}\beta$. Choose finitely many ordinary affinoid neighbourhoods V_i, whose ordinary interiors cover S, such that $\beta|_{V_i} = t_{g_i}^* \widetilde{\beta}_i$. Their union Q is a compact analytic neighbourhood of S. For each nonempty index set I, put $V_I = \bigcap_{i \in I} V_i$, and use one $i \in I$ to restrict its tuple and representative to V_I. The coefficient module is the actual $M_I = \Gamma(V_I, \mathcal{F})$. Define

$$\mathcal{B}_{X,\mathcal{F}}(\beta) = \sum_{I \neq \varnothing} (-1)^{|I|+1} \Phi_{\mathcal{T}_{g_i|V_I}(M_I)}(\widetilde{\beta}_i). \tag{I.1.6}$$

Empty intersections contribute zero. Domain purity supplies the correct analytic dimension for every nonempty intersection, even if its moment image has lower dimension.

Apply (I.1.3) term by term. Exterior differentiation is well-defined on ordinary forms by NAG-002424, and $\mathrm{Supp}(d'\beta) \subset S$. The finite compact integration theorem NAG-002416 identifies the sum with $\mathcal{I}_{Q,\mathcal{F}}(d'\beta)$, and NAG-002425 identifies this with the right side of (I.1.4). Those theorems prove independence of the choice of i, all representatives, subdivisions, the presenting cover, and the compact neighbourhood. Thus the explicit facet construction has the claimed intrinsic value. A single global moment representative on an affinoid recovers (I.1.2) itself.

The kernel of (I.1.5) consists exactly of the compact sections zero on an ordinary neighbourhood of the actual analytic boundary. Such a section has compact support in $\mathrm{Int}(X/K)$. R-F7 makes its right side in (I.1.4) zero. Elementary linear factorization proves the boundary-germ assertion. This factor is defined only on the image of the trace map; it does not assign values to arbitrary incompatible germs along the boundary. □

Theorem I.1.6 (Full trace inclusion–exclusion; F8AG-C)**.** *Let Q be a finite union of affinoid domains in a good Hausdorff pure n-dimensional analytic space. Let $\mathcal{F}$ be coherent on Q_G and let β be compatible finite moment-chart data of degree $(n-1,n)$ on Q. Its facet construction defines $\mathcal{B}_{Q,\mathcal{F}}(\beta) = \mathcal{I}_{Q,\mathcal{F}}(d'\beta)$. For any finite affinoid-domain cover $V_1,\dots,V_s$ of Q,*

$$\mathcal{B}_{Q,\mathcal{F}}(\beta) = \sum_{\varnothing \neq I \subseteq \{1,\dots,s\}} (-1)^{|I|+1} \mathcal{B}_{V_I,\mathcal{F}|V_I}(\beta|V_I), \qquad V_I = \bigcap_{i\in I} V_i. \tag{I.1.7}$$

This holds with different unit tuples on different members. Every term is a full trace on its own actual domain. It does not require or imply that the individual terms vanish. It is unchanged under finite refinements and applies to the flux in Green.

Proof. All intersections are actual affinoid domains, by NAG-002411; all nonempty ones are pure of dimension n, by NAG-002413. The coefficient on an intersection is the single actual restriction of $\mathcal{F}$, equivalently $M_i \otimes_{\mathcal{O}(V_i)} \mathcal{O}(V_I)$ in a finite-module presentation. It is not an independently chosen multiplicity.

For a cover on which β has one representative per member, apply frozen finite inclusion–exclusion NAG-002416 to $d'\beta$, then (I.1.3) on each intersection. This is (I.1.7). On V_I, the two unit tuples from any two chosen members are both defined and can be adjoined. Their joint pullbacks represent the same ordinary form; their derivatives have equal tangential top restrictions

by NAG-002422/002424. Global chain coordinate augmentation NAG-002405 gives the same integral. No unit is extended outside its actual domain.

For two arbitrary finite covers (V_i) and (W_j), refine on $V_I \cap W_J$. The same frozen formula gives the finite double sum with coefficient $(-1)^{|I|+|J|}$. Its forms, derivatives, and coherent modules are restrictions of the same data on every actual intersection. Interchanging these finite sums recovers either cover. This also permits a nonpresenting member to be replaced by its finite presenting cover, proving the formula as stated. Triple overlaps are handled by the same actual restrictions; no unmatched jets are identified across unrelated closed pieces.

If one tuple and representative are defined on all of an affinoid Q, NAG-002401 gives equality of the weighted top chains on one common real subdivision. Applying the linear normal expression (I.1.1) gives facetwise cancellation there. With different tuples, the invariant assertion is (I.1.7); joint-coordinate comparisons are made on actual overlaps. Intersecting unrelated moment images is not a substitute for intersecting the source domains. □

Fixed-stage logarithmic truncation

Write U for a good Hausdorff analytic space over a complete non-Archimedean field, with arbitrary valuation. For an application with interior tests, take U to be an ordinary open subset of the true analytic interior of the ambient space. Put $E = \mathcal{A}_c^{p,q}(U)$, with the frozen ordinary moment sheaf and actual ordinary compact supports. The two final topologies of NAG-002472 are equal by NAG-002473; denote their common value by τ_a. A cutoff generator has the form

$$Q_i : E^{p,q}(\Omega_i) \longrightarrow E, \qquad Q_i\eta = \operatorname{ext}_0(\chi_i t_{f_i}^* \eta), \qquad t_{f_i} = -\log|f_i|.$$

Here V_i is affinoid, f_i is a unit tuple, Ω_i contains $t_{f_i}(V_i)$, and $\operatorname{supp}\chi_i$ is compact in $\operatorname{Int}_{\mathrm{top}}(V_i, U)$. The coefficient space has all compact C^∞ seminorms. A seminorm on E is τ_a-continuous exactly when all its composites with Q_i are continuous. The images of the Q_i span E.

Assume now that U is pure of dimension $n \geq 1$ and $\mathcal{F}$ is a coherent analytic module. Let $g, h \in \mathcal{O}_U(U)$ be non-zero-divisors in every local ring, set $Z = V(g) \cup V(h)$, and assume the exact condition (R_{n-1}) of NAG-002479: for each $z \in Z$ there is an ordinary neighbourhood W_z, a good Hausdorff

analytic space Y_z of local dimension at most $n-1$, a closed immersion into W_z whose image contains z, and an analytic retraction onto Y_z. The image need not contain the rest of $Z \cap W_z$. This is a stated hypothesis, not an existence assertion for arbitrary singular divisors.

Put $u = \log|g/h|$ off Z. Use the smooth truncation of the frozen NAG-002480: choose $0 \le \rho \le 1$ smooth, zero on $(-\infty, 0]$ and one on $[1, \infty)$, and define

$$\lambda_M(t) = -M + (t+M)\rho(t+M+1).$$

Extend $\lambda_M(\log|a|)$ by the constant $-M$ at zeros of a; write the result as $\ell_M(a)$. Set $u_M = \ell_M(g) - \ell_M(h)$ and

$$T(\alpha) = \int_{U\setminus Z,\mathcal{F}} u\alpha, \quad T_M(\alpha) = \int_{U,\mathcal{F}} u_M\alpha, \quad D_M = T_M - T.$$

Here $E = \mathcal{A}_c^{n,n}(U)$, and the integral is the frozen native continuous coefficient integral. No arbitrary measurable multiplier is assigned a grid integral.

Theorem I.1.7 (Strong truncation in the frozen retraction scope)**.** *Under the preceding exact hypotheses, T_M and T are continuous for τ_a, and $T_M = T$ on each τ_a-bounded test set for all sufficiently large real M.*

More quantitatively, every cutoff presentation admits constants $C_i, H_{i,g}, H_{i,h} \ge 0$, with $H_i = \max(H_{i,g}, H_{i,h})$, such that

$$|D_M Q_i\eta| \le A_i(M)p_i(\eta), \qquad A_i(M) \le C_i\big((H_{i,g}-M)_+ + (H_{i,h}-M)_+\big), \tag{I.1.8}$$

where $p_i(\eta) = \max_{I,J} \sup_{t_{f_i}(V_i)} |\eta_{I,J}|$. For each $a > 0$, the weights

$$w_i = 1 + C_i(H_{i,g} + H_{i,h})e^{aH_i}$$

define a continuous gauge P_a by (8.15.6) and give

$$\sup_{\alpha\in B} |(T_M - T)(\alpha)| \le e^{-aM} \sup_{\alpha\in B} P_a(\alpha) \tag{I.1.9}$$

on every τ_a-bounded set B. All coherent-cycle weights, lattice volumes, cutoff coefficients, pullback minors and finite inclusion–exclusion terms enter C_i.

Proof. The frozen NAG-002480 supplies T, its continuity, and a compact $K_i \subset U \setminus Z$ containing the supports of all $Q_i\eta$. It also supplies eventual equality on the whole presentation, uniformly in its unrestricted coefficients. Multiplication by the fixed smooth u_M and the finite native cell formula give

continuity of $T_M Q_i$; hence T_M is continuous by finality. Theorem 8.15.12 already proves uniform eventual equality on bounded sets.

For the explicit constants, choose a fixed smooth cutoff equal to one near K_i and supported where g, h are units. Use finitely many actual joint moment charts there, adjoining f_i, g, h and representations of this cutoff and χ_i. The native finite cell formula has the form

$$D_M Q_i \eta = \sum_{\sigma,I,J} \epsilon_\sigma m_\sigma \int_\sigma b_{\sigma,I,J}(w)\eta_{I,J}(P_\sigma w)\big(\delta_M(l_{g,\sigma}(w)) - \delta_M(l_{h,\sigma}(w))\big)\, d\lambda_\sigma, \tag{I.1.10}$$

where $\delta_M(t) = \lambda_M(t) - t$, the cells are fixed and compact, $\epsilon_\sigma \in \{1, -1\}$ are the finite cover signs, m_σ are the actual integer coherent-cycle multiplicities, λ_σ is lattice-normalized volume, and the smooth fixed coefficients b include the covector pullback minors. The maps P_σ are the coefficient-coordinate projections, with images in $t_{f_i}(V_i)$, and $l_{g,\sigma}, l_{h,\sigma}$ are the finite affine logarithms of the adjoined unit coordinates. The formula uses only the frozen native integration leaves NAG-002389, NAG-002405, NAG-002416 and NAG-002425, on the fixed zero-avoiding support.

Writing $s = t + M$ shows directly that

$$0 \le \delta_M(t) \le (-t - M)_+.$$

Indeed it vanishes for $s \ge 0$, and for $s < 0$ it is $(-s)(1 - \rho(s+1))$. Define

$$A_i(M) = \sum_{\sigma,I,J} |m_\sigma| \int_\sigma |b_{\sigma,I,J}|\big((-l_{g,\sigma} - M)_+ + (-l_{h,\sigma} - M)_+\big) d\lambda_\sigma,$$

$$C_i = \sum_{\sigma,I,J} |m_\sigma| \int_\sigma |b_{\sigma,I,J}| d\lambda_\sigma, \qquad H_{i,a} = \max_{\sigma, w\in\sigma} (-l_{a,\sigma}(w))_+ \quad (a = g, h).$$

Empty sums have $C_i = H_{i,g} = H_{i,h} = 0$. Every integrand in these definitions is continuous on a compact cell. Thus these are legitimate frozen real integrals; no measure extension or cancellation of total variations is needed. Taking absolute values term by term in (I.1.10) proves (I.1.8). If $M > H_i$ the bound is zero. If $M \le H_i$, then

$$A_i(M) e^{aM} \le C_i (H_{i,g} + H_{i,h}) e^{aH_i} < w_i.$$

Proposition 8.15.13 with $\varepsilon(M) = e^{-aM}$ gives (I.1.9). The constants use actual multiplicities rather than an unweighted support or a claimed uniform comparison of charts. □

Projected fibres, sectors and Gauss lifts.

Let K be complete for its specified real-valued non-Archimedean absolute value, possibly trivial. All affinoid algebras and domains are actual generalized affinoid objects with their given Banach structures. A finite map is required to have the canonical finite-module topology. Fibre algebras retain their nilpotents. Purity refers to analytic dimension.

Proposition I.1.8 (One whole degree is not one cone weight)**.** *For every $d \geq 1$ there is an interior analytic point with a pure d-dimensional pointed fan and an entire moment-fibre algebra finite with its canonical norm for two projections injective on every top-cone lattice, for which the whole-fibre Gauss degree divided by the index on a specified cone has different values. The intrinsic cone weights do exist in this example; the two whole degrees measure projected sums, not individual cone weights.*

Proof. Take a trivially valued field K, for example $\mathbb{Q}$ with the trivial norm; it is complete. On

$$Z = \operatorname{Spec} K[t, t^{-1}, (1-t)^{-1}, z_2^{\pm 1}, \dots, z_d^{\pm 1}]$$

use $f = (t, 1-t, z_2, \dots, z_d)$. Let x have the trivial norm on $K(t, z_2, \dots, z_d)$, and let X be the affinoid domain where the coordinates of t_f lie in $[-\epsilon, \epsilon]$, with $\epsilon > 0$. These bounds are strict at x. It is a rational domain in a sufficiently large closed d-polydisc, with the inverses of t, $1-t$, and the z_j imposed by Laurent inequalities. Thus 5.35.3 gives purity of dimension d. The polydisc coordinates and these inequalities are strict at x, giving absolute interior. Its moment initials generate $K(t, z_2, \dots, z_d)$, of transcendence degree d.

The pointed fan is

$$C = (\mathbb{R}_{\geq 0}(1,0) \cup \mathbb{R}_{\geq 0}(0,1) \cup \mathbb{R}_{\geq 0}(-1,-1)) \times \mathbb{R}^{d-1}.$$

The ultrametric inequality for $t + (1-t) = 1$ allows exactly these possibilities for the first two norms. Weighted Gauss norms in t, $1-t$, or t^{-1}, together with the z_j, realize every one. This is also the graded-relation fan at x, so 5.55.8 gives this pointed germ for every sufficiently small ordinary neighbourhood. Write $\sigma_1, \sigma_2, \sigma_3$ for its maximal cones.

The entire moment fibre at zero has algebra

$$D = K\langle T, T^{-1}, U, U^{-1}, Z_2^{\pm 1}, \dots, Z_d^{\pm 1} \rangle / (T + U - 1).$$

For a trivial norm restricted series are finite sums; this is the coordinate ring of Z with its trivial Banach norm. No analytic point or component of the fibre is omitted.

For positive integers a, b set

$$Q_{a,b}(u, v, \lambda) = (au + bv, \lambda), \qquad S = T^a(1 - T)^b.$$

The three saturated cone lattices have indices $a, b, a + b$; each restriction of $Q_{a,b}$ is injective. The exact target is $R = K\langle S^{\pm 1}, Z_2^{\pm 1}, \dots, Z_d^{\pm 1}\rangle$. There is an actual algebra identification

$$D = R[T]/(T^a(1 - T)^b - S).$$

Indeed the relation makes both T and $1 - T$ invertible, with

$$T^{-1} = S^{-1}T^{a-1}(1 - T)^b, \qquad (1 - T)^{-1} = S^{-1}T^a(1 - T)^{b-1}.$$

The polynomial is monic up to a unit and has degree $a + b$. Polynomial division gives the basis $1, T, \dots, T^{a+b-1}$. The given norm and the canonical finite-module norm are both trivial, so this is also a bounded analytic identification. The entire Gauss fibre consequently has degree

$$\delta_{a,b} = a + b.$$

There are genuine full finite collars as well. Use the same monic presentation over a target polyannulus in (S, Z); it is norm-admissible finite by 2.33.18. If $|t| < 1$, then $|S| = |t|^a$; if $|1 - t| < 1$, then $|S| = |1 - t|^b$; and if $|t| > 1$, then $|S| = |t|^{a+b}$. The remaining case has both norms one. Hence for a sufficiently small target logarithmic box its entire inverse image lies in X, with strict defining bounds. The finite analytification theorem 2.33.26, followed by actual domain base change, identifies this full inverse image with the constructed finite analytic algebra. The obstruction is not a failure of norm-admissible finiteness.

On σ_1, both $Q_{1,1}$ and $Q_{1,2}$ have index one; their whole-fibre degrees are 2 and 3. Thus the proposed quotients differ. The actual generic weight of each cone is one: on σ_1 its entire fibre is the exact polytorus in (t, z), because $|t| < 1$ forces $|1 - t| = 1$. Projection to (t, z) has degree and index one, and 5.37.1 gives weight one. Use $(1 - t, z)$ on σ_2 and (t, z) on σ_3.

For a small positive target first weight s, the full projected level has two original moment weights

$$(s/a, 0, \lambda), \quad (0, s/b, \lambda),$$

with contributions a and b. At a negative first weight only σ_3 contributes, with index $a+b$. These explain the whole degree $a+b$. Thickening by $\epsilon'^h = 0$ multiplies all degrees and all cone weights by h with unchanged support; its generic nilpotent length has not disappeared. □

Throughout, K is complete for its specified real-valued non-Archimedean absolute value, possibly trivial. All affinoids, domains and completed tensors are the actual generalized analytic objects with their given Banach structures. Finite maps must carry the canonical finite-module topology. All fibre algebras keep their nilpotents.

