Arithmetic Conjectures for Beilinson–Bloch Heights on Products of Curves

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Abstract

We formulate an arithmetic programme for height pairings on the homologically trivial Chow groups of products of curves. Exact Chow projectors reduce arithmetic hard Lefschetz to off-central vanishing and reduce primitive integral Northcott to two central assertions: injectivity of the algebraic Abel–Jacobi map and Northcott on a residual Griffiths quotient. The latter assertion contains real positivity but also requires integral and varying-field information. We separate the proposed arithmetic inputs into scalar detection by continuous étale cohomology, a common rational lattice, and ramification control for the minimal fields of cycle classes. The stated implications specify where each input enters.

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1 Introduction

The expected positivity of Beilinson–Bloch heights is an arithmetic Hodge–Riemann principle. A Northcott property is stronger: it asks for finiteness of integral cycle classes of bounded degree and height. Even a positive quadratic form does not give this finiteness unless the integral classes are arithmetically discrete. Allowing the field of definition to vary adds a further descent problem.

For products of curves, tensor Chow projectors separate these questions. The all-h^1 part of a subproduct carries a central height precisely when its dimension is $2c-1$ in codimension c. The other parts are responsible for arithmetic Lefschetz. Within the central part, curve correspondences separate a Néron–Tate summand and its residual Griffiths quotient. This paper formulates the resulting conjectures and organizes the arithmetic inputs to their study. The complete reduction proofs are given in the companion manuscript [1].

These conjectures are expressed within the Bloch–Beilinson and arithmetic standard-conjecture framework. The arithmetic Lefschetz and primitive positivity statements are those recorded in [6, Conjecture 2.3.1]. The purpose of the formulation here is to specify the residual objects, their integral images and the exact implications connecting the different arithmetic questions.

1.1 The fixed product and height data

Let F be a number field, and let

$$X=\prod_{i=1}^{n} C_i, \qquad o_i\in C_i(F), \qquad H_i=p_i^*[o_i], \qquad \ell=\sum_{i=1}^{n} a_iH_i, \quad a_i\in\mathbb{Z}_{>0},$$

where the curves are smooth, projective and geometrically connected. For a finite extension E/F, write

$$V^r(X_E)=\mathrm{CH}^r(X_E)_{\mathrm{hom},\mathbb{Q}}.$$

Fix an algebraic closure $\overline{F}$ of F and an embedding into $\mathbb{C}$. Homological triviality is defined by geometric rational Betti cohomology. All Chow groups below have rational coefficients unless otherwise indicated. We regard $V^r(X_E)$ as a subspace of $V^r(X_{\overline{F}})$: restriction is injective because a rational equivalence descends to a finite extension and trace composed with restriction multiplies by its degree.

Definition 1.1. A curve-normalized height system consists of pairings on all smooth projective varieties used below, including the subproducts and the source curves of correspondences. For such a variety X of dimension n, these are

$$h_X : V^r(X_E)\times V^{n+1-r}(X_E)\longrightarrow\mathbb{R}$$

over every finite extension E/F, satisfying:

(H1) bilinearity over $\mathbb{Q}$ and symmetry under exchange of the complementary degrees;

(H2) invariance under extension in the absolute normalization, and covariance under isomorphisms, including Galois conjugation;

(H3) correspondence adjunction

$$h_X(\Gamma_*z,w)=h_Y(z,{}^t\Gamma_*w);$$

(H4) on a pointed curve C, the form $-h_C$ on $\mathrm{Pic}^0(C)_{\mathbb{Q}}$ is the positive Néron–Tate form for the canonical principal polarization, in the corresponding Poincaré normalization.

These hypotheses will be understood in every statement involving h. A real positive form means positive definiteness after tensoring the rational group over a fixed number field with $\mathbb{R}$. For codimension r on X, the symbol $\mathrm{AJ}_{\mathbb{Q}}$ denotes the complex Abel–Jacobi map with target $J^r(X_{\mathbb{C}})(\mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let $\Lambda^r_X(E)$ be the image in $V^r(X_{\overline{F}})$ of the integral-coefficient cycles over E with zero geometric rational cycle class. Thus torsion and rational equivalence are removed before counting. For a fixed rational projector P, we use its projected integral image $P_*\Lambda^r_X(E)$; for an integral projector this equals $\Lambda^r_X(E) \cap P_*V^r(X_E)$.

