Normalized Determinants and Height Metrics

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Abstract

We compare normalized determinant-of-cohomology metrics on Jacobian families. For a perfect complex on the product of a pointed curve and a smooth projective variety, the derived projection formula, for product Kähler metrics, induces an exact isometry after normalization at the origin. In virtual rank zero, the resulting line is the Deligne pairing with the determinant of the pushforward. Filtered Grothendieck–Riemann–Roch identifies the line attached to a curve correspondence and a homologically trivial cycle. At a finite place where the pointed source curve has good reduction, a model-perfect complex of generic virtual rank zero and degree-zero determinant yields the canonical rigidified Poincaré model metric.

Contents

1 Introduction **1**

2 Normalized determinants on Jacobian families **2**
2.1 The Poincaré metric and normalization 2
2.2 The normalized projection-formula isometry 2
2.3 Virtual rank zero and Deligne pairings 3
2.4 Curve correspondences in arbitrary codimension 4
2.5 The source-curve good-reduction comparison 4

3 The remaining metric comparisons **5**

1 Introduction

The comparison between a cycle-theoretic height and a determinant construction has two parts. One must identify the underlying line, and one must identify its metric with the correct normalization. The second part contains a genuine constant ambiguity even when the two curvature forms agree. A Jacobian family provides a useful setting in which that ambiguity can be removed exactly.

Let C be a pointed smooth projective complex curve and $J = \mathrm{Pic}^0(C)$. The standard Poincaré metric on $C \times J$ has purely mixed curvature. In a determinant family obtained by twisting a fixed complex with the Poincaré bundle, the family index formula therefore depends only on the virtual rank of its pushforward to C. Dividing by the determinant fibre at the origin equips the line with a unit rigidification. Equality of curvature forms then gives equality of metrics: their logarithmic ratio is pluriharmonic on the compact connected variety J, and its value at the origin is zero.

This argument proves the projection-formula isometry in Theorem 2.1. Its rank-zero specialization is Theorem 2.2. Applied to a curve correspondence, the two theorems give Corollary 2.3; the relevant rank and determinant are computed by the leading terms of

Grothendieck–Riemann–Roch. The good-reduction analogue is Proposition 2.4. There the quotient of two model lines has a vertical divisor, and the unit rigidification forces its coefficient to vanish.

The full arguments appear in the next section in this order. Throughout, $\mathrm{CH}^j(X)_{\mathbf{Q}}$ denotes the rational Chow group, and

$$V^j(X) = \ker\big(\mathrm{CH}^j(X)_{\mathbf{Q}} \longrightarrow H^{2j}(X, \mathbf{Q})\big)$$

over $\mathbf{C}$. The cochain determinant convention and the Poincaré normalization are fixed explicitly before the comparison theorems. The determinant–biextension identifications of [3, 4] explain how this comparison enters local height formulas.

2 Normalized determinants on Jacobian families

2.1 The Poincaré metric and normalization

Let C be a smooth projective connected complex curve with basepoint o, and let $J = \mathrm{Pic}^0(C)$. Let $\mathcal{P}$ be the Poincaré line bundle on $C \times J$, rigidified on both axes. Equip it with the standard smooth metric for which these rigidifications have norm one. Its Chern form $\Omega = c_1(\mathcal{P})$ is purely mixed: it has real degree one in C and degree one in J.

One way to see this is to pull back the canonical Poincaré bundle on $J \times J^\vee$ by the Abel map and the principal polarization, with the convention realizing $\mathcal{P}$. For a translation-invariant theta curvature θ, the corresponding Poincaré curvature is the mixed form obtained from $m^*\theta - p_1^*\theta - p_2^*\theta$, up to the chosen duality sign. The pure terms cancel. Pullback by the Abel map preserves the mixed-degree assertion.

For a fixed smooth projective complex variety Z, use the cochain determinant convention

$$\lambda_Z(K) = \det R\Gamma(Z, K) = \bigotimes_i \det H^i(Z, K)^{(-1)^i}.$$

A perfect complex is given a bounded vector-bundle representative with smooth hermitian metrics; its determinant metric is the alternating tensor product of the Quillen metrics of the terms. Choose fixed Kähler metrics on the varieties involved.