Definition I.1.9 (Two saturated finite collars with common Gauss lifts)**.** Let $X = \mathcal{M}(A)$ be nonempty and pure of analytic dimension $d \geq 1$, let $f = (f_1, \dots, f_n) \in (A^\times)^n$, and put $t = t_f = -\log|f|$. Let M be a finite A-module. Fix $w_0 \in \mathbb{R}^n$ and distinct integral maps $Q_i : \mathbb{Z}^n \to \mathbb{Z}^d$, $i = 1, 2$. For each i, put $u_i^0 = Q_i w_0$ and choose a full-dimensional closed rectangular box P_i with u_i^0 in its interior. Assume that

$$C_i = (Q_i t)^{-1}(P_i) \subset X, \qquad \psi_i : C_i \longrightarrow \mathbb{T}_{P_i}$$

is norm-admissible finite, where ψ_i is given by the actual units $f^{q_{i1}}, \dots, f^{q_{id}}$. The equality defining C_i is for the entire inverse image in X; it is the *saturation hypothesis*.

Let η_i be the target Gauss point at u_i^0. Assume the equality of actual point sets

$$\psi_1^{-1}(\eta_1) = \psi_2^{-1}(\eta_2) = S = \{x_1, \dots, x_r\} \subset t^{-1}(w_0), \qquad 1 \leq r < \infty.$$

These are analytic Gauss lifts, not selected lifts after scalar extension. The fibre algebras may have arbitrary finite lengths and residue-field degrees. No isomorphism of the two central fibre algebras is assumed. Require $x_j \in \mathrm{Int}(X/K)$ and graded moment rank d at every x_j.

Let C_{x_j} be the pure d-dimensional pointed fan supplied by 5.55.8, and choose a common rational fan subdivision Σ of their union so that each C_{x_j} is a subfan. Assume that both Q_i are injective on the real span of every d-cone of Σ. For a top cone σ, write

$$N_\sigma = \mathbb{Z}^n \cap \mathrm{span}_{\mathbb{R}}(\sigma), \qquad e_{i,\sigma} = [\mathbb{Z}^d : Q_i N_\sigma].$$

These saturated-lattice indices are finite.

These are hypotheses for a local class, not an existence theorem for such pairs of collars at arbitrary moment points. In particular, separate finite presentations and equal numbers of Gauss lifts do not supply the displayed equality of point sets inside the same X. The functions and the coefficient module must also be the same original-field data.

Choose pairwise disjoint ordinary open sets O_j about x_j. After shrinking, choose ordinary affinoid neighbourhoods $V_j \subset O_j \cap C_1 \cap C_2$ such that

$$t(V_j) \subset w_0 + C_{x_j}, \tag{I.1.11}$$

and the pointed-image equality holds for every smaller ordinary neighbourhood of x_j. Here the collar inequalities are strict at x_j, so $C_1 \cap C_2$ is an ordinary neighbourhood of S. The stated choice follows from the full-stabilization theorem, continuity of t, and the ordinary affinoid-neighbourhood basis recorded in 5.52.3: first restrict to a real ball where the upper fan containment holds, then take a smaller ordinary affinoid neighbourhood inside its inverse image. No new pointed-support theorem is being proved.

Lemma I.1.10 (Whole projected levels in a common union of neighbourhoods)**.** *For each j, let $U_{1j}, U_{2j} \subset V_j$ be ordinary affinoid neighbourhoods of x_j. There are ordinary affinoid neighbourhoods $W_j \subset U_{1j} \cap U_{2j}$ of x_j, open real neighbourhoods $B_i \subset \operatorname{int}(P_i)$ of u_i^0, and finite unions H_i of proper integral-normal affine hyperplanes in $\mathbb{R}^d$ such that*

$$\{y \in X : Q_i t(y) = u\} \subset \bigcup_{j=1}^{r} \operatorname{Int}_{\mathrm{top}}(W_j, X) \qquad (u \in B_i \setminus H_i). \tag{I.1.12}$$

This includes every analytic point of each projected tropical level. Its intersections with the W_j form a disjoint finite affinoid cover and hence give a bounded product decomposition of its actual algebra and its finite coefficient modules.

Proof. Choose the W_j using the ordinary affinoid-neighbourhood basis, and set $G = \bigcup_j \operatorname{Int}_{\mathrm{top}}(W_j, X)$. For each i, the set $C_i \setminus G$ is compact. Its image under ψ_i is closed in the Hausdorff target and misses η_i, because the *entire* analytic fibre there is $S \subset G$. Compactness and the evaluation topology are 2.24.1. Thus

$$O_i' = \mathbb{T}_{P_i} \setminus \psi_i(C_i \setminus G)$$

is an ordinary open neighbourhood of η_i with full inverse image inside G. Apply the exact reviewed lemma C.1.43 to O_i'. It puts the entire target polytorus $\mathbb{T}_u$ inside O_i' for $u \in B_i \setminus H_i$. Saturation of C_i then gives (I.1.12).

The level is an actual affinoid, with algebra

$$B_i(u) = \mathcal{O}(C_i)\widehat{\otimes}_{\mathcal{O}(\mathbb{T}_{P_i})} R_i(u),$$

where $R_i(u) = \mathcal{O}(\mathbb{T}_u)$. It is a canonical finite $R_i(u)$-module by 2.33.19. Intersections with the W_j are actual affinoid domains by 2.36.9 and 2.36.10. They cover by (I.1.12) and are disjoint because $W_j \subset O_j$. Apply 2.38.1 to this finite cover. All pairwise intersections have zero algebra, so its norm-strict degree-zero equalizer is a bounded isomorphism from $B_i(u)$ to the product of the restriction algebras. For a finite module the same theorem gives the corresponding module product. The characteristic idempotents are actual elements of $B_i(u)$, and each factor has the quotient finite-module topology. No reduction or selection of a maximal ideal has occurred. □

Theorem I.1.11 (Sectorwise full-fibre overlap and projection comparison)**.** *Use Definition I.1.9, the V_j, and the neighbourhoods U_{ij}, W_j of Lemma I.1.10. After shrinking the real neighbourhoods, the following hold.*

For each top cone σ there is a finite union E_σ of proper affine hyperplanes in the affine plane $w_0 + \operatorname{span}_{\mathbb{R}} \sigma$ such that, whenever $v \in w_0 + \operatorname{relint} \sigma$ is sufficiently close to w_0 and $v \notin E_\sigma$, the whole original moment fibre $F_v = t^{-1}(v)$ satisfies

$$F_v = \coprod_j F_{j,v}, \qquad F_{j,v} = F_v \cap W_j = F_v \cap U_{1j} = F_v \cap U_{2j}.$$

Every nonempty $F_{j,v}$ lies in the absolute interior of all three domains in the displayed equality. The equalities are equalities of actual analytic domains, so their full algebras are canonically the same algebra $D_{j,v}$.

For $u = Q_i v$, let $\Lambda_i(u)$ be the finite set of original moment weights on the entire projected level $\{y \in X : Q_i t(y) = u\}$. For every such regular u, restriction gives a bounded isomorphism with bounded inverse

$$B_i(u) \;\simeq\; \prod_{v' \in \Lambda_i(u)} \prod_{j=1}^{r} D_{j,v'}. \tag{I.1.13}$$

Zero factors are omitted. The corresponding idempotent $e_{i,j,v'}$ is uniquely determined by the actual labelled domain $F_{j,v'}$, not by a numerical match of

lengths. All original moment weights and all Gauss-lift neighbourhoods are included.

Let $H_i(v)$ be the Gauss field of $R_i(Q_i v)$ and put

$$E_{i,j}(v) = D_{j,v} \widehat{\otimes}_{R_i(Q_i v)} H_i(v), \qquad L_{i,j}(v) = M \otimes_A E_{i,j}(v).$$

Then $D_{j,v}$ is norm-admissible finite over both indicated target algebras, and

$$\frac{\dim_{H_1(v)} L_{1,j}(v)}{e_{1,\sigma}} = \frac{\dim_{H_2(v)} L_{2,j}(v)}{e_{2,\sigma}} = m_{M|W_j,f}(v) \in \mathbb{Z}_{\geq 0}. \tag{I.1.14}$$

The same value is obtained using U_{1j} or U_{2j}. An empty factor has value zero.

For either projection the numerator retains the full arithmetic:

$$\dim_{H_i(v)} L_{i,j}(v) = \sum_{\mathfrak{m} \in \operatorname{Max} E_{i,j}(v)} \operatorname{length}_{E_{i,j}(v)_{\mathfrak{m}}} L_{i,j}(v)_{\mathfrak{m}} \, [\kappa(\mathfrak{m}) : H_i(v)]. \tag{I.1.15}$$

This includes all local factors, full inseparable degrees and nilpotent lengths. Generic lengths of the restricted module enter through the inherited module-weight formula.

Proof. We give the choices uniformly; there is no choice of an analytic sheet for an individual projection.

Control of all inverse weights. Since Q_i is injective on the span of each cone of the finite fan Σ, compactness of its intersection with a unit sphere gives $c_i > 0$ such that

$$\|Q_i a\| \geq c_i \|a\| \qquad (a \in |\Sigma|).$$

Use (I.1.11) for every W_j and U_{ij}. The full-stabilization theorem gives one $\rho > 0$ on which all their moment images agree with their respective $w_0 + C_{x_j}$. Shrink B_i to have $\|u - u_i^0\| < c_i \rho$. By (I.1.12), every point of a projected level is in some W_j, and its original moment weight is consequently in $w_0 + |\Sigma|$ and within distance ρ of w_0. Thus no distant inverse weight remains in that projected level.

Let J_i contain, for every cone τ of Σ of dimension less than d, an affine hyperplane containing $u_i^0 + Q_i \tau$. These hyperplanes are proper. Replace H_i by $H_i \cup J_i$. For $u \in B_i \setminus H_i$, every inverse weight belongs to the relative interior of a top cone. There is at most one inverse weight on each top cone, by injectivity there. Hence $\Lambda_i(u)$ is finite. For a fixed σ, pull back

both exceptional sets along $Q_i : w_0 + \operatorname{span}\sigma \to \mathbb{R}^d$. This map is an affine isomorphism, so the pullbacks are proper hyperplanes. They give E_σ.

Common whole factors. Take v as in the statement. Saturation implies that the entire F_v belongs to both C_i. The retention lemma puts it inside the union of the ordinary interiors of the W_j. Because $U_{ij} \subset O_j$ and the O_j are disjoint, intersecting with U_{ij} gives exactly $F_v \cap W_j$. These are actual domain identities, not identifications of reduced spectra. The domain universal property and 2.36.9 identify their actual algebras and their restrictions of M.

For the entire projected level at u, first split its finite set of moment values $\Lambda_i(u)$, then intersect with the W_j. Each exact moment fibre is an affinoid domain obtained by imposing all the equal-radius conditions for the f_k. The intersections are affinoid by the actual domain theorems. This is a finite disjoint affinoid cover of the entire projected level. For any $v' \in \Lambda_i(u)$, retention from this one projection already proves equality of its factors in W_j, U_{1j} and U_{2j}. The flat-germ argument below therefore compares both projections on that factor, even if $Q_k v'$ lies in the exceptional set for the other projection k. Apply 2.38.1, exactly as in Lemma I.1.10. It gives (I.1.13) including the module product and the bounded inverse. The characteristic idempotents are unique because the restriction isomorphism fixes every factor. Since $B_i(u)$ is a canonical finite $R_i(u)$-module, its idempotent factors have the canonical finite $R_i(u)$-module topology. The finite-module comparisons are 1.18.3 and 1.18.7.

Interior and the two normalized degrees. All points of the projected level have target weights strictly inside P_i. They are absolute interior in C_i: the target polyannulus bounds are strict, the finite map is relatively interior by 2.33.18, and interior composes by 2.33.21. By the source-factor inclusion and ordinary local testing they are absolute interior in X. Each point labelled by j lies in the ordinary interiors of W_j, U_{1j} and U_{2j}. The domain interior theorem 2.44.3 and composition then make them absolute interior in these domains as well.

Every nonempty W_j and U_{ij} is pure of dimension d by 5.35.3. At v in the relative interior of σ, their local moment images are either empty or contained in the affine plane $v + (N_\sigma)_{\mathbb{R}}$, by the chosen fan subdivision and the uniform stabilization radius. Apply 5.40.2 to each nonempty whole fibre in W_j and to its restrictions from the two U_{ij}. All of these computations use the same $D_{j,v}$

and the same module, so that theorem gives (I.1.14). We have not compared two sums with their original-weight labels erased.

Lengths and field normalization. The algebra $E_{i,j}(v)$ is finite dimensional over $H_i(v)$ and hence Artinian. Decompose it into its local factors and use a composition series of the finite module in each factor. Every simple quotient is its residue field; taking dimensions proves (I.1.15). For $A_j = \mathcal{O}(W_j)$ and $M_j = M \otimes_A A_j$, the generic version is exactly the inherited formula

$$m_{M_j,f}(v) = \sum_{\mathfrak{p} \in \operatorname{Min} A_j} \operatorname{length}_{(A_j)_{\mathfrak{p}}}(M_j)_{\mathfrak{p}}\, m_{A_j/\mathfrak{p},f}(v).$$

Finite tensors are already complete by 1.18.7.

If radii are normalized after a common complete isometric extension and $|a_k| = e^{-v_k}$, use $\prod_k a_k^{q_{i\ell k}}$ for the projected unit $f^{q_{i\ell}}$. The idempotents in (I.1.13) base change as actual idempotents; retain every factor into which they split. The finite Gauss-field comparison 5.36.2 and model normalization 5.36.3 preserve the total dimensions and their full length-degree formulas. This is a finite-fibre calculation, with no selected geometric lift and no assertion about scalar extension of an integral. □

Proposition I.1.12 (Point labels and sector labels are both necessary)**.** *The hypotheses of Definition I.1.9 occur in every dimension with $r = 2$, a nonflat three-cone fan, and different nilpotent lengths on the two Gauss lifts. For this example $Q_{1,1}$ and $Q_{1,2}$ have whole degrees 6 and 9, and even after separation by Gauss lift their degrees are respectively $(2, 4)$ and $(3, 6)$. After separation by original moment weight, the normalized coefficients on each cone are $(1, 2)$ for both projections.*

Moreover, on either individual lift there is no analytic idempotent on a full projected neighbourhood of the central Gauss fibre that selects the first positive sector and excludes the second on every sufficiently close positive Gauss fibre. Thus the sector idempotents need not extend across the vertex.

Proof. Use the exact reviewed example I.1.8 over trivially valued $K = \mathbb{Q}$:

$$Z = \operatorname{Spec} K[t, t^{-1}, (1-t)^{-1}, z_2^{\pm 1}, \dots, z_d^{\pm 1}], \qquad f = (t, 1-t, z_2, \dots, z_d).$$

Take its bounded moment-box affinoid Y about zero and form the disjoint union of the two actual finite thickenings

$$X = Y \coprod \mathcal{M}\big(\mathcal{O}(Y)[\epsilon]/(\epsilon^2)\big).$$

Use the same moment functions on both components and $M = A$. The two points x_1, x_2 are the trivial generic norms on the two reductions; their thicknesses are $h_1 = 1, h_2 = 2$. The reviewed example proves the pure dimension, interior, full pointed fan and norm-admissible saturated finite collars. Finite products and these canonical thickenings preserve them. The fan on each component is

$$\big(\mathbb{R}_{\geq 0}(1,0) \cup \mathbb{R}_{\geq 0}(0,1) \cup \mathbb{R}_{\geq 0}(-1,-1)\big) \times \mathbb{R}^{d-1}.$$

For $Q_{a,b}(u, v, \lambda) = (au + bv, \lambda)$, $a, b > 0$, the whole finite algebra in the reviewed example has degree $a + b$. Its central analytic Gauss algebra on the jth component is

$$K(t, z_2, \ldots, z_d)[\epsilon]/(\epsilon^{h_j}) \quad \text{over} \quad K(t^a(1-t)^b, z_2, \ldots, z_d).$$

For $h_j = 1$ this means the field itself. These fields have their trivial complete norms; the full degree is $a + b$. Each algebra is local, so the central analytic Gauss fibre consists of precisely x_j. Thus both projections have the same full two-point set S, although their central algebras have different dimensions.