Definition 1.2. For compatible integral images $\Lambda(E)$ in one geometric rational group and a quadratic form q, bounded-degree Northcott means that, for every $B \in \mathbb{R}$ and integer $D \geq 1$,

$$\{\xi \in \bigcup_{[E:F] \leq D} \Lambda(E) : q(\xi) \leq B\}$$

is finite. Fixed-field Northcott means the same assertion with E fixed. Bounded denominators mean the union of $d^{-1}\Lambda(E)$ for $1 \leq d \leq d_0$, with d_0 fixed.

Northcott for integral images implies the bounded-denominator version by scaling. A fixed initial extension F_0/F is harmless for bounded-degree assertions: after replacing E by EF_0, $[EF_0 : F_0] \leq [E : F]$, the integral image remains integral, and the absolute height is unchanged.

1.2 Central cores and the residual pairing

For a subset $S \subseteq \{1, \ldots, n\}$, put

$$Y_S = \prod_{i \in S} C_i, \qquad \Pi_S = \bigotimes_{i \in S} \pi_{1,i}, \qquad U^c_S(E) = \Pi_{S*}V^c(Y_{S,E}).$$

Here $\pi_{1,i} = \Delta_{C_i} - o_i \times C_i - C_i \times o_i$. The empty subproduct has zero homologically trivial Chow groups. The core is central if $|S| = 2c - 1$.

Fix such an odd subproduct. Let

$$A_{S,E} = U^c_S(E) \cap \mathrm{CH}^c(Y_{S,E})_{\mathrm{alg},\mathbb{Q}}, \qquad G_{S,E} = U^c_S(E)/A_{S,E}.$$

The maximal curve-visible construction in [1, Section 3] gives an actual self-transpose Chow projector $e_S \leq \Pi_S$, with $R_S = \Pi_S - e_S$. Choose one finite extension F_0/F over which these projectors are defined for all odd subproducts. For every finite E/F_0,

$$U^c_S(E) = e_S U^c_S(E) \oplus R_S U^c_S(E), \qquad e_S U^c_S(E) \simeq B_S(E)_{\mathbb{Q}}$$

for a fixed abelian variety B_S/F_0. The decomposition is height-orthogonal, and the form $q_{e,S}(u) = (-1)^c h_{Y_S}(u,u)$ on the first summand is a positive Néron–Tate form. Moreover,

$$R_S A_{S,E} = \ker(\mathrm{AJ}_{\mathbb{Q}}|_{A_{S,E}}) = A_{S,E} \cap \mathrm{rad}(h_{Y_S}|_{U^c_S(E)}).$$

For $\xi = [z]$, $\eta = [w]$ in $G_{S,E}$, define

$$h_{G,S}(\xi,\eta) = h_{Y_S}(R_S z, R_S w), \qquad q_{G,S}(\xi) = (-1)^c h_{G,S}(\xi,\xi).$$

The radical equality makes this pairing well defined, and the projector construction proves its independence of the maximal curve-visible lift. These facts hold for every height system in Definition 1.1.

The integral structures are

$$\Lambda_{U,S}(E) = \Pi_{S*}\Lambda^c_{Y_S}(E), \qquad \Lambda_{G,S}(E) = \mathrm{im}(\Lambda_{U,S}(E) \to G_{S,E})\,.$$

The projector Π_S is integral. The residual projector R_S is rational, with one fixed denominator; that denominator is independent of the cycle and of E. Since algebraic triviality is geometric, all these rational quotients embed in their counterparts over $\overline{F}$.

1.3 The three central assertions

Conjecture 1.3 (Off-central vanishing)**.** For every subproduct Y_S, every finite E/F_0, and every c with $|S| \neq 2c-1$,

$$U^c_S(E) = 0.$$

Conjecture 1.4 (Central algebraic Abel–Jacobi injectivity)**.** For every $|S| = 2c-1$ and finite E/F_0, the map

$$\mathrm{AJ}_{\mathbb{Q}} : A_{S,E} \longrightarrow J^c(Y_{S,\mathbb{C}})(\mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective.

Conjecture 1.5 (Residual integral Northcott)**.** For every $|S| = 2c-1$, every $B \in \mathbb{R}$, and every integer $D \geq 1$, the set of geometric classes

$$\left\{ \xi \in \bigcup_{[E:F_0]\leq D} \Lambda_{G,S}(E) : q_{G,S}(\xi) \leq B \right\}$$

is finite.