2.2 The normalized projection-formula isometry

Let X be a smooth projective connected complex variety, $M = C \times X$, and $q : M \to C$. For a hermitian perfect complex $\mathcal{F}$ on M, put $K = Rq_*\mathcal{F}$ and choose any hermitian vector-bundle representative of K. Define line bundles on J by

$$\mathcal{D}_M(\mathcal{F}) = \det Rp_{J*}(\mathcal{P}_M \otimes \mathcal{F}_J) \otimes \big(\lambda_M(\mathcal{F}) \otimes \mathcal{O}_J\big)^{-1},$$
$$\mathcal{D}_C(K) = \det Rp_{J*}(\mathcal{P} \otimes K_J) \otimes \big(\lambda_C(K) \otimes \mathcal{O}_J\big)^{-1}.$$

The subscripts indicate pullbacks to the appropriate products with J. Each constant denominator has the Quillen norm of the same complex and metrics that occur in the numerator at 0. Thus each line has a canonical origin rigidification of norm one.

Theorem 2.1. *Use product Kähler metrics on $C \times X$. The derived projection-formula isomorphism gives an exact isometry of rigidified line bundles*

$$\mathcal{D}_M(\mathcal{F}) \xrightarrow{\sim} \mathcal{D}_C(Rq_*\mathcal{F}).$$

Both Chern forms equal

$$\frac{\mathrm{rk}\, K}{2} \int_C \Omega^2. \tag{1}$$

Under the canonical rigidified identifications, these normalized metrics are independent of the hermitian representatives and the fixed product Kähler metrics used to construct them.

Proof. Derived projection formula and base change identify the numerator complexes and their restrictions at 0. Applying determinants to these two identifications gives the asserted algebraic isomorphism, respecting the origin rigidifications.

The Bismut–Gillet–Soulé curvature formula [1, Theorem 0.1], in the cochain determinant convention, gives the Chern form of the first line as the base-degree-two component of

$$\int_{C\times X} \operatorname{Td}(T_C)\operatorname{Td}(T_X)\operatorname{ch}(\mathcal{F})(e^{\Omega}-1).$$

Every factor except Ω is pulled back from $C\times X$. Since Ω is purely mixed, only $\Omega^2/2$ can contribute degree two on J. It already has degree two on C, so the remaining factor contributes its degree-zero-in-C part. Integration over X gives

$$\int_X \operatorname{Td}(T_X)\operatorname{ch}(\mathcal{F}|_{\{x\}\times X}) = \chi(X,\mathcal{F}|_{\{x\}\times X}) = \operatorname{rk} K \qquad (x\in C).$$

The final equality follows by derived base change and the constant Euler characteristic of the perfect pushforward. This proves (1) for $\mathcal{D}_M$. The curve calculation gives the same form: all positive-degree forms from K or T_C have too much C-degree to contribute.

The determinant lines with their Quillen metrics are smooth on the whole J, including cohomology-jump loci. Under the algebraic isomorphism, the logarithm of the ratio of their metrics is therefore a smooth pluriharmonic function on the compact connected manifold J. It is constant, and the norm-one origin rigidifications make that constant zero. Changing the hermitian representatives or the fixed product metrics preserves both the curvature computation and this normalization, proving the last assertion. □

2.3 Virtual rank zero and Deligne pairings

Theorem 2.2. *Let K be a perfect complex of virtual rank zero on C, and put $L=\det K$. There is a unique rigidified isomorphism*

$$\mathcal{D}_C(K)\xrightarrow{\sim}\langle\mathcal{P},p_C^*L\rangle_{C\times J/J}.$$

It is an isometry for the normalized Quillen metric and the Deligne metric. Both metrics are smooth and flat. The right side can be formed using any smooth hermitian metric on L.