The cone indices are $a, b, a + b$. At a small positive first projected weight s, the two original moment weights are

$$(s/a, 0, \lambda), \qquad (0, s/b, \lambda).$$

Their Gauss dimensions on component j are $h_j a$ and $h_j b$, by the reviewed cone-weight calculation and finite thickening. At a negative projected weight only the third cone contributes, with dimension $h_j(a + b)$. Consequently the full degrees are $(h_1 + h_2)(a + b)$, giving 6 and 9, and the point-labelled degrees are as stated. Dividing each *sector* factor by its own index gives h_j, hence $(1, 2)$ for both projections. Dividing a mixed point-labelled degree by the first cone index still gives 2 versus 3 on the first component. Matching the two point labels by an arbitrary permutation would likewise compare 1 with 2. The same points and the same original weights, rather than counts or degree multisets, fix the correct matching.

Finally fix a component j and one $Q_{a,b}$. Its central analytic Gauss fibre is the singleton x_j. Suppose an analytic idempotent on the full inverse image of an ordinary target neighbourhood of the central Gauss point made the proposed separation for all small $s > 0$. Its value at x_j is 0 or 1, and continuity makes it constant on an ordinary neighbourhood of x_j. Shrink to an affinoid

target neighbourhood to use compactness. The image of the compact complement of that source neighbourhood misses the central Gauss point. The target Gauss points of weights $(s, 0, \dots, 0)$ converge to the central Gauss point, so their *entire analytic fibres* eventually lie in that one constant-idempotent locus. Both positive-sector factors are nonzero: their Gauss dimensions are $h_j a > 0$ and $h_j b > 0$. The idempotent therefore cannot be 1 on one and 0 on the other. This contradiction proves the nonextension assertion. It does not prevent the actual sector idempotents at each nonzero regular projected level. □

The ground field K is complete for its specified real-valued non-Archimedean absolute value, which may be trivial. Every finite analytic map below is required to carry the canonical finite-module norm, up to uniform equivalence. All fibres are actual analytic fibres over K, with their full algebras and nilpotents. No scalar extension is used in the construction.

Theorem I.1.13 (Simultaneous collars after one fixed finite cover)**.** *Use Definition C.1.46. Suppose that*

$$\pi : X = \mathcal{M}(B) \longrightarrow Y_\varepsilon$$

is norm-admissible finite, that X is nonempty and pure of dimension d, and that $S = \pi^{-1}(\eta)$ is nonempty. Let $f = \pi^* g$ and let M be any finite B-module.

For any two distinct positive arrays a, a'*, choose boxes satisfying (2), separately for the two arrays. Then*

$$C_a = (Q_a t_f)^{-1}(P_a), \qquad C_{a'} = (Q_{a'} t_f)^{-1}(P_{a'})$$

are saturated norm-admissible finite collars in the same X. Their entire central analytic Gauss fibres are the same finite point set $S \subset t_f^{-1}(0)$. Every $x \in S$ is absolutely interior and has graded moment rank d.

The full pointed fan at every $x \in S$ is

$$F = F_1 \times \cdots \times F_p \times \mathbb{R}^k, \tag{7}$$

where F_ℓ is (5). The same rational subdivision of $\mathbb{R}^k$ may be used for both projections. On each top cone σ the projections are injective on its span, and their saturated-lattice indices are

$$e_{a,\sigma} = [\mathbb{Z}^d : Q_a N_\sigma] = \prod_{\ell=1}^{p} \begin{cases} a_{\ell j}, & \sigma \text{ uses the puncture ray } j, \\ A_\ell, & \sigma \text{ uses the last ray.} \end{cases} \qquad N_\sigma = \mathbb{Z}^n \cap \operatorname{span}_{\mathbb{R}} \sigma. \tag{8}$$

Thus these data satisfy all hypotheses of the independently reviewed definition I.1.9 and theorem I.1.11, including its common entire Gauss set and saturation requirements. If some $m_\ell \geq 2$, F is not a d-plane.

Proof. Write $Y_a = q_a^{-1}(\mathbb{T}_{P_a})$. Lemma C.1.47 gives $Y_a \subset Y_\varepsilon$ as the full inverse image, with its actual finite algebra. Domain base change and finite base change, 2.36.9 and 2.33.19, identify

$$C_a = \pi^{-1}(Y_a), \qquad \mathcal{O}(C_a) = B \otimes_{\mathcal{O}(Y_\varepsilon)} \mathcal{O}(Y_a).$$

The tensor is already complete with its canonical finite norm. The map $C_a \to Y_a$ is finite, as is $Y_a \to \mathbb{T}_{P_a}$. Composing finite free quotient presentations proves finiteness with the canonical norm for the composite. This construction also proves the displayed saturation equality for all of X. The same argument applies to a'.

By (3), a point of C_a maps to the central target Gauss point if and only if its image under π is η. The entire fibre is therefore exactly S, independently of a. Its finiteness follows from the actual finite-dimensional algebra over $\mathcal{H}(\eta)$; all local factors are kept. The map π is everywhere relatively interior by 2.33.18. Absolute interiority of η and the composition inclusion 2.33.21 give absolute interiority of every $x \in S$.

We identify the fan before transferring it to X. The norm calculation in the preceding lemma gives global containment $t_g(Z^{\mathrm{an}}) \subset F$. Every vector of F sufficiently near 0 is realized as follows. For a puncture ray of size $s \geq 0$, use the Gauss norm in $t_\ell - c_{\ell j}$ of radius e^{-s}. For the last ray use the Gauss norm in t_ℓ of radius e^{s}. Give each z_b its prescribed Gauss radius. Take the simultaneous Gauss norm in these d independent variables. These points have precisely the stated moments. As all parameters tend to zero, every such family tends to η. Indeed at radius 1 translation by an element of K° preserves the unit Gauss norm, and polynomial Gauss evaluations are continuous in the radii. The same is true for the inverses of the puncture coordinates, hence, by uniform approximation, for the affinoid functions on a fixed surrounding domain. There are only finitely many choices of rays. Every ordinary neighbourhood of η therefore realizes every sufficiently small vector in F. The frozen full-stabilization theorem identifies its pointed fan with F.

For $x \in S$, the isometric field embedding $\mathcal{H}(\eta) \hookrightarrow \mathcal{H}(x)$ induces an injective map of graded residue fields. The graded relations of the initial forms of $f(x)$ over $\widetilde{K}^{\mathrm{gr}}$ are consequently *exactly* the relations of $g(\eta)$, not a larger

ideal. The initial forms of $t_\ell - c_{\ell 1}$ and z_b are algebraically independent, and all other initial forms are obtained from them by the displayed translations. Thus the graded moment rank is d. In 5.55.5 the fan is defined from precisely this homogeneous relation ideal. Together with 5.55.8, this proves that every C_x equals F, even when X is nonreduced.

For a top cone, its saturated tangent lattice is the direct sum of the primitive ray lattice of each block and the full $\mathbb{Z}^k$. Each primitive block vector maps to $a_{\ell j}$ or $-A_\ell$ times the corresponding target basis vector. The matrix is diagonal, with the identity on the last k directions. This proves injectivity and (8). If $m_\ell \geq 2$, the span of F_ℓ has dimension $m_\ell > 1$, while that fan has dimension one. Thus the span of F has dimension greater than d, and F is not a d-plane.

We have verified, individually, purity, actual units and module, both saturated finite collars, the same full finite Gauss set, absolute interiority, graded rank, and simultaneous genericity on every top cone. These are exactly the hypotheses of the reviewed sector comparison. □

Fix a complete non-Archimedean field K with its specified real-valued absolute value, possibly trivial. All affinoids are generalized affinoids, with arbitrary positive real presentation radii. Finite maps are norm-admissible finite: their given Banach structures are equivalent to the canonical finite-module structures. All domains, fibres and modules below have their actual analytic structures; nilpotents are retained. Write $t_g = -\log|g|$ for a tuple of analytic units. For a closed rectangular box $P \subset \mathbb{R}^m$, $\mathbb{T}_P$ is the entire inverse image of P in the analytic m-torus. The point η_s is its weighted Gauss point at $s \in P$.

For a point x and an actual unit tuple f, write

$$r_f(x) = \operatorname{trdeg}_{\widetilde{K}^{\mathrm{gr}}} \widetilde{K}^{\mathrm{gr}}\big(\widetilde{f_1(x)}, \dots, \widetilde{f_n(x)}\big).$$

The full pointed fan C_x is the one supplied by 5.55.8. It is pure of dimension $r_f(x)$.

Lemma I.1.14 (Interior Gauss recognition under a finite fan projection)**.** *Let Z be a generalized affinoid, let $x \in \operatorname{Int}(Z/K)$, and let f be an actual unit tuple with $t_f(x) = w$. Suppose that C_x is contained in the support of a finite pure d-dimensional rational fan $\Sigma \subset \mathbb{R}^n$, where $d \geq 1$. Let $Q : \mathbb{Z}^n \to \mathbb{Z}^d$ be*

integral and injective on the real span of every top cone of Σ*. For the actual monomial tuple* $g = f^Q$*,*

$$g(x) = \eta_{Qw} \quad \Longleftrightarrow \quad r_f(x) = d. \tag{7}$$

This is an equality criterion for analytic points of the target torus. It does not identify the different projected fibre algebras with one another.

Proof. First, for any d-tuple g, the target point $g(x)$ is the weighted Gauss point at $t_g(x)$ if and only if its d initial forms are algebraically independent over $\widetilde{K}^{\mathrm{gr}}$. Here is the original-field argument. For a Laurent polynomial $\sum_\nu a_\nu T^\nu$, put

$$R = \max_\nu |a_\nu| \prod_j |g_j(x)|^{\nu_j}.$$

The terms attaining R define a nonzero homogeneous polynomial of degree R in the graded variables. Independent initial forms make its value nonzero; consequently the evaluated polynomial has norm exactly R. These equalities identify the seminorm with the weighted Gauss seminorm. Conversely, lift the finitely many homogeneous coefficients of a proposed graded relation to K. A nonzero homogeneous component of that relation has one common weighted degree. Its lifted Laurent polynomial has that Gauss norm, whereas the relation would force a strictly smaller evaluated norm. This is impossible at a Gauss point. This argument uses neither realizability of the radii in $|K^\times|$ nor a change of the given absolute value.

The monomial initial forms of g belong to the graded field generated by those of f. Hence $r_g(x) \le r_f(x)$. The containment $C_x \subset |\Sigma|$ and the full pointed-fan theorem give $r_f(x) = \dim C_x \le d$. If $g(x)$ is Gauss, the preceding criterion gives $r_g(x) = d$, and therefore $r_f(x) = d$.

Now suppose $r_f(x) = d$. Choose a d-cone τ of the pointed fan. Its intersections with the finitely many cones of Σ cover it. At least one such intersection has dimension d: a finite union of proper linear subspaces cannot contain the relative interior of a d-cone. Thus some d-dimensional subcone $\tau' \subset \tau$ lies in a top cone of Σ. The map Q is injective on its span, so $Q\tau'$ has dimension d.

For every sufficiently small ordinary neighbourhood W of x, pointed stabilization supplies all sufficiently small vectors of $w + \tau'$ in $t_f(W)$. Their images occur in $t_g(W) = Qt_f(W)$. Applying full pointed stabilization also to g, and choosing W small enough for both tuples, shows that its pointed image fan contains a d-dimensional conical germ. Its dimension is $r_g(x)$; hence

$r_g(x) \geq d$. There are only d generators, so $r_g(x) = d$. The Gauss criterion proved in the first paragraph completes the proof. This uses a dimension lower bound from the actual neighbourhood images, not a claim that arbitrary projection commutes with an uncontrolled image germ. □

Proposition I.1.15 (A nonflat pointed germ with no finite saturated box)**.** *Let $K = \mathbb{Q}$ with its trivial absolute value, $d \geq 1$, and use one puncture block $0, 1$, with $k = d-1$. Fix $\varepsilon > 0$ and $0 < r < 1$, and define the affinoid domain*

$$X_r = Y_\varepsilon \cap \{|t-2| \geq r\}, \qquad f = (t, t-1, z_1, \ldots, z_{d-1}).$$

Let $x = \eta$. Then X_r is pure of dimension d, $x \in \mathrm{Int}(X_r/K)$, and the pointed moment fan at x is the nonflat three-ray fan times $\mathbb{R}^{d-1}$, with graded rank d.

For every $a, b > 0$, the map

$$\psi_{a,b} = (t^a(t-1)^b, z_1, \ldots, z_{d-1})$$

is finite at x in the ordinary-neighbourhood sense. Its full fibre over the target unit Gauss point is $\{x\}$. Nevertheless, for no *full-dimensional closed rectangular box P with $0 \in \mathrm{int}(P)$ is the saturated map*

$$(Q_{a,b}t_f)^{-1}(P) \longrightarrow \mathbb{T}_P \tag{11}$$

norm-admissible finite. This failure already occurs separately for $Q_{1,1}$ and $Q_{1,2}$; hence the same fan, common analytic Gauss set, interior moment point, and ordinary local finiteness do not imply the two finite-collar hypotheses.

Proof. The lower bound on $t-2$ is a Laurent-domain condition. Domain purity gives the asserted dimension. At x its inequality is strict, since $|t-2|(x) = 1 > r$; all original moment bounds are also strict. Adjoining the inverse of $t-2$ with radius $r^{-1} > 1$ to the presentation in Lemma C.1.47 witnesses absolute interiority of x. As X_r is an ordinary neighbourhood of x in Y_ε, full pointed stabilization and Theorem I.1.13 with the identity cover give the same rank and fan. The fibre assertion follows from (3), since x remains in the domain.

For completeness, ordinary local finiteness can be exhibited using only the frozen local ingredients. Choose a small box P_0 satisfying (2), and let $Y_0 = q_{a,b}^{-1}(\mathbb{T}_{P_0}) \subset Y_\varepsilon$. It is compact and finite over the full polyannulus. Inside Y_0, the open set $G = \{|t-2| > r\}$ contains its *entire* central analytic Gauss fibre $\{x\}$. The image of $Y_0 \backslash G$ is compact, hence closed in the Hausdorff

target, and misses the target Gauss point. Choose an ordinary affinoid target neighbourhood V of that point, inside the complementary open set and the ordinary interior of $\mathbb{T}_{P_0}$. The ordinary affinoid-neighbourhood basis is the one used in 5.52.3. Finite base change gives $q_{a,b}^{-1}(V) \to V$ finite, and its full inverse image lies in $G \subset X_r$. It is an ordinary neighbourhood of x. The domain-interior and composition formulae (with ordinary openness for this finite free map) identify the ordinary interior of $q_{a,b}^{-1}(V)$ in Y_0 with the inverse image of the ordinary interior of V in $\mathbb{T}_{P_0}$: compute relative interior over $\mathbb{T}_{P_0}$ using the two finite factorizations. Since V lies in the ordinary interior of that polyannulus, these are also ordinary open neighbourhoods in the ambient source and target. Restricting gives a finite map between them, as required by finiteness at x. Only this ordinary target neighbourhood, not a tropical box, has been constructed.

Suppose (11) were norm-admissible finite. Restrict its target to a sufficiently small closed box about zero. Finite base change preserves the supposed property. We may therefore assume that P satisfies (2). Set

$$Y_P = q_{a,b}^{-1}(\mathbb{T}_P), \qquad C = X_r \cap Y_P.$$

Then C is an affinoid domain of Y_P, and is the entire inverse image in (11).

For $0 < \rho < 1$, let y_ρ be the simultaneous Gauss point in the coordinates $(t-2, z_1, \dots, z_{d-1})$, of radii $(\rho, 1, \dots, 1)$. The trivial ground norm gives

$$|t(y_\rho)| = |t(y_\rho) - 1| = |z_b(y_\rho)| = 1, \qquad |t(y_\rho) - 2| = \rho.$$

Every y_ρ lies in Y_P and has original and projected weight zero. The map $\rho \mapsto y_\rho$ is continuous: polynomial Gauss norms are continuous maxima, the puncture inverses have norm 1, and uniform approximation in an ambient affinoid presentation handles every analytic function. But $y_\rho \in C$ exactly when $\rho \geq r$. Thus $y_r \in C$ is not in the ordinary interior of C inside Y_P.

The supposed norm-admissible finiteness of $C \to \mathbb{T}_P$ makes every point relatively interior, by 2.33.18. Apply the frozen composition identity (using the source-factor implication) to $C \hookrightarrow Y_P \to \mathbb{T}_P$:

$$\operatorname{Int}(C/\mathbb{T}_P) = \operatorname{Int}(C/Y_P) \cap (C \hookrightarrow Y_P)^{-1} \operatorname{Int}(Y_P/\mathbb{T}_P).$$

It forces $y_r \in \operatorname{Int}(C/Y_P)$. The domain-interior theorem 2.44.3 says this is exactly ordinary interior, contradicting the preceding path. This proves the obstruction. □

The end cycle of a finite balanced graph

Use the finite graph datum of definition C.1.22, the cylinder description of lemma C.1.23, and the interleaved convention of lattice normalization. The smooth real profile is the one used in the compact-cutoff proof. Write $N = \mathbb{Z}^r$, $E = N_{\mathbb{R}} \times \mathbb{R}$, with coordinates (y, t), projection $\pi : E \to N_{\mathbb{R}}$, and ambient lattice $N \oplus \mathbb{Z}$. A polyhedron below is closed and given by finitely many affine linear inequalities, with rational directions and arbitrary real constants. For its tangent space L_σ, its lattice is the saturated lattice $N_\sigma = L_\sigma \cap (N \oplus \mathbb{Z})$.