These assertions address, respectively, the degrees in which the height pairing should be nondegenerate, its radical on the algebraically trivial part, and the arithmetic size of the remaining integral quotient.

2 From the central assertions to the whole product

2.1 Arithmetic Lefschetz and the primitive form

For $1 \leq r \leq \lfloor (n+1)/2 \rfloor$, put

$$P^r_\ell(X_E) = \ker\big(\ell^{n+2-2r} : V^r(X_E) \longrightarrow V^{n+2-r}(X_E)\big)\,,$$
$$q_{\ell,r}(z) = (-1)^r h_X(z, \ell^{n+1-2r} z).$$

The primitive integral image is $\Lambda^r_X(E) \cap P^r_\ell(X_E)$.

Theorem 2.1 (Equivalence of the arithmetic programmes)**.** *For the fixed pointed product, positive integers a_i, and a height system satisfying Definition 1.1, the following conditions are equivalent.*

(i) Conjectures 1.3 to 1.5 hold for all subproducts of X.

(ii) For every finite E/F_0 and every $1 \le r \le \lfloor (n+1)/2 \rfloor$,

$$\ell^{n+1-2r} : V^r(X_E) \xrightarrow{\sim} V^{n+1-r}(X_E),$$

and $q_{\ell,r}$ has bounded-degree Northcott on the primitive integral images.

Either condition implies real positive definiteness of all primitive forms and nondegeneracy, also after real scalar extension, of the complementary height pairings over every fixed number field.

The full proof is the principal reduction theorem of [1], completed in its Section 7. Its two mechanisms are the exact central-block identity and the central radical decomposition. We recall them because they determine which conjecture is needed at each step.

The Chow block indexed by S, c is

$$U_S^c(E) \otimes_{\mathbb{Q}} \mathbb{Q}[x_i : i \notin S]/(x_i^2),$$

with ℓ acting as multiplication by $\sum_{i \notin S} x_i$. Writing $I = \{1, \dots, n\} \setminus S$, the identification sends $u \otimes x_T$ to $a_T j_T u$, where $a_T = \prod_{i \in T} a_i$ and j_T inserts $[o_i]$ on T and $[C_i]$ on $I \setminus T$. The lowering operator is

$$F x_T = \sum_{i \in T} x_{T \setminus \{i\}}.$$

The arithmetic hard Lefschetz maps are isomorphisms exactly when the noncentral cores vanish. This equivalence is independent of the choice of the positive weights a_i, with the points o_i fixed.

In a central block let $m = n - |S|$, $t = r - c$, $I = \{1, \dots, n\} \setminus S$, and $z_S = \sum_{|T|=t} u_T x_T$. For $0 \le t \le m/2$, primitivity means $F z_S = 0$; the signed height is then

$$q_{\ell,r}(z_S) = (m-2t)! \prod_{i \in I} a_i \sum_{|T|=t} (-1)^c h_{Y_S}(u_T, u_T). \tag{2.1}$$

This central-block identity uses only adjunction and the square-zero calculation. Off-central vanishing enters when all of $V^r(X_E)$ is reduced to these central blocks. If z is integral, the single integer $N_a = \prod_i a_i$ clears the denominators of every u_T.

Finally, central integral Northcott is equivalent to

$$\ker(\mathrm{AJ}_{\mathbb{Q}} |_{A_{S,E}}) = 0 \quad \text{for every finite } E/F_0, \qquad \text{and residual integral Northcott on } \Lambda_{G,S}.$$

Indeed

$$(-1)^c h_{Y_S}(z, z) = q_{e,S}(e_S z) + q_{G,S}([z]).$$

The visible summand has classical bounded-degree Northcott through fixed correspondences with bounded denominators. A nonzero algebraic radical class would have infinitely many integral multiples of height zero. After this radical is removed, the pair $(e_S z, [z])$ detects z.

2.2 Residual positivity and its precise role

Conjecture 2.2 (Real residual positivity)**.** For every central subproduct and every finite E/F_0, $q_{G,S}$ extends to a positive definite quadratic form on $G_{S,E} \otimes_{\mathbb{Q}} \mathbb{R}$.