Proof. On a nonsingular curve, rank and determinant identify $K_0(C)$ with $\mathbf{Z}\oplus\operatorname{Pic}(C)$. Thus $[K]=[L]-[\mathcal{O}_C]$. At the level of line-bundle isomorphism classes, the normalized determinant becomes

$$\lambda_C(\mathcal{P}L)\lambda_C(\mathcal{P})^{-1}\lambda_C(L)^{-1}\lambda_C(\mathcal{O}_C),$$

the determinant expression for the Deligne pairing. Here the first two factors are relative determinants over J. Normalize an isomorphism at 0. It becomes unique because an invertible holomorphic function on the connected projective J is constant.

The Quillen curvature is zero by (1). The Deligne curvature is $\int_C \Omega\, c_1(L)=0$, since the integrand has degree three on C. At 0, $\mathcal{P}$ is the trivially metrized line with unit rigidification; the Deligne unit has norm one. The two flat metrics therefore agree by the same compactness argument as above. The determinant and Deligne curvature formulas are also discussed in [2, Section 2]. □

2.4 Curve correspondences in arbitrary codimension

Let $\dim X = n$, take $\Gamma \in \mathrm{CH}^r(C \times X)_{\mathbf{Q}}$, and let $W \in V^{n+1-r}(X)$. Choose rational K_0-lifts E_Γ, E_W whose Chern characters start with Γ, W in their indicated codimensions. Virtual sums of structure sheaves give such lifts on these smooth varieties. Set

$$\mathcal{F} = E_\Gamma \otimes^{\mathbf{L}} p_X^* E_W, \qquad K = Rq_*\mathcal{F}.$$

The product Chern character starts in codimension $n+1$. Filtered Grothendieck–Riemann–Roch for q gives

$$\operatorname{rk} K = 0, \qquad c_1(\det K) = q_*(\Gamma \cdot p_X^* W) = {}^t\Gamma_* W. \tag{2}$$

Positive-degree Todd terms do not contribute to these two components. The divisor class on the right has degree zero because W is homologically trivial.

Corollary 2.3. *There is a rigidified isometry of rational metrized line bundles on J,*

$$\mathcal{D}_{C\times X}(E_\Gamma \otimes^{\mathbf{L}} p_X^* E_W) \xrightarrow{\sim} \langle \mathcal{P}, \mathcal{O}_C({}^t\Gamma_* W)\rangle.$$

The metric is the normalized flat Poincaré metric attached to the degree-zero divisor class ${}^t\Gamma_* W$.

Proof. Combine Theorems 2.1 and 2.2 with (2). For rational classes, take a common tensor power throughout and then the positive real root of the norm. Changing the chosen cycle lifts by higher filtration changes their product only in codimension at least $n+2$, which is zero on $C \times X$; the rational Chern character then gives the same normalized line. □

The relation to the classical archimedean height uses the determinant–cycle–Hodge biextension comparison of [3, Propositions 4.3, 4.29, 4.31 and Corollary 4.32]. Its disjoint-cycle trivialization gives the local height [3, Remark 4.7]. The relative comparison over a smooth base is given in [4, Theorem 10.9]; that theorem includes an inversion of the central $\mathbf{G}_m$ under its conventions. Fixing the cochain determinant convention above and the corresponding Poincaré convention fixes this sign.

An incidence locus can be a zero or pole locus of the chosen meromorphic frame without being a singularity of the line metric. If $s = fe$, where e is a nonvanishing holomorphic frame, then

$$-\log\|s\| = -\log|f| - \log\|e\|.$$

The second term is smooth across incidence in the Jacobian family just constructed. This global line comparison resolves the normalization and continuity issue for that determinant model.

2.5 The source-curve good-reduction comparison

Let $\mathcal{O}$ be a complete discrete valuation ring with fraction field K_v, equipped with a fixed non-Archimedean absolute value inducing the given discrete valuation. Let $\mathcal{C}/\mathcal{O}$ be a smooth proper geometrically connected curve with a section, let $\mathcal{J}/\mathcal{O}$ be its Jacobian, and let $\mathcal{P}$ be the axis-rigidified Poincaré bundle. For a perfect complex K on $\mathcal{C}$, form the normalized determinant line $\mathcal{D}_{\mathcal{C}}(K)$ on $\mathcal{J}$ by the same algebraic construction.