After a common subdivision of the polytopes Q_σ, define the weighted end chain

$$D_\infty(C) = \sum_{\sigma \text{ unbounded},\, \dim \sigma = n} m_\sigma [Q_\sigma]. \tag{2.2}$$

Coincident projected pieces are added with all their weights. Equation (2.1) fixes the vertical index at one; degrees already present in m_σ must still be retained.

Let $W \subset N_{\mathbb{R}}$ be open. Assume that the restriction of C to $\pi^{-1}W$ is a balanced local n-cycle, with the indicated integer weights, exactly as in NAG-002384. This means that every relevant codimension-one normal sum is tangent to its facet; it includes all finite lower and side faces meeting this open set.

Theorem I.1.16 (Weak end-cycle identity; F8LA-P3R)**.** *For every $\eta \in \mathcal{A}_c^{n-1,n-1}(W)$, extended by zero outside W,*

$$I_C\big((-t)d'd''\pi^*\eta\big) = I_{D_\infty(C)}(\eta). \tag{3.1}$$

The left side is absolutely convergent in the cell sense of definition C.1.22.

Choose $\rho \in C_c^\infty((0,1))$, $\rho \geq 0$, $\int \rho = 1$, and define

$$h_R(t) = -t + \int_{\mathbb{R}} (t - R - s)_+ \rho(s)\, ds. \tag{3.2}$$

Then $h_R = -t$ for $t \leq R$, $h_R = -R - \mu$ for $t \geq R + 1$, where $\mu = \int s\rho(s)\, ds$, and $h_R'' = \rho(t - R)$. For $R \geq H$,

$$I_C\big(d'd''h_R \wedge \pi^*\eta\big) = I_{D_\infty(C)}(\eta). \tag{3.3}$$

Thus these pairings stabilize exactly, on this prescribed class of tests.

Proof. For completeness, the smooth-profile construction in the proof of NAG-002427 gives $E(t) = e^{-1/t}$ for $t > 0$ and $E(t) = 0$ otherwise. One may take $\theta(t) = 1 - E(t)/(E(t) + E(1-t))$. To obtain a density with support strictly inside $(0,1)$, normalize $E(t-\frac{1}{4})E(\frac{3}{4}-t)$ by its positive continuous integral. These formulas supply the θ and ρ used here. The integral in (3.2) has a continuous, compactly supported coefficient; the established one-dimensional integral and fundamental theorem give the stated derivatives.

Put $\beta = \pi^*\eta$. Since $d'd''\beta = \pi^*(d'd''\eta)$ has bidegree (n,n), theorem C.1.24 proves absolute convergence in (3.1).

Choose $\theta \in C^\infty(\mathbb{R})$ equal to one on $(-\infty, 0]$ and zero on $[1, \infty)$, and put $w_R(t) = -t\theta(t-R)$. The forms $w_R d''\beta$ and $d'w_R \wedge \beta$ have compact support on the local cycle over W: horizontally their supports lie in the fixed compact support of η, and vertically they lie in $[a, R+1]$. Two applications of `v037:integration:balancedcurrent` give

$$0 = I_C\big(d'(w_R d''\beta)\big) = I_C(d'w_R \wedge d''\beta) + I_C(w_R d'd''\beta),$$
$$0 = I_C\big(d''(d'w_R \wedge \beta)\big) = -I_C(d'd''w_R \wedge \beta) - I_C(d'w_R \wedge d''\beta).$$

Consequently

$$I_C(w_R d'd''\beta) = I_C(d'd''w_R \wedge \beta). \tag{3.4}$$

For $R \geq H$, the left side is the left side of (3.1). The right side is supported in the cylindrical region $R \leq t \leq R+1$. On a cylinder choose an integral basis of N_{Q_σ} followed by e_t. The product lattice has covolume one. Moving the pair $d't \wedge d''t$ past the even-degree horizontal top form gives sign $+1$. Iterated real integration therefore yields

$$I_C(d'd''w_R \wedge \beta) = \left(\int_R^{R+1} w_R''(t)\,dt\right) I_{D_\infty(C)}(\eta) = I_{D_\infty(C)}(\eta),$$

because $w_R'(R) = -1$ and $w_R'(R+1) = 0$. This proves (3.1). Replacing w_R'' by $h_R'' = \rho(t-R)$, whose integral is also one, proves (3.3). □

The ordinary tube and its absolute-boundary image

The ordinary image-tube scope also applies over each actual Gauss residue field used below. The dimension calculation uses the valued-field tower formula and the equality of completed residue fields in the actual boundary cover.

Lemma I.1.17 (Uniform tube and boundary defect; F8RC-T)**.** *Let $V = \mathcal{M}(A)$ be affinoid, $f \in A$, and let the nonempty actual zero scheme $D = \mathcal{M}(A/(f))$ have dimension at most d. For any unit tuple g, put $P_0 = t_g(D)$. There are $\rho > 0$ and a finite integral affine polyhedral set $\Delta \subset P_0$, with $\dim \Delta \leq d-1$, such that*

$$t_g(\{|f| < \rho\}) \subset P_0, \tag{I.1.16}$$

$$t_g(\partial_K V \cap \{|f| < \rho\}) \subset \Delta. \tag{I.1.17}$$

Here $\Delta = \varnothing$ if $d = 0$. If D is empty, a sufficiently small tube is empty. The same assertions hold over any specified complete base field and with arbitrary radii.

Proof. By NAG-002358, P_0 is a compact finite integral affine polyhedral set of dimension at most d. The inverse image of each compact cell is an actual affinoid Laurent domain: integral affine inequalities become bounds on monomials in the units g_i, with positive real radii. Their finite union $W_0 = t_g^{-1}(P_0)$ is an analytic domain, obtained by gluing its affinoid pieces along their actual intersections. It contains the closed subspace D. Lemma H.0.3 gives an ordinary neighbourhood O of D contained in W_0. On the compact complement $V \setminus O$, $|f|$ has a positive minimum, if the complement is nonempty. This proves (I.1.16) for a sufficiently small positive ρ.

Use NAG-002356 to write the entire absolute boundary as a finite union of actual compact analytic fibres V_j over fields H_j, where $d(H_j/K) = 1$ and the maps $V_j \to V$ preserve completed residue fields. Let D_j be the actual zero scheme of $f|_{V_j}$. Its points are exactly $D \cap V_j$. For such a point z, the tower formula and the maximum dimension bound give

$$d_{H_j}(z) = d_K(z) - 1 \leq d - 1.$$

Consequently $\dim_{H_j} D_j \leq d-1$, by NAG-001591. If $d = 0$, no such point exists. Each nonempty D_j has a finite polyhedral image $\Delta_j = t_g(D_j)$ of dimension at most $d-1$, by NAG-002358 over H_j.

Apply the first paragraph over H_j, with V_j, D_j, to get $t_g(V_j \cap \{|f| < \rho_j\}) \subset \Delta_j$. If D_j is empty, compactness gives a positive minimum of $|f|$ on V_j, so its small tube is empty. Shrink ρ to work for all finitely many j, and take $\Delta = \bigcup_j \Delta_j$. The maps preserve the absolute values of the original units, so this is a subset of P_0. It gives (I.1.17) for the actual absolute boundary, not merely for a polyhedral boundary of a chosen chart. □

Example I.1.18 (A higher-dimensional local application)**.** Let A be a generalized K-affinoid algebra and let

$$X = \mathcal{M}\big(A\{r_1^{-1}T_1, \dots, r_m^{-1}T_m\}\big), \qquad r_j > 0,$$

be good and pure of analytic dimension n, and assume that each coordinate hyperplane $Y_j = \{T_j = 0\}$ used below has analytic dimension at most $n - 1$. Evaluation at $T_j = 0$ and the inclusion of series independent of T_j are bounded algebra maps whose composition is the identity. The actual quotient by (T_j) is the corresponding algebra in the other variables, by its power-series coefficients and `fnd:generalizedaffinoidquotients`. The induced spectral maps give the closed immersion $Y_j \hookrightarrow X$ and retraction $X \to Y_j$. For an analytic unit u and integers a_j, choose

$$g = u \prod_{a_j>0} T_j^{a_j}, \qquad h = \prod_{a_j<0} T_j^{-a_j}.$$

Assume these chosen g, h satisfy the local non-zero-divisor hypotheses of Theorem G.0.15. Their zero and pole set is contained in the union of the coordinate hyperplanes. At every point of that set, choose one coordinate hyperplane containing it. Its coordinate retraction supplies (R_{n-1}). Thus the theorem gives the logarithmic current and the uniform stationarity on each top-degree presentation. The original absolute value and arbitrary positive radii are retained.

Fix a complete field K with its specified real-valued non-Archimedean absolute value. Trivial valuation and generalized affinoid radii are allowed. Spaces in this supplement are good and Hausdorff. Coherent modules live on the analytic G-site. A line bundle means a coherent module which is free of rank one on ordinary open neighbourhoods. The symbol q^* on coherent modules denotes ordinary tensor pullback; it is not a derived functor.

Proposition I.1.19 (Coherent tensor projection)**.** *Let $q : Y \to X$ be norm-admissible and finite. For $\mathcal{F} \in \mathrm{Coh}(Y_G)$ and $\mathcal{E} \in \mathrm{Coh}(X_G)$ there is a canonical isomorphism*

$$q_*(\mathcal{F} \otimes_{\mathcal{O}_Y} q^*\mathcal{E}) \simeq q_*\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}. \tag{I.1.18}$$

It commutes with affinoid-domain restriction and analytic base change. Neither $\mathcal{F}$ nor $\mathcal{E}$ is required to be locally free. If X, Y are pure of the same analytic dimension d and $\alpha \in \mathcal{A}^{d,d}_{X,c}(X)$ is an ordinary smooth moment form, then

$$\int_{Y/K,\,\mathcal{F}\otimes q^*\mathcal{E}} q^*\alpha = \int_{X/K,\,q_*\mathcal{F}\otimes\mathcal{E}} \alpha. \tag{I.1.19}$$

Both integrals use the fixed ground field K.

Proof. On $V = \mathcal{M}(A)$ and its full inverse image $\mathcal{M}(B)$, let M and N represent $\mathcal{E}$ and $\mathcal{F}$. The map

$$N \otimes_B (B \otimes_A M) \longrightarrow N \otimes_A M, \qquad n \otimes (b \otimes m) \longmapsto bn \otimes m$$

has inverse $n \otimes m \mapsto n \otimes (1 \otimes m)$. These are maps of finite A-modules, so they are bounded, with bounded inverses, for the canonical norms. The finite-tensor theorem identifies the modules with the analytic completed constructions.

For a bounded affinoid base change $A \to C$, put $D = B \otimes_A C$. The same formulas identify both base changes with $N \otimes_A M \otimes_A C$. In particular they respect domain restrictions. The finite coherent direct-image theorem and affinoid coherent equivalence therefore glue them to (I.1.18). This argument concerns an isomorphism of tensors, and asserts no exactness for tensoring with an arbitrary $\mathcal{E}$. Finally apply the fixed-field finite projection theorem to the coherent module $\mathcal{F} \otimes q^*\mathcal{E}$. It supplies compact support of $q^*\alpha$ and the integral equality with coefficient $q_*(\mathcal{F} \otimes q^*\mathcal{E})$. Substitute (I.1.18). □

Fix a complete field K with its specified real-valued non-Archimedean absolute value. Trivial valuation and arbitrary positive affinoid radii are allowed. All analytic spaces below are good and Hausdorff. A finite morphism is norm-admissible finite: full affinoid inverse images have finite algebras with their canonical finite-module topology. No algebra is replaced by its reduction. Line bundles are coherent modules with frames on ordinary open neighbourhoods.

Let $q : Y \to X$ be finite, and assume that $E = q_*\mathcal{O}_Y$ is locally free of constant rank $n \geq 1$ on ordinary opens. Fix an analytic line bundle $\mathcal{M}$ on X with a continuous metric. The working class upstairs consists of the *specified* pullback $\mathcal{L} = q^*\mathcal{M}$ with its pulled-back metric. For a frame e of $\mathcal{M}$, put $\phi_e = -\log\|e\|$. Call the metric ordinary moment-smooth when these weights belong to $\mathcal{A}_X^{0,0}$ in the frozen ordinary smooth moment sheaf. This condition is independent of the frame: $\phi_{ue} = \phi_e - \log|u|$, and the logarithm of an analytic unit is a smooth moment function.

The reviewed determinant projection isomorphism is

$$\iota_q : \mathrm{Nm}_q(q^*\mathcal{M}) := \det(q_*q^*\mathcal{M}) \otimes \det(E)^\vee \xrightarrow{\sim} \mathcal{M}^{\otimes n}. \tag{I.1.20}$$

Transport the tensor-power metric through ι_q. This defines what this note calls the *projection-compatible norm metric* on this class. It is specified by the downstairs metrized line and its pullback presentation. No construction for an arbitrary metrized line upstairs is presumed.

For a section s of $q^*\mathcal{M}$ on a full inverse image $q^{-1}U$, the map $a \mapsto as$ from $E|_U$ to $(q_*q^*\mathcal{M})|_U$ gives a determinant section $\mathrm{Nm}_q(s)$. If e is a frame on U and $s = b\,q^*e$, its coefficient under (I.1.20) is

$$\iota_q(\mathrm{Nm}_q(s)) = N(b)e^{\otimes n}, \qquad N(b) = \det(m_b : E|_U \to E|_U). \tag{I.1.21}$$

The useful link is between the vanishing order of this coefficient and the ordinary local intersection lengths of s.

Theorem I.1.20 (Compatibility on the pulled-back metric class)**.** *Use the finite locally free analytic map and the metrized line $\mathcal{M}$ of Section 1.*

1. *The projection-compatible metric on $\mathrm{Nm}_q(q^*\mathcal{M})$ is continuous. If the metric on $\mathcal{M}$ is ordinary moment-smooth, so is this metric. In the norm frame corresponding to $e^{\otimes n}$, its weight is exactly $n\phi_e$.*

2. *For $s = b\,q^{*}e$, the determinant section satisfies (I.1.21). Wherever $N(b)$ is an analytic unit,*

$$-\log\|\operatorname{Nm}_q(s)\| = -\log|N(b)| + n\phi_e. \tag{I.1.22}$$

In the smooth case this is an ordinary smooth moment function on that locus. The line, section, metric and this formula commute with analytic base change between good Hausdorff spaces, with the full finite inverse image and its actual norm retained.

3. *Fix $x \in U$ such that the ordinary local ring $A = \mathcal{O}_{X,x}$ is a DVR. Write $k = \kappa(x)$ for its ordinary residue field. Suppose the coefficient germ $b_y \in \mathcal{O}_{Y,y}$ is a non-zero-divisor for every $y \in q^{-1}(x)$. Then the local Cartier multiplicity of the determinant section is*

$$\operatorname{ord}_x \operatorname{Nm}_q(s) = \sum_{y \in q^{-1}(x)} [\kappa(y) : \kappa(x)]\, \operatorname{length}_{\mathcal{O}_{Y,y}}(\mathcal{O}_{Y,y}/b_y\mathcal{O}_{Y,y}). \tag{I.1.23}$$

The field degrees here are those of the ordinary local residue fields, not a count of analytic points or embeddings. The lengths include every nilpotent layer. This local multiplicity is independent of the chosen metric and frame.

The section s need not be the pullback of a downstairs section. The DVR and non-zero-divisor hypotheses in part (3) are additional; they are not assertions about all analytic points or arbitrary base changes.

Proof. The isomorphism (I.0.11) sends the norm of the pulled-back frame to $e^{\otimes n}$. The tensor-power metric therefore gives weight $n\phi_e$. If $e' = ue$ for an analytic unit u, then

$$\phi_{e'} = \phi_e - \log|u|, \qquad (e')^{\otimes n} = u^n e^{\otimes n}.$$

These are exactly the metric transition rules. They prove the construction glues and is continuous. Scalar multiplication preserves the ordinary smooth moment sheaf, so it also preserves the asserted regularity. A new frame of the norm line multiplies this frame by an analytic unit v; subtracting $\log|v|$ keeps the weight smooth. This verifies regularity for every frame, not just the displayed one.

Locally trivialize E. The map $a \mapsto as$ is then the matrix m_b tensored with e, so exterior multiplication gives (I.1.21). Under $e' = ue$, the coefficient becomes $b' = (q^{*}u)^{-1}b$. Multiplication by $q^{*}u$ on E is uI_n; hence

$N(b') = u^{-n}N(b)$. The displayed determinant section and (I.1.22) are therefore independent of e. For example, the two changes in the right side are $+n\log|u|$ and $-n\log|u|$, which cancel. Where $N(b)$ is a unit, its negative logarithm is the one-coordinate linear moment representative $t \mapsto t$ for that unit. The frozen sheaf theorem consequently proves the last regularity claim.