This is a weaker consequence of Conjecture 1.5. More generally, if a subgroup Λ spans a rational vector space V, finiteness of all height balls in Λ forces the real extension of q to be positive definite. A negative direction gives a negative rational vector and its integer

multiples. A real null direction, in a finite-dimensional rational subspace, gives infinitely many integral vectors of bounded height by rounding its scalar multiples. This is the proof of the real-positivity lemma in [1, Section 5.1].

The converse is an integral problem. For example,

$$\Lambda = \sum_{p \text{ prime}} \mathbb{Z}\, p^{-1} \subset \mathbb{Q}, \qquad q(x) = x^2$$

has a positive real form and a finite free local completion $\Lambda \otimes \mathbb{Z}_p = p^{-1}\mathbb{Z}_p$ at every prime. Nevertheless $q(1/p) \to 0$. The extra input needed for finiteness is one common rational lattice, rather than a separate local bound at each prime.

With Conjectures 1.4 and 2.2, the central form is real positive definite by its orthogonal splitting. Adding Conjecture 1.3 propagates this positivity and nondegeneracy to the product through (2.1). The associated primitive bilinear form then satisfies

$$|b_{\ell,r}(z,w)|^2 \le q_{\ell,r}(z) q_{\ell,r}(w).$$

The finiteness assertion in Conjecture 1.5 requires the additional arithmetic inputs below.

3 Arithmetic detection and the integral lattice

Fix one central subproduct Y of dimension $k = 2c-1$, and abbreviate $G_{S,E}, R_S, q_{G,S}$ to G_E, R, q_G.

3.1 Scalar detection by a residual regulator

For every prime p, let

$$V_p = R_* H^{2c-1}_{\text{ét}}(Y_{\overline{F}}, \mathbb{Q}_p(c)).$$

The actual projector R has geometric cohomology in one degree. Consequently, its part of Hochschild–Serre has one row and gives a canonical identification

$$R_* H^{2c}_{\text{cont}}(Y_E, \mathbb{Q}_p(c)) \simeq H^1(E, V_p).$$

The continuous cycle class of Rz annihilates A_E. It therefore defines a residual arithmetic Abel–Jacobi map

$$a_{p,E} : G_E \longrightarrow H^1(E, V_p),$$

compatible with restriction and trace and independent of the curve-visible lift under the common cohomological identification. The construction is proved in [1, Section 4.1], in the continuous-cohomology framework of [4, 2].

The representation V_p has weight -1. For one fixed set S_0 of places of F_0, containing the bad-reduction places, inflation identifies

$$H^1(G_{E,S_E}, V_p) \simeq H^1(E, V_p), \qquad S_E = \{v : v \mid S_0 \text{ or } v \mid p\},$$

with archimedean places included. This whole target is finite-dimensional over $\mathbb{Q}_p$.

Conjecture 3.1 (Residual scalar detection)**.** For every finite E/F_0 and every prime p, the scalar-extended map

$$G_E \otimes_{\mathbb{Q}} \mathbb{Q}_p \longrightarrow H^1(E, V_p)$$

is injective.

The word *scalar* specifies the strength needed here. Injectivity of the $\mathbb{Q}$-linear map $a_{p,E}$ alone would not imply finite rational rank: a finite-dimensional $\mathbb{Q}_p$-space is infinite-dimensional over $\mathbb{Q}$. In contrast, scalar-extended injectivity at one prime gives

$$\dim_{\mathbb{Q}} G_E \leq \dim_{\mathbb{Q}_p} H^1(E, V_p) < \infty.$$

For every fixed E, p, the integral cycle map and one denominator clearing R place $a_{p,E}(\Lambda_G(E))$ in a finitely generated $\mathbb{Z}_p$-module. Under scalar-extended injectivity,

$$\Lambda_G(E) \otimes_{\mathbb{Z}} \mathbb{Z}_p \quad \text{is finitely generated over } \mathbb{Z}_p, \qquad \Lambda_G(E)/p\Lambda_G(E) \quad \text{is finite.}$$

The proof uses the integral image theorem of [2, Theorem 2.4]; it is given in [1, Section 4.3].

3.2 A common lattice across coefficient primes

Conjecture 3.2 (Residual integral discreteness)**.** For every finite E/F_0, the space G_E is finite-dimensional, and there are a rational basis $v_1, \ldots, v_s$ and an integer $N_E \geq 1$ such that

$$\Lambda_G(E) \subseteq N_E^{-1} \sum_{i=1}^{s} \mathbb{Z} v_i.$$

For a finite-rank torsion-free group and a real positive definite quadratic form, the following conditions are equivalent: finite height balls, a positive height gap, finite generation, a common denominator bound in a rational basis, and relative compactness of the diagonal image in the finite adeles. The proof in [1, Section 5.2] is an elementary lattice argument. It follows that Conjectures 2.2 and 3.2 imply fixed-field Northcott for every E. Conversely, fixed-field Northcott together with finite rational rank implies Conjecture 3.2 and real positivity. Finite rank is an explicit premise in this converse.