Proposition 2.4. *Suppose the generic virtual rank of K is zero and* $\deg\det K_{K_v} = 0$. *The model metric of $\mathcal{D}_{\mathcal{C}}(K)$ on the generic fibre is the canonical rigidified Poincaré metric associated with* $\det K_{K_v}$.

Proof. The degree of $\det K$ is zero on every fibre. After rigidification along the section, it defines a point of the relative Pic^0, and hence a rigidified Poincaré extension on $\mathcal{J}$. The rank-zero determinant identity on the generic curve identifies its generic line with $\mathcal{D}_{\mathcal{C}}(K)$, respecting the origin rigidification.

Regard this isomorphism as a rational section of the quotient of the two model lines. Its divisor is vertical. The abelian scheme $\mathcal{J}$ is regular and its special fibre is geometrically integral, so this divisor is $a[\mathcal{J}_s]$ for an integer a. Pullback to the zero section has valuation a, but the two integral unit rigidifications make that valuation zero. Thus $a = 0$. Normality extends the isomorphism and its inverse over the whole model. The model metrics coincide. The Poincaré multiplication laws extend over the abelian scheme and are isometries, identifying this model metric with the canonical one. □

If an ambient model $\mathcal{X}$ and a model-perfect complex on $\mathcal{C} \times \mathcal{X}$ have a perfect derived pushforward to $\mathcal{C}$, the determinant projection formula reduces their model comparison to this proposition. At a bad-reduction source curve the vertical correction involves the geometry of a semistable or Néron model; this is a separate comparison problem.

3 The remaining metric comparisons

The comparisons above provide the complex-place and good-reduction inputs for a local height comparison. The next step is to identify the vertical correction at a place of semistable reduction.

Problem 3.1 (Semistable source curves)**.** Fix a complete discretely valued field K_v with its original absolute value, a pointed smooth projective geometrically connected curve C/K_v, and a semistable model $\mathcal{C}$. Let K be a perfect complex on the model with generic virtual rank zero and $\deg \det K_{K_v} = 0$. Construct the admissible correction to its normalized determinant model metric and identify the resulting rigidified metric with the canonical Poincaré metric attached to $\det K_{K_v}$. For a determinant arising from a curve correspondence, compare this metric with the actual-form Green construction, retaining the finite local lengths, prescribed cycle representatives and native finite-extension degree.

In the smooth-model case, Proposition 2.4 supplies this comparison. At semistable reduction the correction must account for the components and intersection graph. Passage to an actual-form height also uses its trace, correspondence covariance and normalization on the complete actual test space.

The corresponding family problem concerns both the boundary and the growth of the resulting height. One can formulate a precise target without asserting that every family has positive height growth.

Problem 3.2 (Height domination in a family)**.** Let F be a number field, S/F a smooth quasi-projective variety, and $f : \mathcal{X} \to S$ a smooth projective family of relative dimension $2r - 1$. Let Z be a relative codimension r cycle whose fibres are homologically trivial, and assume the fibrewise pairings h_s have been constructed with the fixed absolute height convention, invariant under extension of number fields. Determine geometric hypotheses on (f, Z) under which the function

$$q_Z(s) = (-1)^r h_s(Z_s, Z_s)$$

is the height of a canonically metrized biextension line, has controlled boundary asymptotics on a specified projective compactification $\overline{S}$, and satisfies

$$q_Z(s) \geq \varepsilon h_A(s) - C \qquad (s \in U(\overline{F}))$$

for a nonempty open $U \subset S$, an ample line bundle A on $\overline{S}$, and constants $\varepsilon > 0$, C.

Such an inequality would give finiteness of bounded-height points of bounded degree on U by the classical Northcott theorem for h_A. It is a statement about the parameter space, distinct from Northcott for all integral classes in one fixed Chow group. The relative biextension comparison over a smooth base is [4, Theorem 10.9]. Its singular-fibre extension in Theorem 11.2 requires a modified cycle with a specified cohomological property; Lemma 11.1 obtains that modification under the Hodge conjecture. These conditions belong to the boundary comparison in Problem 3.2.

References

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