Finite coherent direct image commutes with analytic base change by `NAG-001752`. On an affinoid base change, the new finite algebra is the full ordinary tensor with its canonical complete finite-module norm, not its reduction. The matrix of multiplication and its determinant are the base changes of the old matrix and determinant. The reviewed projection isomorphism has the same compatibility. Pullback of metric weights is composition on points, and ordinary smooth pullback is provided by `NAG-002424`. Thus both constructions have weight n times the pulled-back ϕ_e, proving the claimed isometry and section compatibility. No claim that DVR orders are unchanged by base change is made.

For the local identity, frozen `NAG-001735` gives the actual ordinary-stalk decomposition

$$B = (q_*\mathcal{O}_Y)_x \simeq \prod_{y\in q^{-1}(x)} \mathcal{O}_{Y,y}.$$

The module B is free of rank n over A by hypothesis. Each factor is finite over A, and the decomposition includes the whole fibre. The assumptions on b_y say precisely that multiplication by b is injective on this product. Lemma C.1.95 therefore applies. The germ of $N(b)$ is its determinant, and $\operatorname{length}_A A/(N(b)) = \operatorname{ord}_A N(b)$ is the local Cartier multiplicity at the DVR. The Artinian factors in the lemma are exactly the nonzero quotients of the displayed local-ring factors. This proves (I.1.23) with the stated residue fields. Changing the frame multiplies $N(b)$ by a unit, whose order is zero. The metric enters only the regular term $n\phi_e$ in (I.1.22), not the determinant or any quotient ring. □

Fix a complete field K with its specified real-valued non-Archimedean absolute value. In the new propositions the value may be trivial. All finite algebras and maps retain their canonical finite-module topology, their full fibres, and their nilpotents. A finite analytic map means a norm-admissible finite map. Analytic spaces are good and Hausdorff. All analytic integrals are taken over this *same* K.

For an ordinary locally framed analytic line bundle and a metric h, write

$$\phi_{e,h} = -\log \|e\|_h, \qquad h' = e^{-u}h \quad \Longleftrightarrow \quad \|v\|_{h'} = e^{-u}\|v\|_h.$$

Thus $\phi_{e,h'} - \phi_{e,h} = u$. We use $v(a) = -\log|a|$, without a factor 2, 2π, $[L : K]^{-1}$, or a rescaling of $|\,|$. For an ordinary moment-smooth metric, put $\Theta_h = d'd''\phi_{e,h}$. This form is independent of the frame: if $e' = ae$, then $\phi_{e',h} = \phi_{e,h} - \log|a|$; the unit a can be adjoined as a moment coordinate, on which $-\log|a|$ is affine and has $d'd''$ equal to zero. The frozen germ sheaf makes restriction, wedge product, and pullback compatible with this calculation.

For bounded variation in positive dimension we use a compact space X pure of dimension d, ordinary moment-smooth metrics $h_1, \dots, h_d$, and

$$\theta = \Theta_{h_1} \wedge \cdots \wedge \Theta_{h_d}.$$

When an estimate or a uniform extension is asserted, we explicitly assume that θ is positive in the frozen top-form sense. We do not identify this hypothesis with a model-theoretic nef condition. The initial variation metric h_0 may be any continuous metric; the ratio u must be smooth or belong to the uniform closure specified below. No assertion that every model metric satisfies these conditions is made.

For a finite projection $q : Y \to X$ we additionally require $E = q_*\mathcal{O}_Y$ to be locally free of constant rank $n \geq 1$ on ordinary opens when a determinant norm or a scalar degree formula is used. This is the precise finite locally free hypothesis here. It is not inferred from a proper-finite collar. Both spaces have the same pure dimension in the analytic integral statements. The metrics upstairs are the specified pullbacks of the downstairs metrics.

Proposition I.1.21 (Coherent bounded projection)**.** *Let $q : Y \to X$ be finite between good Hausdorff spaces pure of the same dimension d, with X compact. Let $\mathcal{G}$ be coherent on Y_G, let $u \in \mathcal{A}_X^{0,0}(X)$, and let $\theta \in \mathcal{A}_X^{d,d}(X)$. Put*

$$V_{X,\mathcal{F}}(u;\theta) = \int_{X,\mathcal{F}} u\theta.$$

Then Y is compact and

$$V_{Y,\mathcal{G}}(q^*u; q^*\theta) = V_{X,q_*\mathcal{G}}(u;\theta). \tag{I.1.24}$$

If $q_*\mathcal{G}$ is locally free of constant rank ρ on the analytic G-site, this is $\rho V_{X,\mathcal{O}_X}(u;\theta)$.

If θ is positive, then, for every coherent $\mathcal{F}$,

$$|V_{X,\mathcal{F}}(u;\theta)| \le \|u\|_\infty\, m_{X,\mathcal{F}}(\theta), \qquad m_{X,\mathcal{F}}(\theta) = \int_{X,\mathcal{F}} \theta \ge 0. \tag{I.1.25}$$

This functional extends uniquely and continuously to $u \in \mathcal{B}(X)$, retaining (I.1.25). For positive θ, pullback sends $\mathcal{B}(X)$ into $\mathcal{B}(Y)$ and (I.1.24), including the rank case, holds for these extensions.

Proof. A finite affinoid cover of compact X has full inverse images that are affinoid; their finite union is Y, so Y is compact. Multiplication and pullback in NAG-002424 give $q^*(u\theta) = q^*u\, q^*\theta$. Apply NAG-002430 to the compactly supported form $u\theta$. Its rank clause gives the second assertion.

Let $C = \|u\|_\infty$. The two forms $(C+u)\theta$ and $(C-u)\theta$ are positive. Linearity and positivity in NAG-002425 therefore give $-Cm_{X,\mathcal{F}}(\theta) \le V_{X,\mathcal{F}}(u;\theta) \le Cm_{X,\mathcal{F}}(\theta)$. Applying this inequality to differences shows that the integral values of a uniformly Cauchy smooth sequence are Cauchy. Their limits depend only on the uniform limit, by the same bound. This constructs the unique continuous extension to the specified closure, including the zero-mass case.

Pullback is continuous for the uniform norm since $\|q^*u\|_\infty \le \|u\|_\infty$. The pullback of a positive top form is positive: on a common moment presentation the coefficient on a source tangent d-plane is the corresponding target coefficient multiplied by the square of the real determinant of the induced tangent map; a rank drop gives zero. This is a pointwise superform calculation. Thus the same bounds apply to $q^*\theta$ on Y. Taking limits in the proved smooth identity gives the asserted identity for $\mathcal{B}(X)$ and its rank specialization. □

Proposition I.1.22 (Norm and bounded metric variation)**.** *Assume X is compact, $\dim X = \dim Y = d$ purely, and $q_*\mathcal{O}_Y$ is ordinary-locally free of constant rank n. Let h_0, h_0' be continuous metrics on the same line L_0, with $h_0' = e^{-u}h_0$ and $u \in \mathcal{B}(X)$. Let θ be a positive ordinary smooth top form on X, in particular the positive wedge of smooth curvature forms specified in Section 1. Then*

$$(h_0')^N = e^{-nu}h_0^N$$

and

$$V_{Y,\mathcal{O}_Y}(q^*u; q^*\theta) = V_{X,\mathcal{O}_X}(nu; \theta) = nV_{X,\mathcal{O}_X}(u; \theta). \tag{I.1.26}$$

The absolute value of each expression is at most $n\|u\|_\infty m_{X,\mathcal{O}_X}(\theta)$. For smooth u, the equalities hold without the positivity hypothesis; the stated bound and uniform extension use positivity.

Proof. In a frame e of L_0, the two norm-frame weights are $n\phi_{e,h_0}$ and $n\phi_{e,h_0'}$. Their difference is nu. It is independent of e, so it gives the displayed global metric ratio. Proposition I.1.21 with $\mathcal{G} = \mathcal{O}_Y$ gives the first and third expressions in (I.1.26); linearity gives the middle one. Its bound is (I.1.25) multiplied by n. □

I.2 A common class and the two comparison objects

Let K be an algebraically closed, complete, nontrivially valued non-Archimedean field. Set $v(a) = -\log|a|$, without rescaling it. Fix $d \geq 1$ and integers $m_i \geq 1$, and put

$$X = (\mathbf{P}^1_K)^d, \qquad T = (\mathbf{G}_{m,K})^d, \qquad L_i = \mathrm{pr}_i^*\, \mathcal{O}_{\mathbf{P}^1}(m_i \infty).$$

Use the usual analytification of this explicit smooth projective variety, with its good Hausdorff structure and pure dimension d. Write $t_i = -\log|T_i|$ on T^{an}. The continuous extension

$$\bar{t} : X^{\mathrm{an}} \longrightarrow \overline{\mathbf{R}}^d, \qquad \overline{\mathbf{R}} = [-\infty, +\infty],$$

sends $T_i = 0$ to $+\infty$ and $T_i = \infty$ to $-\infty$. The lattice is $N = \mathbf{Z}^d$, dual to the character lattice $M = \mathbf{Z}^d$. Lebesgue measure dt gives N covolume one.

Choose convex functions $\phi_i \in C^\infty(\mathbf{R})$ and numbers $A_i < B_i$ with

$$\phi_i(t) = -m_i t + c_i^- \quad (t \leq A_i), \qquad \phi_i(t) = c_i^+ \quad (t \geq B_i). \tag{I.2.1}$$

In the frame $e_{i,0} = 1$ on the finite coordinate chart, give L_i the metric h_i with weight $-\log\|e_{i,0}\| = \phi_i(t_i)$. In the frame $e_{i,\infty} = T_i^{m_i} e_{i,0}$ its weight is $\phi_i(t_i) + m_i t_i$. The latter is constant near ∞; the former is constant near zero. Thus these formulas define an actual global metric. They give ordinary smooth moment functions and smooth functions on algebraic tropical charts in the primary theory. At a boundary point, the weight is constant in that coordinate, so a very affine coordinate chart avoiding two other points of $\mathbf{P}^1$ suffices. There is no logarithm of a vanishing coordinate in that local representative.

This class is nonempty. If $\rho_i \in C_c^\infty((A_i, B_i))$ is nonnegative and has integral one, then

$$\phi_i(t) = m_i \int_t^\infty (s - t)\rho_i(s)\, ds \tag{I.2.2}$$

satisfies (I.2.1) and $\phi_i'' = m_i \rho_i$. Put

$$\theta_i = d'd''\phi_i(t_i), \qquad \theta = \theta_1 \wedge \cdots \wedge \theta_d, \qquad W(t) = \prod_{i=1}^d \phi_i''(t_i), \qquad D = \prod_{i=1}^d m_i. \tag{I.2.3}$$

The curvature expressions glue because frame changes add affine functions of logarithmic absolute values. In the primary smooth δ-metric convention, the same smooth representative has zero residual corner term. This is a comparison of the explicit local representatives, not a claim identifying the two general sheaves of forms.

We use the positive generator

$$\omega_d = d't_1 \wedge d''t_1 \wedge \cdots \wedge d't_d \wedge d''t_d, \qquad \int a\,\omega_d = \int a(t)\,dt.$$

With grouped primed and double-primed factors the coefficient changes by $(-1)^{d(d-1)/2}$. This is the convention of the frozen polyhedral integral. In particular $\theta = t^*(W\omega_d)$ on T^{an}, with no factorial in this mixed wedge.

Let $\mathcal{S}$ denote the smooth functions F on $\mathbf{R}^d$ which are eventually independent of each coordinate at both ends, uniformly in all the other coordinates. Explicitly, for each i there exist $a_i < b_i$ such that F is independent of t_i on each region $t_i < a_i$ and $t_i > b_i$. They extend continuously to $\overline{\mathbf{R}}^{,d}$. This class contains 1 and all compactly supported smooth functions. The function $u_F = F \circ \bar{t}$ is globally smooth in both stated chart conventions: near a boundary point ignore the coordinates in which F is constant, and use the remaining units. Define also the larger, explicitly limited test space

$$\mathcal{C}_{\mathrm{rad}} = \{u_F = F \circ \bar{t} : F \in C(\overline{\mathbf{R}}^{,d})\}.$$

No assertion that these are all continuous functions on X^{an} is made.

Let $S_i = \mathrm{supp}(\phi_i'')$ and $S = \prod_i S_i$. Choose a finite box $B = \prod_i [a_i', b_i']$ whose interior contains S. Then

$$Q = t^{-1}(B) = \{e^{-b_i'} \le |T_i| \le e^{-a_i'} \text{ for all } i\} \tag{I.2.4}$$

is a compact affinoid polyannulus. Its ordinary interior in X^{an} contains $\mathrm{supp}(\theta)$, and θ is zero off Q. Every fibre used below is the whole fibre in this polyannulus. In particular Q is not replaced by a set of Gauss points.

Write $I_K(u\theta) = \int_{X/K,\mathcal{O}_X} u\theta$ for the frozen compact smooth integral. The primary comparison object is the smooth Gubler–Künnemann pairing $J_{\mathrm{GK}}(u\theta)$: on a very affine chart of integration its value is the integral of the representing top form on the weighted tropicalization. In our case the single chart T already contains the support. If μ_{GK} denotes the signed Radon measure for these smooth metrics, its defining action is

$$\int u\, d\mu_{\mathrm{GK}} = J_{\mathrm{GK}}(u\theta) \quad \text{for smooth } u.$$

Lemma I.2.1 (The full torus chart has weight one)**.** *For an interior weight w of the box B in (I.2.4), the frozen structure-module weight of Q with moment coordinates $T_1, \dots, T_d$ is one. Its saturated tangent lattice is $\mathbf{Z}^d$. The primary algebraic tropicalization of T has the same lattice and coefficient one. Consequently both chart integrals of $a\omega_d$, for every $a \in C_c^\infty(\operatorname{Int} B)$, equal $\int_B a(t)\,dt$.*

Proof. Write $r_i = e^{-w_i}$. The entire weight fibre has the actual algebra

$$T_w = K\{r^{-1}Z, rZ^{-1}\} = \left\{ \sum_{\nu \in \mathbf{Z}^d} a_\nu Z^\nu : |a_\nu| r^\nu \longrightarrow 0 \right\}, \qquad \|a\|_w = \max_\nu |a_\nu| r^\nu.$$

This is the relation quotient, with its exact quotient norm, of NAG-001553; the fibre identification is NAG-002081. The Laurent bounds of Q impose no further relation at this weight. All points of the fibre satisfy the strict upper bounds for both Z_i and Z_i^{-1} in the presentation of Q, hence lie in $\operatorname{Int}(Q/K)$. The image germ is open in $\mathbf{R}^d$, whose saturated lattice is $\mathbf{Z}^d$. Use the identity lattice projection in NAG-002362. The source fibre algebra and target polytorus algebra are both T_w. If H_w is its Gauss field, multiplicativity of the Gauss norm and the actual tensor identification give

$$T_w \widehat{\otimes}_{T_w} H_w = H_w, \qquad [\mathbf{Z}^d : \mathbf{Z}^d] = 1.$$

Thus the full Gauss-module dimension divided by the lattice index is one. No reduction, radical quotient, separated quotient losing a module, or count of analytic points has entered this calculation.

For the primary torus chart the defining ideal in its own Laurent coordinate ring is zero. Its initial ideal is again zero at every weight. Over a valued extension realizing that weight, its initial scheme is a Laurent torus, with one reduced generic component and length one. Its tangent lattice is exactly $\mathbf{Z}^d$. This verifies its tropical coefficient directly, without a general multiplicity or pushforward theorem. Finally NAG-002405 computes the frozen integral with these weights, NAG-002389 gives the covolume-one real integral, and the defining primary chart integral uses the same coefficient and the same positive generator. This proves the claim. □

Theorem I.2.2 (Identity (M) on every continuous radial test)**.** *For the class in Section I.2, Identity* (M_{tor}) *holds for every* $F \in \mathcal{S}$*:*

$$I_K(u_F\theta) = J_{\mathrm{GK}}(u_F\theta) = \int_{X^{\mathrm{an}}} u_F \, d\mu_{\mathrm{GK}} = \int_{\mathbf{R}^d} F(t)W(t)\,dt. \tag{I.2.5}$$

Both restricted functionals have a unique continuous extension from $\mathcal{S}$ *to all* $F \in C(\overline{\mathbf{R}}^{,d})$ *in the uniform norm, and* (I.2.5) *holds for every such* F*, with* I_K *understood as this extension. These tests belong to the uniform closure of the ordinary smooth moment functions on* X^{an}*. The common functional is positive, vanishes if* $F|_S = 0$*, and satisfies*

$$|I_K(u_F\theta)| \le D\|F\|_\infty, \qquad I_K(\theta) = D. \tag{I.2.6}$$

Here D *is the algebraic mixed degree of* $L_1, \dots, L_d$ *on* X*, including the local intersection lengths. If* $L = L_1 \otimes \cdots \otimes L_d$ *has the product metric, then* $c_1(L)^d = d!\theta$*, so its mass and algebraic self-degree are* $d!D$*.*

Proof. The local weights and their affine frame transitions give the same curvature representative $d'd''\phi_i$ in each of the two smooth chart conventions. Multiplication and wedge product therefore represent $u_F\theta$ on T by $FW\omega_d$. This representative has support in $S \subset \operatorname{Int} B$ even when F is not compactly supported. On X^{an} the whole form has support inside the ordinary interior of Q. NAG-002425 consequently permits its frozen integral to be computed on this one affinoid neighbourhood. Lemma I.2.1 computes it as $\int FW\,dt$. In the primary definition, T is already a very affine chart of integration for the very same compact representative; the same lemma computes its defining pairing. The equality with the smooth action of μ_{GK} is its specified definition.