A route to Conjecture 3.2 is an almost-everywhere saturation comparison. Fix E, put $V = G_E$, and suppose that V is finite-dimensional. Seek a full rational lattice $\Lambda_0 \subset V$ and full local lattices $M_p \subset H^1(E, V_p)$ such that the scalar-extended regulators D_p are injective and

$$D_p(\Lambda_G(E)) \subseteq M_p \quad \text{for every } p, \qquad D_p^{-1}(M_p) = \Lambda_0 \otimes \mathbb{Z}_p \quad \text{for almost every } p. \tag{3.1}$$

Then the inverse lattices at the finitely many exceptional primes can be absorbed into one integer N_E. Thus (3.1) proves Conjecture 3.2. The fixed-prime local bounds above supply the first type of control; the equality with one common rational lattice is the comparison still required.

Integral detection is formulated on the stated image in a rational quotient. The unrestricted integral continuous cycle class map has torsion counterexamples over number fields in codimension three [5, Theorems 1.3–1.4]. The torsion kernels in those statements and the torsion-free group $\Lambda_G(E)$ are different arithmetic objects.

3.3 The height-gap route

There is a second route that can start from one coefficient prime. Suppose $\sqrt{q_G}$ is the norm of a real positive quadratic form, and, for some integer $m \geq 2$,

$$s = |\Lambda_G(E)/m\Lambda_G(E)| < \infty, \qquad q_G(\lambda) \geq \varepsilon_E > 0 \quad (0 \neq \lambda \in \Lambda_G(E)).$$

The quantitative descent criterion of [1, Section 5.4] gives finite generation and fixed-field Northcott. If $B \geq 0$, $j \in \mathbb{Z}_{\geq 0}$, and $m^{2j}\varepsilon_E > 4B$, then

$$\#\{\lambda \in \Lambda_G(E) : q_G(\lambda) \leq B\} \leq s^j.$$

The reason is that two classes in the ball congruent modulo $m^j\Lambda_G(E)$ would differ by $m^j z$ with $q_G(z) < \varepsilon_E$. The usual height descent then generates the group from a finite ball and representatives modulo m.

Scalar detection at one prime supplies a finite quotient with $m = p$, through the local integral bound. A genuine positive height gap is the additional input. Real positive definiteness alone does not supply this gap on a subgroup with unbounded denominators.

4 Minimal class fields and bounded-degree Northcott

4.1 Trace and the class field

View the residual groups inside the geometric quotient with its discrete $\mathrm{Gal}(\overline{F}/F_0)$-action. For a class ξ, define

$$F_0(\xi) = \overline{F}^{\mathrm{Stab}_{\mathrm{Gal}(\overline{F}/F_0)}(\xi)}.$$

If $\xi \in \Lambda_G(E)$, its minimal class field $K = F_0(\xi)$ is contained in E, and trace gives

$$[E : K]\xi \in \Lambda_G(K). \tag{4.1}$$

Indeed the pushforward of an integral representative has geometric class equal to the sum over the K-embeddings of E, and every summand is ξ. Thus the multiplier is at most D when $[E : F_0] \leq D$. This concerns the field of the class; a particular cycle representative need not descend to K.

Conjecture 4.1 (Residual class-field ramification)**.** For every $B \geq 0$ and integer $D \geq 1$, there is a finite set $S(B, D)$ of finite places of F_0 such that the fields $F_0(\xi)/F_0$, for

$$\xi \in \Lambda_G(E), \qquad [E : F_0] \leq D, \qquad q_G(\xi) \leq B,$$

are unramified outside $S(B, D)$.

Under fixed-field Northcott for every finite extension, Conjecture 4.1 is equivalent to bounded-degree Northcott. Hermite–Minkowski leaves finitely many possible class fields K of degree at most D. Equation (4.1) then reduces the classes to the finitely many fixed-field balls

$$m\xi \in \Lambda_G(K), \qquad 1 \leq m \leq D, \qquad q_G(m\xi) \leq D^2 B.$$

The converse follows by taking the union of ramification sets of the finitely many classes in a ball. The full proof is in [1, Section 6.2]; the finite-ramification form of Hermite–Minkowski is [3, Theorem 10.2.2].