Since each ϕ_i'' is nonnegative and compactly supported, ordinary one-variable calculus and (I.2.1) give

$$\int_{\mathbf{R}} \phi_i''(t)\,dt = \phi_i'(+\infty) - \phi_i'(-\infty) = m_i.$$

Iterated integration gives $\int W\,dt = D$. This proves positivity, the support assertion for radial tests, and (I.2.6).

For completeness, the density required for the declared continuous tests is explicit. Identify $\overline{\mathbf{R}}$ with $[-1,1]$ by $t \mapsto \tanh t$. For any $F \in C(\overline{\mathbf{R}}^{,d})$, its restriction to $\mathbf{R}^d$ is bounded and uniformly continuous. Convolution with a compactly supported smooth approximate identity gives smooth bounded F_ϵ with $\|F_\epsilon - F\|_{\infty,\mathbf{R}^d} \to 0$. Choose a smooth clamp $c_R : \mathbf{R} \to \mathbf{R}$ equal to the

identity on $[-R,R]$ and constant on both ends outside $[-R-1,R+1]$, with $c_R(t)\ge R$ for $t\ge R$ and $c_R(t)\le -R$ for $t\le -R$. Such a clamp is obtained by integrating a smooth cutoff of the derivative from one to zero on each end. Let C_R be its coordinatewise product. The function $F_\epsilon\circ C_R$ belongs to $\mathcal{S}$. Moreover

$$\|F_\epsilon\circ C_R-F\|_\infty\le\|F_\epsilon-F\|_\infty+\|F\circ C_R-F\|_\infty.$$

The last term tends to zero as $R\to\infty$, because each of $\tanh(c_R(t))$ and $\tanh t$ lies in the same end interval of length $1-\tanh R$ whenever $|t|>R$. This estimate holds at the endpoints as well. Uniform continuity on $[-1,1]^d$ proves the assertion. Composing with $\bar t$ now gives uniform approximation on X^{an} by the specified globally smooth functions.

The bound (I.2.6) gives unique extensions of the two restricted functionals. The right real integral has the same bound, so their extensions still equal it. The action of the given finite signed primary measure is also continuous in the uniform norm, hence it has the same values on these limits. If the previously reviewed bounded extension is subsequently adopted, uniqueness identifies it with this extension on $\mathcal{C}_{\mathrm{rad}}$; no further density theorem is being assumed.

Finally choose $a_i\in K^\times$ and the effective divisor $m_i\{T_i=a_i\}$ representing L_i. These divisors meet properly at one point. Writing $z_i=T_i-a_i$, their local intersection quotient is

$$K[z_1,\dots,z_d]_{(z_1,\dots,z_d)}/(z_1^{m_1},\dots,z_d^{m_d}).$$

The equations are a regular sequence, since successively the next variable is a polynomial variable over the previous quotient. The residue monomials $z_1^{e_1}\cdots z_d^{e_d}$ with $0\le e_i<m_i$ form a K-basis. The quotient has length D and residue degree one; this computes the mixed degree in this explicit complete intersection. It retains all the nilpotent layers. Since $\theta_i\wedge\theta_i=0$ and $(1,1)$-forms commute with one another, expansion gives $(\sum_i\theta_i)^d=d!\theta$. Independently, the Hilbert polynomial of L is $\prod_i(m_ik+1)$, whose leading coefficient is D; the algebraic self-degree is $d!D$. This verifies both normalizations. □

Proposition I.2.3 (Full degree in the common class)**.** *Let $q : X' = (\mathbf{P}^1_K)^d \to X$ be given by $T_i = S_i^{n_i}$, where $n_i \geq 1$, and put $N = \prod_i n_i$. The map is finite locally free of rank N. Its analytic map, with full inverse images and their canonical finite norms, has the same rank. The pulled-back profiles are $\phi_i(n_i s_i)$, where $s_i = -\log|S_i|$; they belong to the class of Section I.2 with degrees $n_i m_i$. For every $F \in C(\overline{\mathbf{R}}^{,d})$,*

$$\begin{aligned} I_K(q^*u_F\, q^*\theta) &= \int_{X'^{\mathrm{an}}} q^*u_F \, d\mu_{\mathrm{GK},q^*h_1,\dots,q^*h_d} \\ &= N \int_{\mathbf{R}^d} F(t)W(t)\,dt. \end{aligned} \tag{I.2.7}$$

The common absolute value is at most $ND\|F\|_\infty$. Here N includes inseparable degree, without a division by it.

For a line bundle M on X^{an} with continuous metric h_0, give $\mathrm{Nm}_q(q^*M)$ the projection-compatible metric from its canonical isomorphism with $M^{\otimes N}$. If $h_0' = e^{-u_F}h_0$, the logarithmic ratio of the two norm metrics is Nu_F. Its bounded pairing against θ is precisely the value in (I.2.7). This assertion uses the specified pulled-back metric class only.

Proof. On any product of the finite and infinite standard coordinate charts, the algebraic map is a tensor product of maps $K[T_i] \to K[S_i]$, $T_i \mapsto S_i^{n_i}$ (using inverse coordinates at infinity). Division of exponents gives the free basis $\prod_i S_i^{e_i}$, $0 \leq e_i < n_i$. These charts cover the target and their whole inverse images, proving finite local freeness of rank N in every characteristic.

The same assertion is analytic, including its norm. On a closed polydisc, or on a closed polyannulus in these coordinates, expand a convergent series upstairs and group its exponent vectors modulo $(n_1, \dots, n_d)$. This gives a unique sum $\sum_{0 \leq e_i < n_i} S^e a_e(T)$ of N convergent downstairs series. Convergence follows by grouping the weighted coefficient families; shifting by one of the finitely many exponents e changes the coefficient norms by fixed positive factors only. Thus both this map and its inverse are bounded. It is the canonical finite-module topology of NAG-001701. Its spectra use the whole inverse domains; for (I.2.4) the inverse box is $\prod_i [a_i'/n_i, b_i'/n_i]$. No point or branch is dropped.

For the pulled-back metrics, direct differentiation gives

$$q^*\theta_i = n_i^2 \phi_i''(n_i s_i)\, d's_i \wedge d''s_i.$$

Their affine-tail degrees are $n_i m_i$. Theorem I.2.2 applied upstairs gives, for smooth tests first,

$$\int_{\mathbf{R}^d} F(n_1 s_1, \dots, n_d s_d) \prod_i n_i^2 \phi_i''(n_i s_i)\, ds.$$

Substitute $t_i = n_i s_i$. The ordinary Jacobian removes one factor $\prod_i n_i$ from $\prod_i n_i^2$, leaving exactly N. The continuous-test assertion follows either from the same integral or from the already proved uniform approximation and bound. If some n_i is divisible by the characteristic, this algebraic rank and real substitution are unchanged. In particular $T = S^p$ has factor p, although it is purely inseparable and has only one geometric point over each point. Nilpotents in inseparable fibres are retained by the free algebra calculation.

NAG-002539 identifies $\mathrm{Nm}_q(q^*M)$ with $M^{\otimes N}$. In a local frame the chosen metric is the tensor-power metric, so changing its downstairs weight by u_F changes the norm weight by Nu_F, exactly as in NAG-002554. Applying the bounded functional of Theorem I.2.2 gives $NI_K(u_F\theta)$, equal to (I.2.7). This uses no determinant section with a zero or pole, and does not construct a norm metric for an arbitrary upstairs metrized line. □

Let K be algebraically closed, complete and nontrivially valued, with its fixed absolute value and $v = -\log|\cdot|$. Throughout,

$$d \geq 2, \qquad X = (\mathbf{P}^1_K)^d, \qquad T = (\mathbf{G}_{m,K})^d, \qquad L_i = \mathrm{pr}_i^*\,\mathcal{O}(m_i\infty), \quad m_i \geq 1.$$

We use the usual good Hausdorff analytification, pure of dimension d. Put $t_i = -\log|T_i|$ on T^{an}, and let $\bar{t} : X^{\mathrm{an}} \to \overline{\mathbf{R}}^{,d}$ be its continuous extension, where $\overline{\mathbf{R}} = [-\infty, +\infty]$; zero maps to $+\infty$ and infinity to $-\infty$. The lattices are $N = M = \mathbf{Z}^d$ in the coordinate dual bases, and dt has covolume one.

Choose $\phi_i \in C^\infty(\mathbf{R})$ convex, with exact tails

$$\phi_i(t) = -m_i t + c_i^- \quad (t \leq A_i), \qquad \phi_i(t) = c_i^+ \quad (t \geq B_i), \qquad A_i < B_i. \tag{I.2.8}$$

The frame $e_{i,0} = 1$ of L_i has weight $\phi_i(t_i)$, and $e_{i,\infty} = T_i^{m_i} e_{i,0}$ has weight $\phi_i(t_i) + m_i t_i$. Both are constant near their respective boundary points. These define global metrics h_i in both the ordinary moment-smooth convention and the primary algebraic tropical smooth convention. Explicit examples are $\phi_i(t) = m_i \int_t^\infty (s-t)\rho_i(s)\,ds$, with $\rho_i \in C_c^\infty((A_i, B_i))$, $\rho_i \geq 0$, $\int \rho_i = 1$.

Set

$$\theta_i = d'd''\phi_i(t_i), \quad \theta = \bigwedge_{i=1}^d \theta_i, \quad W(t) = \prod_i \phi_i''(t_i), \quad D = \prod_i m_i, \quad S = \prod_i \mathrm{supp}(\phi_i''). \tag{I.2.9}$$

Use the positive generator $\omega_d = \bigwedge_i (d't_i \wedge d''t_i)$, so that $\theta = t^*(W\omega_d)$ and $\int a\omega_d = \int a(t)dt$. There is no $d!$ in this mixed wedge and no rescaling of the valuation. The curvature is zero near $X^{\mathrm{an}} \backslash T^{\mathrm{an}}$. If a finite box B contains S in its interior, then the whole compact polyannulus $Q = t^{-1}(B)$ is an ordinary neighbourhood of its support. All frozen chart weights use full fibres and actual canonical Banach algebras. For $w \in \mathbf{R}^d$, let $\rho(w)$ be the Gauss point

$$\left| \sum_{m \in \mathbf{Z}^d} a_m T^m \right|_{\rho(w)} = \max_m |a_m| e^{-\langle m, w\rangle} \quad \text{on finite Laurent polynomials.} \tag{I.2.10}$$

Write $\Gamma = \rho(\mathbf{R}^d)$. Let $\mathcal{A}$ be the real algebra of global ordinary smooth moment functions on X^{an}. Let $\mathcal{A}_{\mathrm{com}}$ be its subalgebra of functions also smooth on primary algebraic tropical charts. Write

$$L_K(u) = \int_{X/K, \mathcal{O}_X} u\theta \quad (u \in \mathcal{A}), \qquad \nu = W(t)dt, \qquad \mu_\Gamma = \rho_*\nu.$$

The last expression is an explicit positive Radon measure: W has compact support S, and $\rho|_S$ is continuous into the Hausdorff space X^{an}. Its Radon regularity also follows directly by transporting regularity from the compact set S through this homeomorphism onto its image.

For the specified primary smooth measure, take as given its defining action: on a common algebraic tropical chart, it integrates the representing smooth curvature form with its chart weights. No formal-measure comparison enters.

Theorem I.2.4 (Identity (M) on all continuous analytic tests)**.** *Under the standing hypotheses, for every $u \in \mathcal{A}$,*

$$L_K(u) = \int_{\mathbf{R}^d} u(\rho(t))W(t)\,dt, \qquad |L_K(u)| \le D \sup_{\rho(S)} |u|. \tag{I.2.11}$$

It has a unique continuous extension to all $u \in C(X^{\mathrm{an}})$ in the uniform norm. For every such test, including nonradial tests, this extension and the primary smooth Gubler–Künnemann measure satisfy

$$\overline{L}_K(u) = \int_{X^{\mathrm{an}}} u\,d\mu_{\mathrm{GK}} = \int_{X^{\mathrm{an}}} u\,d\mu_\Gamma = \int_{\mathbf{R}^d} u(\rho(t))W(t)\,dt. \tag{I.2.12}$$

The measures have support exactly $\rho(S)$ and mass $D = \prod_i m_i$, the explicit algebraic mixed degree. The product line $L_1 \otimes \cdots \otimes L_d$ has self-measure $d!\mu_\Gamma$ and self-degree $d!D$. No logarithmic test with a zero or pole is asserted to be integrable by this theorem.

Proof. First recover the needed radial calculation directly. On a whole interior fibre of the polyannulus Q, the actual Laurent algebra is $T_w = K\{e^w T, e^{-w}T^{-1}\}$, with its quotient norm (NAG-001553 and NAG-002081). For the identity projection of the saturated lattice $\mathbf{Z}^d$, its actual Gauss algebra is $T_w \widehat{\otimes}_{T_w} H_w = H_w$. Thus the full module dimension and lattice index in NAG-002362 are both one. The frozen chain integral NAG-002405, with NAG-002389's positive generator, gives

$$L_K(F \circ \bar{t}) = \int F(t)W(t)dt \qquad (F \in C_c^\infty(\mathbf{R}^d)). \tag{I.2.13}$$

The same calculation gives $L_K(1) = D$, since $\int \phi_i'' = \phi_i'(+\infty) - \phi_i'(-\infty) = m_i$. The primary torus chart has defining ideal zero, hence initial Laurent torus of generic length one, with the same saturated lattice and positive normalization. Its defining smooth pairing gives the identical formulas. There is no factorial in these mixed calculations.

Let $u \in \mathcal{A}$. The trace $u \circ \rho$ is continuous on $\mathbf{R}^d$. On a compact neighbourhood of S, convolution and a smooth compact cutoff give $F_\epsilon \in C_c^\infty(\mathbf{R}^d)$ with

$$\sup_{t \in S} |u(\rho(t)) - F_\epsilon(t)| < \epsilon.$$

For example, first multiply the trace by a cutoff equal to one near S, convolve with a sufficiently small smooth approximate identity, and multiply by a second cutoff equal to one on that neighbourhood. The function $v = u - F_\epsilon \circ \bar{t}$ is globally ordinary smooth. It satisfies $|v| < \epsilon$ on $\rho(S)$ and hence on an open neighbourhood of this compact set. By Lemma G.0.22, θ has zero germ outside $\rho(S)$. Therefore both $(\epsilon + v)\theta$ and $(\epsilon - v)\theta$ are globally positive sections: on that neighbourhood their scalar coefficients are positive, and on its complement they are zero germs. Positivity of NAG-002425 and $L_K(1) = D$ imply

$$|L_K(u) - L_K(F_\epsilon \circ \bar{t})| \le \epsilon D.$$

Using (I.2.13) and the same error bound for the real integral proves (I.2.11). This proof uses the trace only in the approximation, without pretending that $u \circ \rho \circ \bar{t}$ is itself smooth.

The ordinary smooth algebra is uniformly dense by Lemma G.0.23; the bound in (I.2.11) is at most $D\|u\|_\infty$. Thus L_K has a unique continuous extension. The explicit functional $u \mapsto \int u \, d\mu_\Gamma$ already extends it, so the first and last expressions in (I.2.12) agree for every continuous u. This constructs the full measure in the frozen class without invoking a general Riesz existence result for all smooth forms.

For a common smooth function $u \in \mathcal{A}_{\mathrm{com}}$, apply exactly the same approximation to the primary smooth pairing. The primary zero-germ statement was proved separately in Lemma G.0.22. Its smooth chart integral is nonnegative on a nonnegative smooth top form: after restriction to faces its coefficients are integrated against the nonnegative tropical multiplicities, with the fixed positive normalization. This is the chart definition, and is all the positivity required here. Its radial value and mass were computed above. Hence its value at $u\theta$ is $\int u \circ \rho \, W \, dt$. The defining smooth action of the given finite signed measure μ_{GK} has these values. The common algebra is uniformly dense, and integration against either finite signed Radon measure is continuous in that norm. Thus the equality extends to every continuous u, proving (I.2.12) and, in particular, positivity and support of μ_{GK} in this class. The nonnegative continuous density W has support exactly S: every product neighbourhood of a point of $\prod_i \operatorname{supp}(\phi_i'')$ contains a product of points where all factors are

positive. Thus its measure has support S, and the section homeomorphism gives measure support exactly $\rho(S)$.

For the degree normalization choose $a_i \in K^\times$ and the divisors $m_i\{T_i = a_i\}$. They meet at one point with regular sequence $z_1^{m_1}, \dots, z_d^{m_d}$ in the local polynomial ring, where $z_i = T_i - a_i$. The quotient basis is $\prod_i z_i^{e_i}$ with $0 \le e_i < m_i$, so its full length is D, including nilpotent layers; its residue degree is one. This computes the mixed degree. Finally $\theta_i \wedge \theta_i = 0$ and the $(1,1)$-factors commute, so $(\sum_i \theta_i)^d = d!\theta$. The Hilbert polynomial of the product line is $\prod_i (m_i k + 1)$, confirming self-degree $d!D$. □

Proposition I.2.5 (Power projection and the specified norm metric)**.** *Let $q : X' = (\mathbf{P}^1_K)^d \to X$ be $T_i = S_i^{n_i}$, with $n_i \ge 1$ and $N = \prod_i n_i$. Give q^*L_i the pulled-back metrics and denote their full mixed measure from Theorem I.2.4 by μ'. For every $u \in C(X^{\mathrm{an}})$,*

$$\int_{X'^{\mathrm{an}}} q^*u \, d\mu' = N \int_{X^{\mathrm{an}}} u \, d\mu_\Gamma, \qquad \left| \int q^*u \, d\mu' \right| \le ND\|u\|_\infty. \tag{I.2.14}$$

This is the same identity for the extended frozen integrals and the primary smooth measures. The rank N includes inseparable degree. For a continuously metrized analytic line (M, h_0) on X^{an}, use only the projection-compatible metric on $\mathrm{Nm}_q(q^*M) \simeq M^{\otimes N}$. Replacing h_0 by $e^{-u}h_0$ changes its norm-metric weight by Nu; its bounded curvature pairing is the value in (I.2.14).

Proof. On the finite and infinite coordinate charts, q is finite free with basis $\prod_i S_i^{e_i}$, $0 \le e_i < n_i$. On analytic polydiscs and polyannuli, grouping the convergent coefficient families by these exponent classes gives the same basis with bounded inverse coefficient maps. Finite shifts change the weighted norms only by fixed positive factors; thus this is the actual canonical finite topology of NAG-001701. All inverse domains are retained. This proves rank N in every characteristic, including nilpotents in inseparable fibres.

For $s_i = -\log|S_i|$, the upstairs profiles are $\phi_i(n_i s_i)$. Their mixed density and Gauss map satisfy

$$W'(s) = \prod_i n_i^2 \phi_i''(n_i s_i),$$
$$q(\rho'(s)) = \rho(n_1 s_1, \dots, n_d s_d).$$

The second equality follows by evaluating Laurent polynomials in the Gauss norm: multiplying distinct exponents by (n_i) does not identify them. Hence the full-measure formula yields

$$\int q^{*}u\, d\mu' = \int u(\rho(n_1 s_1, \dots, n_d s_d)) \prod_i n_i^2 \phi_i''(n_i s_i)\, ds.$$

The real substitution $t_i = n_i s_i$ removes one product of the n_i and leaves N. This proves (I.2.14) for every continuous u. The computation still gives p for $T = S^p$ in characteristic p; a point count or separable degree would be wrong. Finally the frozen norm isomorphism NAG-002539 and the specified metric of NAG-002554 identify the norm frame weight with N times the downstairs weight. Its change is Nu, giving the stated bounded pairing. □

Let K be complete and nontrivially discretely valued, with valuation ring R, residue field k, uniformizer π, and the fixed number $\lambda = -\log|\pi| > 0$. Let X be an integral proper K-variety of dimension $d \geq 2$. A model in this note is proper, flat, integral and normal over R, of finite type, with generic fibre identified with X. Retain explicitly the ordinary DVR local rings at all height-one points used below. Its special components have dimension d. For a special component V put

$$A_V = \mathcal{O}_{\mathcal{X},\eta_V}, \qquad b_V = \operatorname{ord}_V(\pi) > 0, \qquad |f(\xi_V)| = |\pi|^{\operatorname{ord}_V(f)/b_V} \quad (f \in K(X)^\times).$$

The valuation norm defines an analytic point by NAG-001707. Its kernel is zero, and its centre on the model is η_V. Distinct components have distinct points. No assertion that an ordinary analytic stalk at ξ_V is a DVR is made.

For lines $\mathcal{L}_i$ extending $L_i^{\otimes e_i}$, $1 \leq i \leq d$, write

$$E = \prod_{i=1}^{d} e_i, \qquad \delta_V = \deg_{\mathcal{L}_1|_V,\dots,\mathcal{L}_d|_V,k}(V), \qquad \mu_{\mathcal{X}} = \sum_V \frac{b_V \delta_V}{E} \delta_{\xi_V}. \quad \text{(I.2.15)}$$

This is the specified finite model measure, with the ordinary mixed component degree over k. All root exponents, nilpotent lengths and full residue-field degrees are retained. It can be signed. The root model metric has $\|s\| = |a|^{1/e}$ when $s^{\otimes e} = a e_{\mathcal{M}}$ in a model frame of $\mathcal{M}$. The finite measure is the object compared here; its general independence is not assumed in its definition.

A *displayed modification* is a proper morphism $p : \mathcal{Y} \to \mathcal{X}$ inducing the identity on X, together with an open $U \subseteq \mathcal{X}$ containing X and every height-one point of $\mathcal{X}$, such that $p^{-1}(U) \to U$ is an isomorphism. This open is part of the hypotheses. We do not infer it merely from the equality of generic fibres. The complement has codimension at least two. A component W of $\mathcal{Y}_k$ is called exceptional here if it does not dominate a component of $\mathcal{X}_k$.

Lemma I.2.6 (Codimension-one coefficients under a modification)**.** *Let p be a displayed modification. For each prime divisor V of $\mathcal{X}$ there is exactly one prime divisor $\widetilde{V}$ of $\mathcal{Y}$ dominating it, and their ordinary local DVRs are identified inside $K(X)$. For a Cartier divisor D with rational section presentation,*

$$p_* \operatorname{cyc}(p^* D) = \operatorname{cyc}(D). \quad \text{(I.2.16)}$$

The pushforward uses full residue-field degrees on equal-dimensional images and zero on dimension-decreasing images. In particular, for special components, $b_{\widetilde{V}} = b_V$ and $\xi_{\widetilde{V}} = \xi_V$. Every exceptional special component has image of dimension at most $d - 1$.

The codimension-one isomorphism hypothesis is checkable for a blowup of a finite-type ideal I with $I|_X = \mathcal{O}_X$: it holds wherever I is locally principal and nonzero, hence on an open containing every height-one point of the normal model. This assertion concerns the blowup itself; any additional normalization or domination needs its own compatibility check.

Proof. Over the generic point of a prime divisor V, the morphism is an isomorphism. Taking the closure of its inverse-image point gives $\widetilde{V}$ and identifies its local ring and residue field with those of V. No second divisor can dominate V, since it would also meet this isomorphic inverse image. Any other prime divisor lies over the complement of U, so its image has smaller dimension. Properness makes that image closed. The coefficient of p^*D at $\widetilde{V}$ is the coefficient of D at V, because the same rational function is measured in the same DVR. Its pushforward degree is one, and the other coefficients push forward to zero. This proves (I.2.16). Apply it locally to π and to arbitrary rational functions to obtain the assertions about b and ξ.

At a height-one point v, I_v is a nonzero finitely generated ideal in a DVR and thus is principal, say (a). After shrinking a neighbourhood, a is a section of I and every chosen finite generator of I is a multiple of a; clearing the finitely many localization denominators proves this assertion on an open. On that open, the Rees algebra is $\bigoplus_{n\geq 0}(a^n)T^n \simeq A[t]$, since a is a non-zero-divisor. Its Proj is $\operatorname{Spec} A$. The union of these opens and the generic fibre gives the required U for the blowup. □

Theorem I.2.7 (Exact component-degree criterion for model measures)**.** *Use $p : \mathcal{Y} \to \mathcal{X}$ as above and the pulled-back model lines $p^*\mathcal{L}_i$ with the same root exponents. Then the following are equivalent:*

1. *The two finite measures on the common X^{an} are equal.*
2. *Their component degrees satisfy*

$$\delta_{\widetilde{V}} = \delta_V \quad \text{for every } V, \qquad \delta_W = 0 \quad \text{for every exceptional } W. \tag{I.2.17}$$

Under this criterion all continuous tests, and all Borel tests finite on the common finite support, have equal integrals. In particular, if M has a model $\mathcal{M}$ extending $M^{\otimes e_0}$, the pullback model metric is the same metric on M. Every nonzero rational section s satisfies

$$\int -\log\|s\|\, d\mu_{\mathcal{Y}} = \int -\log\|s\|\, d\mu_{\mathcal{X}}. \tag{I.2.18}$$

The logarithm is used off its zero/pole support and may be extended arbitrarily across that null set for these atomic measures. Both are finite total-variation integrals. If $a_V = \operatorname{ord}_V(s^{\otimes e_0})$ *in a model frame, the common downstairs expression is*

$$\frac{\lambda}{e_0 E}\sum_V a_V \delta_V, \qquad -\log\|s(\xi_V)\| = \frac{\lambda a_V}{e_0 b_V}. \tag{I.2.19}$$

Replacing s *by* πs *changes it by* $+\lambda\mu_{\mathcal{X}}(X^{\mathrm{an}})$.

Proof. The strict-transform atoms and their b's agree by Lemma I.2.6. An exceptional atom cannot equal one of these: its centre on $\mathcal{X}$ is in $\mathcal{X}\setminus U$, whereas the strict-transform atom has centre $\eta_V \in U$. Distinct components of $\mathcal{Y}$ also give distinct atoms. Comparing the coefficients of each Dirac mass in (I.2.15), and using $b_W > 0$, proves precisely the equivalence with (I.2.17). This argument allows signed degrees; no cancellation between distinct atoms is possible. Equality of the finite measures proves equality for every stated test.

A frame of $\mathcal{M}$ pulls back to a frame of $p^*\mathcal{M}$ at every point reducing over that frame. The coefficient of a generic section is the same rational function, evaluated at the same analytic point. The root metric formula therefore proves equality of these metrics on all of X^{an}. The reduction maps commute with p, as follows directly by pulling back the inequalities $|f| \le 1$ and $|f| < 1$ defining a centre. Every atom has generic underlying scheme point, so a nonzero rational section is regular and nonvanishing there. The root norm gives the second formula in (I.2.19); multiplication by its mass $b_V\delta_V/E$ gives the first. This proves finiteness without any singular limiting argument. Finally a_V becomes $a_V + e_0 b_V$ for πs, giving the positive change $\lambda\sum_V b_V\delta_V/E$. □

Proposition I.2.8 (Two semipositive systems on the point blowup)**.** *The special fibre is* $V' + F$ *with both multiplicities one. Its two component DVR norms are*

$$|f(\xi_{V'})| = |f(\xi_0)|, \qquad \left|\sum_m a_m z^m\right|(\xi_F) = \max_m |a_m|\,|\pi|^{|m|},$$

where $z_i = Z_i/Z_0$ *and* ξ_0 *is the standard unit-radius Gauss point. Put* $A = H \otimes \mathcal{O}(-F)$*. Both* H *and* A *are generated by global sections and have*

semipositive model metrics. For positive integers $m_1, \dots, m_d$ and $D = \prod_i m_i$, their mixed degrees are

	V'	F
$H^{m_1}, \dots, H^{m_d}$	D	0
$A^{m_1}, \dots, A^{m_d}$	0	D

(I.2.20)

Consequently the pullbacks of the standard model lines give $\mu_H = D\delta_{\xi_0}$ on both models, verifying (I.2.17). The other semipositive system gives $\mu_A = D\delta_{\xi_F}$. Although H and A have the same generic line bundle, A is not being treated as the pullback of that standard metrized line.

Proof. Near P, the ideal is $(\pi, z_1, \dots, z_d)$. The π-chart is $R[u_1, \dots, u_d]$ with $z_i = \pi u_i$. Its exceptional generic DVR has uniformizer π and residue field $k(u_1, \dots, u_d)$. The z_j-chart has equation

$$R[z_j, u_0, (u_i)_{i\neq j}]/(\pi - z_j u_0), \qquad z_i = z_j u_i \ (i \neq j).$$

These are the usual regular charts of the blowup of the coordinate regular closed point: at a crossing over a codimension-c point of $k[(u_i)_{i\neq j}]$, the local ring has dimension $c+2$ and its maximal ideal is generated by z_j, u_0 and lifts of c regular parameters. Away from the crossing one of z_j, u_0 is a unit, giving a localization of a polynomial chart over R. Thus the total space is regular, in particular normal. The blowup is projective over the proper model. These charts also show explicitly that the special fibre consists of $z_j = 0$ and $u_0 = 0$, each with multiplicity one. The component V' is $\mathrm{Bl}_P \mathbf{P}^d_k$, while $F = \mathbf{P}^d_k$. The generic DVR of V' is unchanged from $\mathcal{X}_k$. In the π-chart, the reduction of a polynomial in the independent u_i cannot cancel its distinct monomials at the generic point. Its order is therefore $\min_m(\mathrm{ord}_R\, a_m + |m|)$, which gives the stated Gauss norm at F.

The sections $Z_0, \dots, Z_d$ generate H. The sections $\pi Z_0, Z_1, \dots, Z_d$ generate A: their ideal is the pullback of the centre ideal times H, hence becomes the invertible ideal $\mathcal{O}(-F)$ times H on the blowup. On F the line H is trivial, and the restrictions of those divided generators identify $A|_F$ with $\mathcal{O}_{\mathbf{P}^d_k}(1)$. On V' the first divided generator πZ_0 restricts to zero and the remaining d generators have no common zero. These descriptions can also be read from the two displayed charts. Global generation gives nonnegative degree on every special-fibre curve: a nonzero generating section on an integral curve has an effective zero divisor. Thus the two metrics are semipositive.

For H on V', use $Z_0 + Z_i$, $1 \le i \le d$. Each avoids P, and their common zero is the single rational point $Q = [1 : -1 : \cdots : -1] \neq P$, where p is an isomorphism. Raising the ith section to m_i gives the local algebra $k[t_1, \ldots, t_d]/(t_1^{m_1}, \ldots, t_d^{m_d})$, of length D by its monomial basis. On F, choose nonzero constant sections of the trivial restrictions of H; there is no intersection. This proves the first row of (I.2.20).

For A on $F = \mathbf{P}_k^d$, d coordinate hyperplanes meet at a rational point and the same powers give length D. On V', the remaining divided coordinate generators define the resolution of projection from P to $\mathbf{P}_k^{d-1}$. In the z_j-chart this is $[z_1 : \cdots : z_d] = [u_1 : \cdots : 1 : \cdots : u_d]$; its fibre direction is the line through P. The d coordinate hyperplanes of $\mathbf{P}^{d-1}$ have empty total intersection. Their pullbacks intersect properly through each nonempty stage (the chart fibres are one-dimensional), and the final intersection is empty. Taking powers preserves that support and gives degree zero. This proves the second row. Substituting the degrees and the two multiplicities into (I.2.15) proves all measure assertions. □

Proposition I.2.9 (Two ways that forgetting support loses information)**.** *In every dimension $d \ge 2$ the following hold.*

1. *On $\mathbf{P}_R^d$, a closed rational special-fibre point has class zero in the ordinary Chow group $CH_0(\mathbf{P}_R^d)$, but has vertical degree one. Thus a degree homomorphism on that ordinary Chow group cannot recover the required vertical intersection degree. The support must be retained.*

2. *On the blowup of Section 3, $p_*[F] = 0$ as a d-cycle, but*

$$\deg_{A|_F, \ldots, A|_F, k}(F) = 1.$$

The two semipositive model metrics h_H, h_A on the same generic $\mathcal{O}(1)$ agree outside the tube of P. On the standard unit polydisc around P their difference of weights is

$$u = \log(h_A/h_H) = -\log \max\{|\pi|, |z_1|, \ldots, |z_d|\}. \tag{I.2.21}$$

It satisfies $u(\xi_0) = 0$ and $u(\xi_F) = \lambda$. With the semipositive model measure $\mu_A = \delta_{\xi_F}$ one therefore has $\int u \, d\mu_A = \lambda > 0$. Agreement outside that tube and equality of generic line-bundle classes do not imply zero metric correction. If the supported model values of the preceding proposition are available, changing the first model line from A to $H = A(F)$, with the other d lines equal to A, changes that value by $+\lambda$.