4.2 The extra information needed as the field varies

A fixed set S_E for $H^1(E, V_p)$ controls ramification over E. It does not bound the ramification of the class field $F_0(\xi)/F_0$. The distinction persists even when fixed-field estimates are uniform in the degree. For example, choose infinitely many distinct quadratic extensions K_i/F_0, with characters χ_i, and set

$$\mathcal{G} = \bigoplus_i \mathbb{Q}e_i, \qquad \sigma e_i = \chi_i(\sigma)e_i, \qquad \Lambda(E) = \bigoplus_{K_i \subseteq E} \mathbb{Z}e_i, \qquad q\left(\sum a_i e_i\right) = \sum a_i^2.$$

These integral images are compatible with trace. Their ranks, height gaps and quotients modulo any integer $m \geq 2$ have degree-dependent uniform bounds, and all fixed-field balls are finite. Yet the infinitely many $e_i \in \Lambda(K_i)$ have degree two and height one. This example isolates the field-control hypothesis in the ramification criterion.

A quantitative strengthening asks for constants $A_D, B_D \geq 0$, depending only on the fixed residual data and D, such that

$$\log N_{F_0/\mathbb{Q}}\mathfrak{d}_{F_0(\xi)/F_0} \leq A_D q_G(\xi) + B_D \qquad ([E : F_0] \leq D,\ \xi \in \Lambda_G(E)). \tag{4.2}$$

This is a research question stronger than Conjecture 4.1. A bound of the displayed form would imply the required finite ramification set at every bounded height.

5 A proof programme and its arithmetic obstacles

The reduction theorem provides two complementary routes. One seeks the three central assertions directly. The other constructs arithmetic detectors for the residual quotient and then proves integral and field-of-definition control.

5.1 The geometric and central steps

The first target is Conjecture 1.3. The tensor Chow decomposition gives an exact test: a nonzero off-central core would produce a misplaced Lefschetz string. A proof must therefore eliminate the core in the rational Chow group, rather than merely its cohomological realization.

The next target is the algebraic kernel in Conjecture 1.4. The projector construction already identifies it as $R_S A_{S,E}$, and adjunction places it in the full central height radical. Establishing its vanishing is a separate arithmetic cycle statement. Once it vanishes, the visible Néron–Tate factor has all the required fixed-field and bounded-degree finiteness properties.

For the residual factor, real positivity is an initial target. It supplies Cauchy–Schwarz and, together with the algebraic kernel statement, central nondegeneracy. The primitive sum-of-squares identity transfers these consequences uniformly to every degree of the product.

5.2 A detector-to-Northcott route

For each fixed finite E/F_0, the sequence of inputs is:

1. Prove scalar-extended injectivity of one residual regulator. The finite-dimensional cohomological target then gives finite rational rank.
2. Prove one common denominator bound for the integral image, for example through the all-prime saturation condition (3.1). Alternatively, combine a finite Kummer quotient with a positive height gap.
3. Establish real residual positivity. Together with the common lattice, this gives fixed-field Northcott.
4. Control the ramification of minimal class fields in bounded degree and height. The trace criterion then gives bounded-degree Northcott.

The alternatives in the second step have different arithmetic content. All-prime saturation compares integral cohomology across coefficient primes. The height-gap route instead requires a uniform positive lower bound on nonzero integral classes over the fixed field.

Under real positivity, scalar detection and the almost-everywhere lattice comparison for every finite E/F_0, the first three steps establish fixed-field Northcott. Adding Conjecture 4.1 proves Conjecture 1.5. Alternatively, a finite Kummer quotient and a genuine gap over every fixed field, together with real positivity and Conjecture 4.1, give the same conclusion without a preliminary finite-rank assumption. Finally, Conjectures 1.3 and 1.4 and Theorem 2.1 give the product programme.

The local-to-global construction of a height system and these arithmetic properties meet at Definition 1.1. A construction of local Green forms must supply the global pairing, adjunction and curve normalization before the arithmetic reductions apply. The present programme starts from that explicitly stated height system; its subsequent tasks concern vanishing, positivity, integral discreteness and fields of cycle classes.

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