Proof. For (1), choose the horizontal closed section $C = \{Z_1 = \cdots = Z_d = 0\} \simeq \operatorname{Spec} R$. The divisor of π on this integral one-dimensional scheme is its closed point with coefficient $\operatorname{length}_R(R/\pi R) = 1$. Its pushforward is a principal-divisor relation in $CH_0(\mathbf{P}^d_R)$, so that point has ordinary class zero. Its degree as a zero-cycle on the special fibre is $[k : k] = 1$. Rational equivalence inside the vertical support cannot use the horizontal curve C; this is exactly the information lost by forgetting support.

For (2), the image of F is the point P, so cycle pushforward in dimension d is zero. But $A|_F = \mathcal{O}_{\mathbf{P}^d_k}(1)$ by Proposition I.2.8, whose top mixed degree is one. This does not contradict projection: the factors A are not the pulled-back line H. On the unit polydisc, a generator of the pulled-back centre ideal has absolute value $\max\{|\pi|, |z_1|, \ldots, |z_d|\}$. The canonical generic section of $\mathcal{O}(F)$ has that norm. Since $H = A \otimes \mathcal{O}(F)$, the ratio of the metrics gives (I.2.21). Away from the tube $\{\operatorname{red}_{\mathcal{X}}(x) = P\}$ the centre ideal is generated by a unit of norm one, so the metrics agree there. At ξ_0 one of the displayed coordinate norms is one; at ξ_F they all equal $|\pi|$. This proves the two values and the integral. Finally, compatible generic sections of H and A have model divisors differing by F. Additivity of the stated supported interface gives $\lambda \deg(c_1(A|_F)^d[F]) = \lambda$, with the asserted positive sign. □

Let R be a DVR, $S = \operatorname{Spec} R$, η its generic point, and k its residue field. For analytic assertions assume that R is the valuation ring of a complete nontrivially discretely valued field K with its fixed absolute value. Use the dimension function $\delta(\eta) = 1$, $\delta(s) = 0$ at the closed point. On a finite-type S-scheme, dimensions below are δ-dimensions. In particular, a horizontal integral model of relative dimension d has dimension $d + 1$, and a vertical integral k-scheme has its usual dimension. For a closed subset T of a finite-type S-scheme $\mathcal{X}$ set

$$C_r(\mathcal{X}; T) = \mathrm{CH}_r(T_{\mathrm{red}}).$$

Rational equivalences use integral subschemes *inside* T. This group is not defined as the image in $\mathrm{CH}_r(\mathcal{X})$. Reduction identifies Chow groups: integral subschemes, their rational functions, and their divisor relations are unchanged. This observation does not replace the fundamental cycle of a nonreduced scheme by that of its reduction; its generic lengths remain coefficients of the cycle. We freely use a scheme-theoretic intersection to construct a cycle and the canonical reduction identification to place its class in the group on its closed support.

Use the existing supported Gysin operation $G_{(\mathcal{L},s),T} : C_r(\mathcal{X}; T) \to C_{r-1}(\mathcal{X}; T \cap |Z(s)|)$ for an invertible sheaf $\mathcal{L}$ and any global section s, including the zero section or a zero-divisor. Use the selected signed presentations, with their signed operation defined by (C.1.50). For proper integral W/k, the mixed degree $\delta(W; M_1, \dots, M_d)$ is the degree of $c_1(M_1) \cdots c_1(M_d) \cap [W]$ as in the selected algebraic component-degree formula. Now let K be complete and nontrivially discretely valued with ring R, and put $\lambda = -\log|\pi| > 0$. Let $\mathcal{X}$ be an integral normal proper flat model of relative dimension $d \geq 2$. For $0 \leq i \leq d$, let $\mathcal{L}_i$ extend $L_i^{\otimes e_i}$, $e_i \geq 1$, and let $\sigma_i = s_i^{\otimes e_i}$ be a nonzero rational model section. Specify signed presentations on closed supports Z_i satisfying

$$\bigcap_{i=0}^{d} Z_i \cap X = \varnothing, \qquad X = \mathcal{X}_K. \tag{I.2.22}$$

Write I_i for the supported operations, $E = \prod_{i=0}^{d} e_i$, and $T = \bigcap_i Z_i$. Define

$$J_{\mathcal{X}} = \frac{\lambda}{E} \deg_k\big(I_0 \cdots I_d[\mathcal{X}]\big), \qquad I_0 \cdots I_d[\mathcal{X}] \in \mathrm{CH}_0(T_{\mathrm{red}}). \tag{I.2.23}$$

The value is defined on this model data. This construction also applies to the horizontal integral cycles occurring below, using the remaining pulled-back factor presentations, even when a factor vanishes identically.

Theorem I.2.10 (Refined operations on every prescribed support)**.** *The maps $G_{(\mathcal{L},s),T}$ are well-defined on Chow groups and satisfy all the following assertions.*

1. *For an integral $W \subset T$, if $s|_W$ is nonzero, the result is the divisor cycle of $s|_W$, with its actual local lengths. If $s|_W = 0$, it is the image of $c_1(\mathcal{L}|_W) \cap [W]$ in the stated target, not an instruction to discard W.*
2. *Let $p : \mathcal{Y} \to \mathcal{X}$ be proper, let $T' \subset \mathcal{Y}$ and $T \subset \mathcal{X}$ be closed with $p(T') \subset T$, and pull back the actual line and section. Then*

$$p_* G_{(p^*\mathcal{L},p^*s),T'}(\alpha) = G_{(\mathcal{L},s),T}(p_*\alpha) \quad \text{in } C_{r-1}(\mathcal{X}; T \cap |D|). \tag{I.2.24}$$

   This includes naturality under enlargement of the prescribed support.
3. *Two such operations commute in the group on $T \cap |Z(s)| \cap |Z(t)|$. No proper-intersection hypothesis is needed.*
4. *For $(\mathcal{M}, t)$ a second pair, set $B = |Z(s)| \cup |Z(t)|$. After pushing the three targets into $T \cap B$,*

$$G_{(\mathcal{L}\otimes\mathcal{M},s\otimes t),T} = G_{(\mathcal{L},s),T} + G_{(\mathcal{M},t),T}. \tag{I.2.25}$$

   After pushing into T, the operation is $c_1(\mathcal{L}|_T) \cap -$. If a coherent $\mathcal{F}$ on $\mathcal{X}$ is supported in T, has $\dim_\delta \operatorname{Supp}(\mathcal{F}) \le r$, and $s : \mathcal{F} \otimes \mathcal{L}^{-1} \to \mathcal{F}$ is injective, then

$$G_{(\mathcal{L},s),T}[\mathcal{F}]_r = [\mathcal{F}/s(\mathcal{F} \otimes \mathcal{L}^{-1})]_{r-1} \quad \text{in } C_{r-1}(\mathcal{X}; T \cap |D|).$$

   All generic and quotient lengths are retained.

Proof. The definition gives (1), and Stacks, Tag 02TO proves descent through rational equivalence. Apply Stacks, Tag 02TA to the proper map $T'_{\mathrm{red}} \to T_{\mathrm{red}}$ and the restricted line and section. Its target is the zero scheme of the restricted section on T, so its pushforward identity is exactly (I.2.24) after reduction. Applying this also to closed immersions proves the support-enlargement statement. It does not pass through the image of $\mathrm{CH}(T)$ in the ambient Chow group.

For clarity, the proper-pushforward calculation retains the factor $[K(W') : K(p(W'))]$ when dimensions agree. If an integral W' is contracted, choose a rational section of the relevant line on its integral image W. Its pullback divisor is supported over a proper closed subset of W. Every component then

has image dimension less than $\dim_\delta W' - 1$, and its cycle pushforward is zero. This also handles the contained-component Chern term by its divisor representative. Thus dimension-decreasing images do not create an unrecorded coefficient.

Apply Stacks, Tag 0B73 on T_{red}. The rational equivalence in its proof lies on the intersection of both zero schemes; in particular its target is precisely the support in (3). Reduction and the naturality just proved identify the successive targets used here with those of that lemma.

To prove (4), work on an integral cycle $[W]$. If neither restricted section is zero, divisors of their tensor product add as cycles. Locally this is $\operatorname{ord}(fg) = \operatorname{ord}(f) + \operatorname{ord}(g)$; at a one-dimensional local domain one can see the length equality from $0 \to A/(f) \xrightarrow{g} A/(fg) \to A/(g) \to 0$. If one or both sections vanish identically on W, then $W \subset B$. The product contributes $c_1((\mathcal{L} \otimes \mathcal{M})|_W)[W]$; each nonzero restricted section represents its own first Chern class on W. Additivity of that class, Stacks, Tag 02SP, proves equality already in $\mathrm{CH}(W)$, and hence in the required target. Linearity proves (I.2.25). Finally Stacks, Tag 02T9 identifies the push into T with the first Chern operation, and the coherent assertion uses the actual annihilator scheme $Z = V(\operatorname{Ann} \mathcal{F})$, with $|Z| \subset T$. Keep the full coherent module on Z. Tensor the asserted injection with the invertible line so that Stacks, Tag 02TB(2), with $k = r - 1$, applies to the actual module and its restriction to the zero scheme. This restriction is exactly $\mathcal{F}/s(\mathcal{F} \otimes \mathcal{L}^{-1})$. The closed inclusion $Z_{\mathrm{red}} \to Z$ and its base-changed zero scheme give the canonical identifications of the Chow groups: the cycle and rational-function generators are unchanged. Apply Stacks, Tag 02TA to these inclusions and then apply Stacks, Tag 02TA to the proper closed map $Z_{\mathrm{red}} \to T_{\mathrm{red}}$ with its actual zero support. The coherent cycles are computed on the full modules before these reduction identifications, so all lengths are retained. Injectivity remains essential. □

Theorem I.2.11 (The missing component-degree projection identity)**.** *Let $q : W \to V$ be a proper morphism of proper integral k-schemes, with $\dim W = d$ and $\dim V \le d$, and let $M_1, \dots, M_d$ be invertible sheaves on V. If $\dim q(W) < d$, then*

$$\delta(W; q^*M_1, \dots, q^*M_d) = 0.$$

If q is dominant and $\dim V = d$, then

$$\delta(W; q^*M_1, \dots, q^*M_d) = [k(W) : k(V)]\delta(V; M_1, \dots, M_d). \tag{I.2.26}$$

In particular, let $p : \mathcal{Y} \to \mathcal{X}$ be a supplied proper map of integral normal proper flat DVR models, inducing the identity on the generic fibre, and an isomorphism over an open containing every height-one point of $\mathcal{X}$. For arbitrary model lines $\mathcal{L}_i$, each strict-transform special component retains its mixed degree and each exceptional special component has mixed degree zero. The finite model measures with masses $b_V \delta_V / \prod_i e_i$, where $b_V = \mathrm{ord}_V(\pi)$, are consequently equal on every continuous test and every Borel test finite on their finite support. The conclusion concerns these supplied pullback models.

Proof. Use the global pairs $(M_i, 0)$ on V. Their Gysin operations are exactly $c_1(M_i)$, since their zero schemes are all of V. Iterating (C.1.48) gives

$$q_*\big(c_1(q^*M_1)\cdots c_1(q^*M_d)\cap[W]\big) = c_1(M_1)\cdots c_1(M_d)\cap q_*[W].$$

In the contracted case $q_*[W] = 0$. In the dominant equal-dimensional case it is $[k(W) : k(V)][V]$, with the full field degree. Proper pushforward preserves degree over k, proving both formulas.

For the models, the supplied open identifies the ordinary DVRs at a special component and its unique strict transform. The proper map between these integral special components has degree one. An exceptional component has image dimension less than d, so the two formulas give respectively the old mixed degree and zero. The identified DVRs have the same b_V and the same valuation point on the common generic analytic fibre. The additional exceptional terms have zero weight. Substitution in the specified finite atomic measure proves equality, and hence equality for all the stated tests. Neither existence of the map nor an analytic integral comparison enters this argument. □

Theorem I.2.12 (Supported projection and horizontal–vertical induction)**.** *The value* (I.2.23) *is independent of the permitted signed presentations and symmetric in all $d+1$ inputs. Let $p : \mathcal{Y} \to \mathcal{X}$ be a supplied proper dominant map of integral proper flat models of the same relative dimension, and put $N = [K(\mathcal{Y}) : K(\mathcal{X})]$. With the actual pulled-back factor data and supports on $\mathcal{Y}$, one has*

$$J_{\mathcal{Y}} = N J_{\mathcal{X}}. \tag{I.2.27}$$

Normality of $\mathcal{Y}$ is not needed for this supported equality.

At the ordinary height-one DVRs of $\mathcal{X}$, write

$$\mathrm{div}_{\mathcal{L}_0}(\sigma_0) = \sum_{H \text{ horizontal}} a_H[H] + \sum_{V \text{ vertical}} a_V[V].$$

Let J_H be the construction (I.2.23) on H using the remaining d pulled-back presentations and denominator $\prod_{i=1}^{d} e_i$. Put $\delta_V = \delta(V; \mathcal{L}_1|_V, \dots, \mathcal{L}_d|_V)$. Then

$$J_{\mathcal{X}} = \frac{1}{e_0} \sum_H a_H J_H + \frac{\lambda}{e_0 \prod_{i=1}^{d} e_i} \sum_V a_V \delta_V. \tag{I.2.28}$$

Thus the horizontal coefficients are exactly those of $\operatorname{div}(s_0)$ on the generic fibre. For the root model metric and the specified finite model measure, the second summand is

$$\int -\log \|s_0\| \, d\mu, \qquad -\log \|s_0(\xi_V)\| = \frac{\lambda a_V}{e_0 b_V}, \qquad \mu = \sum_V \frac{b_V \delta_V}{\prod_{i=1}^{d} e_i} \delta_{\xi_V}. \tag{I.2.29}$$

It is a finite total-variation integral, regardless of signs of the mixed degrees. At fixed metrics, replacing any s_i by πs_i changes J by $+\lambda$ times the corresponding model-measure mass. Only (I.2.22) is required; no pairwise proper intersection or generic nonvanishing on every intermediate cycle is imposed.

Proof. The operations produce a zero-class on T after $d+1$ steps. Condition (I.2.22) makes T_{red} a proper k-scheme, so its degree is defined even if T has positive dimension. Presentation independence and commutation prove the first assertions. For the supplied proper map, repeated supported projection gives

$$p_*(I_{p^*0} \cdots I_{p^*d}[\mathcal{Y}]) = I_0 \cdots I_d p_*[\mathcal{Y}] = N I_0 \cdots I_d[\mathcal{X}].$$

The supports remain the displayed inverse images; degree compatibility proves (I.2.27). This proves a statement about a supplied map and compatible data, not existence of a common model.

Apply I_0 first. Each factor section in its signed presentation is nonzero on $\mathcal{X}$. Its Gysin image of $[\mathcal{X}]$ is therefore its divisor cycle, and subtraction gives the displayed cycle of σ_0 on Z_0. A positive coefficient at a height-one point is the length of the quotient by a local equation in the ordinary DVR; differences give the signed coefficient. Apply the remaining operators to each component, using proper compatibility for its closed immersion. On a horizontal H, the remaining generic supports have empty total intersection: $H \subset Z_0$ and (I.2.22) applies. This contribution is $(a_H/e_0)J_H$. The value J_H uses the restricted factor presentations, not a purported restriction of a rational section that vanishes identically on H.

On a vertical V, push the remaining supported intersection class into $\mathrm{CH}_0(V)$. The last assertion of Theorem C.1.101, its signed extension, and

proper compatibility show that the result is $c_1(\mathcal{L}_1|_V)\cdots c_1(\mathcal{L}_d|_V)[V]$. This push stays inside the proper vertical scheme V, so it preserves the relevant degree. It gives $a_V\delta_V$ and proves (I.2.28). Localization at π identifies the horizontal DVR with its generic-fibre DVR. Since $\sigma_0 = s_0^{e_0}$, its horizontal coefficient is e_0 times the coefficient of $\operatorname{div}(s_0)$.

For (I.2.29), the point ξ_V has norm $|f(\xi_V)| = |\pi|^{\operatorname{ord}_V(f)/b_V}$ and generic scheme support. A model frame gives the stated root norm. Multiplying its negative logarithm by the mass cancels b_V, giving exactly the vertical summand of (I.2.28). All these finitely many values are finite; the zero/pole locus is null for the finite atomic measure. This proves the integral assertion by a finite sum, without a limit.

Finally $s_i \mapsto \pi s_i$ changes the model divisor by $e_i \operatorname{div}(\pi)$ and may enlarge its support by the special fibre. Its first supported operation on $[\mathcal{X}]$ gives $e_i \sum_V b_V[V]$. Commutation and the same vertical calculation give

$$\Delta J = \frac{\lambda}{\prod_{j\neq i} e_j} \sum_V b_V \delta(V; (\mathcal{L}_j|_V)_{j\neq i}),$$

with positive sign. This is the stated model mass; identifying it with a generic-fibre degree would require the separate specialization result. □